Value-at-Risk and Expected Shortfall

Daniele Bianchi[1]

whitesphd.com

[1]School of Economics and Finance
Queen Mary, University of London

Summary

This week we introduce measures of financial risk management such as the Value-at-Risk (VaR) and the Expected Shortfall (ES). These measures are heavily based on the material covered so far, in particular volatility modeling.

Contents

Defining risk

Defining risk

Portfolios are exposed to multiple distinct sources of risk. The most important sources of risk can be classified into one of six categories:

- **Market risk:** Uncertainty about the future price of an asset due to changes in fundamentals or beliefs.
- **Liquidity risk:** Measure the loss involved if a position must be rapidly unwound. Liquidity risk is distinct from market risk since it represents a transitory distortion due to buying/selling pressure.
- **Credit risk:** Credit risk, also known as default risk, covers cases where a 3^{rd} party is unable to pay per previously agreed terms.

Defining risk

- **Counterparty risk:** Counterparty risk extends credit risk to instruments other than bonds and captures the event that a counterparty to a transaction, for example, the seller of an option contract is unable to complete the transaction at expiration.
- **Model risk:** Model risk represents an econometric form of risk that measures the uncertainty about the correct form of the model used to compute the price of the asset or the asset's riskiness.
- **Estimation risk:** Estimation risk captures an aspect of risk that is present whenever estimated parameters are used in econometric models to price securities or assess risk. Estimation risk is distinct from model risk since it is present even if a model is correctly specified. However, model and estimation risk are always present and are generally substitutes.

Value-at-Risk

Value-at-Risk defined

The most widely reported measure of risk is Value-at-Risk (VaR).

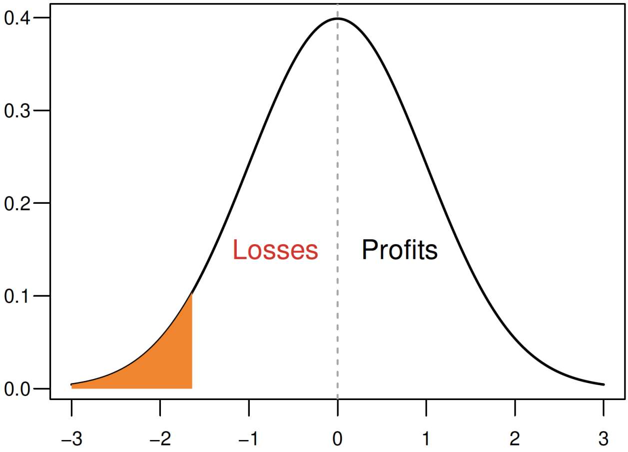
The VaR of a portfolio is a measure of the risk in the left tail of portfolio's return over some period, often a day or a week.

It provides a more sensible measure of the risk of the portfolio than variance since it focuses on losses.

The VaR of a portfolio measures the value (eg., in USD/EUR/GBP) which an investor would lose with some small probability, usually between 1 and 10%, over a specified horizon.

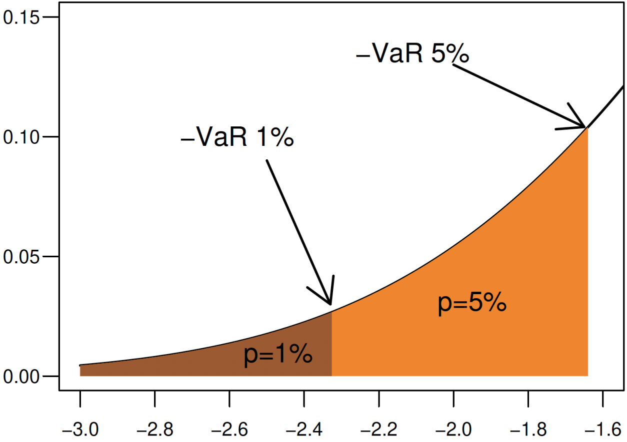
Value-at-Risk defined

Figure 1: VaR graphically



Value-at-Risk defined

Figure 2: VaR graphically



Value-at-Risk defined

Definition (Value-at-Risk)

The α Value-at-Risk (VaR) of a portfolio is defined as the largest change in the portfolio such that the probability that the loss in portfolio value over a specified horizon is greater than the VaR is α,

$$\Pr\left(R_t < -VaR\right) = \alpha,$$

where $R_t = W_t - W_{t-1}$ is the change in the value of the portfolio, W_t and the time span depends on the application.

