Algorithms Lab

Dynamic Programming

Example problem: A, B two arrays of positive numbers. You can remove *a* last elements from A and *b* last elements from B (both *a* and *b* have to be at least 1), and the cost of such operation is X * Y, where X (Y) is the sum of the removed elements from A (B).

Your **goal** is to **remove all elements from A and B** by repeatedly applying such operation, with the **minimal total cost**!

2 5 10 2 6

7 1 9 4 2

Cost:

Total cost:

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2 5 10 ~~2 6~~

7 1 ~~9 4 2~~

Cost: (2 + 6) * (9 + 4 + 2) = 120

Total cost: 120

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2 ~~5 10~~ ~~2 6~~

7 ~~1~~ ~~9 4 2~~

Cost: (5 + 10) * (1) = 15

Total cost: 135

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Cost:

Total cost: 135

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Your **goal** is to **remove all elements from A and B** by repeatedly applying such operation, with the **minimal total cost**!

~~2~~ ~~5 10~~ ~~2 6~~

~~7~~ ~~1~~ ~~9 4 2~~

Cost: (2) * (7) = 14

Total cost: 149

First approach - brute force

Recursively try all possible removals

```
rec_try(i, j)  // consider only first i elements of
               // A and j elements of B
```

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```
rec_try(i, j)   // consider only first i elements of
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    if (i == 1) return
```

$$A[1] \cdot (B[1] + \ldots + B[j])$$

```
    if (j == 1) return
```

$$(A[1] + \ldots + A[i]) \cdot B[1]$$

First approach - brute force

Recursively try all possible removals

```
rec_try(i, j)  // consider only first i elements of
               // A and j elements of B

    if (i == 1) return   A[1]·(B[1]+...+B[j])

    if (j == 1) return   (A[1]+...+A[i])·B[1]

    best = (A[1]+...+A[i])·(B[1]+...+B[j])  // take all
```

$$A[1] \cdot (B[1] + \ldots + B[j])$$

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   best = (A[1]+...+A[i])·(B[1]+...+B[j])  // take all

   for a = 1 to i - 1
     for b = 1 to j - 1
```

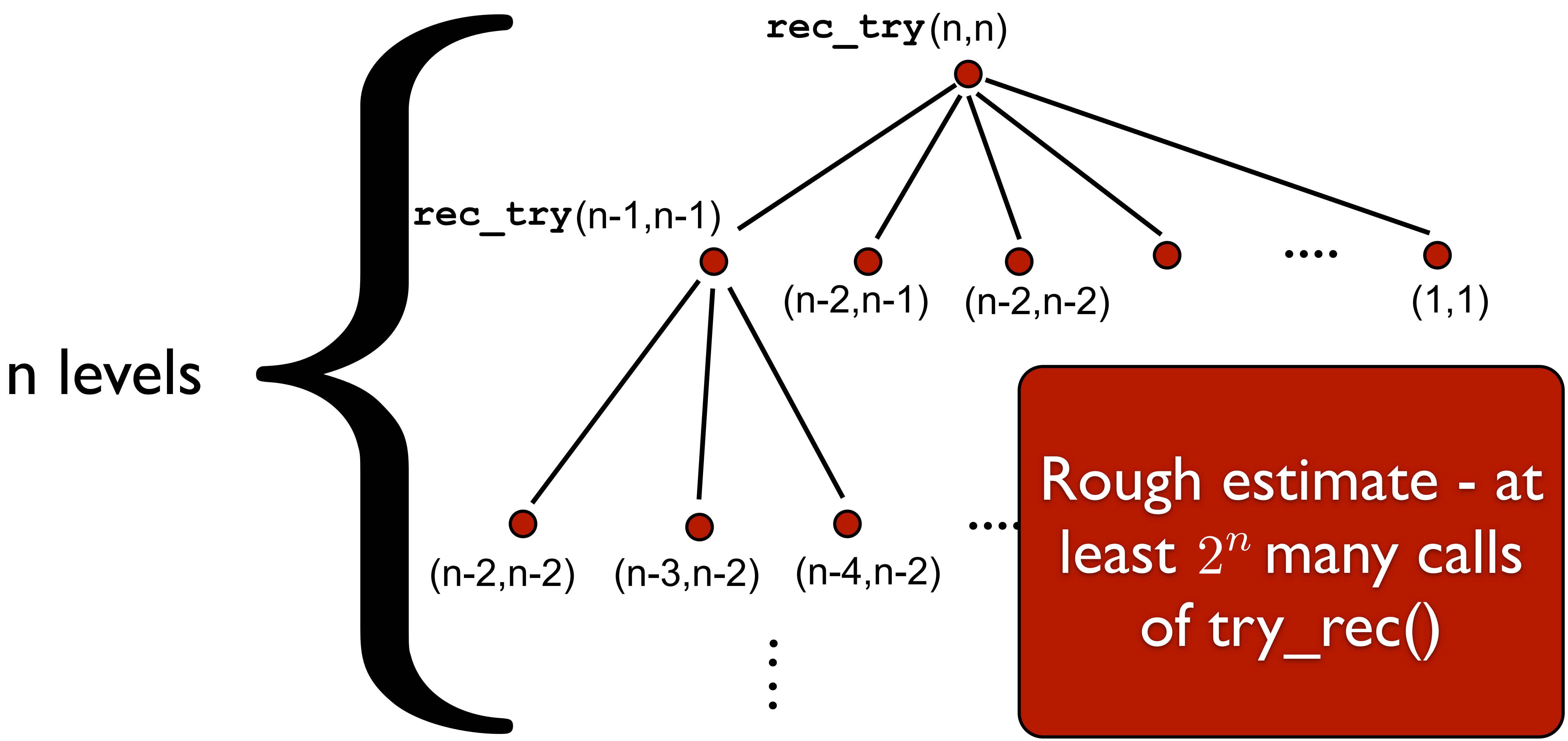
$$\text{cost} = \left(\sum_{t=i-a+1}^{i} A[t]\right) \cdot \left(\sum_{t=j-b+1}^{j} B[t]\right)$$

```
       if (cost + rec_try(i - a, j - b) < best)
           best = cost + rec_try(i - a, j - b)

   return best
```

