Homogeneous Functions

Definition
A function $f(x, y)$ of two independent variables x and y is said to be **Homogeneous** if degree of variables in each terms are equal

$$
\begin{aligned}
f(x, y) &= a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + ... + a_{n-1} x y^{n-1} + a_n y^n \\
&= x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + ... + a_n \left(\frac{y}{x}\right)^n \right] \\
f(x, y) &= x^n \ \phi\left(\frac{y}{x}\right)
\end{aligned}
$$

where n is an integer and it is also known as **degree of Homogeneous Function**
Condition:
A two variable function $f(x, y)$ is called **Homogeneous function** if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Euler's Theorem (Two variables)
Statement
If $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Proof
Given $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n
$\therefore$ by definition

$$u = f(x, y) = x^n \ \phi\left(\frac{y}{x}\right) \text{..............}\mathbf{(1)}$$

Partially differentiating (1) w.r.t x we have

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= x^n \frac{\partial}{\partial x}\left[\phi\left(\frac{y}{x}\right)\right] + \phi\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(x^n\right) \\
&= x^n\left[\phi'\left(\frac{y}{x}\right)\right]\ \frac{\partial}{\partial x}\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)\left(nx^{n-1}\right) \\
&= x^n\left[\phi'\left(\frac{y}{x}\right)\right]\left(\frac{-y}{x^2}\right) + \phi\left(\frac{y}{x}\right)\left(nx^{n-1}\right) \\
\frac{\partial u}{\partial x} &= nx^{n-1}\ \phi\left(\frac{y}{x}\right) - x^{n-2}y\ \phi'\left(\frac{y}{x}\right) \\
\implies x\ \frac{\partial u}{\partial x} &= nx^n\ \phi\left(\frac{y}{x}\right) - x^{n-1}y\ \phi'\left(\frac{y}{x}\right) \text{.........}\mathbf{(2)}
\end{aligned}
$$

Again Partially differentiating (1) w.r.t y we have

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= x^n \frac{\partial}{\partial y}\left[\phi\left(\frac{y}{x}\right)\right] + \phi\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(x^n\right) \\
&= x^n\left[\phi'\left(\frac{y}{x}\right)\right]\ \frac{\partial}{\partial y}\left(\frac{y}{x}\right) + 0 \\
&= x^n\left[\phi'\left(\frac{y}{x}\right)\right]\left(\frac{1}{x}\right) \\
\frac{\partial u}{\partial y} &= x^{n-1}\ \phi'\left(\frac{y}{x}\right) \\
\implies y\ \frac{\partial u}{\partial y} &= x^{n-1}y\ \phi'\left(\frac{y}{x}\right) \text{.........}\mathbf{(3)}
\end{aligned}
$$

Adding (2) and (3) and using (1),we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\ \phi\left(\frac{y}{x}\right) = nu$$

Corollary 1 (Deduction 1)
Statement
If $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Proof
Given $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n
$\therefore$ by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\ \phi\left(\frac{y}{x}\right) = nu.............\textbf{(1)}$$

Partially differentiating (1) w.r.t x we have

$$\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[y\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[nu\right]$$
$$x\left[\frac{\partial^2 u}{\partial x^2}\right] + \frac{\partial u}{\partial x} + y\left[\frac{\partial^2 u}{\partial x \partial y}\right] = n\frac{\partial u}{\partial x}.........\textbf{(2)}$$

Again Partially differentiating (1) w.r.t y we have

$$\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial y}\left[nu\right]$$
$$x\left[\frac{\partial^2 u}{\partial y \partial x}\right] + y\left[\frac{\partial^2 u}{\partial y^2}\right] + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}.........\textbf{(3)}$$

Adding (2) and (3) and using (1),we have

$$x^2\left[\frac{\partial^2 u}{\partial x^2}\right] + xy\left[\frac{\partial^2 u}{\partial x \partial y}\right] + xy\left[\frac{\partial^2 u}{\partial y \partial x}\right] + y^2\left[\frac{\partial^2 u}{\partial y^2}\right] = (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

