Derivation of $\triangle_0$

Yash Lokare[1*†]

[1] *Department of Physics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, 100016, India*

Abstract

Presented herein is the derivation of a self-consistent relation for the superconducting energy gap as a function of temperature.

1 Derivation

For the fermionic creation and annihilation operators, we define new operators $\gamma_{k\sigma}$ and coefficients u_k , v_k as (*note*: we apply the Bogoliubov transformation)

$$c_{k\uparrow} = u_k^* \gamma_{k\uparrow} + v_k \gamma_{-k\downarrow}^\dagger, \tag{1}$$

and

$$c_{-k\downarrow}^\dagger = u_k \gamma_{-k\downarrow}^\dagger - v_k^* \gamma_{k\uparrow}. \tag{2}$$

Note that, for the fermionic commutation relations to be satisfied, the following condition must hold

$$|u_k|^2 + |v_k|^2 = 1. \tag{3}$$

To determine the gap function Δ_k , we apply the Bogoliubov transformations again. Note that the BCS theory gives the following self-consistent relation for the gap function Δ_k

$$\Delta_k = -\frac{1}{N} \sum_{k'} V_{kk'} \langle c_{-k'\downarrow}^\dagger c_{k'\uparrow} \rangle, \tag{4}$$

where $V_{kk'}$ are the matrix elements for the two-body interaction term in the BCS Hamiltonian. From Eq. (4) and the Bogoliubov transformations we define in Eqs. (1) and (2), we have for the gap function

$$\Delta_k = -\frac{1}{N} \sum_{k'} V_{kk'} \langle (u_{k'}^* \gamma_{-k'\downarrow} - v_{k'} \gamma_{k'\uparrow}^\dagger)(u_{k'}^* \gamma_{k'\uparrow} + \gamma_{k'\downarrow}^\dagger) \rangle, \tag{5}$$

*Corresponding author.

†E-mail: ph1180857@iitd.ac.in

and

$$\Delta_k = -\frac{1}{N}\sum_{k'} V_{kk'} u^*_{k'} v_{k'} (\langle \gamma_{-k'\downarrow}\gamma^\dagger_{-k'\downarrow}\rangle - \langle \gamma^\dagger_{k'\uparrow}\gamma_{k'\uparrow}\rangle). \tag{6}$$

We know that the Bogoliubons follow a Fermi-Dirac distribution, for which we have for the energy dispersion E_k

$$\langle \gamma^\dagger_{k'\uparrow}\gamma_{k'\uparrow}\rangle = \langle \gamma^\dagger_{-k'\downarrow}\gamma_{k'\downarrow}\rangle = \frac{1}{e^{\beta E_{k'}} + 1}, \tag{7}$$

which yields

$$\langle \gamma_{-k'\downarrow}\gamma^\dagger_{-k'\downarrow}\rangle - \langle \gamma^\dagger_{k'\uparrow}\gamma_{k'\uparrow}\rangle = 1 - \frac{2}{e^{\beta E_{k'}} + 1} = \tanh\left(\frac{E_{k'}}{2k_B T}\right). \tag{8}$$

The BCS theory gives us closed-form expressions for $|u_k|$ and $|v_k|$, as follows

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}}\right), \tag{9}$$

and

$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}}\right). \tag{10}$$

From Eqs. (9) and (10), we have

$$u^*_{k'} v_{k'} = |u_k|^2 \frac{v_{k'}}{u_{k'}} = \frac{\Delta_{k'}}{2\sqrt{\xi_{k'}^2 + |\Delta_{k'}|^2}}, \tag{11}$$

which finally yields a self-consistent relation for the gap function Δ_k

$$\Delta_k = -\frac{1}{N}\sum_{k'} \frac{V_{kk'}\Delta_{k'}}{2E_{k'}} \tanh\left(\frac{E_{k'}}{2k_B T}\right). \tag{12}$$