Q.1)

1a) Show that $\operatorname{sech}^{-1}(\sin\theta)=\log\left(\cot\frac{\theta}{2}\right)$. (Chp: Hyperbolic Functions) (3)

Ans. We know, $\cosh^{-1}x=\log\left(x+\sqrt{x^2-1}\right)\rightarrow(1)$

$$\text{LHS}=\operatorname{sech}^{-1}(\sin\theta)$$

$$=\cosh^{-1}\left(\frac{1}{\sin\theta}\right)$$

$$=\cosh^{-1}(\operatorname{cosec}\theta)$$

$$=\log\left(\operatorname{cosec}\theta+\sqrt{\operatorname{cosec}^2\theta-1}\right)\text{ (From 1)}$$

$$=\log(\operatorname{cosec}\theta+\cot\theta)$$

$$=\log\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$=\log\left[\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right]$$

$$=\log\left[\frac{\cos(\theta/2)}{\sin(\theta/2)}\right]$$

$$=\log\left(\cot\frac{\theta}{2}\right)$$

$$=\text{RHS}$$

1b) Show that the matrix $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$ is unitary. (Chp: Rank of Matrix) (3)

Ans. $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

$\therefore\ A'=\frac{1}{2}\begin{bmatrix}\sqrt{2} & i\sqrt{2} & 0\\ -i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

$\therefore\ \overline{A}'=A^{\theta}=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

Consider,

$AA^{\theta}=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}\times\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

$=\frac{1}{4}\begin{bmatrix}2-2i^2+0 & -2i+2i+0 & 0+0+0\\ 2i-2i+0 & -2i^2+2+0 & 0+0+0\\ 0+0+0 & 0+0+0 & 0+0+4\end{bmatrix}$

$=\frac{1}{4}\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$

$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\mathrm{I}$

Similarly, we can prove, $A^{\theta}A=I$

$\therefore\ A^{\theta}A=AA^{\theta}=I$

$\therefore$ A is unitary.

$\therefore\ A^{-1}=A^{\theta}=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

1c) Evaluate $\lim_{x \to 0} \sin x \log x$. (Chp: Indeterminate Forms) (3)

Ans. Let $L = \lim_{x \to 0} \sin x \log x$

$$= \lim_{x \to 0} \frac{1}{\operatorname{cosec} x} \times \log x \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cdot \cot x} \quad \text{(L' Hospital's Rule)}$$

$$= \lim_{x \to 0} \left(\frac{-\sin x}{x} \cdot \tan x\right)$$

$$= -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \tan x$$

$$= -1 \times \tan 0$$

$$= 0$$

Hence, $\boxed{\lim_{x \to 0} \sin x \log x = 0}$

1d) Find the nth derivative of $y = e^{ax}\cos^2 x\sin x$. (Chp: Successive Differentiation) (3)

Ans. $y = e^{ax}\cos^2 x\sin x$

$$= e^{ax}\left(\frac{1+\cos 2x}{2}\right)\sin x \quad \left\{\because 1+\cos 2A = 2\cos^2 A\right\}$$

$$= \frac{1}{2}e^{ax}\left(\sin x + \cos 2x\sin x \times \frac{2}{2}\right)$$

$$= \frac{1}{2}e^{ax}\left\{\sin x + \frac{1}{2}\left[\sin(2x+x) - \sin(2x-x)\right]\right\} \quad \left\{\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)\right\}$$

$$= \frac{1}{2}e^{ax} \times \frac{2\sin x + \sin 3x - \sin x}{2}$$

$$= \frac{1}{4}e^{ax}(\sin 3x + \sin x)$$

$$\therefore\ y = \frac{1}{4}\left[e^{ax}\sin 3x + e^{ax}\sin x\right]$$

Taking nth order derivative, $y_n = \frac{1}{4}\left\{\frac{d^n}{dx^n}\left(e^{ax}\sin 3x\right) + \frac{d^n}{dx^n}\left(e^{ax}\sin x\right)\right\} \rightarrow (1)$

We know, If $y = e^{ax}\sin(bx+c)$, $y_n = r^n e^{ax}\sin(bx+c+n\phi) \rightarrow (2)$

Here, c = 0, $b_1 = 3$ and $b_2 = 1$

$$\therefore\ r_1 = \sqrt{a^2+b_1^2} = \sqrt{a^2+3^2} = \sqrt{a^2+9} \text{ and } r_2 = \sqrt{a^2+b_2^2} = \sqrt{a^2+1^2} = \sqrt{a^2+1} \rightarrow (3)$$

And, $\phi_1 = \tan^{-1}\frac{b_1}{a} = \tan^{-1}\frac{3}{a}$ & $\phi_2 = \tan^{-1}\frac{b_2}{a} = \tan^{-1}\frac{1}{a} \rightarrow (4)$

$\therefore$ From (1), (2), (3) and (4),

$$y_n = \frac{1}{4}\left\{\sqrt{a^2+9}\,e^{ax}\sin\left(3x+0+n\phi_1\right) + \sqrt{a^2+1}\,e^{ax}\sin\left(x+0+n\phi_2\right)\right\}$$

$$\therefore\ y_n = \frac{1}{4}\left\{\sqrt{a^2+9}\,e^{ax}\sin\left(3x+n\phi_1\right) + \sqrt{a^2+1}\,e^{ax}\sin\left(x+n\phi_2\right)\right\}, \text{ where } \phi_1 = \tan^{-1}\frac{3}{a} \ \&\ \phi_2 = \tan^{-1}\frac{1}{a}$$

1e) If x = r cos θ and y = r sin θ then prove that J.J' = 1. (Chp: Jacobian) (4)

Ans. Given, x = r cos θ

$$\therefore\ x_r = \frac{\partial x}{\partial r} = \cos\theta \text{ and } x_\theta = \frac{\partial x}{\partial \theta} = -r\sin\theta \rightarrow (1)$$

Given, y = r sin θ

$$\therefore\ y_r = \frac{\partial y}{\partial r} = \sin\theta \text{ and } y_\theta = \frac{\partial y}{\partial \theta} = r\cos\theta \rightarrow (2)$$

$$\therefore\ J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= x_r\, y_\theta - x_\theta\, y_r$$

$$= \cos\theta \cdot r\cos\theta + r\sin\theta \cdot \sin\theta \text{ (From 1 \& 2)}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$\therefore\ J = \frac{\partial(x,y)}{\partial(r,\theta)} = r \rightarrow (3)$$

