COMPLEX NUMBERS

PREFACE

- The real number system had limitations that were at first accepted and later overcome by a series of improvements in both concepts and mechanics.
- In connection with, quadratic, equations we encountered the the situation where there is no real numbers astifying specific quadratic equations that requires sruareroot of negative numbers.
- as a result, a broader number system was devised so that such equations possess its solution in that number system, which was eventually named as **Complex Numbers** by Carl Friedrich Gauss.

INTRODUCTION

- A complex number is a number of the form $a + bi$ where a and b are real and$i^2 = -1 or i = \sqrt{-1}$. The letter $'a'$ is called the real part and 'b' is called the imaginary part of $a + bi$
- If $a = 0$, the number ib is said to be a purely imaginary number and if $b = 0$ the number a is real.
  Hence, real numbers and pure imaginary numbers are special cases of complex numbers.
- The complex numbers are denoted by Z , i.e., $Z = a + bi$. In coordinate form,$Z = (a, b)$

Note : Every real number is a complex number with 0 as its imaginary part.

PROPERTIES OF COMPLEX NUMBERS

- **The two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$**
  for example if.

$$(x\text{–}2) + 4yi = 3 + 12i$$
$$x\text{–}2 = 3, \quad y = 3$$
$$x = 5 \quad y = 3$$

- **If any complex number vanishes then its real and imaginary parts will separately vanish**
  for example if

$$a + ib = 0$$

then

$$a = -i\ b$$

Squaring both sides, we have

$$a^2 = -b^2$$
$$a^2 + b^2 = 0$$

which hold true only when

$$a = 0, b = 0$$

CONJUGATE OF A COMPLEX NUMBER

Two complex numbers are called the conjugates of each other if their real parts are equal and their imaginary parts differ only in sign

- If $Z = a + bi$ the complex number $a–bi$ is called the conjugate of Z it is denoted by $\overline{Z}$ i.e.,

$$\overline{Z} = \overline{a + bi} = a–bi$$

- Moreover $Z.\overline{Z} = (a + bi).(a–bi) = a^2 + b^2$
- $Z + \overline{Z} = 2a$ and $Z - \overline{Z} = 2bi$

Result

If Z_1 and Z_2 are complex numbers, then

(1) $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

(2) $\overline{Z_1.Z_2} = \overline{Z_1}.\overline{Z_2}$

(3) $\overline{(\frac{Z_1}{Z_2})} = \frac{\overline{Z_1}}{\overline{Z_2}}$

POWER OF i

we know that

$i^3 = i^2.i = (-1).i = -i$

$i^4 = (i^2)^2 = ((-1)^2) = 1$

$i^5 = (i^2)^2.i = ((-1)^2).i = i$

$i^6 = (i^4)^2 = ((-1)^3) = -1$

Also, we have

$i^{-1} = \frac{1}{i}.\frac{i}{i} = \frac{i}{-1} = -i$

$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$

$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = \frac{1}{-i}.\frac{i}{i} = i$

$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$

In general, for any integer k

$i^4k = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

ALGEBRA OF COMPLEX NUMBERS

There are four algebraic operations of complex numbers
If $Z_1 = a_1 + b_1 i$ and $Z_2 = a_2 + b_2 i$ then

(1) Addition

$$Z_1 + Z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

(2) Subtraction

$$Z_1 - Z_2 = (a_1 + b_1 i) – (a_2 + b_2 i)$$

$$= (a_1 – a_2) + i(b_1 – b_2)$$

(3)Multiplication

$$Z_1 * Z_2 = (a_1 + b_1 i).(a_2 + b_2 i)$$

$$= a_1 a_2 + b_1 b_i 2 + a_1 b_2 i + b_1 a_2 i$$

$$= (a_1 a_2 – b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

(4)Division where $Z_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

Multiply Numerator and denominator by the number $a_2 – b_2 i$ in order to make the denominator real.

$$\frac{z_1}{z_2} = \frac{(a_1 + b_1 i)}{(a_2 + b_2 i)} * \frac{a_2 – b_2 i}{a_2 – b_2 i}$$

$$= \frac{a_1a_2 + b_1b_2}{(a_2)^2 + (b_2)^2} + i\frac{b_1a_2 - a_1b_2}{(a_2)^2 + (b_2)^2}$$

Generally result will be expressed in the form $a + ib$

EXAMPLES Example 1

Prove that if the sum and product of two complex numbers are real , then the two numbers must be real or complex conjugates of each other
OR
Example 1
Show that $z_1 + z_2$ and $z_1.z_2$ are both real , then either z_1 and z_2 are both real or $z_2 = \overline{z_1}$

Solution
Let z_1 and z_2 be two complex numbers defined as $z_1 = a_1 + i\ b_1$ and $z_2 = a_2 + i\ b_2$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

Given $z_1 + z_2$ is real

$b_1 + b_2 = 0$
hence

$b_1 = -b_2 \ldots..(1)$ Also

$$z_1.z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

Given $z_1.z_2$ is real

$a_1b_2 + a_2b_1 = 0.\ldots..(2)$

Using (1) in(2)

$a_1b_2 - a_2b_2 = 0$

$b_2(a_1 - a_2) = 0$

$b_2 = 0$ or $(a_1 - a_2) = 0$

Case (i) $b_2 = 0$ hence $b_1 = 0$ (By (1))

$z_1 = a_1$ and $z_2 = a_2$

Two Complex Numbers are real

Case (ii) $(a_1 - a_2) = 0$

$a_1 = a_2$ and by (1) $b_1 = -b_2$

$z_1 = a_1 + i\ b_1$ and $z_2 = a_2 + i\ b_2 = a_1 - i\ b_1$

Two Complex Numbers are complex conjugates of each other

Hence Proved

ARGAND PLANE

- We know that corresponding to each ordered pair of real numbers (x, y) we get a unique point in the XY plane and vice-versa with reference to a set of mutually perpendicular lines known as the $x - axis$ and the $y - axis$
- The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the$XY - plane$ and vice-versa

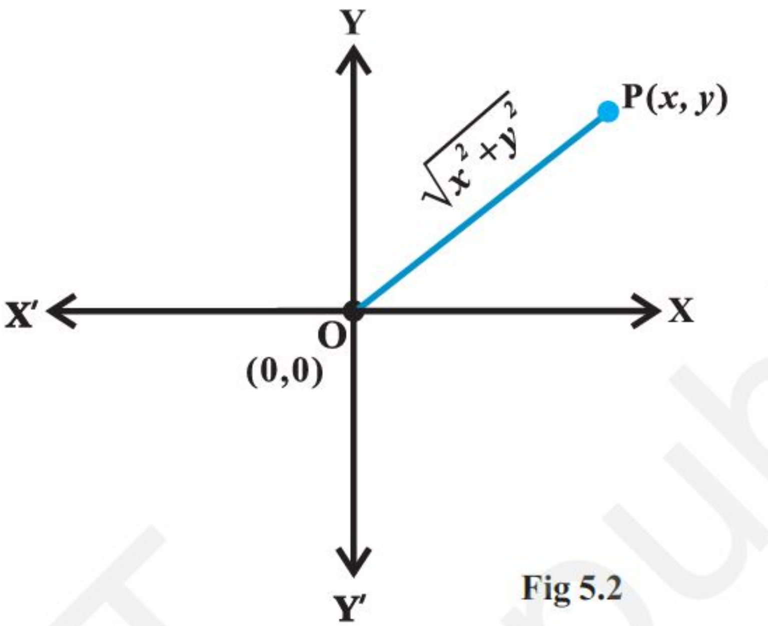


Fig 5.2

- The plane having complex number assigned to each of its poits is called the ***Complex plane*** or ***Argand Plane***
- The $x - axis$ and $y - axis$ in the Argand plane are called ***Real axis*** and ***Imaginary axis*** respectively
- In the Argand Plane **the Modulus** of a complex number $x + iy = \sqrt{x^2 + y^2}$ is defined as the distance of a point $P(x, y)$ from

the origin $O(0, 0)$

- The points on real axis corresponds to complex number of the form $x+i0$ and points on imaginary axis corresponds to complex number of the form $0 + iy$

