Q.1) Answer the following (20)

1a) Separate into real parts and imaginary of $\cos^{-1}\left(\frac{3i}{4}\right)$. (Chp: Hyperbolic Functions) (3)

Ans. Let $a+ib=\cos^{-1}\left(\frac{3i}{4}\right) \rightarrow (1)$

$\therefore\ \cos(a+ib)=\frac{3i}{4}$

$\therefore\ \cos a\cos(ib)-\sin a\sin(ib)=\frac{3i}{4}$

$\therefore\ \cos a\cdot\cosh b-i\sin a\cdot\sinh b=0+\frac{3i}{4}$

$\{\because \cos(ix)=\cosh x;\sin(ix)=i\sinh x;\}$

Comparing Real and Imaginary terms on both sides,

$\cos a\cosh b=0 \rightarrow (2)$ & $-\sin a\sinh b=\frac{3}{4} \rightarrow (3)$

From (2), $\cos a=0$ or $\cosh b=0$

$\therefore\ a=\frac{\pi}{2} \rightarrow (4)$

From (3) & (4), $-\sin\frac{\pi}{2}\sinh b=\frac{3}{4}$

$\therefore\ 1\cdot\sinh b=\frac{-3}{4}$

$\therefore\ b=\sinh^{-1}\left(\frac{-3}{4}\right)$

$=\log\left[\left(\frac{-3}{4}\right)+\sqrt{\left(\frac{-3}{4}\right)^2+1}\right]$

$\left\{\because \sinh^{-1}z=\log\left(z+\sqrt{z^2+1}\right)\right\}$

$=\log\left[\frac{-3}{4}+\sqrt{\frac{9}{16}+1}\right]$

$=\log\left[\frac{-3}{4}+\frac{5}{4}\right]$

$=\log\frac{1}{2}$

$=\log 2^{-1}$

$\therefore\ b=-\log 2 \rightarrow (5)$

Substituting (4) & (5) in (1), $\cos^{-1}\left(\frac{3i}{4}\right)=\frac{\pi}{2}-i\cdot\log 2$

Comparing Real and Imaginary terms on both sides,

Real part = $a=\frac{\pi}{2}$

Imaginary part = b = – log 2

1b) Show that the matrix $A=\begin{bmatrix}\alpha+iy & \beta+i\delta \\ \beta+i\delta & \alpha+iy\end{bmatrix}$ is unitary if $\alpha^2+y^2+\beta^2+\delta^2=1$. (Chp: Rank of Matrix) (3)

Ans. (Question is Wrong) $A=\begin{bmatrix}\alpha+iy & -\beta+i\delta \\ \beta+i\delta & \alpha-iy\end{bmatrix}$

$$\therefore\ A^T=\begin{bmatrix}\alpha+iy & \beta+i\delta \\ -\beta+i\delta & \alpha-iy\end{bmatrix}$$

$$\therefore\ A^\theta=\overline{A^T}=\begin{bmatrix}\alpha-iy & \beta-i\delta \\ -\beta-i\delta & \alpha+iy\end{bmatrix}$$

$$\therefore\ AA^\theta=\begin{bmatrix}\alpha+iy & -\beta+i\delta \\ \beta+i\delta & \alpha-iy\end{bmatrix}\times\begin{bmatrix}\alpha-iy & \beta-i\delta \\ -\beta-i\delta & \alpha+iy\end{bmatrix}$$

$$\therefore\ AA^\theta=\begin{bmatrix}(\alpha+iy)(\alpha-iy)+(-\beta+i\delta)(-\beta-i\delta) & (\alpha+iy)(\beta-i\delta)+(-\beta+i\delta)(\alpha+iy) \\ (\beta+i\delta)(\alpha-iy)+(\alpha-iy)(-\beta-i\delta) & (\beta+i\delta)(\beta-i\delta)+(\alpha-iy)(\alpha+iy)\end{bmatrix}\rightarrow(1)$$

Consider,

$$(\alpha+iy)(\alpha-iy)+(-\beta+i\delta)(-\beta-i\delta)=\alpha^2-i^2y^2+(-\beta)^2-i^2\delta^2=\alpha^2+y^2+\beta^2+\delta^2\ \rightarrow(2)$$

$$(\alpha+iy)(\beta-i\delta)+(-\beta+i\delta)(\alpha+iy)=(\cancel{\alpha\beta}-\cancel{i\alpha\delta}+\cancel{iy\beta}-\cancel{i^2y\delta})+(-\cancel{\alpha\beta}-\cancel{iy\beta}+\cancel{i\alpha\delta}+\cancel{i^2y\delta})=0\ \rightarrow(3)$$

