Q.1)

1a) If $\tan\frac{x}{2}=\tanh\frac{u}{2}$, show that $u=\log\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$. (Chp: Hyperbolic Functions) (3)

Ans. $\tan\frac{x}{2}=\tanh\frac{u}{2}$

$$=\frac{\sinh u/2}{\cosh u/2}$$

$$=\frac{\left(e^{u/2}-e^{-u/2}\right)/\cancel{2}}{\left(e^{u/2}+e^{-u/2}\right)/\cancel{2}}\quad\left\{\because \sinh x=\frac{e^{x}-e^{-x}}{2};\cosh x=\frac{e^{x}+e^{-x}}{2}\right\}$$

$$=\frac{\left(e^{u/2}-e^{-u/2}\right)}{\left(e^{u/2}+e^{-u/2}\right)}\times\frac{e^{u/2}}{e^{u/2}}$$

$$=\frac{e^{u}-1}{e^{u}+1}$$

$$\therefore\ \tan\frac{x}{2}=\frac{e^{u}-1}{e^{u}+1}$$

$$\therefore\ \frac{1}{\tan x/2}=\frac{e^{u}+1}{e^{u}-1}\quad\text{(By Invertendo)}$$

$$\therefore\ \frac{1+\tan x/2}{1-\tan x/2}=\frac{\left(e^{u}+\cancel{1}\right)+\left(e^{u}-\cancel{1}\right)}{\left(\cancel{e^{u}}+1\right)-\left(\cancel{e^{u}}-1\right)}\quad\text{(By Componendo and Dividendo)}$$

$$\therefore\ \frac{\tan\pi/4+\tan x/2}{1-(\tan\pi/4)(\tan x/2)}=\frac{2e^{u}}{2}\quad\left\{\because 1=\tan\frac{\pi}{4}\right\}$$

$$\therefore\ \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=e^{u}$$

$$\therefore\ \boxed{u=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

1b) Prove that the following matrix is orthogonal & hence find A^{-1}. $A=\frac{1}{3}\begin{bmatrix}-2 & 1 & 2\\ 2 & 2 & 1\\ 1 & -2 & 2\end{bmatrix}$.(Chp: Rank of Matrix)(3)

Ans. $A=\frac{1}{3}\begin{bmatrix}-2 & 1 & 2\\ 2 & 2 & 1\\ 1 & -2 & 2\end{bmatrix}$

Taking transpose, $A'=\frac{1}{3}\begin{bmatrix}-2 & 2 & 1\\ 1 & 2 & -2\\ 2 & 1 & 2\end{bmatrix}$

$$\therefore\ AA'==\frac{1}{9}\begin{bmatrix}4+1+4 & -4+2+2 & 4-2-2\\ -4+2+2 & 4+4+1 & 2-4+2\\ -2-2+4 & 2-4+2 & 1+4+4\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}9 & 0 & 9\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\therefore$ A A' = I

Similarly, we can prove, A' A = I

$\therefore$ A is orthogonal.

For Orthogonal matrix, $A^{-1}=A'=\frac{1}{3}\begin{bmatrix}-2 & 2 & 1\\ 1 & 2 & -2\\ 2 & 1 & 2\end{bmatrix}$.

1c) State Euler's theorem on Homogeneous function of two variables & if $u = \dfrac{x+y}{x^2+y^2}$ then evaluate $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

(Chp: Homogenous Functions) (3)

Ans. Part I:

Euler's theorem on Homogeneous function of two variables:

If u (x, y) is homogenous function of degree 'n' then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu \rightarrow (1)$

Part II:

Let $u(x, y) = \dfrac{x+y}{x^2+y^2} \rightarrow (2)$

$\therefore\ u(X, Y) = \dfrac{X+Y}{X^2+Y^2}$

Put X = xt and Y = yt

$\therefore\ u(X, Y) = \dfrac{xt+yt}{(xt)^2+(yt)^2}$

$= \dfrac{xt+yt}{x^2t^2+y^2t^2}$

$= \dfrac{t(x+y)}{t^2(x^2+y^2)}$

$= t^{-1} \times \dfrac{(x+y)}{(x^2+y^2)}$

$\therefore\ u(X, Y) = t^{-1} \times u(x, y)$ (From 2)

$\therefore$ 'u' is a homogenous function of degree '−1'

$\therefore$ From (1), $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -1u$

$\therefore\ \boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{-(x+y)}{x^2+y^2}}$

1d) If $u = r^2 \cos 2\theta$; $v = r^2 \sin 2\theta$ find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$. (Chp: Jacobian) (3)

Ans. Given, $u = r^2 \cos 2\theta$ and $v = r^2 \sin 2\theta \rightarrow (1)$

Partially differentiating w.r.t. 'r', we get

$$\therefore\ u_r = \frac{\partial u}{\partial r} = 2r\cos 2\theta \text{ and } v_r = \frac{\partial v}{\partial r} = 2r\sin 2\theta \rightarrow (2)$$

Partially differentiating (1) w.r.t. 'θ', we get

$$\therefore\ u_\theta = \frac{\partial u}{\partial \theta} = r^2 \cdot -\sin 2\theta \cdot 2 \text{ and } v_\theta = \frac{\partial v}{\partial \theta} = r^2 \cdot \cos 2\theta \cdot 2 \rightarrow (3)$$