RESULT For complex numbers Z_1 and Z_2

(1) $|Z_1.Z_2| = |Z_1||Z_2|$

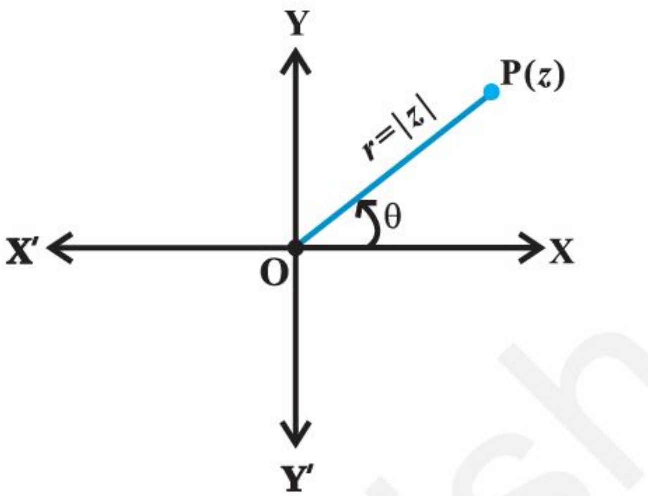
(2) $|\frac{Z_1}{Z_2}| = \frac{|Z_1|}{|Z_2|}$

(3) $|Z_1 + Z_2| < |Z_1| + |Z_2|$

(4) $|Z_1 – Z_2| > |Z_1| – |Z_2|$

POLAR REPRESENTATION

- Consider a point $P(x, y)$ representing a complex number $z = x + iy$
- The point P can be located in polar co-ordinate system by its distance from the pole say $r = |z|$ and its angular position with respect to $+ve$ side of $X - axis$ as θ



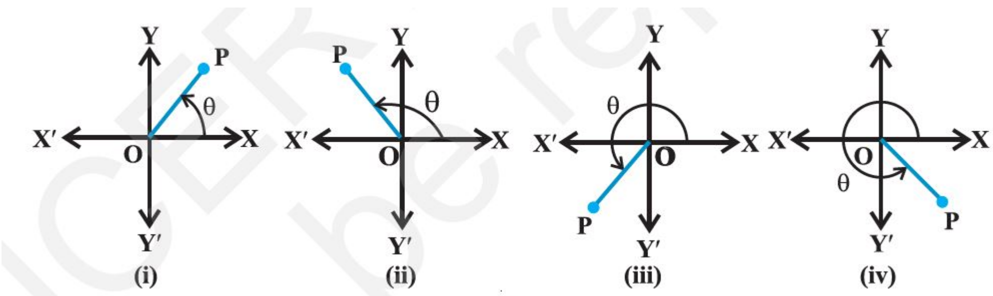
- Considering polar coordinates for P
  $x = r\ cos\ \theta$ and $y = r\ sin\ \theta$
  $x^2 + y^2 = r^2$
- Hence for complex Number $z = x + iy$
  $z = x + iy = r\ cos\ \theta + r\ sin\ \theta = r(cos\ \theta + i\ sin\ \theta)$
- This expression is known as **Polar Form** of the complex Number z where

  (1) $|z| = r$ is known as **modulus** of a complex number,defined as $r = \sqrt{x^2 + y^2}$

(2) θ is known as **argument or amplitude** of complex number defined as $\theta = tan^{-1}|\frac{y}{x}|$

- The argument θ can have infinite number of values, the values lies in the interval $(-\pi, \pi]$ is known as **Principal value** of the argument

- For any complex number $z \neq 0$, there corresponds only one value of θ in $[0, 2\pi)$



However for any interval of length 2π , say $(-\pi, \pi]$, can be such interval wherewe have Principal value of the argument.

- In four quadrants, arguments of a complex numbers are defined as follows
  Let $P(x, y)$ be a point representing a complex number $z = x+iy$ and let α be the angle that $\overline{OP}$ makes with positive x axis then

- For $x > 0, y > 0$ (i.e. Point in first quadrant) $arg(z) = \theta = \alpha = tan^{-1}|\frac{y}{x}|$
  fig(i)

- For $x < 0, y > 0$ (i.e. Point in second quadrant) $arg(z) = \theta = \pi - \alpha = \pi - tan^{-1}|\frac{y}{x}|$

fig(ii)

- For $x < 0, y < 0$ (i.e. Point in third quadrant) $arg(z) = \theta = \pi + \alpha = \pi + tan^{-1}|\frac{y}{x}|$
  fig(iii)

- For $x > 0, y < 0$ (i.e. Point in fourth quadrant) $arg(z) = \theta = 2\pi - \alpha = 2\pi - tan^{-1}|\frac{y}{x}| = -tan^{-1}|\frac{y}{x}| = -\alpha$
  fig(iv)

Use of Polar Form in Multiplication and Division Let $z_1 = r_1(cos\ \theta_1 + i\ sin\ \theta_1)$ and $z_2 = r_2(cos\ \theta_2 + i\ sin\ \theta_2)$ then

(1)

$$arg(z_1.z_2) = arg(z_1).arg(z_2)$$

OR

$$z_1.z_2 = r_1.r_2[cos(\theta_1 + \theta_2) + i\ sin(\theta_1 + \theta_2)]$$

(2)

$$arg(\frac{z_1}{z_2}) = \frac{arg(z_1)}{arg(z_2)}$$

where

$$z_2 \neq 0$$

OR

$$(\frac{z_1}{z_2}) = \frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + i\ sin(\theta_1 - \theta_2)]$$

EXPONENTIAL FORM

The polar form of complex number is given by

$$z = r(cos\ \theta + i\ sin\ \theta)$$

Also Euler's formula is given by

$$e^{i\ \theta} = cos\ \theta + i\ sin\ \theta$$

and

$$e^{-i\ \theta} = cos\ \theta - i\ sin\ \theta$$

Using Euler's formula in polar form of complex number, we have

$$z = r\ e^{i\ \theta}$$

which is known as **exponential form** of complex number

Example 2 Prove that, for any complex numbers z_1 and z_2

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Solution

Let $z_1 = a + i\ b$ and $z_2 = c + i\ d$ be two complex numbers

$|z_1 + z_2|^2 = (a + c)^2 + (b + d)^2$ and $|z_1 - z_2|^2 = (a - c)^2 + (b - d)^2$

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (a + c)^2 + (b + d)^2 + (a - c)^2 + (b - d)^2$

$= 2a^2 + 2b^2 + 2c^2 + 2d^2$

$= 2[a^2 + b^2 + c^2 + d^2] \ldots (1)$

Also $|z_1|^2 = a^2 + b^2$ and $|z_2|^2 = c^2 + d^2$

$2[|z_1|^2 + |z_2|^2] = 2[a^2 + b^2 + c^2 + d^2]....(2)$

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Example 3
If $arg(z+1) = \frac{\pi}{6}$ and $arg(z-1) = \frac{2\pi}{3}$, then find Complex Number z

Solution
Given $arg(z+1) = \frac{\pi}{6}$
Let $z = a + i\ b$

Then

$$z + 1 = (a + i\ b) + 1 = (a+1) + i\ b$$

$arg(z+1) = tan^{-1}(\frac{b}{a+1}) = \frac{\pi}{6}$

$\frac{b}{a+1} = \frac{1}{\sqrt{3}}$

$\sqrt{3}b = a + 1$

$a - \sqrt{3}b = -1.......(1)$

Also,$arg(z-1) = \frac{2\pi}{3}$ Let $z = a + i\ b$

Then $z - 1 = (a + i\ b) - 1 = (a - 1) + i\ b$

$arg(z-1) = tan^{-1}(\frac{b}{a-1}) = \frac{2\pi}{3}$

$\frac{b}{a-1} = tan(\frac{2\pi}{3})$

$\frac{b}{a-1} = -\sqrt{3}$

$b = -\sqrt{3}a + \sqrt{3}$........(2)