Now, x = r cos θ, y = r sin θ

$$\therefore\ x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta \ \&$$

$$\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \tan\theta$$

$$\therefore\ x^2 + y^2 = r^2\left(\cos^2 v + \sin^2 v\right) \ \&\ \theta = \tan^{-1}\frac{y}{x} \rightarrow (4)$$

$$\therefore\ x^2 + y^2 = r^2$$

$$\therefore\ r = \sqrt{x^2 + y^2} \rightarrow (5)$$

$$\therefore\ r_x = \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}} \rightarrow (6)$$

$$\text{Similarly, } r_y = \frac{y}{\sqrt{x^2 + y^2}} \rightarrow (7)$$

And, From (4),

$$\theta_x = \frac{1}{1 + (y/x)^2} \cdot y \cdot \frac{-1}{x^2} = \frac{x^2}{x^2 + y^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2 + y^2} \rightarrow (8)$$

$$\theta_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} \rightarrow (9)$$

$$\therefore\ J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$$= r_x\,\theta_y - r_y\,\theta_x$$

$$= \frac{x}{\sqrt{x^2 + y^2}} \times \frac{x}{x^2 + y^2} - \frac{y}{\sqrt{x^2 + y^2}} \times \frac{-y}{x^2 + y^2}$$

(From 6, 7, 8 & 9)

$$= \frac{\cancel{\left(x^2 + y^2\right)}}{\cancel{\left(x^2 + y^2\right)}\sqrt{x^2 + y^2}}$$

$$\therefore\ J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r} \rightarrow (10) \text{ (From 5)}$$

$$\text{From (3) and (10), } J \cdot J' = \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r}$$

$$\therefore\ \boxed{J \cdot J' = \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1}$$

Hence proved.

1f) Using coding matrix $A=\begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix}$ encode the message THE CROW FLIES AT MIDNIGHT. (Chp: Coding) (4)

Ans. We use following numerical values of each alphabet for coding

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	*
15	16	17	18	19	20	21	22	23	24	25	26	27

Step 1:

Message: THE CROW FLIES AT MIDNIGHT

As per the above table, the numerical values of each alphabet in the message are

T	H	E	*	C	R	O	W	*	F	L	I	E	S	*	A	T	*	M	I	D	N	I	G	H	T
20	8	5	27	3	18	15	23	27	6	12	9	5	19	27	1	20	27	13	9	4	14	9	7	8	20

Step 2:

Writing the above values column-wise in a 2-row matrix we get,

$$A=\begin{bmatrix}20 & 5 & 3 & 15 & 27 & 12 & 5 & 27 & 20 & 13 & 4 & 9 & 8\\ 8 & 27 & 18 & 23 & 6 & 9 & 19 & 1 & 27 & 9 & 14 & 7 & 20\end{bmatrix}$$

Encoding matrix $A=\begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix} \rightarrow (1)$

$$\text{Now, } EA=\begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix}\times\begin{bmatrix}20 & 5 & 3 & 15 & 27 & 12 & 5 & 27 & 20 & 13 & 4 & 9 & 8\\ 8 & 27 & 18 & 23 & 6 & 9 & 19 & 1 & 27 & 9 & 14 & 7 & 20\end{bmatrix}$$

$$=\begin{bmatrix}40+8 & 10+27 & 6+18 & 30+23 & 54+6 & 24+9 & 10+19 & 54+1 & 40+27 & 26+9 & 8+14 & 18+7 & 16+20\\ 60+8 & 15+27 & 9+18 & 45+23 & 81+6 & 36+9 & 15+19 & 81+1 & 60+27 & 39+9 & 12+14 & 27+7 & 24+20\end{bmatrix}$$

$$\therefore\ EA=\begin{bmatrix}48 & 37 & 24 & 53 & 60 & 33 & 29 & 55 & 67 & 35 & 22 & 25 & 36\\ 68 & 42 & 27 & 68 & 87 & 45 & 34 & 82 & 87 & 48 & 26 & 34 & 44\end{bmatrix}$$

Writing the numbers in EA matrix column wise gives the encoded message.

$\therefore$ Encoded Message =

48 68 37 42 24 27 53 68 60 87 33 45 29 34 55 82 67 87 35 48 22 26 25 34 36 44

This encoded message is transmitted.

Q.2)

2a) Find all the value of $(1+i)^{1/3}$ & show that their continued product is (1 + i). (Chp: Complex - DMT) (6)

Ans. Let $z=(1+i)^{1/3}$

$\therefore\ z^3=1+i=x+iy$

$\therefore$ x = y = 1

$\therefore\ z^3$ lies in first quadrant.

$\therefore$ Argument (θ) $=\tan^{-1}\left|\dfrac{y}{x}\right|=\tan^{-1}\left|\dfrac{1}{1}\right|=\dfrac{\pi}{4}$

$\therefore$ Modulus $r=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2}$

$\therefore$ In polar form, $z^3=re^{-i\theta}$

$\therefore\ 1+i=\sqrt{2}\,e^{-i\pi/4}\rightarrow(1)$ (Principal Value)

$=\sqrt{2}\,e^{-i\left(\frac{\pi}{4}+2n\pi\right)}$ (General Value)

$=2^{1/2}e^{-i\left(\frac{\pi+8n\pi}{4}\right)}$

$\therefore\ z^3=2^{\frac{1}{2}}e^{-i(1+8n)\frac{\pi}{4}}$

$\therefore\ z=\left(2^{\frac{1}{2}}e^{-i(1+8n)\frac{\pi}{4}}\right)^{\frac{1}{3}}$

$\therefore\ z=2^{1/6}e^{-i(1+8n)\frac{\pi}{12}}$

Put n = 0, $z_1=2^{1/6}e^{-i\frac{\pi}{12}}=2^{1/6}\left(\cos\dfrac{\pi}{12}-i\sin\dfrac{\pi}{12}\right)$

Put n = 1, $z_2=2^{1/6}e^{-i\frac{9\pi}{12}}=2^{1/6}e^{-i\frac{3\pi}{4}}$

$=2^{1/6}\left(\cos\dfrac{3\pi}{4}-i\sin\dfrac{3\pi}{4}\right)$

$=2^{1/6}\left(\dfrac{-1}{\sqrt{2}}-i\dfrac{1}{\sqrt{2}}\right)$

$=2^{1/6}\times\dfrac{-1}{\sqrt{2}}(1+i)$

$=2^{1/6}\times -2^{-1/2}(1+i)$

$=-2^{-1/3}(1+i)$

$\therefore\ z_2=2^{1/6}e^{-i\frac{3\pi}{4}}=-2^{-1/3}(1+i)$

Put n = 2, $z_3=2^{1/6}e^{-i\frac{17\pi}{12}}$

$=2^{1/6}e^{-i\left(2\pi-\frac{7\pi}{12}\right)}$

$=2^{1/6}e^{-i2\pi}e^{i\frac{7\pi}{12}}$

$=2^{1/6}\times 1\times\left(\cos\dfrac{7\pi}{12}+i\sin\dfrac{7\pi}{12}\right)$

$\therefore\ z_3=2^{1/6}e^{-i\frac{17\pi}{12}}=2^{1/6}\left(\cos\dfrac{7\pi}{12}+i\sin\dfrac{7\pi}{12}\right)$

