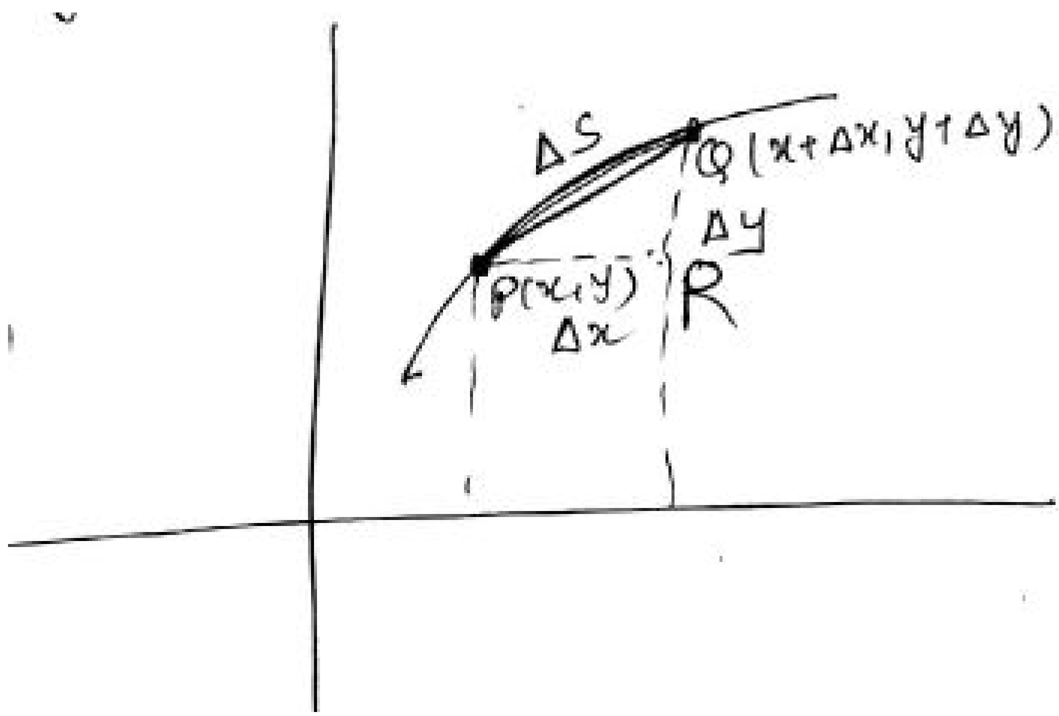
RECTIFICATION OF CURVE

Concept:Rectification of cartesian curve

Determining the length of a curve in plane or space is known as **Rectification**

Type I Curve of the form $y = f(x)$

Let length of the curve AB; $y = f(x)$ between the points $x = a$ and $x = b$ is S and let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ on the curve AB



$\therefore$ in ΔPQR

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\therefore (\Delta s)^2 = (\Delta x)^2 \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

$$\therefore \left(\frac{\Delta s}{\Delta x}\right)^2 = \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

$$\therefore \lim_{\Delta x \to 0} \left(\frac{\Delta s}{\Delta x}\right)^2 = \lim_{\Delta x \to 0} \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \left(\frac{ds}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore S = \int\limits_a^b \frac{ds}{dx}\, dx = \int\limits_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\therefore S = \int\limits_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

which is arc length of the cartesian curve $y = f(x)$ between $x = a$ and $x = b$,similarly arc length of the cartesian curve $x = g(y)$ between $y = c$ and $y = d$ is given by

$$S=\int_{c}^{d}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Working rule for examples for the curves $y=f(x)$ or $x=g(y)$:

1. Draw the given curves and find limits of integration by finding points of intersection or using given conditions.(In case if limits are given no need to draw the curves.
2. Find $\frac{dy}{dx}$ or $(\frac{dx}{dy})$ and simplify
3. Find $1+\left(\frac{dy}{dx}\right)^2$ or$1+\left(\frac{dx}{dy}\right)^2$ and simplify
4. Find length using formula $S=\int\limits_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $S=\int\limits_{y_1}^{y_2}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ where x_1 and x_2 are Xcoordinates and y_1 and y_2 are Y coordinates respectively

