SLOPE And Its Role In Design For Generalization

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Joint work with Jiwei Zhao and Hyunseung Kang

ACIC 2025

Two-Slide Overview: Motivation

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Effect generalization	Baseline	Potential outcome	Peru	India	$\mathrm{E}_{Q_{O,X}}(O)$
LLM [Mirzadeh et al. 2024]	Math question	Correct/Incorrect (1/0)	Sophie	Julia	$\mathrm{pr}_{Q_O}(O=1)$

P = Peru

Q = India

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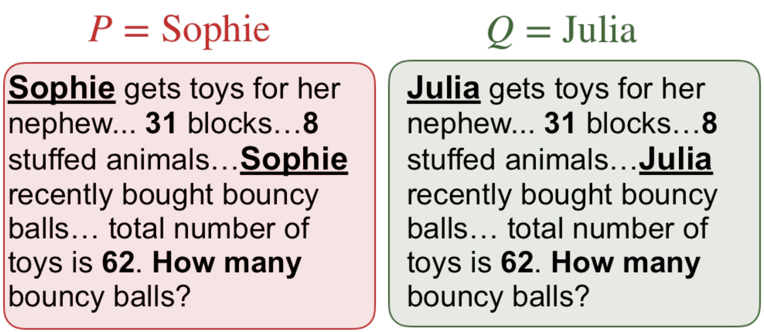

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$$Q_{O|X}(\cdot \mid x) = P_{O|X}(\cdot \mid x), \text{ a.e. } Q_X. \tag{1}$$

For example,

$$\begin{aligned}\psi(Q_{O,X}) &= \mathrm{E}_{Q_O}(O) \\ &= \mathrm{E}_{Q_X}\left[\mathrm{E}_{Q_{O|X}}(O \mid X)\right] \\ &= \mathrm{E}_{Q_X}\left[\mathrm{E}_{P_{O|X}}(O \mid X)\right].\end{aligned}$$

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However, the conditional exchangeability is made on unobserved data and cannot be verified.

Two-Slide Overview: Contribution

- Our work: If conditional exchangeability is **implausible**, should we (a) still borrow from $P_{O,X}$? (b) still report the mean?
  Are all **source distributions** ($P_{O,X}$) and **estimands** (ψ) equally "good/bad" when conditional exchangeability doesn't hold?

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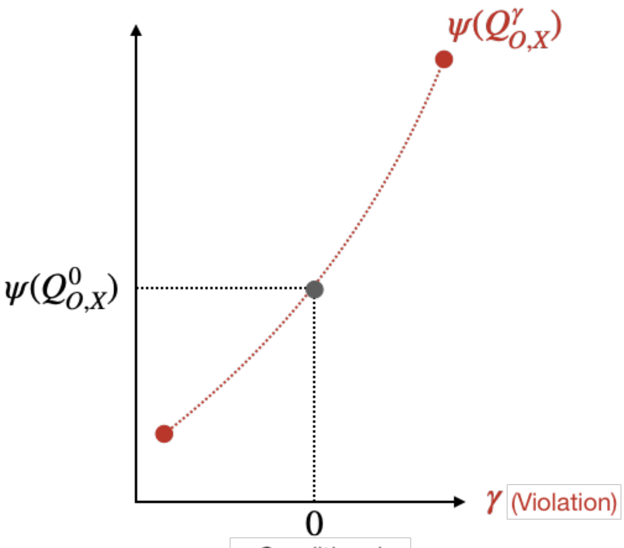

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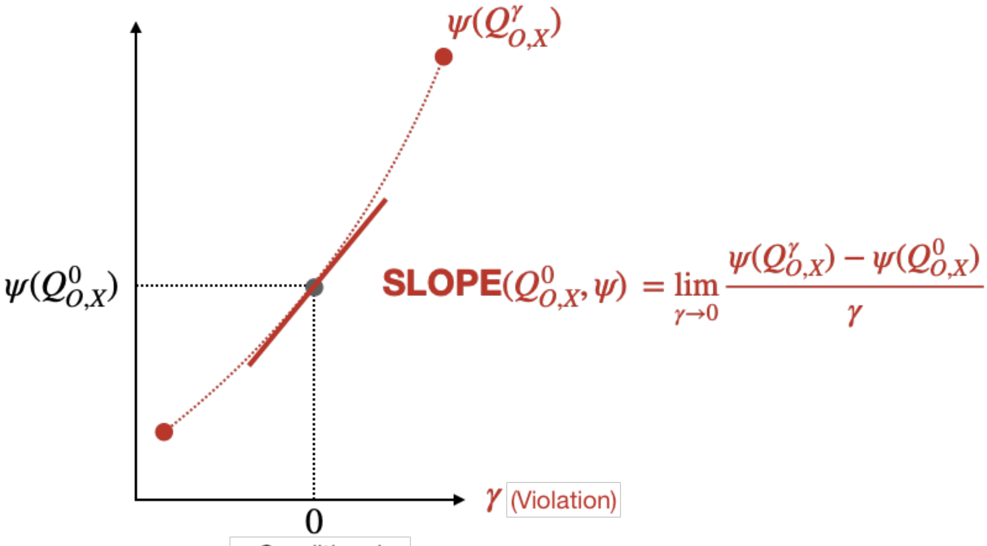
- We propose a scalar summary, SLOPE. Higher $|\text{SLOPE}| \Rightarrow$ more sensitive to violations.



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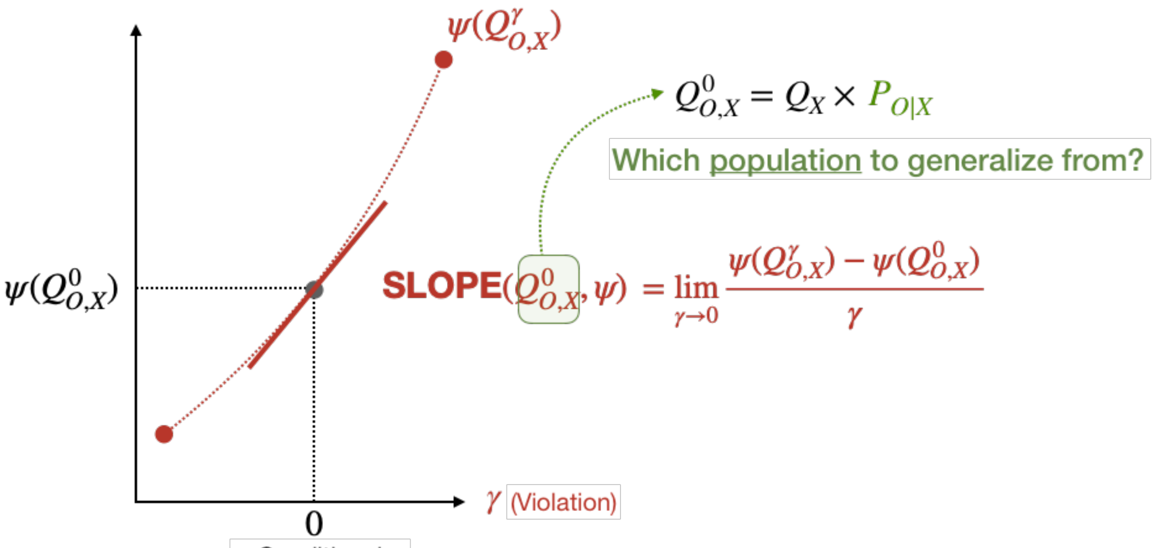