First approach - brute force

Recursively try all possible removals



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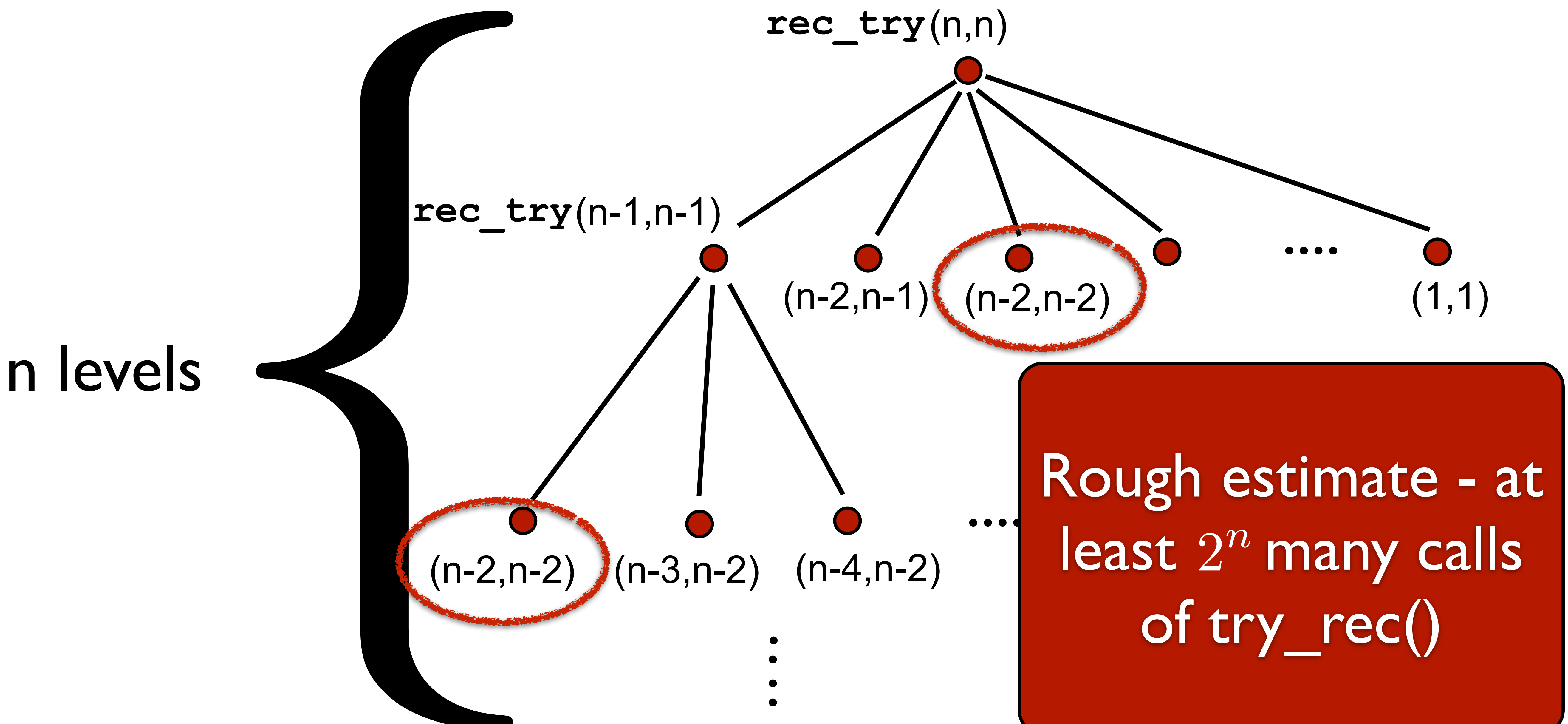

First approach - brute force

Recursively try all possible removals

Problem: for some (i,j), we call **try_rec(i,j)** many times

Observation: for a fixed (i,j), **try_rec(i,j)** always returns the same value!

Solution: for each (i,j), store the value which **try_rec(i,j)** returns! Only call **try_rec(i,j)** if this value is not stored yet!

What is the running time of try_rec(i,j) ?

```
rec_try(i, j)  // consider only first i elements of
               // A and j elements of B

  if (i == 1) return   A[1]·(B[1] + ... + B[j])

  if (j == 1) return   (A[1] + ... + A[i])·B[1]

  best = (A[1] + ... + A[i])·(B[1] + ... + B[j])  // take all

  for a = 1 to i - 1
    for b = 1 to j - 1
      cost = (sum_{t=i-a+1}^{i} A[t]) · (sum_{t=j-b+1}^{j} B[t])
      if (not stored(i - a, j - b)) rec_try(i-a, j-b)
      if (cost + stored(i - a, j - b) < best)
         best = cost + stored(i - a, j - b)

  store(i,j) <- best
```

Where the math in the listing is:

$$A[1]\cdot(B[1]+\ldots+B[j])$$

$$(A[1]+\ldots+A[i])\cdot B[1]$$

$$(A[1]+\ldots+A[i])\cdot(B[1]+\ldots+B[j])$$

$$\left(\sum_{t=i-a+1}^{i} A[t]\right)\cdot\left(\sum_{t=j-b+1}^{j} B[t]\right)$$

What is the running time of try_rec(i,j) ?

