Algolab 2015
Winter Games

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Objective

Learn how to solve a problem given by a textual description.
This includes:

- appropriate problem modeling
- choice of suitable (combinatorial) algorithms, and
- implementation.

Your Task:

- read the 3 problem descriptions
- sketch your approach (modeling, algorithms)

Use them all!

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- no overlap — only "close neighbors" matter
- maximize radius — optimization problem?

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- only the closest cannon pair matters

Use them all! – Solution

Try them all!

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- Compute the Delaunay triangulation $\rightarrow \Theta(n \log n)$
- Iterate over the edges of the triangulation $\rightarrow \Theta(n)$

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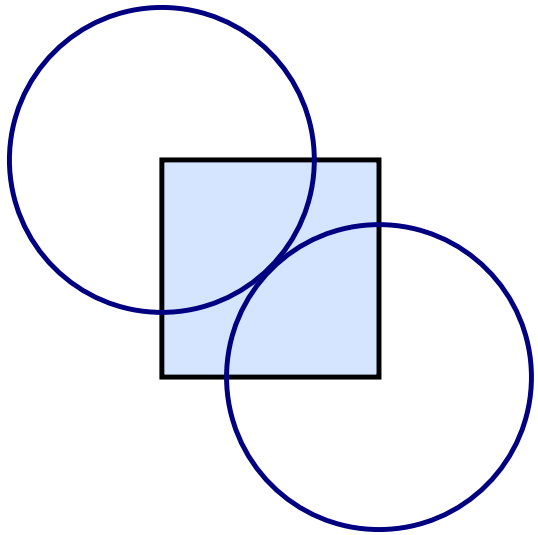
Implementation details

- the distance fits into a `double`
  radius $\leq \sqrt{2^{50} + 2^{50}} < 2^{50}$



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- find maximum independent set
  - in general — NP-complete
  - bipartite graphs — König's Theorem, Matching
  - special cases — trivial

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Construct a graph G to model the dependencies: vertices are cannons and there is an edge between two vertices if the repective operation ranges overlap

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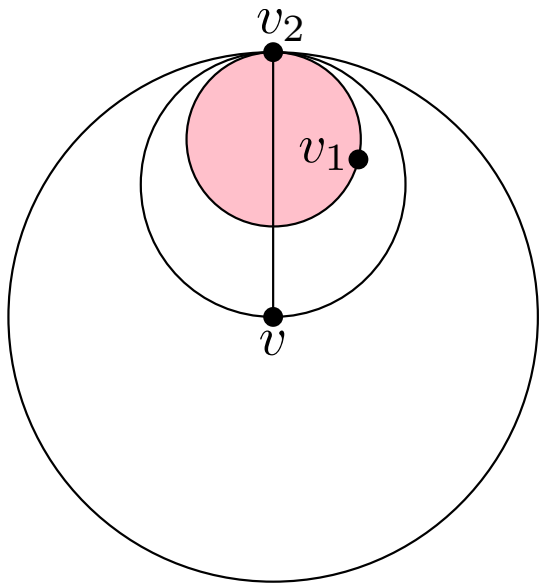

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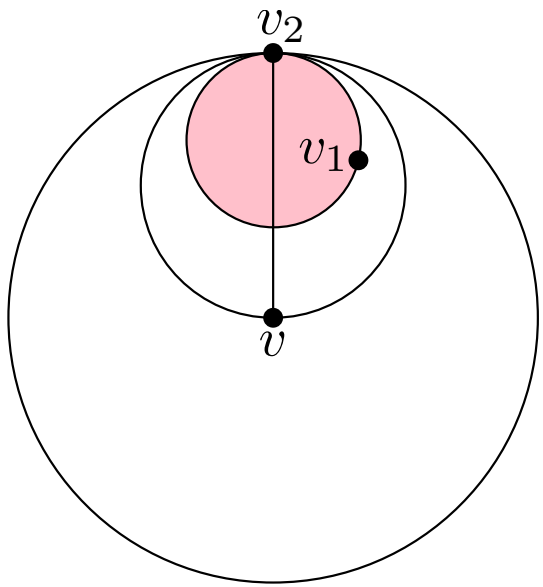

Lemma

Let v_1 be a nearest neighbor and v_2 be a second nearest neighbor of v. Then at least one of vv_2 or v_1v_2 is an edge of the Delaunay triangulation.

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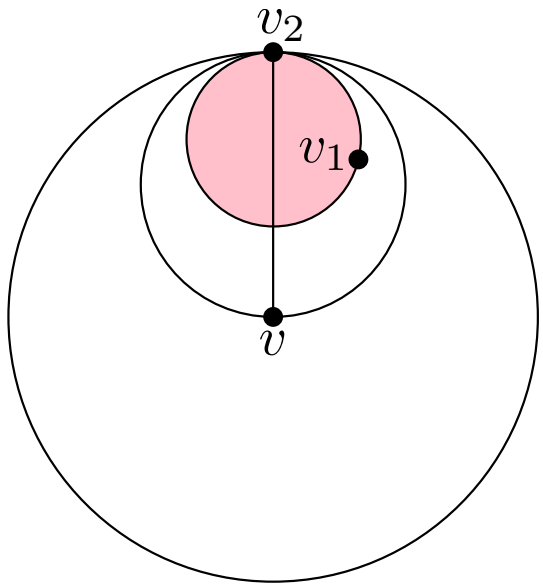

- maximal degree 2

$\Rightarrow$ the graph G must be a disjoint union of paths and cycles

$\Rightarrow$ greedy/ad-hoc solution for every component $\rightarrow \Theta(n)$

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Key ideas

- optimization problem — find a larger radius for each cannon
  - the improvement is linear in the radius
  - lower bounds for the radii
  - implicit upper bounds: no overlap

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Linear program with n variables and $n + \binom{n}{2}$ constraints:

- *Variables*: Operation range (radius) of every snow cannon

$$r_i \geq \frac{\texttt{closest\_pair\_dist}}{2}, \quad i = 1, \ldots, n$$

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