Using (2) in (1)
$a - \sqrt{3}(-\sqrt{3}a + \sqrt{3}) = -1$

$a + 3a - 3 = -1$

$4a = -2$

$a = \frac{1}{2}$

$b = \frac{\sqrt{3}}{2}$.......................$(By(2))$
Hence required complex number is

$$z = \frac{1}{2} + i\ \frac{\sqrt{3}}{2}$$

DE MOIVRE'S THEOREM
For any two complex numbers with $r_1 = r_2 = 1$

$$(cos\ \theta_1 + isin\ \theta_1).(cos\ \theta_2 + isin\ \theta_2)$$

$$= cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)$$

In particular, if $\theta = \theta_1 = \theta_2$ we have

$$(cos\ \theta + isin\ \theta)^2 = cos\ 2\theta + isin\ 2\theta$$

For any positive integer n , by induction on n , the result may be generalized as

$$(cos\ \theta + isin\ \theta)^n = cos\ n\theta + isin\ n\theta$$

and this is known as the **DeMoivre's Theorem for integer index**
For any negative integer $n = -m$, where m is positive
$(cos\ \theta + isin\ \theta)^n = (cos\ \theta + isin\ \theta)^{-m}$
$= \frac{1}{(cos\ \theta + isin\ \theta)^m}$

$= \frac{1}{cos\ m\theta + isin\ m\theta}$

$= \frac{cos\ m\theta - isin\ m\theta}{cos^2\ m\theta + sin^2\ m\theta}$

$= cos\ m\theta - isin\ m\theta$

$= cos\ (-m)\theta + isin\ (-m)\theta$

$= cos\ n\theta + isin\ n\theta$

Hence result also holds for negative integers n

For any rational number $n = \frac{p}{q}$, where $p, q \in Z$

$$(cos\ \theta + isin\ \theta)^n = cos\ n\theta + isin\ n\theta$$

Hence result also holds for rational number n
In general, for any real number θ,rational number n , one of the

solution of $(cos\ \theta + isin\ \theta)^n$ is $cos\ n\theta + i\ sin\ n\theta$

APPLICATION OF DE MOIVRE'S THEOREM

- Power of complex Numbers

- Expansion of $cos^n\theta$ and $sin^n\theta$ in terms of sines and cosines multiples of θ

- Expansion of $cos\ (n\theta)$ and $sin\ (n\theta)$ in terms of powers of $sin\theta$ and $cos\theta$

- Roots of complex Numbers

Powers of complex Numbers
Properties

- $z^m.z^n = z^{m+n}$

- $\frac{z^m}{z^n} = z^{m-n}$

- $(z^m)^n = z^{mn}$

Example 1

Prove that $(1+cos\theta+isin\theta)^n+(1+cos\theta-isin\theta)^n = 2^{n+1}\,cos^n(\frac{\theta}{2})\,cos(\frac{n\theta}{2})$

Solution

$(1 + cos\theta + isin\theta)^n + (1 + cos\theta - isin\theta)^n$

$= [2cos^2(\frac{\theta}{2}) + 2icos(\frac{\theta}{2})sin(\frac{\theta}{2})]^n + [2cos^2(\frac{\theta}{2}) - 2icos(\frac{\theta}{2})sin(\frac{\theta}{2})]^n$

$= [2cos(\frac{\theta}{2})(cos(\frac{\theta}{2}) + isin(\frac{\theta}{2})]^n + [2cos(\frac{\theta}{2})(cos(\frac{\theta}{2}) - isin(\frac{\theta}{2})]^n$

$= 2^n cos^n(\frac{\theta}{2})[\ [cos(\frac{\theta}{2}) + isin(\frac{\theta}{2})]^n + [cos(\frac{\theta}{2}) - isin(\frac{\theta}{2})]^n]$

$= 2^n cos^n(\frac{\theta}{2})[cos(\frac{n\theta}{2}) + isin(n\frac{\theta}{2})] + [cos(\frac{n\theta}{2}) - isin(\frac{n\theta}{2})]$

(By De Moivre's Theorem)

$= 2^n cos^n(\frac{\theta}{2})(2cos(\frac{n\theta}{2}))$

$= 2^{n+1} cos^n(\frac{\theta}{2})(2cos(\frac{n\theta}{2}))$

Example 2

If 2 $cos\theta = x + \frac{1}{x}$; Prove that $x^r + \frac{1}{x^r} = 2\ cos\ r\theta$

Solution

Given

$$2\cos\theta = x + \frac{1}{x}$$

$$\frac{x^2+1}{x} = 2\cos\theta$$

$$x^2 - (2\cos\theta)x + 1 = 0$$

which is quadratic equation in x whose roots are given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = cos\ \theta \pm i\ sin\ \theta$$

Case(i)
$x = cos\ \theta + i\ sin\ \theta$

$x^r = (cos\ \theta + i\ sin\ \theta)^r$

$x^r = cos\ r\theta + i\ sin\ r\theta$...(By De Moivre's theorem)
Also

$x^{-r} = (cos\ \theta + i\ sin\ \theta)^{-r}$
$x^{-r} = cos\ (-r)\theta + i\ sin\ (-r)\theta$....(By De Moivre's theorem)

$x^{-r} = cos\ r\theta - i\ sin\ r\theta$
Hence

$x^r + \frac{1}{x^r} = cos\ r\theta + i\ sin\ r\theta + cos\ r\theta - i\ sin\ r\theta$
$x^r + \frac{1}{x^r} = 2\ cos\ r\theta$

Hence result holds true in this case. **Case(ii)**

$x = cos\ \theta - i\ sin\ \theta$
$x^r = (cos\ \theta - i\ sin\ \theta)^r$

$x^r = cos\ r\theta - i\ sin\ r\theta$...(By De Moivre's theorem)
Also

$x^{-r} = (cos\ \theta - i\ sin\ \theta)^{-r}$
$x^{-r} = cos\ (-r)\theta - i\ sin\ (-r)\theta$....(By De Moivre's theorem)

$x^{-r} = cos\ r\theta + i\ sin\ r\theta$
Hence

$x^r + \frac{1}{x^r} = cos\ r\theta - i\ sin\ r\theta + cos\ r\theta + i\ sin\ r\theta$
$x^r + \frac{1}{x^r} = 2\ cos\ r\theta$

Hence result holds true in this case.

Example 3
If $\alpha = 1 + i; \beta = 1 - i$ and $cot\ \theta = x + 1$, Prove that

$(x+\alpha)^n - (x+\beta)^n = (\alpha-\beta)\ sin\ n\theta\ cosec^n\theta$

Solution

Given

$\alpha = 1+i; \beta = 1-i$ and $cot\ \theta = x+1$ hence $x = cot\ \theta - 1$

$x+\alpha$

$= cot\ \theta - 1 + 1 + i$

$= cot\ \theta + i$

$= \frac{cos\ \theta + i\ sin\ \theta}{sin\ \theta}$

Also $x+\beta$

$= cot\ \theta - 1 + 1 - i$

$= cot\ \theta - i$

$= \frac{cos\ \theta - i\ sin\ \theta}{sin\ \theta}$

Hence $(x+\alpha)^n - (x+\beta)^n$

$= [\frac{cos\ \theta + i\ sin\ \theta}{sin\ \theta}]^n - [\frac{cos\ \theta - i\ sin\ \theta}{sin\ \theta}]^n$

$= cosec^n\theta[(cos\ \theta + i\ sin\ \theta)^n - (cos\ \theta - i\ sin\ \theta)^n]$

$= cosec^n\theta[cos\ n\theta + i\ sin\ n\theta - cos\ n\theta + i\ sin\ \theta]$

(By De Moivre's theorem)

$= cosec^n\theta[2i\ sin\ n\theta]$

$= 2i\ sin\ n\theta\ cosec^n\theta$

$= (\alpha - \beta) sin\ n\theta\ cosec^n\theta$ (By given data)

Hence proved.