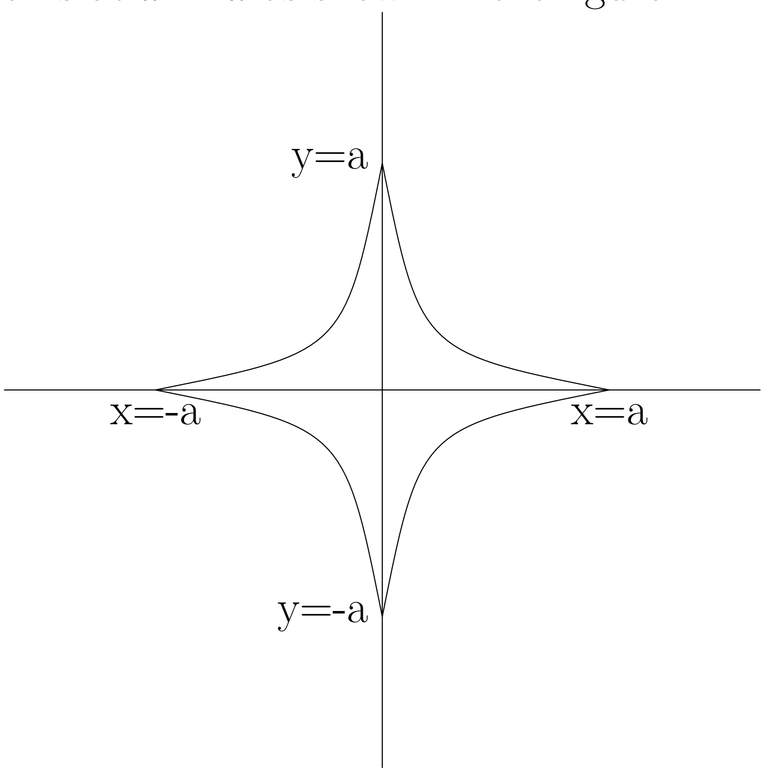
EXAMPLES

1. Find perimeter of the curve astroid given by

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

Solution:

Given curve is symmetric about both the axes and intersects x axis at $x = a$ as shown in the figure



Now differentiating given equation with respect to x we have

$$
\begin{aligned}
x^{\frac{2}{3}} + y^{\frac{2}{3}} &= a^{\frac{2}{3}} \\
\therefore \frac{2}{3}\ x^{\frac{-1}{3}} + \frac{2}{3}\ y^{\frac{-1}{3}} \frac{dy}{dx} &= 0 \\
\therefore \frac{dy}{dx} &= -\left(\frac{y}{x}\right)^{\frac{1}{3}} \\
Perimeter &= 4 \int\limits_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= 4 \int\limits_0^a \sqrt{1 + \left(\frac{y}{x}\right)^{\frac{2}{3}}} dx \\
&= 4 \int\limits_0^a \left[\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right]^{\frac{1}{2}} dx \\
&= 4 \int\limits_0^a \left[\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right]^{\frac{1}{2}} dx \\
&= 4\ a^{\frac{1}{3}} \frac{3}{2} \left[x^{\frac{2}{3}}\right]_0^a \\
&= 6\ a^{\frac{1}{3}} a^{\frac{2}{3}} \\
&= 6\ a .....Ans
\end{aligned}
$$

2. If L is the length of the curve $y^2 = x\left(1 - \frac{x}{3}\right)^2$ measured from the origin to the ordinate $x = a$ show that $9L^2 = a(a+3)^2$

Solution:

