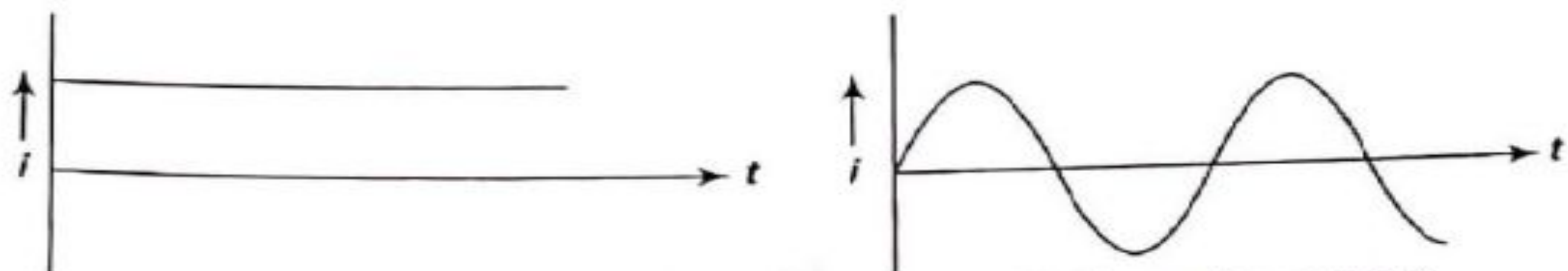
AC CIRCUITS

2.1 AC Fundamentals

In the earlier chapter, we have dealt with direct currents, which flow continuosly in one direction only. If the applied voltage and the circuit resistance are kept constant, the magnitude of the current flowing through this circuit remains constant over time. When the current flowing in the circuit varies in magnitude as well as direction periodically, it is called **alternating current**. Thus, an alternating current or voltage is one that periodically passes through a definite cycle, consisting of two half cycles—during one of which the current or voltage varies in one direction and during the other half cycle varies in the opposite direction. The circuits in which alternating currents flow are called **ac circuits.**

The graphical representations of both dc and ac are shown in Figs 2.1(a) and (b). In order to produce an ac through an electric circuit, a source capable of reversing the emf periodically is required, e.g., an ac generator. On the other hand, to produce a dc, a source capable of developing a constant emf is necessary, e.g., a battery and a dc generator.



(a) Direct current (b) Alternating current

Fig. 2.1 Waveforms of dc and ac

Advantages of ac

1. The voltages in this system can be raised or lowered with the help of a transformer.
2. High-voltage ac transmission is possible and economical through the use of transformers.
3. AC motors are simple, cheap and require less attention from maintenance point of view.
4. An ac supply can be easily converted into a dc supply. This is required as dc is very much essential for the applications such as battery charging, printing process, cranes, and telephone systems.

Due to these advantages, ac is used extensively in practice.

2.1.1 Alternating Voltage and Current

A voltage that changes its polarity at regular intervals of time is called an **alternating voltage.** When an alternating voltage is applied, an alternating current is produced in the circuit. Figure 2.2 shows an alternating voltage source (ac generator) connected to a resistor *R*. In Fig. 2.2(a), the upper terminal of alternating voltage source is positive and the lower terminal is negative so that the current flows from positive to negative terminal. After some time (a fraction of second), the polarities of the voltage source are reversed [see Fig. 2.2(b)] so that the current now flows in the opposite direction. It is called an ac because the current changes its direction periodically in the circuit.

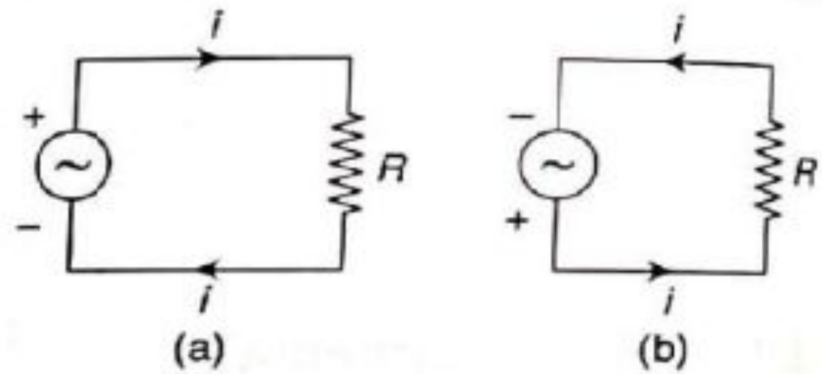
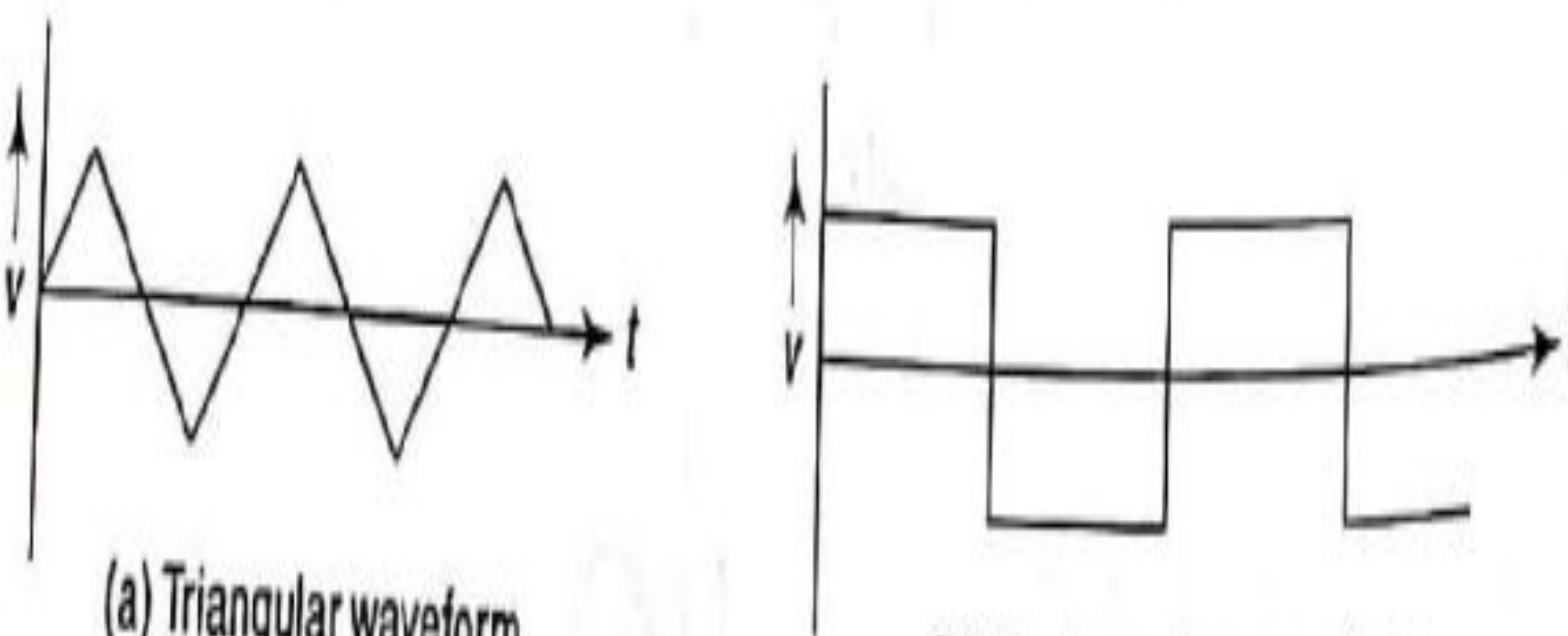


Fig. 2.2 AC circuit

The graph showing the manner in which an alternating voltage or current changes with time is known as its **waveform.** Generally, the instantaneous values of the alternating quantity are taken along *y*-axis and time along *x*-axis.

There can be several waveforms of alternating voltages and currents, e.g., triangular waveform and rectangular waveforms [Figs 2.3(a) and (b)]. But in electrical engineering, we use sinusoidal alternating voltages and currents.

... alternating voltages and currents



(a) Triangular waveform

(b) Rectangular waveform

Fig. 2.3 Alternating quantity

2.1.2 Sinusoidal Alternating Voltage and Current

A sinusoidal alternating quantity is one whose instantaneous value varies according to the sine function of time, i.e., it produces a sine wave. A sinusoidal alternating voltage can be produced by rotating a coil (winding) with a constant angular velocity (say ω rad/sec) in a uniform magnetic field. Figures 2.4(a) and (b) show the waveforms of sinusoidal alternating voltage and current respectively.

The sinusoidal alternating voltage shown in Fig. 2.4(a) can be expressed mathematically as:

$$v = V_m \sin\omega t$$

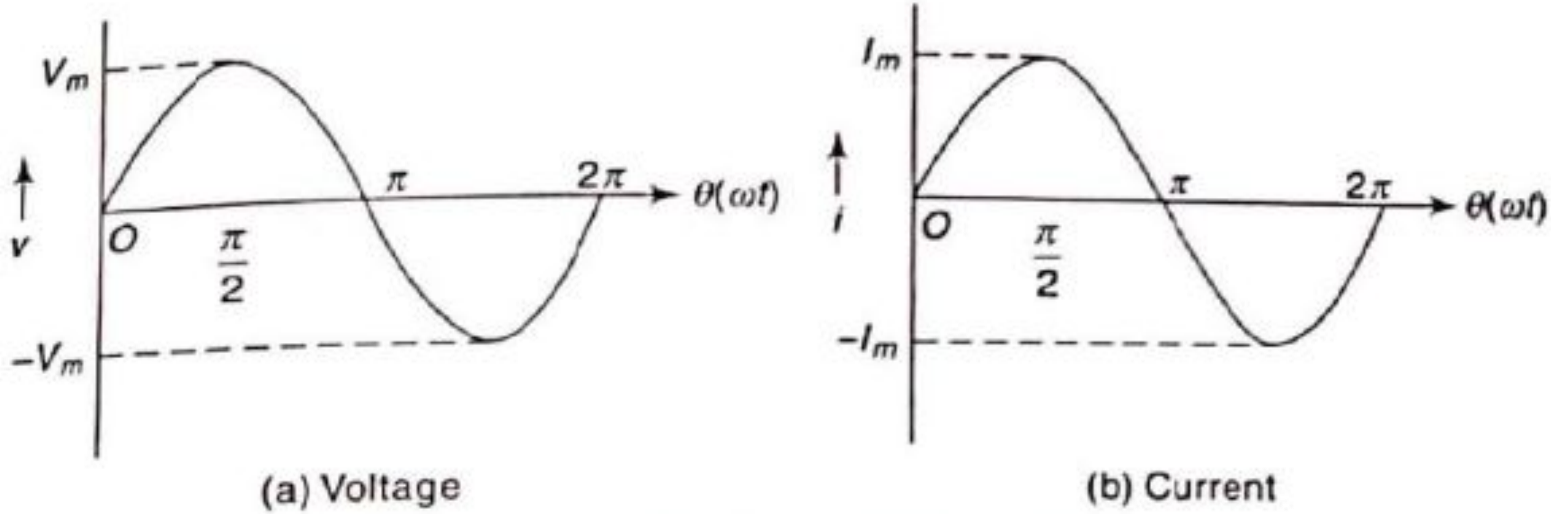


Fig. 2.4 Sinusoidal alternating quantity

where v = Instantaneous value of alternating voltage
V_m = Maximum value of alternating voltage
ω = Angular velocity of coil (winding)

It may be noted that in the above equation of sinusoidal alternating voltage, V_m and ω are constant. Therefore, the instantaneous value v (i.e., value at any instant) changes with time according to the sine function. Similarly, an alternating current varying sinusoidally can be expressed as

$$i = I_m \sin \omega t$$

Out of all these types of alternating waveforms, a sinusoidal alternating waveform is preferred for ac system.

2.1.3 Generation of Alternating Voltage

The sinusoidal alternating voltage can be generated by rotating a coil (winding) in a magnetic field or by rotating a magnetic field in a stationary coil (winding). The machine which is used to generate a sinusoidal alternating voltage is called an **ac generator.**

Figure 2.5 shows the elementary simple ac generator. It consists of a permanent magnet of two poles. A single rectangular coil which is placed in the vicinity of the permanent magnet. The coil has two conductors *AB* and *CD* connected at one end to form a coil. The coil is so placed that it can be rotated about its own axis in clockwise or anticlockwise direction. '*P*' and '*Q*' are the two ends of the coil.

When the coil is rotated in anticlockwise direction with constant angular velocity (by means of prime mover) in a uniform magnetic field, its conductors *AB* and *CD* cut the magnetic lines of flux and according to Faraday's laws of electromagnetic induction, an emf gets induced in them (see Section C.5.1, Appendix C).

Let

ω = Angular velocity of coil in rad/sec
l = Active length of each conductor
B = Flux density of the magnetic field
V = Linear velocity of coil conductors
θ = Angle between plane of the coil and x-axis

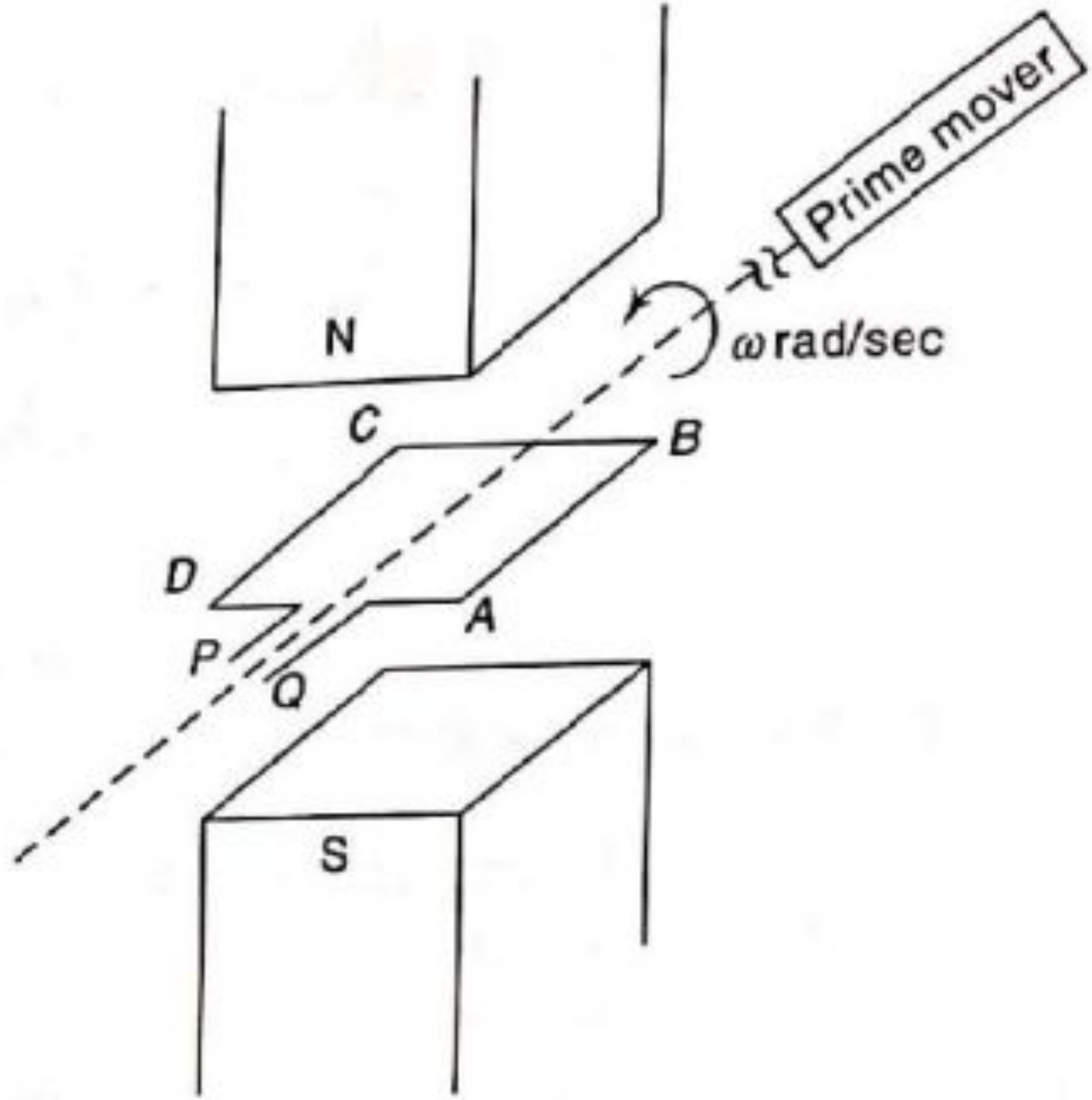


Fig. 2.5 Simple ac generator

According to Faraday's laws of electromagnetic induction,

The emf induced in one conductor = $BlV\sin\theta$

In a coil, two conductors are identical and emf's induced in them are additive. So, the induced emf in a coil,

$$v = 2\ BlV\sin\theta \tag{2.1}$$

When $\theta = 90°$, induced emf is maximum (maximum value of induced emf is expressed by V_m).