For Homogeneous functions mixed partial derivatives are equal hence

$$x^2\left[\frac{\partial^2 u}{\partial x^2}\right] + xy\left[\frac{\partial^2 u}{\partial x \partial y}\right] + xy\left[\frac{\partial^2 u}{\partial y \partial x}\right] + y^2\left[\frac{\partial^2 u}{\partial y^2}\right] = (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

by(1)

$$x^2\left[\frac{\partial^2 u}{\partial x^2}\right]+2xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]+y^2\left[\frac{\partial^2 u}{\partial y^2}\right]=n(n-1)u$$

Hence Proved.

Corollary 2 (Deduction 2)
Statement
If $z(x, y) = f(u)$ is a homogeneous function of two independent variables x and y of degree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}$$

Proof
Given $z(x, y) = f(u)$ is a homogeneous function of two independent variables x and y of degree n
$\therefore$ by Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.............\mathbf{(1)}$$

Given $z = f(u)$
Partially differentiating (1) w.r.t x and y we have

$$\frac{\partial z}{\partial x}=f^{'}(u)\frac{\partial u}{\partial x}.....\mathbf{(2)}$$

and

$$\frac{\partial z}{\partial y}=f^{'}(u)\frac{\partial u}{\partial y}.....\mathbf{(3)}$$

Using (2) and (3) in (1), we have

$$
\begin{aligned}
x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} &= nz \\
\therefore x\ f^{'}(u)\frac{\partial u}{\partial x}+y\ f^{'}(u)\frac{\partial u}{\partial y} &= n\ f(u) \\
\therefore f^{'}(u)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right] &= n\ f(u) \\
\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}
\end{aligned}
$$

Corollary 3 (Deduction 3)
Statement
If $z(x,y)=f(u)$ is a homogeneous function of two independent variables x and y of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g^{'}(u)-1]$$

where $g(u)=n\ \frac{f(u)}{f'(u)}$ **Proof**
Given $z(x,y)=f(u)$ is a homogeneous function of two independent variables x and y of degree n
$\therefore$ by Deduction to Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}\text{.............}\mathbf{(1)}$$

Let $g(u)=n\ \frac{f(u)}{f'(u)}$
Substituting in (1) we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ g(u)\text{.............}\mathbf{(2)}$$

Partially differentiation (2),w.r.t. x, we have

$$\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial x}\left[y\frac{\partial u}{\partial y}\right]=\frac{\partial}{\partial x}\left[g(u)\right]$$
$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g^{'}(u)\frac{\partial u}{\partial x}\text{.............}\mathbf{(3)}$$

Partially differentiation (2),w.r.t. y, we have

$$\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[y\frac{\partial u}{\partial y}\right]=\frac{\partial}{\partial y}\left[g(u)\right]$$
$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=g^{'}(u)\frac{\partial u}{\partial y}\text{.............}\mathbf{(4)}$$

$x(3)+y(4)$ we have

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+2xy\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=xg^{'}(u)\frac{\partial u}{\partial x}+yg^{'}(u)\frac{\partial u}{\partial y}$$
$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(g^{'}(u)-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$
$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)(g^{'}(u)-1)$$

Hence Proved

Corollary 4 (Deduction 4)
Statement
If $u=f(v)$ where v is a homogeneous function of two independent variables x and y of degree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nvf^{'}(v)$$

Proof
Given $u=f(v)$ where v is a homogeneous function of two independent variables x and y of degree n

$\therefore$ by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv.....\mathbf{(1)}$$

Also

$$u = f(v)$$

$$\therefore \frac{\partial u}{\partial x} = f^{'}(v)\frac{\partial v}{\partial x}.......\mathbf{(2)}$$

and

$$\frac{\partial u}{\partial y} = f^{'}(v)\frac{\partial v}{\partial y}.....\mathbf{(3)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf^{'}(v)\frac{\partial v}{\partial x} + yf^{'}(v)\frac{\partial v}{\partial y}$$

$$= f^{'}(v)\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right]$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf^{'}(v)$$