For example, if an investor had a portfolio value of £10,000,000 and had a daily portfolio return which was $N\left(0.001, 0.015^2\right)$ (annualised mean of 25%, volatility of 23.8%), the daily α VaR of this portfolio is

$$\text{£}10,000,000\left(-0.001 - 0.015\Phi^{-1}\left(\alpha\right)\right) = \text{£}236,728.04$$

where $\Phi^{-1}\left(\cdot\right)$ is the inverse CDF of a standard normal.

Value-at-Risk defined

Definition (% Value-at-Risk)

The α percentage VaR of a portolio (%VaR) is defined as the largest return such that the probability that the return on the portfolio over a specified horizon is less than $-1 \times \%$VaR is α,

$$\Pr\left(r_t < -\%\text{VaR}\right) = \alpha,$$

where $r_t = \left(W_t - W_{t-1}\right)/W_{t-1}$ is the return on the portfolio. %VaR can be equivalently defined as %VaR$/W_{t-1}$.

Since %VaR and VaR only differ by the current value of the portfolio, the remainder of the lecture focuses on %VaR.

Conditional VaR

Most applications of VaR are used to measure risk over short horizons, and so require a conditional VaR.

Definition (Conditional VaR)

The conditional α VaR is defined as

$$\Pr\left(r_{t+1} < -\text{VaR}_{t+1|t}|\mathcal{F}_t\right) = \alpha,$$

where $r_{t+1} = \left(W_{t+1} - W_t\right)/W_t$ is the time $t+1$ return on a portfolio.

Conditioning employs information up to time t to produce a VaR in period $t+h$.

Most conditional models for VaR forecast the density directly, although some only attempt to estimate the rquired quantile.

Four standard methods are presented in the order of strength of the assumptions required, from strongest to weakest:

– RiskMetrics (JP Morgan 1997)
– Parametric GARCH models
– Filtered historical simulation
– Cornish-Fisher approximation

RiskMetrics

The RiskMetrics group has produces a simple, yet robust method for producing conditional VaR.

The estimate of the portfolios's variance is

$$\sigma_{t+1}^2 = (1 - \lambda)\, r_t^2 + \lambda \sigma_t^2,$$

where r_t is the (percentage) return on the portolio period t.

The VaR is constructed from the α-quantile of a normal distribution,

$$\text{VaR}_{t+1} = -\sigma_{t+1} \Phi^{-1}(\alpha),$$

where $\Phi^{-1}(\cdot)$ is the inverse normal CDF.

RiskMetrics

Few things are worth to mention:

- The RiskMetrics does not include a conditional mean of returns, and so is applicable to assets with returns that are close to zero.

- The restriction limits the applicability to applications where the risk-measurement horizon is short, e.g., daily to monthly.

- λ has been calibrated to 0.94 for daily data, 0.97 for weekly data, and 0.99 for monthly data.

- This model can also be extended to multiple assets by replacing the squared return with the outer product of returns $\boldsymbol{r}_t\boldsymbol{r}_t'$, and σ^2_{t+1} with a matrix Σ_{t+1}.

Parametric GARCH models

Parametric GARCH-family models provide a complete description of the future return distribution, and so can be applied to estimate the VaR of a portfolio.

For example, let's consider a simple GARCH(1,1),

$$
\begin{aligned}
r_{t+1} &= \mu + \epsilon_{t+1}, \qquad \text{with} \qquad \epsilon_{t+1} = \sigma_{t+1} e_{t+1} \\
\sigma^2_{t+1} &= \omega + \alpha \epsilon^2_t + \beta \sigma^2_t,
\end{aligned}
$$

The conditional VaR at time t can be calculated as

$$
\mathsf{VaR}_{t+1} = -\hat{\mu} - \hat{\sigma}_{t+1} F^{-1}_{\alpha},
$$

where F^{-1}_{α} is the α-quantile of the distribution of e_{t+1}.

Filtered Historical Simulation

Filtered historical simulation mixes parametric mean and variance models with non-parametric estimators of the distribution.