So, $\quad V_m = 2\ BlV\sin 90°$

or $\quad V_m = 2\ BlV$

Equation (2.1) now becomes

$$v = V_m\sin\theta \tag{2.2}$$

Thus, Eq. (2.2) gives the instantaneous value of the voltage induced in a coil.

Note: In the above equation, θ is the angle between motion of the conductor and direction of the magnetic field. As motion of the conductor is always perpendicular to the plane of the coil and the direction of the magnetic field is perpendicular to x-axis (see Fig. 2.5), at any instant the angle θ is same between plane of the coil and the x-axis.

From Eq. (2.2), it is clear that the instantaneous value of the induced voltage in a coil depends on 'θ'. As the coil continuously rotates with constant angular velocity (i.e., θ continuously varies from 0° to 360°), the magnitude of induced emf in a coil (v) continuously varies.

Let us consider different positions of the coil. When the initial position of the coil is as shown in Fig. 2.5, i.e., the plane of the coil coincides with the x-axis, $\theta = 0°$. Hence, from Eq. (2.2), no emf is generated in the coil. Thus, when $\theta = 0$, $v = 0$.

When the coil is rotated in anticlockwise direction, the angle θ increases. As coil completes its one rotation, θ varies from 0° to 360°. Taking different instants, i.e., positions of the coil, the induced emf in the coil can be calculated as follows:

When	$\theta = 0°$,	$v = 0$ volt
	$\theta = 45°$,	$v = 0.707\ V_m$ volt
	$\theta = 90°$,	$v = V_m$ volt
	$\theta = 180°$,	$v = 0$ volt
	$\theta = 270°$,	$v = -V_m$ volt
	$\theta = 360°$,	$v = 0$ volt

By plotting the above different instantaneous values of voltages against the posi-

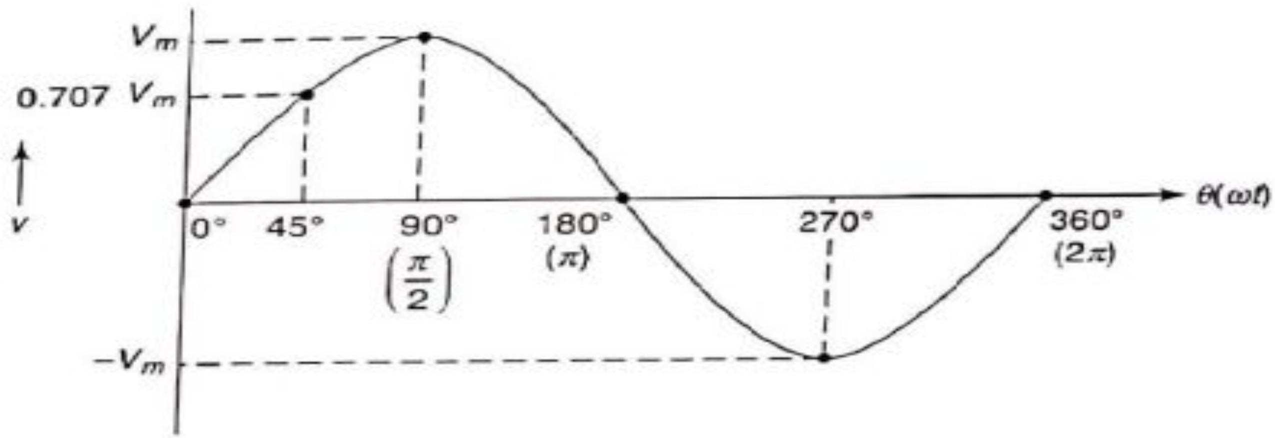


Fig. 2.6 Sinusoidal alternating voltage

tions of the coil (i.e., angle θ), we get Fig. 2.6.

From Fig. 2.6, it is clear that the waveform generated by the instantaneous values of the induced emf in a coil is purely sinusoidal in nature. Thus, sinusoidal alternating voltage can be generated by rotating a coil with constant angular velocity in a uniform magnetic field. As coil completes its one rotation, one cycle of alternating voltage is generated.

2.1.4 AC Terminology

Before further analysis of alternating quantity, it is necessary to be familiar with the different terms which are very frequently used related to the alternating quantities.

Instantaneous value (v) The value of the alternating quantity at a particular instant of time is known as its instantaneous value, e.g., v_1 and v_2 are the instantaneous values of alternating voltages at the instants t_1 and t_2 respectively shown in Fig. 2.7.

Cycle One complete set of positive and negative values of an alternating quantity is known as cycle. A cycle can be defined as each repetition of a set of positive and negative instantaneous values of an alternating quantity. One such cycle of an alternating quantity is shown in Fig. 2.7.

Time period (T) The time taken by an alternating quantity to complete its one cycle is known as its time period. It is denoted by T.

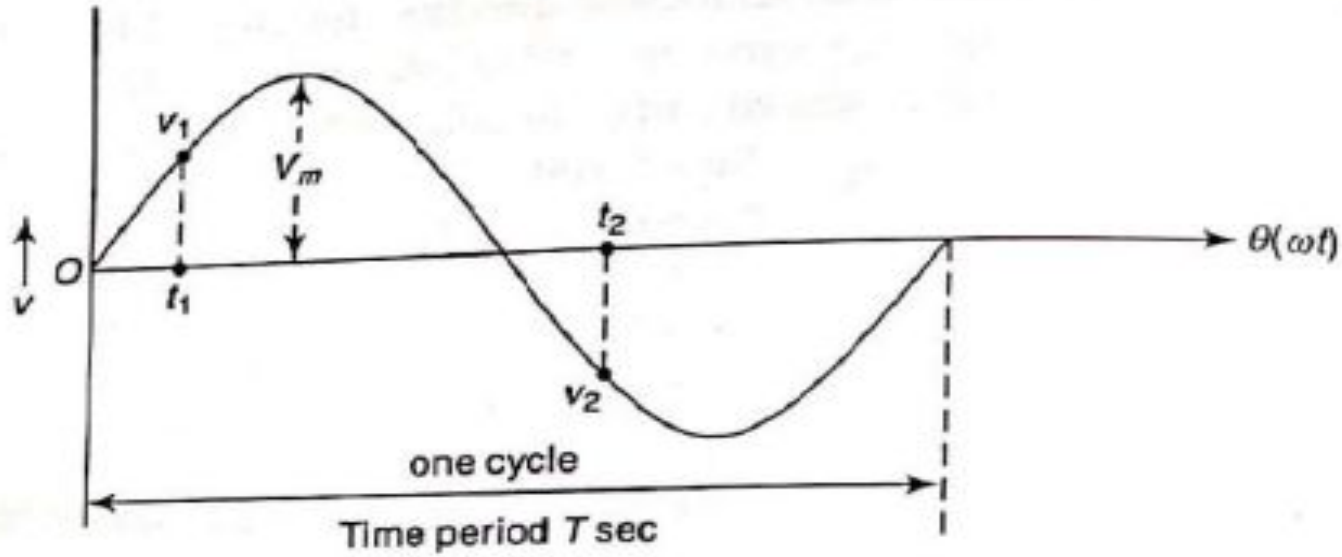


Fig. 2.7 One cycle of sinusoidal alternating voltage

Frequency (f) The number of cycles completed by an alternating quantity per second is known as its frequency. It is denoted by f and it is measured in cycles/second, also known as hertz (Hz).

As time period T is the time for one cycle, i.e., seconds/cycle and frequency is cycles/second, we can say that the frequency is reciprocal of the time period.

$$f = \frac{1}{T} \text{ Hz}$$

Amplitude (V_m) The maximum value attained by an alternating quantity during positive or negative half cycle is called its amplitude. It is denoted by V_m.

2.1.5 Standard Forms of Alternating Quantity

In an ac generator, if the coil is rotated with constant angular velocity ω rad/sec, the angle turned by the coil is given by

Angle turned, $\theta = \omega t$ rad

In one revolution of coil, the angle turned is 2π radian and the voltage wave completes one cycle. The time taken to complete one cycle is the time period T of the alternating voltage.

So, Angular velocity $\omega = \dfrac{\text{Angle turned}}{\text{Time taken}} = \dfrac{2\pi}{T}$

or

$$\omega = 2\pi f \qquad \left(\because f = \frac{1}{T}\right)$$

The standard forms of a sinusoidal alternating voltage are given by

$$v = V_m \sin\theta$$

$$v = V_m \sin\omega t \qquad (\because \theta = \omega t)$$

$$v = V_m \sin 2\pi f t \qquad (\because \omega = 2\pi f)$$

$$v = V_m \sin \frac{2\pi}{T} t \qquad \left(\because f = \frac{1}{T}\right)$$

Similarly, the standard forms of a sinusoidal alternating current are given by

$$i = I_m \sin\theta$$

$$i = I_m \sin\omega t \qquad (\because \theta = \omega t)$$

$$i = I_m \sin 2\pi f t \qquad (\because \omega = 2\pi f)$$

$$i = I_m \sin \frac{2\pi}{T} t \qquad \left(\because f = \frac{1}{T}\right)$$

Example 2.1 An alternating current i is given by $i = 141.4 \sin 314t$. Find (i) the maximum value, (ii) the frequency, (iii) the time period, and (iv) the instantaneous value when time is 3 msec.

Solution

Comparing the given equation of alternating current with the standard form $i = I_m \sin \omega t$, we have

(i) Maximum value, $I_m = 141.4$ A

(ii) Frequency, $f = \dfrac{\omega}{2\pi} = \dfrac{314}{2\pi} = 50$ Hz

(iii) Time period, $T = \dfrac{1}{f} = \dfrac{1}{50} = 0.02$ sec

(iv) Given $i = 141.4 \sin 314t$

When $t = 3$ msec $= 3 \times 10^{-3}$ sec, we get

$$i = 141.4 \sin(314 \times 3 \times 10^{-3}) = 141.4 \sin\left[(314 \times 3 \times 10^{-3}) \times \frac{180}{\pi}\right]^\circ$$

So, $i = 114.36$ A

Example 2.2 An alternating current of frequency 60 Hz has a maximum value of 12A. (i) Write down the equation of the instantaneous value, (ii) find the value of current after $\frac{1}{360}$ sec, and (iii) find the time taken to reach 9.6 A for the first time.

Solution

(i) $i = I_m \sin \omega t = 12 \sin 2\pi \times 60t = 12 \sin 377t$

(ii) Value of current after $\frac{1}{360}$ sec,

$$i = 12 \sin 377t$$

or $$i = 12 \sin 377\left(\frac{1}{360}\right)$$

or $$i = 12 \sin 1.047$$

or $$i = 12 \sin\left(1.047 \times \frac{180}{\pi}\right)^{\circ}$$

or $\quad i = 12 \sin 60^\circ$

Hence, $\quad i = 10.39\ \text{A}$

(iii) Time taken to reach 9.6 A for the first time is given by

$$i = I_m \sin 377t$$

$$9.6 = 12 \sin 377t$$

or

$$\sin 377t = 0.8$$

or

$$\sin\left(377t \times \frac{180}{\pi}\right)^\circ = 0.8$$

or

$$377t \times \frac{180}{\pi} = \sin^{-1} 0.8$$

or

$$377t \times \frac{180}{\pi} = 53.13$$

Hence,

$$t = 2.46 \times 10^{-3}\ \text{sec}$$

2.1.6 Values of Alternating Voltage and Current

In a dc system, the voltage and current are constant so that there is no problem of specifying their magnitudes. However, an alternating voltage or current varies from instant to instant. The magnitude of an alternating voltage or current is expressed by three ways, namely

1. peak value
2. average value
3. rms value or effective value

1. Peak value

It is the maximum value attained by an alternating quantity. The peak or maximum value of alternating voltage or current is represented by V_m or I_m.

2. Average value

The arithmetical average of all the values of an alternating quantity over one cycle is called its average value, i.e.,

$$\text{Average value} = \frac{\text{Area under the curve}}{\text{Base}}$$

The waveforms can be classified into two types: symmetrical waveform and unsymmetrical waveform. A symmetrical waveform has positive half cycle exactly equal to the negative half cycle. If positive half cycle is not equal to the negative half cycle, then the waveform is said to be unsymmetrical.

In case of symmetrical waveforms (e.g., sinusoidal voltage or current), the average value over one cycle is zero. It is because the positive half is exactly equal to the negative half so that the net area is zero. However, the average value of positive or negative half is not zero. Hence, in case of symmetrical waveforms, average value means the average value of half cycle or one alternation. In case of unsymmetrical waveform, average value is taken over full cycle.

Thus, for symmetrical waveforms (+ve half = –ve half),

$$\text{Average value} = \frac{\text{Area of half cycle}}{\text{Base length of half cycle}}$$

Similarly, for unsymmetrical waveforms (+ve half $\neq$ −ve half),

$$\text{Average value} = \frac{\text{Area of full cycle}}{\text{Base length of full cycle}}$$

Average value of sinusoidal ac

The equation of sinusoidal ac is given by

$$i = I_m \sin\theta$$

Figure 2.9 shows the waveform of sinusoidal ac.

As the given waveform is symmetrical,

$$\text{Average value} = \frac{\text{Area of half cycle}}{\text{Base length of half cycle}}$$

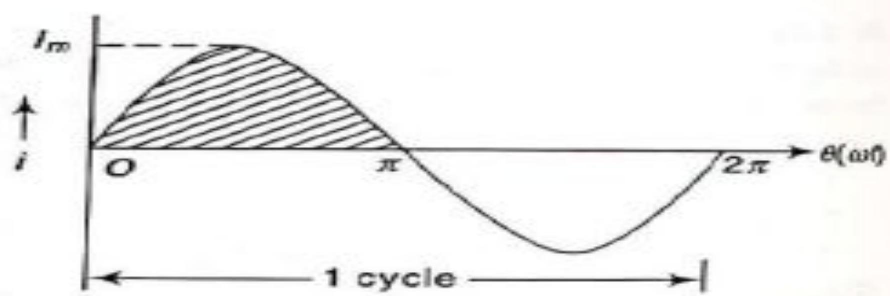


Fig. 2.9

So, $$I_{\text{average}} = \frac{\int_0^{\pi} i\,d\theta}{\pi}$$

$$= \frac{1}{\pi}\int_0^{\pi} I_m \sin\theta\, d\theta$$

$$= \frac{I_m}{\pi}[-\cos\theta]_0^{\pi}$$

$$= \frac{I_m}{\pi}[-\cos\pi - (-\cos 0)]$$

$$= \frac{I_m}{\pi}[-(-1) - (-1)]$$

$$= \frac{2I_m}{\pi}$$

or $$I_{\text{average}} = 0.637\, I_m$$

Similarly, it can be proved that for sinusoidal alternating voltage,

$$V_{\text{average}} = 0.637\, V_m$$

Example 2.4 Find the average value of the waveform shown in Fig. 2.10.