Similarly, $(\beta+i\delta)(\alpha-iy)+(\alpha-iy)(-\beta-i\delta)=0\ \rightarrow(4)$ and,

$$(\beta+i\delta)(\beta-i\delta)+(\alpha-iy)(\alpha+iy)=\alpha^2+y^2+\beta^2+\delta^2\ \rightarrow(5)$$

Substituting (2), (3), (4) & (5) in (1), $AA^\theta=\begin{bmatrix}\alpha^2+y^2+\beta^2+\delta^2 & 0 \\ 0 & \beta^2+\delta^2+\alpha^2+y^2\end{bmatrix}$

A is unitary if and only if $AA^\theta=I$

$$\therefore\begin{bmatrix}\alpha^2+y^2+\beta^2+\delta^2 & 0 \\ 0 & \beta^2+\delta^2+\alpha^2+y^2\end{bmatrix}=\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

$$\therefore\ \boxed{\alpha^2+y^2+\beta^2+\delta^2=1}$$

1c) If $z=\tan(y+ax)+(y-ax)^{3/2}$ then show that $\dfrac{\partial^2 z}{\partial x^2}=a^2\dfrac{\partial^2 z}{\partial y^2}$. (Chp: Partial Differentiation) (3)

Ans. $z=\tan(y+ax)+(y-ax)^{3/2} \rightarrow (1)$

Differentiate partially w.r.t. x, $\dfrac{\partial z}{\partial x}=\sec^2(y+ax)\cdot a+\dfrac{3}{2}(y-ax)^{1/2}\cdot -a$

$$\therefore\ \frac{\partial z}{\partial x}=a\sec^2(y+ax)-\frac{3a}{2}(y-ax)^{1/2}$$

Again, differentiate partially w.r.t. x, $\dfrac{\partial^2 z}{\partial x^2}=a\cdot 2\sec(y+ax)\cdot\sec(y+ax)\tan(y+ax)\cdot a-\dfrac{3a}{2}\cdot\dfrac{1}{2}(y-ax)^{-1/2}\cdot -a$

$$=2a^2\sec^2(y+ax)\tan(y+ax)+\frac{3}{4}a^2(y-ax)^{-1/2}$$

$$\therefore\ \frac{\partial^2 z}{\partial x^2}=a^2\left[2\sec^2(y+ax)\tan(y+ax)+\frac{3}{4}(y-ax)^{-1/2}\right]\rightarrow (2)$$

Differentiate (1) partially w.r.t. y, $\dfrac{\partial z}{\partial y}=\sec^2(y+ax)\cdot 1+\dfrac{3}{2}(y-ax)^{1/2}\cdot 1$

Again, differentiate partially w.r.t. y, $\dfrac{\partial^2 z}{\partial y^2}=2\sec(y+ax)\cdot\sec(y+ax)\tan(y+ax)+\dfrac{3}{2}(y-ax)^{-1/2}$

$$\therefore\ \frac{\partial^2 z}{\partial y^2}=2\sec^2(y+ax)\tan(y+ax)+\frac{3}{4}(y-ax)^{-1/2}\rightarrow (3)$$

From (2) & (3), $\dfrac{\partial^2 z}{\partial x^2}=a^2\dfrac{\partial^2 z}{\partial y^2}$

1d) If $x = uv$, $y = \frac{u}{v}$. Prove that $JJ' = 1$. (Chp: Jacobian) (3)

Ans. $x = uv \rightarrow (1)$

$\therefore\ x_u = \frac{\partial x}{\partial u} = v$ and $x_v = \frac{\partial x}{\partial v} = u \rightarrow (2)$

And, $y = \frac{u}{v} \rightarrow (3)$

$\therefore\ y_u = \frac{\partial y}{\partial u} = \frac{1}{v}$ and $y_v = \frac{\partial y}{\partial v} = u \cdot \frac{-1}{v^2} \rightarrow (4)$

$$\therefore\ J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$= x_u\, y_v - x_v\, y_u$

$= v \cdot \frac{-u}{v^2} - u \cdot \frac{1}{v}$ (From 2 & 4)