$\therefore$ Continued Product of all the roots $=z_1\cdot z_2\cdot z_3$

$=2^{1/6}e^{-i\frac{\pi}{12}}\cdot 2^{1/6}e^{-i\frac{9\pi}{12}}\cdot 2^{1/6}e^{-i\frac{17\pi}{12}}$

$=\left(2^{1/6}\right)^3e^{-i\left(\frac{\pi}{12}+\frac{9\pi}{12}+\frac{17\pi}{12}\right)}$

$=2^{1/2}e^{-i\frac{27\pi}{12}}$

$=\sqrt{2}\,e^{-i\frac{9\pi}{4}}$

$=\sqrt{2}\,e^{-i\left(2\pi+\frac{\pi}{4}\right)}$

$=\sqrt{2}\,e^{-i\frac{\pi}{4}}$

$=1+i$ (From 1)

Hence, Continued Product of all the roots is (1 + i)

Q.6)

6a) Find non-singular Matrices P & Q such that $A=\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&0&5&-10\end{bmatrix}$ is reduced to normal form. Also find Rank.

(Chp: Rank of Matrix) (6)

Ans. Let $A_{3\times4}=I_{3\times3}\times A_{3\times4}\times I_{4\times4}$

$$\therefore\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&0&5&-10\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_2-2R_1;\ R_3-3R_1\Rightarrow$

$$\therefore\begin{bmatrix}1&2&3&4\\0&-3&-2&-5\\0&-6&-4&-22\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_2-2C_1;\ C_3-C_1;\ C_4-4C_1\Rightarrow$

$$\therefore\begin{bmatrix}1&0&0&0\\0&-3&-2&-5\\0&-6&-4&-22\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&0&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-4\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$\frac{1}{-3}C_2\Rightarrow$

$$\therefore\begin{bmatrix}1&0&0&0\\0&1&-2&-5\\0&2&-4&-22\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&0&1\end{bmatrix}A\begin{bmatrix}1&2/3&-3&-4\\0&-1/3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_3-2R_2\Rightarrow$

$$\therefore\begin{bmatrix}1&0&0&0\\0&1&-2&-5\\0&0&0&-12\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\1&-2&1\end{bmatrix}A\begin{bmatrix}1&2/3&-3&-4\\0&-1/3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_3+2C_2;\ C_4+5C_2\Rightarrow$

$$\therefore\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-12\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\1&-2&1\end{bmatrix}A\begin{bmatrix}1&2/3&-5/3&-2/3\\0&-1/3&-2/3&-5/3\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$\frac{1}{-12}R_3\Rightarrow$

$$\therefore\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1/12&1/6&-1/12\end{bmatrix}A\begin{bmatrix}1&2/3&-5/3&-2/3\\0&-1/3&-2/3&-5/3\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_3\leftrightarrow C_4\Rightarrow$

$$\therefore\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1/12&1/6&-1/12\end{bmatrix}A\begin{bmatrix}1&2/3&-2/3&-5/3\\0&-1/3&-5/3&-2/3\\0&0&0&1\\0&0&1&0\end{bmatrix}\to(1)$$

which is the required PAQ form.

Here, $P=\begin{bmatrix}1&0&0\\-2&1&0\\-1/12&1/6&-1/12\end{bmatrix}$ and

$Q=\begin{bmatrix}1&2/3&-2/3&-5/3\\0&-1/3&-5/3&-2/3\\0&0&0&1\\0&0&1&0\end{bmatrix}$

Rank = Number of non-zero rows on LHS of (1) = 3

2c) Find the maximum & minimum values of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. (Chp: Maxima and Minima)(8)

Ans. S1: Let $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 - 30x + 72$

Let $r = f_{xx} = 6x - 30 \rightarrow (1)$

$f_y = 6xy - 30y$

Let $t = f_{yy} = 6x - 30 \rightarrow (2)$

Let $s = f_{xy} = 6y \rightarrow (3)$

S2: Put $f_x = 0$ and $f_y = 0$

$\therefore\ 3x^2 + 3y^2 - 30x + 72 = 0$

$\therefore\ x^2 + y^2 - 10x + 24 = 0 \rightarrow (4)$ (Dividing by 3)

And, $6xy - 30y = 0$

$\therefore\ 6y(x-5) = 0$

Substituting y = 0 in (4), $x^2 - 10x + 24 = 0$

$\therefore$ x = 4 or x = 6

Substituting x = 5 in (4), $5^2 + y^2 - 10 \times 5 + 24 = 0$

$\therefore\ y^2 - 1 = 0$

$\therefore$ y = 1 or y = –1

$\therefore$ Stationary Points are (4, 0); (6, 0); (5, 1); (5, –1)

S3:

(i) At (4, 0)

From (1), r = 6 (4) – 30 = –6 < 0

From (2), t = 6 (4) – 30 = –6 < 0

From (3), s = 6 (0) = 0

$\therefore\ rt - s^2 = (-6)(-6) - 0 = 36 > 0$

$\therefore$ f has maximum at (4, 0)

$\therefore\ f_{\max} = (4)^3 + 0 - 15(4)^2 - 0 + 72(4)$

$\therefore$ Maximum value $f_{\max} = 112$

(ii) At (6, 0)

From (1), r = 6 (6) – 30 = 6 > 0

From (2), t = 6 (6) – 30 = 6 > 0

From (3), s = 6 (0) = 0

$\therefore\ rt - s^2 = (6)(6) - 0 = 36 > 0$

$\therefore$ f has minimum at (6, 0)

$\therefore\ f_{\min} = (6)^3 + 0 - 15(6)^2 - 0 + 72(6)$

Minimum value $f_{\min} = 108$

(iii) At (5, ± 1)

From (1), r = 6 (5) – 30 = 0

From (2), t = 6 (5) – 30 = 0

From (3), s = 6 (± 1) = ± 6

$\therefore\ rt - s^2 = (0)(0) - (\pm 1)^2 = -1$

$\therefore$ Maximum or minimum cannot be found at (5, ± 1)

Q.3)

3a) If $u=e^{xyz}f\left(\frac{xy}{z}\right)$ then prove that $x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=2xyz\cdot u$ and $y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2xyz\cdot u$.

