Univariate volatility modeling: Part II

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Summary

This week we expand the analysis to univariate volatility models. In particular, we are going to investigate how to build volatility models, how to forecast volatility as well as both realised and implied volatility.

Contents

Model building

Model building

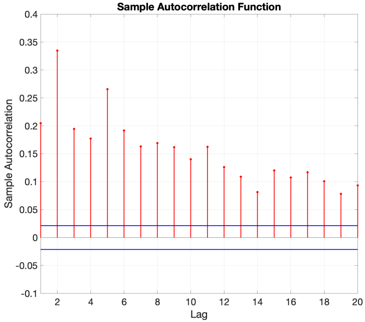
Since ARCH and GARCH models are similar to AR and ARMA models, the Box-Jenkins methodology is a natural way to approach the problem.

The first step is to analyze the sample ACF and PACF of the squared returns.

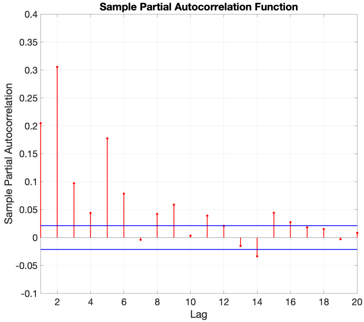
Alternatively, if the model for the conditional mean is non-trivial, the sample ACF and PACF of the estimated residuals, $\hat{\epsilon}_t$, should be examined for heteroskedasticity.

Model building

Squared of market returns



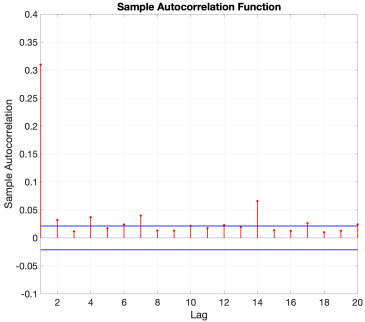
(a) ACF



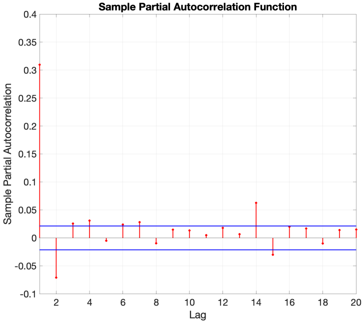
(b) PACF

Model building

Squared WTI returns

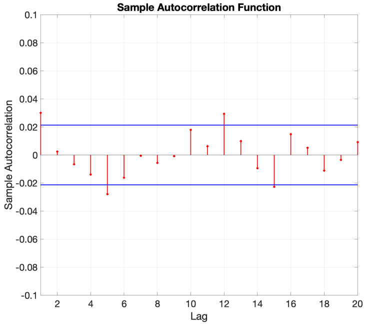


(c) ACF

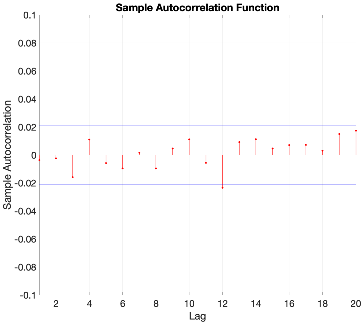


(d) PACF

Squared of standardized returns $z_t = \epsilon_t / \sigma_t$



(e) S&P500



(f) WTI

Testing for (G)ARCH

Although conditional heteroskedasticity can often be identified by graphical inspection, a formal test of conditional homoskedasticity is also helpful.

The standard method to test for ARCH is to use the ARCH-LM test which is implemented as a regression of *squared* residuals on lagged residuals.

The test directly exploits the AR representation of an ARCH process (Engle 1982) and is computed as T times the $R^2 (LM = T \times R^2)$ from the regression:

$$\hat{\epsilon}_t^2 = \phi_0 + \phi_1 \hat{\epsilon}_{t-1}^2 + \ldots + \phi_p \hat{\epsilon}_{t-p}^2 + \eta_t,$$

The test statistic is asymptotically distributed as a χ_p^2 where $\hat{\epsilon}_t$ are residuals constructed from the returns by subtracting the conditional mean.

The null hypothesis is $H_0 : \phi_1 = \ldots = \phi_p = 0$ which corresponds to no persistence in the conditional variance.

Forecasting volatility

Forecasting volatility with ARCH

Forecasting conditional variances with ARCH-family models ranges from simple for ARCH and GARCH processes, to difficult for non-linear specific-ations.

