Interference in Small Cell Networks

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1 Mathematical Background

The purpose of this document is to provide the mathematical foundation to understand analytical network capacity analysis under spatial structures using point processes (PP). To understand this analysis requires a significant amount of background understanding in point processes and how they have been used in the literature to model wireless network interference. Using spatial geometry has significant advantages for interference modeling as it provides analytic tools for characterization. Networks can be modeled over spatial distributions rather than single realizations. This provides more generalized results, over any network configuration. This does come at the cost of mathematical convenience.

We will first begin with an introduction to PP and how we relate them to the physical structure of the network. Next we will introduce a mathematical description of the network interference through a concept called shot noise (SN), based on the original work of Schottkey in 1918. Then with SN based interference we will derive network wide performance metric and bounds. Finally we will introduce medium access control (MAC), and the associated mathematically model in this PP framework. Slotted Aloha, CSMA, FDMA, and TDMA will be discussed and compared.

Tractability has been a primary concern of existing work, resulting in limited availability of many realistic medium access control scenarios. This work will only utilize Poisson PP since they can provide accurate results for many scenarios, and are less mathematically cumbersome then their hardcore (HC) counter parts which have been used in other works [1,2]. HCPP's can provide more elegant, or accurate, solutions to such interference modeling, but the analysis can become overwhelming. In this work, since we are more interested in the relative performance of certain MAC protocols. Therefore we will trade-off modeling accuracy for a more direct comparison of protocols. Secondly, we only require use of such models to provide guidance into design of interference mitigation techniques, no exact performance of existing networks.

1.1 Point Process Networks

The network we are modeling contains an infinite number of transmitters denoted by the homogeneous Poisson PP (PPP) Φ, with intensity measure λ. A point process is simply a collection of random variables, $x_1, ..., x_n \in \Phi$, spatial distributed on a plain according to specified distribution. A PPP can be informally understood as a PP with uniformly spaced points in an area B, with the number of points in areas of size B being Poisson distributed with the mean number of points in an area $\lambda|B|$, or equivalently(1).

$$\mathbf{P}(\Phi(B) = k) = \frac{1}{k!} e^{-\lambda|B|} (\lambda|B|)^k \tag{1}$$

For simplicity we will only be working with two dimensional plains, points $x_i \in \Phi$ and $x \in \mathbb{R}^2$, avoiding much of the confusion introduced by heavy measure theory based work. λ is a first order statistical property of a PP, defined as the mean number of points/events N per unit area/volume v in any set B [3]. Equation (2) show this relation, where $\mathbf{E}$ represents the expectation operation. This is analogous to density of points in region or area.

$$\lambda = \frac{\mathbf{E}[N(B)]}{v(B)} \tag{2}$$

We will discuss the application of Small Cells (SC) in the SG framework. We will only be considering co-channel interference among SC, assuming they are operating in disjoint channels from other possible interferers. The SC's are spatial arrays by the points of the PPP previously discussed. SC's are assumed to be uncoordinated nodes that are uniformly spread across the environment. Such nodes are uncoordinated having no centralized decision making. Each SC will have a single associated receiver, and only downlinks to those receivers considered in this work. The spacing between SC and associated receiver is at distance u, and from the uniformly random spacing of the SC's, it may not necessarily be the closest SC to that receiver. An example realization of this process can be seen in Figure 1.1.

1.2 Interference

Now that we have an understanding on the physical placement of nodes in the network we will address the concept of interference. A common metric used to cellular networks is signal to interference plus noise ratio (SINR). Assuming constant power P of each network node and noise power N, the SINR at each receiver is shown in equation (3).

$$SINR(\mathbf{x}_i) = \frac{S}{I(\mathbf{x}_i) + N} \tag{3}$$

S and I, the power received from the desired transmitted and combined interferers respectively, are provided in equation (4). I is a SN process, formed by the sum of all spatially random points of Φ. To remove the path loss singularity around the receiver itself we will assume a minimum distance ϵ between the receiver and all transmitters. The channel h is a positive random variable with unit mean.

$$S = Pl_{\alpha,\epsilon}(u) = Ph_i u^{-\alpha}, \quad I(\mathbf{x}_i) = \sum_{k \in \Phi, k \neq i} Ph_k l_{\alpha,\epsilon}(|x_i - x_k|) \tag{4}$$

