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Vivekanand Education Society's Institute of Technology

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Subject: Engineering Mathematics- I

Semester: I

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Assignment NO :- *2*

TOPIC:- ***Partial Differenciation, Homogeneous Functions.***

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Assignment 2

Topic: Partial Differentiation, Homogenous Functions

I] If $u(x,y,z) = f(x^2+y^2+z^2)$ where $x = r\cos\theta\cos\varphi$, $y = r\cos\theta\sin\varphi$, $z = r\sin\theta$;
Find $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial u}{\partial \varphi}$

→ Now,

$$x^2+y^2+z^2 = r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta$$
$$= r^2\cos^2\theta(\cos^2\varphi+\sin^2\varphi) + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta+\sin^2\theta)$$
$$= r^2$$

i.e. $u = f(x^2+y^2+z^2)$
$u = f(r^2)$

partially differentiating w.r.t. r,

$$\frac{\partial u}{\partial r} = f'(r^2) \cdot \frac{\partial r^2}{\partial r}$$

$$\boxed{\frac{\partial u}{\partial r} = 2r f'(r^2)}$$

Since, θ and φ are not related to u, i.e. u is independent of θ and φ

$$\therefore \quad \boxed{\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \varphi} = 0}$$

2] If $u = \log(\tan x + \tan y + \tan z)$, P.T. $\sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} = 2$

→

$u = \log(\tan x + \tan y + \tan z)$

partially differentiating w.r.t x,

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \frac{\partial(\tan x + \tan y + \tan z)}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

Multiplying both sides by $\sin(2x)$, we get,

$$\sin(2x) \cdot \frac{\partial u}{\partial x} = \frac{\sin 2x \sec^2 x}{\tan x + \tan y + \tan z}$$

$$\therefore \sin(2x)\frac{\partial u}{\partial x} = \frac{2\sin x \cos x \sec^2 x}{\tan x + \tan y + \tan z}$$

$$\therefore \sin(2x)\frac{\partial u}{\partial x} = \frac{2\tan x}{\tan x + \tan y + \tan z} \quad \text{--- (i)}$$

Similarly,

$$\sin(2y)\frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z} \quad \text{--- (ii)}$$

& $$\sin(2z)\frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z} \quad \text{--- (iii)}$$

Adding ~~From~~, (i), (ii) & (iii)

$$\therefore \sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} =$$

$$\frac{2\tan x}{\tan x + \tan y + \tan z} + \frac{2\tan y}{\tan x + \tan y + \tan z} + \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$\therefore \sin(2x)\dfrac{\partial u}{\partial x} + \sin(2y)\dfrac{\partial u}{\partial y} + \sin(2z)\dfrac{\partial u}{\partial z} = \dfrac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$

$\therefore \boxed{\sin(2x)\dfrac{\partial u}{\partial x} + \sin(2y)\dfrac{\partial u}{\partial y} + \sin(2z)\dfrac{\partial u}{\partial z} = 2}$

Hence, proved.

3] If $x^2 = au + bv$, $y^2 = au - bv$, Prove that $\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial u} = \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v} = \dfrac{1}{2}$

$\rightarrow$ $x^2 = au + bv$ — (i)

Partially differentiating w.r.t. u,

$\therefore 2x \dfrac{\partial x}{\partial u} = a$

$\therefore \dfrac{\partial x}{\partial u} = \dfrac{a}{2x}$ — (ii)

Now, $y^2 = au - bv$ — (iii)

Partially differentiating w.r.t. v,

$\therefore 2y \dfrac{\partial y}{\partial v} = -b$

$\therefore \dfrac{\partial y}{\partial v} = \dfrac{-b}{2y}$ — (iv)

Adding eq^n (i) & (iii),

$2au = x^2 + y^2$

partially differentiating w.r.t. x

$\therefore 2a \dfrac{\partial u}{\partial x} = 2x$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{x}{a}$ — (v)

