Lecture notes

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Linear Differential equations with constant coefficients

Definition

An ordinary Differential Equation of order n is of the form

$$P_n \frac{d^n y}{dx^n} + P_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + P_1 \frac{dy}{dx} + P_0 y = X$$

where $P_0, P_1, ..., P_n$ are Constant coefficients and X is a function of x only or constant is known as **Linear differential Equation with constant coefficients**.

For $X = 0$, equation is known as **Homogeneous equation** otherwise it is known as **non- homogeneous equation**

- The compact form of above equation is given by

$$f(D)y = X$$

  where $f(D) = P_n D^n + P_{n-1} D^{n-1} ... + P_0$ is polynomial (Differential Operator) of degree n and equation $f(D) = 0$ is called **characteristic equation** or **Auxiliary equation**

- The **general solution** of Non homogeneous linear differential equation with constant coefficient consist of **Complementary function (C.F) and Particular integral (P.I)**

- **Complimentary function** is defined using roots of auxiliary equation $f(D) = 0$ and **Particular integral** is defined as $y = \frac{1}{f(D)} X$ where $\frac{1}{f(D)}$ is an integral operator which is independent of arbitrary constants.

Rules to find complementary Function

Case 1:Roots of Auxiliary Equation are real and distinct

Let $m_1, m_2, ..., m_n$ be n distinct roots of $f(D) = 0$, then Complementary Function is given by

$$C.F. = y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$$

where $c_1, c_2, ...c_n$ are arbitrary constants

Concept

Given $m_1, m_2, ..., m_n$ be n distinct roots of $f(D) = 0 \therefore [(D - m_1)(D - m_2)...(D - m_n)]y = 0$ Consider

$$\begin{aligned}
(D - m_1)y &= 0 \\
\therefore \frac{dy}{dx} - m_1 y &= 0 \\
\therefore \frac{dy}{y} &= m_1 dx \\
\therefore log(y) &= m_1 x + c \qquad (integrating) \\
\therefore y &= c_1 e^{m_1 x} \\
where\ c_1 = e^c
\end{aligned}$$

Similarly applying each operators one by one we have roots are of the form $y = c_i e^{m_i x}$ Hence

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$$

Case 2:Roots of Auxiliary Equation are real and repeated

Let $m_1, m_2, ..., m_n$ be n distinct roots of $f(D) = 0$,where m_1 and m_2 are repeated twice then Complementary Function is given by

$$C.F. = y = [c_1 + c_2 x]e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

where $c_1, c_2, ...c_n$ are arbitrary constants.

Concept
Let $m_1 = m_2$ be repeated roots of A.E. and remaining roots are real and distinct
$\therefore [(D-m_1)^2(D-m_3)...(D-m_n)]y = 0$ Consider

$$(D-m_1)^2 y = 0$$
$$\therefore (D-m_1)(D-m_1)y = 0$$
$$\therefore \frac{dP}{dx} - m_1 P = 0 \quad (Let (D-m_1)y = P)$$
$$\therefore P = c_1 e^{m_1 x} \quad (By Case(1))$$
$$\therefore (D-m_1)y = c_1 e^{m_1 x}$$
$$\therefore \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

which is linear Differential equation in x and y

$$\therefore y(e^{-m_1 x}) = \int c_1 e^{m_1 x} e^{-m_1 x} dx + c_2$$
$$\therefore y = (c_1 x + c_2) e^{m_1 x}$$

Considering all the roots

$$y = (c_1 x + c_2)e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

Case 3:Roots of Auxiliary Equation are Complex conjugates of each other

Let $m_1, m_2, ..., m_n$ be n distinct roots of $f(D) = 0$,where $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ then Complementary Function is given by

$$C.F. = y = e^{\alpha x}[c_1 cos\beta x + c_2 sin\beta x] + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

where $c_1, c_2, ...c_n$ are arbitrary constants.

Concept
Let $m_1 and m_2$ be complex cojugates of each other and remaining roots are real and distinct
Let $m_1 = \alpha + i\beta \quad ; \quad m_2 = \alpha - i\beta$