$O(n)$

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rec_try(i, j)  // consider only first i elements of
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    if (i == 1) return    A[1]·(B[1] + ... + B[j])

    if (j == 1) return    (A[1] + ... + A[i])·B[1]

    best = (A[1] + ... + A[i])·(B[1] + ... + B[j])  // take all

    for a = 1 to i - 1
      for b = 1 to j - 1
         cost = (sum_{t=i-a+1}^{i} A[t]) · (sum_{t=j-b+1}^{j} B[t])
         if (not stored(i - a, j - b)) rec_try(i-a, j-b)
         if (cost + stored(i - a, j - b) < best)
            best = cost + stored(i - a, j - b)

    store(i,j) <- best
```

$$\text{cost} = \left(\sum_{t=i-a+1}^{i} A[t]\right) \cdot \left(\sum_{t=j-b+1}^{j} B[t]\right)$$

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    if (i == 1) return   A[1]·(B[1] + ... + B[j])

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    best = (A[1] + ... + A[i])·(B[1] + ... + B[j])  // take all

    for a = 1 to i - 1
      for b = 1 to j - 1
        cost = (sum_{t=i-a+1}^{i} A[t]) · (sum_{t=j-b+1}^{j} B[t])
        if (not stored(i - a, j - b)) rec_try(i-a, j-b)
        if (cost + stored(i - a, j - b) < best)
           best = cost + stored(i - a, j - b)

    store(i,j) <- best
```

$$A[1] \cdot (B[1] + \ldots + B[j])$$

$$(A[1] + \ldots + A[i]) \cdot B[1]$$

$$\text{best} = (A[1] + \ldots + A[i]) \cdot (B[1] + \ldots + B[j])$$

$$\text{cost} = \left(\sum_{t=i-a+1}^{i} A[t]\right) \cdot \left(\sum_{t=j-b+1}^{j} B[t]\right)$$

What is the running time of try_rec(i,j) ?

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rec_try(i, j)   // consider only first i elements of
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    if (i == 1) return
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$A[1] \cdot (B[1] + \ldots + B[j])$

```
    if (j == 1) return
```
$(A[1] + \ldots + A[i]) \cdot B[1]$

best = $(A[1] + \ldots + A[i]) \cdot (B[1] + \ldots + B[j])$ `// take all`

$O(n^2)$

```
    for a = 1 to i - 1
      for b = 1 to j - 1
```

cost = $\left(\sum_{t=i-a+1}^{i} A[t] \right) \cdot \left(\sum_{t=j-b+1}^{j} B[t] \right)$

```
         if (not stored(i - a, j - b)) rec_try(i-a, j-b)
         if (cost + stored(i - a, j - b) < best)
            best = cost + stored(i - a, j - b)

    store(i,j) <- best
```

What is the running time of try_rec(i,j) ?

rec_try(i, j) // consider only first i elements of
// A and j elements of B

$O(n)$

if (i == 1) return $A[1]\cdot(B[1]+\ldots+B[j])$

if (j == 1) return $(A[1]+\ldots+A[i])\cdot B[1]$

best = $(A[1]+\ldots+A[i])\cdot(B[1]+\ldots+B[j])$ // take all

$O(n^3)$

for a = 1 to i - 1
  for b = 1 to j - 1

cost = $\left(\sum_{t=i-a+1}^{i} A[t]\right)\cdot\left(\sum_{t=j-b+1}^{j} B[t]\right)$

```
        if (not stored(i - a, j - b)) rec_try(i-a, j-b)
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What is the running time of try_rec(i,j) ?

```
rec_try(i, j)  // consider only first i elements of
               // A and j elements of B
```

$O(n^3)$

```
if (i == 1) return
```
$A[1] \cdot (B[1] + \ldots + B[j])$

```
if (j == 1) return
```
$(A[1] + \ldots + A[i]) \cdot B[1]$

best = $(A[1] + \ldots + A[i]) \cdot (B[1] + \ldots + B[j])$ `// take all`

```
for a = 1 to i - 1
  for b = 1 to j - 1
```
cost = $\left(\sum_{t=i-a+1}^{i} A[t]\right) \cdot \left(\sum_{t=j-b+1}^{j} B[t]\right)$

```
    if (not stored(i - a, j - b)) rec_try(i-a, j-b)
    if (cost + stored(i - a, j - b) < best)
      best = cost + stored(i - a, j - b)

store(i,j) <- best
```

Second approach - running time

try_rec(i,j) will be executed at most once for each pair (*i*,*j*), $1 \leq i, j \leq n$

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$O(n^3)$ per **try_rec(i,j)**

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$O(n^3)$ per **try_rec(i,j)**

Total running time: $O(n^5)$

This was an example of
Dynamic Programming!

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Dynamic Programming!

Let’s improve it!

Cheap improvement.

```
rec_try(i, j)   // consider only first i elements of
                // A and j elements of B

    if (i == 1) return   A[1]·(B[1]+...+B[j])

    if (j == 1) return   (A[1]+...+A[i])·B[1]

    best = (A[1]+...+A[i])·(B[1]+...+B[j])  // take all

    for a = 1 to i - 1
      for b = 1 to j - 1
          cost = (sum_{t=i-a+1}^{i} A[t]) · (sum_{t=j-b+1}^{j} B[t])
          if (not stored(i - a, j - b)) rec_try(i-a, j-b)
          if (cost + stored(i - a, j - b) < best)
             best = cost + stored(i - a, j - b)

    store(i,j) <- best
```

$$A[1] \cdot (B[1] + \ldots + B[j])$$

$$(A[1] + \ldots + A[i]) \cdot B[1]$$

$$(A[1] + \ldots + A[i]) \cdot (B[1] + \ldots + B[j])$$

$$O(n^3)$$

$$\text{cost} = \left(\sum_{t=i-a+1}^{i} A[t]\right) \cdot \left(\sum_{t=j-b+1}^{j} B[t]\right)$$

Cheap improvement.