Subtracting eq^n (iii) from (i),

$2bv = x^2 - y^2$

partially differentiating w.r.t. y,

$\therefore 2b \dfrac{\partial v}{\partial y} = -2y$

$$\therefore \quad \frac{\partial v}{\partial y} = \frac{-y}{b} \quad -(vi)$$

Multiplying eqn (ii) & (v),

$$\frac{\partial x}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{a}{2x} \cdot \frac{x}{a} = \frac{1}{2} \quad -(vii)$$

Multiplying eqn (iv) & (vi),

$$\frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial y} = -\frac{b}{2y} \cdot \frac{-y}{b} = \frac{1}{2} \quad -(viii)$$

From eqn (vii) & (viii) we can conclude that,

$$\boxed{\frac{\partial x}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{1}{2}}$$

4] If $u = x^2+y^2+z^2$, $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$; find $\frac{\partial u}{\partial t}$.

→ $u = x^2+y^2+z^2$

$$\therefore \frac{\partial u}{\partial x} = 2x \ -(i); \quad \frac{\partial u}{\partial y} = 2y \ -(ii); \quad \frac{\partial u}{\partial z} = 2z \ -(iii)$$

Now, $x = e^{2t}$

Differentiating w.r.t. t,

$$\therefore \frac{dx}{dt} = 2e^{2t} \quad -(iv)$$

Also, $y = e^{2t}\cos 3t$

Differentiating w.r.t. t,

$$\frac{dy}{dt} = e^{2t}\frac{d}{dt}(\cos 3t) + \cos 3t \frac{d}{dt}(e^{2t})$$

$$\therefore \frac{dy}{dt} = -3e^{2t}\sin 3t + 2e^{2t}\cos 3t \quad \text{— (v)}$$

Also, $z = e^{2t}\sin 3t$

Differentiating w.r.t. $\cancel{x}$ t,

$$\frac{dz}{dt} = e^{2t}\frac{d(\sin 3t)}{dt} + \sin 3t\frac{d(e^{2t})}{dt}$$

$$\therefore \frac{dz}{dt} = 3e^{2t}\cos 3t + 2e^{2t}\sin 3t \quad \text{— (vi)}$$

Now,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$\therefore \frac{\partial u}{\partial t} = (2x)(2e^{2t}) + (2y)(-3e^{2t}\sin 3t + 2e^{2t}\cos 3t) + (2z)(3e^{2t}\cos 3t + 2e^{2t}\sin 3t)$$

— (from (i) (ii) (iii) (iv) (v) & (vi))

$$\therefore \frac{\partial u}{\partial t} = \cancel{2}(2e^{2t})(2e^{2t}) + 2(e^{2t}\cos 3t)(-3e^{2t}\sin 3t + 2e^{2t}\cos 3t) + 2(e^{2t}\sin 3t)(3e^{2t}\cos 3t + 2e^{2t}\sin 3t)$$

— (from given)

$$\therefore \frac{\partial u}{\partial t} = 4e^{4t} - 6e^{4t}\cos 3t\sin 3t + 4e^{4t}\cos^2 3t + 6e^{4t}\cos 3t\sin 3t + 4e^{4t}\sin^2 3t$$

$$\therefore \frac{\partial u}{\partial t} = 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t)$$

$$\therefore \frac{\partial u}{\partial t} = 4e^{4t} + 4e^{4t}$$

$$\therefore \boxed{\frac{\partial u}{\partial t} = 8e^{4t}}$$

5] If $z = f(x, y)$, $x = e^u\cos v$, $y = e^u\sin v$, show that

(i) $xz_v + yz_u = e^{2u}z_y$

(ii) $(z_x)^2 + (z_y)^2 = e^{-2u}\left((z_u)^2 + (z_v)^2\right)$

→ Now,

$x = e^u \cos v$ | $y = e^u \sin v$

$\frac{\partial x}{\partial u} = e^u \cos v$ | $\frac{\partial y}{\partial u} = e^u \sin v$