Assume $e_{t+1} \sim G(0,1)$, where $G(0,1)$ is an unknown distribution with mean zero and variance one,

The conditional VaR at time t can be calculated as

$$\mathsf{VaR}_{t+1} = -\hat{\mu} - \hat{\sigma}_{t+1} G_{\alpha}^{-1},$$

where G_{α}^{-1} is the empirical α-quantile of the standardized returns e_{t+1}.

To estimate this quantile, define $\hat{e}_{t+1} = \hat{\epsilon}_{t+1}/\hat{\sigma}_{t+1}$, and sort the errors such that

$$\hat{e}_1 < \hat{e}_2 < \ldots < \hat{e}_{n-1} < \hat{e}_n$$

The estimate of G^{-1} is the α-quantile of the empirical distribution $\hat{e}_{t+1}$ which is the value in position αn of the ordered standardized residuals.

Filtered Historical Simulation

Filtered historical simulation provide one clear advantage over the parametric GARCH,

- The quantile, and hence the VaR, is consistent under weaker conditions since the density of the standardized residuals does not have to be assumed.

The primary disadvantage of the semiparametric approach is that $\hat{G}_{\alpha}^{-1}$ may be poorly estimated, especially if α is very small.

Cornish-Fisher Approximation

The Cornish-Fisher estimator of VaR lies between fully parametric model and the FHS. The setup is identical to that of the FHS.

The CF approximation is a Taylor-series-like expansion of the α-quantile around the α-quantile of a Normal and is given by

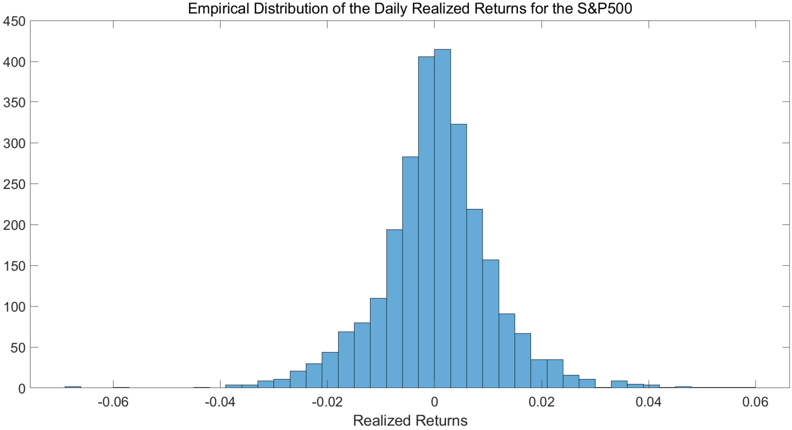
$$\mathsf{VaR}_{t+1} = -\mu - \sigma_{t+1} G_{CF}^{-1}(\alpha),$$

with

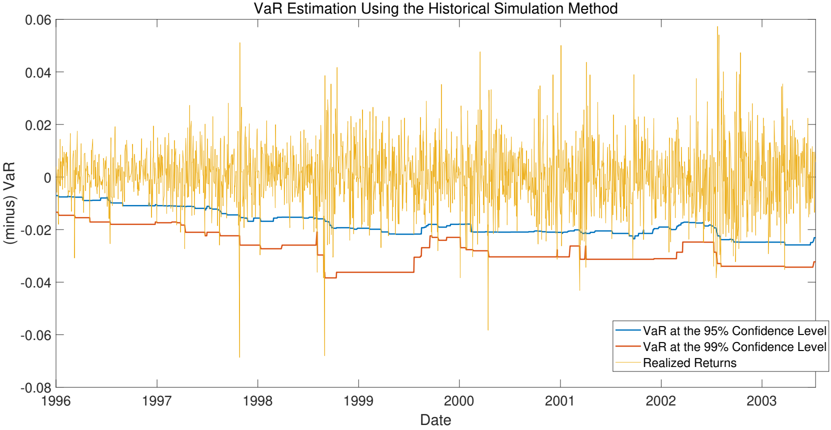
$$G_{CF}^{-1}(\alpha) = \Phi^{-1}(\alpha) + \frac{\zeta}{6}\left(\left[\Phi^{-1}(\alpha)\right]^2 - 1\right) + \frac{\kappa - 3}{24}\left(\left[\Phi^{-1}(\alpha)\right]^3 - 3\Phi^{-1}(\alpha)\right) - \frac{\zeta^2}{36}\left(2\left[\Phi^{-1}(\alpha)\right]^3 - 5\Phi^{-1}(\alpha)\right),$$

where ζ and κ are the skewness and kurtosis of $\hat{e}_{t+1}$, respectively.