Hence

$$
\begin{aligned}
y &= a_1 e^{m_1 x} + a_2 e^{m_2 x} + a_3 e^{m_3 x} + ... + a_n e^{m_n x} \\
&= a_1 e^{\alpha + i\beta} + a_2 e^{\alpha - i\beta} + a_3 e^{m_3 x} + ... + a_n e^{m_n x} \\
&= a_1 \left\{ e^{\alpha} e^{i\beta} \right\} + a_2 \left\{ e^{\alpha} e^{-i\beta} \right\} + a_3 e^{m_3 x} + ... + a_n e^{m_n x} \\
&= a_1 \left\{ e^{\alpha} (cos\beta + isin\beta) \right\} + a_2 \left\{ e^{\alpha} (cos\beta - isin\beta) \right\} + a_3 e^{m_3 x} + ... + a_n e^{m_n x} \\
&= e^{\alpha} \left\{ (a_1 + a_2) cos\beta + i(a_1 - a_2) sin\beta) \right\} + a_3 e^{m_3 x} + ... + a_n e^{m_n x}
\end{aligned}
$$

Considering all the roots

$$e^{\alpha x}[c_1 cos\beta x + c_2 sin\beta x] + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

Case 4:Roots of Auxiliary Equation are Complex and repeated

Let $m_1, m_2, ..., m_n$ be n distinct roots of $f(D) = 0$,where $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ are repeated twic then Complementary Function is given by

$$C.F. = y = e^{\alpha x}[(c_1 + c_2 x) cos\beta x + (c_3 + c_4 x) sin\beta x] + c_5 e^{m_5 x} + ... + c_n e^{m_n x}$$

where $c_1, c_2, ...c_n$ are arbitrary constants.

Rules to find Particular Integral

Particular integral is defined as

$$y = \frac{1}{f(D)}X$$

where $\frac{1}{f(D)}$ is an integral operator independent of arbitrary constants.
The general solution of Homogeneous linear Differential equation with constant coefficients ($f(D)y = 0$) is given by

$$G.S. = C.F.$$

and
The general solution of Non-homogeneous linear Differential equation with constant coefficients ($f(D)y = X$) is given by

$$G.S. = C.F. + P.I.$$

Example: Homogeneous case

Solve

$$1)\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$2)\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 28y = 0$$

Solution (1)
The Compact form of given equation is

$$(D^2 - 3D + 2)y = 0$$

A.E.

$$f(D) = 0$$

$$(D^2 - 3D + 2) = 0$$

$$D = 1, D = 2$$

Roots are real and distinct for given homogeneous equation

$$G.S. = C.F. + P.I.$$

$$y = c_1e^x + c_2e^{2x}$$

Solution (1)
The Compact form of given equation is

$$(D^4 - 5D^2 + 12D + 28)y = 0$$

A.E.

$$f(D) = 0$$

$$(D^4 - 5D^2 + 12D + 28) = 0$$

since $f(-2) = 0$; $D = -2$ is one of the factor, by synthetic division

$$\begin{array}{r|rrrrr} & 1 & 0 & -5 & 12 & 28 \\ -2 & & -2 & 4 & 2 & -28 \\ \hline & 1 & -2 & -1 & 14 & 0 \end{array}$$

$$\begin{array}{r|rrrr} & 1 & -2 & -1 & 14 \\ -2 & & -2 & 8 & -14 \\ \hline & 1 & -4 & 7 & 0 \end{array}$$

Hence,

$$(D^4 - 5D^2 + 12D + 28) = (D + 2)^2(D^2 - 4D + 7) = 0$$

Now .

$$D^2 - 4D + 7 = 0$$

$$\implies D = \frac{-(-4) \pm \sqrt{16 - 28}}{2}$$

$$\implies D = \frac{4 \pm 2i\sqrt{3}}{2}$$

$$\implies D = 2 \pm \sqrt{3}i$$

Hence roots of A.E.$f(D)y = 0$ are
$D = -2, -2, 2 \pm \sqrt{3}i$
Two roots are real and repeatatives and two roots are complex conjugates of each other for given homogeneous equation

$$\therefore G.S. = C.F. + P.I.$$

$$\therefore y = (c_1 + c_2 x)e^{-2x} + e^{2x}[c_3 \ cos\sqrt{3}x + c_4 \ sin\sqrt{3}x]$$

Examples

Solve

(1) $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} + 12y = 0$

(2) $\dfrac{d^4x}{dt^4} = m^4x$

(3) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$

(4) $\dfrac{d^3y}{dx^3} - 5\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} - 4y = 0$

(5) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$

Answers

(1) $y = c_1e^{-4x} + c_2e^{x} + c_3e^{3x}$

(2) $x = c_1e^{-mt} + c_2e^{mt} + c_3\ cosmt + c_4\ sinmt$

(3) $y = [c_1 + c_2x]e^{3x}$

(4) $y = c_1e^{x} + [c_2 + c_3x]e^{2x}$

(5) $y = e^{-2x}[c_1\ cosx + c_2\ sinx]$

Non-homogeneous equation

In this case ,Particular integral is defined as

$$y = \frac{1}{f(D)}X$$

where $\frac{1}{f(D)}$ is an integral operator is that function of x when acted upon by the differential operator $f(D)$ gives X and it is independent of arbitrary constants.
Thus by this definition