Given

$$\begin{aligned}
y^2 &= x\left(1 - \frac{x}{3}\right)^2 \\
\implies y &= \sqrt{x}\left(1 - \frac{x}{3}\right) \\
\implies \frac{dy}{dx} &= \sqrt{x}\left(-\frac{1}{3}\right) + \frac{1}{2\sqrt{x}}\left(1 - \frac{x}{3}\right) \\
\implies \frac{dy}{dx} &= \sqrt{x}\left[-\frac{1}{3} + \frac{1}{2x}\left(1 - \frac{x}{3}\right)\right] \\
\implies \frac{dy}{dx} &= \sqrt{x}\left[\frac{1}{2x} - \frac{1}{2}\right] \\
\implies \frac{dy}{dx} &= \frac{1-x}{2\sqrt{x}}
\end{aligned}$$

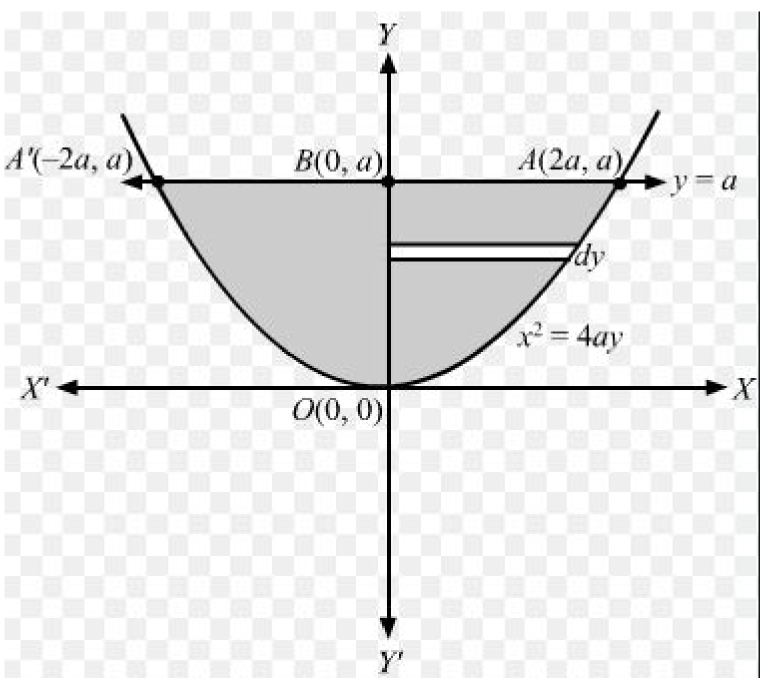
Since L is length of the curve

$$
\begin{aligned}
L &= \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\
&= \int_0^a \sqrt{1+\left(\frac{1-x}{2\sqrt{x}}\right)^2}dx \\
&= \int_0^a \sqrt{1+\frac{(1-x)^2}{4x}}dx \\
&= \int_0^a \sqrt{\frac{(x+1)^2}{4x}}dx \\
&= \int_0^a \frac{x+1}{2\sqrt{x}}dx \\
&= \frac{1}{2}\int_0^a \left[x^{\frac{1}{2}}+x^{\frac{-1}{2}}\right]dx \\
&= \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}\right]_0^a \\
&= \sqrt{a}\left[\frac{a}{3}+1\right] \\
&= \sqrt{a}\left[\frac{a+3}{3}\right] \\
\implies L^2 &= \frac{a(a+3)^2}{9} \\
\implies 9L^2 &= a\ (a+3)^2....proved
\end{aligned}
$$

3. Show that length of the arc of parabola $x^2 = 4ay$ cut off by its latus rectum is $2a\left[\sqrt{2} + log(1+\sqrt{2})\right]$

Solution:

Here limits are not given so we need to sketch the curve and find points of intersection
Here parabola $x^2 = 4ay$ is symmetric about yy axisand has vertex at $(0, 0)$. The latus rectum is straight line passingthrough the focus $(0, a)$ and perpendicular to the axis of parabola.
$\therefore$ equation of the latus rectum for given parabola is line $y = a$ at the Point of intersection of parabola and its latus rectum we have $x = -2a, 2a$
$\therefore$ poins of intersection are $(-2a, a)$ and $(2a, a)$



