Univariate volatility modeling: Part I

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Summary

This week we introduce volatility as a meaningful concept and then describes a widely used framework for volatility analysis: the ARCH model. The material will describe the most widely used members of the ARCH family, fundamental properties of each, estimation and inference.

Contents

Understanding volatility

Why does volatility change?

Time-varying volatility is a pervasive empirical regularity in financial time series.

It is difficult to find an asset return series which does NOT exhibit time-varying volatility.

Many explanations have been given to explain this phenomenon:

- *News announcements*: The arrival of unanticipated news forces agents to update their beliefs.
- *Leverage*: A higher debt-equity ratio makes firm's value more volatile.
- *Illiquidity*: Short-run spells of illiquidity may produce time-varying volatility even when shocks are i.i.d.

What is volatility?

Volatility comes in many shapes and forms. It is critical to distinguish between related but different uses of “volatility”

Volatility is often defined as the standard deviation and is preferred to variance as it is measured in the same *units* as the original data.

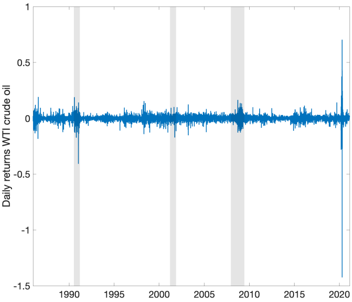
- For e.g., when using percentage returns, the volatility is also measured in percentages.

Other concepts such as **Realised volatility** and **Implied volatility** have been used:

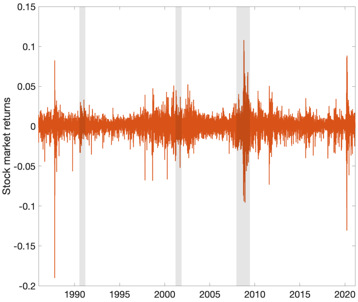
- **Realised volatility** indicates the volatility measured of a given time interval.
- **Implied volatility** indicates the volatility that correctly prices an option.

Stock returns and WTI crude oil

Daily returns on the WTI Crude oil and the stock market



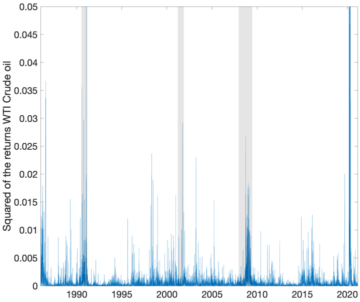
(a) WTI Crude oil



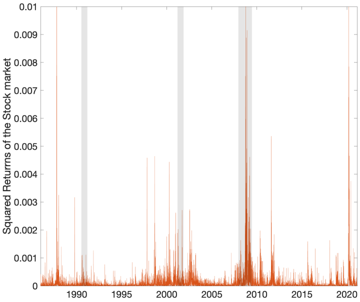
(b) Stock market returns

Stock returns and WTI crude oil

Daily squared returns on the WTI Crude oil and the stock market



(c) WTI Crude oil



(d) Stock market returns

RiskMetrics model

One of the simplest volatility model is the so-called Exponential Weighted Moving Average (EWMA), popularized by RiskMetrics.

The recursive form of an EWMA is

$$\sigma_t^2 = (1-\lambda)\,\epsilon_{t-1}^2 + \lambda\sigma_{t-1}^2$$

which can be equivalently expressed as

$$\sigma_t^2 = (1-\lambda)\sum_{i=0}^{\infty}\lambda^i\epsilon_{t-1-i}^2$$

The weights on the lagged squared returns decay exponentially so that the ratio of two consecutive weights is λ.

The parameter λ is typically set to 0.94 when using monthly returns, 0.97 when using weekly returns and 0.99 when using daily returns.

ARCH and GARCH models

The ARCH model

The complete ARCH(p) model (Engle 1982) relates the current level of volatility to the past P squared shocks.

Definition (ARCH(p))

A p_{th} order ARCH process is given by

$$r_t = \mu_t + \epsilon_t \qquad \text{with} \qquad \epsilon_t = \sigma_t e_t \qquad \text{and} \qquad e_t \sim N(0,1)$$
$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \ldots + \alpha_p \epsilon_{t-p}^2$$

where μ_t can be any adapted model for the conditional mean.

The key feature of this model is that the variance of the shock, ϵ_t, is time varying and depends on past p shocks $\epsilon_{t-1}, \epsilon_{t-2}, \ldots, \epsilon_{t-p}$, thrhough their square value.

The ARCH model

A common alternative description of an ARCH(p) model is

$$
\begin{aligned}
r_t|\mathcal{F}_{t-1} \sim N\left(\mu_t, \sigma_t^2\right), \\
\sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \ldots + \alpha_p \epsilon_{t-p}^2 \\
\epsilon_t &= r_t - \mu_t
\end{aligned}
$$

where μ_t can be any adapted model for the conditional mean.

This reads "r_t given the information set at time $t-1$ is conditionally normal with mean μ_t and variance σ_t^2".