Example 4
Prove that

$(1-e^{i\theta})^{-\frac{1}{2}}+(1-e^{-i\theta})^{-\frac{1}{2}}=[1+cosec(\frac{\theta}{2})]^{\frac{1}{2}}$

Solution
By Euler's Formula

$$e^{i\theta}=cos\ \theta+i\ sin\ \theta$$

and

$$e^{-i\theta}=cos\ \theta-i\ sin\ \theta$$

Hence $(1-e^{i\theta})^{-\frac{1}{2}}+(1-e^{-i\theta})^{-\frac{1}{2}}$

$=(1-cos\ \theta-i\ sin\ \theta)^{-\frac{1}{2}}+(1-cos\ \theta+i\ sin\ \theta)^{-\frac{1}{2}}$

$=[2sin^2(\frac{\theta}{2})-2i\ sin(\frac{\theta}{2})cos(\frac{\theta}{2})]^{-\frac{1}{2}}+[2sin^2(\frac{\theta}{2})+2i\ sin(\frac{\theta}{2})cos(\frac{\theta}{2})]^{-\frac{1}{2}}$

$=[2sin(\frac{\theta}{2})(sin(\frac{\theta}{2})-i\ cos(\frac{\theta}{2})]^{-\frac{1}{2}}+[2sin(\frac{\theta}{2})(sin(\frac{\theta}{2})+i\ cos(\frac{\theta}{2})]^{-\frac{1}{2}}$

$=[2sin(\frac{\theta}{2})(cos(\frac{\pi}{2}-\frac{\theta}{2})-i\ sin(\frac{\pi}{2}-\frac{\theta}{2})]^{-\frac{1}{2}}+$

$[2sin(\frac{\theta}{2})(cos(\frac{\pi}{2}-\frac{\theta}{2})+i\ sin(\frac{\pi}{2}-\frac{\theta}{2})]^{-\frac{1}{2}}$

Let $\frac{\pi}{2}-\frac{\theta}{2}=\alpha$ then above result can be written as

$(1-e^{i\theta})^{-\frac{1}{2}} + (1-e^{-i\theta})^{-\frac{1}{2}}$
$= [2sin(\frac{\theta}{2})(cos(\alpha) - i\ sin(\alpha)]^{-\frac{1}{2}} + [2sin(\frac{\theta}{2})(cos(\alpha) + i\ sin(\alpha)]^{-\frac{1}{2}}$

$= [2sin(\frac{\theta}{2})]^{-\frac{1}{2}}[cos(\frac{\alpha}{2}) + i\ sin(\frac{\alpha}{2}) + cos(\frac{\alpha}{2}) - i\ sin(\frac{\alpha}{2})]$

$= \frac{1}{2sin(\frac{\theta}{2}))^{\frac{1}{2}}}[2\ cos(\frac{\alpha}{2})]$

$= \frac{[4\ cos^2(\frac{\alpha}{2})]^{\frac{1}{2}}}{[2sin(\frac{\theta}{2})]^{\frac{1}{2}}}$

$= \left[\frac{1+cos\ \alpha}{sin(\frac{\theta}{2})}\right]^{\frac{1}{2}}$

$= \left[\frac{1+cos\ (\frac{\pi}{2}-\frac{\theta}{2})}{sin(\frac{\theta}{2})}\right]^{\frac{1}{2}}$

$= \left[\frac{1+sin(\frac{\theta}{2})}{sin(\frac{\theta}{2})}\right]^{\frac{1}{2}}$

$= [1 + cosec(\frac{\theta}{2})]^{\frac{1}{2}}$
Hence proved

Example 5
Find 10^{th} Power of $\sqrt{3}+i$ Using De Moivre's Theorem and express

your answer in standard form
Solution
Given $z = \sqrt{3}+i$

So point (x, y) lies in first quadrant
$r = \sqrt{3+1} = 2$ and $\theta = tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$
Hence Polar form : $z = r(cos\theta + i\ sin\theta) = 2(cos(\frac{\pi}{6}) + isin(\frac{\pi}{6}))$
Taking 10^{th} power on both the sides, by De Moivre's theorem
$z^{10} = [2(cos(\frac{\pi}{6}) + isin(\frac{\pi}{6}))]^{10}$
$= 2^{10}[cos(\frac{10\pi}{6}) + i\ sin(\frac{10\pi}{6})]$

$= 2^{10}[cos(\frac{5\pi}{3}) + i\ sin(\frac{5\pi}{3})]$

$= 2^{10}[cos(2\pi - \frac{\pi}{3}) + i\ sin(2\pi - \frac{\pi}{3})]$

$= 2^{10}[cos(\frac{\pi}{3}) - i\ sin(\frac{\pi}{3})]$

$= 2^{10}[\frac{1}{2} - i\frac{\sqrt{3}}{2}]$

$= 2^{9}[1 - \sqrt{3}i]$

Hence $z^{10} = 512[1 - \sqrt{3}i]$

Example 6

Express $z = (1+7i)(2-i)^{-2}$ in polar form and prove that second

power of z is an imaginary number and fourth power is a negative real number.

Solution

Let

$z = (1+7i)(2-i)^{-2}$
$= \frac{1+7i}{(2-i)^2}$

$= \frac{1+7i}{3-4i}$

$= \frac{1+7i}{3-4i} \frac{3+4i}{3+4i}$

$= \frac{-25+25i}{25}$

$z = -1 + i$

Hence Point lies in second quadrant

so $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
and $arg(z) = \theta = \pi - tan^{-1}|\frac{1}{1}| = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence polar form of z is given by

$z = r(cos\ \theta + i\ sin\ \theta)$
$= \sqrt{2}(cos(\frac{3\pi}{4}) + isin(\frac{3\pi}{4}))$

$= \sqrt{2}(cos(\pi - \frac{\pi}{4}) + isin(\pi - \frac{\pi}{4}))$

$z = \sqrt{2}(cos(\frac{\pi}{4}) + isin(\frac{\pi}{4}))$

By De Moivre's Theorem

$z^2 = \left[\sqrt{2}(cos(\frac{\pi}{4}) + isin(\frac{\pi}{4}))\right]^2$

$z^2 = (\sqrt{2})^2 \left[(cos(\frac{2\pi}{4}) + isin(\frac{2\pi}{4}))\right]$
$z^2 = 2 \left[(cos(\frac{\pi}{2}) + isin(\frac{\pi}{2}))\right]$

$z^2 = 2(0 + i)$

$z^2 = 2i$ ....**which is purely imaginary number**

Also
$z^4 = \left[\sqrt{2}(cos(\frac{\pi}{4}) + isin(\frac{\pi}{4}))\right]^4$
$z^4 = (\sqrt{2})^4 \left[(cos(\frac{4\pi}{4}) + isin(\frac{4\pi}{4}))\right]$

$z^4 = 4\left[(cos(\pi) + isin(\pi)\right]$

$z^4 = 4(-1 + 0)$

$z^4 = -4$ ....**which is negative real number**

Expansion of $cos^n\theta$ and $sin^n\theta$ in terms of sine and Cosine multiples of θ(Concept)

- By Euler's Formula

$$z = e^{i\theta} = cos\ \theta + i\ sin\ \theta$$
$$\frac{1}{z} = e^{-i\theta} = cos\ \theta - i\ sin\ \theta$$

- Also by De Moivre's theorem

$$z^m = e^{im\theta} = cos\ m\theta + i\ sin\ m\theta$$

$$\frac{1}{z^m} = e^{-im\theta} = cos\ m\theta - i\ sin\ m\theta$$

- Then

$$z + \frac{1}{z} = 2\ cos\ \theta ...(1) \quad z - \frac{1}{z} = 2i\ sin\ \theta ...(2)$$

$$z^m + \frac{1}{z^m} = 2\ cos\ m\theta ...(3) \quad z^m - \frac{1}{z^m} = 2i\ sin\ m\theta ...(4)$$

- Expressing $\left(z + \frac{1}{z}\right)^n$ or $\left(z - \frac{1}{z}\right)^n$ using (1) or (2) and expanding by Binomial expansion, we can merge the result in $z^m + \frac{1}{z^m}$ or $z^m - \frac{1}{z^m}$ form

- Using (3) and (4) we can get expansion of $cos^n\theta$ and $sin^n\theta$ in terms of sine and Cosine multiples of θ

Expansion of $cos\ n\theta$ and $sin\ n\theta$ in terms of powers of $sin\ \theta$ and $cos\ \theta$(Concept)

- To find expansion of $cos\ n\theta$ and $sin\ n\theta$ in terms of powers of $sin\ \theta$ and $cos\ \theta$, we write De Moivre's theorem in reverse order i.e $cos\ n\theta + i\ sin\ n\theta = (cos\ \theta + i\ sin\ \theta)^n$

- We expand $(cos\ \theta + i\ sin\ \theta)^n$ by Binomial expansion and write the expression in standard form $A + iB$

- Equating real and imaginary parts on both sides we get expansion of $cos\ n\theta$ and $sin\ n\theta$ respectively.