Hence show that $x\frac{\partial^2 u}{\partial z\partial x}=y\frac{\partial^2 u}{\partial z\partial y}$. (Chp: Partial Differentiation) (6)

Ans. $u=e^{xyz}f\left(\frac{xy}{z}\right)\to(1)$

Partially Differentiate 'u' w.r.t. 'x',

$$\frac{\partial u}{\partial x}=e^{xyz}\cdot\frac{\partial}{\partial x}f\left(\frac{xy}{z}\right)+f\left(\frac{xy}{z}\right)\cdot\frac{\partial}{\partial x}e^{xyz}$$

$$=e^{xyz}\cdot f'\left(\frac{xy}{z}\right)\frac{\partial}{\partial x}\left(\frac{xy}{z}\right)+f\left(\frac{xy}{z}\right)\cdot e^{xyz}\frac{\partial}{\partial x}(xyz)$$

$$\therefore\ \frac{\partial u}{\partial x}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot\frac{y}{z}+f\left(\frac{xy}{z}\right)\cdot yz\right]\to(2)$$

Similarly, Partially Differentiate 'u' w.r.t. 'y',

$$\frac{\partial u}{\partial y}=e^{xyz}\cdot\frac{\partial}{\partial y}f\left(\frac{xy}{z}\right)+f\left(\frac{xy}{z}\right)\cdot\frac{\partial}{\partial y}e^{xyz}$$

$$=e^{xyz}\cdot f'\left(\frac{xy}{z}\right)\frac{\partial}{\partial y}\left(\frac{xy}{z}\right)+f\left(\frac{xy}{z}\right)\cdot e^{xyz}\frac{\partial}{\partial y}(xyz)$$

$$\therefore\ \frac{\partial u}{\partial y}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot\frac{x}{z}+f\left(\frac{xy}{z}\right)\cdot xz\right]\to(3)$$

And, Partially Differentiate 'u' w.r.t. 'z',

$$\frac{\partial u}{\partial z}=e^{xyz}\cdot\frac{\partial}{\partial z}f\left(\frac{xy}{z}\right)+f\left(\frac{xy}{z}\right)\cdot\frac{\partial}{\partial z}e^{xyz}$$

$$=e^{xyz}\cdot f'\left(\frac{xy}{z}\right)\frac{\partial}{\partial z}\left(\frac{xy}{z}\right)+f\left(\frac{xy}{z}\right)\cdot e^{xyz}\frac{\partial}{\partial z}(xyz)$$

$$\therefore\ \frac{\partial u}{\partial z}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot xy\cdot\frac{-1}{z^2}+f\left(\frac{xy}{z}\right)\cdot xy\right]\to(4)$$

Now, from (2) and (4)

$$x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=x\cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot\frac{y}{z}+f\left(\frac{xy}{z}\right)\cdot yz\right]+ z\cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot xy\cdot\frac{-1}{z^2}+f\left(\frac{xy}{z}\right)\cdot xy\right]$$

$$=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot\frac{xy}{z}+f\left(\frac{xy}{z}\right)\cdot xyz\right]+ e^{xyz}\left[-f'\left(\frac{xy}{z}\right)\cdot xy\cdot\frac{1}{z}+f\left(\frac{xy}{z}\right)\cdot xyz\right]$$

$$=e^{xyz}\left\{\cancel{f'\left(\frac{xy}{z}\right)\cdot\frac{xy}{z}}+f\left(\frac{xy}{z}\right)\cdot xyz \cancel{-f'\left(\frac{xy}{z}\right)\cdot\frac{xy}{z}}+f\left(\frac{xy}{z}\right)\cdot xyz\right\}$$

$$=e^{xyz}\times 2f\left(\frac{xy}{z}\right)\cdot xyz$$

$$\therefore\ x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=2xyz\cdot u \text{ (From 1) }\to(5)$$

Similarly, from (3) and (4)

$$y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=y\cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot\frac{y}{z}+f\left(\frac{xy}{z}\right)\cdot yz\right]+ z\cdot e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot xy\cdot\frac{-1}{z^2}+f\left(\frac{xy}{z}\right)\cdot xy\right]$$

$$=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\cdot\frac{xy}{z}+f\left(\frac{xy}{z}\right)\cdot xyz\right]+ e^{xyz}\left[-f'\left(\frac{xy}{z}\right)\cdot xy\cdot\frac{1}{z}+f\left(\frac{xy}{z}\right)\cdot xyz\right]$$

$$=e^{xyz}\left\{\cancel{f'\left(\frac{xy}{z}\right)\cdot\frac{xy}{z}}+f\left(\frac{xy}{z}\right)\cdot xyz \cancel{-f'\left(\frac{xy}{z}\right)\cdot\frac{xy}{z}}+f\left(\frac{xy}{z}\right)\cdot xyz\right\}$$

$$=e^{xyz}\times 2f\left(\frac{xy}{z}\right)\cdot xyz$$

$$\therefore\ y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2xyz\cdot u \text{ (From 1) }\to(6)$$

From (5) & (6), $x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2xyz\cdot u$

$$\therefore\ x\frac{\partial u}{\partial x}=2xyz\cdot u-z\frac{\partial u}{\partial z} \text{ and } y\frac{\partial u}{\partial y}=2xyz\cdot u-z\frac{\partial u}{\partial z}$$

$$\therefore\ x\frac{\partial u}{\partial x}=y\frac{\partial u}{\partial y}$$

Partially Differentiating w.r.t. 'z', $x\frac{\partial^2 u}{\partial z\partial x}=y\frac{\partial^2 u}{\partial z\partial y}$

3b) By using Regular Falsi method solve 2x – 3 sin x – 5 = 0 correct to three decimal places.