We know $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix}$

$= u_r v_\theta - v_r u_\theta$

$= (2r\cos 2\theta)(2r^2\cos 2\theta) - (2r\sin 2\theta)(-2r^2\sin 2\theta)$ (From 2 & 3)

$= 4r^3\cos^2 2\theta + 4r^3\sin^2 2\theta$

$= 4r^3(\cos^2 2\theta + \sin^2 2\theta)$

$= 4r^3 \times 1$

$= 4r^3$

$$\therefore\ \boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

1e) Find the nth derivative of cos 5x · cos 3x · cos x. (Chp: Successive Differentiation) (4)

Ans. Let $y=\cos 5x\cdot\cos 3x\cdot\cos x\times\frac{2}{2}$

$$=\frac{1}{2}\cos 5x\left(\cos 4x+\cos 2x\right)\times\frac{2}{2}\quad\left\{\text{Using, } 2\cos A\cos B=\cos\left(A+B\right)+\cos\left(A-B\right)\right\}$$

$$=\frac{1}{4}\left(2\cos 5x\cos 4x+2\cos 5x\cos 2x\right)$$

$$=\frac{1}{4}\left(\cos 9x+\cos x+\cos 7x+\cos 3x\right)\quad\left\{\text{Using, } 2\cos A\cos B=\cos\left(A+B\right)+\cos\left(A-B\right)\right\}$$

We know, if $y=\cos\left(ax+b\right)$ then $y_n=a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$

Taking n^{th} order derivative

$$y_n=\frac{1}{4}\left[9^n\cos\left(9x+\frac{n\pi}{2}\right)+7^n\cos\left(7x+\frac{n\pi}{2}\right)+3^n\cos\left(3x+\frac{n\pi}{2}\right)+\cos\left(x+\frac{n\pi}{2}\right)\right]$$

1f) Evaluate $\lim_{x \to 0} \left(\frac{2x+1}{x+1} \right)^{\frac{1}{x}}$. (Chp: Indeterminate Forms) (4)

Ans. Let $L = \lim_{x \to 0} \left(\frac{2x+1}{x+1} \right)^{\frac{1}{x}}$

$$= \lim_{x \to 0} \left[\frac{(2x+1)^{1/x}}{(x+1)^{1/x}} \right]$$

$$= \frac{\lim_{x \to 0} (1+2x)^{1/x}}{\lim_{x \to 0} (1+x)^{1/x}}$$

$$= \frac{\left[\lim_{x \to 0} (1+2x)^{\frac{1}{2x}} \right]^{2x \times \frac{1}{x}}}{\lim_{x \to 0} (1+x)^{1/x}}$$

$$= \frac{e^2}{e} \quad \left\{ \because \lim_{x \to 0} (1+x)^{1/x} = e \right\}$$

= e

$$\therefore \boxed{L = \lim_{x \to 0} \left(\frac{2x+1}{x+1} \right)^{\frac{1}{x}} = e}$$

Q.2)

2a) Solve: $x^4 - x^3 + x^2 - x + 1 = 0$. (Chp: Complex - DMT) (6)

Ans. $x^4 - x^3 + x^2 - x + 1 = 0$

But, $x^4 - x^3 + x^2 - x + 1 = \frac{x^5 + 1}{x + 1}$

$\therefore \frac{x^5 + 1}{x + 1} = 0$

Consider, $x^5 + 1 = 0$

$\therefore x^5 = -1 = e^{i\pi}$ (Principal Form)

$\therefore x^5 = e^{i(2n\pi + \pi)}$ (General Form)

$\therefore x = e^{i(2n+1)\pi/5}$

Put n = 0, $x_1 = e^{i\pi/5}$

Put n = 1, $x_2 = e^{i3\pi/5}$

Put n = 2, $x_3 = e^{i5\pi/5} = e^{i\pi} = -1$ $\left\{\because e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1\right\}$

Put n = 3, $x_4 = e^{i7\pi/5} = e^{i(2\pi - 3\pi/5)} = e^{i2\pi}e^{-i3\pi/5} = e^{-i3\pi/5}$ $\left\{\because e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1 + 0i = 1\right\}$

Put n = 4, $x_4 = e^{i9\pi/5} = e^{i(2\pi - \pi/5)} = e^{i2\pi}e^{-i\pi/5} = e^{-i\pi/5}$

But, given equation is of degree 4. So, it will have four roots.

The extra root is of x + 1 = 0 i.e. x = –1

$\therefore$ Roots of $x^4 - x^3 + x^2 - x + 1 = 0$ are $x = e^{\pm i\pi/5} = \cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}$ and $x = e^{\pm i3\pi/5} = \cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$

4b) If $y = e^{\tan^{-1}x}$, prove that $\left(1+x^2\right)y_{n+2} + \left[2(n+1)x - 1\right]y_{n+2} + n(n+1)y_n = 0$. (Chp: Successive Differentiation)(6)

Ans. $y = e^{\tan^{-1}x} \rightarrow (1)$

Differentiating w.r.t. x, $\dfrac{dy}{dx} = e^{\tan^{-1}x} \cdot \dfrac{d}{dx}\tan^{-1}x$

$\therefore \dfrac{dy}{dx} = y \cdot \dfrac{1}{1+x^2}$ (From 1)

$\therefore \left(1+x^2\right)\dfrac{dy}{dx} = y$

Again, differentiating w.r.t. x, $\left(1+x^2\right) \cdot \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] + \dfrac{dy}{dx} \cdot \dfrac{d}{dx}\left(1+x^2\right) = \dfrac{d}{dx}y$