EXAMPLES

Example 1

Prove that $cos^6\theta - sin^6\theta = \frac{1}{16}[cos\ 6\theta + 15cos\ 2\theta]$

Solution

Using Euler's Formula , define

$$z = e^{i\theta} = cos\ \theta + i\ sin\ \theta$$
$$\frac{1}{z} = e^{-i\theta} = cos\ \theta - i\ sin\ \theta$$

Also by De Moivre's theorem

$$z^m = e^{im\theta} = cos\ m\theta + i\ sin\ m\theta$$
$$\frac{1}{z^m} = e^{-im\theta} = cos\ m\theta - i\ sin\ m\theta$$

Then

$$z + \frac{1}{z} = 2\ cos\ \theta ...(1) \quad z - \frac{1}{z} = 2i\ sin\ \theta ...(2)$$
$$z^m + \frac{1}{z^m} = 2\ cos\ m\theta ...(3) \quad z^m - \frac{1}{z^m} = 2i\ sin\ m\theta ...(4)$$

Consider

$(2\ cos\ \theta)^6 = \left(z + \frac{1}{z}\right)^6$ ......**[by (1)]**

$= z^6 +^6 C_1\ z^5\ \left(\frac{1}{z}\right) +^6 C_2\ z^4\ \left(\frac{1}{z^2}\right) +^6 C_3\ z^3\ \left(\frac{1}{z^3}\right) +^6 C_4\ z^2\ \left(\frac{1}{z^4}\right)$

$+^6C_5\ z\ \left(\frac{1}{z^5}\right) +^6 C_6 \left(\frac{1}{z^6}\right)$

$\therefore 2^6\ cos^6\theta = z^6 + 6\ z^5\ \left(\frac{1}{z}\right) + 15\ z^4\ \left(\frac{1}{z^2}\right) +20\ z^3\ \left(\frac{1}{z^3}\right) + 15\ z^2\ \left(\frac{1}{z^4}\right)$
$+6\ z\ \left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$
$\therefore 2^6\ cos^6\theta = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$\therefore cos^6\theta = \frac{1}{2^6}\left[2\ cos\ 6\theta + 6 * 2\ cos\ 4\theta + 15 * 2\ cos\ 2\theta + 20\right] ...\textbf{[By (3)]}$

$cos^6\theta = \frac{1}{2^5}\left[cos\ 6\theta + 6\ cos\ 4\theta + 15\ cos2\theta + 10\right] ...\textbf{(5)}$

Also Consider

$(2i\ sin\ \theta)^6 = \left(z - \frac{1}{z}\right)^6 ......\textbf{[by (2)]}$

$= z^6 -^6 C_1\ z^5\ \left(\frac{1}{z}\right) +^6 C_2\ z^4\ \left(\frac{1}{z^2}\right)$
$-^6C_3\ z^3\ \left(\frac{1}{z^3}\right) +^6 C_4\ z^2\ \left(\frac{1}{z^4}\right)$
$-^6C_5\ z\ \left(\frac{1}{z^5}\right) +^6 C_6 \left(\frac{1}{z^6}\right)$

$\therefore (2i)^6\ sin^6\theta = z^6 - 6\ z^5\ \left(\frac{1}{z}\right) + 15\ z^4\ \left(\frac{1}{z^2}\right) -20\ z^3\ \left(\frac{1}{z^3}\right) + 15\ z^2\ \left(\frac{1}{z^4}\right)$
$-6\ z\ \left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$

$\therefore 2^6\ i^6\ sin^6\theta = \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$
$\therefore 2^6\,(-1)\,sin^6\theta = \left[2\ cos\ 6\theta - 6 * 2\ cos\ 4\theta + 15 * 2\ cos\ 2\theta - 20\right] ...\textbf{[By (3)]}$

$\therefore sin^6\theta = \frac{-1}{2^5}\left[cos\ 6\theta - 6\ cos\ 4\theta + 15\ cos2\theta - 10\right] ...\textbf{(6)}$

Hence (5)-(6) gives

$cos^6\theta - sin^6\theta$
$= \frac{1}{2^5}[cos\ 6\theta + cos\ 6\theta + 6\ cos\ 4\theta - 6\ cos\ 4\theta + 15\ cos\ 2\theta + 15\ cos\ 2\theta +$

$10 - 10]$

$= \frac{1}{32}\,[2\ cos\ 6\theta + 30\ cos\ 2\theta]$

$cos^6\theta - sin^6\theta = \frac{1}{16}\,[cos\ 6\theta + 15\ cos\ 2\theta]$

Hence Proved.

Example 2

Prove that

$$2^6\ sin^4\theta\ cos^3\theta = [cos\ 7\theta - cos\ 5\theta - cos\ 3\theta + cos\ \theta]$$

Solution

Using Euler's Formula , define

$$z = e^{i\theta} = cos\ \theta + i\ sin\ \theta$$
$$\frac{1}{z} = e^{-i\theta} = cos\ \theta - i\ sin\ \theta$$

Also by De Moivre's theorem

$$z^m = e^{im\theta} = cos\ m\theta + i\ sin\ m\theta$$
$$\frac{1}{z^m} = e^{-im\theta} = cos\ m\theta - i\ sin\ m\theta$$

Then

$$z+\frac{1}{z}=2\ cos\ \theta...(1) \qquad z-\frac{1}{z}=2i\ sin\ \theta...(2)$$

$$z^m+\frac{1}{z^m}=2\ cos\ m\theta...(3) \qquad z^m-\frac{1}{z^m}=2i\ sin\ m\theta...(4)$$

Consider $(2i\,sin\,\theta)^4\,(2\,cos\,\theta)^3=\left(z-\frac{1}{z}\right)^4\left(z+\frac{1}{z}\right)^3$ ......**[by (1) and (2)]**

$\therefore 2^4\ i^4\ sin^4\theta\ 2^3\ cos^3\theta=\left(z-\frac{1}{z}\right)\left[\left(z-\frac{1}{z}\right)^3\left(z+\frac{1}{z}\right)^3\right]$

$\therefore 2^7\ sin^4\theta\ cos^3\theta$

$=\left(z-\frac{1}{z}\right)\left(z^2-\frac{1}{z^2}\right)^3$

$=\left(z-\frac{1}{z}\right)\ \left[(z^2)^3-3(z^2)^2\frac{1}{z^2}+3z^2\frac{1}{(z^2)^2}-\left(\frac{1}{z^2}\right)^3\right]$

$=\left(z-\frac{1}{z}\right)\left[z^6-3z^2+3\frac{1}{z^2}-\frac{1}{z^6}\right]$

$=z^7-3z^3+3\frac{1}{z}-\frac{1}{z^5}-z^5-3z-3\frac{1}{z^3}+\frac{1}{z^7}$

$=\left(z^7+\frac{1}{z^7}\right)-3\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)-\left(z^5+\frac{1}{z^5}\right)$

$=2\ cos\ 7\theta-3*2\ cos\ 3\theta+3*2\cos\ \theta-2\ cos\ 5\theta$

$\therefore 2^6\ sin^4\theta\ cos^3\theta=cos\ 7\theta-cos\ 5\theta-3\ cos\ 3\theta+3\ cos\ \theta$

Hence Proved.