(Chp: Transcendental equations) (6)

Ans. Let $f(x) = 2x - 3\sin x - 5 \rightarrow (1)$

Let a = 2 and b = 3

$\therefore f(2) = 2(2) - 3\sin(2) - 5 = -3.73 < 0$ and

$f(3) = 2(3) - 3\sin(3) - 5 = 0.57 > 0$

$\therefore$ Root of $f(x)$ lies between 2 and 3

By Regula Falsi Method $x = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \rightarrow (2)$

Method I:

Iteration	a	b	f(a)	f(b)	x	f(x)
1)	2	3	–3.7279	0.5766	2.8660	–0.0842
2)	2.8660	3	–0.0842	0.5766	2.8831	–0.00067
3)	2.8831	3	–0.00067	0.5766	2.8832	–0.000005
4)	2.8832	3	–0.000005	0.5766	2.8832	

Method II:

Iteration I:

Let a = 2, b = 3, $f(a) = -3.7279$ and $f(b) = 0.5766$

$\therefore$ From (2), $x_1 = \dfrac{2(0.5766) - 3(-3.7279)}{(0.5766) - (-3.7279)} = 2.8660$

$\therefore$ From (1),

$f(2.8660) = 2(2.8660) - 3\sin(2.8660) - 5$

$= -0.0842 < 0$

Iteration II:

Let a = 2.8660, b = 3, $f(a) = -0.0842$ and $f(b) = 0.5766$

$\therefore$ From (2),

$x_2 = \dfrac{2.8660(0.5766) - 3(-0.0842)}{(0.5766) - (-0.0842)} = 2.8831$

$\therefore$ From (1),

$f(2.8831) = 2(2.8831) - 3\sin(2.8831) - 5$

$= -0.00067 < 0$

Iteration III:

Let a = 2.8831, b = 3, $f(a) = -0.00067$ and

$f(b) = 0.5766$

$\therefore$ From (2),

$x_3 = \dfrac{2.8831(0.5766) - 3(-0.00067)}{(0.5766) - (-0.00067)} = 2.8832$

$\therefore$ From (1),

$f(2.8832) = 2(2.8832) - 3\sin(2.8832) - 5$

$= -0.000005 < 0$

Iteration IV:

Let a = 2.8832, b = 3, $f(a) = -0.000005$ and

$f(b) = 0.5766$

$\therefore$ From (2),

$x_4 = \dfrac{2.8832(0.5766) - 3(-0.000005)}{(0.5766) - (-0.000005)} = 2.8832$

Hence, by Regula Falsi Method, Root of the equation $f(x) = 2x - 3\sin x - 5$ is 2.8832

3c) If $y=\sin\left[\log\left(x^2+2x+1\right)\right]$ then prove that $(x+1)^2 y_{n+2}+(2n+1)(x+1)y_{n+1}+\left(n^2+4\right)y_n=0$.

(Chp: Successive Differentiation) (8)

Ans. $y=\sin\left[\log\left(x^2+2x+1\right)\right]$

$=\sin\left[\log(x+1)^2\right]$

$=\sin\left[2\log(x+1)\right] \rightarrow (1)$

Differentiating w.r.t. x, $y_1=\cos\left[2\log(x+1)\right]\cdot 2\cdot\frac{1}{x+1}$

Multiplying by (x + 1), $(x+1)y_1=2\cos\left[2\log(x+1)\right]$

Again differentiating w.r.t. x,

$\therefore\ (x+1)\cdot y_2+y_1\cdot 1=2\cdot -\sin\left[2\log(x+1)\right]\cdot 2\cdot\frac{1}{x+1}$

Multiplying by (x + 1), $(x+1)^2 y_2+(x+1)y_1=-4y$ (From 1)

Applying Leibnitz theorem,

$\therefore\ \left[(x+1)^2 y_{n+2}+n\cdot 2(x+1)y_{n+1}+\frac{n(n-1)}{2}\cdot 2(1)y_n\right]+\left[(x+1)y_{n+1}+n\cdot(1)y_n\right]=-4y_n$

$\therefore\ (x+1)^2 y_{n+2}+2n(x+1)y_{n+1}+n(n-1)y_n+(x+1)y_{n+1}+ny_n+4y_n=0$

$\therefore\ (x+1)^2 y_{n+2}+(2n+1)(x+1)y_{n+1}+\left[n(n-1)+n+4\right]y_n=0$

$\therefore\ (x+1)^2 y_{n+2}+(2n+1)(x+1)y_{n+1}+\left[n^2-n+n+4\right]y_n=0$

$\therefore\ (x+1)^2 y_{n+2}+(2n+1)(x+1)y_{n+1}+\left(n^2+4\right)y_n=0$

Q.4)

4a) State and prove Euler's Theorem for three Variables. (Chp: Homogenous Functions) (6)

Ans. Euler's theorem:

Statement: If 'u' is a homogenous function of three variables x, y, z of degree 'n' then Euler's theorem states that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

Proof:

Let u = f(x, y, z) be the homogenous function of degree 'n'.

Let X = xt, Y = yt , Z = zt

$$\therefore \frac{\partial X}{\partial t}=x;\ \frac{\partial Y}{\partial t}=y\ \ \&\ \frac{\partial Z}{\partial t}=z \rightarrow (1)$$

At t = 1, → (2)

X = x, Y = y and Z = z

$$\therefore \frac{\partial f}{\partial X}=\frac{\partial f}{\partial x};\ \frac{\partial f}{\partial Y}=\frac{\partial f}{\partial y}\ \&\ \frac{\partial f}{\partial Z}=\frac{\partial f}{\partial z} \rightarrow (3)$$

Now, $f(X,Y,Z)=t^n f(x,y,z) \rightarrow (4)$

$\therefore f \rightarrow$ X, Y, Z $\rightarrow$ x, y, z, t

Differentiating (4) partially w.r.t. 't', $\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial t}+\frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial t}+\frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial t}=nt^{n-1}f(x,y,z)$

$$\therefore \frac{\partial f}{\partial x}\cdot x+\frac{\partial f}{\partial y}\cdot y+\frac{\partial f}{\partial z}\cdot z=n(1)^{n-1}f(x,y,z)\ \text{(From 1, 2 \& 3)}$$

$$\therefore \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu} \rightarrow (5)$$

Q.4)

4a) Obtain tan 5θ in terms of tan θ and show that $1-10\tan^2\frac{\pi}{10}+5\tan^4\frac{\pi}{10}=0$. (Chp: Complex - DMT) (6)

Ans. By De Moivre's Theorem, $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

Let c = cos θ and s = sin θ

$\therefore\ \cos 5\theta + i\sin 5\theta = (c+is)^5$

$= c^5 + 5c^4 \cdot is + 10c^3 \cdot i^2 s^2 + 10c^2 \cdot i^3 s^3 + 5c \cdot i^4 s^4 + i^5 s^5$

$= c^5 + i5c^4 s - 10c^3 s^2 - i10c^2 s^3 + 5cs^4 + is^5$

$\therefore\ \cos 5\theta + i\sin 5\theta = \left(c^5 - 10c^3 s^2 + 5cs^4\right) + i\left(5c^4 s - 10c^2 s^3 + s^5\right)$

Comparing real and Imaginary parts, $\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$ & $\sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5 \rightarrow (1)$

Now, $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$\therefore\ \tan 5\theta = \dfrac{5c^4 s - 10c^2 s^3 + s^5}{c^5 - 10c^3 s^2 + 5cs^4}$ (From 1)