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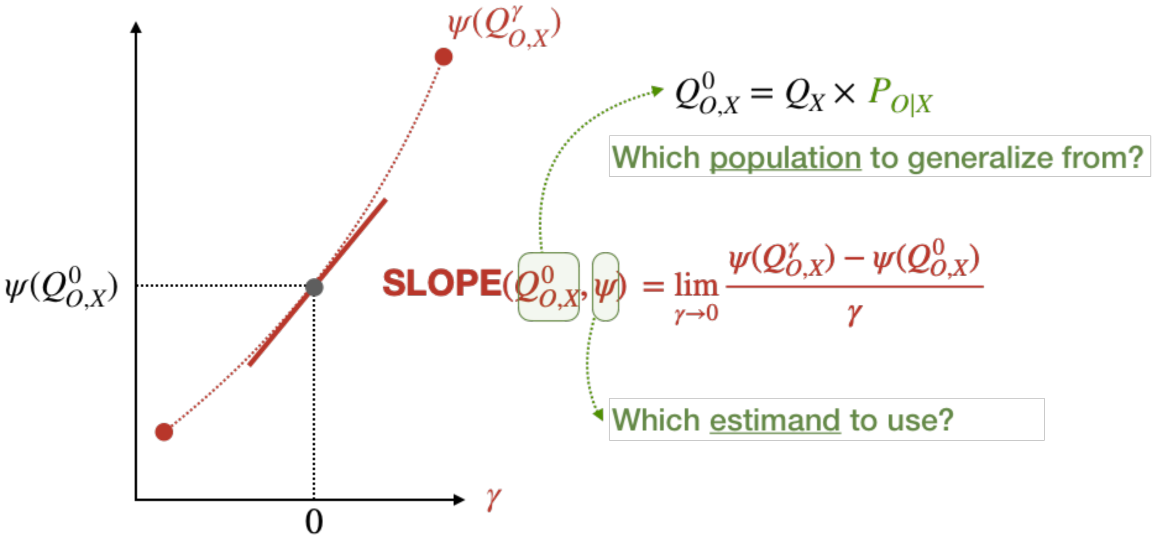

Outline

1 Proposal: SLOPE

2 Data Application

3 Summary

Notation and Review of Key Concepts

- Target distribution: $Q_{O,X}$ of (O, X) where $O \in \mathbb{R}$ and $X \in \mathcal{X} \subset \mathbb{R}^d$.
- Scientific interest: a low dimensional $\psi(Q_{O,X})$ with $\psi(\cdot)$ defined by investigator.
- We only have access to Q_X and a related source distribution $P_{O,X}$.

In order to identify the functional $\psi(Q_{O,X})$ using the random sample from $P_{O,X}$ and Q_X, two assumptions are often imposed.

Assumption 1 (Overlap)

Q_X is absolute continuous with respect to P_X.

Assumption 2 (Conditional Exchangeability)

$Q_{O|X}(\cdot \mid x) = P_{O|X}(\cdot \mid x)$ *almost everywhere* Q_X.

Under Assumptions 1-2, the target functional can be identified:

$$\psi(Q_{O,X}) = \psi(Q_X \times Q_{O|X}) = \psi(Q_X \times P_{O|X}).$$

Example functionals include the mean ψ^{MEAN}, median ψ^{MED}, and OLS coefficient ψ^{OLS}.

What If exchangeability Doesn't Hold? Sensitivity Analysis

Conditional exchangeability is made on the unobserved data and cannot be empirically verified [Nguyen et al. 2017; Nie, Imbens, and Wager 2021; Huang 2024]. Assuming it blindly can lead to distorted conclusions [Jin, Egami, and Rothenhäusler 2024].

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$$\frac{\mathrm{d}Q^{\gamma}_{O|X}(o \mid x)}{\mathrm{d}P_{O|X}(o \mid x)} \propto \exp(\gamma \cdot o), \tag{2}$$

where we assume $\int \exp(\gamma o)\mathrm{d}P_{O|X}(o, x) < \infty$.

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- $\gamma \in \mathbb{R}$ is a sensitivity parameter that measures the unobserved differences between $Q_{O|X}$ and $P_{O|X}$.
- $\gamma = 0 \implies$ conditional exchangeability holds.
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Examples:

- If $P_{O|X} \sim N\left(\mu(X), \sigma^2(X)\right)$, then (2) elicits a location shift, $Q^{\gamma}_{O|X} \sim N\left(\mu(X) + \gamma\sigma^2(X), \sigma^2(X)\right)$.
- If O is binary, then γ has an odds ratio interpretation, $\exp(\gamma) = \dfrac{Q^{\gamma}(O = 1 \mid X)/\{1 - Q^{\gamma}(O = 1 \mid X)\}}{P(O = 1 \mid X)/\{1 - P(O = 1 \mid X)\}}$.

SLOPE

Let $Q^\gamma_{O,X} = Q_X \times Q^\gamma_{O|X}$.