$= \frac{-u}{v} - \frac{u}{v}$

$= \frac{-2u}{v}$

$\therefore\ J = -2y \rightarrow (5)$

From (3), $u = v\,y \rightarrow (6)$

Substituting 'u' in (1) we get, $x = (v\,y)v$

$\therefore\ \frac{x}{y} = v^2$

$\therefore\ v = \frac{\sqrt{x}}{\sqrt{y}} = x^{1/2} y^{-1/2} \rightarrow (7)$

$\therefore\ v_x = y^{-1/2} \cdot \frac{1}{2} x^{-1/2}$ and $v_y = x^{1/2} \cdot \frac{-1}{2} y^{-3/2} \rightarrow (8)$

From (6) and (7), $u = \left(x^{1/2} y^{-1/2}\right) y$

$\therefore\ u = x^{1/2} y^{1/2}$

$\therefore\ u_x = y^{1/2} \cdot \frac{1}{2} x^{-1/2}$ and $u_y = x^{1/2} \cdot \frac{1}{2} y^{-1/2} \rightarrow (9)$

$$\therefore\ J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$= u_x v_y - u_y v_x$

$$= \left(y^{1/2} \cdot \frac{1}{2} x^{-1/2}\right) \times \left(x^{1/2} \cdot \frac{-1}{2} y^{-3/2}\right) - \left(x^{1/2} \cdot \frac{1}{2} y^{-1/2}\right) \times \left(y^{-1/2} \cdot \frac{1}{2} x^{-1/2}\right) \text{ (From 8 \& 9)}$$

$= \frac{-1}{4} x^{\frac{-1}{2}+\frac{1}{2}} \cdot y^{\frac{1}{2}-\frac{3}{2}} - \frac{-1}{4} x^{\frac{1}{2}-\frac{1}{2}} \cdot y^{\frac{-1}{2}-\frac{1}{2}}$

$= \frac{-1}{4} y^{-1} - \frac{1}{4} y^{-1}$

$= \frac{-2}{4} y^{-1}$

$J' = \frac{-1}{2y} \rightarrow (10)$

From (5) and (10), $J \cdot J' = -\cancel{2y} \cdot \frac{-1}{\cancel{2y}}$

$\therefore\ J.J' = 1$

1e) Find the n[th] derivative of $\dfrac{x^3}{(x+1)(x-2)}$. (Chp: Successive Differentiation) (4)

Ans. Let $y=\dfrac{x^3}{(x+1)(x-2)}=\dfrac{x^3}{x^2-x-2}$

Consider,

$$\begin{array}{r|l} & \quad\quad\quad\quad x+1 \\ x^2-x-2 & x^3+0x^2+0x+0 \\ & \underline{x^3-\ x^2-2x} \\ & \quad\quad x^2+2x+0 \\ & \quad\quad \underline{x^2-\ x-2} \\ & \quad\quad\quad\quad 3x+2 \end{array}$$

$$\therefore\ y=x+1+\frac{3x+2}{x^2-x-2}$$

$$\therefore\ y=x+1+\frac{3x+2}{(x+1)(x-2)}$$

$$\therefore\ y=x+1+\frac{1/3}{x+1}+\frac{8/3}{x-2}$$ (By Partial Fractions)

Taking n[th] order derivative, $y_n=0+0+\dfrac{1}{3}\times\dfrac{n!\times 1^n(-1)^n}{(x+1)^{n+1}}+\dfrac{8}{3}\times\dfrac{n!\times 1^n(-1)^n}{(x-2)^{n+1}}$ $\left\{\text{If } y=\dfrac{1}{ax+b}\text{ then } y_n=\dfrac{n!\,a^n(-1)^n}{(ax+b)^{n+1}}\right\}$

$$\therefore\ \boxed{y_n=\frac{n!(-1)^n}{3}\left[\frac{1}{(x+1)^{n+1}}+\frac{8}{(x-2)^{n+1}}\right]}$$

1f) Using the matrix $A=\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix}$ decode the message of matrix $C=\begin{bmatrix}4 & 11 & 12 & -2\\ -4 & 4 & 9 & -2\end{bmatrix}$. (Chp: Coding) (4)

Ans. Encoding matrix $A=\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix} \rightarrow (1)$

Given, $C=\begin{bmatrix}4 & 11 & 12 & -2\\ -4 & 4 & 9 & -2\end{bmatrix}$

Step 1:

Writing the numbers in C matrix column wise gives the encoded message.

$\therefore$ Encoded Message = 4 –4 11 4 12 9 –2 –2

This encoded message is transmitted.