Explain shot noise

ADD stuff about performance metrics here

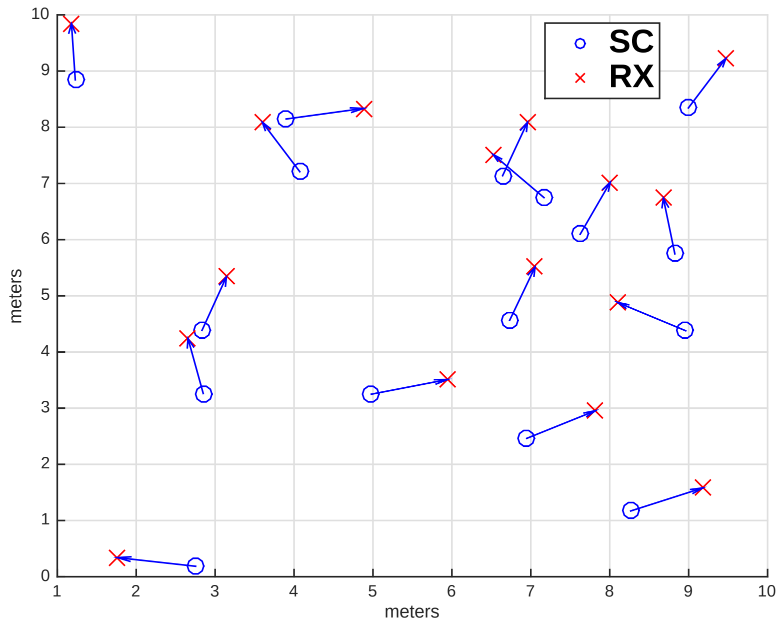

Figure 1: Example realization of PPP network arrangement.

1.3 Interference

Now that we have a definition for interference in the SG model, we will provide the densities for this interference and how these densities can be used for SIR calculation and performance metrics. The case we are considering is only the Rayleigh fading case with path loss exponent $\alpha = 4$. It has been shown in [4] that in non-fading cases the expected interference is infinite for $\alpha \geq 2$. We selected $\alpha = 4$, since this is commonly used for indoor applications, which is the assumed environment of any SC deployment.

The performance metric we will study is called probability of successful transmission p_{st}. This is simply the probability of the SINR being greater than some threshold T.

$$p_{st}(\mathrm{T}) = \mathbf{P}(SINR > \mathrm{T}) \tag{5}$$

Equation (6) can be rewritten in the case of Rayleigh fading where S is exponentially distributed and $\mathbf{E}[S] = M_s$:

$$\begin{aligned} p_{st}(\mathrm{T}) &= \mathbf{P}\Big(\frac{S}{I+N} > \mathrm{T}\Big) \\ &= \mathbf{P}(S > \mathrm{T}(I+N)) \\ &= \exp(-M_s\mathrm{T}I)\exp(-M_s\mathrm{T}N) \end{aligned} \tag{6}$$

Here $\exp(-\mathrm{T}I)$ and $\exp(-\mathrm{T}N)$ are the Laplace transforms of I and N respectively. Therefore p_{st} can be directly calculated from these transforms, rather than having to compute the distribution of p_{st}. This is a common technique in the literature [4,5]. For the sake of simplicity we will ignore noise in the system, can just consider interference. We will first calculate the Laplace transforms of our interference and then provide densities for a specific case of (6).

Assuming fading for h_i in (4), and using the probability generating functional (PGFL) to determine the interference densities. The Laplace transform of the second equation in (4), assuming the PP is stationary, with points independent, and two-dimensional:

$$\begin{aligned}\mathcal{L}_I(s) &= \exp(-\lambda c_d \mathbf{E}[h^\delta]\Gamma(1-\delta)s^\delta) \\ &= \exp(-\lambda \pi^{3/2}\sqrt{s}\mathbf{E}[\sqrt{h}]))\end{aligned} \tag{7}$$