$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

hence satisfies the equation $f(D)y = X$
Thus particular integral is symbollically given by

$$P.I. = y = \frac{1}{f(D)}X$$

Thus considering both cases
The general solution of Homogeneous linear Differential equation with constant coefficients ($f(D)y = 0$) is given by

$$G.S. = C.F.$$

and
The general solution of Non-homogeneous linear Differential equation with constant coefficients ($f(D)y = X$) is given by

$$G.S. = C.F. + P.I.$$

Methods to find Particular Integral

There are three methods to find Particular Intergral $P.I. = y = \frac{1}{f(D)}X$
1) General Method
2) Shortcut Methods
3) Methods of variation of parameters

Method 1: General Method

- This methods are useful when shortcut methods are not applicable to find particular Integral

- **(I)** $\dfrac{1}{(D-m)}X$

  By definition, $P.I. = y = \frac{1}{(D-m)}X$ is the solution of equation $(D-m)y = X$

- Now $(D-m)y = X$ is linear D.E. whose I.F is given by $I.F = e^{-mx}$ and G.S. is given by

$$y(I.F) = \int Xe^{-mx}dx + c_1$$
$$\therefore y = (c_1 e^{mx}) + \left(e^{mx}\int Xe^{-mx}dx\right)$$

- Since first term contains arbitrary constant it is the C.F. and second term which is independent of arbitrary constant must be P.I.

Hence

$$P.I. = y = \frac{1}{D-m}\ X = e^{mx}\int Xe^{-mx}dx$$

Similarly

$$P.I. = y = \frac{1}{D+m}\ X = e^{-mx}\int Xe^{mx}dx$$

For $m = 0$

$$P.I. = y = \frac{1}{D}\ X = \int Xdx$$

Also

$$P.I. = y = \frac{1}{D^2}\ X = \frac{1}{D}\left[\frac{1}{D}\ X\right]$$
$$P.I. = y = \frac{1}{D^2}\ X = \int\left[\int Xdx\right]dx$$

(II) $\dfrac{1}{(D-m_1)(D-m_2)}X$

By above formula

$$\begin{aligned}\frac{1}{(D-m_1)(D-m_2)}X &= \frac{1}{(D-m_1)}\left[\frac{1}{(D-m_2)}X\right]\\ &= \frac{1}{(D-m_1)}\left[e^{m_2x}\int Xe^{-m_2x}dx\right]\\ &= e^{m_1x}\int e^{-m_1x}\left[e^{m_2x}\int Xe^{-m_2x}dx\right]dx\end{aligned}$$

OR we can factorize $\dfrac{1}{(D-m_1)(D-m_2)}$ using partial fraction and apply the formula simultaneously.

Example: Direct Method
Solve

$$1)\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{e^x}$$

Solution
The Compact form of given equation is

$$f(D)y=X$$

$$(D^2+3D+2)y=e^{e^x}$$

Now for Complimentary function(C.F.)
A.E.

$$f(D)=0$$

$$D^2+3D+2=0$$

$$D=-1, D=-2$$

Roots of auxillary equation are real and distinct

$$C.F. = y_c = c_1e^{-2x} + c_2e^{-x}$$

Now for Particular integral (P.I.)

$$\begin{aligned}
P.I. = y_p = \frac{1}{f(D)}X &= \frac{1}{D^2+3D+2}X \\
&= \frac{1}{(D+2)(D+1)}\left(e^{e^x}\right) \\
&= \frac{1}{(D+2)}\left[\frac{1}{(D+1)}\left(e^{e^x}\right)\right] \\
&= \frac{1}{(D+2)}\left[e^{-x}\int e^xe^{e^x}dx\right]
\end{aligned}$$

Put $e^x = t \implies e^xdx = dt$
Substituting in above integral we have
Now for Particular integral (P.I.)

$$\begin{aligned}
P.I. = y_p = \frac{1}{f(D)}X &= \frac{1}{(D+2)}\left[e^{-x}\int e^xe^{e^x}dx\right] \\
&= \frac{1}{(D+2)}\left[e^{-x}\int e^tdt\right] \\
&= \frac{1}{(D+2)}\left[e^{-x}e^{e^x}\right] \\
&= e^{-2x}\int e^{2x}e^{-x}e^{e^x}dx \\
&= e^{-2x}\int e^xe^{e^x}dx \\
P.I. = y_p &= e^{-2x}e^{e^x}
\end{aligned}$$

