Method of Variation of Parameters

This method is derived by mathematician **Lagrange** by evolving the P.I. from its C.F.only by assuming temporarily the constantsas some variable functions.

Due to this assumptions this method is called **Variation of Parameters or Variation of constants**

Concept

Consider a second order non homogeneous equation with constant coefficients

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = X ....\mathbf{(1)}$$

Let y_1 and y_2 be the solution of corresponding Homogeneous equation

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = 0 ....\mathbf{(2)}$$

then complimentary function is given by

$$y_c = c_1\ y_1 + c_2\ y_2$$

Suppose Particular Integral is obtained by above C.F. by considering c_1 and c_2 as variable functions say $v_1(x)$ and $v_2(x)$ and is given by

$$y_p = v_1\ y_1 + v_2\ y_2$$

Goal:To determine v_1 and v_2
Now

$$\begin{aligned}
y_p &= v_1\ y_1 + v_2\ y_2 \\
y_p' &= v_1'\ y_1 + v_1\ y_1' + v_2'\ y_2 + v_2\ y_2' \\
y_p'' &= v_1''\ y_1 + v_1'\ y_1' + v_1'\ y_1' + v_1\ y_1'' + v_2''\ y_2 + v_2'\ y_2' + v_2'\ y_2' + v_2\ y_2'' \\
&= v_1''\ y_1 + 2v_1'\ y_1' + v_1\ y_1'' + v_2''\ y_2 + 2v_2'\ y_2' + v_2\ y_2''
\end{aligned}$$

if y_p is particular solution of (1), it must satisfy the equation

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = X$$

$$\frac{d^2y_p}{d^2x} + P(x)\frac{dy_p}{dx} + Q(x)y_p = X$$

$$[v_1''\ y_1 + 2v_1'\ y_1' + v_1\ y_1'' + v_2''\ y_2 + 2v_2'\ y_2' + v_2\ y_2'']$$
$$+P(x)[v_1'\ y_1 + v_1\ y_1' + v_2'\ y_2 + v_2\ y_2'] + Q(x)[v_1\ y_1 + v_2\ y_2] = X$$

$$v_1\ [y_1'' + P(x)y_1' + Q(x)y_1] + v_2\ [y_2'' + P(x)y_2' + Q(x)y_2]$$
$$+v_1''\ y_1 + 2v_1'\ y_1' + v_2''\ y_2 + 2v_2'\ y_2' + P(x)[v_1'\ y_1 + v_2'\ y_2] = X$$

Hence ,by (2)

$$v_1[0] + v_2[0] + v_1''\ y_1 + 2v_1'\ y_1' + v_2''\ y_2 + 2v_2'\ y_2' + P(x)[v_1'\ y_1 + v_2'\ y_2] = X$$

$$v_1''\ y_1 + 2v_1'\ y_1' + v_2''\ y_2 + 2v_2'\ y_2' + P(x)[v_1'\ y_1 + v_2'\ y_2] = X$$

Now

$$\frac{d}{dx}[v_1'\ y_1 + v_2'\ y_2] = v_1''\ y_1 + v_1'\ y_1' + v_2''\ y_2 + v_2'\ y_2'$$

Using in last expression we have

$$\frac{d}{dx}[v_1'\ y_1 + v_2'\ y_2] + v_1'\ y_1' + v_2'\ y_2' + P(x)[v_1'\ y_1 + v_2'\ y_2] = X$$

Hence assumption for P.I. holds true if v_1 and v_2 satisfies

$$v_1'\ y_1 + v_2'\ y_2 = 0$$

and

$$v_1'\ y_1' + v_2'\ y_2' = X$$

which are two linear equation with two unknowns v_1' and v_2' and can be easily determined by various solution methods.

Working Rule (Second order)

(1) For given second order non homogeneous equation , if y_1 and y_2 are solutions of corresponding homogeneous equations then write C.F. as

$$y_c = c_1y_1 + c_2y_2$$

(2) From C.F. assume P.I. as

$$y_p = v_1y_1 + v_2y_2$$

satisfying

$$v_1'\ y_1 + v_2'\ y_2 = 0$$

and

$$v_1'\ y_1' + v_2'\ y_2' = X$$

(3) Solve above equation in v_1' and v_2' by either Cramer's Rule and determine v_1 and v_2
where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$; $W_1 = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix}$ and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix}$
which gives gives $v_1' = \dfrac{W_1}{W}$ and $v_2' = \dfrac{W_2}{W}$
Hence $v_1 = \displaystyle\int v_1'\ dx$ and $v_2 = \displaystyle\int v_2'\ dx$
or determine v_1 and v_2 by the formula

$$v_1 = \int \frac{-y_2X}{W}dx \quad , \quad v_2 = \int \frac{y_1X}{W}dx$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

Note(Third Order)

For given third order non homogeneous equation , if y_1 , y_2 and y_3 are solutions of corresponding homogeneous equations then write C.F. as

$$y_c = c_1y_1 + c_2y_2 + c_3y_3$$

From C.F. assume P.I. as

$$y_p = v_1y_1 + v_2y_2 + v_3y_3$$

satisfying

$$v_1'\ y_1 + v_2'\ y_2 + v_3'\ y_3 = 0$$
$$v_1'\ y_1' + v_2'\ y_2' + v_3'\ y_3' = 0$$

and

$$v_1'\ y_1'' + v_2'\ y_2'' + v_3'\ y_3'' = X$$

Solve above equation in v_1',v_2' and v_3' by either Cramer's Rule and determine v_1 , v_2 and v_3

Solve by Method of Variation of parameter

(1) $\dfrac{d^2y}{dx^2} + y = xsinx$

(2) $[(D^2 - 4D + 4)]y = e^{2x}sec^2x$

(3) $\dfrac{d^2y}{dx^2} + y = sec\ x\ tan\ x$

<u>Answers</u>

(1) $y = c_1cos\ x + c_2sin\ x + \frac{x}{2}\ sin\ x - \frac{x^2}{4}\ cos\ 3x$

(2) $y = [(c_1x + c_2 + log(secx)]e^{2x}$

(3) $y = c_1cos\ x + c_2sin\ x + xcosx - sinx + sinxlog(secx)$