Assume there is no corruption of data, the message at the receiving end is 4 –4 11 4 12 9 –2 –2

This message is decoded

Step 2:

We know, if $P=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then $P^{-1}=\frac{1}{|P|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

From (1), | A | = –1 + 2 = 1 → (2)

$\therefore$ Decoding matrix $A^{-1}=\frac{1}{1}\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}$ (From 2) → (3)

From (2) & (3), $A^{-1}C=\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}\times\begin{bmatrix}4 & 11 & 12 & -2\\ -4 & 4 & 9 & -2\end{bmatrix}$

$$\therefore\ A^{-1}C=\begin{bmatrix}4+8 & 11-8 & 12-18 & -2+4\\ 4+4 & 11-4 & 12-9 & -2+2\end{bmatrix}$$

$$\therefore\ A^{-1}C=\begin{bmatrix}12 & 3 & -6 & 2\\ 8 & 7 & 3 & 0\end{bmatrix}$$

Step 3:

Considering the numbers column-wise we get,

12 8 3 7 –6 3 2 0

Decoded Message = $\boxed{12\ \ 8\ \ 3\ \ 7\ \ -6\ \ 3\ \ 2\ \ 0}$ or $\boxed{\begin{bmatrix}12 & 3 & -6 & 2\\ 8 & 7 & 3 & 0\end{bmatrix}}$

Q.2)

2a) If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$ then find a, b, c, d. (Chp: Complex - DMT) (6)

Ans. We know, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \rightarrow (1)$

Consider, $\sin^4\theta\cos^3\theta = \left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right)^4 \times \left(\dfrac{e^{i\theta} + e^{-i\theta}}{2}\right)^3$ (From 1)

$$= \frac{1}{2^4 i^4 \times 2^3} \times \left(e^{i\theta} - e^{-i\theta}\right)\left(e^{i\theta} - e^{-i\theta}\right)^3 \left(e^{i\theta} + e^{-i\theta}\right)^3$$

$$= \frac{1}{2^7 \times 1} \times \left(e^{i\theta} - e^{-i\theta}\right)\left[\left(e^{i\theta}\right)^2 - \left(e^{-i\theta}\right)^2\right]^3$$

$$= \frac{1}{2^7}\left(e^{i\theta} - e^{-i\theta}\right)\left(e^{2i\theta} - e^{-2i\theta}\right)^3$$

$$= \frac{1}{2^7}\left(e^{i\theta} - e^{-i\theta}\right)\left[\left(e^{2i\theta}\right)^3 - 3\left(e^{2i\theta}\right)^2\left(e^{-2i\theta}\right) + 3\left(e^{2i\theta}\right)\left(e^{-2i\theta}\right)^2 - \left(e^{-2i\theta}\right)^3\right]$$

$$= \frac{1}{2^7}\left(e^{i\theta} - e^{-i\theta}\right)\left[e^{6i\theta} - 3e^{2i\theta} + 3e^{-2i\theta} - e^{-6i\theta}\right]$$

$$= \frac{1}{2^7}\left[e^{7i\theta} - 3e^{3i\theta} + 3e^{-i\theta} - e^{-5i\theta} - e^{5i\theta} + 3e^{i\theta} - 3e^{-3i\theta} + e^{-7i\theta}\right]$$

$$= \frac{1}{2^7}\left[\left(e^{7i\theta} + e^{-7i\theta}\right) - \left(e^{5i\theta} + e^{-5i\theta}\right) - 3\left(e^{3i\theta} + e^{-3i\theta}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right)\right]$$

$$= \frac{1}{128}\left[2\cos7\theta - 2\cos5\theta - 3\times2\cos3\theta + 3\times2\cos\theta\right] \text{ (From 1)}$$

$$= \frac{1}{128}\times2\cos7\theta - \frac{1}{128}\times2\cos5\theta - \frac{1}{128}\times6\cos3\theta + \frac{1}{128}\times6\cos\theta$$

$$\therefore \sin^4\theta\cos^3\theta = \frac{3}{64}\cos\theta - \frac{3}{64}\cos3\theta - \frac{1}{64}\cos5\theta + \frac{1}{64}\cos7\theta \rightarrow (2)$$

But, given, $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta \rightarrow (3)$

Comparing (2) and (3), $a = \dfrac{3}{64}$; $b = \dfrac{-3}{64}$; $c = \dfrac{-1}{64}$; $d = \dfrac{1}{64}$;

2b) Using Newton Raphson method solve 3x – cos x – 1 = 0. Correct to 3 decimal places.
(Chp: Transcendental equations) (6)

Ans. Let $f(x) = 3x - \cos x - 1$

$\therefore\ f'(x) = 3 + \sin x - 0$

When x = 0, $f(0) = 3(0) - \cos 0 - 1 = -2$

When x = 1, $f(1) = 3(1) - \cos 1 - 1 = 1.4597$

$\therefore$ Root of f (x) lies between 0 and 1.