Hence
$G.S. = C.F. + P.I. = c_1e^{-2x} + c_2e^{-x} + e^{-2x}e^{e^x}$

Method 2: Shortcut Methods

Case I:$f(x) = e^{ax}$

If $f(x) = e^{ax}, a$ is any constant then

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}; \quad provided \quad f(a) \neq 0$$

If $f(a) = 0$

$$\frac{1}{f(D)}e^{ax} = x.\frac{1}{f'(a)}e^{ax}; \quad provided \quad f'(a) \neq 0$$

In general

$$\frac{1}{f(D)}e^{ax} = x^n \frac{1}{f^{(n)}(a)}e^{ax}; \quad provided \quad f^{(n)}(a) \neq 0$$

Case I: Proof

Given

$$\begin{aligned} f(x) &= e^{ax} \\ \implies D(e^{ax}) &= ae^{ax} \\ D^2(e^{ax}) &= a^2e^{ax} \\ Hence\ D^n(e^{ax}) &= a^ne^{ax} \\ i.e. f(D)(e^{ax}) = f(a)(e^{ax}) & \end{aligned}$$

Operating $\frac{1}{f(D)}$ on both sides

$$\frac{1}{f(D)}\left[f(D)(e^{ax})\right] = \frac{1}{f(D)}\left[f(a)(e^{ax})\right]$$

$$e^{ax} = f(a)\left[\frac{1}{f(D)}(e^{ax})\right]$$

dividing by $f(a)$

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}; \quad provided f(a) \quad \neq 0$$

If $f(a) = 0$ then $(D - a)$ must be a factor of $f(D)$

Let $f(D) = (D-a)\phi(D)$ where $\phi(a) \neq 0$, Then

$$\begin{aligned}
\frac{1}{f(D)}e^{ax} &= \frac{1}{(D-a)\phi(D)}e^{ax} \\
&= \frac{1}{(D-a)}\frac{e^{ax}}{\phi(D)} \\
&= \frac{1}{\phi(a)}\frac{1}{(D-a)}e^{ax} \\
&= \frac{1}{\phi(a)}e^{ax}\int e^{-ax}e^{ax}dx \\
&= x.\frac{1}{\phi(a)}e^{ax} \\
\frac{1}{f(D)}e^{ax} &= x.\frac{1}{f'(a)}e^{ax} \\
provided \quad f'(a) \neq 0
\end{aligned}$$

Case II: $f(x) = sin(ax+b) or cos(ax+b)$
If $f(x) = sin(ax+b) or cos(ax+b)$ then

$$\frac{1}{f(D^2)}[sin(ax+b)] = \frac{1}{f(-a^2)}(sin(ax+b)); \quad provided \quad f(-a^2) \neq 0$$

OR

$$\frac{1}{f(D^2)}[cos(ax+b)] = \frac{1}{f(-a^2)}(cos(ax+b)); \quad provided \quad f(-a^2) \neq 0$$

If $f(-a^2) = 0$

$$\frac{1}{f(D^2)}[sin(ax+b)] = x \ \frac{1}{f'(-a^2)}[sin(ax+b)]$$

$$\left\{\frac{1}{f(D^2)}[cos(ax+b)] = x \ \frac{1}{f'(-a^2)}[cos(ax+b)]\right\}$$

provided $f'(-a^2) \neq 0$
In general If $f^{(n)}(-a^2) = 0$

$$\frac{1}{f(D^2)}[sin(ax+b)] = x^n \ \frac{1}{f^{(n)}(-a^2)}[sin(ax+b)]$$

$$\left\{\frac{1}{f(D^2)}[cos(ax+b)] = x^n \ \frac{1}{f^{(n)}(-a^2)}[cos(ax+b)]\right\}$$

provided $f^{(n)}(-a^2) \neq 0$

Case II: Proof

Given

$$f(x) = sin(ax+b)$$
$$\implies D[sin(ax+b)] = acos(ax+b)$$
$$\implies D^2[sin(ax+b)] = -a^2 sin(ax+b)$$
$$Hence f(D^2)[sin(ax+b)] = f(-a^2)[sin(ax+b)]$$

Operating $\frac{1}{f(D^2)}$ on both sides

$$\frac{1}{f(D^2)}\left\{f(D^2)[sin(ax+b)]\right\} = \frac{1}{f(D^2)}\left\{f(-a^2)[sin(ax+b)]\right\}$$
$$[sin(ax+b)] = f(-a^2)\left\{\frac{1}{f(D^2)}[sin(ax+b)]\right\}$$

dividing by $f(-a^2)$

$$\frac{1}{f(D^2)}[sin(ax+b)] = \frac{1}{f(-a^2)}[sin(ax+b)]; \quad provided \quad f(-a^2) \ \neq 0$$