The derivation can be seen in Appendix E In the case for Rayleigh fading where the δ^{th} moment can be written as $E[h^\delta] = \Gamma(1+\delta) = \sqrt{\pi}$, and with uses of Euler's reflection formula

$$\begin{aligned}\mathcal{L}_I(s) &= \exp(-\lambda c_d \Gamma(1+\delta)\Gamma(1-\delta)s^\delta) \\ &= \exp(-\lambda c_d \delta\Gamma(\delta)\Gamma(1-\delta)s^\delta) \\ &= \exp(-\lambda c_d \frac{\delta\pi}{sin(\pi\delta)}s^\delta) \\ &= \exp(-\lambda\pi^2\sqrt{s}))\end{aligned} \tag{8}$$

The resulting probability density function (PDF) of the last equation in (9) can be found by the inverse Laplace transform as follows, where t is real:

There is a factor of 2 missing here

$$f_I(t) = \frac{\lambda}{4}\left(\frac{\pi}{t}\right)^{3/2}\exp\left(\frac{-\lambda^2\pi^4}{16t}\right) \tag{9}$$

1.4 Performance Metric

Now that we have the Laplace transforms for interference in the case of Rayleigh fading we can return back to our performance metric p_{st}. The Laplace transform of the interference is p_{st} (assuming only SIR), and s is related as follows from $\mathcal{L}(s) = exp(-sX)$.

$$\begin{aligned}p_{st}(\mathrm{T}) &= \mathbf{P}(S > I\mathrm{T}), \quad \mathbf{E}[S] = r^{-\alpha} \\ &= \exp\big(-I\mathrm{T}r^{-\alpha}\big)\end{aligned} \tag{10}$$

Therefore $s = \mathrm{T}r^{-\alpha}$. Now plugging that into (7), and then applying the Rayleigh fading assumption. In the case of a PPP network of intensity λ with all nodes transmitting, the probability of success can be written as follows:

$$\begin{aligned}p_{st}(\mathrm{T}) &= \exp(-\lambda c_d \mathbf{E}[h^\delta]\Gamma(1-\delta)s^\delta) \\ &= \exp(-\lambda c_d \Gamma(1+\delta)\Gamma(1-\delta)s^\delta) \\ &= \exp(-\lambda c_d \Gamma(1+\delta)\Gamma(1-\delta)(r^{-\alpha}\mathrm{T}))^\delta) \\ &= \exp(-\lambda c_d \Gamma(1+\delta)\Gamma(1-\delta)r^{-d}\mathrm{T}^\delta) \\ &= \exp\Big(-\lambda\pi^2\frac{\sqrt{\mathrm{T}}}{2r^2}\Big), \quad d=2, \alpha=4\end{aligned} \tag{11}$$

This metric is

Recheck last line

1.5 Aloha

1.6 CSMA

2 Notation

e_i	the medium access indicator of node i; ($e_i = 1$ if node i is allowed to transmit in the considered time slot and 0 otherwise). The random variables e_i are hence *i.i.d.* and independent of everything else, with $\mathbf{P}](e_i = 1) = p$ (p is the MAP).
F_i^j	denotes the virtual power emitted by node i (provided $e_i = 1$) towards receiver y_j
$x : x > 0$	the set of numbers x such that $x > 0$

A Palm Probability

Palm characteristics are probabilities or means that refer to individual points in a PP. Meaning that we want statistics from individual points perspectives. The usual approach considers a point at the origin $\mathbf{0}$. However, the probability that a stationary point process has a point exactly at $\mathbf{0}$ is zero. Therefore, the probability that a PP Φ has some property provided that it has a point at $\mathbf{0}$ is a difficult quantity.

We determine the mean and probability related to an event of having a point at $\mathbf{0}$ as follows. Consider an observation window W in which Φ has $\Phi(W) = n$ points. These points $x_1, .., x_n$ are taken in turn and Φ is shifted such that the relevant point lies at the origin $\mathbf{0}$. This process is repeated for each x_i. For each shift, the number of points is counted within r i.e. $b(\mathbf{0}, r)$. Their average yields an estimate of the mean number of points in a sphere of radius r centered at a PP point, where in all cases the point x_i itself is never counted. Second, each point is then marked. If within the radius r, it will receive a mark 1 and 0 otherwise. Then all the marks are then added and divided by n, the number of points in Φ. This is an estimate of the probability that a point in the PP has its nearest neighbor at a distance less than r [3].