$\therefore$ length of the arc of parabola cut off by its latus rectum is

$$S = \int_{x=-2a}^{x=2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$= 2 \int_{x=0}^{x=2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

where

$$
\begin{aligned}
y &= \frac{x^2}{4a} \\
\therefore \frac{dy}{dx} &= \frac{x}{2a} \\
\therefore 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{x^2}{4a^2} = \frac{4a^2 + x^2}{4a^2} \\
\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{\frac{4a^2 + x^2}{4a^2}} = \frac{1}{2a}\sqrt{4a^2 + x^2} \\
\therefore S &= 2\int\limits_{x=0}^{x=2a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= 2\int\limits_{x=0}^{x=2a} \frac{1}{2a}\sqrt{4a^2 + x^2} dx \\
&= \frac{1}{a}\left[\frac{x}{2}\sqrt{4a^2 + x^2} + \frac{4a^2}{2} log\left(x + \sqrt{4a^2 + x^2}\right)\right]_0^{2a} \\
&= \frac{1}{a}\left[2\sqrt{2}a^2 + 2a^2 log\left(2a + 2\sqrt{2}a\right) - 2a^2 log(2a)\right] \\
&= \frac{1}{a}\left[2\sqrt{2}a^2 + 2a^2 log\left(\frac{2a(1 + \sqrt{2})}{2a}\right)\right] \\
&= \frac{1}{a}\left[2\sqrt{2}a^2 + 2a^2 log\left(1 + \sqrt{2}\right)\right] \\
\therefore S &= 2a\left[\sqrt{2} + log\left(1 + \sqrt{2}\right)\right]
\end{aligned}
$$

4. If L is the length of the curve $y^2 = x\left(1 - \frac{x}{3}\right)^2$ measured from $(0,0)$ to (x,y) then show that $L^2 = y^2 + \frac{4}{3}x^2$

5. Find length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$

6. Find length of the curve $y = \frac{3}{4}x^{\frac{4}{3}} - \frac{3}{8}x^{\frac{2}{3}} + 5$ where $1 \leq x \leq 8$

7. Find length of the loop of the curve $9ay^2 = x(x - 3a)^2$

8. Find length of the loop of the curve $3ay^2 = x(x - a)^2$

9. Find arc length of parabola $x^2 = 4y$ lies inside the circle $x^2+y^2 = 6y$

10. Find the length of the arc of parabola $y^2 = 8x$ cut off by its latus rectum.

Type II Curve of the form $r = f(\theta)$

In polar coordinates equation of the curve is given by $r = f(\theta)$ where $\alpha \leq \theta \leq \beta$
Consider polar coordinates

$x = rcos(\theta) = f(\theta)\ cos(\theta)$ and $y = rsin(\theta) = f(\theta)\ sin(\theta)$ then

$\frac{dx}{d\theta} = f'(\theta)\,cos(\theta) - f(\theta)\,sin(\theta)$ and $\frac{dy}{d\theta} = f'(\theta)\,sin(\theta) + f(\theta)\,cos(\theta)$

$\therefore$ arc length formula for given polar curve between angles $\theta = \alpha$ and $\theta = \beta$ is given by

$$
\begin{aligned}
S &= \int\limits_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta \\
&= \int\limits_{\alpha}^{\beta} \sqrt{\left(f'(\theta)\ cos(\theta) - f(\theta)\ sin(\theta)\right)^2 + \left(f'(\theta)\ sin(\theta) + f(\theta)\ cos(\theta)\right)^2}\ d\theta \\
&= \int\limits_{\alpha}^{\beta} \sqrt{(f'(\theta))^2\left[cos^2(\theta) + sin^2(\theta)\right] + (f(\theta))^2\left[cos^2(\theta) + sin^2(\theta)\right]}\ d\theta \\
&= \int\limits_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2}\ d\theta \\
\therefore S &= \int\limits_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta
\end{aligned}
$$

Working rule for examples for the curves $r = f(\theta)$:

1. Draw the given curves and find limits of integration by finding points of intersection or using given conditions.(In case if limits are given no need to draw the curves.