$\frac{\partial x}{\partial v} = -e^u \sin v$ | $\frac{\partial y}{\partial v} = e^u \cos v$

$$Z_u = \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\therefore Z_u = e^u \cos v \frac{\partial z}{\partial x} + e^u \sin v \frac{\partial z}{\partial y}$$

$$\therefore Z_u = e^u \cos v (Z_x) + e^u \sin v (Z_y)$$

Also,

$$Z_v = \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\therefore Z_v = -e^u \sin v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial z}{\partial y}$$

$$\therefore Z_v = e^u \cos v (Z_y) - e^u \sin v (Z_x)$$

(i) $$xZ_v + yZ_u = x\left(e^u \cos v (Z_y) - e^u \sin v (Z_x)\right) + y\left(e^u \sin v (Z_y) + e^u \cos v (Z_x)\right)$$

$$\therefore xZ_v + yZ_u = e^u \left[Z_y (x \cos v + y \sin v) + Z_x (y \cos v - x \sin v)\right]$$

$$\therefore xZ_v + yZ_u = e^u \left[Z_y (e^u \cos^2 v + e^u \sin^2 v) + Z_x (e^u \sin v \cos v - e^u \sin v \cos v)\right]$$

$$\therefore xZ_v + yZ_u = e^u (Z_y \cdot e^u)$$

$$\therefore \boxed{xZ_v + yZ_u = e^{2u} Z_y}$$

(ii) $e^{-2u}[(z_u)^2+(z_v)^2] = e^{-2u}[(e^u \sin v(z_y) + e^u \cos v(z_x))^2 + (e^u \cos v(z_y) - e^u \sin v(z_x))^2]$

$\therefore e^{-2u}[(z_u)^2+(z_v)^2] = e^{-2u}[e^{2u}(\sin v(z_y) + \cos v(z_x))^2 + e^{2u}(\cos v(z_y) - \cancel{e^u}\sin v(z_x))^2]$

$\therefore e^{-2u}[(z_u)^2+(z_v)^2] = e^{-2u} \cdot e^{2u}[(\sin v(z_y) + \cos v(z_x))^2 + (\cos v(z_y) - \cancel{e^u}\sin v(z_x))^2]$

$\therefore e^{-2u}[(z_u)^2+(z_v)^2] = \sin^2 v\, z_y^2 + \cos^2 v\, z_x^2 + 2\sin v \cos v\, z_y z_x + \cos^2 v\, z_y^2 + \sin^2 v\, z_x^2 - 2\sin v\cos v\, z_y z_x$

$\therefore e^{-2u}[(z_u)^2+(z_v)^2] = z_y^2(\sin^2 v + \cos^2 v) + z_x^2(\sin^2 v + \cos^2 v)$

$\therefore \boxed{e^{-2u}[(z_u)^2+(z_v)^2] = (z_y)^2 + (z_x)^2}$

6] If $u = f(e^{y-x}, e^{z-x}, e^{x-y})$, find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

→ let, $p = e^{y-x}$, $q = e^{z-x}$, $r = e^{x-y}$

$\therefore u = f(p, q, r)$

Now,

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(e^{z-x})(-1) + \frac{\partial u}{\partial r}(e^{x-y})$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(e^{x-y}) - \frac{\partial u}{\partial q}(e^{z-x})$ — (i)

Also,

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$

$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(e^{y-x}) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(e^{x-y})(-1)$

$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(e^{y-x}) - \frac{\partial u}{\partial r}(e^{x-y})$ — (ii)

Also,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(e^{y-z})(-1) + \frac{\partial u}{\partial q}(e^{z-x}) + \frac{\partial u}{\partial r}(0)$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial q}(e^{z-x}) - \frac{\partial u}{\partial p}(e^{y-z}) \quad \text{--- (iii)}$$

Adding eq^n (i), (ii) & (iii),

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(e^{x-y}) - \frac{\partial u}{\partial q}(e^{z-x})$$
$$+ \frac{\partial u}{\partial p}(e^{y-z}) - \frac{\partial u}{\partial r}(e^{x-y})$$
$$+ \frac{\partial u}{\partial q}(e^{z-x}) - \frac{\partial u}{\partial p}(e^{y-z})$$

$$\therefore \boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

7] If $x = u+v+w$, $y = uv+vw+wu$, $z = uvw$ show that $xf_x + 2yf_y + 3zf_z = uf_u + vf_v + wf_w$ where f is a function of (x,y,z)