$\therefore\ \tan 5\theta = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$

Dividing N and D by $\cos^5\theta$, $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

Put $\theta = \dfrac{\pi}{10}$,

$\therefore\ \tan\left(\dfrac{5\pi}{10}\right) = \dfrac{5\tan(\pi/10) - 10\tan^3(\pi/10) + \tan^5(\pi/10)}{1 - 10\tan^2(\pi/10) + 5\tan^4(\pi/10)}$

$\therefore\ \dfrac{1}{0} = \dfrac{5\tan(\pi/10) - 10\tan^3(\pi/10) + \tan^5(\pi/10)}{1 - 10\tan^2(\pi/10) + 5\tan^4(\pi/10)}$ $\left\{\because \tan\left(\dfrac{5\pi}{10}\right) = \tan\dfrac{\pi}{2} = \text{ N.D.} = \dfrac{1}{0}\right\}$

$\therefore$ On cross multiplication, $1 - 10\tan^2\dfrac{\pi}{10} + 5\tan^4\dfrac{\pi}{10} = 0$

Q.3)

3a) Investigate for what values of λ and μ the system of equations: x + y + z = 6; x + 2y + 3z = 10; x + 2y + λz = μ has (i) no solution, (ii) a unique solution, (iii) an infinite no. of solutions. (Chp: Linear Equations) (6)

Ans. Writing the equations in the matrix form, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

$R_3 - R_2;\ R_2 - R_1;$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \mu - 10 \end{bmatrix}$

Augmented Matrix [A | B] $= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$

Number of unknowns = n = 3

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when μ ≠ 10 and λ = 3.

We then have, rank of A (r_A) = 2 and rank of [A | B] (r_{AB}) = 3

Case II: Unique Solution

For which, $r_A = r_{AB} = n$

This is only possible, when λ ≠ 3 and μ has any value.

We then have, rank of A (r_A) = rank of [A | B] (r_{AB}) = 3

Case III: Infinite Solution

For which, $r_A = r_{AB} < 3$ (i.e. < 3)

This is only possible, when μ = 10 and λ = 3.

We then have, rank of A (r_A) = rank of [A | B] (r_{AB}) = 2

Hence,

No Solution	μ ≠ 10, λ = 3
Unique Solution	μ = any value, λ ≠ 3
Infinite Solution	μ = 10, λ = 3

Q.5)

5a) Find nth derivative of $y=\frac{1}{x^2+a^2}$. (Chp: Successive Differentiation) (6)

Ans. $y=\frac{1}{x^2+a^2}=\frac{1}{(x+ai)(x-ai)}$

$=\frac{1/2ai}{(x-ai)}-\frac{1/2ai}{(x+ai)}$ (By Partial Fractions)

Taking n^{th} derivative,

$$y_n=\frac{1}{2ai}\times\frac{n!1^n(-1)^n}{(x-ai)^{n+1}}-\frac{1}{2ai}\times\frac{n!1^n(-1)^n}{(x+ai)^{n+1}}$$

$$\left\{\text{If } y=\frac{1}{ax+b} \text{ then } y_n=\frac{n!a^n(-1)^n}{(ax+b)^{n+1}}\right\}$$

$$\therefore\ y_n=\frac{1}{2ai}\times n!(-1)^n\left[\frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}\right]$$

Let $x+ai=re^{i\theta}$, where,

$$r=\sqrt{x^2+a^2}\rightarrow(1)\ \&\ \theta=\tan^{-1}\frac{a}{x}\rightarrow(2)$$

$$\therefore\ x-ai=re^{-i\theta}$$

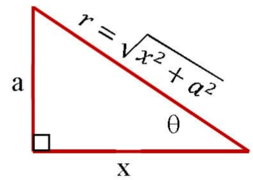
$$\therefore\ y_n=\frac{n!}{2ai}(-1)^n\left[\frac{1}{\left(re^{i\theta}\right)^{n+1}}-\frac{1}{\left(re^{-i\theta}\right)^{n+1}}\right]$$

$$=\frac{n!}{2ai}(-1)^n\left[\frac{1}{r^{n+1}e^{i\theta(n+1)}}-\frac{1}{r^{n+1}e^{-i\theta(n+1)}}\right]$$

$$=\frac{n!}{2ai}(-1)^n\times\frac{-1}{r^{n+1}}\left[-e^{-i\theta(n+1)}+e^{i\theta(n+1)}\right]$$

$$=\frac{n!}{\cancel{2i}a}(-1)^{n+1}\times\frac{1}{r^{n+1}}\times\cancel{2i}\sin(n+1)\theta\rightarrow(3)$$

From (2), $\tan\theta=\frac{a}{x}$



$\therefore$ From figure, $\sin\theta=\frac{a}{r}$

$$\therefore\ r=\frac{a}{\sin\theta}$$

$$\therefore\ \frac{1}{r}=\frac{\sin\theta}{a}\rightarrow(4)$$

$\therefore$ From (3) & (4),

$$y_n=\frac{n!}{a}(-1)^{n+1}\times\left(\frac{\sin\theta}{a}\right)^{n+1}\sin(n+1)\theta$$

$$\therefore\ \boxed{y_n=\frac{n!}{a^{n+2}}(-1)^{n+1}\sin^{n+1}\theta\sin(n+1)\theta}$$

5b) If z = f (x, y), $x = e^{u} + e^{-v}$, $y = e^{u} - e^{-v}$. then prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$. (Chp: Partial Differentiation) (6)

Ans. Given, $x = e^{u} + e^{-v}$ and $y = e^{u} - e^{-v} \rightarrow (1)$

$$\frac{\partial x}{\partial v} = -e^{-v};\ \frac{\partial x}{\partial u} = e^{u};\ \frac{\partial y}{\partial v} = e^{-v};\ \frac{\partial y}{\partial u} = e^{u} \rightarrow (2)$$

Now, z $\rightarrow$ x, y $\rightarrow$ u, v

Differentiate partially w.r.t. x, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$

$$\therefore\ \frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} + e^{-v}\frac{\partial z}{\partial y} \text{ (From 2)} \rightarrow (3)$$

Similarly, $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$

$$\therefore\ \frac{\partial z}{\partial u} = e^{u}\frac{\partial z}{\partial x} + e^{u}\frac{\partial z}{\partial y} \text{ (From 2)} \rightarrow (4)$$

$$\text{LHS} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$= \left(e^{u}\frac{\partial z}{\partial x} + e^{u}\frac{\partial z}{\partial y}\right) - \left(-e^{-v}\frac{\partial z}{\partial x} + e^{-v}\frac{\partial z}{\partial y}\right) \text{ (From 3 \& 4)}$$

$$= \frac{\partial z}{\partial x}\left(e^{u} + e^{-v}\right) + \frac{\partial z}{\partial y}\left(e^{u} - e^{-v}\right)$$