Let initial value $x_0 = 0$

By Newton-Raphson's Method $x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n(3 + \sin x_n) - (3x_n - \cos x_n - 1)}{3 + \sin x_n}$$

$$= \frac{\cancel{3x_n} + x_n \sin x_n - \cancel{3x_n} + \cos x_n + 1}{3 + \sin x_n}$$

$$\therefore\ x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \rightarrow (1)$$

Iteration 1: Put n = 0 in (1)

$$\therefore\ x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0 + \cos 0 + 1}{3 + \sin 0} = 0.6667$$

Iteration 2: Put n = 1 in (1)

$$\therefore\ x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6667 \sin(0.6667) + \cos(0.6667) + 1}{3 + \sin(0.6667)} = 0.6075$$

Iteration 3: Put n = 2 in (1)

$$\therefore\ x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{3 + \sin x_2} = \frac{0.6075 \sin(0.6075) + \cos(0.6075) + 1}{3 + \sin(0.6075)} = 0.6071$$

Iteration 4: Put n = 3 in (1)

$$\therefore\ x_4 = \frac{x_3 \sin x_3 + \cos x_3 + 1}{3 + \sin x_3} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} = 0.6071$$

Hence, Root of 3x – cos x – 1 = 0 is 0.6071

2c) Find the stationary points of the function $x^3+3xy^2-3x^2-3y^2+4$ & also find maximum and minimum values of the function. (Chp: Maxima and Minima) (8)

Ans. Let $f(x,y)=x^3+3xy^2-3x^2-3y^2+4 \rightarrow (1)$

$\therefore\ f_x=3x^2+3y^2-6x-0+0$

$\therefore\ r=f_{xx}=6x-6 \ \rightarrow (2)$

Also, $f_y=0+6xy-0-6y+0$

$\therefore\ t=f_{yy}=6x-6 \rightarrow (3)$

$\therefore\ s=f_{xy}=0+6y-0 \ \rightarrow (4)$

Put $f_x=0$ and $f_y=0$

$\therefore\ 3x^2+3y^2-6x=0$

$\therefore\ x^2+y^2-2x=0 \rightarrow (5)$

And, $6xy-6y=0$

$\therefore\ 6y(x-1)=0$

$\therefore$ y = 0 or x = 1

Case I: When x = 1

From (5), $1^2+y^2-2(1)=0$

$\therefore\ y^2-1=0$

$\therefore$ y = ± 1

Case II: When y = 0

From (5), $x^2+0-2x=0$

$\therefore$ x (x – 2) = 0

$\therefore$ x = 0 or x = 2

$\therefore$ Stationary points are (1, 1); (1, –1); (0, 0); (2, 0);

(i) At (1, 1)

From (2), r = 6 (1) – 6 = 0

$\therefore$ f is neither maximum or minimum at (1, 1)

(ii) At (1, –1)

From (2), r = 6 (1) – 6 = 0

$\therefore$ f is neither maximum or minimum at (1, –1)

(iii) At (0, 0)

From (2), r = 6 (0) – 6 = –6 < 0

From (3), t = 6 (0) – 6 = –6

From (4), s = 6 (0) = 0

$\therefore\ rt-s^2$ = (–6) (–6) – 0 = 36 > 0

$\therefore$ f has maximum at (0, 0)

From (1), Maximum value of

$f=(0)^3+3(0)(0)^2-3(0)^2-3(0)^2+4=4$

(iv) At (2, 0)

From (2), r = 6 (2) – 6 = 6 > 0

From (3), t = 6 (2) – 6 = 6

From (4), s = 6 (0) = 0

$\therefore\ rt-s^2$ = (6) (6) – 0 = 36 > 0

$\therefore$ f has minimum at (2, 0)

From (1), Minimum value

$f=(2)^3+3(2)(0)^2-3(2)^2-3(0)^2+4=0$

Hence, the Function has

Maximum at (0, 0) and Maximum value = 4

Minimum at (2, 0) and Maximum value = 0

Q.3)

3a) Show that $x\operatorname{cosec} x = 1+\frac{x^2}{6}+\frac{7x^4}{360}+\ldots$ (Chp: Expansion) (6)