Consider the simple ARCH(1) process,

$$\epsilon_t = \sigma_t e_t, \qquad \text{with} \qquad e_t \overset{iid}{\sim} N(0,1),$$
$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2,$$

Iterating forward, $\sigma_{t+1}^2 = \omega + \alpha_1 \epsilon_t^2$, and taking conditional expectations, $E_t\left[\sigma_{t+1}^2\right] = E_t\left[\omega + \alpha_1 \epsilon_t^2\right] = \omega + \alpha_1 \epsilon_t^2$, since all of these quantities are known at time t.

Forecasting volatility with ARCH

The 2-step ahead forecasts follows from the law of iterated expectations,

$$E_t\left[\sigma^2_{t+2}\right] = \omega + \alpha_1 E_t\left[\epsilon^2_{t+1}\right] = \omega + \alpha_1\omega + \alpha_1^2\epsilon_t^2,$$

The expression for an h-step ahead forecast can be constructed by repeated substitution and is given by

$$E_t\left[\sigma^2_{t+h}\right] = \sum_{i=0}^{h-1}\alpha_1^i\omega + \alpha_1^h\epsilon_t^2,$$

An ARCH(1) *is* an AR(1), and this formula is identical to the expression for the multi-step forecast of an AR(1).

Forecasting volatility with GARCH

Forecast from GARCH(1,1) models are constructed following the same steps. The one-step ahead forecast is,

$$E_t\left[\sigma^2_{t+1}\right] E_t\left[\omega + \alpha_1\epsilon^2_t + \beta_1\sigma^2_t\right] = \omega + \alpha_1\epsilon^2_t + \beta_1\sigma^2_t,$$

The two-step ahead forecast is

$$E_t\left[\sigma^2_{t+2}\right] = \omega + (\alpha_1 + \beta_1) E_t\left[\sigma^2_{t+1}\right],$$

Repeated substitution reveals a pattern in the multi-step forecasts,

$$E_t\left[\sigma^2_{t+h}\right] = \sum_{i=0}^{h-1} (\alpha_1 + \beta_1)^i \omega + (\alpha_1 + \beta_1)^{h-1} \left(\alpha_1\epsilon^2_t + \beta_1\sigma^2_t\right),$$

Evaluating volatility forecasts

In standard time series models, once time $t + h$ has arrived, the value of the variable being forecast is known.

However, the value of σ^2_{t+h} is always unknown in volatility model evaluation and so the realization must be replaced by a proxy.

The standard choice is to use the squared return, r_t^2. An alternative choice is the *Realised Variance*, $RV_t^{(m)}$ (see later in the slides).

Once a choice of proxy has been made, the Generalized Mincer-Zarnowitz regressions can be used to assess forecast optimality,

$$r^2_{t+h} - \hat{\sigma}^2_{t+h|t} = \gamma_0 + \gamma_1 \hat{\sigma}^2_{t+h|t} + \gamma_2 z_{1t} + \ldots + \gamma_{K+1} z_{Kt} + \eta_t,$$

where z_{jt} are any instruments known at time t.

Evaluating volatility forecasts

Common choices for z_{jt} include $r_t^2, |r_t|, r_t$ or indicator variables for the sign of the lagged return.

The GMZ regression is testing one key property of a well-specified model:

$$E\left[r_{t+h}^2 - \hat{\sigma}_{t+h|t}^2\right] = 0,$$

If one use the Realized Variance as the proxy, the GMZ regression is:

$$RV_{t+h} - \hat{\sigma}_{t+h|t}^2 = \gamma_0 + \gamma_1 \hat{\sigma}_{t+h|t}^2 + \gamma_2 z_{1t} + \ldots + \gamma_{K+1} z_{Kt} + \eta_t,$$

Evaluating volatility forecasts

Diebold-Mariano tests can also be used to test the relative performance of two models.

A loss function must be specified when implementing a DM test.