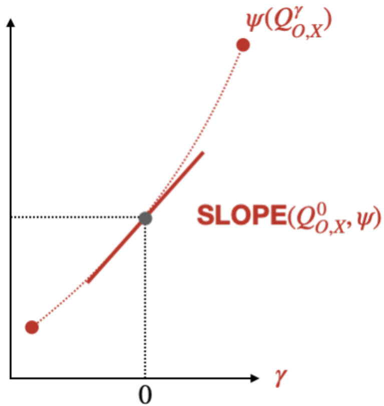
Definition 1

The ***S****ensitivity to* ***LO****cal* ***P****erturbation from* ***E****xchangeability (SLOPE) of a functional ψ with respect to the perturbation* (2) *is*

$$\text{SLOPE}(Q^0_{O,X}, \psi) = \lim_{\gamma \to 0} \frac{\psi(Q^\gamma_{O,X}) - \psi(Q^0_{O,X})}{\gamma}, \tag{3}$$

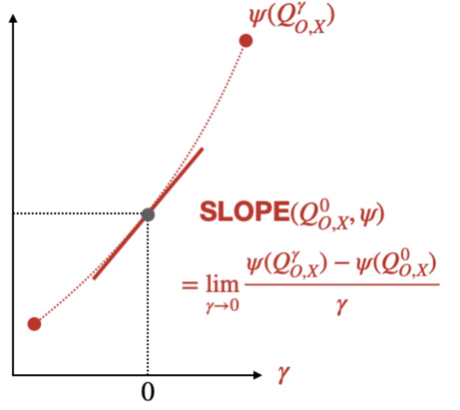
provided the limit exists.

- SLOPE is a (usually) scalar summary to summarize sensitivity of an **estimand** when conditional exchangeability is violated in a particular "direction".
- Higher $|\text{SLOPE}| \Longrightarrow$ estimand changes drastically under local violations $\Longrightarrow$ more sensitive.



REMARKS ON THE SLOPE (1/2)

- SLOPE($Q^0_{O,X}, \psi$) depends on the distribution $Q^0_{O,X} = Q_X \times P_{O|X}$ and the estimand ψ.
  - SLOPE can be applied to choose source ($P_{O|X}$) and estimand (ψ) in order to be less sensitive to violations of conditional exchangeability.



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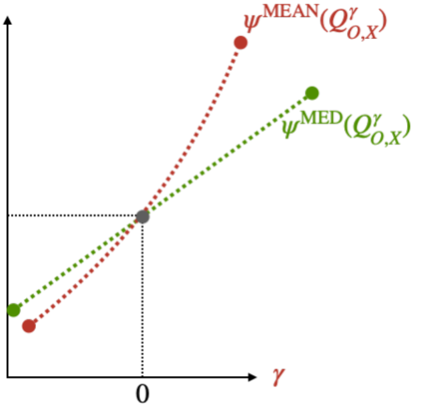

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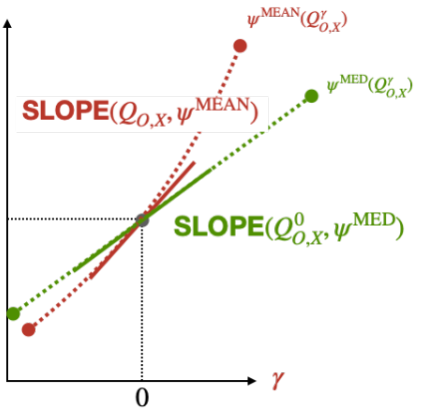

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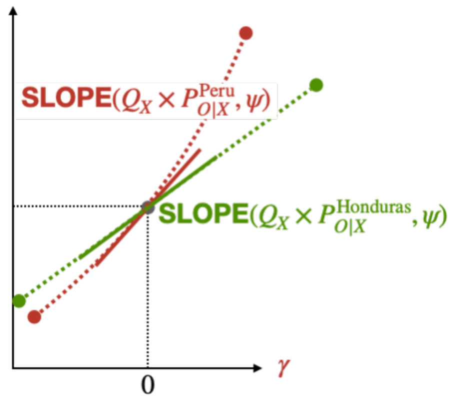

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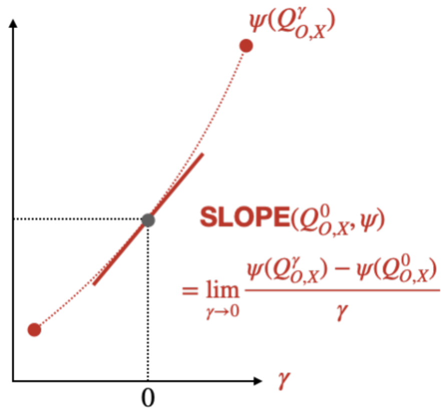
  - SLOPE is a population level quantity which reflects the design instead of estimation; it does not depend on any estimation procedure.
  - We think this way of summarizing sensitivity can be broadly helpful.



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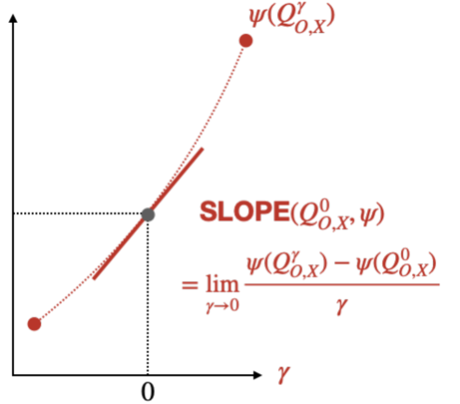
- Additional remarks on interpretation.
  - The unit of the SLOPE matches the unit of the estimand.
  - SLOPE is a first-order violation of conditional exchangeability, i.e.,

$$\psi(Q^\gamma_{O,X}) \approx \psi(Q^0_{O,X}) + \gamma \cdot \text{SLOPE}(Q^0_{O,X}, \psi).$$

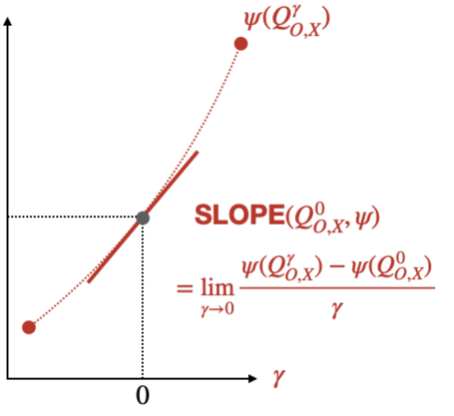
  - SLOPE cannot identify the bias under violation (which is unknown).



Remarks on The SLOPE (2/2)

Technical remarks:

- Finding the form of the SLOPE is not difficult; it's an exercise in calculus if we can freely interchange integration and differentiation.
- SLOPE can be consistently estimated under regularity conditions; not covered in this talk.
- SLOPE does not exist for every estimand; e.g., $\mathrm{sign}_{Q_O}(O)$.



Example: SLOPE for Mean and Median

Let $\sigma^2(X) = \mathrm{var}_{P_{O|X}}(O \mid X)$ be the conditional variance on the source population.