2. Find $\frac{dr}{d\theta}$

3. Find $r^2 + \left(\frac{dr}{d\theta}\right)^2$ and simplify

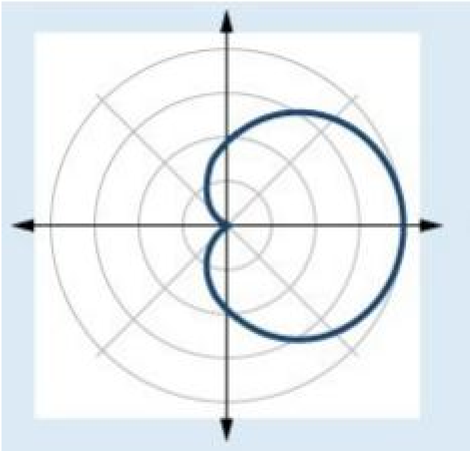
4. Find length using formula $S = \int\limits_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

EXAMPLES

1. Find perimeter or the whole length of the cardioid given by $r = a\,(1 + \cos\theta)$

Solution:

Given curve is symmetric about the initial line $\theta = 0$, as shown in the figure where $a = b$



$r = a + b\cos\theta$

Now because of symmetry total length or perimeter of the curve=2* length of the curve in upper half

$$\therefore S = 2 \int_{\theta=0}^{\theta=\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta$$

Now,

$$r = a\ (1\ +\ cos\theta)$$

$$\therefore \frac{dr}{d\theta} = a\ sin\ \theta$$

$$\therefore r^2\ +\ \left(\frac{dr}{d\theta}\right)^2 = a^2\ (1 + cos\theta)^2 + a^2\ sin^2\theta$$

$$= a^2\ (1 + cos^2\theta + 2cos\theta) + a^2\ sin^2\theta$$

$$= a^2\ (1 + cos^2\theta + 2cos\theta +\ sin^2\theta)$$

$$= a^2\ (1 + 2cos\theta + 1)$$

$$= a^2\ (2\ +\ 2cos\theta)$$

$$= 2a^2\ (1\ +\ cos\theta)$$

$$= 2a^2 \left[2cos^2\left(\frac{\theta}{2}\right)\right]$$

$$= 4a^2\ cos^2\left(\frac{\theta}{2}\right)$$

$$\therefore \sqrt{r^2\ +\ \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4a^2\ cos^2\left(\frac{\theta}{2}\right)} = 2acos\left(\frac{\theta}{2}\right)$$

$$\therefore S = 2 \int\limits_{\theta=0}^{\theta=\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$= 2 \int\limits_0^{\pi} 2a\ cos\left(\frac{\theta}{2}\right)\ d\theta$$