```
rec_try(i, j)   // consider only first i elements of
                // A and j elements of B
```

...urn $A[1] \cdot (B[1] + \ldots + B[j])$

$(A[1] + \ldots + A[i]) \cdot B[1]$

Can you improve me to $O(n^2)$?

be... $+ A[i]) \cdot (B[1] + \ldots + B[j])$ `// take all`

```
for a = 1 to i - 1
  for b = 1 to j - 1
```

cost = $\left(\sum_{t=i-a+1}^{i} A[t] \right) \cdot \left(\sum_{t=j-b+1}^{j} B[t] \right)$

```
    if (not stored(i - a, j - b)) rec_try(i-a, j-b)
    if (cost + stored(i - a, j - b) < best)
      best = cost + stored(i - a, j - b)

store(i,j) <- best
```

Cheap improve

```
rec_try(i, j)   // cons
                // A an
              urn    A[
```

$(A[1] + \ldots + A[i]) \cdot B[1]$

$+ A[i]) \cdot (B[1] + \ldots + B[j])$ `// take all`

```
    for a = 1 to i - 1
      for b = 1 to j - 1
```

`cost =` $\left(\sum_{t=i-a+1}^{i} A[t]\right) \cdot \left(\sum_{t=j-b+1}^{j} B[t]\right)$

```
        if (not stored(i - a, j - b)) rec_try(i-a, j-b)
        if (cost + stored(i - a, j - b) < best)
            best = cost + stored(i - a, j - b)