The SLOPE for the mean is

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Under the conditional Gaussian model, $P_{O|X} \sim N\left(\mu(X), \sigma^2(X)\right)$, the SLOPE for the median is

$$\mathrm{SLOPE}(Q^0_{O,X}, \psi^{\mathrm{MED}}) = \mathrm{E}_{Q_X}\left[\sigma^2(X) \cdot \underbrace{\frac{f_{P_{O|X}}(m_{1/2})}{\mathrm{E}_{Q_X}\left\{f_{P_{O|X}}(m_{1/2})\right\}}}_{\text{Weight}}\right], \quad \text{(SLOPE for Median)}$$

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SLOPE for location parameters depends heavily on heterogeneity of O in the source population. This matches with existing insights that homogeneous ATE is easier to generalize [Tipton 2014].

Connection To Literature

Prior works are mostly on estimators rather than estimands.

- Robust statistics: "...robustness signifies insensitivity to small deviations from the assumptions..." [Huber 1981]
  - SLOPE is analogous to the influence function (IF) [Hampel 1974] as a local derivative.

$$\mathrm{SLOPE}(Q^0_{O,X}, \psi) = \lim_{\gamma \to 0} \frac{\psi(Q^\gamma_{O,X}) - \psi(Q^0_{O,X})}{\gamma}$$

$$\mathrm{IF}(Q^0_{O,X}, \psi, o, x) = \lim_{\varepsilon \to 0^+} \frac{\psi((1-\varepsilon)Q^0_{O,X} + \varepsilon \delta_{o,x}) - \psi(Q^0_{O,X})}{\varepsilon}$$

  - For parameters defined through Z-estimation, SLOPE can be re-expressed with the IF (next slide).

Connecting SLOPE with IF in Z-Estimation

Suppose the finite-dimensional parameter ψ is defined through

$$\mathrm{E}_{Q^0_{O,X}}\{s(O,X,\psi)\} = 0, \tag{4}$$

where s is a known function.

Theorem 1 (SLOPE and Influence Function for Z-Estimation)

The SLOPE for ψ in (4) is

$$\mathsf{SLOPE}(Q^0_{O,X},\psi) = \iint \underbrace{\mathsf{IF}(O,X,\psi(Q^0_{O,X}))}_{\text{Influence function}} \underbrace{\{O-\mu(X)\}}_{\text{Direction of sensitivity}} dP_{O|X}dQ_X,$$

where $\mu(X) = E_{P_{O|X}}(O \mid X)$, and $\mathsf{IF}(O,X,\psi(Q^0_{O,X}))$ is the influence function for $\psi(Q^0_{O,X})$ under $Q^0_{O,X}$.

SLOPE is a recentered integral of the IF, where the re-centering is determined by $O-\mu(X)$.

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Running Example: Transporting Potential Outcomes

- [Banerjee et al. 2015] conducted a multi-site RCT to evaluate the effect of the "Graduation Program" on improving livelihoods of poor households in six countries (Ethiopia, Ghana, Honduras, India, Pakistan, Peru).
    - "...provides a holistic set of services, including the grant of a productive asset, to the poorest households in a village"
- We treat one country as the source (P) and another as the target (Q) and investigate the sensitivity to violations of conditional exchangeability between $Q_{O|X}$ and $P_{O|X}$.
- X is a categorical baseline measurement prior to intervention.
- $O = Y(1)$ is the potential outcome under the intervention from "Graduation Program".
- $P = P_{O,X} = P_{Y(1),X}$: households in the source country (e.g., Honduras).
- $Q = Q_{O,X} = Q_{Y(1),X}$: households in the target country (e.g., Ethiopia), where we only observe the baseline Q_X.
- Identification of the SLOPE requires additional assumptions on $P_{O,X}$ (SUTVA, strong ignorability).

Application I: The Choice of Source Country

- Outcome: log-transformed total asset index (standardized total asset value in terms of goats in each country). It ranges from -1.3 to 2.97.

Question: Which source country is the least sensitive?

Estimand (ψ)	Source ($P_{O\|X}$)	Target (Q_X) Ethiopia
Mean	Ethiopia	
	Ghana	0.13
	Honduras	$\boxed{0.09}$
	India	0.16
	Pakistan	0.16
	Peru	0.20

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Mean	Ethiopia		0.15	0.18	0.15	0.14	0.13
	Ghana	0.13		$\boxed{0.10}$	0.10	0.11	0.13
	Honduras	$\boxed{0.09}$	$\boxed{0.08}$		$\boxed{0.07}$	$\boxed{0.08}$	$\boxed{0.10}$
	India	0.16	0.15	0.16		0.16	0.16
	Pakistan	0.16	0.15	0.14	0.15		0.16
	Peru	0.20	0.21	0.21	0.21	0.21	

Application I: The Choice of Source Country

- Outcome: log-transformed total asset index (standardized total asset value in terms of goats in each country). It ranges from -1.3 to 2.97.

Question: Which source country is the least sensitive? **Answer**: Honduras.

Estimand (ψ)	Source ($P_{O\|X}$)	Target (Q_X)					
		Ethiopia	Ghana	Honduras	India	Pakistan	Peru
Mean	Ethiopia		0.15	0.18	0.15	0.14	0.13
	Ghana	0.13		$\boxed{0.10}$	0.10	0.11	0.13
	Honduras	$\boxed{0.09}$	$\boxed{0.08}$		$\boxed{0.07}$	$\boxed{0.08}$	$\boxed{0.10}$
	India	0.16	0.15	0.16		0.16	0.16
	Pakistan	0.16	0.15	0.14	0.15		0.16
	Peru	0.20	0.21	0.21	0.21	0.21	
Median	Ethiopia		0.15	0.18	0.15	0.14	0.13
	Ghana	0.11		$\boxed{0.10}$	0.09	0.09	0.11
	Honduras	$\boxed{0.07}$	$\boxed{0.06}$		$\boxed{0.05}$	$\boxed{0.06}$	$\boxed{0.07}$
	India	0.15	0.15	0.16		0.15	0.16
	Pakistan	0.16	0.15	0.14	0.15		0.16
	Peru	0.19	0.21	0.21	0.22	0.21	

Application I: The Choice of Source Country

- Outcome: log-transformed total asset index (standardized total asset value in terms of goats in each country). It ranges from -1.3 to 2.97.

Question: Which source country / estimand is the least sensitive? **Answer**: Honduras / Median.

