Module 3.2:Differentiation Under Integral Sign

By Dr. Keya Doshi

Introduction

Differentiation under the integral sign is an operation in calculus used to evaluate certain integrals.
Under some conditions on the function being integrated, differentiation under the integral sign allows to interchange the order of integration and differentiation.
This is also called the **Leibniz integral rule**, differentiation under the integral sign makes the following equation valid under light assumptions on f:

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial}{\partial t} f(x,t)\,dx$$

where $f(x,t)$ is a continuous function of x and t
t is a parameter
$\frac{\partial}{\partial t}$ is a continuous function throughout the interval $[a,b]$ where a and b are independent of t
Hence If

$$I = \int_a^b f(x,t)\,dx$$

Then

$$\frac{dI}{dt} = \int_a^b \frac{\partial}{\partial t} f(x,t)\,dx$$

Integrals with constant limits of integration having one parameter

Ex 1 : Prove that

$$\int_0^\infty \frac{e^{-xt} sinx}{x}\ dx = cot^{-1}x$$

Hence deduce that

$$\int_0^\infty \frac{sinx}{x}\ dx = \frac{\pi}{2}$$

Solution

Consider

$$I(t) = \int_0^\infty \frac{e^{-xt} sinx}{x}\ dx$$

Using DUIS rule, w.r.t t we get

$$\begin{aligned}
\frac{dI}{dt} &= \int_0^\infty \frac{\partial}{\partial t}\left(\frac{e^{-xt} sinx}{x}\right)\ dx \\
&= \int_0^\infty \frac{-xe^{-xt} sinx}{x}\ dx \\
&= -\int_0^\infty e^{-xt} sinx\ dx \\
&= -\left\{\frac{e^{-xt}}{t^2+1}\left[-tsinx - cosx\right]\right\}_0^\infty \\
&= 0 - \frac{e^0}{t^2+1}\left[-tsin0 - cos0\right] \\
\therefore \frac{dI}{dt} &= -\frac{1}{t^2+1}
\end{aligned}$$

Integrating w.r.t. t we get

$$I = \int -\frac{1}{t^2+1} dt$$

$$I = -cot^{-1}t + c$$

As $t \to \infty$

$$I(t \to \infty) = -cot^{-1}\infty + c$$

$$\implies c = 0$$

Hence

$$I = cot^{-1}t$$

Also as $t \to 0$

$$\int_0^\infty \frac{sinx}{x}\ dx = \frac{\pi}{2}$$

Hence proved

Ex 2 : Prove that

$$\int_0^\infty \frac{tan^{-1}(ax)}{x(1+x^2)}\ dx = \frac{\pi}{2}log(1+a); a > 0, a \neq 1$$

Solution
Consider

$$I(a) = \int_0^\infty \frac{tan^{-1}(ax)}{x(1+x^2)}\ dx$$

Using DUIS rule, w.r.t a we get

$$\begin{aligned}
\frac{dI}{da} &= \int_0^\infty \frac{\partial}{\partial a}\left(\frac{tan^{-1}(ax)}{x(1+x^2)}\right)\ dx \\
&= \int_0^\infty \left(\frac{1}{1+a^2x^2}\right)\left(\frac{x}{x(1+x^2)}\right)\ dx \\
&= \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)} dx....\mathbf{(1)}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{(1+x^2)(1+a^2x^2)} &= \frac{A}{1+x^2} + \frac{B}{1+a^2x^2} \\
&= \frac{A}{1+t} + \frac{B}{1+a^2t}\ where\ t = x^2 \\
&= A(1+a^2t) + B(1+t)
\end{aligned}$$

For $t = \frac{-1}{a^2}$

$$B(1 - \frac{1}{a^2}) = 1$$

$$\implies B = \frac{-a^2}{1 - a^2}$$

For $t = -1$

$$A(1 - a^2) = 1$$

$$\implies A = \frac{1}{1 - a^2}$$

Substituting in (1)

$$\begin{aligned}
\frac{dI}{da} &= \int_0^\infty \frac{1}{(1 + x^2)(1 + a^2x^2)} dx \\
&= \int_0^\infty \frac{1}{(1 - a^2)} \left[\frac{1}{1 + x^2} - \frac{a^2}{1 + a^2x^2} \right] dx \\
&= \frac{1}{(1 - a^2)} \int_0^\infty \frac{dx}{1 + x^2} - \frac{a^2}{(1 - a^2)} \int_0^\infty \frac{dx}{1 + a^2x^2} \\
&= \frac{1}{(1 - a^2)} \left[tan^{-1}x \right]_0^\infty - \frac{a^2}{(1 - a^2)} \left[\frac{tan^{-1}ax}{a} \right]_0^\infty \\
&= \frac{1}{(1 - a^2)} \left[\frac{\pi}{2} \right] - \frac{a}{(1 - a^2)} \left[\frac{\pi}{2} \right] \\
&= \frac{1}{(1 - a^2)} \left[\frac{\pi}{2} - \frac{a\pi}{2} \right] \\
\frac{dI}{da} &= \frac{\pi/2}{1 + a}
\end{aligned}$$

Integrating w.r.t. a we get

$$I = \int \frac{\pi/2}{1 + a} da$$

$$I = \frac{\pi}{2} log(1 + a) + c$$

As $a = 0$

$$I(0) = \frac{\pi}{2} log(1 + 0) + c$$

$$\implies c = 0$$

Hence

$$I = \frac{\pi}{2} log(1 + a)$$