    store(i,j) <- best
```

Yes!

Calculating `cost` can be done in constant time.

(Hint: Precomputing partial sums)

Can you improve me to $O(n^2)$?

From exponential to $O(n^5)$ and then to $O(n^4)$.
Not bad!

From exponential to $O(n^5)$ and then to $O(n^4)$.
Not bad!

But we can do even better!

Can we improve the running time of the previous algorithm?

Let us observe the cost function more closely

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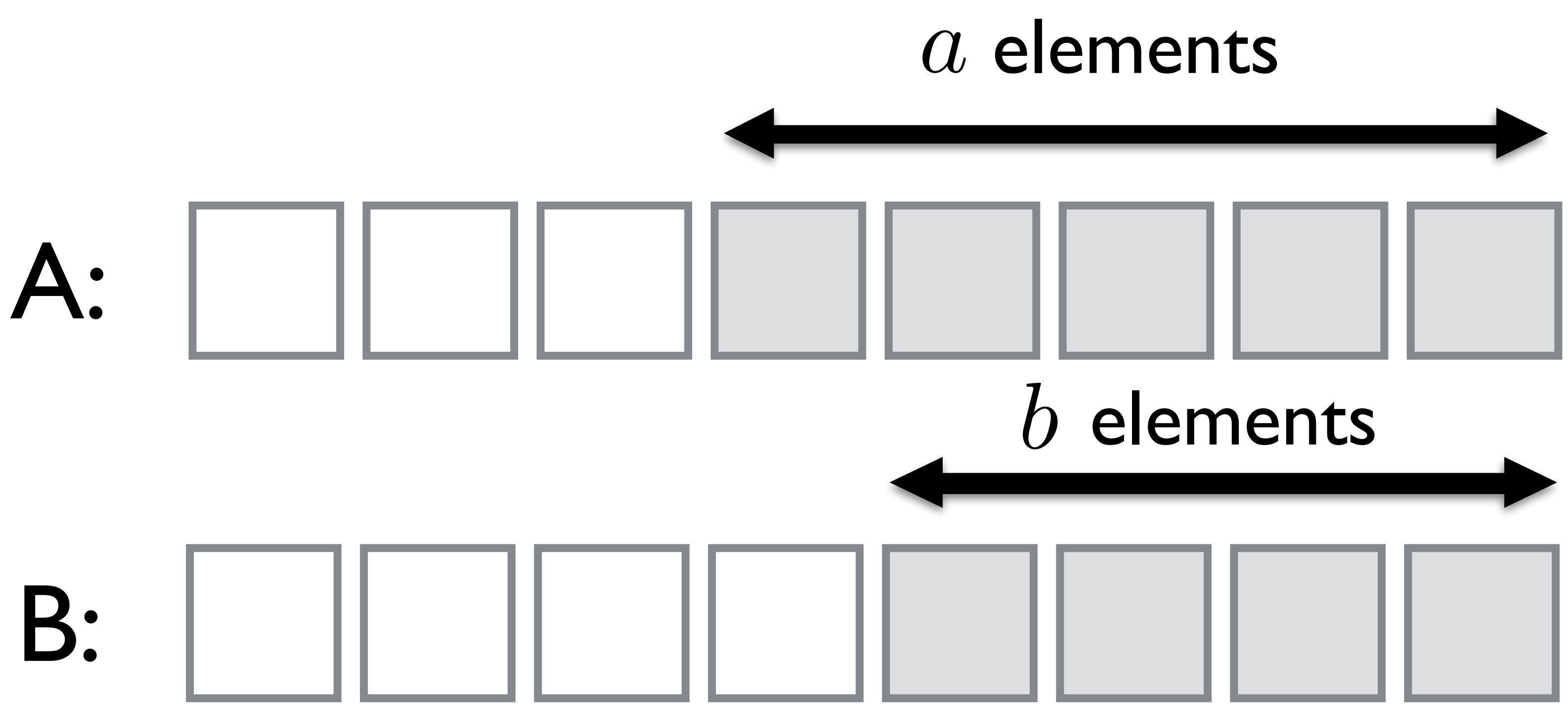

Cost 1: $(A[1] + A[2] + \cdots + A[a]) \cdot (B[1] + B[2] + \cdots + B[b])$

* The order of elements in A/B is reversed here, for the convenience of presentation.

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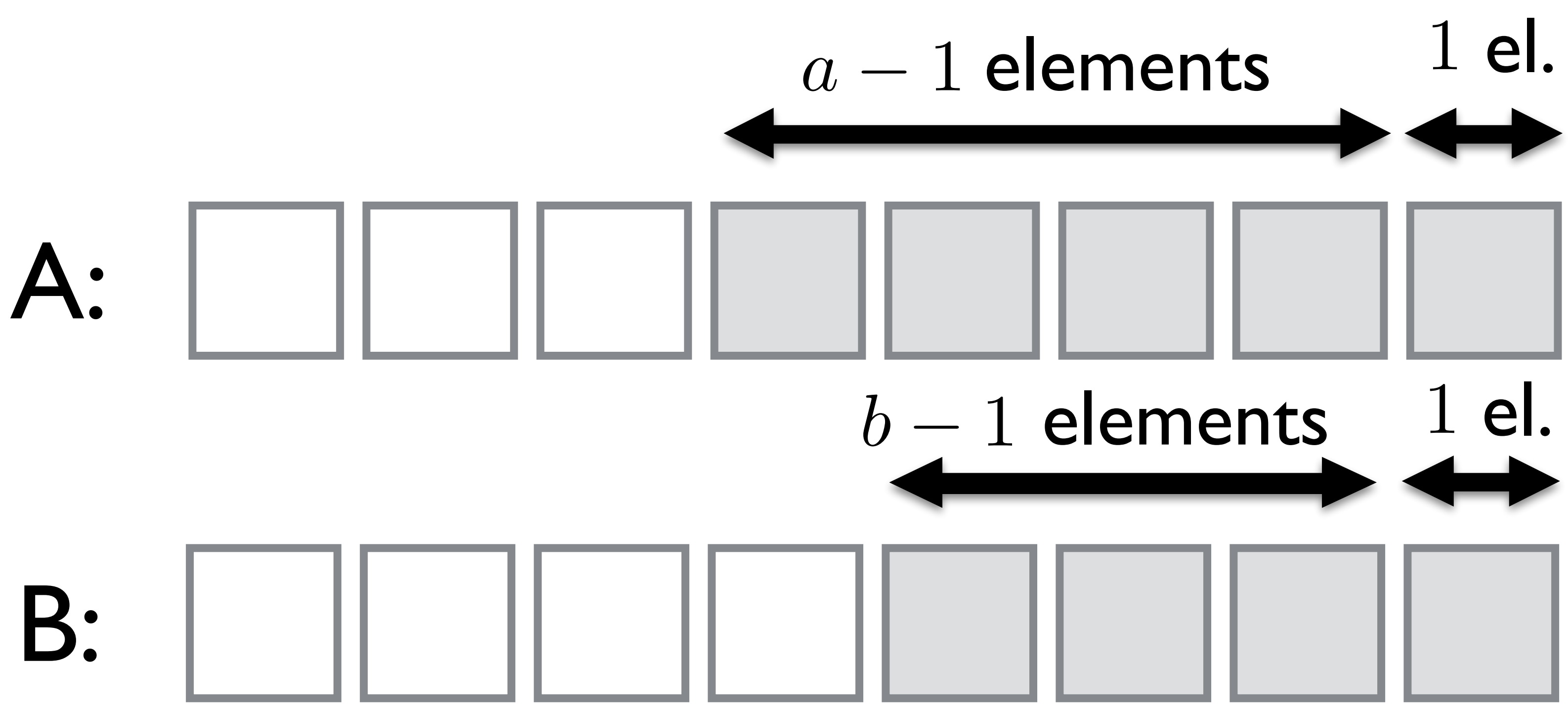

Cost 1: $(A[1]+A[2]+\cdots+A[a])\cdot(B[1]+B[2]+\cdots+B[b])$

Cost 2: $A[1]B[1]+(A[2]+\cdots+A[a])(B[2]+\cdots+B[b])$

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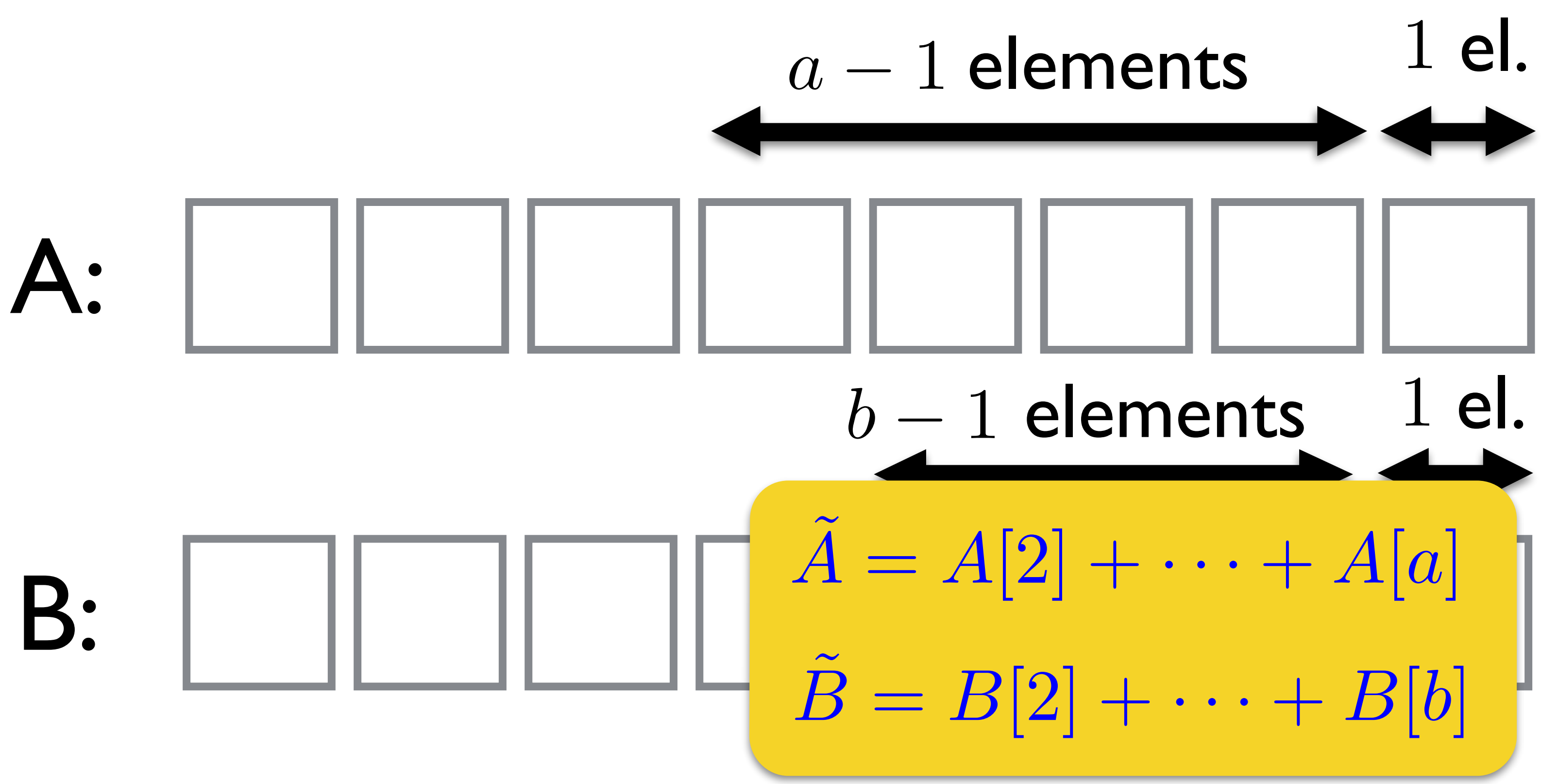

$$\tilde{A} = A[2] + \cdots + A[a]$$

$$\tilde{B} = B[2] + \cdots + B[b]$$