Estimand (ψ)	Source ($P_{O\|X}$)	Target (Q_X) Ethiopia	Ghana	Honduras	India	Pakistan	Peru
Mean	Ethiopia		0.15	0.18	0.15	0.14	0.13
	Ghana	0.13		$\boxed{0.10}$	0.10	0.11	0.13
	Honduras	$\boxed{0.09}$	$\boxed{0.08}$		$\boxed{0.07}$	$\boxed{0.08}$	$\boxed{0.10}$
	India	0.16	0.15	0.16		0.16	0.16
	Pakistan	0.16	0.15	0.14	0.15		0.16
	Peru	0.20	0.21	0.21	0.21	0.21	
Median	Ethiopia		0.15	0.18	0.15	0.14	0.13
	Ghana	0.11		$\boxed{0.10}$	0.09	0.09	0.11
	Honduras	$\boxed{0.07}$	$\boxed{0.06}$		$\boxed{0.05}$	$\boxed{0.06}$	$\boxed{0.07}$
	India	0.15	0.15	0.16		0.15	0.16
	Pakistan	0.16	0.15	0.14	0.15		0.16
	Peru	0.19	0.21	0.21	0.22	0.21	

- The median is usually slightly better than the mean (i.e., has smaller SLOPE).

Application II: The Choice of Linear Combinations of a Multi-variate Outcome

[Banerjee et al. 2015] reported the "physical health index", which is the average of z-scores of three variables:

- Did not miss work due to illness (`work`), Z_{work} between -4.3 and 7.0;
- Activities of daily living score (`act`), Z_{act} between -3.2 and 1.1;
- Perception of health status (`perc`), Z_{perc} between -2.3 and 4.5.

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- Perception of health status (`perc`), Z_{perc} between -2.3 and 4.5.

> **Question**: Is there another way to define a physical health index so that it's less sensitive during generalization?

- Consider linear combinations:

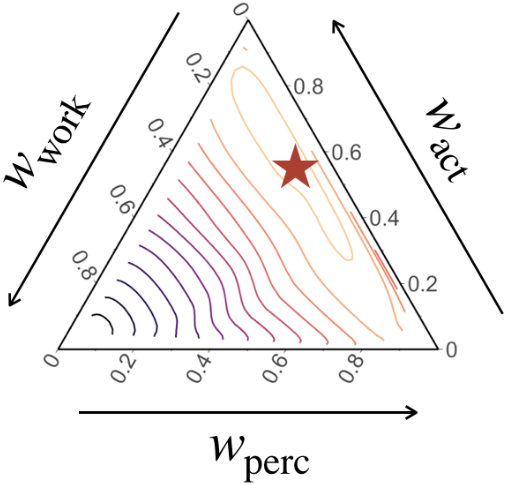
$$w_{\text{work}} \cdot Z_{\text{work}} + w_{\text{act}} \cdot Z_{\text{act}} + w_{\text{perc}} \cdot Z_{\text{perc}},$$

  where the original index has $w_{\text{work}} = w_{\text{act}} = w_{\text{perc}} = 1/3$.
- On the simplex of $w = [w_{\text{work}}, w_{\text{act}}, w_{\text{perc}}]^{\mathrm{T}}$, we want to find the weight that yields the smallest SLOPE.

Application II: The Choice of Linear Combinations of a Multi-variate Outcome

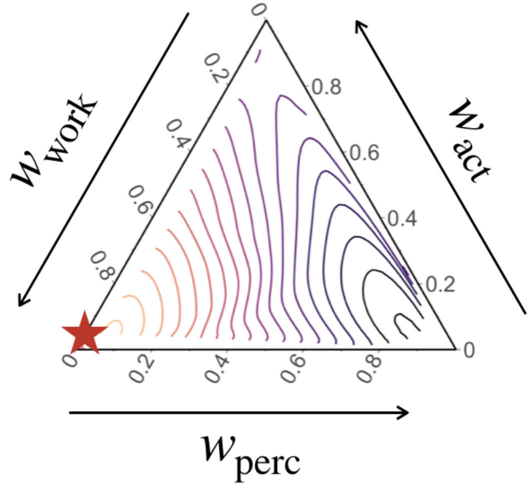
India (P) ⇒ Ethiopia (Q)

★ $[w_{\text{work}}, w_{\text{act}}, w_{\text{perc}}]^{\mathrm{T}} = [0.10, 0.55, 0.35]^{\mathrm{T}}$



Peru (P) ⇒ Ethiopia (Q)

★ $[w_{\text{work}}, w_{\text{act}}, w_{\text{perc}}]^{\mathrm{T}} = [1, 0, 0]^{\mathrm{T}}$



Outline

1 Proposal: SLOPE

2 Data Application

3 **Summary**

Summary and contact

We propose a measure, sensitivity to local perturbation from exchangeability (SLOPE), that describes the change in a functional ψ with respect to a local perturbation (2):

$$\text{SLOPE}(Q^0_{O,X}, \psi) = \lim_{\gamma \to 0} \frac{\psi(Q^\gamma_{O,X}) - \psi(Q^0_{O,X})}{\gamma},$$

provided the limit exists.

- SLOPE is a scalar summary for sensitivity to violation of the conditional exchangeability assumption.
- SLOPE guides two choices in design:
  - Choosing the source population P.
  - Choosing the estimand $\psi(\cdot)$.
- SLOPE connects to influence function (both conceptually and mathematically).
- Main limitations:
  - SLOPE assesses the local sensitivity (in a linear approximation).
  - SLOPE is not unit-less.

Contact: at ACIC or xinran.miao@wisc.edu

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Example Estimands $\psi(\cdot)$

- Outcome mean: $\psi^{\text{MEAN}}(Q_{O,X})$ which is

$$\begin{aligned}\psi^{\text{MEAN}}(Q_{O,X}) = \mathrm{E}_{Q_{O,X}}(O) &= \mathrm{E}_{Q_X}\left[\mathrm{E}_{Q_{O|X}}(O)\right] \\ &= \mathrm{E}_{Q_X}\left[\mathrm{E}_{P_{O|X}}(O)\right] \qquad \text{(By Assumptions 1-2).}\end{aligned}$$

- Outcome median: $\psi^{\text{MED}}(Q_{O,X})$ such that

$$\begin{aligned}1/2 = \int_{-\infty}^{\psi^{\text{MED}}} \mathrm{d}Q_O &= \int \int_{-\infty}^{\psi^{\text{MED}}} \mathrm{d}Q_{O|X}\mathrm{d}Q_X \\ &= \int \int_{-\infty}^{\psi^{\text{MED}}} \mathrm{d}P_{O|X}\mathrm{d}Q_X \qquad \text{(By Assumptions 1-2).}\end{aligned}$$