Ans. LHS $= x\operatorname{cosec} x$

$$= \frac{x}{\sin x}$$

$$= \frac{x}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots}$$

$$= \frac{\cancel{x}}{\cancel{x}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots\right)}$$

$$= \left[1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}-\ldots\right)\right]^{-1}$$

$$= 1+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}-\ldots\right)+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}-\ldots\right)^2+\ldots \quad \left\{\because (1-y)^{-1}=1+y+y^2+y^3+\ldots\right\}$$

$$= 1+\frac{x^2}{3!}-\frac{x^4}{5!}+\left(\frac{x^2}{3!}\right)^2+\ldots$$

$$= 1+\frac{x^2}{6}+\frac{7x^4}{360}+\ldots$$

$$\therefore \boxed{x\operatorname{cosec} x = 1+\frac{x^2}{6}+\frac{7x^4}{360}+\ldots}$$

3b) Reduce matrix to PAQ normal form and find 2 non-Singular matrices P & Q. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$.

(Chp: Rank of Matrix) (6)

Ans. Let $A_{3\times4} = I_{3\times3} \times A_{3\times4} \times I_{4\times4}$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1;\ R_3 - R_1;$ $$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1;\ C_3 + C_1;\ C_4 - 2C_1;$ $$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 + C_2;\ \frac{1}{2}C_3;$ $$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

LHS is the required PAQ form.

Here, $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Q.4)

4a) State and prove Euler's Theorem for three Variables. (Chp: Homogenous Functions) (6)

Ans. Euler's theorem:

Statement: If 'u' is a homogenous function of three variables x, y, z of degree 'n' then Euler's theorem states that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

Proof:

Let u = f(x, y, z) be the homogenous function of degree 'n'.

Let X = xt, Y = yt , Z = zt

$$\therefore\ \frac{\partial X}{\partial t}=x;\ \frac{\partial Y}{\partial t}=y\ \ \&\ \ \frac{\partial Z}{\partial t}=z\ \rightarrow(1)$$

At t = 1, $\rightarrow$ (2)

X = x, Y = y and Z = z

$$\therefore\ \frac{\partial f}{\partial X}=\frac{\partial f}{\partial x};\ \frac{\partial f}{\partial Y}=\frac{\partial f}{\partial y}\ \ \&\ \ \frac{\partial f}{\partial Z}=\frac{\partial f}{\partial z}\ \rightarrow(3)$$

Now, $f(X,Y,Z)=t^n f(x,y,z)\ \rightarrow(4)$

$\therefore f\rightarrow$ X, Y, Z $\rightarrow$ x, y, z, t

Differentiating (4) partially w.r.t. 't', $\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial t}+\frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial t}+\frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial t}=nt^{n-1}f(x,y,z)$

$$\therefore\ \frac{\partial f}{\partial x}\cdot x+\frac{\partial f}{\partial y}\cdot y+\frac{\partial f}{\partial z}\cdot z=n(1)^{n-1}f(x,y,z)\ \text{(From 1, 2 \& 3)}$$

$$\therefore\ \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu}\ \rightarrow(5)$$

4b) Show that all the roots of $(x+1)^6+(x-1)^6=0$ are given by $-i\cot\frac{(2k+1)n}{12}$ where k= 0, 1, 2, 3, 4, 5.

(Chp: Complex - DMT) (6)

Ans. $(x+1)^6+(x-1)^6=0$

$\therefore\ (x+1)^6=-(x-1)^6$

$\therefore\ \frac{(x+1)^6}{(x-1)^6}=-1$

$\therefore\ \left(\frac{x+1}{x-1}\right)^6=e^{i\pi}\ \left\{\because e^{i\pi}=\cos\pi+i\sin\pi=-1+i(0)=-1\right\}$ (Principal Value)

$\therefore\ \left(\frac{x+1}{x-1}\right)^6=e^{i(\pi+2k\pi)}$, k = 0, 1, 2, 3, 4, 5 (General value)

$\therefore\ \frac{x+1}{x-1}=e^{i\pi(1+2k)/6}\rightarrow(1)$

Let $2\theta=\frac{\pi(1+2k)}{6}\rightarrow(2)$

$\therefore$ From (1) & (2), $\frac{x+1}{x-1}=e^{i2\theta}$

$\therefore$ By Componendo – Dividendo, $\frac{(x+\cancel{1})+(x-\cancel{1})}{(\cancel{x}+1)-(\cancel{x}-1)}=\frac{e^{i2\theta}+1}{e^{i2\theta}-1}$

$\therefore\ \frac{\cancel{2}x}{\cancel{2}}=\frac{\cancel{e^{i\theta}}\left[e^{i\theta}+e^{-i\theta}\right]}{\cancel{e^{i\theta}}\left[e^{i\theta}-e^{-i\theta}\right]}$