$\rightarrow$

$$x = u+v+w$$

$\frac{\partial x}{\partial u} = 1$ | $\frac{\partial x}{\partial v} = 1$ | $\frac{\partial x}{\partial w} = 1$

$$y = uv+vw+wu$$

$\frac{\partial y}{\partial u} = v+w$ | $\frac{\partial y}{\partial v} = u+w$ | $\frac{\partial y}{\partial w} = u+v$

$$z = uvw$$

$$\frac{\partial z}{\partial u} = vw \quad \Bigg| \quad \frac{\partial z}{\partial v} = uw \quad \Bigg| \quad \frac{\partial z}{\partial w} = uv$$

Let $\quad a = f(x, y, z)$

$$f_u = \frac{\partial a}{\partial u} = \frac{\partial a}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial a}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial a}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$\therefore f_u = f_x(1) + f_y(v+w) + f_z(vw)$$

$$\therefore u f_u = u[f_x + (v+w) f_y + (vw) f_z] \quad \text{— (i)}$$

Similarly,

$$v f_v = v[f_x + (u+w) f_y + (uw) f_z] \quad \text{— (ii)}$$

$$\& \quad w f_w = w[f_x + (u+v) f_y + (uv) f_z] \quad \text{— (iii)}$$

Adding eq^n (i), (ii) & (iii),

$$u f_u + v f_v + w f_w = u[f_x + (v+w) f_y + (vw) f_z] + v[f_x + (u+w) f_y + (uw) f_z] + w[f_x + (u+v) f_y + (uv) f_z]$$

$$\therefore u f_u + v f_v + w f_w = (u+v+w) f_x + (uv + uw + vu + vw + wu + wv) f_y + (uvw + vuw + wuv) f_z$$

$$\therefore u f_u + v f_v + w f_w = (u+v+w) f_x + 2(uv + vw + wu) f_y + 3(uvw) f_z$$

$$\therefore \boxed{u f_u + v f_v + w f_w = x f_x + 2y f_y + 3z f_z}$$

8] If $u = f(y/x) + \sqrt{x^2+y^2}$, P.T. $xu_x + yu_y = \sqrt{x^2+y^2}$

$\rightarrow$ Let, $v = f(y/x)$ & $w = \sqrt{x^2+y^2}$ $\therefore u = v+w$

Putting $X = xt$ and $Y = yt$ in v, we get,

$$f_1(X,Y) = f_1(yt/xt) = t^0 f_1(y/x)$$

Thus, v is a homogenous function of degree 0.
Similarly, putting $X = xt$, $Y = yt$ in w, we get,

$$f_2(X,Y) = \sqrt{X^2+Y^2} = \sqrt{x^2t^2+y^2t^2} = t\sqrt{x^2+y^2} = tf_2(x,y)$$

Thus, w is a homogenous function of degree 1.
Hence, by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0.v = 0$$

$$\& \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 1.w = w$$

Adding above results, we get,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0 + w$$

$$\therefore x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = w$$

$$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2+y^2}}$$

9] If $z = x^n f(y/x) + y^{-n} f(x/y)$, P.T.

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

→ let, $u = x^n f(y/x)$ & $v = y^n f(x/y)$ so that $z = u+v$
Now, u & v are homogenous functions of degree n and -n, hence by Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{— (i)}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \quad \text{— (ii)}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u \quad \text{— (iii)}$$

$$\& \; x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = -n(-n-1)v = n(n+1)v \quad \text{— (iv)}$$

Adding (i) & (ii), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] + \left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right]$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nu - nv = n(u-v) \quad \text{— (v)}$$

Adding (iii) & (iv), weget,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = \left[x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right] + \left[x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2}\right]$$

$$\therefore x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)u + n(n+1)v$$

$$\therefore x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n^2(u+v) - n(u-v) \quad \text{— (vi)}$$