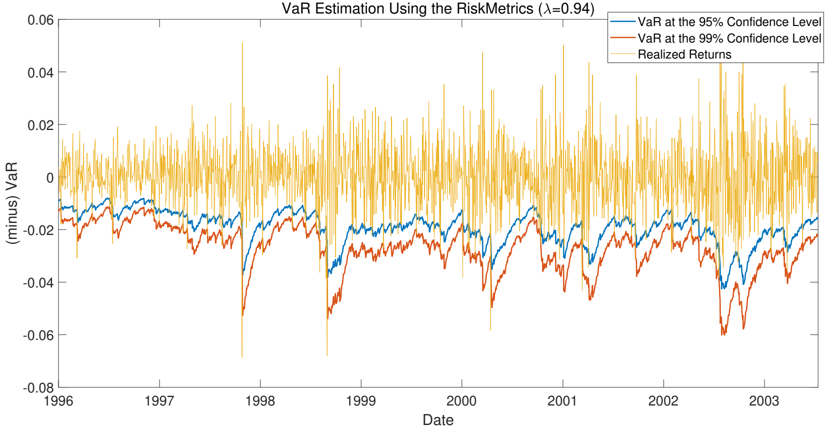
Example: Conditional VaR for the S&P500



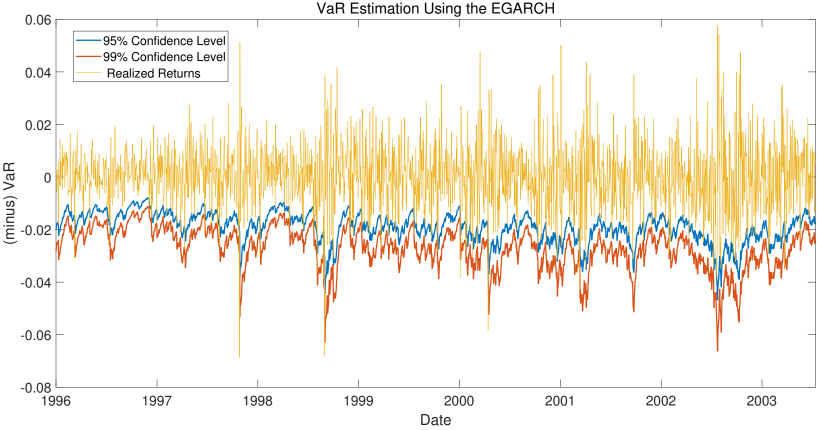
Example: Conditional VaR for the S&P500



Example: Conditional VaR for the S&P500



Example: Conditional VaR for the S&P500



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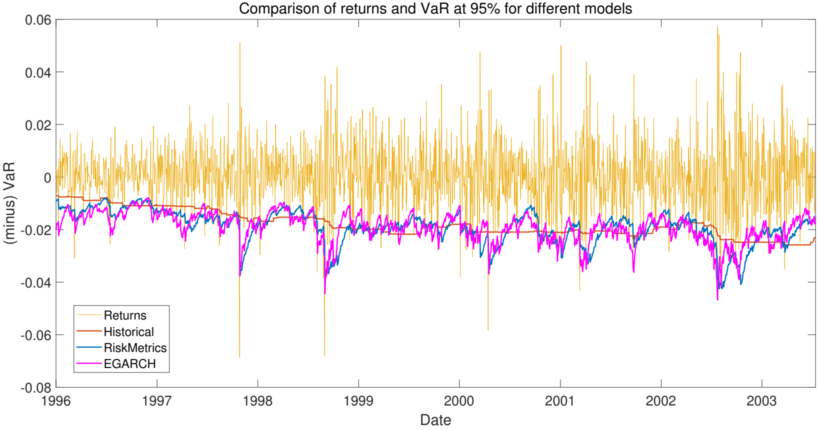
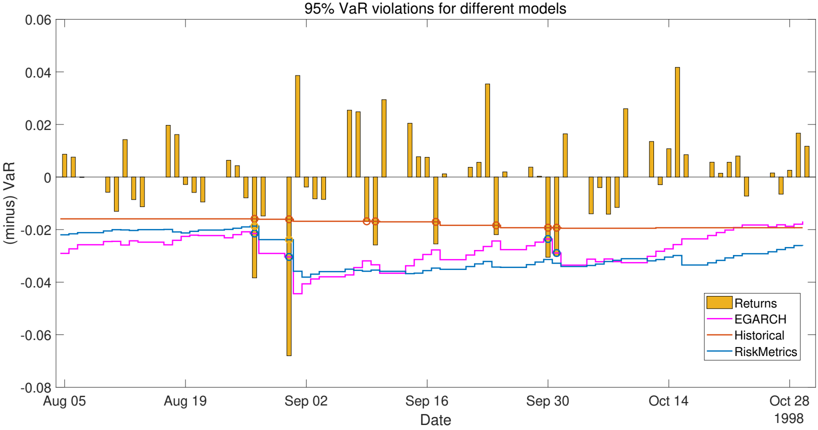