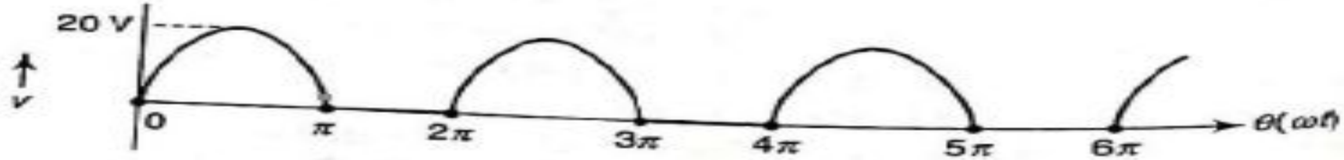
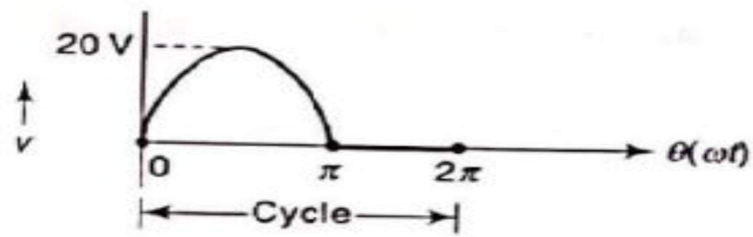


Fig. 2.10

Solution

Consider a full cycle of the given waveform. (Each repetition of a set of positive and negative values of a quantity is known as cycle.)



In a cycle, as positive half is not equal to the negative half, the waveform is unsymmetrical. So, we have to consider a full cycle for average value.

Now, mathematical equation of a cycle is required. It is not possible to write the single equation of a waveform. The given voltage waveform can be divided into two intervals.

First interval is $0 < \theta < \pi$, where the voltage is sinusoidal. Therefore, equation of the voltage is $v = 20 \sin\theta$.

Second interval is $\pi < \theta < 2\pi$, where the voltage remains zero. Therefore, equation of the voltage is $v = 0$.

Thus,

Interval	Equation
$0 < \theta < \pi$	$v = 20 \sin\theta$
$\pi < \theta < 2\pi$	$v = 0$

So, $$V_{\text{average}} = \frac{\text{Area of full cycle}}{\text{Base length of full cycle}}$$

or $$V_{\text{average}} = \frac{\int_0^{2\pi} v d\theta}{2\pi}$$

or $$V_{\text{average}} = \frac{1}{2\pi}\left\{\int_0^{\pi} v d\theta + \int_{\pi}^{2\pi} v d\theta\right\}$$

$$= \frac{1}{2\pi}\int_0^{\pi} 20 \sin\theta \, d\theta$$

$$= \frac{20}{2\pi}\int_0^{\pi} \sin\theta \, d\theta$$

$$= \frac{20}{2\pi}[-\cos\theta]_0^{\pi}$$

$$= \frac{20}{2\pi}[-\cos\pi - (-\cos 0)]$$

$$= \frac{20}{2\pi}[2]$$

$$= 6.366 \text{ V}$$

Example 2.5 Find the average value of the waveform shown in Fig. 2.11.

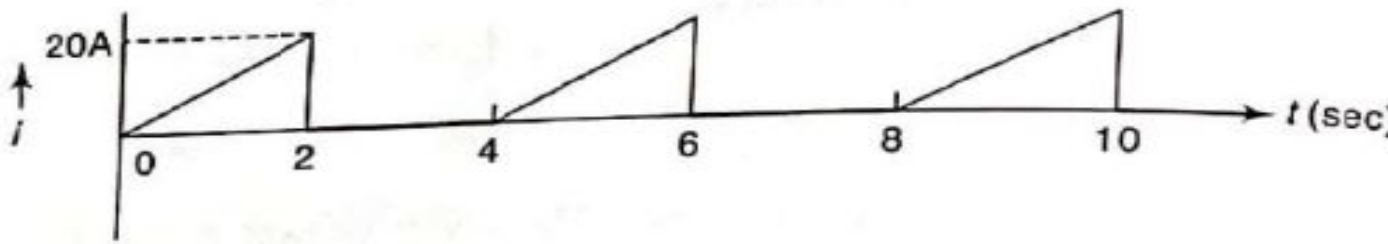
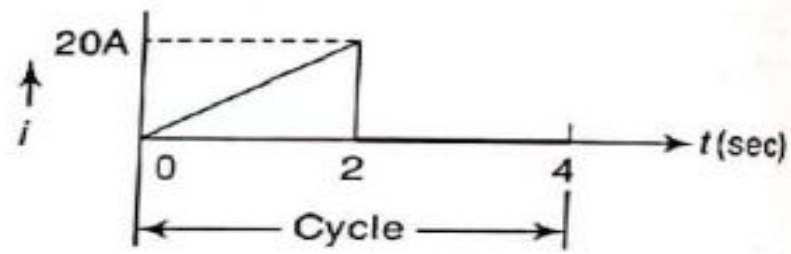


Fig. 2.11

Solution

Consider one cycle of a given waveform.

As positive half is not equal to negative half, given waveform is unsymmetrical. Therefore, for average value, we have to consider a full cycle.



In interval $0 < t < 2$, current increases with constant slope of 10 A/sec. Therefore, the equation of current is $i = 10t$.

In interval $2 < t < 4$, current remains zero. Therefore, the equation of current is $i = 0$.

Thus,

Interval	Equation
$0 < t < 2$	$i = 10t$
$2 < t < 4$	$i = 0$

So, $$I_{\text{average}} = \frac{\int_0^4 i\,dt}{4}$$

$$= \frac{1}{4}\left\{\int_0^2 i\,dt + \int_2^4 i\,dt\right\}$$

$$= \frac{1}{4}\int_0^2 10t\,dt + 0$$

$$= \frac{10}{4}\left[\frac{t^2}{2}\right]_0^2$$

$$= \frac{10}{4}\left[\frac{(2)^2}{2} - \frac{(0)^2}{2}\right]$$

$$= 5\text{ A}$$

Example 2.6 Find the average value of the waveform shown in Fig. 2.12.

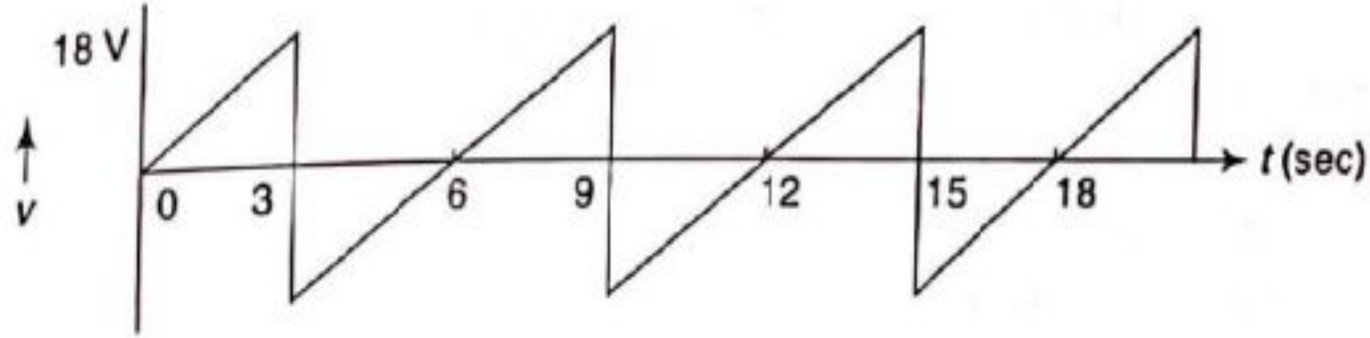
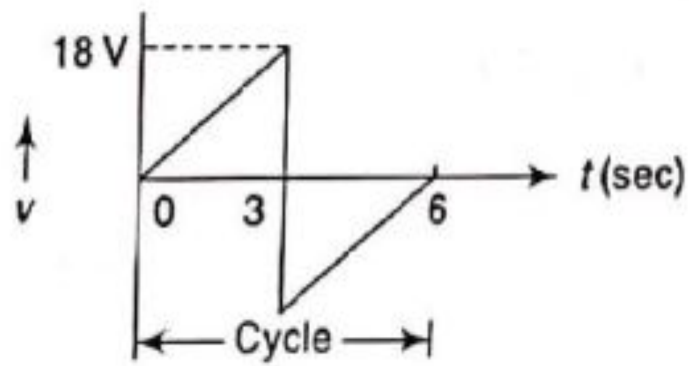


Fig. 2.12

Solution

Consider one cycle of the given waveform. As positive half is exactly equal to negative half, the given waveform is symmetrical. Therefore, for average value, we have to consider a half cycle, i.e., duration of 3 sec ($0 < t < 3$).



In interval $0 < t < 3$, voltage increases with constant slope of 6 V/sec. Therefore, the equation of voltage is $v = 6t$.

So, $$V_{\text{average}} = \frac{\int_0^3 v\,dt}{3}$$

$$= \frac{1}{3}\int_0^3 6t\,dt$$

$$= \frac{6}{3}\left[\frac{t^2}{2}\right]_0^3$$

$$= \frac{6}{3}\left[\frac{(3)^2}{2} - \frac{(0)^2}{2}\right]$$

$$= 9\ \text{V}$$

$= 9\ V$

Example 2.7 Find the average value of the waveform shown in Fig. 2.13.

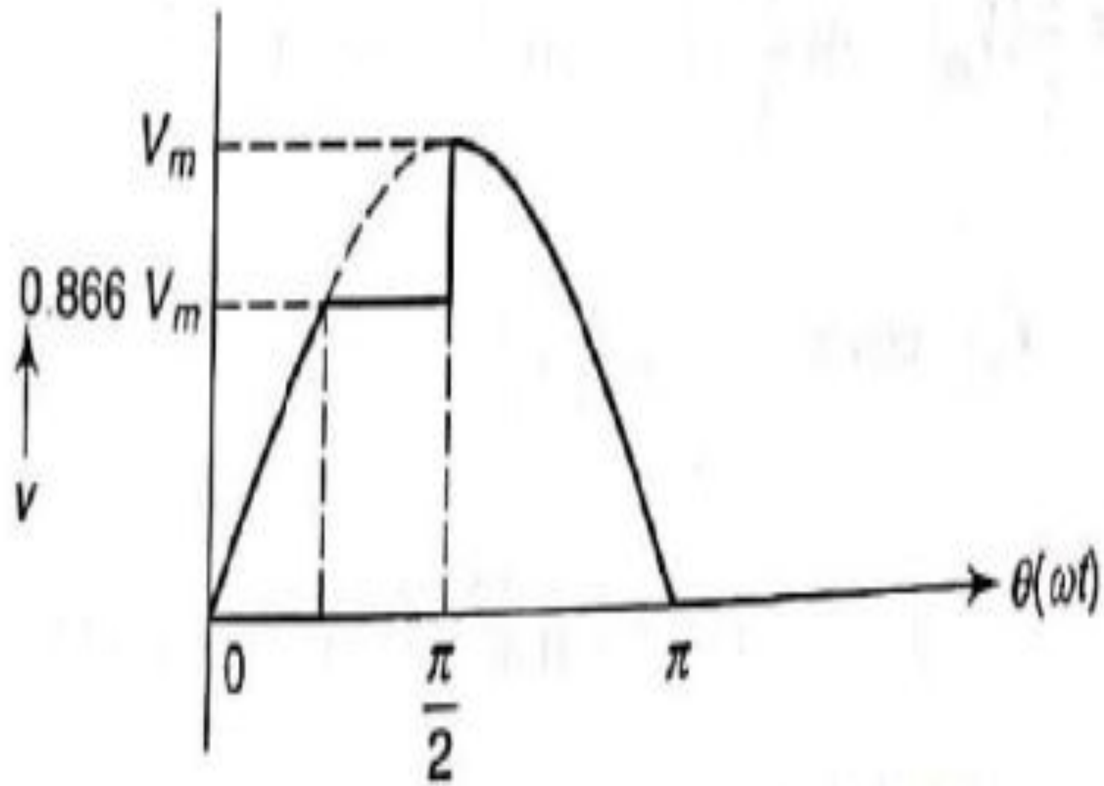


Fig. 2.13

Solution

The angle 'θ' at which the instantaneous value of the voltage becomes equal to 0.866 V_m is required. The angle 'θ' can be calculated as given below.

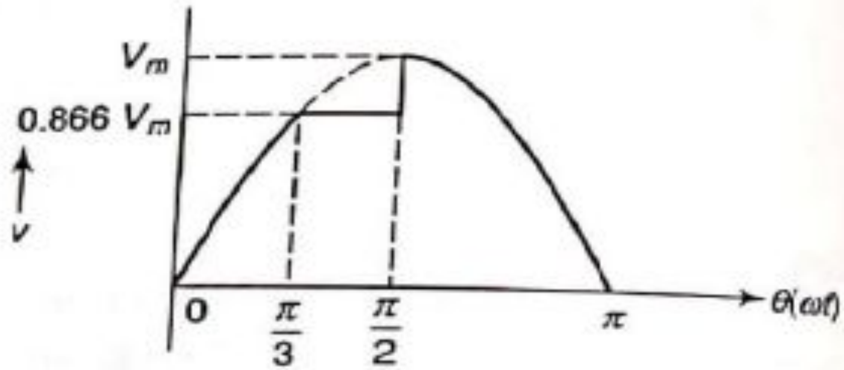
The equation of the waveform is given by

$$v = V_m \sin\theta$$

When $v = 0.866\ V_m$, $\theta = ?$

So, $\quad 0.866\ V_m = V_m \sin\theta$

Hence, $\theta = \dfrac{\pi}{3}$ rad



Now, the voltage waveform is given as follows:

Interval	Equation
$0 < \theta < \dfrac{\pi}{3}$	$v = V_m \sin\theta$
$\dfrac{\pi}{3} < \theta < \dfrac{\pi}{2}$	$v = 0.866\ V_m$
$\dfrac{\pi}{2} < \theta < \pi$	$v = V_m \sin\theta$

So, $\quad V_{\text{average}} = \dfrac{\int_0^{\pi} v\, d\theta}{\pi}$

$$V_{\text{average}} = \frac{\int_0^{\pi} v\, d\theta}{\pi}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/3} v\, d\theta + \int_{\pi/3}^{\pi/2} v\, d\theta + \int_{\pi/2}^{\pi} v\, d\theta\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/3} V_m \sin\theta\, d\theta + \int_{\pi/3}^{\pi/2} 0.866\, V_m\, d\theta + \int_{\pi/2}^{\pi} V_m \sin\theta\, d\theta\right\}$$

$$= \frac{1}{\pi}\left\{V_m[-\cos\theta]_0^{\pi/3} + 0.866\, V_m[\theta]_{\pi/3}^{\pi/2} + V_m[-\cos\theta]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{V_m\left[-\cos\frac{\pi}{3} - (-\cos 0)\right] + 0.866\, V_m\left[\frac{\pi}{2} - \frac{\pi}{3}\right]\right.$$
$$\left. + V_m\left[-\cos\pi - \left(-\cos\frac{\pi}{2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\{V_m[-0.5 - (-1)] + 0.866\ V_m[1.57 - 1.047]$$
$$+ V_m[-(-1) - (-0)]\}$$

$$= \frac{1}{\pi} \{0.5\ V_m + 0.45\ V_m + V_m\}$$

$$= 0.621\ V_m$$

3. RMS or effective value

The value of alternating current changes continuously with time, whereas the direct current remains constant with respect to time. The rms value is the criterion to measure the effectiveness of an alternating current (or voltage). The evident choice would be to measure it in terms of direct current that would do work (or produce heat) at the same average rate under similar conditions.