Example 3
Expand $\frac{sin\ 7\theta}{sin\ \theta}$ in powers of $sin\ \theta$
Solution
By De Moivre's theorem
$cos\ 7\theta + i\ sin\ 7\theta = (cos\ \theta + i\ sin\ \theta)^7$

$= (cos\ \theta)^7 +^7 C_1\ (cos\ \theta)^6\ (i\ sin\ \theta) + {}^7C_2\ (cos\ \theta)^5\ (i\ sin\ \theta)^2 +^7 C_3\,(cos\,\theta)^4\,(i\,sin\,\theta)^3 + {}^7C_4\,(cos\,\theta)^3\,(i\,sin\,\theta)^4 + {}^7C_5\,(cos\,\theta)^2\,(i\,sin\,\theta)^5 + {}^7C_6\ (cos\ \theta)\ (i\ sin\ \theta)^6 +^7 C_7\ (i\ sin\ \theta)^7$

$= cos^7\theta + 7i\ cos^6\theta\ sin\theta - 21\ cos^5\theta\ sin^2\theta\ - 35i\ cos^4\theta\ sin^3\theta + 35\ cos^3\theta\ sin^4\theta + 21i\ cos^2\theta\ sin^5\theta - 7\ cos\ \theta\ sin^6\theta - i\ sin^7\theta$

$= (cos^7\theta - 21\ cos^5\theta\ sin^2\theta + 35\ cos^3\theta\ sin^4\theta - 7\ cos\ \theta\ sin^6\theta) + i\ (7\ cos^6\theta\ sin\theta - 35\ cos^4\theta\ sin^3\theta + 21\ cos^2\theta\ sin^5\theta - sin^7\theta)$

$\therefore \frac{sin\ 7\theta}{sin\ \theta} = 7\ cos^6\theta - 35\ cos^4\theta\ sin^2\theta + 21\ cos^2\theta\ sin^4\theta - sin^6\theta$

$= 7\,(1 - sin^2\theta)^3 - 35\,(1 - sin^2\theta)^2\,sin^2\theta + 21\,(1 - sin^2\theta)\,sin^4\theta - sin^6\theta$

$\therefore \frac{sin\ 7\theta}{sin\ \theta} = 7\ (1 - 3\ sin^2\theta + 3\ sin^4\theta - sin^6\theta) - 35\ (1 - 2\ sin^2\theta + sin^4\theta) sin^2\theta + 21\ (sin^4\theta - sin^6\theta) - sin^6\theta$

$= 7\ \ - 21\ sin^2\theta + 21\ sin^4\theta - 7\ sin^6\theta - 35\ sin^2\theta + 70\ sin^4\theta - 35\ sin^6\theta + 21\ sin^4\ \theta - 21\ sin^6\theta - sin^6\theta$

$\therefore \frac{sin\ 7\theta}{sin\ \theta} = 7\ \ - 56\ sin^2\theta + 112\ sin^4\theta - 64\ sin^6\theta$

which is required expansion.

Example 4

Using De Moivre's theorem show that

$$2(1 + cos\ 8\theta) = (x^4 - 4x^2 - 2)^2$$

where $x = 2\ cos\ \theta$

Solution

Consider

$$2(1 + cos\ 8\theta) = 2 * 2\ cos^2\ (4\theta) = (2\ cos\ 4\theta)^2 ...(1)$$

By De Moivre's theorem

$cos\ 4\theta + i\ sin\ 4\theta$

$= (cos\ \theta + i\ sin\ \theta)^4$

$= (cos\,\theta)^4 + ^4C_1\,(cos\,\theta)^3(i\,sin\,\theta) + theta)(i\,sin\,\theta)^3 + {}^4C_2\,(cos\,\theta)^2(i\,sin\,\theta)^2 + ^4$
$C_3\ (cos\ \theta)(i\ sin\ \theta)^3 + sin^4\theta$

$= cos^4\theta + 4i\ cos^3\theta\ sin\theta - 6\ cos^2\theta\ sin^2\theta\ - 4i\ cos\ \theta\ sin^3\theta + sin^4\theta$

$\therefore cos\ 4\theta + i\ sin\ 4\theta$

$= (cos^4\theta - 6\ cos^2\theta\ sin^2\theta + +sin^4\theta)$

$i\ (4\ cos^3\theta\ sin\theta - 4\ cos\ \theta\ sin^3\theta)$

Equating real part on bothe sides, we have

$cos\ 4\theta = cos^4\theta - 6\ cos^2\theta\ sin^2\theta + +sin^4\theta$

$\therefore 2\ cos\ 4\theta = 2\ cos^4\theta - 12\ cos^2\theta\ sin^2\theta + +2\ sin^4\theta$

$= 2\ cos^4\theta - 12\ cos^2\theta\ (1 - cos^2\theta) + 2\ (1 - cos^2\theta)^2$

$= 2\ cos^4\theta - 12\ cos^2\theta + 12\ cos^4\theta + 2\ (1 - 2\ cos^2\theta + cos^4\ \theta)$

$= 2\ cos^4\theta - 12\ cos^2\theta + 12\ cos^4\theta + 2 - 4\ cos^2\ \theta$

$= 16\ cos^4\theta - 16\ cos^2\theta + 2$

By (1) $2(1 + cos\ 8\theta) = [16\ cos^4\theta - 16\ cos^2\theta + 2]^2$

$= [(2\ cos\ \theta)^4 - 4(2\ cos\ \theta)^2 + 2]^2$

$\therefore 2(1 + cos\ 8\theta) = (x^4 - 4x^2 - 2)^2$

where $x = 2\ cos\ \theta$

Application of De Moirve's theorem to find Roots of Complex Numbers(Concept)

- The nth roots of a complex number z are the n values of w which satisfy the equation $w^n = z$.

- write $z = cos\ \theta + i\ sin\ \theta$ and assuming that the equation is satisfied by $w = cos\ \phi + i\ sin\ \phi$, then

  $$w^n = z \implies (cos\,\phi + i\,sin\,\phi)^n = cos\,n\phi + i\,sin\,n\phi = cos\,\theta + i\,sin\,\theta$$

  Equating real and imaginary parts on both sides, we have

  $$cos\ \theta = cos\ n\phi \qquad\qquad sin\ \theta = sin\ n\phi$$

  $$\therefore n\phi = 2k\pi + \theta \implies \phi = \frac{2k\pi + \theta}{n} \qquad k \in Z$$

- We obtain n distinct complex roots of z with the values of ϕ obtained above for $k = 0, 1, 2...n-1$ as for $k < 0$ and $k > n-1$ the root obtained is one of the root, mentioned above.

- **Hence, the equation** $w^n = z = r(cos\ \theta + i\ \ \text{sin}\ \ \theta)$ **has** n **distinct complex roots, given by**

  $$w = r^{\frac{1}{n}} \left[cos \left(\frac{2k\pi + \theta}{n} \right) + i\ sin \left(\frac{2k\pi + \theta}{n} \right) \right] \qquad k = 0, 1, 2...n-1$$

  where θ **is the principal argument.**

Working Rule

- To find n distinct roots of Complex number by De Moirve's theorem, equation should be in one of the following form
  x^n=complex number....(1)
  OR
  $[f(x)]^n$=complex Number....(2)

- Write the complex number in polar form using its modulus and principal argument

- write the generalized polar form of complex number as discussed before

- By (1) , n distinct roots of complex number is given by

  $$x = r^{\frac{1}{n}} \left[cos(2k\pi + \theta) + i\ sin(2k\pi + \theta)\right]^{\frac{1}{n}} \qquad k = 0, 1, 2...n - 1$$
  $$\therefore x = r^{\frac{1}{n}} \left[cos\left(\frac{2k\pi + \theta}{n}\right) + i\ sin\left(\frac{2k\pi + \theta}{n}\right)\right] \qquad k = 0, 1, 2...n-1$$
  (By De Moivre's theorem)

- substituting values of k we, get required n distinct roots

EXAMPLES

Example 1

Find the cube roots of unity (or 1) Show that if ω is complex cube root of unity, then $a)1+\omega+\omega^2=0 \quad b)(1-\omega)^6=-27$