Two natural choices for the loss function are MSE,

$$\left(r_{t+h}^2 - \hat{\sigma}_{t+h|t}^2\right)^2,$$

and a loss function based on the kernel from a normal log-likelihood

$$\left(\ln\left(\hat{\sigma}_{t+h|t}^2\right) + \frac{r_{t+h}^2}{\hat{\sigma}_{t+h|t}^2}\right),$$

Evaluating volatility forecasts

The DM statistic is a t-test of the null $H_0 : E\left[\delta_t\right] = 0$ where:

$$\delta_t = \left(r_{t+h}^2 - \hat{\sigma}_{A,t+h|t}^2\right)^2 - \left(r_{t+h}^2 - \hat{\sigma}_{B,t+h|t}^2\right)^2,$$

in the case of the MSE loss or

$$\delta_t = \left(\ln\left(\hat{\sigma}_{A,t+h|t}^2\right) + \frac{r_{t+h}^2}{\hat{\sigma}_{A,t+h|t}^2}\right)^2 - \left(\ln\left(\hat{\sigma}_{B,t+h|t}^2\right) + \frac{r_{t+h}^2}{\hat{\sigma}_{B,t+h|t}^2}\right)^2,$$

when using the alternative loss function.

Statistically significant positive values of $\overline{\delta} = R^{-1}\sum_{r=1}^{R}\delta_r$ indicate that B is a better model than A, while negative values indicate the opposite.

Realised variance

Realized Variance

Realised Variance (RV) is a relatively new econometric methodology for measuring the variance of asset returns.

RV differs from ARCH-models since it does not require a specific model to measure the volatility.

RV instead uses a non-parametric estimator of the variance that is computed *using ultra high-frequency data*.

Define $p_t = \ln(P_t)$ as the log-price. The RV is estimated by sampling p_t throughout the trading day.

Suppose that prices on day t were sampled on a regular grid of $m+1$ points, $0, 1, \ldots m$ and let p_{it} denote the ith observation of the log price. The m-sample RV on day t is defined

$$RV_t^{(m)} = \sum_{i=1}^{m} \left(p_{i,t} - p_{i-1,t}\right)^2 = \sum_{i=1}^{m} r_{it}^2,$$

Implementing the RV

The most pronounced challenge is that observed prices are contaminated by noise; there is no singles price and traded prices are only observed at the bid and the ask.

Consider a simple model of the bid-ask bounce where returns are computed as the log difference in observed prices composed of the true efficient prices, p^*_{it} contaminated by an independent mean zero shock, v_{it},

$$p_{it} = p^*_{it} + v_{it},$$

The shock v_{it} captures the difference between the efficient price and the observed prices which are always on the bid or ask price. This the ith observed return, r_{it} can be decomposed as

$$r_{it} = r^*_{it} + \eta_{it},$$

where r^*_{it} the actual return and $\eta_{it} = v_{it} - v_{i-1,t}$ the independent noise.

Implementing the RV

Computing the RV from returns contaminated by noise has an unambiguous effect on RV, it is biased upward,

$$
\begin{aligned}
RV_t^{(m)} &= \sum_{i=1}^{m} r_{it}^2 = \sum_{i=1}^{m} \left(r_{it}^* + \eta_{it}\right)^2, \\
&= \sum_{i=1}^{m} r_{it}^* + 2r_{it}^*\eta_{it} + \eta_{it}^2 \approx \widehat{RV}_t + m\tau^2,
\end{aligned}
$$

where τ^2 is the variance of η_{it} and $\widehat{RV}_t$ is the RV that would be computed if the efficient returns could be observed.

The bias is increasing in the number of samples (m) and can be substantial for assets with large bid-ask spreads.

Implementing the RV

Two simple methods can be used to mitigate the bias. First, one can filter the noise from the data using an MA(1).

Transaction data contain a strong negative MA due to the bid-ask bounce, and so RV computed using the errors $\hat{\epsilon}_{it}$ from a model,

$$r_{it} = \theta\epsilon_{i-1t} + \epsilon_{it},$$

eliminates much of the bias.

A better method to remove the bias is to use an estimator known as RV^{AC1} which is similar to Newey-West estimator,

$$RV_t^{AC1(m)} = \sum_{i=1}^{m} r_{it}^2 + 2\sum_{i=2}^{m} r_{it}r_{i-1t},$$

Modeling RV

If RV is observable, then it can be modeled using standard time series tools such as ARMA models.

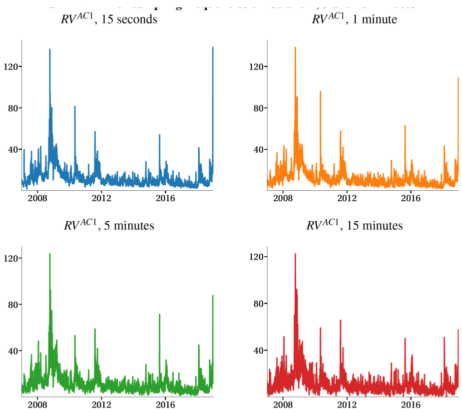
Corsi (2009) proposed the heterogeneous autoregression (HAR) as a simple method to capture the dynamics in RV in a parsimonious model.