Example: Conditional VaR for the S&P500



Unconditional VaR

While the Conditional VaR is often the object of interest, there may be situations which call for the unconditional VaR.

Unconditional VaR expands the set of choices from the conditional to include models that do not make use of conditioning information to estimate the VaR directly from the historical return data.

We look at two different methods:

- Parametric estimation
- Historical simulation

Parametric estimation

The simplest form of unconditional VaR specifies a complete parametric model for the unconditional distribution of returns.

The VaR is then computed from the α-quantile of this distribution.

For example, if $r_t \sim N\left(\mu, \sigma^2\right)$, then the α-VaR is

$$\text{VaR} = -\mu - \sigma\Phi^{-1}\left(\alpha\right),$$

The parameters of the distribution are estimated using Maximum likelihood with the usual estimators,

$$\hat{\mu} = T^{-1}\sum_{t=1}^{T} r_t, \qquad \hat{\sigma}^2 = T^{-1}\sum_{t-1}^{T}\left(r_t - \hat{\mu}\right)^2,$$

In a general parametric VaR model, some distribution of returns which depends on a set of unknown parameters θ is assumed, $r_t \sim F\left(\theta\right)$.

Historical simulation

At the other hand of the spectrum is a simple nonparametric estimate of the unconditional VaR known as Historical Simulation (HS).As was the case for the FHS in the conditional VaR case, the first step is to sort returns so that

$$r_1 < r_2 < \ldots < r_{n-1} < r_n$$

The estimate of the VaR is the α-quantile of the empirical distribution of r_t,

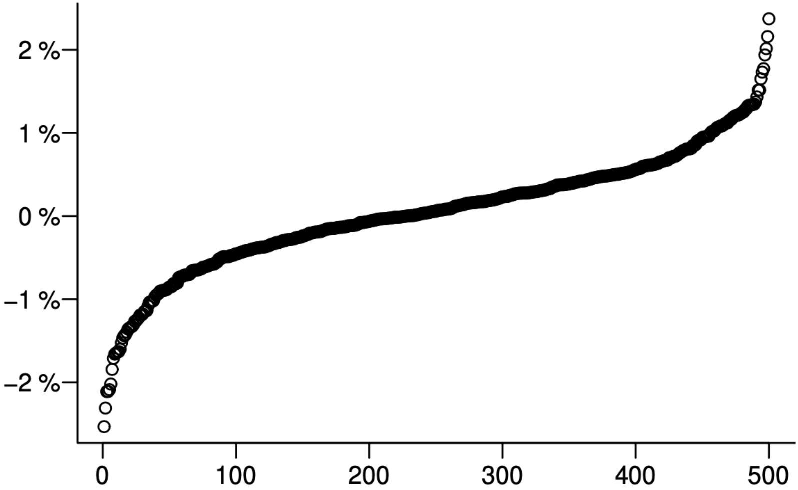
$$\text{VaR} = -\hat{G}_\alpha^{-1}$$

where $\hat{G}_\alpha^{-1}$ is the estimated quantile.

The pros is that the HS is easy to estimate, the cons are that the HS is sensible to outliers and you often need large samples.