In Fig. 2.16(a), direct current of I A is passed through the resistance R for some time. In Fig. 2.16(b), an alternating current is passed through the same resistance for the same time. If the same amount of heat is produced in both cases, then the rms value of alternating current is said to be equal to direct current, i.e., I A. Thus, from the above example, the rms or effective value can be stated as follows:

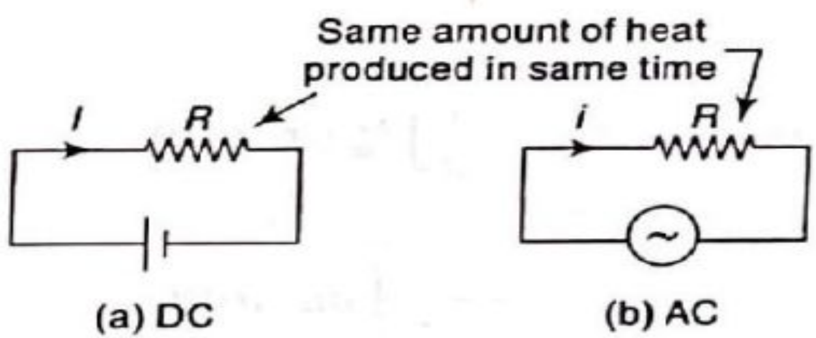


Fig. 2.16 Illustration of rms value

The effective or rms value of an alternating current is equal to that direct current which when flowing through a given resistance for a given time produces the same amount of heat as produced by the ac when flowing through the same resistance for the same time.

For symmetrical waveform, the rms or effective value can be found by considering half cycle or full cycle. However, for unsymmetrical waves, full cycle should be considered.

For symmetrical waveform (+ve half = –ve half),

$$\text{RMS value} = \sqrt{\frac{\text{Area of half/full cycle of squared wave}}{\text{Base length of half/full cycle}}}$$

For unsymmetrical waveform (+ve half ≠ –ve half),

$$\text{RMS value} = \sqrt{\frac{\text{Area of full cycle of squared wave}}{\text{Base length of full cycle}}}$$

RMS value of sinusoidal alternating current The equation of an alternating current varying sinusoidally is given by

$$i = I_m \sin\theta$$

Figure 2.17 shows the waveform of sinusoidal alternating current. The waveform of squared wave is shown by dotted line.

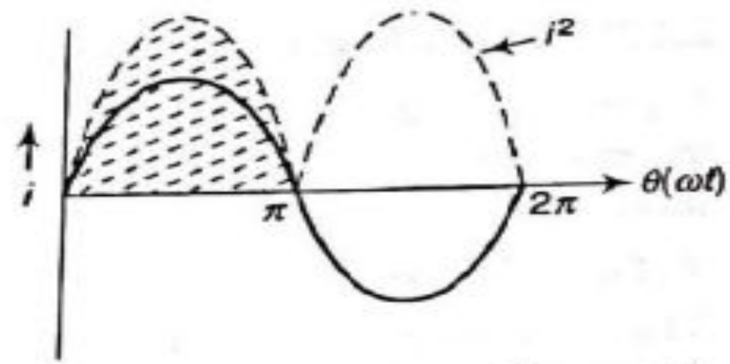


Fig. 2.17 Sinusoidal alternating current

As the given waveform is symmetrical, for rms value, we can consider half or full cycle.

So, $$\text{rms value} = \sqrt{\frac{\text{Area of half cycle of squared wave}}{\text{Base length of half cycle}}}$$

or $$I_{rms} = \sqrt{\frac{\int_0^{\pi} i^2\, d\theta}{\pi}}$$

or $$I^2_{rms} = \frac{1}{\pi}\int_0^{\pi} I_m^2 \sin^2\theta\, d\theta$$

$$= \frac{I_m^2}{\pi}\int_0^{\pi} \sin^2\theta\, d\theta$$

$$= \frac{I_m^2}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{I_m^2}{2\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{I_m^2}{2\pi}\left[\pi - \frac{\sin 2\pi}{2} - \left(0 - \frac{\sin(2\times 0)}{2}\right)\right]$$

$$= \frac{I_m^2}{2\pi}[\pi]$$

Hence, $I_{rms} = \dfrac{I_m}{\sqrt{2}} = 0.707\, I_m$

Similarly, it can be proved that for sinusoidal alternating voltage,

$$V_{rms} = \frac{V_m}{\sqrt{2}} = 0.707\, V_m$$

2.1.7 Form Factor and Peak Factor

There exists a definite relationship among the peak value, the average value, and the rms value of an alternating quantity. The relationship is expressed by two factors, namely form factor and peak factor.

1. Form factor

The ratio of rms value to the average value of an alternating quantity is known as **form factor.**

Thus, $\text{Form factor} = \dfrac{\text{rms value}}{\text{Average value}}$

The value of form factor depends upon the waveform of the alternating quantity. The form factor for an alternating voltage or current varying sinusoidally is 1.11, i.e., for a sinusoidal voltage or current,

$$\text{Form factor} = \frac{\text{rms value}}{\text{Average value}} = \frac{0.707 \times \text{Maximum value}}{0.637 \times \text{Maximum value}} = 1.11$$

The form factor gives a measure of the 'peakiness' of the waveform. The peakier the wave, the greater is its form factor and vice versa. The form factor is useful in rectifier service.

2. Peak factor

The ratio of maximum value to the rms value of an alternating quantity is known as peak factor, i.e.,

$$\text{Peak factor} = \frac{\text{Maximum value}}{\text{rms value}}$$

The value of peak factor also depends upon the waveform of the alternating quantity. The peak factor for an alternating voltage or current varying sinusoidally is 1.414, i.e., for a sinusoidal voltage or current,

$$\text{Peak factor} = \frac{\text{Maximum value}}{\text{rms value}} = \frac{\text{Maximum value}}{0.707 \times \text{Maximum value}} = 1.414$$

Knowledge of this factor is important in dieletric insulation testing because the dielectric stress to which the insulation is subjected, is proportional to the maximum or peak value of the applied voltage.

Example 2.10 Find the rms value of the waveform shown in Fig. 2.18.

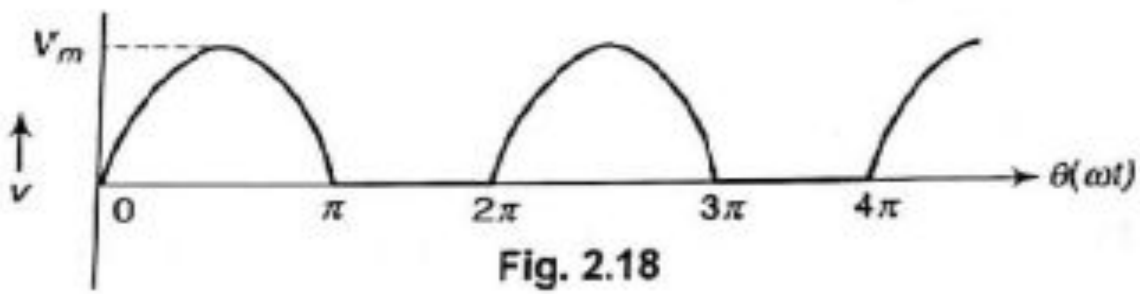
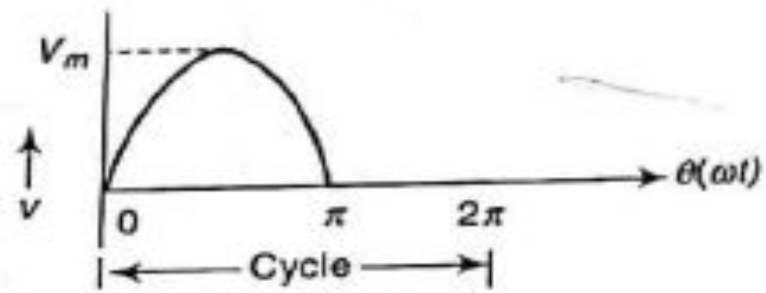


Fig. 2.18

Solution

As given waveform is unsymmetrical, for rms value, we have to consider full cycle. The equations of the voltage waveform are given by

$$v = V_m \sin\theta, \qquad 0 < \theta < \pi$$
$$v = 0, \qquad \pi < \theta < 2\pi$$



So, $$V_{\text{rms}} = \sqrt{\frac{\text{Area of full cycle of squared wave}}{\text{Base length of full cycle}}}$$

or $$V_{\text{rms}} = \sqrt{\frac{\int_0^{2\pi} v^2\, d\theta}{2\pi}}$$

or $$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{2\pi} v^2 \, d\theta$$

$$= \frac{1}{2\pi}\left\{\int_0^{\pi} v^2 \, d\theta + \int_{\pi}^{2\pi} v^2 \, d\theta\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^{\pi} (V_m \sin\theta)^2 \, d\theta + \int_{\pi}^{2\pi} (0)^2 \, d\theta\right\}$$

$$= \frac{1}{2\pi}\int_0^{\pi} V_m^2 \sin^2\theta \, d\theta$$

$$= \frac{V_m^2}{2\pi}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{V_m^2}{2\pi \times 2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{V_m^2}{2\pi \times 2}\left\{\pi - \frac{\sin 2\pi}{2} - \left[0 - \frac{\sin(2\times 0)}{2}\right]\right\}$$

$$= \frac{V_m^2}{2\pi \times 2}\{\pi\}$$

Hence, $V_{\text{rms}} = \dfrac{V_m}{2} = 0.5\ V_m$

Example 2.11 Find the rms value of the waveform shown in Fig. 2.19.

Solution

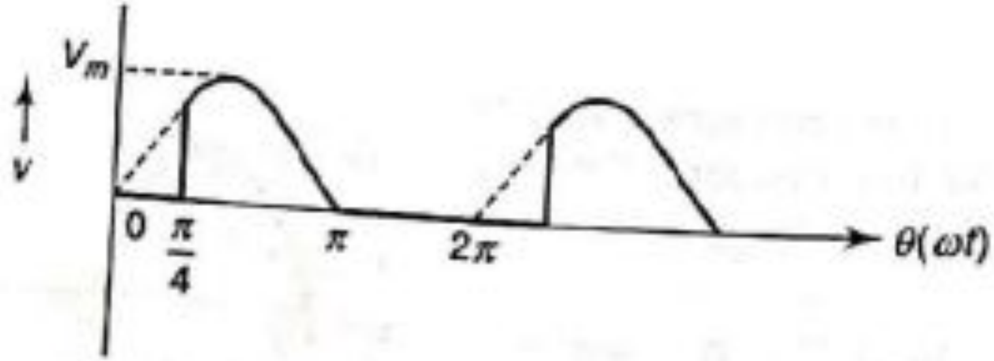


Fig. 2.19

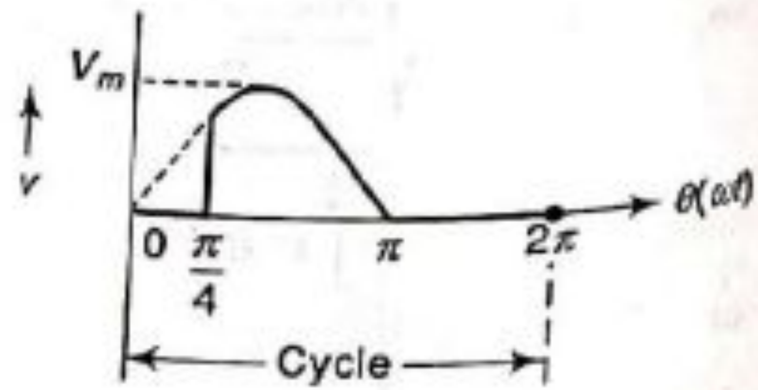
As given waveform is unsymmetrical, for rms value we have to consider full cycle.

The equations of the voltage waveform are given as

$v = 0, \qquad 0 < \theta < \frac{\pi}{4}$

$v = V_m \sin\theta, \qquad \frac{\pi}{4} < \theta < \pi$

$v = 0, \qquad \pi < \theta < 2\pi$



So, $$V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2\, d\theta}$$

or $$V_{\text{rms}}^2 = \frac{1}{2\pi}\left\{\int_0^{\pi/4} v^2\, d\theta + \int_{\pi/4}^{\pi} v^2\, d\theta + \int_{\pi}^{2\pi} v^2\, d\theta\right\}$$

$$= \frac{1}{2\pi}\int_{\pi/4}^{\pi} (V_m \sin\theta)^2\, d\theta$$

$$= \frac{1}{2\pi}\int_{\pi/4}^{\pi} V_m^2 \sin^2\theta\, d\theta$$

$$= \frac{V_m^2}{2\pi}\int_{\pi/4}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{V_m^2}{2\pi\times 2}\left\{\theta - \frac{\sin 2\theta}{2}\right\}_{\pi/4}^{\pi}$$

$$= \frac{V_m^2}{2\pi\times 2}\left\{\pi - \frac{\sin 2\pi}{2} - \left[\frac{\pi}{4} - \frac{\sin\left(2\times\frac{\pi}{4}\right)}{2}\right]\right\}$$

$$= \frac{V_m^2}{2\pi\times 2}\left\{3.14 - \frac{0}{2} - \left[0.785 - \frac{1}{2}\right]\right\}$$

$$= 0.227\ V_m^2$$

Hence, $V_{\text{rms}} = 0.476\ V_m$

Example 2.12 Find the rms value of the waveform shown in Fig. 2.20.

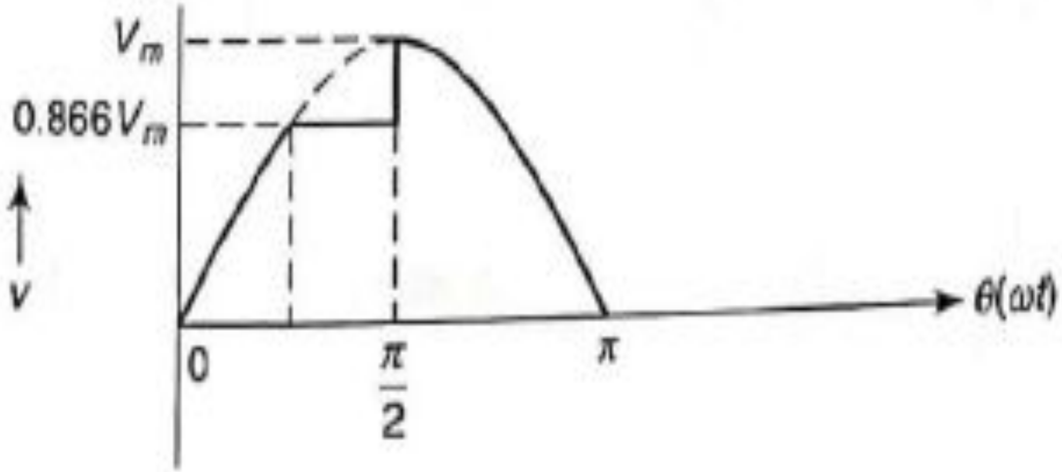


Fig. 2.20

Solution

The angle 'θ', at which the instantaneous value of the voltage becomes equal to 0.866 V_m, is required. This angle 'θ' can be calculated as given below.

The equation of the waveform is given by

$$v = V_m \sin\theta$$

When $v = 0.866\ V_m,\ \theta = ?$

we have $0.866\ V_m = V_m \sin\theta$

or $\theta = \sin^{-1} 0.866$

or $\theta = 60°$

Hence, $\theta = \dfrac{\pi}{3}$ rad

Now, the voltage waveform is shown in Fig. 2.21.