If $f(-a^2) = 0$ then

$$\frac{1}{f(D^2)}[sin(ax+b)] = I.P.\ of \frac{1}{f(D^2)}[e^{i(ax+b)}]$$
$$= x\ I.P.\ of \frac{1}{f'(-a^2)}[e^{i(ax+b)}]$$
$$\therefore \frac{1}{f(D^2)}[sin(ax+b)] = x\ \frac{1}{f'(-a^2)}[sin(ax+b)]$$
$$provided \quad f'(-a^2) \ \neq 0$$

In general If $f^{(n)}(-a^2) = 0$

$$\frac{1}{f(D^2)}[sin(ax+b)] = x^n\ \frac{1}{f^{(n)}(-a^2)}[sin(ax+b)]$$

provided $f^{(n)}(-a^2) \neq 0$

Examples

Solve
$(D-1)^3y = e^x + 2^x - \frac{3}{2} + sinx$
Solution
Given $(D-1)^3y = e^x + 2^x - \frac{3}{2} + sinx$
A.E. $f(D) = 0 \implies (D-1)^3 = 0$
$D = 1(Repeated \ \ thrice)$

$$\therefore C.F. = y_c = (c_1x^2 + c_2x + c_3)e^x$$

Now

$$\begin{aligned}
P.I. &= \frac{1}{(D-1)^3}[e^x + 2^x - \frac{3}{2} + sinx] \\
&= \frac{1}{(D-1)^3}(e^x) + \frac{1}{(D-1)^3}(2^x) - \frac{1}{(D-1)^3}(\frac{3}{2}) + \frac{1}{(D-1)^3}(sinx) \\
&= \frac{1}{(D-1)^3}(e^x) + \frac{1}{(D-1)^3}(e^x log2) - \frac{1}{(D-1)^3}(\frac{3}{2}e^{0x}) \\
&\quad + \frac{1}{(D-1)^3}(sinx) \\
&= (P.I.)_1 + (P.I.)_2 + (P.I.)_3 + (P.I.)_4
\end{aligned}$$

$$\begin{aligned}
(P.I.)_1 &= \frac{1}{(D-1)^3}(e^x) \\
&= x \ \frac{1}{3(D-1)^2}(e^x) \\
&= x^2 \ \frac{1}{6(D-1)}(e^x) \\
&= x^3 \ \frac{1}{6}(e^x) \\
(P.I.)_1 &= \frac{x^3}{3!}(e^x)
\end{aligned}$$

$$
\begin{aligned}
(P.I.)_2 &= \frac{1}{(D-1)^3}(2^x) \\
&= \frac{1}{(D-1)^3}(e^x \ log \ 2) \\
&= \frac{1}{(D-1)^3}e^{(x \ log \ 2)} \\
&= \frac{1}{(log \ 2-1)^3}e^{(x \ log \ 2)} \\
(P.I.)_2 &= \frac{1}{(log \ 2-1)^3}2^x
\end{aligned}
$$

$$
\begin{aligned}
(P.I.)_3 &= \frac{1}{(D-1)^3}(\frac{3}{2}e^{0x}) \\
&= \frac{3}{2}\frac{1}{(D-1)^3}(e^{0x}) \\
&= \frac{3}{2}\frac{1}{(0-1)^3}(e^{0x}) \\
&= -\frac{3}{2} \\
(P.I.)_3 &= -\frac{3}{2}
\end{aligned}
$$

$$
\begin{aligned}
(P.I.)_4 &= \frac{1}{(D-1)^3}(sin\ x) \\
&= \frac{1}{D^3 - 1 - 3D^2 + 3D}(sin\ x) \\
&= \frac{1}{D^2.D - 1 - 3D^2 + 3D}(sin\ x) \\
&= \frac{1}{-1.D - 1 - 3(-1) + 3D}(sin\ x) \\
&= \frac{1}{2D+2}(sin\ x) \\
&= \frac{2-2D}{4-4D^2}(sin\ x) \\
&= \frac{1}{8}[2sin\ x - 2D(sin\ x)] \\
(P.I.)_4 &= \frac{1}{4}[sin\ x - cos\ x]
\end{aligned}
$$