Cost 1: $(A[1] + A[2] + \cdots + A[a]) \cdot (B[1] + B[2] + \cdots + B[b])$

Cost 2: $A[1]B[1] + (A[2] + \cdots + A[a])(B[2] + \cdots + B[b])$

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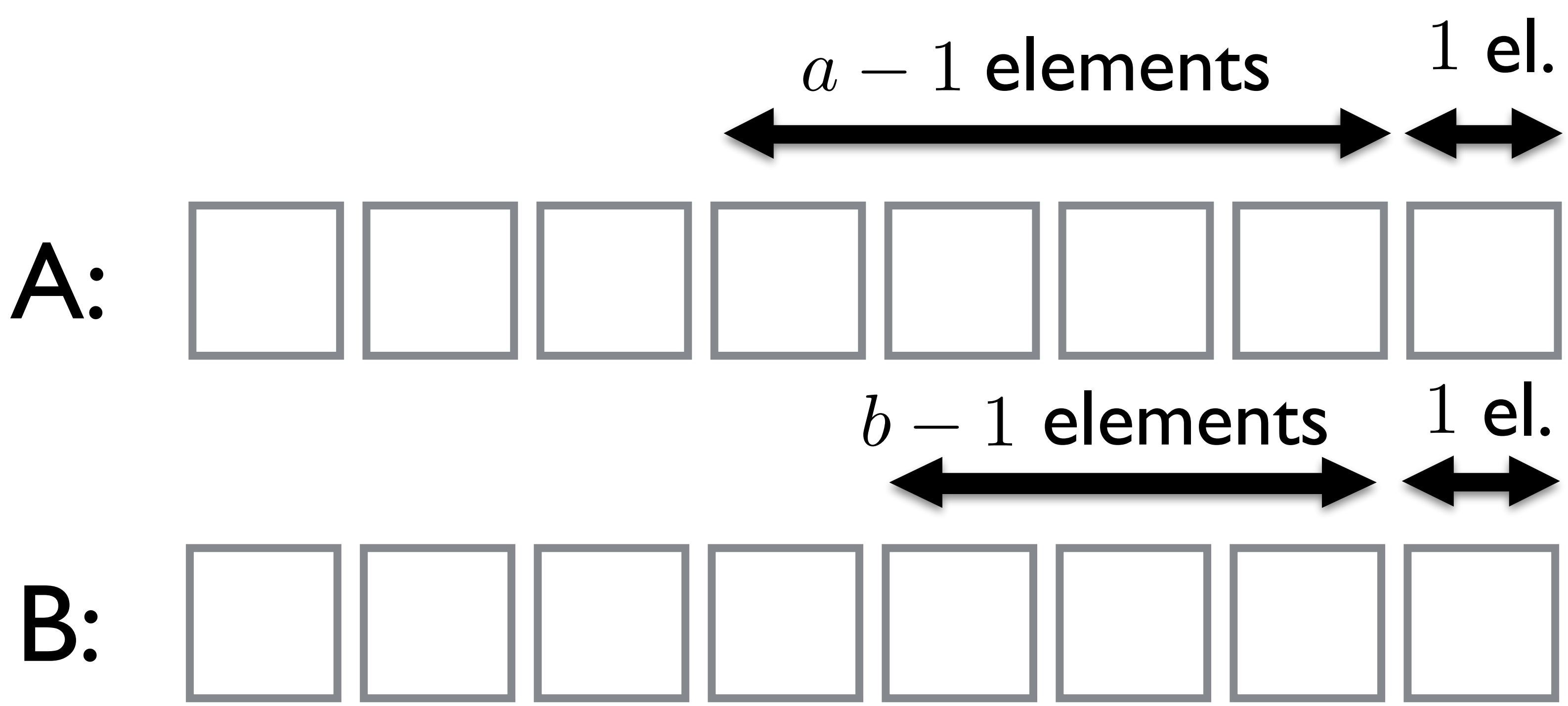

Cost 1: $(A[1]+\tilde{A})(B[1]+\tilde{B})$

Cost 2: $A[1]B[1]+\tilde{A}\tilde{B}$

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Can we improve the running time of the previous algorithm?

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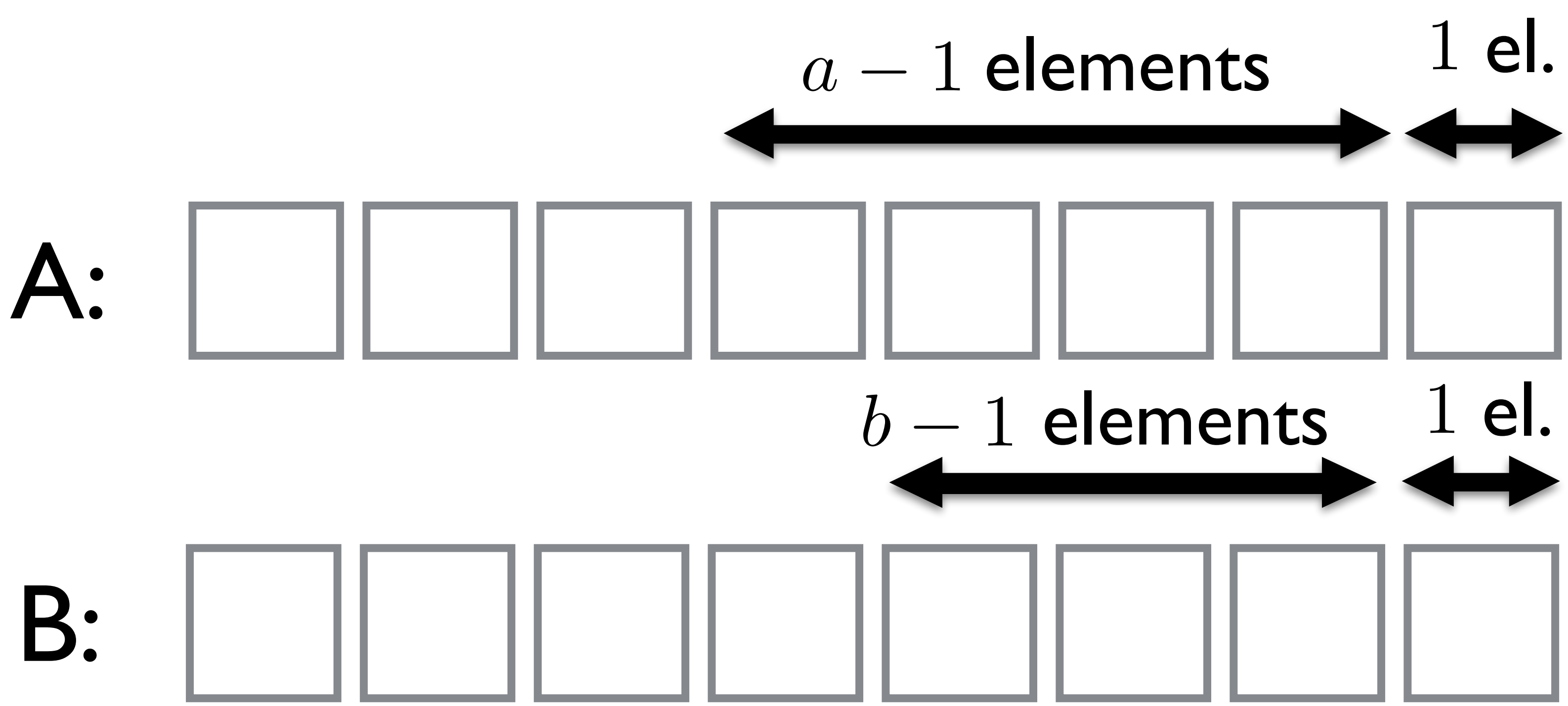

Cost 1: $(A[1]+\tilde{A})(B[1]+\tilde{B}) = A[1]B[1]+\tilde{A}\tilde{B}+A[1]\tilde{B}+\tilde{A}B[1]$

Cost 2: $A[1]B[1]+\tilde{A}\tilde{B}$

Can we improve the running time of the previous algorithm?

Let us observe the cost function more closely

$$A[1]B[1] + \tilde{A}\tilde{B} + A[1]\tilde{B} + \tilde{A}B[1] \geq A[1]B[1] + \tilde{A}\tilde{B}$$

Cost 1 Cost 2

B:

Cost 1: $(A[1] + \tilde{A})(B[1] + \tilde{B}) = A[1]B[1] + \tilde{A}\tilde{B} + A[1]\tilde{B} + \tilde{A}B[1]$