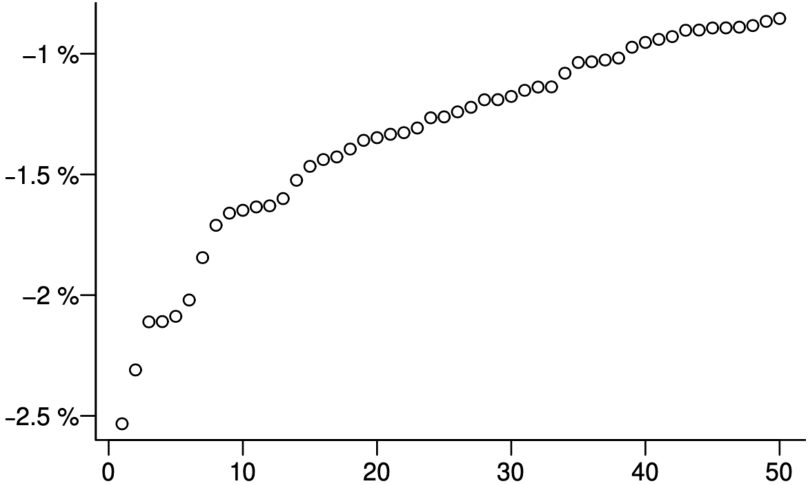
Example: Unconditional VaR for the S&P500

Figure 3: Sorted returns



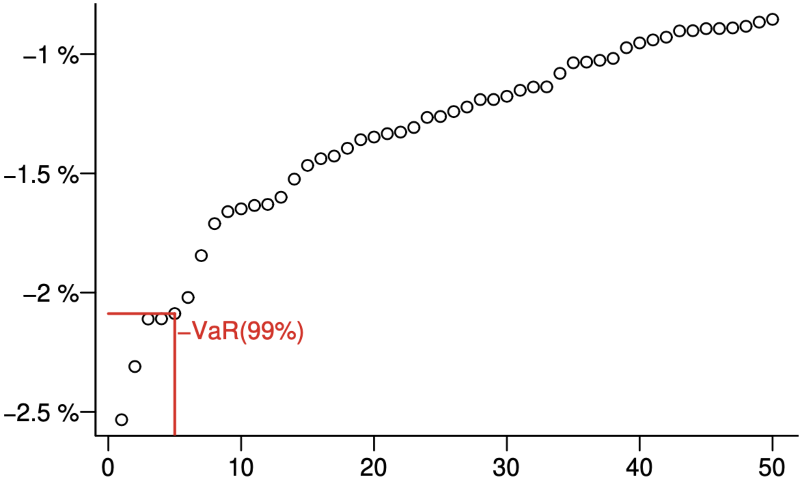
Example: Unconditional VaR for the S&P500

Figure 4: Zoom into sorted returns



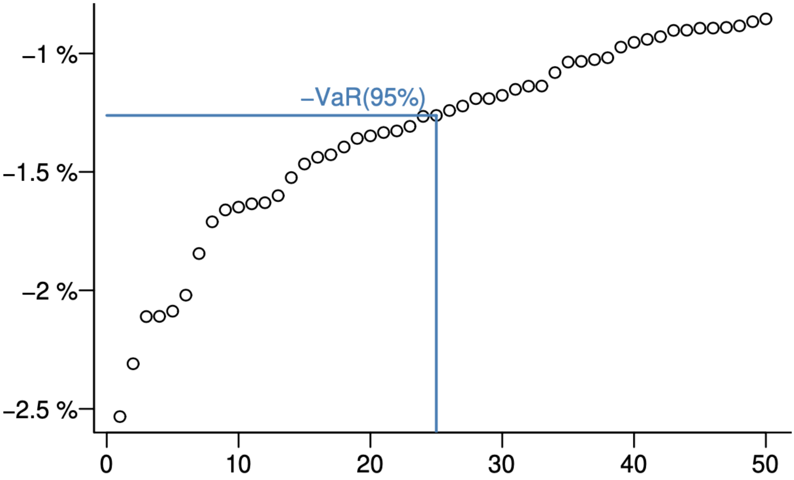
Example: Unconditional VaR for the S&P500

Figure 5: Zoom into sorted returns



Example: Unconditional VaR for the S&P500

Figure 6: Zoom into sorted returns



Expected shortfall and QQ-plots

Expected shortfall

The Expected Shortfall (ES) – also known as tail VaR – combines aspects of VaR with additional information about the distribution of returns in the tail.