Hence

$$
\begin{aligned}
P.I. &= (P.I.)_1 + (P.I.)_2 + (P.I.)_3 + (P.I.)_4 \\
P.I. &= \frac{x^3}{3!}(e^x) + \frac{1}{(log\ 2-1)^3}2^x - \frac{3}{2} + \frac{1}{4}[sin\ x - cos\ x]
\end{aligned}
$$

$\therefore G.S. = C.F + P.I.$

$$
G.S. = (c_1x^2 + c_2x + c_3)e^x + \frac{x^3}{3!}(e^x) + \frac{1}{(log\ 2-1)^3}2^x - \frac{3}{2} + \frac{1}{4}[sin\ x - cos\ x]
$$

Case III:$f(x) = x^m or P_m(x)\ a\ polynomial\ \ in\ \ x$

- If $f(x) = x^m or P_m(x)$ then

$$
\frac{1}{f(D)}[x^m or P_m(x)] = [1+\phi(D)]^{-1} x^m or P_m(x)
$$

- where $[1+\phi(D)]$ is obtained by considering lowest power of D common in denominator
- Then expand $[1+\phi(D)]^{-1}$ using binomial expansion in ascending Powers of D

- P.I. is obtained by applying Powers of D one by one on X.

Case IV:$f(x) = e^{ax}.V \ where \ V \ is \ a \ function \ \ in \ \ x$
If $f(x) = e^{ax}.V$ then

$$\frac{1}{f(D)}[e^{ax}.V] = e^{ax}\frac{1}{f(D+a)}V$$

Case IV: Proof
Given

$$\begin{aligned} f(x) &= e^{ax}U \\ \implies D[e^{ax}U] &= e^{ax}DU + ae^{ax}U \\ &= e^{ax}(D+a)U \\ D^2[e^{ax}U] &= e^{ax}D^2U + 2ae^{ax}U + a^2e^{ax}U \\ &= e^{ax}(D+a)^2U \\ Generalizing & \\ D^n[e^{ax}U] &= e^{ax}(D+a)^nU \\ \implies f(D)[e^{ax}.U] &= e^{ax}f(D+a)U.....\mathbf{(1)} \end{aligned}$$

Let $f(D+a)U = V \implies U = \dfrac{1}{f(D+a)}V$
substituting in (1) and applying $\frac{1}{f(D)}$ on both sides, we have

$$\frac{1}{f(D)}[e^{ax}.V] = e^{ax}\frac{1}{f(D+a)}V$$

Case V:$f(x) = x^m.V \ where \ V \ is \ sin(ax)orcos(ax)$
If $f(x) = x^m \ sin(ax)$ then

$$\begin{aligned} \frac{1}{f(D)}[x^m \ sin(ax)] &= imaginary \ part \ of \ \frac{1}{f(D)}[x^m.e^{iax}] \\ &= imaginary \ part \ of \ e^{iax}\frac{1}{f(D+ia)}[x^m] \end{aligned}$$

If $f(x) = x^m \ cos(ax)$ then

$$\begin{aligned} \frac{1}{f(D)}[x^m \ cos(ax)] &= Real \ part \ of \ \frac{1}{f(D)}[x^m.e^{iax}] \\ &= Real \ part \ of \ e^{iax}\frac{1}{f(D+ia)}[x^m] \end{aligned}$$

Examples
Solve
$(D^2 - 4D + 3)y = x^3e^{2x} + 3x^2 - 1$

Solution
Given $(D^2 - 4D + 3)y = x^3e^{2x} + 3x^2 - 1$
A.E. $f(D) = 0 \implies (D^2 - 4D + 3) = 0$
$D = 1, 3$

$$\therefore C.F. = y_c = c_1e^x + c_2e^{3x}$$

Now

$$\begin{aligned} P.I. &= \frac{1}{(D^2 - 4D + 3)}[x^3e^{2x} + 3x^2 - 1] \\ &= \frac{1}{(D^2 - 4D + 3)}(x^3e^{2x}) + \frac{1}{(D^2 - 4D + 3)}(3x^2 - 1) \\ &= (P.I.)_1 + (P.I.)_2 \end{aligned}$$