$$= 4a\left[2sin\left(\frac{\theta}{2}\right)\right]_0^{\pi}$$

$$= 8a\left[sin\left(\frac{\pi}{2}\right) - sin(0)\right] = 8a...Ans$$

2. Find length of the cardioid $r = a(1-cos\theta)$ lies outside the circle $r = acos\theta$

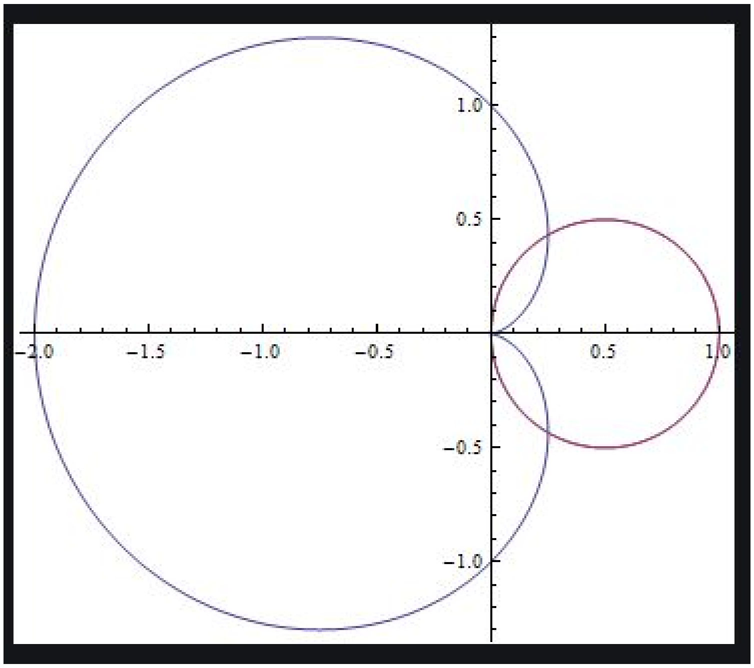
Solution:

The circle $r = acos\theta$ is having center at $(\frac{a}{2}, 0)$ and radius $\frac{a}{2}$

Now intersecting points of cardioid $r = a(1 - cos\theta)$ and circle $r = acos\theta$ is obtained by

$$\begin{aligned} a(1 - cos\theta) &= acos\theta \\ \implies 1 - cos\theta &= cos\theta \\ \implies 2cos\theta &= 1 \\ \implies cos\theta &= \frac{1}{2} \\ \implies \theta &= \pm\frac{\pi}{3} \end{aligned}$$

Region is shown in the figure and both the curves are symmetric about the initial line $\theta = 0$



Hence Required length L is given by

$$L = 2\int_{\frac{\pi}{3}}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Now

$$\begin{aligned}
r &= a(1 - cos\theta) \\
\implies \frac{dr}{d\theta} &= asin\theta \\
\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 &= a^2\ (1 - cos\theta)^2 + a^2\ sin^2\theta \\
&= a^2\ (1 + cos^2\theta - 2cos\theta) + a^2\ sin^2\theta \\
&= a^2\ (1 + cos^2\theta - 2cos\theta + \ sin^2\theta) \\
&= a^2\ (1 - 2cos\theta + 1) \\
&= a^2\ (2\ -\ 2cos\theta) \\
&= 2a^2\ (1\ -\ cos\theta) \\
&= 2a^2 \left[2sin^2\left(\frac{\theta}{2}\right)\right] \\
&= 4a^2\ sin^2\left(\frac{\theta}{2}\right) \\
\therefore \sqrt{r^2\ +\ \left(\frac{dr}{d\theta}\right)^2} &= \sqrt{4a^2\ sin^2\left(\frac{\theta}{2}\right)} = 2asin\left(\frac{\theta}{2}\right) \\
\therefore S &= 2 \int_{\theta=\frac{\pi}{3}}^{\theta=\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta \\
&= 2 \int_{\frac{\pi}{3}}^{\pi} 2a\ sin\left(\frac{\theta}{2}\right)\ d\theta \\
&= 4a \left[-2cos\left(\frac{\theta}{2}\right)\right]_{\frac{\pi}{3}}^{\pi} \\
&= 8a \left[-cos\left(\frac{\pi}{3}\right) + cos(\frac{\pi}{6})\right] \\
&= -8a \left[0 - \frac{\sqrt{3}}{2}\right] = 4a\sqrt{3} \ldots .Ans
\end{aligned}$$

3. Find perimeter or the whole length of the cardioid given by $r = a\,(1 - cos\ \theta)$

4. Find perimeter or the whole length of the cardioid given by $r = a\,(1 \pm sin\ \theta)$

5. Find perimeter or the whole length of the lemniscate given by $r^2 = a^2\ cos2(\theta)$

6. Show that length of an ace part of a cardioid $r = a\,(1 + cos\ \theta)$ which lies on the side of a line $4r = 3asec\theta$ remote from the pole is equal to $4a$