Definition (Expected shortfall)

Expected Shortfall (ES) is defined as the expected value of the portfolio loss *given* a Value-at-Risk exceedance has occured, i.e.,

$$\mathsf{ES} = E\left[r_{t+1} | r_{t+1} < -\mathsf{VaR}\right],$$

Expected shortfall

The conditional, and generally more useful, Expected Shortfall is similarly defined.

Definition (Conditional Expected shortfall)

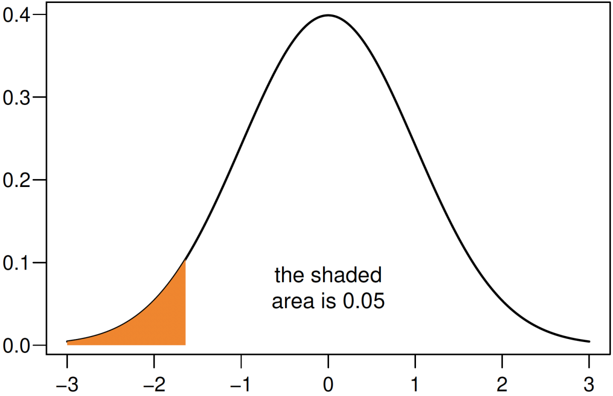
Conditional Expected Shortfall is defined as

$$\mathrm{ES}_{t+1} = E_t\left[r_{t+1} | r_{t+1} < -\mathrm{VaR}\right],$$

where r_{t+1} return on a portfolio at time $t+1$.

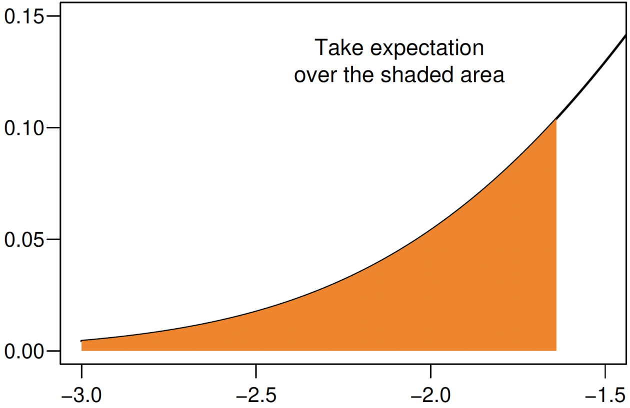
Expected shortfall

Figure 7: VaR graphically



Expected shortfall

Figure 8: VaR graphically



Quantile-quantile plots

A Quantile-Quantile, or QQ, plot is a graphical tool that is used to assess the fit of a density.

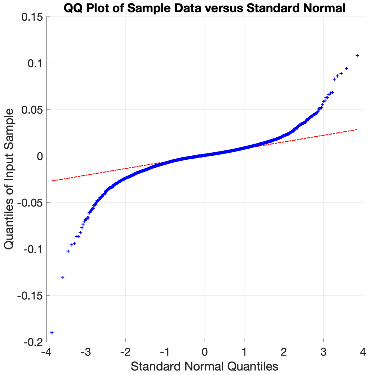
Suppose a set of standardized residuals $\hat{e}_t$ are assumed to have a distribution F.

The QQ plot is generated by ordering the standardized residuals,

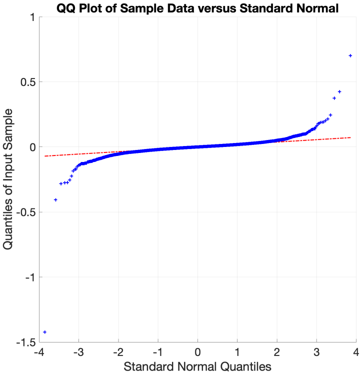
$$\hat{e}_1 < \hat{e}_2 < \ldots < \hat{e}_{n-1} < \hat{e}_n$$

and then plotting the ordered residuals $\hat{e}_j$ (x-axis) against its hypothetical value (y-axis) if the correct distribution were F.

Quantile-quantile plots



(a) S&P500



(b) WTI Crude oil