Now

$$
\begin{aligned}
(P.I.)_1 &= \frac{1}{(D^2-4D+3)}(x^3e^{2x}) \\
&= e^{2x}\ \frac{1}{(D+2)^2-4(D+2)+3}(x^3) \\
&= e^{2x}\ \frac{1}{D^2+4D+4-4D-8+3}(x^3) \\
&= e^{2x}\ \frac{1}{D^2-1}(x^3) \\
&= \frac{e^{2x}}{-1}[1-D^2]^{-1}(x^3) \\
&= \frac{e^{2x}}{-1}[1+D^2+D^4+...](x^3) \\
&= \frac{e^{2x}}{-1}[1+D^2](x^3) \\
&= \frac{e^{2x}}{-1}[x^3+D^2(x^3)] \\
&= -e^{2x}[x^3+6x] \\
(P.I.)_1 &= -e^{2x}[x^3+6x]
\end{aligned}
$$

$$
\begin{aligned}
(P.I.)_2 &= \frac{1}{(D^2-4D+3)}(3x^2-1)\\
&= \frac{1}{3}(1+\frac{D^2}{3}-\frac{4D}{3})(3x^2-1)\\
&= \frac{1}{3}\left[1+\frac{1}{3}(D^2-4D)\right]^{-1}(3x^2-1)\\
&= \frac{1}{3}[1-\frac{1}{3}(D^2-4D)+\frac{1}{9}(D^2-4D)^2-...](3x^2-1)\\
&= \frac{1}{3}[1-\frac{1}{3}(D^2-4D)+\frac{1}{9}(16D^2)](3x^2-1)\\
&= \frac{1}{3}[1+\frac{4}{3}D+\frac{13}{3}(D^2)](3x^2-1)\\
&= \frac{1}{3}[3x^2+8x+25]\\
(P.I.)_2 &= \frac{1}{3}[3x^2+8x+25]
\end{aligned}
$$

Hence

$$
\begin{aligned}
P.I. &= (P.I.)_1+(P.I.)_2\\
P.I. &= -e^{2x}[x^3+6x]+\frac{1}{3}[3x^2+8x+25]
\end{aligned}
$$

$\therefore G.S. = C.F + P.I.$

$$G.S. = c_1e^x + c_2e^{3x} + -e^{2x}[x^3+6x]+\frac{1}{3}[3x^2+8x+25]$$

Examples
Solve
$(D^2+1)y = x^2 sin2x$
Solution
Given $(D^2+1)y = x^2 sin2x$
A.E. $f(D)=0 \implies (D^2+1)=0$
$D=\pm i$

$$\therefore C.F. = y_c = c_1cosx + c_2sinx$$

Now

$$
\begin{aligned}
P.I. &= \frac{1}{(D^2+1)}[x^2 sin2x] \\
&= Imaginary Part of \left\{ \frac{1}{(D^2+1)}[x^2 e^{i2x}] \right\} \\
&= I.P.of \left\{ e^{i2x} \frac{1}{(D+2i)^2+1)}[x^2] \right\} \\
&= I.P.of \left\{ e^{i2x} \frac{1}{(D^2+4iD-4+1)}[x^2] \right\} \\
&= I.P.of \left\{ e^{i2x} \frac{1}{(D^2+4iD-3)}[x^2] \right\} \\
&= I.P.of \left\{ e^{i2x} \frac{1}{(-3)\left[1+\frac{D^2+4iD}{-3}\right]}[x^2] \right\} \\
&= I.P.of \left\{ \frac{e^{i2x}}{(-3)} \left[1+\frac{D^2+4iD}{-3}\right]^{-1} [x^2] \right\} \\
&= I.P.of \left\{ \frac{e^{i2x}}{(-3)} \left[1-\frac{D^2+4iD}{3}\right]^{-1} [x^2] \right\} \\
&= I.P.of \left\{ \frac{e^{i2x}}{(-3)} \left[1+\frac{D^2+4iD}{3}+\frac{D^4+8iD^3-16D^2}{9}+...\right] [x^2] \right\} \\
&= I.P.of \left\{ \frac{e^{i2x}}{(-3)} \left[1+\frac{D^2+4iD}{3}+\frac{-16D^2}{9}\right] [x^2] \right\} \\
&= I.P.of \left\{ \frac{e^{i2x}}{(-3)} \left[1+\frac{4}{3}iD-\frac{13}{9}D^2\right] [x^2] \right\} \\
&= I.P.of \left\{ \frac{e^{i2x}}{(-3)} \left[x^2+\frac{8}{3}ix-\frac{26}{9}\right] \right\} \\
&= I.P.of \left\{ \frac{(cos2x+isin2x)}{(-3)} \left[x^2+\frac{8}{3}ix-\frac{26}{9}\right] \right\} \\
&= I.P.of \left\{ \frac{1}{(-3)} \left[x^2cos2x+\frac{8}{3}ixcos2x-\frac{26}{9}cos2x+x^2isin2x+\frac{8}{3}i^2xsin2x-\frac{26}{9}isin2x\right] \right\} \\
&= I.P.of \left\{ \frac{1}{(-3)} \left[(x^2cos2x-\frac{8}{3}xsin2x-\frac{26}{9}cos2x)+i(\frac{8}{3}xcos2x+x^2sin2x-\frac{26}{9}sin2x)\right] \right\} \\
P.I. &= \frac{1}{(-3)} \left[\frac{8}{3}xcos2x+x^2sin2x-\frac{26}{9}sin2x\right]
\end{aligned}
$$