$\therefore \left(1+x^2\right)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} \cdot (0+2x) = \dfrac{dy}{dx}$

$\therefore \left(1+x^2\right)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - \dfrac{dy}{dx} = 0$

$\therefore \left(1+x^2\right)y_2 + (2x-1)y_1 = 0$

Using Leibnitz theorem the nth order derivative is,

$$\therefore \left\{\left(1+x^2\right) \cdot y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{\cancel{2}} \cdot \cancel{2} \cdot y_n\right\} + \left\{(2x-1) \cdot y_{n+1} + n \cdot (2-0) \cdot y_n\right\} = 0$$

$$\therefore \left(1+x^2\right)y_{n+2} + 2nxy_{n+1} + \left(n^2 - n\right)y_n + (2x-1)y_{n+1} + 2ny_n = 0$$

$$\therefore \left(1+x^2\right)y_{n+2} + \left[2nx + 2x - 1\right]y_{n+1} + \left(n^2 - n + 2n\right)y_n = 0$$

$$\therefore \left(1+x^2\right)y_{n+2} + \left[2(n+1)x - 1\right]y_{n+1} + n(n+1)y_n = 0$$

Hence, proved.

2c) Examine the function $f(x, y) = xy(3 - x - y)$ for extremes values & also find maximum and minimum values of f(x, y). (Chp: Maxima and Minima) (8)

Ans. Let $f(x, y) = xy(3 - x - y)$

$f(x, y) = 3xy - x^2y - xy^2 \rightarrow (1)$

Partially differentiating f w.r.t. x, $f_x = 3y - 2xy - y^2$

Again, partially differentiating w.r.t. x,

$r = f_{xx} = 0 - 2y - 0 \rightarrow (2)$

Partially differentiating f w.r.t. y, $f_y = 3x - x^2 - 2xy$

Again, partially differentiating w.r.t. y,

$t = f_{yy} = 0 - 0 - 2x \rightarrow (3)$

Partially differentiating f_y w.r.t. x,

$s = f_{xy} = 3 - 2x - 2y \rightarrow (4)$

Put $f_x = 0$ and $f_y = 0$

$\therefore 3y - 2xy - y^2 = 0$

$\therefore y(3 - 2x - y) = 0$

$\therefore$ y = 0 or 3 – 2 x – y = 0

$\therefore$ y = 0 or 2 x + y = 3 $\rightarrow$ (5)

And, $3x - x^2 - 2xy = 0$

$\therefore x(3 - x - 2y) = 0$

$\therefore$ x = 0 or 3 – x – 2 y = 0

$\therefore$ x = 0 or x + 2 y = 3 $\rightarrow$ (6)

Put x = 0 in (5), 0 + y = 3 $\therefore$ y = 3

Put y = 0 in (6), x + 0 = 3 $\therefore$ x = 3

Solving (5) & (6) simultaneously, we get,

x = 1 and y = 1

$\therefore$ Stationary Points are (0, 3); (3, 0); (0, 0); (1, 1);

(i) At (0, 3)

From (2), r = – 2 (3) = –6

From (3), t = – 2 (0) = 0

From (4), s = 3 – 2 (0) – 2 (3) = –3

$\therefore rt - s^2 = (-6)(0) - (-3)^2 = -9$

$\therefore$ Maxima or minima cannot be found.

(ii) At (3, 0);

From (2), r = – 2 (0) = 0

From (3), t = – 2 (3) = –6

From (4), s = 3 – 2 (3) – 2 (0) = –3

$\therefore rt - s^2 = (0)(-6) - (-3)^2 = -9$

$\therefore$ Maxima or minima cannot be found.

(iii) At (0, 0);

From (2), r = – 2 (0) = 0

From (3), t = – 2 (0) = 0

From (4), s = 3 – 2 (0) – 2 (0) = 3

$\therefore rt - s^2 = (0)(0) - (3)^2 = -9$

$\therefore$ Maxima or minima cannot be found.

(iv) At (1, 1)

From (2), r = – 2 (1) = –2 < 0

From (3), t = – 2 (1) = –2 < 0

From (4), s = 3 – 2 (1) – 2 (1) = –1 < 0

$\therefore rt - s^2 = (-2)(-2) - (-1)^2 = 3 > 0$

$\therefore$ f has maximum at (1, 1)

$\therefore$ From (1), $f(x, y) = 3(1)^3(1) - (1)^2(1) - (1)(1)^2 = 1$

$\therefore$ Maximum value of $f(x, y) = 1$ at (1, 1)

Q.3)

3a) Investigate for what values of λ and μ the system of equations: $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = \mu$ has (i) no solution, (ii) a unique solution, (iii) an infinite no. of solutions. (Chp: Linear Equations) (6)

Ans. Writing the equations in the matrix form, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

$R_3 - R_2;\ R_2 - R_1;$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \mu-10 \end{bmatrix}$

Augmented Matrix [A | B] $= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$

Number of unknowns = n = 3

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when $\mu \neq 10$ and $\lambda = 3$.

We then have, rank of A (r_A) = 2 and rank of [A | B] (r_{AB}) = 3

Case II: Unique Solution

For which, $r_A = r_{AB} = n$

This is only possible, when $\lambda \neq 3$ and μ has any value.

We then have, rank of A (r_A) = rank of [A | B] (r_{AB}) = 3

Case III: Infinite Solution

For which, $r_A = r_{AB} < 3$ (i.e. < 3)

This is only possible, when $\mu = 10$ and $\lambda = 3$.