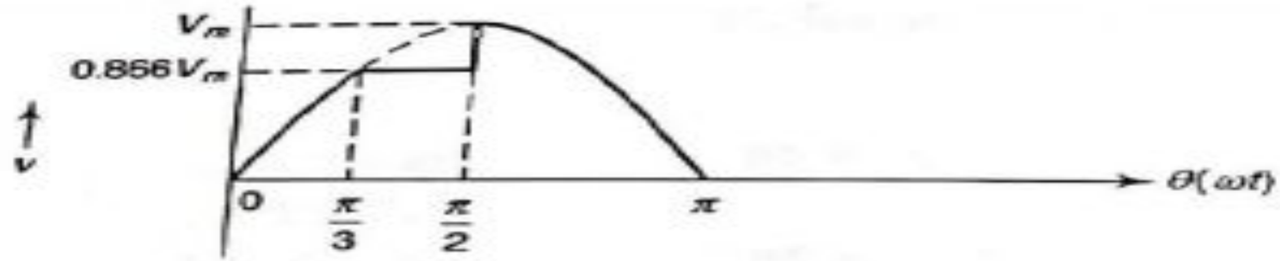


Fig. 2.21

The equations are as follows:

$$v = V_m \sin\theta, \qquad 0 < \theta < \frac{\pi}{3}$$

$$v = 0.866\ V_m, \qquad \frac{\pi}{3} < \theta < \frac{\pi}{2}$$

$$v = V_m \sin\theta, \qquad \frac{\pi}{2} < \theta < \pi$$

So, $$V_{\text{rms}} = \sqrt{\frac{1}{\pi}\int_0^{\pi} v^2\, d\theta}$$

or $$V_{\text{rms}}^2 = \frac{1}{\pi}\int_0^{\pi} v^2\, d\theta = \frac{1}{\pi}\left\{\int_0^{\pi/3} v^2\, d\theta + \int_{\pi/3}^{\pi/2} v^2\, d\theta + \int_{\pi/2}^{\pi} v^2\, d\theta\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/3} (V_m \sin\theta)^2\, d\theta + \int_{\pi/3}^{\pi/2} (0.866\, V_m)^2\, d\theta + \int_{\pi/2}^{\pi} (V_m \sin\theta)^2\, d\theta\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/3} V_m^2 \sin^2\theta\, d\theta + \int_{\pi/3}^{\pi/2} (0.866\, V_m)^2\, d\theta + \int_{\pi/2}^{\pi} V_m^2 \sin^2\theta\, d\theta\right\}$$

$$= \frac{V_m^2}{\pi}\left\{\int_0^{\pi/3} \left(\frac{1-\cos 2\theta}{2}\right) d\theta + \int_{\pi/3}^{\pi/2} (0.866)^2\, d\theta + \int_{\pi/2}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta\right\}$$

$$= \frac{V_m^2}{\pi}\left\{\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/3} + (0.866)^2[\theta]_{\pi/3}^{\pi/2} + \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{V_m^2}{\pi}\left\{\frac{1}{2}\left[\frac{\pi}{3} - \frac{\sin 2\left(\frac{\pi}{3}\right)}{2} - \left(0 - \frac{\sin 2\times 0}{2}\right)\right] + (0.866)^2\left[\frac{\pi}{2} - \frac{\pi}{3}\right]\right.$$

$$\left. + \frac{1}{2}\left[\pi - \frac{\sin 2\pi}{2} - \left(\frac{\pi}{2} - \frac{\sin 2\left(\frac{\pi}{2}\right)}{2}\right)\right]\right\}$$

$$= \frac{V_m^2}{\pi}\left\{\frac{1}{2}[1.047 - 0.433] + (0.866)^2[1.57 - 1.047]\right.$$

$$\left. + \frac{1}{2}[3.14 - 0 - (1.57 - 0)]\right\}$$

$$= \frac{V_m^2}{3.14}(0.307 + 0.392 + 0.785)$$

or $\quad V_{\text{rms}}^2 = 0.4726\ V_m^2$

Hence, $\quad V_{\text{rms}} = 0.687\ V_m$

Example 2.13 Find the rms value of the waveform shown in Fig. 2.22.

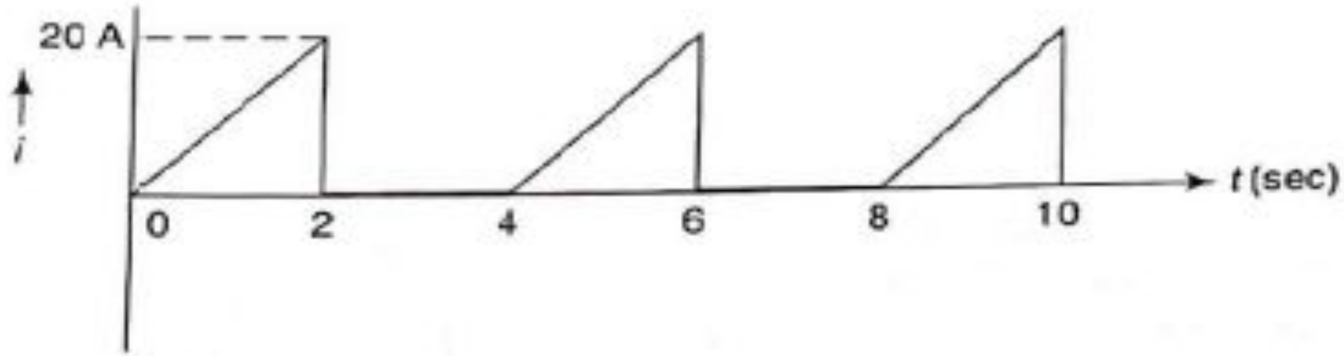
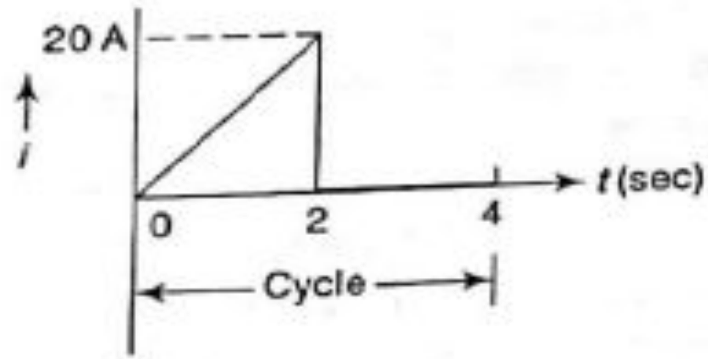


Fig. 2.22

Solution



As the given waveform is unsymmetrical, for rms value, we have to consider full cycle.

The equations of the current waveforms are:

$$i = 10t \qquad 0 < t < 2$$
$$i = 0 \qquad 2 < t < 4$$

So, $$I_{\text{rms}} = \sqrt{\frac{1}{4}\int_0^4 i^2\, dt}$$

or $$I_{\text{rms}}^2 = \frac{1}{4}\int_0^4 i^2\, dt$$

$$= \frac{1}{4}\left\{\int_0^2 i^2\, dt + \int_2^4 i^2\, dt\right\}$$

$$= \frac{1}{4}\int_0^2 (10t)^2\, dt$$

$$= \frac{(10)^2}{4}\left[\frac{t^3}{3}\right]_0^2$$

$$= \frac{(10)^2}{4}\left[\frac{(2)^3}{3} - \frac{(0)^3}{3}\right]$$

or $\quad I_{rms}^2 = 66.66$

Hence, $\quad I_{rms} = 8.16$ A

Example 2.17 The equation of an alternating current is $i = 62.35 \sin 323t$ A. Determine (i) maximum value, (ii) frequency, (iii) rms value, (iv) average value, and (v) form factor.

Solution

(i) $I_m = 62.35$ A

(ii) Frequency, $f = \dfrac{323}{2\pi} = 51.41$ Hz

(iii) rms value, $I_{rms} = \dfrac{I_m}{\sqrt{2}} = \dfrac{62.35}{\sqrt{2}} = 44.1$ A

(iv) Average value, $I_{average} = 0.637\, I_m = 39.7$ A

(v) Form factor $= \dfrac{I_{rms}}{I_{average}}$

$$= \frac{44.1}{39.7}$$

$$= 1.11$$

2.1.8 Phase Angle and Phasor

To understand the concept of phase angle and phasor, consider the following cases.

Case (i) The sinusoidal alternating current represented by the waveform shown in Fig. 2.25.

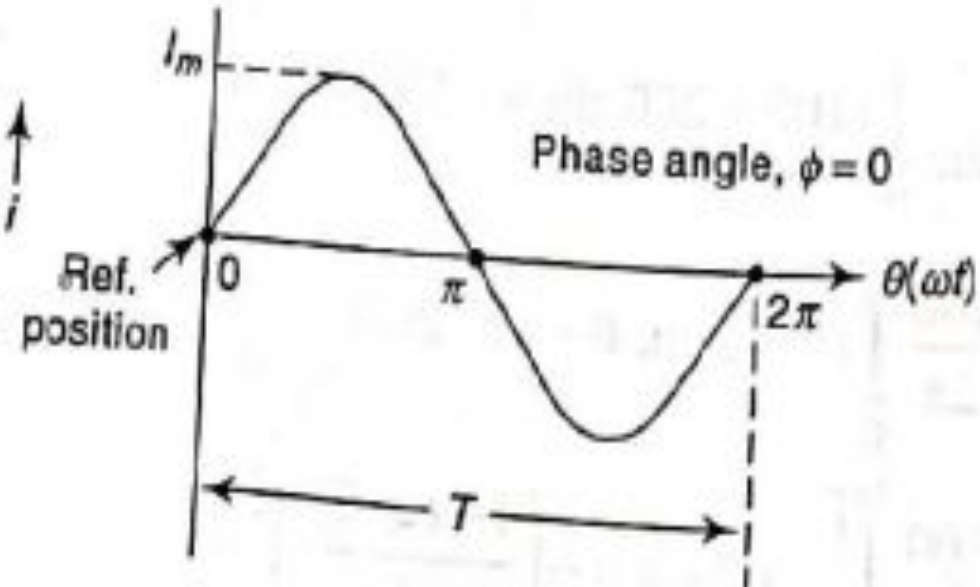


Fig. 2.25 Concept of phase angle ($\phi = 0$)

The X-axis is marked as 'ωt'. As 'ω' is constant, X-axis is the time axis. The instant from which time is measured (counted) is called reference position or zero (origin) position. In Fig. 2.25, the current attains its zero value exactly at reference position (or we can say that time is counted when the instant current is zero). Therefore, the phase angle of the current is said to be zero (i.e., $\phi = 0$). The given sinusoidal alternating current can be expressed by the equation

$$i = I_m \sin(\omega t \pm 0)$$

This equation is known as standard sinusoidal form. The alternating current can be represented by the phasor. Length of the phasor represents magnitude, i.e., rms value, and inclination w.r.t. reference axis such as X-axis is equal to the phase angle of that quantity. Thus, the given alternating current can be represented by phasor as shown in Fig. 2.26.

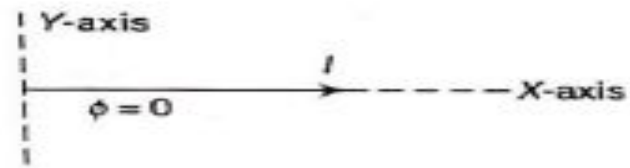


Fig. 2.26 Phasor

An alternating quantity is generally referred by its rms value. As a result, length of the phasor is drawn equal to the rms value (i.e., $I = \dfrac{I_m}{\sqrt{2}}$) instead of the maximum value. In practice, the rms values are denoted by capital letter such as 'I' or 'V', instead of I_{rms} or V_{rms}.

Case (ii) The sinusoidal alternating voltage represented by the waveform shown in Fig. 2.27.

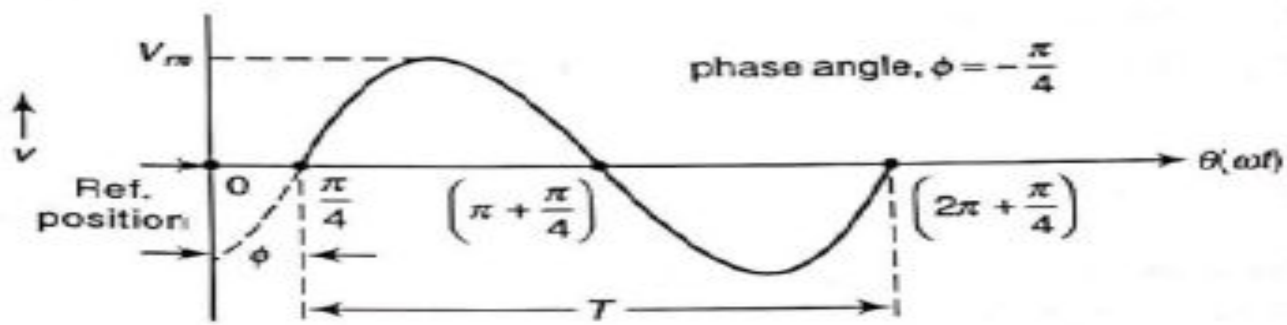


Fig. 2.27 Concept of phase angle ($\phi = -\pi/4$)

The voltage attains its zero value (first time) after a reference position by an angle $\pi/4$ rad or 45° (i.e., it lags behind reference). Therefore, phase angle of the voltage is $-\pi/4$ rad or $-45°$. Here the standard sinusoidal form of the given quantity is

$$v = V_m \sin\left(\omega t - \frac{\pi}{4}\right)$$

or

$$v = V_m \sin(\omega t - 45°)$$

The corresponding voltage phasor is shown in Fig. 2.28.

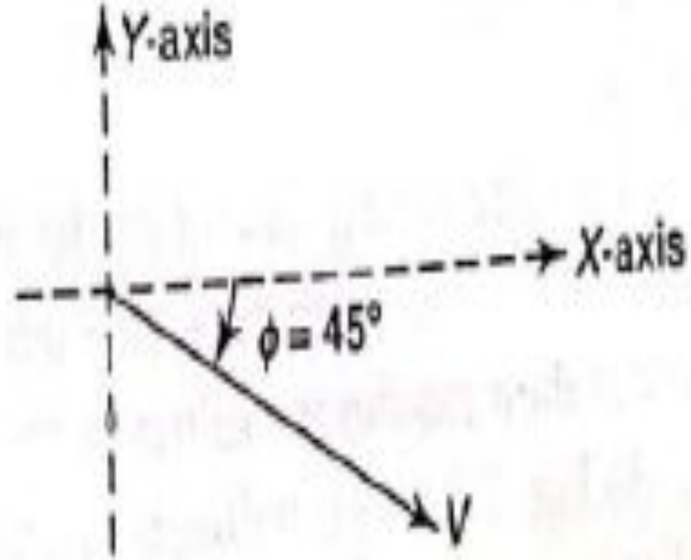


Fig. 2.28 Phasor

Length of phasor, $V = \frac{V_m}{\sqrt{2}}$

The negative phase angle is plotted in clockwise direction from positive X-axis.

Case (iii) The sinusoidal alternating current represented by the waveform shown in Fig. 2.29.