<u>Exercise</u>

Solve

(1) $\dfrac{d^2y}{dx^2} + y = sin\ x\ sin\ 2x$

(2) $[(D-1)^2(D^2+1)^2]y = sin^2(\dfrac{x}{2})$

(3) $\dfrac{d^2y}{dx^2} + a^2y = tan\ ax$

(4) $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 cosx$

(5) $\dfrac{d^4y}{dx^4} + y = 2\ sinhxsinx$

(6) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = e^x x + sinx + cosx$

(7) $(D^5 - D)y = 12e^x + 85mx + 2^x$

(8) $(D^2 - 1)y = (1 + e^{-x})^2$

(9) $(D^2 - 4D + 4)y = 8x^2.e^{2x} sin2x$

(10) $(D^2 - 40D + 8)y = 12e^{-2x} sinxsin3x$

(11) $(D^3 - D^2 - D + 1)y = coshx\ sinx$

(12) $(D^2 + 2D + 5)^2 y = xe^{-x}\ cos2x$

(13) $(D^2 + 3D + 2)y = sine^x$

(14) $(D^2 - 1)y = xsinx + (1 + x^2)e^x$

(15) $(D^3 + 8)y = x^4 + 2x + 1$

Answers

(1) $y = c_1 cos\ x + c_2 sin\ x + \frac{1}{4}x\ sin\ x + \frac{1}{16}\ cos\ 3x$

(2) $y = (c_1x + c_2)e^x + (c_3x + c_4)cos\ x + (c_5x + c_6)sin\ x + \frac{1}{2} - \frac{1}{32}\ x^2\ sin\ x$

(3) $y = c_1 cos\ ax + c_2 sin\ ax - \frac{1}{a^2} cos\ ax\ log[sec\ ax + tan\ ax]$

(4) $y = (c_1x + c_2)cosx + (c_3x + c_4)sinx + \frac{x^3 sinx}{12} - \frac{x^4 - 9x^2}{48} cosx$

(5) $y = e^{\frac{x}{\sqrt{2}}}\left[c_1 cos\frac{x}{\sqrt{2}} + c_2 sin\frac{x}{\sqrt{2}}\right] + e^{\frac{-x}{\sqrt{2}}}\left[c_3 cos\frac{x}{\sqrt{2}} + c_4 sin\frac{x}{\sqrt{2}}\right] - \frac{2}{3} sinxsinhx$

(6) $y = c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{e^x}{2}\left(\frac{x^2}{3} + \frac{3}{2}x\right) - \frac{1}{10}cosx + \frac{1}{10}sinx$

(7) $y = c_1 + c_2e^x + c_3e^{-x} + c_4cosx + c_5sinx + 3xe^x - 35m\frac{x^2}{2} + \frac{2^x}{(log2)^5 - log2}$

(8) $y = c_1e^x + (c_2 - 2)e^{-x} + e^{-x}\left[\frac{(1+e^x)^2}{2} + log1 + e^x\right] - 2$

(9) $y = e^{2x}[c_1 + c_2x + 3sin2x - 2x^2 sin2x - 4xcos2x]$

(10) $y = e^{-2x}(c_1cos2x + c_2sin2x) + \frac{3}{2}xe^{-2x}sin2x + \frac{1}{2}e^{-2x}cos4x$

(11) $y = (c_1x + c_2)e^x + c_3e^{-x} + \frac{e^x}{10}[cosx - 2sinx] - \frac{e^{-x}}{50}[3cosx - 4sinx]$

(12) $y = e^{-x}[(c_1x + c_2)cos2x + (C_3x + c_4)sin2x] - \frac{e^{-x}}{32}[(x^3 - x^2)cos2x - \frac{2}{3}x^3 sin2x]$

(13) $y = c_1e^{-2x} + c_2e^{-x} - e^{-2x}sine^x$

(14) $y = c_1e^x + c_2e^{-x} - \frac{1}{2}(xsinx + cosx) + \frac{e^x}{12}(2x^3 - 3x^2 + 9x)$

(15) $y = c_1e^{-2x} + e^x[c_2cos\sqrt{3}x + c_3sin\sqrt{3}x] + \frac{1}{8}(x^4 - x + 1)$