We then have, rank of A (r_A) = rank of [A | B] (r_{AB}) = 2

Hence,

No Solution	$\mu \neq 10, \lambda = 3$
Unique Solution	μ = any value, $\lambda \neq 3$
Infinite Solution	$\mu = 10, \lambda = 3$

Q.3)

3a) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial z} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$. (Chp: Partial Differentiation) (6)

Ans. (Question is wrong) Correct question is

Show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Let $v = \frac{y-x}{xy}$ and $w = \frac{z-x}{xz}$

$\therefore\ v = \frac{y}{xy} - \frac{x}{xy} = \frac{1}{x} - \frac{1}{y}$ and $w = \frac{z}{xz} - \frac{x}{xz} = \frac{1}{x} - \frac{1}{z}$

$\therefore\ \frac{\partial v}{\partial x} = \frac{-1}{x^2}$; $\frac{\partial v}{\partial y} = \frac{1}{y^2}$; $\frac{\partial v}{\partial z} = 0 \rightarrow (1)$ and

$\frac{\partial w}{\partial x} = \frac{-1}{x^2}$; $\frac{\partial w}{\partial y} = 0$; $\frac{\partial w}{\partial z} = \frac{1}{z^2}$; $\rightarrow (2)$

Now, u $\rightarrow$ v, w $\rightarrow$ x, y, z

$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\times\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\times\frac{\partial w}{\partial x}$

$= u_v \times \frac{-1}{x^2} + u_w \times \frac{-1}{x^2}$ (From 1 and 2)

$\therefore\ \frac{\partial u}{\partial x} = \frac{-u_v - u_w}{x^2}$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\times\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\times\frac{\partial w}{\partial y}$

$= u_v \times \frac{1}{y^2} + u_w \times 0$ (From 1 and 2)

$\therefore\ \frac{\partial u}{\partial y} = \frac{u_v}{y^2}$

And, $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\times\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\times\frac{\partial w}{\partial z}$

$= u_v \times 0 + u_w \times \frac{1}{z^2}$ (From 1 and 2)

$\therefore\ \frac{\partial u}{\partial z} = \frac{u_w}{z^2}$

$\therefore\ x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} =$

$= x^2\left(\frac{-u_v - u_w}{x^2}\right) + y^2\left(\frac{u_v}{y^2}\right) + z^2\left(\frac{u_w}{z^2}\right)$

$= -u_v - u_w + u_v + u_w$

$= 0$

Hence, $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

3c) Prove that $\log\left(\dfrac{a+ib}{a-ib}\right)=2i\tan^{-1}\left(\dfrac{b}{a}\right)$ & $\cos\left[i\log\left(\dfrac{a+ib}{a-ib}\right)\right]=\dfrac{a^2-b^2}{a^2+b^2}$. (Chp: Log of Complex Numbers) (8)

Ans. Part I:

$$\log\left(\frac{a+ib}{a-ib}\right)=\log(a+ib)-\log(a-ib)$$

$$=\left[\cancel{\frac{1}{2}\log\left(a^2+b^2\right)}+i\theta\right]-\left[\cancel{\frac{1}{2}\log\left(a^2+b^2\right)}-i\theta\right]\ \left\{\text{where}, \theta=\tan^{-1}\left(\frac{b}{a}\right)\right\}\rightarrow(1)\ \text{(Principal Form)}$$

$$\therefore\ \log\left(\frac{a+ib}{a-ib}\right)=2i\theta\rightarrow(2)$$

$$\therefore\ \boxed{\log\left(\frac{a+ib}{a-ib}\right)=2i\tan^{-1}\left(\frac{b}{a}\right)}\ \text{(From 1)}$$

Part II:

$$\text{LHS}=\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right]$$

$$=\cos[i\times 2i\theta]\ \text{(From 2)}$$

$$=\cos[-2\theta]$$

$$=\cos[2\theta]$$

$$=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$=\frac{1-(b/a)^2}{1+(b/a)^2}\ \text{(From 1)}$$

$$=\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}$$

$$=\frac{(a^2-b^2)/\cancel{a^2}}{(a^2+b^2)/\cancel{a^2}}$$

$$=\frac{a^2-b^2}{a^2+b^2}$$

$= \text{RHS}$

$$\therefore\ \boxed{\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right]=\frac{a^2-b^2}{a^2+b^2}}$$

6b) If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that (i) $xu_x + yu_y = \dfrac{1}{2}\tan u$ (ii) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \dfrac{-\sin u \cos 2u}{4\cos^3 u}$.

(Chp: Homogenous Functions) (6)

Ans. $u(x,y) = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$\therefore \sin u(x,y) = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} \rightarrow (1)$

$\therefore \sin u(X,Y) = \dfrac{X+Y}{\sqrt{X}+\sqrt{Y}}$

Now, Put X = x t, Y = y t

$\therefore \sin u(X,Y) = \dfrac{xt+yt}{\sqrt{xt}+\sqrt{yt}}$

$= \dfrac{t(x+y)}{\sqrt{t}\left(\sqrt{x}+\sqrt{y}\right)}$

$= \sqrt{t}\sin u(x,y)$

$\sin u(X,Y) = t^{1/2}\sin u(x,y)$ (From 1)

$\therefore$ sin u is homogenous function of degree $(n) = \dfrac{1}{2}$.