$\therefore\ x=\frac{\cancel{2}\cos\theta}{\cancel{2}i\sin\theta}\qquad\left\{\because\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}\text{ and }\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}\right\}$

$\therefore\ x=\frac{1}{i}\cot\theta$

$\therefore\ \boxed{x=-i\cot\left(\frac{2k+1}{12}\right)\pi}$ (From 2) where k = 0, 1, 2, 3, 4, 5

4c) Show that the following equations: –2x + y + z = a, x – 2y + z = b, x + y – 2z = c have no solutions unless a + b + c = 0 in which case they have infinitely many solutions. Find these solutions when a = 1, b = 1, c = – 2. (Chp: Linear Equations) (8)

Ans. Part I:

–2x + y + z = a

x – 2y + z = b

x + y – 2z = c

Writing the equations in the matrix form,

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$R_3 + (R_1 + R_2) \Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b+c \end{bmatrix} \rightarrow (1)$$

Augmented Matrix [A | B] = $\left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$

Number of unknowns = n = 3

Rank of A (r_A) = Number of non-zero rows in A = 2

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when 'a + b + c ≠ 0' upon which,

Rank of [A | B] = (r_{AB}) = 3

Case II: Infinite Solution

For which, $r_A = r_{AB} < n$ (i.e. < 3)

This is only possible, when 'a + b + c = 0' upon which,

Rank of [A | B] = (r_{AB}) = 2

Part II: Put a = 1, b = 1, c = – 2, in (1)

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$R_2 - R_1 ; \Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow (2)$$

Here, $n - r_A = 3 - 2 = 1$

We have to assume one unknown.

Let y = t (≠ 0)

On expanding (2), 3x – 3y = 0

∴ x – y = 0

∴ x = y = t

And, – 2x + y + z = 1

∴ – 2t + t + z = 1

∴ z = 1 + t

Hence, the solution is

x = t; y = t; z = 1 + t (Infinite Solutions)

Q.5)

5a) If z = f(x, y), x = r cos θ, y = r sin θ, prove that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$. (Chp: Partial Differentiation)

(6)

Ans. $x=r\cos\theta$ and $y=r\sin\theta \quad \to (1)$

Differentiating partially w.r.t. 'θ', $\frac{\partial x}{\partial \theta}=-r\sin\theta$; $\frac{\partial y}{\partial \theta}=r\cos\theta$; $\to$ (2)

Differentiating partially w.r.t. 'r', $\frac{\partial x}{\partial r}=\cos\theta$; $\frac{\partial y}{\partial r}=\sin\theta \to$ (3)

Now, z → x, y → r, θ

By Chain Rule, $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial r}$

$\therefore \frac{\partial z}{\partial r}=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}$ (From 3) → (4)

Similarly, By Chain Rule, $\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial \theta}$

$\therefore \frac{\partial z}{\partial \theta}=-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}$ (From 2) → (5)

$$\text{RHS}=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$=\left(\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}\right)^2+\frac{1}{r^2}\left(-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}\right)^2 \quad \text{(From 4 \& 5)}$$

$$=\cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2+2\cos\theta\frac{\partial z}{\partial x}\cdot\sin\theta\frac{\partial z}{\partial y}+\sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2+\frac{1}{r^2}\cdot r^2\left[\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2-2\sin\theta\frac{\partial z}{\partial x}\cdot\cos\theta\frac{\partial z}{\partial y}+\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2\right]$$

$$=\cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2+\cancel{2\sin\theta\frac{\partial z}{\partial x}\cos\theta\frac{\partial z}{\partial y}}+\sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2+\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2-\cancel{2\sin\theta\frac{\partial z}{\partial x}\cos\theta\frac{\partial z}{\partial y}}+\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2$$

$$=\left(\frac{\partial z}{\partial x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{\partial z}{\partial y}\right)^2\left(\cos^2\theta+\sin^2\theta\right)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

= LHS

Hence, $\boxed{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2}$

5b) If cosh x = sec θ prove that (i) x = log (sec θ + tan θ). (ii) $\theta=\frac{\pi}{2}-2\tan^{-1}\left(e^{-x}\right)$. (Chp: Hyperbolic Functions)(6)