- OLS coefficient: $\psi^{\text{OLS}}(Q_{O,X})$ such that

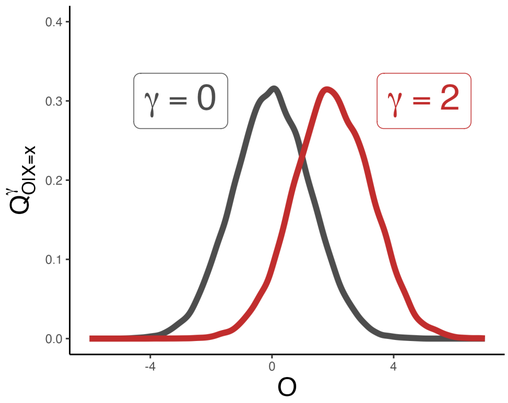
$$\begin{aligned}0 &= \mathrm{E}_{Q_{O,X}}\left[XX^{\mathrm{T}}\psi^{\text{OLS}} - XO\right] \\ &= \mathrm{E}_{Q_X}\left[\mathrm{E}_{P_{O|X}}\left\{XX^{\mathrm{T}}\psi^{\text{OLS}} - XO\right\}\right] \qquad \text{(By Assumptions 1-2).}\end{aligned}$$

Examples of Exponential Tilt

$$\frac{\mathrm{d}Q^{\gamma}_{O|X}(o \mid x)}{\mathrm{d}P_{O|X}(o \mid x)} \propto \exp(\gamma \cdot o), \tag{2}$$

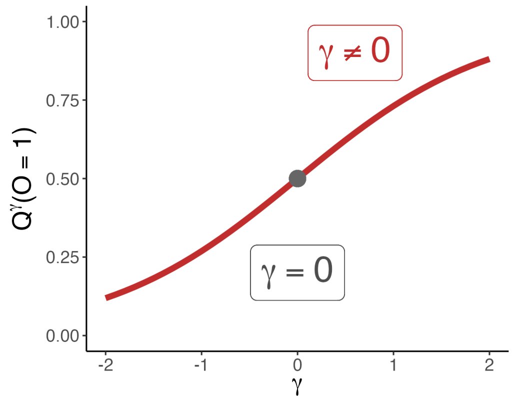
- If $P_{O|X} \sim N\left(\mu(X), \sigma^2(X)\right)$, then (2) elicits a location shift

$$Q^{\gamma}_{O|X} \sim N\left(\mu(X) + \gamma\sigma^2(X), \sigma^2(X)\right).$$



- If O is binary, then γ has an odds ratio interpretation,

$$\exp(\gamma) = \frac{Q^{\gamma}(O=1 \mid X)/\{1 - Q^{\gamma}(O=1 \mid X)\}}{P(O=1 \mid X)/\{1 - P(O=1 \mid X)\}}.$$



Why This Sensitivity Model?

$$\frac{\mathrm{d}Q^{\gamma}_{O|X}(o \mid x)}{\mathrm{d}P_{O|X}(o \mid x)} \propto \exp(\gamma \cdot o), \tag{2}$$

- Used in non-ignorable missing data [Robins, Rotnitzky, and Scharfstein 2000], unmeasured confounding in causal inference [Franks, D'Amour, and Feller 2020; Nabi et al. 2024], and effect generalization [Dahabreh et al. 2022].
- Simple to interprete (i.e., exponential tilting; odds ratio); can reparametrize γ to pseudo-R^2 [Franks, D'Amour, and Feller 2020].
- Does not place restrictions on observed data.
- Makes inference of the SLOPE (next slide) tractable (e.g., asymptotic normality).

Connection To Literature

Prior works are mostly on estimators rather than estimands.

- Robust statistics: "...robustness signifies insensitivity to small deviations from the assumptions..." [Huber 1981]
  - SLOPE is analogous to the influence function (IF) [Hampel 1974] as a local derivative.

$$\mathrm{SLOPE}(Q^0_{O,X}, \psi) = \lim_{\gamma \to 0} \frac{\psi(Q^\gamma_{O,X}) - \psi(Q^0_{O,X})}{\gamma}$$

$$\mathrm{IF}(Q^0_{O,X}, \psi, o, x) = \lim_{\varepsilon \to 0^+} \frac{\psi((1-\varepsilon)Q^0_{O,X} + \varepsilon\delta_{o,x}) - \psi(Q^0_{O,X})}{\varepsilon}$$

  - For parameters defined through Z-estimation, SLOPE can be re-expressed with the IF (next slide).
- Sensitivity analysis: our parametrization of $Q^\gamma_{O|X}$ as deviations from $Q^0_{O,X} = P_{O|X}$ was proposed by [Robins, Rotnitzky, and Scharfstein 2000].
  - This line of work focuses on estimation/inference given $\gamma \neq 0$ [Rotnitzky et al. 2001; Birmingham, Rotnitzky, and Fitzmaurice 2003; Franks, D'Amour, and Feller 2020; Dahabreh et al. 2022; Nabi et al. 2024].
- Other related works: [Rosenbaum 2004; Troxel, Ma, and Heitjan 2004; Ding and VanderWeele 2016; Andrews, Gentzkow, and Shapiro 2017; Oster 2019; Cinelli and Hazlett 2020; Zhao, Small, and Bhattacharya 2019; Colnet et al. 2023; Jin, Egami, and Rothenhäusler 2024].

Example: SLOPE for Transporting the Mean

Suppose $O \in \mathbb{R}$ is an outcome variable. The target estimand is the outcome mean

$$\psi^{\text{MEAN}}(Q_{O,X}) := \int O \mathrm{d}Q_O.$$

Under the sensitivity model (2), the target functional with sensitivity parameter γ is

$$\psi^{\text{MEAN}}(Q^{\gamma}_{O,X}) = \int \frac{\int O \exp(\gamma O) dP_{O|X}}{\int \exp(\gamma O) \mathrm{d}P_{O|X}} Q_X, \tag{5}$$

The SLOPE can be calculated via calculus:

$$\text{SLOPE}(Q^0_{O,X}, \psi^{\text{MEAN}}) = \frac{\partial \psi^{\text{MEAN}}(Q^{\gamma}_{O,X})}{\partial \gamma}\bigg|_{\gamma=0} = \mathrm{E}_{Q_X}\{\text{var}_{P_{O|X}}(O \mid X)\} := \mathrm{E}_{Q_X}\{\sigma^2(X)\}.$$

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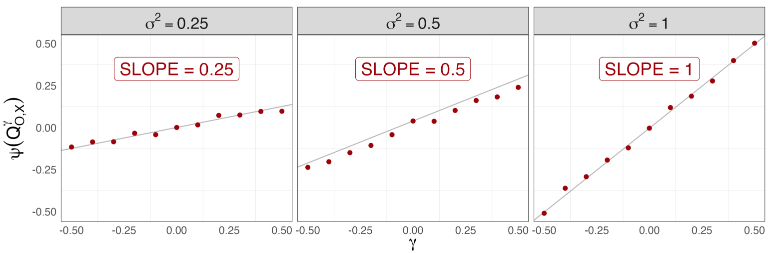
- For a specific target (i.e., Q_X), the choice of the $P_{O|X}$ determines the magnitude of $\sigma^2(X)$, and therefore determines the magnitude of SLOPE.