The conditional variance σ_t^2 is

$$
E_{t-1}\left[\epsilon_t^2\right] = E_{t-1}\left[e_t^2 \sigma_t^2\right] = \sigma_t^2 E_{t-1}\left[e_t^2\right] = \sigma_t^2,
$$

The ARCH model

The unconditional variance of an ARCH(p) model is defined as:

$$E\left[\epsilon_{t+1}^2\right] = \overline{\sigma}^2 = \frac{\omega}{1 - \alpha_1 - \alpha_2 - \ldots - \alpha_p}$$

Simple inspection shows that the condition needed to ensure that the unconditional expectation is finite is $1 - \alpha_1 - \alpha_2 - \ldots - \alpha_p > 0$.

One crucial requirement of any ARCH(p) process is that the parameters of the variance evolution, $\alpha_1, \alpha_2, \ldots, \alpha_p$ must all be positive.

The intuition is that if one αs is negative, eventually a shock would be large enough to produce a negative conditional variance and therefore an ill-defined process.

Finally, it is also necessary that $\omega > 0$ to ensure covariance stationary.

The GARCH model

The Generalised ARCH (GARCH) process, introduced by Bollerslev (1986), improves the original specification adding lagged conditional variance, which acts as a *smoothing* term.

A low-order GARCH model typically fits as well as a high-order ARCH.

Definition (GARCH)

A GARCH(p,q) process is defined as

$$r_t = \mu_t + \epsilon_t, \qquad \text{with} \qquad \epsilon_t = \sigma_t e_t \qquad \text{and} \qquad e_t \sim N(0,1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

where μ_t can be any adapted model for the conditional mean.

The GARCH model

As was the case in the ARCH(p) model, the coefficients of a GARCH model must be restricted to ensure the conditional variances are uniformly positive.

In a GARCH(1,1), these restrictions are $\omega > 0, \alpha_1 \geq 0$ and $\beta_1 \geq 0$.

In a general GARCH(p,q) the unconditional variance is

$$\overline{\sigma}^2 = \frac{\omega}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j},$$

The requirements on the parameters for stationarity in a GARCH(1,1) are $1 - \alpha - \beta > 0$ and $\alpha \geq 0$, $\beta \geq 0$ and $\omega > 0$.

The EGARCH model

The Exponential GARCH (EGARCH) model represents a major shift from the ARCH and GARCH models (Nelson 1991).

Rather than model the variance directly, EGARCH models the natural logarithm of the variance.

Definition (EGARCH)

An EGARCH(p,o,q) process is defined as

$$r_t = \mu_t + \epsilon_t, \qquad \text{with} \qquad \epsilon_t = \sigma_t e_t \qquad \text{and} \qquad e_t \sim N(0,1)$$

$$\ln \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \left(\left| \frac{\epsilon_{t-i}}{\sigma_{t-i}} \right| - \sqrt{\frac{2}{\pi}} \right) + \sum_{l=1}^{o} \gamma_l \frac{\epsilon_{t-l}}{\sigma_{t-l}} + \sum_{j=1}^{q} \beta_j \ln \sigma_{t-j}^2,$$

where μ_t can be any adapte model for the conditional mean.

Notice p and o were assumed equal in the original parametrization of Nelson (1991).

The EGARCH model

Rather than working with the complete specification, consider a simpler version, an EGARCH(1,1,1) with a constant mean,

$$r_t = \mu + \epsilon_t, \qquad \text{with} \qquad \epsilon_t = \sigma_t e_t \qquad \text{and} \qquad e_t \sim N(0,1)$$

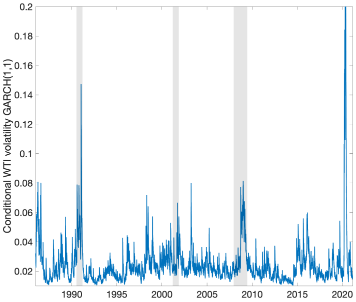
$$\ln \sigma_t^2 = \omega + \alpha \left(\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| - \sqrt{\frac{2}{\pi}} \right) + \gamma \frac{\epsilon_{t-1}}{\sigma_{t-1}} + \beta \ln \sigma_{t-1}^2,$$

In the usual case when $\gamma < 0$, the magnitude of the effect of the past shock $e_{t-1} = \frac{\epsilon_{t-1}}{\sigma_{t-1}}$ is

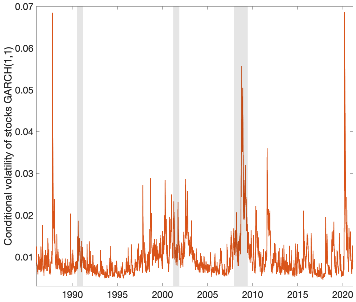
$$\text{Shock coefficient} = \begin{cases} \alpha + \gamma & \text{when} \quad e_{t-1} < 0 \\ \alpha - \gamma & \text{when} \quad e_{t-1} > 0 \end{cases}$$