Let $f(u) = \sin u$

$\therefore f'(u) = \cos u$

Let $g(u) = n\dfrac{f(u)}{f'(u)}$

$= \dfrac{1}{2}\cdot\dfrac{\sin u}{\cos u}$

$= \dfrac{\tan u}{2}$

$\therefore g'(u) = \dfrac{1}{2}\sec^2 u$

Using Euler's Theorem, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = g(u)$

$\therefore x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\tan u}{2}$

$\therefore \boxed{xu_x + yu_y = \dfrac{1}{2}\tan u}$

Using Corollary to Euler's Theorem,

$x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)\cdot\left[g'(u)-1\right]$

$\therefore x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{2}\cdot\left[\dfrac{1}{2}\sec^2 u - 1\right]$

$= \dfrac{\sin u}{2\cos u}\cdot\left[\dfrac{1}{2\cos^2 u} - 1\right]$

$= \dfrac{\sin u}{2\cos u}\cdot\left[\dfrac{1-2\cos^2 u}{2\cos^2 u}\right]$

$= \dfrac{\sin u}{4\cos^3 u}\cdot -\left(2\cos^2 u - 1\right)$

$\therefore \boxed{x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \dfrac{-\sin u\cos 2u}{4\cos^3 u}}$

4b) Using the encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, encode & decode the message 'ALL IS WELL'. (Chp: Coding) (6)

Ans. We use following numerical values of each alphabet for coding

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	*
15	16	17	18	19	20	21	22	23	24	25	26	27

Step 1:

Message: ALL IS WELL

As per the above table, the numerical values of each alphabet in the message are

A	L	L	*	I	S	*	W	E	L	L
1	12	12	27	9	19	27	23	5	12	12

Step 2:

Writing the above values column-wise in a 2-row matrix we get, $A = \begin{bmatrix} 1 & 12 & 9 & 27 & 5 & 12 \\ 12 & 27 & 19 & 23 & 12 & 0 \end{bmatrix}$

Encoding matrix $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \to (1)$

Now, $EA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 12 & 9 & 27 & 5 & 12 \\ 12 & 27 & 19 & 23 & 12 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1+12 & 12+27 & 9+19 & 27+23 & 5+12 & 12+0 \\ 0+12 & 0+27 & 0+19 & 0+23 & 0+12 & 0+0 \end{bmatrix}$

$\therefore\ EA = \begin{bmatrix} 13 & 39 & 28 & 50 & 17 & 12 \\ 12 & 27 & 19 & 23 & 12 & 0 \end{bmatrix} \to (2)$

Writing the numbers in EA matrix column wise gives the encoded message.

$\therefore$ Encoded Message =
13 12 39 27 28 19 50 23 17 12 12 0

This encoded message is transmitted.

Step 3:

Assume there is no corruption of data, the message at the receiving end is

13 12 39 27 28 19 50 23 17 12 12 0

This message is decoded

We know, if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $P^{-1} = \frac{1}{|P|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

From (1), | E | = 1 – 0 = 1 → (3)

$\therefore\ E^{-1} = \frac{1}{|E|}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ Decoding matrix $E^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (From 3) → (4)

From (2) & (4),

$E^{-1}(EA) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 13 & 39 & 28 & 50 & 17 & 12 \\ 12 & 27 & 19 & 23 & 12 & 0 \end{bmatrix}$

$\therefore\ A = \begin{bmatrix} 1 & 12 & 9 & 27 & 5 & 12 \\ 12 & 27 & 19 & 23 & 12 & 0 \end{bmatrix}$

Step 4:

Considering the numbers column-wise we get,

1 12 12 27 9 19 27 23 5 12 12 0

Reconverting each of the above numbers into corresponding alphabet,

Decoded Message = ALL IS WELL

4c) Solve the following equations by Gauss Seidal method:

$10x_1 + x_2 + x_3 = 12$; $2x_1 + 10x_2 + x_3 = 13$; $2x_1 + 2x_2 + 10x_3 = 14$. (Chp: Linear algebraic equations) (8)

Ans From 1st equation, $10x_1 = 12 - x_2 - x_3$

$\therefore\ x_1 = \frac{1}{10}(12 - x_3 - x_2) = 0.1(12 - x_3 - x_2)$

Similarly,

$x_2 = 0.1(13 - x_3 - 2x_1)$ & $x_3 = 0.1(14 - 2x_1 - 2x_2)$

Iteration 1:

Put $x_2 = x_3 = 0$

$\therefore\ x_1' = 0.1(12 - x_3 - x_2)$

$= 0.1(12 - 0 - 0)$

$= 1.2$

Put $x_1' = 1.2$; $x_3 = 0$

$\therefore\ x_2' = 0.1(13 - x_3 - 2x_1')$

$= 0.1[13 - 0 - 2(1.2)]$

$= 1.06$

Put $x_1' = 1.2; x_2' = 1.06$

$\therefore\ x_3' = 0.1(14 - 2x_1' - 2x_2')$

$= 0.1[14 - 2(1.2) - 2(1.06)]$

$= 0.9480$

Iteration 2:

Put $x_2' = 1.06; x_3' = 0.9480;$

$\therefore\ x_1'' = 0.1(12 - x_3' - x_2')$

$= 0.1(12 - 0.948 - 1.06)$

$= 0.9992$

Put $x_1'' = 0.9992; x_3' = 0.9480$

$\therefore\ x_2'' = 0.1(13 - x_3' - 2x_1'')$

$= 0.1[13 - 0.948 - 2(0.9992)]$

$= 1.0054$

Put $x_1'' = 0.9992; x_2'' = 1.0054$

$\therefore\ x_3'' = 0.1(14 - 2x_1'' - 2x_2'')$

$= 0.1[14 - 2(0.9992) - 2(1.0054)]$

$= 0.9991$

Iteration 3:

Put $x_2'' = 1.0054; x_3'' = 0.9991$

$\therefore\ x_1''' = 0.1(12 - x_3'' - x_2'')$

$= 0.1(12 - 0.9991 - 1.0054)$

$= 0.9996$

Put $x_1''' = 0.9996; x_3'' = 0.9991$

$\therefore\ x_2''' = 0.1(13 - x_3'' - 2x_1''')$

$= 0.1[13 - 0.9991 - 2(0.9996)]$

$= 1.0002$

Put $x_1''' = 0.9995; x_2''' = 1.0002$

$\therefore\ x_3''' = 0.1(14 - 2x_1''' - 2x_2''')$

$= 0.1[14 - 2(0.9996) - 2(1.0002)]$

$= 1.001$

Hence, by Gauss-Seidal method, solution of given set of equations is $x = 1, y = 1, z = 1$.

Q.3)

3a) If $u = e^{xyz} f\left(\frac{xy}{z}\right)$ then prove that $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2xyz \cdot u$ and $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyz \cdot u$.

Hence show that $x\frac{\partial^2 u}{\partial z \partial x} = y\frac{\partial^2 u}{\partial z \partial y}$. (Chp: Partial Differentiation) (6)

Ans. $u = e^{xyz} f\left(\frac{xy}{z}\right) \to (1)$

Partially Differentiate 'u' w.r.t. 'x',

$$\frac{\partial u}{\partial x} = e^{xyz} \cdot \frac{\partial}{\partial x} f\left(\frac{xy}{z}\right) + f\left(\frac{xy}{z}\right) \cdot \frac{\partial}{\partial x} e^{xyz}$$

$$= e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \frac{\partial}{\partial x}\left(\frac{xy}{z}\right) + f\left(\frac{xy}{z}\right) \cdot e^{xyz} \frac{\partial}{\partial x}(xyz)$$

$$\therefore \frac{\partial u}{\partial x} = e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot \frac{y}{z} + f\left(\frac{xy}{z}\right) \cdot yz\right] \to (2)$$

Similarly, Partially Differentiate 'u' w.r.t. 'y',

$$\frac{\partial u}{\partial y} = e^{xyz} \cdot \frac{\partial}{\partial y} f\left(\frac{xy}{z}\right) + f\left(\frac{xy}{z}\right) \cdot \frac{\partial}{\partial y} e^{xyz}$$

$$= e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \frac{\partial}{\partial y}\left(\frac{xy}{z}\right) + f\left(\frac{xy}{z}\right) \cdot e^{xyz} \frac{\partial}{\partial y}(xyz)$$

$$\therefore \frac{\partial u}{\partial y} = e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot \frac{x}{z} + f\left(\frac{xy}{z}\right) \cdot xz\right] \to (3)$$

And, Partially Differentiate 'u' w.r.t. 'z',

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot \frac{\partial}{\partial z} f\left(\frac{xy}{z}\right) + f\left(\frac{xy}{z}\right) \cdot \frac{\partial}{\partial z} e^{xyz}$$

$$= e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \frac{\partial}{\partial z}\left(\frac{xy}{z}\right) + f\left(\frac{xy}{z}\right) \cdot e^{xyz} \frac{\partial}{\partial z}(xyz)$$

$$\therefore \frac{\partial u}{\partial z} = e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot xy \cdot \frac{-1}{z^2} + f\left(\frac{xy}{z}\right) \cdot xy\right] \to (4)$$

Now, from (2) and (4)

$$x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = x \cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot \frac{y}{z} + f\left(\frac{xy}{z}\right) \cdot yz\right] + z \cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot xy \cdot \frac{-1}{z^2} + f\left(\frac{xy}{z}\right) \cdot xy\right]$$

$$= e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot \frac{xy}{z} + f\left(\frac{xy}{z}\right) \cdot xyz\right] + e^{xyz}\left[-f'\left(\frac{xy}{z}\right) \cdot xy \cdot \frac{1}{z} + f\left(\frac{xy}{z}\right) \cdot xyz\right]$$

$$= e^{xyz}\left\{\cancel{f'\left(\frac{xy}{z}\right) \cdot \frac{xy}{z}} + f\left(\frac{xy}{z}\right) \cdot xyz \cancel{- f'\left(\frac{xy}{z}\right) \cdot \frac{xy}{z}} + f\left(\frac{xy}{z}\right) \cdot xyz\right\}$$

$$= e^{xyz} \times 2f\left(\frac{xy}{z}\right) \cdot xyz$$

$$\therefore x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2xyz \cdot u \text{ (From 1)} \to (5)$$