Solution

Consider

$$x=\sqrt[3]{1}$$

$$\implies x^3=1$$

$$=cos\ 0+i\ sin\ 0$$

$$=cos\ (0+2k\pi)+i\ sin\ (0+2k\pi)$$

$$\therefore x^3=cos\ (2k\pi)+i\ sin\ (2k\pi)$$

$$\implies x_r=[cos\ (2k\pi)+i\ sin\ (2k\pi)]^{\frac{1}{3}}$$

$$=[cos\ (\tfrac{2k\pi}{3})+i\ sin\ (\tfrac{2k\pi}{3})]....wherek=0,1,2$$

for $k=0$ and $r=0$

$x_0=cos\ 0+i\ sin\ 0$

$x_0=1$

for $k=1$ and $r=1$

$x_1 = cos(\frac{2\pi}{3}) + i\ sin(\frac{2\pi}{3})$......**(1)**

$= cos(\pi - \frac{\pi}{3}) + i\ sin(\pi - \frac{\pi}{3})$

$= (-cos(\frac{\pi}{3}) + i\ sin(\frac{\pi}{3}))$

$x_1 = -\frac{1}{2} + i\ \frac{\sqrt{3}}{2}$

for $k = 2$ and $r = 2$

$x_2 = cos(\frac{4\pi}{3}) + i\ sin(\frac{4\pi}{3})$......**(2)**

$= cos(\pi + \frac{\pi}{3}) + i\ sin(\pi + \frac{\pi}{3})$

$= (-cos(\frac{\pi}{3}) - i\ sin(\frac{\pi}{3}))$

$x_2 = -\frac{1}{2} - i\ \frac{\sqrt{3}}{2}$

Hence required cube roots of unity are $1,\ -\frac{1}{2} + i\ \frac{\sqrt{3}}{2}$ **and** $-\frac{1}{2} - i\ \frac{\sqrt{3}}{2}$

Now by (1)

$x_1 = cos(\frac{2\pi}{3}) + i\ sin(\frac{2\pi}{3})$

$x_1 = \omega \quad$ (say)

Then by (2)

$x_2 = cos(\frac{4\pi}{3}) + i\ sin(\frac{4\pi}{3})$

$x_2 = \left[cos(\frac{2\pi}{3}) + i\ sin(\frac{2\pi}{3})\right]^2$

$x_2 = \omega^2$ (say)

i.e. $x_0 = 1; x_1 = \omega; x_2 = \omega^2$
(1) $1 + \omega + \omega^2$

$= 1 + cos(\frac{2\pi}{3}) + i\ sin(\frac{2\pi}{3}) + cos(\frac{4\pi}{3}) + i\ sin(\frac{4\pi}{3})$

$= 1 - cos(\frac{\pi}{3}) + i\ sin(\frac{\pi}{3}) - cos(\frac{\pi}{3}) - i\ sin(\frac{\pi}{3})$

$= 1 - 1$

Hence $1 + \omega + \omega^2 = 0$

(2)$(1 - \omega)^6$

$= [(1 - \omega)^2]^3$

$= (1 - 2\omega + \omega^2)^3$

$= (-3\omega)^3$...(By above Result)

Hence $(1 - \omega)^6 = -27$

Example 2

Find the roots, $\alpha, \alpha^2, \alpha^3, \alpha^4$ of equation $x^5 = 1$. Hence prove that

(1) $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$

(2) $x^5 - 1 = (x-1)(x^2 + 2\ x\ cos(\frac{\pi}{5}) + 1)(x^2 + 2\ x\ cos(\frac{3\pi}{5}) + 1) = 0$

Solution

Consider

$x^5 = 1$

$= cos\ 0 + i\ sin\ 0$

$= cos\ (0 + 2k\pi) + i\ sin\ (0 + 2k\pi)$

$\therefore x^5 = cos\ (2k\pi) + i\ sin\ (2k\pi)$

$\Longrightarrow\ x_r = [cos\ (2k\pi) + i\ sin\ (2k\pi)]^{\frac{1}{5}}$

$= [cos\ (\frac{2k\pi}{5}) + i\ sin\ (\frac{2k\pi}{5})]$

$....wherek = 0, 1, 2, 3, 4$

for $k = 0$ and $r = 0$

$x_0 = cos\ 0 + i\ sin\ 0$

$x_0 = 1$

for $k = 1$ and $r = 1$

$x_1 = cos(\frac{2\pi}{5}) + i\ sin(\frac{2\pi}{5}) = \alpha \quad$ (say)......**(1)**

for $k = 2$ and $r = 2$

$x_2 = cos(\frac{4\pi}{5}) + i\ sin(\frac{4\pi}{5})$

$x_2 = [cos(\frac{2\pi}{5}) + i\ sin(\frac{2\pi}{5})]^2 = \alpha^2 \quad$ (say)......**(2)**

for $k = 3$ and $r = 3$

$x_3 = cos(\frac{6\pi}{5}) + i\ sin(\frac{6\pi}{5})$

$x_3 = [cos(\frac{2\pi}{5}) + i\ sin(\frac{2\pi}{5})]^3 = \alpha^3 \quad$ (say).....**(3)**

for $k = 4$ and $r = 4$

$x_4 = cos(\frac{8\pi}{5}) + i\ sin(\frac{8\pi}{5})$

$x_4 = [cos(\frac{2\pi}{5}) + i\ sin(\frac{2\pi}{5})]^4 = \alpha^4 \quad$ (say)......**(4)**

Hence (1),(2),(3) and (4) gives required roots of $x^5 - 1 = 0$

Now

$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\frac{x^5-1}{x-1} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$1 + x + x^2 + x^3 + x^4 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

Since $x = 1$ is one of the root, let $x = 1$ in above expression

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5........\mathbf{(A)}$$

Again by (2),(3)and (4)

$\alpha^2 = cos(\frac{4\pi}{5}) + i\ sin(\frac{4\pi}{5})$

$\alpha^2 = cos(\pi - \frac{\pi}{5}) + i\ sin(\pi - \frac{\pi}{5})$

$\alpha^2 = -cos(\frac{\pi}{5}) + i\ sin(\frac{\pi}{5})......\mathbf{(5)}$

$\alpha^3 = cos(\frac{6\pi}{5}) + i\ sin(\frac{6\pi}{5})$

$\alpha^3 = cos(\pi + \frac{\pi}{5}) + i\ sin(\pi + \frac{\pi}{5})$

$\alpha^3 = -cos(\frac{\pi}{5}) - i\ sin(\frac{\pi}{5})......\mathbf{(6)}$

$\alpha^4 = cos(\frac{8\pi}{5}) + i\ sin(\frac{8\pi}{5})$

$\alpha^4 = cos(2\pi - \frac{2\pi}{5}) + i\ sin(2\pi + \frac{2\pi}{5})$

$\alpha^4 = cos(\frac{2\pi}{5}) - i\ sin(\frac{2\pi}{5})$......**(7)**

Now by (2),(5),(6)and (7) , roots of the equation $x^5 - 1 = 0$ are of the form

$x = -cos(\frac{\pi}{5}) \pm\ i\ sin(\frac{\pi}{5})$ and

$x = cos(\frac{2\pi}{5}) \pm\ i\ sin(\frac{2\pi}{5})$

Let Case(1)

$x = -cos(\frac{\pi}{5}) \pm i\ sin(\frac{\pi}{5})$

$x + cos(\frac{\pi}{5}) = \pm i\ sin(\frac{\pi}{5})$

$(x + cos(\frac{\pi}{5}))^2 = i^2 sin^2(\frac{\pi}{5})$

$x^2 + 2\ x\ cos(\frac{\pi}{5}) + cos^2(\frac{\pi}{5}) = -sin^2(\frac{\pi}{5})$

$x^2 + 2\ x\ cos(\frac{\pi}{5}) + cos^2(\frac{\pi}{5}) + sin^2(\frac{\pi}{5}) = 0$