Examples

Example (Type 1)
Find values of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x^2}$$

if

(1) $u = \frac{\sqrt{x}+\sqrt{y}}{x+y}$

(2) $u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Solution
(1)Given

$$\begin{aligned} u(x,y) &= \frac{\sqrt{x}+\sqrt{y}}{x+y} \\ \therefore u(\lambda x, \lambda y) &= \frac{\sqrt{\lambda x}+\sqrt{\lambda y}}{\lambda x+\lambda y} \\ &= \frac{(\lambda)^{\frac{1}{2}}\left(\sqrt{x}+\sqrt{y}\right)}{\lambda(x+y)} \\ &= \lambda^{-\frac{1}{2}}\frac{\left(\sqrt{x}+\sqrt{y}\right)}{(x+y)} \\ \therefore u(\lambda x, \lambda y) &= \lambda^{-\frac{1}{2}}u(x,y) \end{aligned}$$

$\therefore u$ is a homogeneous function of degree $n = -\frac{1}{2}$

$\therefore$ by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(-\frac{1}{2}\right) u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(-\frac{1}{2}\right) \; \frac{\sqrt{x}+\sqrt{y}}{x+y} \text{.......}\mathbf{(1)}$$

Also by deduction to Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x^2} = n(n-1)u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x^2} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right) u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x^2} = \left(\frac{3}{4}\right) \; \frac{\sqrt{x}+\sqrt{y}}{x+y} \text{.......}\mathbf{(2)}$$

(1) and (2) are required results

(2)

$$u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\implies tan\ u = \frac{x^3+y^3}{x-y}$$

Let $z = f(u) = tan\ u$ and

$$z(x,y) = \frac{x^3+y^3}{x-y}$$

$$\therefore z(\lambda x, \lambda y) = \frac{(\lambda x)^3 + (\lambda y)^3}{\lambda x - \lambda y}$$

$$= \frac{(\lambda)^3\,(x^3 + y^3)}{\lambda(x - y)}$$

$$= \lambda^2 \frac{(x^3 + y^3)}{(x - y)}$$

$$\therefore z(\lambda x, \lambda y) = \lambda^2 z(x, y)$$

$\therefore z$ is a homogeneous function in x and y of degree $n = 2$ and $z = f(u)$
$\therefore$ by Deduction to Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\left(\frac{tan\ u}{sec^2\ u}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\,(sin\ u\ cos\ u)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin\ 2u$$

Let $g(u) = sin\ 2u$

Then by deduction to Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}=g(u)\left[g^{'}(u)-1\right]$$
$$=(sin\ 2u)\left[2\ cos\ 2u-1\right]$$
$$=2\ sin\ u\ cos\ u\left[2\ (2\ cos^2\ u-1)-1\right]$$
$$=2\ sin\ u\ cos\ u\left[4\ cos^2\ u-3\right]$$
$$=2\ sin\ u\left[4\ cos^3\ u-3\ cos\ u\right]$$
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}=2\ sin\ u\ cos\ 3u.......\mathbf{(2)}$$

(1) and (2) are required results

Example (Type 2)
Find values of

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}$$

if
$u=e^{x+y}+log(x^3+y^3-x^2y-xy^2)$

Solution
Given

$$u=e^{x+y}+log(x^3+y^3-x^2y-xy^2)$$

Let

$$u(x,y)=v+w$$

where $v(x,y)=e^{x+y}$ and $w(x,y)=log(x^3+y^3-x^2y-xy^2)$
Now

$$v(x,y)=e^{x+y}$$
$$\therefore log\ v=x+y$$

Let $f(v) = log\ v$ and

$$
\begin{aligned}
z(x,y) &= x+y \\
\therefore z(\lambda x, \lambda y) &= \lambda x + \lambda y \\
\therefore z(\lambda x, \lambda y) &= \lambda(x+y) \\
\therefore z(\lambda x, \lambda y) &= \lambda z(x,y)
\end{aligned}
$$