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

= RHS (From 1)

$$\therefore\ \boxed{\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}}$$

5c) Solve-by using Gauss Jacobi Iteration method: 2x + 12y + z – 4w = 13; 13x + 5y – 3z + w = 18; 2x + y – 3z + 9w = 31; 3x – 4y + 10z + w = 29. (Chp: Linear algebraic equations) (8)

Ans Re-arranging the given equations to get a Diagonal Dominant Matrix,

13x + 5y – 3z + w = 18;

2x + 12y + z – 4w = 13;

3x – 4y + 10z + w = 29

2x + y – 3z + 9w = 31;

From 1st equation, 13x = 18 – 5y + 3z – w

$$\therefore\ x=\frac{1}{13}(18-5y+3z-w)=13^{-1}(18-5y+3z-w)$$

From 2nd equation, 12y = 13 – 2x – z + 4w

$$\therefore\ y=12^{-1}(13-2x-z+4w)$$

Similarly, $z=10^{-1}(29-3x+4y-w)$ and, $w=9^{-1}(31-2x-y+3z)$

Iteration 1:

Put $x_0=y_0=z_0=w_0=0$

$$\therefore\ x_1=13^{-1}(18-5y_0+3z_0-w_0)=13^{-1}(18-0+0-0)=1.3846$$

$$\therefore\ y_1=12^{-1}(13-2x_0-z_0+4w_0)=12^{-1}(13-0-0+0)=1.0833$$

$$\therefore\ z_1=10^{-1}(29-3x_0+4y_0-w_0)=10^{-1}(29-0+0-0)=2.9$$

$$\therefore\ w_1=9^{-1}(31-2x_0-y_0+3z_0)=9^{-1}(31-0-0+0)=3.4444$$

Iteration 2:

Put $x_1=1.3846;\ y_1=1.0833;\ z_1=2.9;\ w_1=3.4444$

$$\therefore\ x_2=13^{-1}(18-5y_1+3z_1-w_1)=13^{-1}(18-5\times1.0833+3\times2.9-3.4444)=1.3722$$

$$\therefore\ y_2=12^{-1}(13-2x_1-z_1+4w_1)=12^{-1}(13-2\times1.3846-2.9+4\times3.4444)=1.7590$$

$$\therefore\ z_2=10^{-1}(29-3x_1+4y_1-w_1)=10^{-1}(29-3\times1.3846+4\times1.0833-3.4444)=2.5735$$

$$\therefore\ w_2=9^{-1}(31-2x_1-y_1+3z_1)=9^{-1}(31-2\times1.3846-1.0833+3\times2.9)=3.9831$$

Iteration 3:

Put $x_2=1.3722;\ y_2=1.7590;\ z_2=2.5735;\ w_2=3.9831$

$$\therefore\ x_3=13^{-1}(18-5y_2+3z_2-w_2)=13^{-1}(18-5\times1.7590+3\times2.5735-3.9831)=0.9956$$

$$\therefore\ y_3=12^{-1}(13-2x_2-z_2+4w_2)=12^{-1}(13-2\times1.3722-2.5735+4\times3.9831)=1.9679$$

$$\therefore\ z_3=10^{-1}(29-3x_2+4y_2-w_2)=10^{-1}(29-3\times1.3722+4\times1.7590-3.9831)=2.7936$$

$$\therefore\ w_3=9^{-1}(31-2x_2-y_2+3z_2)=9^{-1}(31-2\times1.3722-1.7590+3\times2.5735)=3.8019$$

Iteration 4:

Put $x_3=0.9953;\ y_3=1.9679;\ z_3=2.7936;\ w_3=3.8019$

$$\therefore\ x_4=13^{-1}(18-5y_3+3z_3-w_3)=13^{-1}(18-5\times1.9679+3\times2.7936-3.8019)=0.9800$$

$$\therefore\ y_4=12^{-1}(13-2x_3-z_3+4w_3)=12^{-1}(13-2\times0.9953-2.7936+4\times3.8019)=1.9519$$

$$\therefore\ z_4=10^{-1}(29-3x_3+4y_3-w_3)=10^{-1}(29-3\times0.9953+4\times1.9679-3.8019)=3.0083$$

$$\therefore\ w_4=9^{-1}(31-2x_3-y_3+3z_3)=9^{-1}(31-2\times0.9953-1.9679+3\times2.7936)=3.9358$$

Solving similarly, we get,

Iteration No.	x_i	y_i	z_i	w_i
5	1.0254	1.9812	2.9932	4.0126
6	1.0047	2.0005	2.9836	3.9942
7	0.9965	1.9986	2.9994	3.9934
8	1.0009	1.9985	3.0012	4.0007
9	1.0008	2.0000	2.9990	4.0004
10	0.9998	2.0001	2.9997	3.9995

Hence, by Gauss Jacobi Iteration Method, the solution is x = 1, y = 2, z = 3, w = 4

1) If $u=\log\left(\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]\right)$, then PT.

(i) cosh u = sec θ; (ii) sinh u = tan θ; (iii) tanh u = sin θ; (iv) $\tanh\frac{u}{2}=\tan\frac{\theta}{2}$. (v) coshu cosh x = 1 (8)

Ans. $u=\log\left(\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]\right)$

$$\therefore\ u=\log\left[\frac{\tan(\pi/4)+\tan(\theta/2)}{1-\tan(\pi/4)\tan(\theta/2)}\right]$$

$$\therefore\ e^u=\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}$$

By Componendo- Dividendo,

$$\frac{e^u+1}{e^u-1}=\frac{\left[1+\cancel{\tan(\theta/2)}\right]+\left[1-\cancel{\tan(\theta/2)}\right]}{\left[\cancel{1}+\tan(\theta/2)\right]-\left[\cancel{1}-\tan(\theta/2)\right]}$$

$$\therefore\ \frac{\cancel{e^{u/2}}\left(e^{u/2}+\frac{1}{e^{u/2}}\right)}{\cancel{e^{u/2}}\left(e^{u/2}-\frac{1}{e^{u/2}}\right)}=\frac{\cancel{2}}{\cancel{2}\tan(\theta/2)}$$

$$\therefore\ \frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}}=\frac{1}{\tan(\theta/2)}$$

$$\therefore\ \frac{\cancel{2}\cosh(u/2)}{\cancel{2}\sinh(u/2)}=\frac{1}{\tan(\theta/2)}$$

$$\therefore\ \frac{\sinh(u/2)}{\cosh(u/2)}=\tan(\theta/2)$$

$$\therefore\ \boxed{\tanh\left(\frac{u}{2}\right)=\tan\left(\frac{\theta}{2}\right)}\rightarrow(1)$$