$x^2 + 2\ x\ cos(\frac{\pi}{5}) + 1 = 0$....**(8)**

Also Case(2)

$$x = cos(\frac{2\pi}{5}) \pm\ i\ sin(\frac{2\pi}{5})$$

$$x - cos(\frac{2\pi}{5}) = \pm i\ sin(\frac{2\pi}{5})$$

$$(x - cos(\frac{2\pi}{5}))^2 = i^2 sin^2(\frac{2\pi}{5})$$

$$x^2 - 2\ x\ cos(\frac{2\pi}{5}) + cos^2(\frac{2\pi}{5}) = -sin^2(\frac{2\pi}{5})$$

$$x^2 - 2\ x\ cos(\frac{2\pi}{5}) + cos^2(\frac{2\pi}{5}) + sin^2(\frac{2\pi}{5}) = 0$$

$$x^2 - 2\ x\ cos(\frac{2\pi}{5}) + 1 = 0$$

$$x^2 - 2\ x\ cos(\pi - \frac{3\pi}{5}) + 1 = 0$$

$$x^2 + 2\ x\ cos(\frac{3\pi}{5}) + 1 = 0.....\textbf{(9)}$$

substituting (8) and (9) in

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = 0$$

$$x^5 - 1 = (x-1)(x^2 + 2x\ cos(\frac{\pi}{5}) + 1)(x^2 + 2x\ cos(\frac{3\pi}{5}) + 1) = 0....\textbf{(B)}$$

(A) and (B) are required results

Example 3 Find continued product of the roots of $\left(\frac{1}{2} - i\ \frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$

Solution

Consider

$x = \left(\frac{1}{2} - i\ \frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$

$\implies\ x^4 = \left(\frac{1}{2} - i\ \frac{\sqrt{3}}{2}\right)^{3}$

$= \left[\left(cos\ \frac{\pi}{3} - i\ sin\ \frac{\pi}{3}\right)^3\right]$

$= \left[\left(cos\ \frac{3\pi}{3} - i\ sin\ \frac{3\pi}{3}\right)\right]$

$x^4 = [cos\ \pi - i\ sin\ \pi]$

$\therefore x_r = [cos\ (\pi + 2k\pi) - i\ sin\ (\pi + 2k\pi)]^{\frac{1}{4}}$

$\therefore x_r = \left[cos\ \left(\frac{\pi+2k\pi}{4}\right) - i\ sin\ \left(\frac{\pi+2k\pi}{4}\right)\right]$
$...where\ k = 0, 1, 2, 3$

$x_r = e^{-i\ \left(\frac{\pi+2k\pi}{4}\right)} ...where\ k = 0, 1, 2, 3$

for $k = 0$ and $r = 0$
$x_0 = e^{-i\ \frac{\pi}{4}}$

for $k = 1$ and $r = 1$

$x_1 = e^{-i\ \frac{3\pi}{4}}$

for $k = 2$ and $r = 2$ $x_2 = e^{-i\ \frac{5\pi}{4}}$

for $k = 3$ and $r = 3$ $x_3 = e^{-i\ \frac{7\pi}{4}}$

$\therefore$ Continued product of roots is defined as

$x_0.x_1.x_2.x_3 = e^{-i\ (\frac{\pi}{4})}.e^{-i\ (\frac{3\pi}{4})}.e^{-i\ (\frac{5\pi}{4})}.e^{-i\ (\frac{7\pi}{4})}$

$= e^{-i\ (\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4})}$

$= e^{-i\ (4\pi)}$

$= cos\ 4\pi - i\ sin\ 4\pi$

$= 1$

Example 4

Show that roots of $(x+1)^7 = (x-1)^7$ are given by

$$\pm\ i\ cot\frac{r\pi}{7}...r = 1, 2, 3, 4, 5, 6$$

Solution

Given

$$(x+1)^7 = (x-1)^7$$

$$\therefore \frac{(x+1)^7}{(x-1)^7} = 1$$

$$\therefore \left(\frac{x+1}{x-1}\right)^7 = cos\ 0 + i\ sin\ 0$$

$$\therefore \left(\frac{x+1}{x-1}\right)^7 = cos\ (2k\pi) + i\ sin\ (2k\pi)$$

$$\therefore \frac{x+1}{x-1} = [cos\ (2k\pi) + i\ sin\ (2k\pi)]^{\frac{1}{7}}$$

$$\therefore \frac{x+1}{x-1} = [cos\ \left(\frac{2k\pi}{7}\right) + i\ sin\ \left(\frac{2k\pi}{7}\right)]..wherek = 0,1,2,3,4,5,6$$

$$\frac{x+1}{x-1} = \frac{cos\ \left(\frac{2k\pi}{7}\right) + i\ sin\ \left(\frac{2k\pi}{7}\right)}{1}..wherek = 0,1,2,3,4,5,6$$

Using Componendo and dividendo

$$\frac{x-1+x+1}{x-1-x-1} = \frac{1+cos\ \left(\frac{2k\pi}{7}\right) + i\ sin\ \left(\frac{2k\pi}{7}\right)}{1-cos\ \left(\frac{2k\pi}{7}\right) - i\ sin\ \left(\frac{2k\pi}{7}\right)}$$

$$\frac{2x}{-2} = \frac{2\ cos^2\left(\frac{k\pi}{7}\right) + i\ 2\ sin\ \left(\frac{k\pi}{7}\right)\ cos\ \left(\frac{k\pi}{7}\right)}{2\ sin^2\left(\frac{k\pi}{7}\right) - i\ 2\ sin\left(\frac{k\pi}{7}\right)\ cos\left(\frac{k\pi}{7}\right)}$$

$$\frac{x}{1} = \frac{2\ cos\ \left(\frac{k\pi}{7}\right)\left[cos\ \left(\frac{k\pi}{7}\right) + i\ sin\ \left(\frac{k\pi}{7}\right)\right]}{-2\ sin^2\left(\frac{k\pi}{7}\right) + i\ 2\ sin\left(\frac{k\pi}{7}\right)\ cos\left(\frac{k\pi}{7}\right)}$$

$$x = \frac{2\ cos\ \left(\frac{k\pi}{7}\right)\left[cos\ \left(\frac{k\pi}{7}\right) + i\ \ sin\ \left(\frac{k\pi}{7}\right)\right]}{i^2\ 2\ sin^2\left(\frac{k\pi}{7}\right) + i\ 2\ sin\left(\frac{k\pi}{7}\right)\ cos\left(\frac{k\pi}{7}\right)}$$

$$x = \frac{1}{i}\ \frac{cos\ \left(\frac{k\pi}{7}\right)\left[cos\ \left(\frac{k\pi}{7}\right) + i\ sin\ \left(\frac{k\pi}{7}\right)\right]}{sin\left(\frac{k\pi}{7}\right)\left[cos\ \left(\frac{k\pi}{7}\right) + i\ sin\ \left(\frac{k\pi}{7}\right)\right]}$$

$$x = \frac{1}{i}\ cot\left(\frac{k\pi}{7}\right)$$

$$x = -i\ cot\left(\frac{k\pi}{7}\right) wherek = 1, 2, 3, 4, 5, 6$$

For $k = 0$ x is undefined

For $k = 1$

$$x_1 = -icot\left(\frac{\pi}{7}\right)$$

For $k = 2$

$$x_1 = -icot\left(\frac{2\pi}{7}\right)$$

For $k = 3$

$$x_1 = -icot\left(\frac{3\pi}{7}\right)$$

For $k = 4$

$$x_1 = -icot\left(\frac{4\pi}{7}\right)$$

$$x_1 = i\ cot\left(\frac{3\pi}{7}\right)$$

For $k = 5$

$$x_1 = -icot\left(\frac{5\pi}{7}\right)$$

$$x_1 = i\ cot\left(\frac{2\pi}{7}\right)$$

For $k = 6$

$$x_1 = -icot\left(\frac{6\pi}{7}\right)$$

$$x_1 = i\ cot\left(\frac{\pi}{7}\right)$$

Hence roots of the equation $(x+1)^7 = (x-1)^7$ are of the form

$$\pm\ i\ cot\left(\frac{r\pi}{7}\right) ......wherer = 1, 2, 3$$