Similarly, from (3) and (4)

$$y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = y \cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot \frac{y}{z} + f\left(\frac{xy}{z}\right) \cdot yz\right] + z \cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot xy \cdot \frac{-1}{z^2} + f\left(\frac{xy}{z}\right) \cdot xy\right]$$

$$= e^{xyz}\left[f'\left(\frac{xy}{z}\right) \cdot \frac{xy}{z} + f\left(\frac{xy}{z}\right) \cdot xyz\right] + e^{xyz}\left[-f'\left(\frac{xy}{z}\right) \cdot xy \cdot \frac{1}{z} + f\left(\frac{xy}{z}\right) \cdot xyz\right]$$

$$= e^{xyz}\left\{\cancel{f'\left(\frac{xy}{z}\right) \cdot \frac{xy}{z}} + f\left(\frac{xy}{z}\right) \cdot xyz \cancel{- f'\left(\frac{xy}{z}\right) \cdot \frac{xy}{z}} + f\left(\frac{xy}{z}\right) \cdot xyz\right\}$$

$$= e^{xyz} \times 2f\left(\frac{xy}{z}\right) \cdot xyz$$

$$\therefore y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyz \cdot u \text{ (From 1)} \to (6)$$

From (5) & (6), $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyz \cdot u$

$$\therefore x\frac{\partial u}{\partial x} = 2xyz \cdot u - z\frac{\partial u}{\partial z} \text{ and } y\frac{\partial u}{\partial y} = 2xyz \cdot u - z\frac{\partial u}{\partial z}$$

$$\therefore x\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$$

Partially Differentiating w.r.t. 'z', $x\frac{\partial^2 u}{\partial z \partial x} = y\frac{\partial^2 u}{\partial z \partial y}$

5b) Prove that $\sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$. (Chp: Complex - DMT) (6)

Ans. Let $x = \cos\theta + i\sin\theta$

$\therefore\ x^n = (\cos\theta + i\sin\theta)^n$

$\therefore\ x^n = \cos n\theta + i\sin n\theta \rightarrow (1)$ (De Moivre's theorem)

Replacing n by –n in (1), $x^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$\therefore\ \frac{1}{x^n} = \cos n\theta - i\sin n\theta \rightarrow (2)$

Subtracting (2) from (1), $x^n - \frac{1}{x^n} = (\cancel{\cos n\theta} + i\sin n\theta) - (\cancel{\cos n\theta} - i\sin n\theta)$

$\therefore\ \sin n\theta = \frac{1}{2i}\left(x^n - \frac{1}{x^n}\right) \rightarrow (3)$

For n = 1, $\sin\theta = \frac{1}{2i}\left(x - \frac{1}{x}\right) \rightarrow (4)$

$\text{LHS} = \sin^5\theta$

$= \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^5$ (From 4)

$= \frac{1}{2^5 i^5}\left[x^5 - 5x^4\cdot\frac{1}{x} + 10x^3\cdot\frac{1}{x^2} - 10x^2\cdot\frac{1}{x^3} + 5x\cdot\frac{1}{x^4} - \frac{1}{x^5}\right]$ (Binomial Expansion)

$= \frac{1}{32i}\left[x^5 - 5x^3 + 10x - 10\cdot\frac{1}{x} + 5\cdot\frac{1}{x^3} - \frac{1}{x^5}\right]$

$= \frac{1}{32i}\left[\left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)\right]$

$= \frac{1}{32i}[2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta]$ (From 3)

$= \frac{1}{32i}\times 2i[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

$= \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

= RHS

$\therefore\ \boxed{\sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]}$

5c) Prove that $\log(\sec x)=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots$. (Chp: Expansion) (4)

Ans. $LHS=\log(\sec x)$

$$=\log\left(\frac{1}{\cos x}\right)$$

$$=\log(\cos x)^{-1}$$

$$=-\log\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots.\right)$$

$$=-\log\left[1-\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots.\right)\right]$$

$$=\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots\right)+\frac{1}{2}\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots\right)^2+\frac{1}{3}\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots\right)^3+\ldots$$

$$=\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{1}{2}\left(\frac{x^2}{2!}\right)^2+\frac{x^6}{6!}-\frac{1}{2}\left(2\cdot\frac{x^2}{2!}\cdot\frac{x^4}{4!}\right)+\frac{1}{3}\left(\frac{x^2}{2!}\right)^3+\ldots$$

$$=\frac{x^2}{2}+x^4\left(\frac{-1}{24}+\frac{1}{8}\right)+x^6\left(\frac{1}{720}-\frac{1}{48}+\frac{1}{24}\right)+\ldots$$

$$=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots$$

$= \text{RHS}$

Hence, $\log(\sec x)=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots$.

5d) Expand $\left(2x^3+7x^2+x-1\right)$ in powers of (x – 2). (Chp: Expansion) (4)

Ans. Let $f(x)=2x^3+7x^2+x-1$

$\therefore\ f'(x)=6x^2+14x+1$

$\therefore\ f''(x)=12x+14$

$\therefore\ f'''(x)=12$

Let a = 2,

$\therefore\ f(a)=f(2)=2(2)^3+7(2)^2+(2)-1=45$

$\therefore\ f'(a)=f'(2)=6(2)^2+14(2)+1=53$

$\therefore\ f''(a)=f''(2)=12(2)+14=38$

$\therefore\ f'''(a)=f'''(2)=12$

By Taylor Series, $f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$

$\therefore\ f(x)=45+(x-2)\cdot 53+\frac{1}{2}(x-2)^2\cdot 38+\frac{1}{6}(x-2)^3\cdot 12$

$\therefore\ 2x^3+7x^2+x-6=45+53(x-2)+19(x-2)^2+2(x-2)^3$

Q.6)