Mathematically in the Palm sense, the mean and probabilities of having a point within r are show in equations (12) and (13) respectively. Where the index 0 in $\mathbf{E}_0$ and P_0 indicates the shifting of the patterns towards $\mathbf{0}$ in the spatial plain.

$$\mathbf{E}_0(\Phi(b(\mathbf{0}, r) \setminus \{\mathbf{0}\})) \tag{12}$$

$$P_0(\Phi(b(\mathbf{0}, r) \setminus \{\mathbf{0}\}) > 0) \tag{13}$$

To provide even futher relation to more common probability tools the Palm probability P_0 can be written in the following way:

$$P_0(\Phi \in A) = \frac{\mathbf{E}\Big(\sum_{x \in \Phi \cap W} (\mathbb{1}_A(\Phi - x))\Big)}{\lambda v(W)} \tag{14}$$

Here W is some test set of positive volume $v(W)$, and $\Phi \in A$ is a general notation for point process Φ has property A. Clearly, A has to be a property which makes sense for a point process with a point at $\mathbf{0}$. An example of this is $\Phi(b(\mathbf{0}, r) \setminus \{\mathbf{0}\}) = 0$. The indicator $\mathbb{1}_A(\Phi - x)$ is 1 if the shifted point process $\Phi - x$ has property A and 0 otherwise. Mecke (1967) showed that P_0 is independent of choice of W.

B
Summary of Existing Work

[1]

- - For CSMA: provides probability of transmission (media access), coverage probabilities, and rough estimates for successful transmission density
  - For Aloha: provides
- A Stochastic Geometry Analysis of Dense IEEE 802.11 Networks[2]
  -

C Campbell's Theorem

The Campbell theorem is a very useful tool for the calculation of mean values of point process characteristics or statistical estimators. Many of these have the form:

$$\sum_{x \in \Phi} f(x) \tag{15}$$

Basically, the Campbell theorem states that the mean of such a sum can be calculated by solving a volume integral. For example, the mean value of a sum S_f with non-negative $f(x)$ ($S_f = \sum_{x \in \Phi} f(x)$) can be calculated using the Campbell theorem as follows:

$$\mathbf{E}[S_f] = \mathbf{E}\Big[\sum_{x \in \Phi} f(x)\Big] = \int f(x)\lambda(x)dx \tag{16}$$

Where $\lambda(x)$ is a density function, and proportional to the intensity of the point density around a location x.

A common usage is with the indicator function were $f(x) = \not\Vdash_A(x)$ and (16) can be rewritten as:

$$\int \not\Vdash_A(x)\lambda(x)dx = \Delta(A) \tag{17}$$

Here $\Delta(A)$ is an intensity measure (deterministic). Much of the time $\lambda(x)$ is constant. When $\lambda(x)$ is constant, then we are in a simple stationary case:

$$\mathbf{E}[S_f] = \mathbf{E}\Big[\sum_{x \in \Phi} f(x)\Big] = \lambda \int_{\mathbb{R}^d} f(x)dx \tag{18}$$

C.1 Campbell–Mecke Formula

The Palm mean appears in a refined form of the Campbell theorem, the Campbell–Mecke formula:

$$\mathbf{E}\Big[\sum_{x \in \Phi} f(x, \Phi)\Big] = \lambda \int \mathbf{E}_0[f(x, \Phi_{-x})]dx = \lambda \mathbf{E}_0\Big[\int f(x, \Phi_{-x})dx\Big] \tag{19}$$

See page 303 of **??** for additional examples of its use.