Stock returns and WTI crude oil

Daily returns volatility on the WTI Crude oil and the stock market



(e) WTI Crude oil



(f) Stock market returns

Volatility estimates for the Stock market

ARCH(5)					
ω	α_1	α_2	α_3	α_4	α_5
0.001	0.104	0.196	0.148	0.168	0.143
(0.000)	(0.001)	(0.006)	(0.012)	(0.010)	(0.011)
GARCH(1,1)					
ω	α_1	β_1			
0.001	0.869	0.112			
(0.000)	(0.004)	(0.003)			
EGARCH(1,1)					
ω	α_1	γ_1	β_1		
-0.282	0.968	-0.116	0.165		
(0.014)	(0.001)	(0.006)	(0.003)		

Volatility estimates for the WTI

ARCH(5)					
ω	α_1	α_2	α_3	α_4	α_5
0.001	0.219	0.153	0.166	0.171	0.129
(0.000)	(0.009)	(0.009)	(0.007)	(0.007)	(0.009)

GARCH(1,1)		
ω	α_1	β_1
0.001	0.875	0.112
(0.000)	(0.004)	(0.003)

EGARCH(1,1)			
ω	α_1	γ_1	β_1
-0.094	0.986	-0.020	0.228
(0.006)	(0.001)	(0.002)	(0.004)

Alternative specifications

Alternative specifications

Many extensions of the ARCH/GARCH models have been introduced to capture important empirical features.

We cover here two important extensions of the ARCH family:

- The GJR-GARCH introduced by Glosten, Jagannathan and Runkle (1993)
- The Threshold ARCH (TARCH)
- The GARCH-in-Mean

Alternative specifications

The GJR-GARCH model extends the standard GARCH(p,q) by adding asymmetric terms that capture common phenomenon in the conditional variance of stocks:

- The propensity of the volatility to rise more after large nagttive shocks than to large positive shocks, i.e., the leverage effect.

Definition (GJR-GARCH)

A GJR-GARCH(1,1,1) process is defined as

$$r_t = \mu_t + \epsilon_t, \qquad \text{with} \qquad \epsilon_t = \sigma_t e_t \qquad \text{and} \qquad e_t \sim N(0,1)$$
$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \gamma\epsilon_{t-1}^2 I_{\{|\epsilon_{t-1}|<0\}} + \beta\sigma_{t-1}^2,$$

where $I_{\{|\epsilon_{t-1}|<0\}}$ is an indicator function that takes value 1 if $\epsilon_{t-1} < 0$ and 0 otherwise.

Parameters restrictions are that $\omega > 0, \alpha \geq 0, \alpha + \gamma \geq 0$ and $\beta \geq 0$.

Alternative specifications

The TARCH makes one fundamental change to the GJR-GARCH, that is it parameterizes the *conditional standard deviation* as a function of the lagged absolute value of the shocks and not the squares.

Definition (TARCH)

A TARCH(1,1,1) process is defined as

$$r_t = \mu_t + \epsilon_t, \qquad \text{with} \qquad \epsilon_t = \sigma_t e_t \qquad \text{and} \qquad e_t \sim N(0,1)$$
$$\sigma_t = \omega + \alpha|\epsilon_{t-1}| + \gamma|\epsilon_{t-1}|I_{\{|\epsilon_{t-1}|<0\}} + \beta\sigma_{t-1},$$

where $I_{\{|\epsilon_{t-1}|<0\}}$ is an indicator function that takes value 1 if $\epsilon_{t-1} < 0$ and 0 otherwise.

Parameters restrictions are that $\omega > 0, \alpha \geq 0, \alpha + \gamma \geq 0$ and $\beta \geq 0$.

Alternative specifications

The GARCH-in-Mean model makes a significant change to the role of time-varying volatility by explicitly relating the level of volatility to the expected return.

A simple GARCH-in-mean model can be specified as

$$r_t = \mu_t + \delta\sigma_t^2 + \epsilon_t, \qquad \text{with} \qquad \epsilon_t = \sigma_t e_t \qquad \text{and} \qquad e_t \sim N(0,1)$$
$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2,$$

Notice that virtually any ARCH-family model could be used to model the conditional variance.

The obvious difference with a standard GARCH(1,1) is that the variance appears in the mean of the return. Yet, because the variance appears in the mean equation, the mean and the variance parameters CANNOT be estimated separately.

The News Impact Curve

With a wide range of volatility models, each with a different specification in the dynamics of conditional variances, it can be difficult to determine the precise effect of a shock to the conditional variance.

News impact curves measure the effect of a shock in the current period on the conditional variance in the subsequent period.

Definition (News Impact Curve)

The News Impact Curve (NIC) of an ARCH-family model is defined as the difference between the variance with a shock e_t and the variance with no shock, i.e.,m $e_t = 0$. The variance in all previous periods is set to the unconditional expectation $\overline{\sigma}^2$,

$$n\left(e_t\right) = \sigma_{t+1}^2\left(e_t | \sigma_t^2 = \overline{\sigma}^2\right),$$
$$NIC(e_t) = n(e_t) - n(0),$$

The News Impact Curve

The NIC for the GARCH only depends on the terms which include ϵ_t^2,

$$n(e_t) = \omega + \alpha\overline{\sigma}^2 e_t^2 + \beta\overline{\sigma}^2$$

such that

$$NIC(e_t) = \alpha\overline{\sigma}^2 e_t^2,$$

Stock returns and WTI crude oil