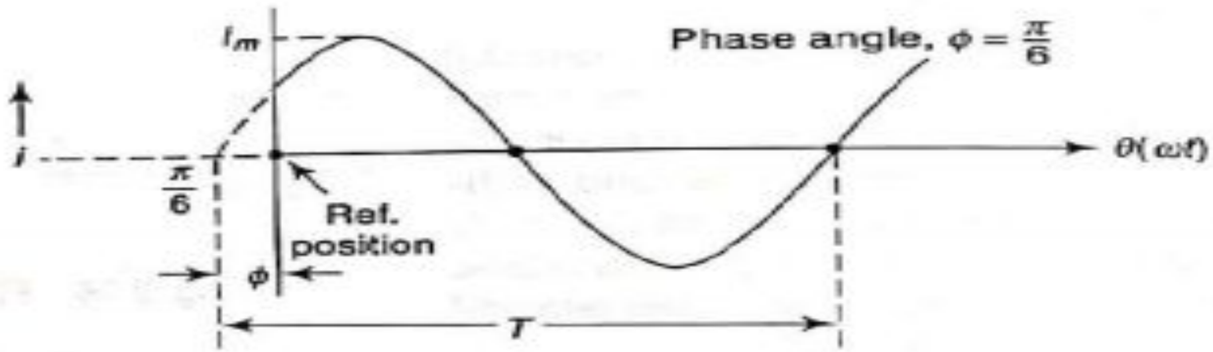


Fig. 2.29 Concept of phase angle ($\phi = \pi/6$)

The current attains its zero value (first time) before a reference position by an angle $\pi/6$ rad or 30° (i.e., it leads reference). Therefore, phase angle of the voltage is positive ($+\pi/6$ rad or 30°). The standard sinusoidal form of the given quantity is

$$i = I_m \sin\left(\omega t + \frac{\pi}{6}\right)$$

or $\quad i = I_m \sin(\omega t + 30)$

The corresponding current phasor is shown in Fig. 2.30.

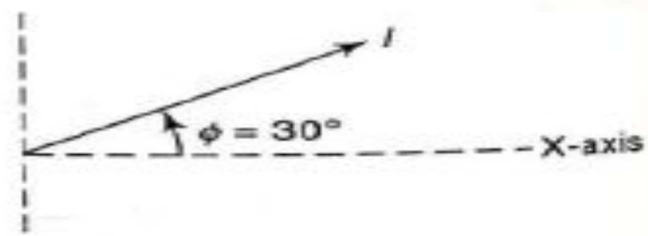


Fig. 2.30 Phasor

Length of phasor, $I = \frac{I_m}{\sqrt{2}}$

The positive phase angle is plotted in anticlockwise direction from positive X-axis.

2.1.9 In Phase, Out of Phase, and Phase Difference

When an alternating voltage is applied to the circuit, an alternating current of the same frequency flows through the circuit. In some circuits, the applied voltage and circuit current remains in phase. In most of the circuits, for reasons we will discuss later, voltage or current are out of phase (i.e., have different phase angles).

Two alternating quantities (let v_1 and v_2) of the same frequency are said to be in phase, when each pass through their zero value at the same instant and also attain their maximum values at the same instant in a given cycle (see Fig. 2.31).

In Fig. 2.31, two voltages v_1 and v_2 are in phase to each other. In other words, there is no phase difference between the two. The phase difference is zero ($\phi = 0$).

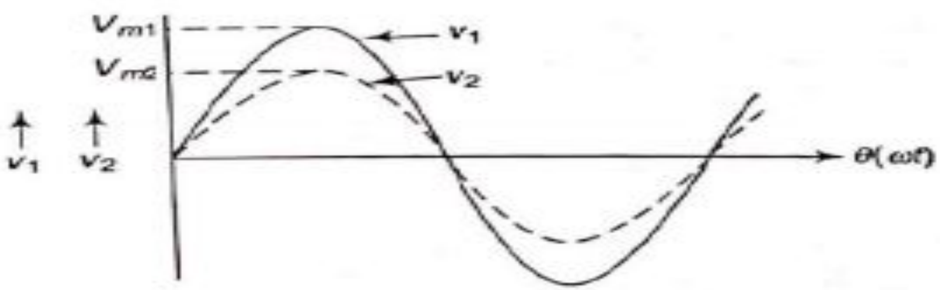


Fig. 2.31 Voltages in phase

The standard sinusoidal forms of the above two quantities are:

$$v_1 = V_{m1} \sin \omega t$$

$$v_2 = V_{m2} \sin \omega t$$

The above two voltages can be shown in the same phasor diagram (see Fig. 2.32).

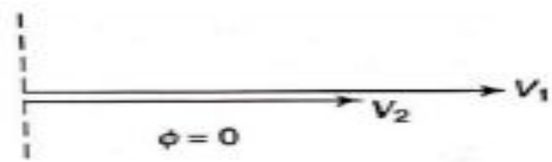


Fig. 2.32 Phasor diagram

When two alternating quantities (let v and i) of the same frequency (same time period) reach their maximum or zero value at the different instants, then the alternating quantities are said to have a phase difference or out of phase (see Fig. 2.33).

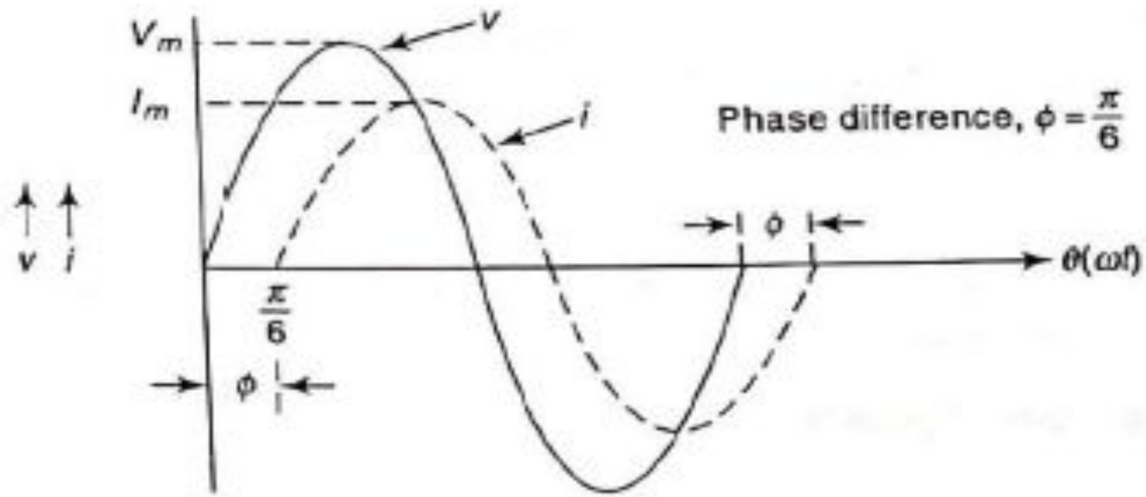


Fig. 2.33 Concept of phase difference

In Fig. 2.33, voltage passes through its zero value earlier than current by an angle ϕ (= $\pi/6$). Therefore, the voltage is said to be leading while the current is said to be lagging. It should be noted that those zero values of alternating quantities are to be considered where they pass in the same direction. Thus, if the voltage has passed through its zero value and is rising in the positive direction, then zero value considered for the current should have similar situation.

The equations (standard sinusoidal forms) of voltage and current are:

$$v = V_m \sin \omega t$$

$$i = I_m \sin\left(\omega t - \frac{\pi}{6}\right)$$

In the current equation, minus sign is used because the current lags behind the voltage.

Figure 2.34 shows the phasor diagram.

In this case, the current lags behind the voltage by an angle $\phi (= \pi/6$ rad or $30^\circ)$.

As phase angle of the voltage is zero, it acts as a reference quantity. Remember that the lagging and leading words are relative to the reference. In this case, if we take the current as reference, we have to say that the voltage leads current by angle ϕ.

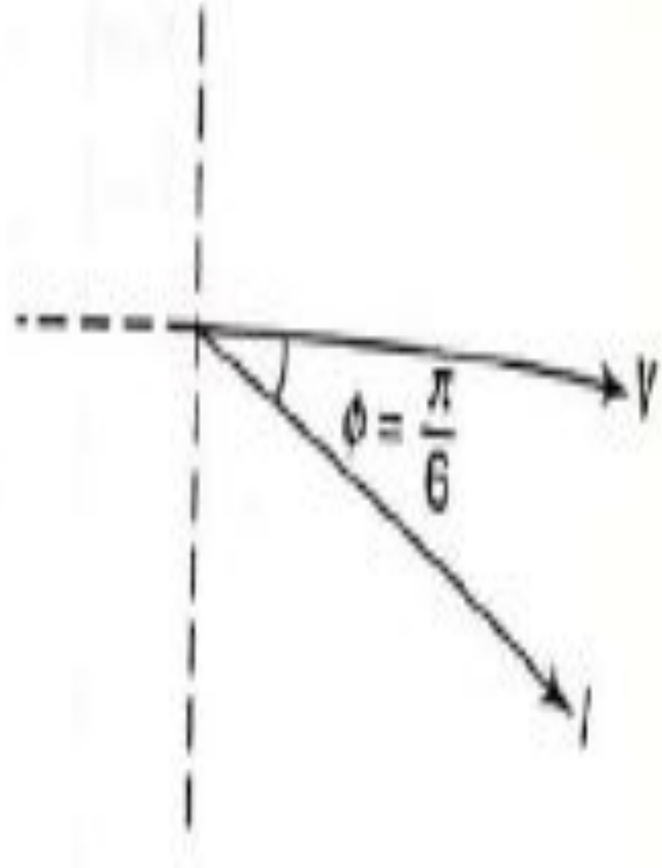


Fig. 2.34 Phasor diagram

2.1.10 Phasor Algebra

Mathematical representation of any phasor (current, voltage, or impedance) is known as **phasor algebra**.

The phasor can be represented mathematically in two ways: (a) rectangular form and (b) polar form.

(a) Rectangular form

It is a complex form in which operator 'j' is used. Operator 'j' is same as operator 'i' in complex form. Consider a current phasor (I) as shown in Fig. 2.35.

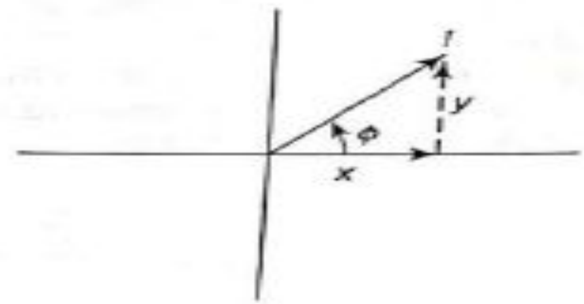


Fig. 2.35 Current phasor

In rectangular form, the phasor is resolved into horizontal (x) and vertical (y) components and expressed in complex form, i.e., $\bar{I} = (x + jy)$ A

where $x =$ Real part, horizontal component, or x-component

$y =$ Imaginary part, vertical component, or y-component

From rectangular form,

$$\text{Magnitude of phasor, } I = \sqrt{x^2 + y^2}$$

Its angle w.r.t. X-axis, $\phi = \tan^{-1} \dfrac{y}{x}$.

This form is used for addition and subtraction of alternating quantities.

(b) Polar form

In this form, current phasor is represented by ($I \angle \pm\phi$) or voltage phasor is represented as ($V \angle \pm\phi$).

Consider a voltage phasor as shown in Fig. 2.36.

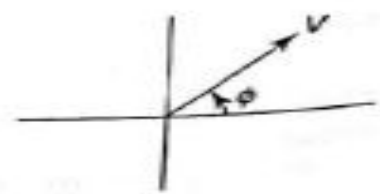


Fig. 2.36 Voltage phasor

The polar form of the voltage phasor, $\bar{V} = (V\angle\phi)$ V

where V is the magnitude (rms value) and ϕ is the phase angle (i.e., angle made with +ve X-axis), which is taken as positive if measured anticlockwise direction while negative if measured clockwise direction w.r.t. positive X-axis. This form is used for multiplication and division of alternating quantities. This form is also used for drawing a phasor diagram of ac circuit.

Remember that the phase angle (ϕ), i.e., angle made by the phasor with +ve X-axis can be measured in two ways. For example, consider a current phasor of magnitude 10 A as shown in Fig. 2.37. The phase angle of current phasor is 190° if measured in clockwise direction, or 170° if measured in anticlockwise direction.

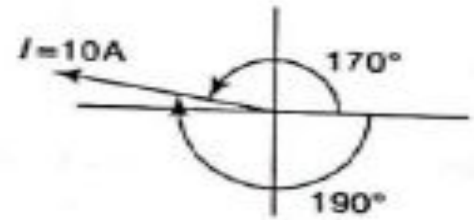


Fig. 2.37 Concept of +ve and –ve phase angles

The polar form of the current phasor is

$$\overline{I} = (10 \angle 170°) \text{ A}$$

or

$$\overline{I} = (10 \angle -190°) \text{ A}$$

2.1.11 Addition and Subtraction of Alternating Quantities

It is often required in ac analysis to add or subtract the two or more alternating quantities with same frequency but with different amplitudes and phases. The addition or subtraction using waveforms is much tedious and cumbersome. So, it is always preferred to add or subtract these quantities by using respective phasors.

Addtion and subtraction of alternating currents and voltages can be accomplished by one of the following methods:

(i) By phasor diagram (graphical method)
(ii) By phasor algebra

Graphical method (by phasor diagram)

Let us take an illustrative example of adding currents:

$$i_1 = 7.07 \sin(\omega t - 45) \quad \text{and} \quad i_2 = 4.24 \sin(\omega t + 30)$$

The rms values are: $I_1 = \dfrac{7.07}{\sqrt{2}} = 5 \text{ A}$

$$I_2 = \frac{4.24}{\sqrt{2}} = 3 \text{ A}$$

In thi...

$\sqrt{2}$

In this method, the phasor diagram is required to be plotted to the scale. Now, the polar forms of the two currents are:

$$\bar{I}_1 = (5 \angle -45)\ \text{A}$$

$$\bar{I}_2 = (3 \angle 30)\ \text{A}$$

Take scale 1 cm = 1 A. Draw $I_1 = 5$ cm at $-45°$ and $I_2 = 3$ cm at $30°$, with respect to positive X-axis (reference) as shown in Fig. 2.38. Complete the parallelogram. Then the diagonal (=6.4cm by measurement) represents the resultant current. Angle made with positive X-axis by the resultant is the phase angle, $\phi = -19°$ (by measurement).

From phasor diagram,

Resultant, $I = 6.4$ A

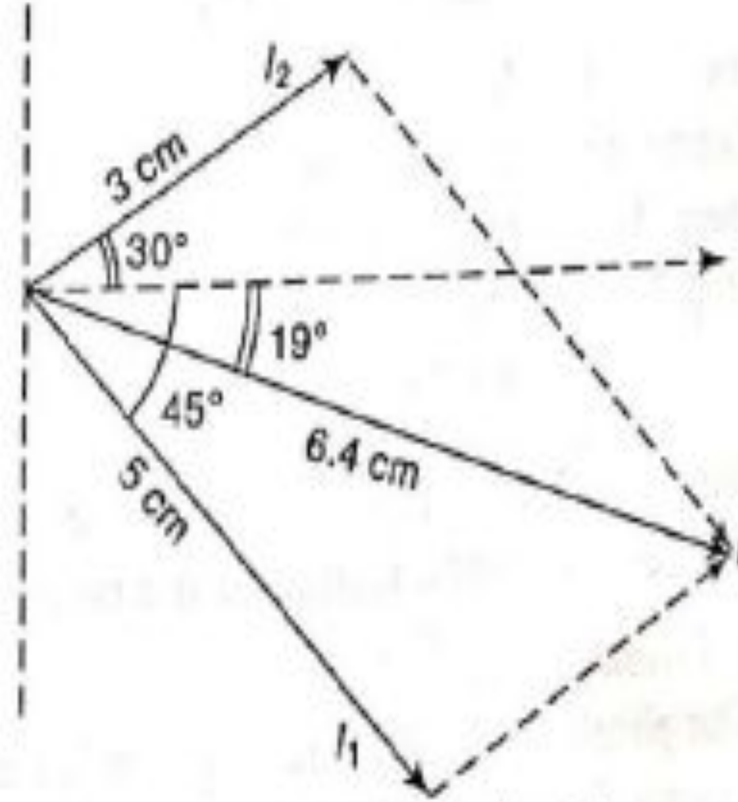


Fig. 2.38 Phasor diagram

Phase angle, $\phi = -19°$

i.e., $\quad \overline{I} = (6.4 \angle -19)$ A

So, $\quad I_m = 6.4 \times \sqrt{2} = 9.05$ A

Hence, equation of the resultant current is given by

$$i = 9.05 \sin(\omega t - 19°)$$

By phasor algebra

Steps to be followed in phasor algebra method are as follows:

Step I Write all the given quantities in standard sinusoidal forms, i.e., for voltage, standard sinusoidal form is $v = V_m \sin(\omega t \pm \phi)$ and for current, it is $i = I_m \sin(\omega t \pm \phi)$.