6a) Prove that $\sin^{-1}(\operatorname{cosec}\theta) = \frac{\pi}{2} + i\log\left(\cot\frac{\theta}{2}\right)$. (Chp: Complex - DMT) (6)

Ans. Let $\sin^{-1}(\operatorname{cosec}\theta) = x + iy$ (Assuming x, y $\neq 0$) $\rightarrow (1)$

$\therefore\ \operatorname{cosec}\theta = \sin(x+iy)$

$\therefore\ \operatorname{cosec}\theta = \sin x\cos(iy) + \cos x\sin(iy)$

$\therefore\ \operatorname{cosec}\theta + 0i = \sin x\cosh y + i\cos x\sinh y$

Comparing Real and Imaginary parts, we get $\operatorname{cosec}\theta = \sin x\cosh y \rightarrow (2)$ and $0 = \cos x\sinh y \rightarrow (3)$

From (3), cos x = 0 or sinh y = 0

$\therefore\ x = (2n-1)\frac{\pi}{2}$ or y = 0

Considering only the principal value, $x = \frac{\pi}{2} \rightarrow (4)$

Put $x = \frac{\pi}{2}$ in (2) we get, $\operatorname{cosec}\theta = \sin\frac{\pi}{2}\cosh y$

$\therefore\ \operatorname{cosec}\theta = \cosh y$

$\therefore\ y = \cosh^{-1}(\operatorname{cosec}\theta)$

$= \log\left(\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right)$ $\left\{\because \cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right)\right\}$

$= \log\left(\operatorname{cosec}\theta + \sqrt{\cot^2\theta}\right)$

$= \log(\operatorname{cosec}\theta + \cot\theta)$

$= \log\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$= \log\left(\frac{1+\cos\theta}{\sin\theta}\right)$

$= \log\left[\frac{\cancel{2}\cos^{\cancel{2}}(\theta/2)}{\cancel{2}\sin(\theta/2)\cancel{\cos(\theta/2)}}\right]$

$\therefore\ y = \log\left(\cot\frac{\theta}{2}\right) \rightarrow (5)$

Substitute (4) and (5) in (1), we get, $\sin^{-1}(\operatorname{cosec}\theta) = \frac{\pi}{2} + i\log\left(\cot\frac{\theta}{2}\right)$

6b) Find non-singular matrices P & Q such that $A=\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}$ is reduced to normal form. Also find its rank.

(Chp: Rank of Matrix) (6)

Ans. Let $A_{3\times4}=I_{3\times3}\times A_{3\times4}\times I_{4\times4}$

$$\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_2-2R_1;R_3-R_1$ $\Rightarrow$ $$\begin{bmatrix}1&2&3&2\\0&-1&-1&-3\\0&1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_2-2C_1;C_3-3C_1;C_4-2C_1$ $\Rightarrow$ $$\begin{bmatrix}1&0&0&0\\0&-1&-1&-3\\0&1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_3+R_2;-R_2$ $\Rightarrow$ $$\begin{bmatrix}1&0&0&0\\0&1&1&3\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_3-C_2;C_4-2C_2$ $\Rightarrow$ $$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}\rightarrow(1)$$

RHS is the required PAQ form.

Here, $P=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}$ and $Q=\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}$

Rank of A = Number of non-zero rows on LHS of (1) = 2

6c) Obtain the root of $x^3 - x - 1 = 0$ by Regula Falsi Method (Take three iterations). (Chp: Transcendental equations)(8)

Ans. Let $f(x) = x^3 - x - 1 \rightarrow (1)$

Let a = 1.3 and b = 1.4

$\therefore\ f(a) = f(1.3) = (1.3)^3 - (1.3) - 1 = -0.103 < 0$ and

$f(b) = f(1.4) = (1.4)^3 - (1.4) - 1 = 0.344 > 0$

$\therefore$ Root of f(x) lies between 1.3 and 1.4

By Regula Falsi Method $x = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \rightarrow (2)$

Method I:

Iteration	a	b	f(a)	f(b)	x	f(x)
1)	1.3	1.4	–0.103	0.344	1.3230	–0.0073
2)	1.3230	1.4	–0.0073	0.344	1.3246	–0.0005
3)	1.3246	1.4	–0.0005	0.344	1.3247	

Method II:

Iteration I:

Let a = 1.3, b = 1.4, f(a) = –0.103 and f(b) = 0.344

$$\therefore \text{From (2)},\ x_1 = \frac{1.3(0.344) - 1.4(-0.103)}{(0.344) - (-0.103)} = 1.3230$$

$\therefore$ From (1), $f(x_1) = f(1.3230) = (1.3230)^3 - (1.3230) - 1 = -0.0073 < 0$

Iteration II:

Let a = 1.3240, b = 1.4, f(a) = –0.0073 and f(b) = 0.344

$$\text{From (2)},\ x_2 = \frac{1.3230(0.344) - 1.4(-0.0073)}{(0.344) - (-0.0073)} = 1.3246$$

$\therefore$ From (1), $f(x_2) = f(1.3246) = (1.3246)^3 - (1.3246) - 1 = -0.0005 < 0$

Iteration III:

Let a = 1.3246, b = 1.4, f(a) = –0.0005 and f(b) = 0.344

$$\text{From (2)},\ x_2 = \frac{1.3246(0.344) - 1.4(-0.0005)}{(0.344) - (-0.0005)} = 1.3247$$

Hence, by Regula Falsi Method, Root of the equation $x^3 - x - 1 = 0$ is 1.3247