$\therefore z$ is a homogeneous function of degree $n = 1$ and $z = f(v) = log\ v$ $\therefore$ by deduction to Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\frac{f(v)}{f'(v)}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1\frac{log\ v}{\frac{1}{v}}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v\ log\ v.......\mathbf{(1)}$$

and
for $g(v) = v\ log\ v$

$$
\begin{aligned}
x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial x^2} &= g(v)\left[g^{'}(v) - 1\right] \\
&= v\ log\ v\left[log\ v + 1 - 1\right] \\
&= v\ (log\ v)^2.......\mathbf{(2)}
\end{aligned}
$$

Also

$$
\begin{aligned}
w(x,y) &= log(x^3 + y^3 - x^2y - xy^2) \\
\therefore e^w &= x^3 + y^3 - x^2y - xy^2
\end{aligned}
$$

Let $g(w) = e^w$ and

$$
\begin{aligned}
p(x,y) &= x^3 + y^3 - x^2y - xy^2 \\
\therefore p(\lambda x, \lambda y) &= (\lambda x)^3 + (\lambda y)^3 + (\lambda x)^2\lambda y - (\lambda y)^2\lambda x \\
\therefore p(\lambda x, \lambda y) &= (\lambda)^3(x^3 + y^3 - x^2y - xy^2) \\
\therefore p(\lambda x, \lambda y) &= \lambda^3\ p(x,y)
\end{aligned}
$$

$\therefore$ p is a homogeneous function of degree $n = 3$ and $p = f(w) = e^w$ $\therefore$ by deduction to Euler's theorem

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = n\frac{f(w)}{f'(w)}$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 3\frac{e^w}{e^w}$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 3.......\mathbf{(3)}$$

and
for $g(w) = 3$

$$\begin{aligned} x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial x^2} &= g(w)\left[g^{'}(w) - 1\right] \\ &= 3\left[0 - 1\right] \\ &= -3.......\mathbf{(4)} \end{aligned}$$

Adding(1),(3)

$$\begin{aligned} x\left[\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right] + y\left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right] &= v\ log\ v + 3 \\ \left[x\frac{\partial}{\partial x}\right](v + w) + \left[y\frac{\partial}{\partial y}\right](v + w) &= e^{(x+y)}(x + y) + 3 \\ \left[x\frac{\partial}{\partial x}\right](u) + \left[y\frac{\partial}{\partial y}\right](u) &= e^{(x+y)}(x + y) + 3 \\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= e^{(x+y)}(x + y) + 3........\mathbf{(5)} \end{aligned}$$

Adding(2),(4)

$$x^2\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right]+2xy\left[\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right]+y^2\left[\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right]=v\ (log\ v)^2-3$$

$$\left[x^2\frac{\partial^2}{\partial x^2}\right](v+w)+\left[2xy\frac{\partial^2}{\partial x\partial y}\right](v+w)+\left[y^2\frac{\partial^2}{\partial y^2}\right](v+w)=e^{(x+y)}(x+y)^2-3$$

$$\left[x^2\frac{\partial^2}{\partial x^2}\right](u)+\left[2xy\frac{\partial^2}{\partial x\partial y}\right](u)+\left[y^2\frac{\partial^2}{\partial y^2}\right](u)=e^{(x+y)}(x+y)^2-3$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=e^{(x+y)}(x+y)^2-3.....\mathbf{(6)}$$

Adding(5) and (6)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}=e^{(x+y)}(x+y)+3+e^{(x+y)}(x+y)^2-3$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}=e^{(x+y)}(x+y)+e^{(x+y)}(x+y)^2$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}-y^2\frac{\partial^2 u}{\partial x^2}=e^{(x+y)}(x+y)\left[1+x+y\right]$$