From (v) & (vi), we get,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n^2(v+u)-n(u-v)+n(u-v)$$

$$=n^2(u+v)$$

$$\therefore \boxed{x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n^2 z}$$

10) If $z=\log(x^2+y^2)+\frac{x^2+y^2}{x+y}-2\log(x+y)$, find xz_x+yz_y and ~~yzy~~ $x^2z_{xx}+2xyz_{xy}+y^2z_{yy}$

$\longrightarrow$ Let, $u=\log(x^2+y^2)$, $v=\frac{x^2+y^2}{x+y}$ & $w=-2\log(x+y)$

Now,

$u=\log(x^2+y^2)$

$\therefore e^u=x^2+y^2$ i.e. $f(u)=e^u$

Let, $X=xt$ & $Y=yt$

$f_1(X,Y)=X^2+Y^2=x^2t^2+y^2t^2=t^2(x^2+y^2)=t^2f_1(x,y)$

$\therefore f(u)$ is a homogenous function of x,y of degree 2.

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n.\frac{f(u)}{f'(u)}=2.\frac{e^u}{e^u}=2 \quad \text{— (i)}$$

Also, $v=\frac{x^2+y^2}{x+y}$

$f_2(X,Y)=\frac{X^2+Y^2}{X+Y}=\frac{x^2t^2+y^2t^2}{x+y}=\frac{t^2}{t}\left(\frac{x^2+y^2}{x+y}\right)=tf_2(x,y)$

$\therefore v$ is a homogenous function of degree 1.

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1.v=\frac{x^2+y^2}{x+y} \quad \text{— (ii)}$$

Also, $\quad w = -2\log(x+y)$

$w = \log\left(\frac{1}{(x+y)^2}\right)$

$\therefore \quad e^{w} = \cancel{\log} \frac{1}{(x+y)^2} \qquad$ i.e. $\quad f(w) = e^{w}$

$$f_3(x,y) = \frac{1}{(x+y)^2} = \frac{1}{(xt+yt)^2} = \frac{1}{t^2(x+y)^2} = t^{-2} f_3(x,y)$$

$\therefore$ f(w) is homogenous function of degree -2.

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -2 \cdot \frac{f(w)}{f'(w)} = -2 \cdot \frac{e^{w}}{e^{w}} = -2 \quad \text{--- (iii)}$$

Adding eqn (i), (ii) & (iii), we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 2 + \frac{x^2+y^2}{x+y} - 2$$

$$\therefore x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = \frac{x^2+y^2}{x+y}$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2+y^2}{x+y}$$

$$\therefore \quad \boxed{x z_x + y z_y = \frac{x^2+y^2}{x+y}}$$

Now,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

We know that,

$$g(u) = n \cdot \frac{f(u)}{f'(u)} = 2 \cdot \frac{e^{u}}{e^{u}} = 2$$

$$\therefore g'(u) = 0$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+\cancel{y^2}\frac{\partial^2 u}{\partial y^2} = 2(0-1) = -2 \quad \text{(iv)}$$

Also,

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)\cancel{\ }V = 1(1-1)V = 0 \quad \text{(v)}$$

Also,

$$x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}+y^2\frac{\partial^2 w}{\partial y^2} = g(w)\left[g'(w)-1\right]$$

We know that,

$$g(w) = n.\frac{f(w)}{f'(w)} = -2.\frac{e^w}{e^w} = -2$$

$$\therefore g'(w) = 0$$

$$\therefore x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}+y^2\frac{\partial^2 w}{\partial y^2} = \cancel{g}-2(0-1) = 2 \quad \text{(vi)}$$

Adding eq^n (iv), (v) & (vi), we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}$$

$$+x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}+y^2\frac{\partial^2 w}{\partial y^2} = -2+0+2$$

$$\therefore x^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)+2xy\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)$$

$$+y^2\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right) = 0$$

$$\therefore x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2} = 0$$

$$\therefore \boxed{xz_{xx}+2xy\,z_{xy}+y^2 z\,z_{yy} = 0}$$