The RHS of the equation must be positive and sin function only. Any non-standard sinusoidal form can be converted into standard sinusoidal form by using the following trigonometric formulae:

$$\cos\alpha = \sin(\alpha + 90°)$$

$$-\sin\alpha = \sin(\alpha + 180°)$$

Step II Convert the standard sinusoidal forms into polar forms.

Step III Convert the polar forms into rectangular forms. Add or subtract as required. The result will be in rectangular form.

Step IV Convert the rectangular form of result into polar form.

Step V Convert the result from polar form into standard sinusoidal form.

Let us take an illustrative example of adding the currents:

$$i_1 = -565.69 \sin(\omega t + 20) \quad \text{and} \quad i_2 = 10 \cos(\omega t + 10)$$

Step I Standard sinusoidal forms of the given quantities are:

$$i_1 = 565.69 \sin(\omega t + 200) \text{ A}$$

$$i_2 = 10 \sin(\omega t + 100) \text{ A}$$

Step II Converting the standard sinusoidal forms into polar forms,

$$\overline{I_1} = (400 \angle 200) \text{ A}$$

$$\overline{I_2} = (7.07 \angle 100) \text{ A}$$

Step III Converting the polar forms into rectangular forms,

$$\overline{I_1} = (-375.88 - j136.81)$$

$$\overline{I_2} = (-1.23 + j6.96)$$

Let $\quad \overline{I} = \overline{I_1} + \overline{I_2}$

Hence, resultant current, $\overline{I} = (-377.11 - j129.85)$ A

Step IV Taking polar form of resultant current, $\overline{I} = (398.84 \angle -161°)$ A

Step V Writing the equation of resultant current,

$$i = 564.04 \sin(\omega t - 161°)$$

Example 2.18 Two sinusoidal currents are given by $i_1 = 10\sin(\omega t + \pi/3)$ and $i_2 = 15\sin(\omega t - \pi/4)$. Calculate the phase difference between them in degrees.

Solution

The phase angle of current i_1 is $\pi/3$, i.e., $60°$, while the phase angle of current $i_2 = -\pi/4$, i.e., $-45°$. This is shown in Fig. 2.39.

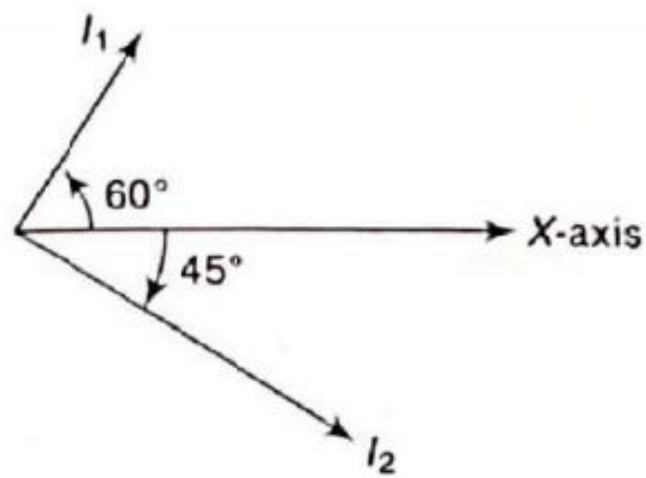


Fig. 2.39

Hence, the phase difference between the two currents is

$$\phi = \phi_1 - \phi_2 = 60 - (-45) = 105°$$

and I_2 lags behind I_1 by ϕ.

Example 2.19 The instantaneous values of two alternating voltages are represented respectively by $v_1 = 60 \sin\theta$ and $v_2 = 40 \sin(\theta + \pi/3)$. Derive the expression for the instantaneous values of (i) the sum and (ii) the difference of these voltages.

Draw neat phasor diagram with each and every value of the quantity marked.

Solution

The given voltages are, $v_1 = 60 \sin \theta$ and $v_2 = 40 \sin\left(\theta + \frac{\pi}{3}\right)$

The rms values of the two voltages are:

$$V_1 = \frac{60}{\sqrt{2}} = 42.43 \text{ V and } V_2 = \frac{40}{\sqrt{2}} = 28.28 \text{ V}$$

Writing the polar forms of the voltages,

$$\overline{V_1} = (42.43 \angle 0°) \text{ V and } \overline{V_2} = (28.28 \angle 60°) \text{ V}$$

Converting the polar forms into rectangular forms,

$$\overline{V_1} = (42.43 + j0) \text{ V and } \overline{V_2} = (14.14 + j24.49) \text{ V}$$

Case (i) Sum of the two voltages

Let the sum of the two voltages, i.e., resultant is V_A.

So, $\overline{V_A} = \overline{V_1} + \overline{V_2}$

or $\overline{V_A} = (42.43 + j0) + (14.14 + j24.49)$

or $\overline{V_A} = (56.57 + j24.49)$ volt

Converting into polar form,

$$\overline{V_A} = (61.64 \angle 23.41) \text{ volt}$$

Maximum value of the resultant voltage, $V_m = 61.64 \times \sqrt{2} = 87.17$ V

Expression for the instantaneous value of sum is

$$v_A = 87.17 \sin(\omega t + 23.41°)$$

Phasor diagram:

$$\overline{V_A} = \overline{V_1} + \overline{V_2}; \qquad \overline{V_1} = (42.43 \angle 0°) \text{ V}$$

$$\overline{V_2} = (28.28 \angle 60°) \text{ V}$$

Example 2.21 Find the resultant of the following voltages:

$$v_1 = 25 \sin \omega t$$
$$v_2 = 10 \sin \left(\omega t + \frac{\pi}{6}\right)$$
$$v_3 = 30 \cos \omega t$$
$$v_4 = 20 \sin \left(\omega t - \frac{\pi}{4}\right)$$

Solution

Writing the standard sinusoidal forms of the given quantities,

$$v_1 = 25 \sin \omega t$$
$$v_2 = 10 \sin \left(\omega t + \frac{\pi}{6}\right)$$
$$v_3 = 30 \sin \left(\omega t + \frac{\pi}{2}\right)$$
$$v_4 = 20 \sin \left(\omega t - \frac{\pi}{4}\right)$$

Converting the standard sinusoidal forms into polar forms,

$$\bar{V}_1 = (17.68 \angle 0) \text{ V}$$
$$\bar{V}_2 = (7.07 \angle 30) \text{ V}$$
$$\bar{V}_3 = (21.21 \angle 90) \text{ V}$$
$$\bar{V}_4 = (14.14 \angle -45) \text{ V}$$

Converting the polar forms into rectangular forms,

$$\bar{V}_1 = (17.68 + j0) \text{ V}$$
$$\bar{V}_2 = (6.12 + j3.54) \text{ V}$$
$$\bar{V}_3 = (0 + j21.21) \text{ V}$$
$$\bar{V}_4 = (10 - j10) \text{ V}$$

Let the resultant voltage is $\bar{V}$.

So, $\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3 + \bar{V}_4$

or $\bar{V} = (17.68 + j0) + (6.12 + j3.54) + (0 + j21.21) + (10 - j10)$

or $\bar{V} = (33.8 + j14.75)$ V

Converting into polar form,

$$\bar{V} = (36.88 \angle 23.58) \text{ V}$$

Converting into standard sinusoidal form,

$$v = 52.16 \sin(\omega t + 23.58)$$

Example 2.24 Three coils are connected in series. Each of them generates an emf of 230 V. The emf of the second coil leads that of the first coil by 120°, and the emf of the third coil lags behind that of the first by the same angle. What is the resultant emf across the series combination of the coils?

Solution

Let the emf generated in the first coil is E_1.

By taking this emf as reference, we can express in polar form as

$$\overline{E}_1 = (230 \angle 0)\ \text{V}$$

Let the emf generated in the second coil is E_2. As E_2 leads E_1 by 120°, we get the polar form as

$$\overline{E}_2 = (230 \angle 120)\ \text{V}$$

Let the emf generated in the third coil is E_3. As E_3 lags behind E_1 by 120°, we get the polar form as

$$\overline{E}_3 = (230 \angle -120°)\ \text{V}$$

Now, the resultant emf E_R across the series combination of the coils is

$$\begin{aligned}\overline{E}_R &= \overline{E}_1 + \overline{E}_2 + \overline{E}_3 \\ &= (230 \angle 0) + (230 \angle 120) + (230 \angle -120) \\ &= (230 + j0) + (-115 + j200) + (-115 - j200)\end{aligned}$$

So, $$\overline{E}_R = 0\ \text{V}$$

2.1.12 Fundamental AC Circuits

The closed path followed by an alternating current is known as **ac circuit**. When a sinusoidal alternating voltage is applied in a circuit, the resultant alternating current is also sinusoidal and has same frequency as that of the applied voltage. However, there is generally a phase difference between the applied voltage and the resulting current. In other words, when the applied voltage is maximum in the positive direction, the resultant current may not be maximum. As we shall see, the phase difference is introduced due to the presence of inductance (L) and capacitance (C) in the circuit. The resistance, inductance, and capacitance are the three basic elements of any electrical network. In order to analyse any ac electric circuit, it is necessary to understand the following three cases:

1. AC through pure resistance
2. AC through pure inductance
3. AC through pure capacitance

In each case, it is assumed that a purely sinusoidal alternating voltage given by the equation $v = V_m \sin \omega t$ is applied to the circuit. The equations of the current, power, and phase shift are developed in each case. The voltage applied having zero phase angle is assumed reference while plotting phasor diagram in each case.

AC Circuit Containing Resistance Only (Resistive Circuit)

Consider a circuit containing a pure resistance of $R\ \Omega$ connected across the alternating voltage source (see Fig. 2.44).

Let the alternating voltage be given by the equation

$$\boxed{v = V_m \sin \omega t} \tag{2.3}$$

As a result of this voltage, an alternating current i will flow in the circuit. According to Ohm's law, we can find the equation of current i as

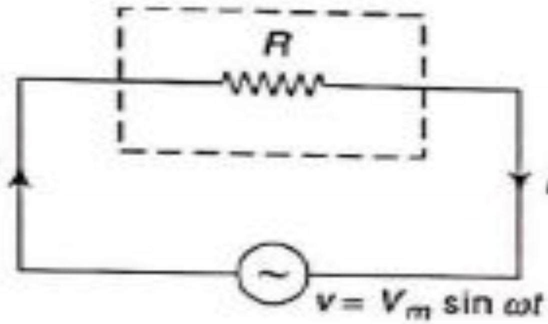


Fig. 2.44 Resistive AC circuit

$$i = \frac{v}{R}$$

Substituting the value of v, we get

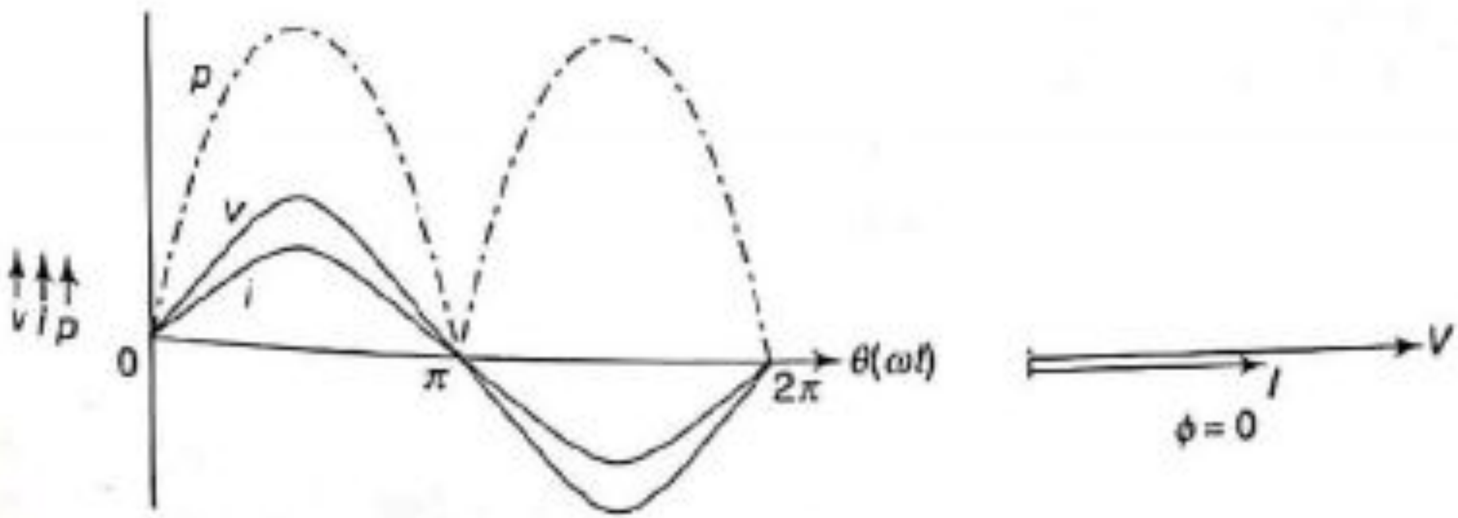
$$i = \frac{V_m}{R} \sin \omega t \tag{2.4}$$

The value of i will be maximum (i.e., I_m) when $\sin \omega t = 1$.

So, $$I_m = \frac{V_m}{R}$$

Thus, Eq. (2.4) becomes $\boxed{i = I_m \sin \omega t}$ (2.5)

Phase angle From Eqs (2.3) and (2.5), it is clear that the applied voltage and the circuit current are in phase with each other. Therefore, nature of the circuit is resistive. The waveforms of voltage and current and the corresponding phasor diagram are shown in Figs 2.45(a) and (b).