Now, $\sin\theta=\frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}$

$$=\frac{2\tanh(u/2)}{1+\tanh^2(u/2)}\ \text{(From 1)}$$

$$=\tanh u$$

$$\therefore\ \boxed{\sin\theta=\tanh u}$$

Also, $\tan\theta=\frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$

$$=\frac{2\tanh(u/2)}{1-\tanh^2(u/2)}\ \text{(From 1)}$$

$$=\sinh u\ \rightarrow(2)$$

$$\therefore\ \boxed{\tan\theta=\sinh u}$$

And, $\sec^2\theta=1+\tan^2\theta$

$$=1+\sinh^2 u$$

$$=\cosh^2 u\ \text{(From 2)}$$

$$\therefore\ \boxed{\sec\theta=\cosh u}$$

$$\therefore\ \frac{1}{\cos\theta}=\cosh u$$

$$\therefore\ \boxed{\cosh u\cdot\cos\theta=1}$$

6b) If $u=\sin^{-1}\left(\dfrac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\tan u}{144}\left(\tan^2 u+13\right)$.

(Chp: Homogenous Functions) (6)

Ans. $u(x,y)=\sin^{-1}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$

$$\therefore\ \sin u(x,y)=\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}\rightarrow(1)$$

$$\therefore\ \sin u(X,Y)=\left(\frac{X^{1/3}+Y^{1/3}}{X^{1/2}-Y^{1/2}}\right)^{1/2}$$

Now, Put X = x t, Y = y t

$$\therefore\ \sin u(X,Y)=\left[\frac{(xt)^{1/3}+(yt)^{1/3}}{(xt)^{1/2}-(yt)^{1/2}}\right]^{1/2}$$

$$=\left[\frac{t^{1/3}\left(x^{1/3}+y^{1/3}\right)}{t^{1/2}\left(x^{1/2}-y^{1/2}\right)}\right]^{1/2}$$

$$=\left[t^{\frac{1}{3}-\frac{1}{2}}\right]^{1/2}\left[\frac{\left(x^{1/3}+y^{1/3}\right)}{\left(x^{1/2}-y^{1/2}\right)}\right]^{1/2}$$

$$=\left[t^{-1/6}\right]^{1/2}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right]^{1/2}$$

$\sin u(X,Y)=t^{-1/12}\sin u(x,y)$ (From 1)

$\therefore$ sin u is homogenous function of degree $(n)=\dfrac{-1}{12}$.

Let $f(u)=\sin u$

$\therefore\ f'(u)=\cos u$

Let $g(u)=n\dfrac{f(u)}{f'(u)}$

$$=\frac{-1}{12}\cdot\frac{\sin u}{\cos u}$$

$$=\frac{-\tan u}{12}$$

$$\therefore\ g'(u)=\frac{-1}{12}\sec^2 u$$

Using formula,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\cdot\left[g'(u)-1\right]$$

$$\therefore\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{-\tan u}{12}\cdot\left[\frac{-1}{12}\sec^2 u-1\right]$$

$$=\frac{\tan u}{12}\cdot\left[\frac{1}{12}\left(1+\tan^2 u\right)+1\right]$$

$$=\frac{\tan u}{12}\cdot\frac{1}{12}\left[1+\tan^2 u+12\right]$$

$$\therefore\ \boxed{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}\left(\tan^2 u+13\right)}$$

3d) Expand $2x^3+7x^2+x-6$ in powers of x – 2. (Chp: Expansion) (4)

Ans. Let $f(x)=2x^3+7x^2+x-6$

$\therefore\ f'(x)=6x^2+14x+1$

$\therefore\ f''(x)=12x+14$

$\therefore\ f'''(x)=12$

Let a = 2,

$\therefore\ f(a)=f(2)=2(2)^3+7(2)^2+(2)-6=40$

$\therefore\ f'(a)=f'(2)=6(2)^2+14(2)+1=53$

$\therefore\ f''(a)=f''(2)=12(2)+14=38$

$\therefore\ f'''(a)=f'''(2)=12$

By Taylor Series, $f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$

$\therefore\ f(x)=40+(x-2)\cdot 53+\frac{1}{2}(x-2)^2\cdot 38+\frac{1}{6}(x-2)^3\cdot 12$

$\therefore\ 2x^3+7x^2+x-6=40+53(x-2)+19(x-2)^2+2(x-2)^3$

6d) Expand sec x by Maclaurins theorem considering upto x^4 term. (Chp: Expansion) (4)

Ans. Let $f(x) = \sec x$

$\therefore\ f'(x) = \sec x \tan x$

$f''(x) = \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$

$= \sec x\left(\sec^2 x + \tan^2 x\right)$

$= \sec x\left(\sec^2 x + \sec^2 x - 1\right)$

$= \sec x\left(2\sec^2 x - 1\right)$

$\therefore\ f''(x) = 2\sec^3 x - \sec x$

$\therefore\ f'''(x) = 2\times 3\sec^2 x \cdot \sec x \tan x - \sec x \tan x$

$\therefore\ f'''(x) = 6\sec^3 x \cdot \tan x - \sec x \cdot \tan x$

$\therefore\ f^{iv}(x) = 6\left(\sec^3 x \cdot \sec^2 x + \tan x \cdot 3\sec^2 x \cdot \sec x \tan x\right) - \left(\sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x\right)$

$\therefore\ f^{iv}(x) = 6\left(\sec^5 x + 3\sec^3 x \tan^2 x\right) - \left(\sec^3 x + \sec x \tan^2 x\right)$

At x = 0,

$f(0) = \sec 0 = 1$

$f'(0) = \sec 0 \cdot \tan 0 = 0$

$f''(0) = 2\sec^3 0 - \sec 0 = 2\times 1 - 1 = 1$

$f'''(x) = 6\sec^3 0 \cdot \tan 0 - \sec 0 \cdot \tan 0 = 0$

$f^{iv}(x) = 6\left(\sec^5 0 + 3\sec^3 0 \tan^2 0\right) - \left(\sec^3 0 + \sec 0 \tan^2 0\right) = 6(1+0) - (1+0) = 5$

By Maclaurin's theorem, $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$

$\therefore\ \sec x = 1 + x(0) + \frac{x^2}{2!}\times(1) + \frac{x^3}{3!}\times(0) + \frac{x^4}{4!}\times(5) + \ldots$

$\therefore\ \boxed{\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots}$