Intuition: Direction of Sensitivity

$$\mathsf{SLOPE}(Q^0_{O,X}, \psi) = \iint \underbrace{\mathsf{IF}(O, X, \psi(Q^0_{O,X}))}_{\text{Influence function}} \underbrace{\{O - \mu(X)\}}_{\text{Direction of sensitivity}} \; dP_{O|X} dQ_X,$$

To understand the "direction of sensitivity", we consider a broader class of tilting,

$$\frac{\mathrm{d}Q^{\gamma}_{O|X}(o, X)}{\mathrm{d}P_{O|X}(o, X)} \propto \rho(o, x, \gamma),$$

where $\rho(o, x, 0) = 1$ and $\int \rho(o, X, \gamma)\mathrm{d}P_{O|X}(o, X) \in (0, \infty)$.

Intuition: Direction of Sensitivity

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To understand the "direction of sensitivity", we consider a broader class of tilting,

$$\frac{\mathrm{d}Q^{\gamma}_{O|X}(o, X)}{\mathrm{d}P_{O|X}(o, X)} \propto \rho(o, x, \gamma),$$

where $\rho(o, x, 0) = 1$ and $\int \rho(o, X, \gamma)\mathrm{d}P_{O|X}(o, X) \in (0, \infty)$.

Then the SLOPE is

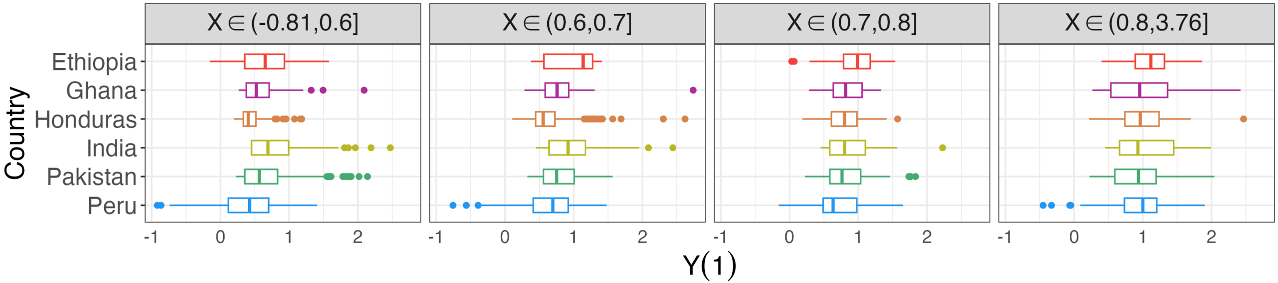
$$\mathsf{SLOPE}(Q^0_{O,X}, \psi) = \iint \mathsf{IF}(D, X, \psi(Q^0_{O,X})) \underbrace{\left[\dot{\rho}(O, X, 0) - \mathrm{E}_{P_{O|X}}\{\dot{\rho}(O, X, 0))\}\right]}_{\text{Direction of sensitivity}} dP_{O|X} dQ_X,$$

where $\dot{\rho}$ is the derivative of ρ with respect to γ.

Application I: The Choice of Source Country

Explanation: why does Honduras have the lowest SLOPE?

- Mathematically, $\sigma^2(X)$ is lower in Honduras.
  - More heterogeneity in the source population $\Rightarrow$ more sensitive to violations of the transportability assumption.
  - It matches existing understanding about ease of generalizability when the treatment effect is homogeneous (Tipton and Olsen 2018).



- Context-wise, treated households can choose an asset at their choice. In Honduras treated housholds made less diverse choice of the asset.

Table. Most common choice of asset transfer among countries.

Ethiopia	Ghana	Honduras	India	Pakistan	Peru
Sheep & goats (62%)	Goats & hens (44%)	Chickens (83%)	Goats (52%)	Goats (56%)	Guinea pigs (64%)

Application III: The Choice of Estimand

For all three outcome variables and all (source, target) pairs, we find the median to be in general less sensitive than the mean, i.e., $\mathrm{SLOPE}(Q^0_{Y(1),X}, \psi^{\mathrm{MED}}) < \mathrm{SLOPE}(Q^0_{Y(1),X}, \psi^{\mathrm{MEAN}})$.

To investigate what if the data generation changes slightly, we consider a semi-synthetic simulation.

- A simplified scenario with binary $X \in \{1, 2\}$ and Gaussian $P_{Y(1)|X}$:

$$\text{Source: } P_{Y(1)|X=1} \sim N(0, 0.5^2), \quad P_{Y(1)|X=2} \sim N(\mu_2, \sigma_2^2),$$
$$\text{Target: } q_1 = Q_X(X = 1), \quad q_2 = Q_X(X = 2)$$

  where μ_2, σ_2^2, q_1, q_2 are estimated from data.
- Source $P_{Y(1)|X}$: Peru.
- Target (Q_X): India ($q_1 = 0.2,\ q_2 = 0.8$) and a synthetic country ($q_1 = q_2 = 0.5$).
- Outcome: per capita consumption.

Application III: The Choice of Estimand

Peru (P) $\Rightarrow$ India (Q), with $P_{Y(1)|X=1} \sim N(0, 0.5^2)$, $P_{Y(1)|X=2} \sim N(\mu_2, \sigma_2^2)$.

Application III: The Choice of Estimand

Peru (P) $\Rightarrow$ India (Q), with $P_{Y(1)|X=1} \sim N(0, 0.5^2)$, $P_{Y(1)|X=2} \sim N(\mu_2, \boxed{\sigma_2^2})$.
Question: how would the choice of estimand change if the data changes a bit?