The standard HAR models the RV as a function of the RV in the previous day, the average RV over the previous week, and the average RV over the previous month (22 days), i.e.,

$$RV_t = \phi_0 + \phi_1 RV_{t-1} + \phi_5 \overline{RV}_{t-5} + \phi_{22} \overline{RV}_{t-22} + \epsilon_t,$$

N.B.: HARs is technically AR(22) models with many parameter restrictions.

Realized Variance of the S&P500



Source: Financial Econometrics Notes by Kevin Shepard.

Implied Volatility

Implied volatility

Implied volatility differs from other measures in that it is both market-based and forward-looking.

Implied volatility was originally conceived as the "solution" to the Black-Scholes options pricing formula where all values except the volatility are observable.

Under some additional assumptions sufficient to ensure no arbitrage, the price of a call option can be shown to be,

$$\begin{aligned} C_t\left(T,K\right) &= S_t\Phi\left(d_1\right) + Ke^{-rT}\Phi\left(d_2\right), \\ d_1 &= \frac{\ln\left(S_t/K\right) + \left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}}, \\ d_2 &= \frac{\ln\left(S_t/K\right) + \left(r-\sigma^2/2\right)T}{\sigma\sqrt{T}}, \end{aligned}$$

where K is the strike price, T is the time to maturity, in years, r is the risk-free interest rate, and $\Phi\left(\cdot\right)$ is the normal CDF.

Implied volatility

The price of a call option is monotonic in the volatility, and so the formula can be inverted to express the volatility as a function of the call price and other observable.

The implied volatility,

$$\sigma_t^{\text{implied}} = g\left(C_t\left(T, K\right), S_t, K, T, r\right),$$

is the expected volatility between t and T under the risk-neutral measure (which is the same as under the physical when volatility is constant).

The VIX

The VIX - volatility index - is a volatility measure produces by the Chicago Board Options Exchange (CBOE).

It is computed using a “model-free” like estimator which uses both call and put prices.

The VIX is computed as

$$\sigma^2 = \frac{2}{T} \exp\left(rT\right) \sum_{i=1}^{N} \frac{\Delta K_i}{K_i^2} Q\left(K_i\right) - \frac{1}{T}\left(\frac{F_0}{K_0} - 1\right)^2,$$

where T is the time to expiration of the options used, F_0 is the forward price which is computed from index option prices, K_i is the strike of the i_{th} out-of-the-money option, $\Delta_{Ki} = \left(K_{i+1} - K_{i-1}\right)/2$, r is the risk-free rate and $Q\left(K_i\right)$ is the mid-point of the bid and ask for the call or put used at strike K_i.

The VIX

The VIX is consistently higher than the forward volatility.

Forward volatility is computed as

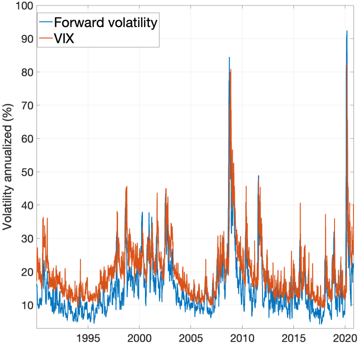
$$\sigma_t^{MA} = \sqrt{\frac{252}{22}\sum_{i=0}^{21} r_{t+i}^2},$$

This relationship highlights both a feature and a drawback of the VIX:

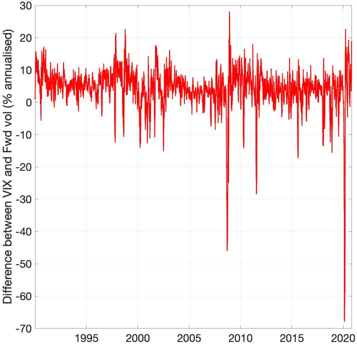
- It captures a (possibly) time-varying risk premium.
- This risk premium captures investor compensation for changes in volatility and jump risks.

The VIX

Forward volatility vs VIX



(g) VIX and Fwd Vol



(h) VIX minus Fwd Vol