D Slivnyak–Mecke Theorem

The Slivnyak–Mecke theorm provides a connection from Palm distributions to traditional probabilistic analysis. The theorem states that the Palm distribution of a homogeneous Poisson process coincides with that of the point process obtained by adding the origin $\mathbf{0}$ to the homogeneous Poisson process. Mathematically:

$$P_0(\Phi \in A) = P(\Phi \cup \{\mathbf{0}\} \in A) \tag{20}$$

The mean is also related as follows:

$$\mathbf{E}_0(\Phi(B)) = \mathbf{E}(\Phi(B)) + \not\Vdash_A(\mathbf{0}) \tag{21}$$

E Laplace Transform

$$
\begin{aligned}
p_{st}(\theta) = \mathcal{L}_I(s) &= \mathbf{E}_I\left[exp(-I\theta)\right] \\
&= \mathbf{E}\Big[\exp\big(-s\sum_{r\in\Phi} h_r r^{-\alpha}\big)\Big]
\end{aligned}
\tag{22}
$$

$$
\mathcal{L}_I(s) = \mathbf{E}_\Phi\Big[\prod_{r\in\Phi}\mathbf{E}_h[\exp(-sh_r r^{-\alpha})]\Big]
\tag{23}
$$

Applying a probability generating functional identity:

$$
\begin{aligned}
\mathcal{L}_I(s) &= \mathbf{E}_\Phi\Big[\prod_{r\in\Phi} v(r)\Big], \quad v(r) = \mathbf{E}_h[\exp(-sh_r r^{-\alpha})] \\
&= \exp\Big(-\mathbf{E}_h(\underbrace{(\int_0^\infty (1-\exp(-shr^{-\alpha}))\lambda^{'}(r)dr)}_{\text{A}}\Big)
\end{aligned}
\tag{24}
$$

Swapping expectation and integral and conditioning on h, the integral is:

$$
\lambda^{'}(r) = \lambda c_d d r^{d-1} \quad \delta = d/\alpha
\tag{25}
$$

$$
\begin{aligned}
A &= \int_0^\infty (1-\exp(-shr^{-\alpha}))\lambda^{'}(r)dr \\
&= \lambda c_d \int_0^\infty (1-\exp(-shr^{-d/\delta}))dr^{d-1}dr
\end{aligned}
\tag{26}
$$

Apply our scenario specification of two-dimensional system:

$$
d = 2 \rightarrow c_d = \pi
\tag{27}
$$

$$
A = 2\lambda\pi\int_0^\infty (1-\exp(-shr^{-2/\delta}))rdr
\tag{28}
$$

$$
\begin{aligned}
r^\alpha &\rightarrow y, \delta = 2/\alpha \\
\alpha r^{\alpha-1}dr &\rightarrow dy \\
rdr &\rightarrow \frac{r^-\alpha}{\alpha}(r^2)dy \\
rdr &\rightarrow \frac{\delta}{2}(y^{\delta-1})dy
\end{aligned}
\tag{29}
$$

$$
A = \lambda\pi\int_0^\infty (1-\exp(-sh/y))\delta y^{\delta-1}dy
\tag{30}
$$

$$
\begin{aligned}
y^{-1} &\rightarrow x \\
-y^{-2}dy &\rightarrow dx \\
dy &\rightarrow -x^{-2x}dx
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
A &= -\lambda\pi\int_0^\infty \underbrace{(1-\exp(-shx))}_{\text{u}}\underbrace{\delta x^{-\delta-1}}_{v'}dx \\
&= uv - \int u'v = [1-\exp(-shx)]x^{-\delta} + \int_0^\infty x^{-\delta}sh\exp(-shx)dx
\end{aligned}
\tag{32}
$$

$$\mathbf{E}_h\Big[[1-\exp(-shx)]x^{-\delta}\Big] = 0 \tag{33}$$

$$A = \lambda\pi\int_0^\infty x^{-\delta} she^{-shx}dx \tag{34}$$

$$\begin{aligned} shx &\to t \\ shdx &\to dt \end{aligned} \tag{35}$$

$$A = \lambda\pi\int_0^\infty (t/sh)^{-\delta}e^{-t}dt \tag{36}$$

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt \tag{37}$$

$$\begin{aligned} A &= \lambda\pi(sh)^\delta\int_0^\infty t^{-\delta}e^{-t}dt \\ &= \lambda\pi(sh)^\delta\Gamma(1-\delta) \end{aligned} \tag{38}$$

Placing A back into $\mathcal{L}_I$:

$$\mathcal{L}_I(s) = \exp(-\lambda\pi\mathbf{E}_h(h^\delta)\Gamma(1-\delta)s^\delta) \tag{39}$$

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