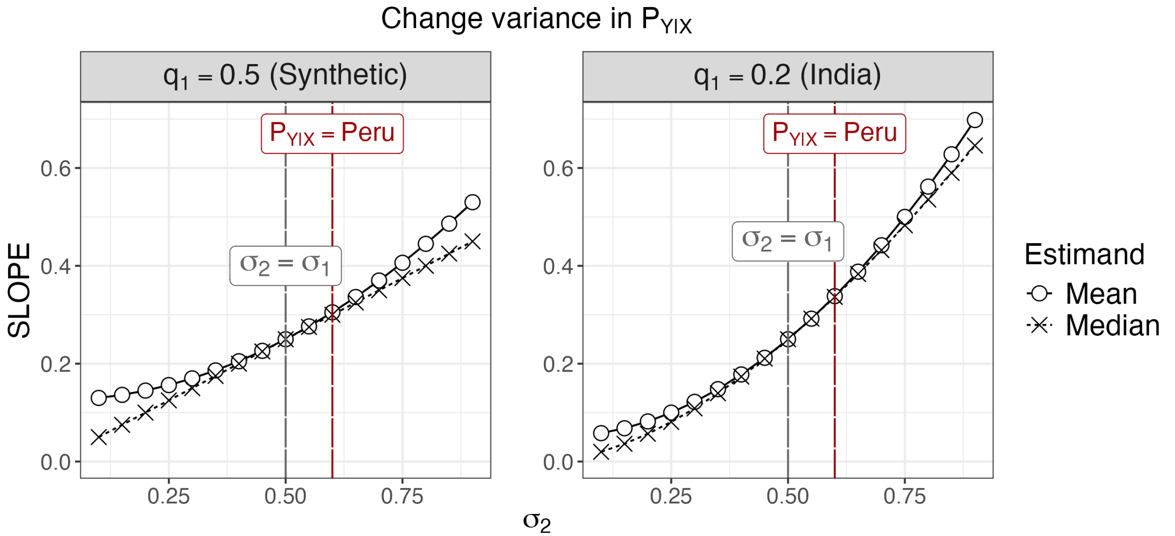
Application III: The Choice of Estimand

Peru (P) $\Rightarrow$ India (Q), with $P_{Y(1)|X=1} \sim N(0, 0.5^2)$, $P_{Y(1)|X=2} \sim N(\mu_2, \boxed{\sigma_2^2})$.

Question: how would the choice of estimand change if the data changes a bit?

- We vary the variance σ_2^2 and calculate the SLOPE in each case.
- For both mean and median, the SLOPE is increasing with σ_2^2.
- SLOPE for median $\leq$ SLOPE for the mean, where "$=$" happens when $\sigma_2 = \sigma_1$.

Answer: we still choose the median.



Application III: The Choice of Estimand

Peru (P) $\Rightarrow$ India (Q), with $P_{Y(1)|X=1} \sim N(0, 0.5^2)$, $P_{Y(1)|X=2} \sim N(\boxed{\mu_2}, \sigma_2^2)$.
Question: how would the choice of estimand change if the data changes a bit?

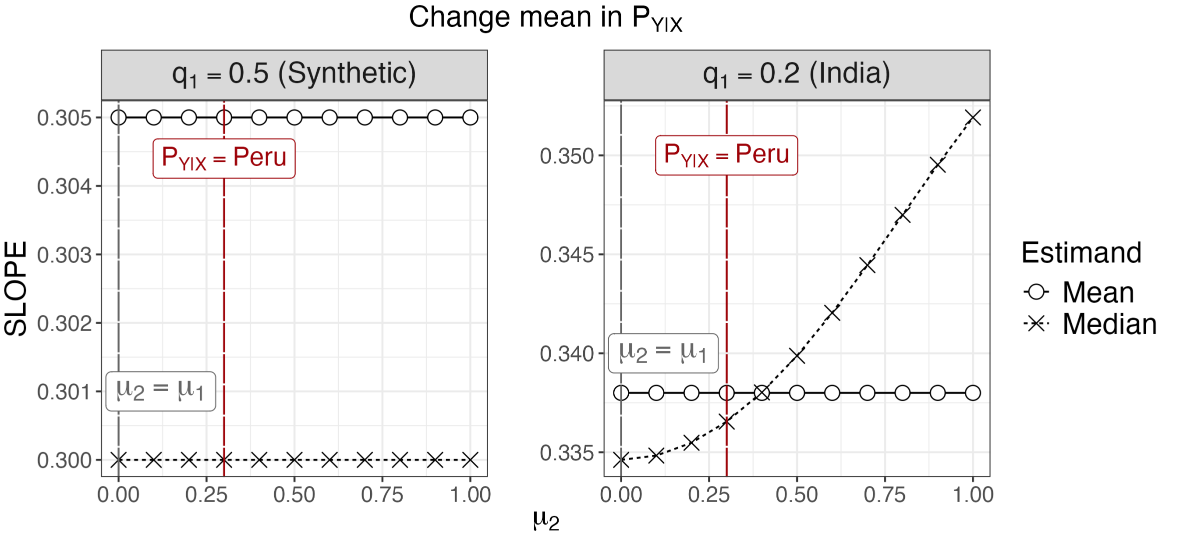
- We vary the mean μ_2 and calculate the SLOPE in each case.

Application III: The Choice of Estimand

Peru (P) $\Rightarrow$ India (Q), with $P_{Y(1)|X=1} \sim N(0, 0.5^2)$, $P_{Y(1)|X=2} \sim N(\boxed{\mu_2}, \sigma_2^2)$.
Question: how would the choice of estimand change if the data changes a bit?

- We vary the mean μ_2 and calculate the SLOPE in each case.
- When $q_1 = q_2 = 0.5$, SLOPEs do not depend on μ_2:

$$\mathrm{SLOPE}(Q^{\gamma}_{Y(1),X}, \psi^{\mathrm{MEAN}}) = (\sigma_1^2 + \sigma_2^2)/2 \geq \sigma_1\sigma_2 = \mathrm{SLOPE}(Q^{\gamma}_{Y(1),X}, \psi^{\mathrm{MED}}).$$



Application III: The Choice of Estimand

Peru (P) $\Rightarrow$ India (Q), with $P_{Y(1)|X=1} \sim N(0, 0.5^2)$, $P_{Y(1)|X=2} \sim N(\boxed{\mu_2}, \sigma_2^2)$.
Question: how would the choice of estimand change if the data changes a bit?

- We vary the mean μ_2 and calculate the SLOPE in each case.
- When $q_1 = 0.2$, recall that SLOPEs for mean and median are both weighted average of σ_1^2 and σ_2^2:

$$\text{SLOPE}(Q^{\gamma}_{Y(1),X}, \psi^{\text{MEAN}}) = q_1\sigma_1^2 + q_2\sigma_2^2 \text{ does not change with } \mu_2$$

$$\text{SLOPE}(Q^{\gamma}_{Y(1),X}, \psi^{\text{MED}}) = w_1\sigma_1^2 + w_2\sigma_2^2 = \sigma_2^2 + w_2\underbrace{(\sigma_2^2 - \sigma_1^2)}_{\geq 0}, \text{ increases with } \mu_2.$$