Cost 2: $A[1]B[1] + \tilde{A}\tilde{B}$

Can we improve the running time of the previous algorithm?

Let us observe the cost function more closely

Conclusion?

Always take one element from each array. Right?

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Let us observe the cost function more closely

Conclusion?

Always take one element from each array. Right? **No!**

Previous analysis only works when $a > 1$ and $b > 1$

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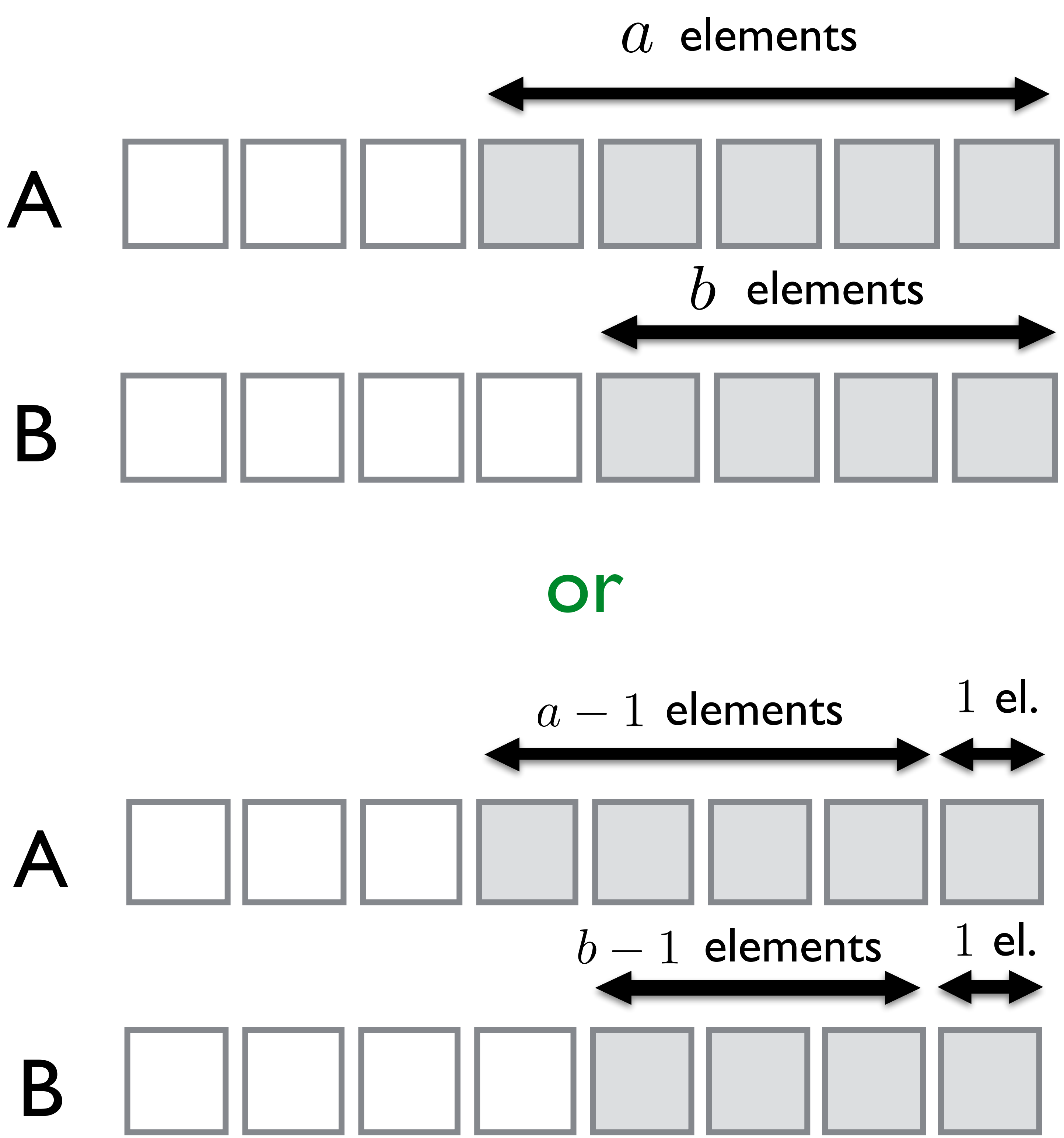

This only makes sense when $a > 1$ and $b > 1$

The optimal strategy will either:

1. Take one element from both sides (our analysis), or

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2. Take one element from A ($a = 1$) and more elements from B (or vice versa)

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1. Take one element from both sides (our analysis), or
2. Take one element from A ($a = 1$) and more elements from B (or vice versa)