(a) Waveforms

(b) Phasor diagram

Fig. 2.45 Voltage and current in phase with each other

In the phasor diagram, the phasors are drawn in phase and there is no phase difference. The angle between voltage across the circuit and current through the circuit is known as phase angle of the circuit. So, in this case

Phase angle of the circuit, $\phi = 0$

Hence, power factor of the circuit, $pf = \cos\phi = 1$ (unity)

Opposition to current (Z) In general, opposition offered by circuit elements to the current flow is called impedance (Z). The impedance of the circuit can be calculated as

$$\frac{V}{I} = Z \qquad \text{(by Ohm's law)}$$

We have seen that in resistive circuit,

$$I_m = \frac{V_m}{R}$$

or
$$\frac{V_m}{I_m} = R$$

Dividing both numerator and denominator by $\sqrt{2}$, we get

$$\frac{\frac{V_m}{\sqrt{2}}}{\frac{I_m}{\sqrt{2}}} = R$$

or
$$\frac{V}{I} = R$$

Thus, $Z = R\,\Omega$

Power (P) In any circuit, electric power consumed at any instant is the product of voltage and current at that instant, i.e.,

$$p = v \times i$$

In the above equation, voltage and current are varying sinusoidally. Therefore, power is also varying w.r.t. time.

$$\begin{aligned}
\text{Instantaneous power, } p &= v \times i \\
&= (V_m \sin \omega t)\,(I_m \sin \omega t) \\
&= V_m I_m \sin^2 \omega t \\
&= V_m I_m \left(\frac{1 - \cos 2\omega t}{2}\right) \\
&= \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos 2\omega t
\end{aligned}$$

Since power is a scalar quantity, average power over a complete cycle is to be considered. So, we have

$$\text{Power consumed, } P = \frac{1}{2\pi} \int_0^{2\pi} p \, d\omega t$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{V_m I_m}{2} - \frac{V_m I_m}{2}\cos 2\omega t\right] d\omega t$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{V_m I_m}{2}\, d\omega t - \frac{1}{2\pi}\int_0^{2\pi}\frac{V_m I_m}{2}\cos 2\omega t\, d\omega t$$

$$= \frac{V_m I_m}{2\pi \times 2}[\omega t]_0^{2\pi} - \frac{V_m I_m}{2\pi \times 2}[\sin 2\omega t]_0^{2\pi}$$

$$= \frac{V_m I_m}{2\pi \times 2}[2\pi - 0] - \frac{V_m I_m}{2\pi \times 2}[0 - 0]$$

$$= \frac{V_m I_m}{2} - 0$$

$$= \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$

So, $P = VI$

where V = rms value of the applied voltage

I = rms value of the circuit current

AC Circuit Containing Pure Inductance Only (Pure Inductive Circuit)

Consider a circuit containing a pure inductance of L henry as shown in Fig. 2.46.

An alternating voltage is applied to a pure inductance. Let the equation of the applied voltage be

$$\boxed{v = V_m \sin \omega t} \tag{2.6}$$

As a result of this voltage, an alternating current 'i' flows through the inductance L. The alternating current sets up an alternating magnetic field around the inductance. This changing flux links the coil and an emf is induced in it, called self- induced emf ($=Ldi/dt$). This emf opposes the applied voltage. At any instant, self-induced emf is equal and opposite to the applied voltage.

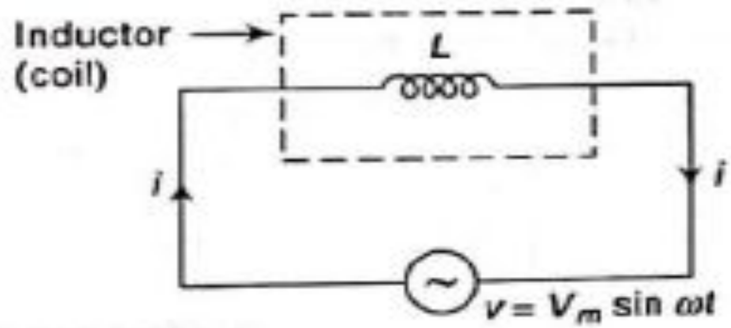


Fig. 2.46 Pure inductive circuit

So, $$v = L\frac{di}{dt}$$

or $$Ldi = vdt$$

or $$Ldi = V_m \sin \omega t \, dt \qquad (\because v = V_m \sin \omega t)$$

or $$di = \frac{V_m}{L} \sin \omega t \, dt$$

Integrating both sides, we get

$$i = \frac{V_m}{L} \int \sin \omega t \, dt$$

$$i = \frac{V_m}{\omega L}\,(-\cos \omega t)$$

$$i = \frac{V_m}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right) \tag{2.7}$$

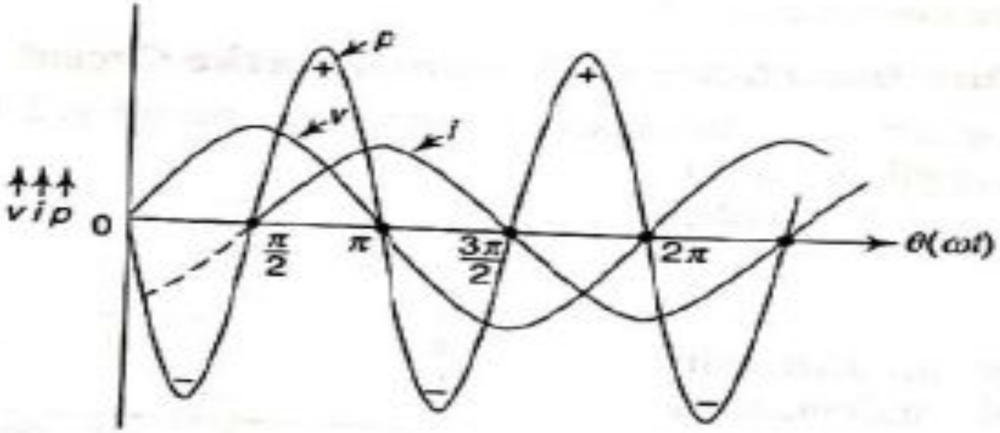
The value of i will be maximum (i.e., I_m) when $\sin\left(\omega t - \frac{\pi}{2}\right) = 1$.

So, $$I_m = \frac{V_m}{\omega L}$$

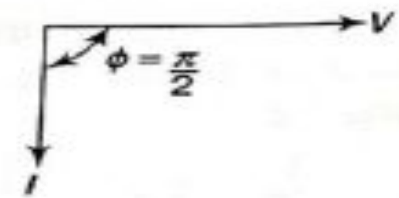
Substituting the value of $\frac{V_m}{\omega L} = I_m$ in Eq. (2.7), we get

$$\boxed{i = I_m \sin\left(\omega t - \frac{\pi}{2}\right)} \tag{2.8}$$

Phase angle and power factor From Eqs (2.6) and (2.8), it is clear that the current lags behind the voltage by $\pi/2$ rad or 90°. Hence, the nature of the circuit is pure inductive. The waveforms of voltage and current and the corresponding phasor diagram are shown in Figs 2.47(a) and (b).



(a) Waveforms



(b) Phasor diagram

Fig. 2.47 Current lagging behind voltage by 90°

Thus,

Phase angle of the circuit, $\phi = 90°$

Hence, power factor of the circuit, pf = $\cos\phi = \cos 90° = 0$ lagging

Note: Power factor has magnitude and nature. Magnitude of the power factor equals to $\cos\phi$. The nature of the power factor is same as nature of current. In this case, circuit current lags to the applied voltage, therefore nature of the power factor is lagging.

Opposition to current (Z) We have seen that

$$I_m = \frac{V_m}{\omega L}$$

or $$\frac{V_m}{I_m} = \omega L$$

Dividing both numerator and denominator by $\sqrt{2}$,

$$\frac{\frac{V_m}{\sqrt{2}}}{\frac{I_m}{\sqrt{2}}} = \omega L$$

So, $$\frac{V}{I} = \omega L$$

or $$Z = \omega L\ \Omega$$

Thus, opposition offered by inductance to current flow is ωL. This quantity ωL is called inductive reactance X_L of the coil.

So, $$X_L = \omega L\ \Omega$$

or $$X_L = 2\pi f L\ \Omega$$

Note: There are two reactive elements: inductor and capacitor. The opposition offered by the reactive element to current flow is called reactance (denoted by X). Thus, the opposition offered by inductor to current flow is called inductive reactance (denoted by X_L), and the opposition offered by capacitor to current flow is called capacitive reactance (denoted by X_C).

Power Instantaneous power,

$$p = vi$$

$$= V_m \sin \omega t \times I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= -V_m I_m \sin \omega t \cos \omega t$$

$$= -\frac{V_m I_m}{2} \sin 2\omega t$$

Power Instantaneous power,

$$p = vi$$

$$= V_m \sin \omega t \times I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= -V_m I_m \sin \omega t \cos \omega t$$

$$= -\frac{V_m I_m}{2} \sin 2\omega t$$

Since power is a scalar quantity, average power over a complete cycle is to be considered.

So, Average power, P = Average of p over one cycle

or $$P = \frac{1}{2\pi}\int_0^{2\pi} p \, d\omega t$$

or $$P = \frac{1}{2\pi}\int_0^{2\pi} -\frac{V_m I_m}{2} \sin 2\omega t \, d\omega t$$

or $$P = 0$$

Hence, power absorbed in pure inductance is zero.

Figure 2.47(a) shows the power curve. During the first 90° of the cycle, the voltage is +ve and the current is –ve. Therefore, the power supplied is negative. It means power is flowing from the inductor to the source. During the next 90° of the cycle, both voltage and current are positive and the power supplied is positive. Therefore, power flows from the source to the inductor. Similarly, for the next 90° of the cycle, power flows from the inductor to the source and during the last 90° of the cycle, power flows from the source to the inductor. An examination of the power curve over one cycle shows that positive power is equal to the negative power. Hence the resultant power over one cycle is zero, i.e., pure inductance consumes no power. When current rises in an inductor, energy is required to build up magnetic field around the inductor. This energy is supplied from the source and is stored in the magnetic field of the inductor. As the current falls, the collapsing magnetic field returns the stored energy to the source. In Fig. 2.47(a), when power is positive; energy is being put into the circuit to build up the magnetic field around the inductor. And when the power is negative, the magnetic energy is being returned to the source. Since power supplied is equal to power returned (positive areas being equal to the negative areas) over one cycle, the net power absorbed by the inductor is zero. In this case power circulates in the circuit. The circulating power is called as reactive power. The power actually consumed in the circuit is called as active power.

AC Circuit Containing Pure Capacitance Only (Pure Capacitive Circuit)

Consider an alternating voltage applied to a capacitor of capacitance C F as shown in Fig. 2.48. Let the equation of the applied alternating voltage be

$$\boxed{v = V_m \sin \omega t} \tag{2.9}$$

As a result of this alternating voltage, alternating current will flow through the circuit. Let at any instant, i is the current and q is the charge on the plates.

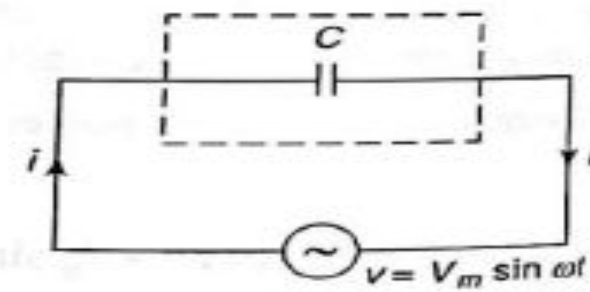


Fig. 2.48 Pure capacitive circuit

Charge on capacitor, $q = Cv$

So, Circuit current, $i = \frac{dq}{dt}$

$$= \frac{dCv}{dt}$$

$$= \frac{d}{dt}(C\,V_m \sin \omega t)$$

$$= \omega C\,V_m \cos \omega t$$

or $$i = \omega C V_m \sin\left(\omega t + \frac{\pi}{2}\right) \tag{2.10}$$

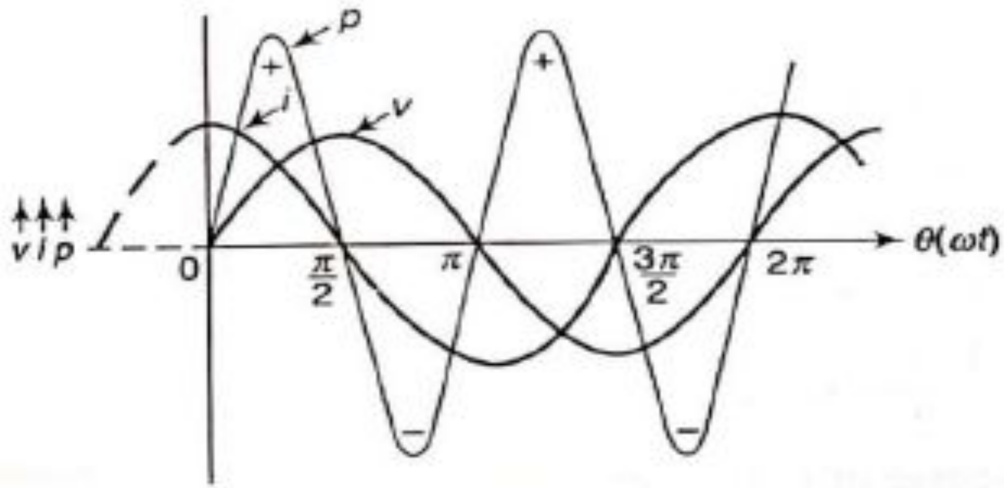
The value of i will be maximum (i.e., I_m) when $\sin\left(\omega t + \frac{\pi}{2}\right)$ is unity.

So, $I_m = \omega C V_m$

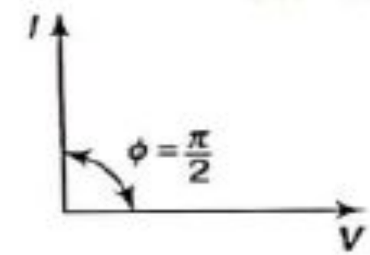
Substituting the value $\omega C V_m = I_m$ in Eq. (2.10), we get

$$\boxed{i = I_m \sin\left(\omega t + \frac{\pi}{2}\right)} \tag{2.11}$$

Phase angle and power factor From Eqs (2.9) and (2.11), it is clear that the current leads the voltage by $\pi/2$ rad or 90°. Hence, in a pure capacitance, the current leads the voltage by 90°. The nature of the circuit is said to be pure capacitive. The waveforms of voltage and current and the corresponding phasor diagram are shown in Figs 2.49(a) and (b).



(a) Waveforms



(b) Phasor diagram

Fig. 2.49 Current leads voltage by 90°

Thus, Phase angle of the circuit, $\phi = 90°$

So, power factor of the circuit, pf = $\cos \phi = \cos 90° = 0$ leading

Opposition to current (Z) We have seen that

$$I_m = \omega C V_m$$

or $$\frac{V_m}{I_m} = \frac{1}{\omega C}$$

Dividing both numerator and denominator by $\sqrt{2}$,

$$\frac{\frac{V_m}{\sqrt{2}}}{\frac{I_m}{\sqrt{2}}} = \frac{1}{\omega C}$$

So, $$\frac{V}{I} = \frac{1}{\omega C}$$

or $$Z = \frac{1}{\omega C}\,\Omega$$

Thus, opposition offered by capacitance to current flow is $\frac{1}{\omega C}$. This quantity

$\frac{1}{\omega C}$ is called capacitive reactance X_C of the capacitor.

So, $$X_C = \frac{1}{\omega C}\,\Omega$$

So, $$X_C = \frac{1}{2\pi f C}\,\Omega$$

Power (p) Instantaneous power, $p = vi$

$$= V_m \sin \omega t \times I_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$= V_m I_m \sin \omega t \cos \omega t$$

$$= \frac{V_m I_m}{2} \sin 2\omega t$$

So, Average power, P = Average of p over one cycle

or $$P = \frac{1}{2\pi}\int_0^{2\pi} p \, d\omega t$$

or $$P = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{V_m I_m}{2}\sin 2\omega t\right] d\omega t$$

or $$P = 0$$

Hence, power absorbed in a pure capacitance is zero.

Figure 2.49(a) shows th...

power absorbed in a pure capacitance is zero.

Figure 2.49(a) shows the power curve. The power curve is similar to that for a pure inductor because now the current leads the voltage by 90°. It is clear that positive power is equal to the negative power over one cycle. Hence, the net power absorbed in a capacitor is zero. When voltage rises across the plates of a capacitor, energy is required to build up electrostatic field between the plates of the capacitor. This energy is supplied from the source and is stored in the capacitor in the form of electrostatic field energy. As the voltage falls, the collapsing electrostatic field returns the stored energy to the source. Since the power supplied is equal to the power returned (positive areas being equal to the negative areas) over one cycle, the net power absorbed in a capacitor is zero.

Example 2.26