Therefore: there exists an optimal strategy such that each removal takes either exactly one element from A or exactly one element from B!!!

Third approach

```
rec_try(i, j)  // consider only first i elements of
               // A and j elements of B

   if (i == 1) return  A[1]·(B[1]+...+B[j])

   if (j == 1) return  (A[1]+...+A[i])·B[1]

   best = (A[1]+...+A[i])·(B[1]+...+B[j]) // take all

   for a = 1 to i - 1 // take one element from B
      cost = (sum_{t=i-a+1}^{i} A[t]) B[j]
      if (not stored(i - a, j - 1)) rec_try(i-a, j - 1)
        if (cost + stored(i - a, j - 1) < best)
           best = cost + stored(i - a, j - 1)

   for b = 1 to j - 1
     similar ...

   store(i,j) <- best
```

Where:

$A[1] \cdot (B[1] + \ldots + B[j])$

$(A[1] + \ldots + A[i]) \cdot B[1]$

$(A[1] + \ldots + A[i]) \cdot (B[1] + \ldots + B[j])$

$\text{cost} = \left(\sum_{t=i-a+1}^{i} A[t] \right) B[j]$

Third approach

try_rec(i,j) will be executed at most once for each pair (*i*,*j*), $1 \leq i, j \leq n$

two **non-nested** loops: one up to *i*, one up to *j*

Total running time: $\mathcal{O}(n^3)$