Ans. (i) cosh x = sec θ

$$\therefore\ \frac{e^{x}+e^{-x}}{2}=\sec\theta \quad \left\{\because \cosh x=\frac{e^{x}+e^{-x}}{2}\right\}$$

$$\therefore\ e^{x}+\frac{1}{e^{x}}=2\sec\theta$$

$$\therefore\ \left(e^{x}\right)^{2}+1=2\sec\theta\, e^{x}$$

$$\therefore\ \left(e^{x}\right)^{2}-2\sec\theta\, e^{x}+1=0$$

$$\therefore\ e^{x}=\frac{2\sec\theta\pm\sqrt{4\sec^{2}\theta-4\times1\times1}}{2\times1}$$

$$\left\{\because \text{Using, } x=\frac{-b\pm\sqrt{b^{2}-4ac}}{2a}\right\}$$

$$\therefore\ e^{x}=\frac{2\sec\theta\pm\sqrt{4\left(\sec^{2}\theta-1\right)}}{2}$$

$$\therefore\ e^{x}=\frac{2\sec\theta\pm2\tan\theta}{2}$$

$$\therefore\ e^{x}=\sec\theta\pm\tan\theta$$

Considering only positive root,

$$\therefore\ e^{x}=\sec\theta+\tan\theta\rightarrow(1)$$

∴ x = log (sec θ + tan θ)

(ii) From (1), $e^{x}=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}$

$$\therefore\ e^{x}=\frac{1+\sin\theta}{\cos\theta}$$

$$\therefore\ \frac{1}{e^{x}}=\frac{\cos\theta}{1+\sin\theta}$$

$$\therefore\ e^{-x}=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)}$$

Put $\alpha=\frac{\pi}{2}-\theta\rightarrow(2)$

$$\therefore\ e^{-x}=\frac{\sin\alpha}{1+\cos\alpha}$$

$$\therefore\ e^{-x}=\frac{\cancel{2}\sin(\alpha/2)\cancel{\cos(\alpha/2)}}{\cancel{2}\cos^{\cancel{2}}(\alpha/2)}$$

$$\left\{\because 2\sin A\cos A=\sin2A;1+\cos2A=2\cos^{2}A\right\}$$

$$\therefore\ e^{-x}=\tan\left(\frac{\alpha}{2}\right)$$

$$\therefore\ \tan^{-1}\left(e^{-x}\right)=\frac{\alpha}{2}$$

$$\therefore\ 2\tan^{-1}\left(e^{-x}\right)=\alpha$$

$$\therefore\ 2\tan^{-1}\left(e^{-x}\right)=\frac{\pi}{2}-\theta \text{ (From 2)}$$

$$\therefore\ \theta=\frac{\pi}{2}-2\tan^{-1}\left(e^{-x}\right)$$

5c) Solve by Gauss Jacobi Iteration Method: 5x – y + z = 10, 2x + 4y = 12, x + y + 5z = –1.
(Chp: Linear algebraic equations) (8)

Ans From 1st equation, 5x = 10 + y – z

$\therefore\ x = \frac{1}{5}(10 + y - z) = 0.2(10 + y - z)$

Similarly,

From 2nd equation, x + 2y = 6

$\therefore$ 2y = 6 – x

$y = \frac{1}{2}(6 - x) = 0.5(6 - x)$ and,

$z = 0.2(-1 - x - y)\ z = -0.2(1 + x + y)$

Iteration 1:

Put $x_0 = y_0 = z_0 = 0$

$\therefore\ x_1 = 0.2(10 + y_0 - z_0) = 0.2(10 + 0 - 0) = 2$

$\therefore\ y_1 = 0.5(6 - x_0) = 0.5(6 - 0) = 3$

$\therefore\ z_1 = -0.2(1 + x_0 + y_0) = -0.2(1 + 0 + 0) = -0.2$

Iteration 2:

Put $x_1 = 2;\ y_1 = 3;\ z_1 = -0.2$

$\therefore\ x_2 = 0.2(10 + y_1 - z_1) = 0.2(10 + 3 + 0.2) = 2.64$

$\therefore\ y_2 = 0.5(6 - x_1) = 0.5(6 - 2) = 2$

$\therefore\ z_2 = -0.2(1 + x_1 + y_1) = -0.2(1 + 2 + 3) = -1.2$

Iteration 3:

Put $x_2 = 2.64;\ y_2 = 2;\ z_2 = -1.2$

$\therefore\ x_3 = 0.2(10 + y_2 - z_2) = 0.2(10 + 2 + 1.2) = 2.64$

$\therefore\ y_3 = 0.5(6 - x_2) = 0.5(6 - 2.64) = 1.68$

$\therefore\ z_3 = -0.2(1 + x_2 + y_2) = -0.2(1 + 2.64 + 2)$

$= -1.128$

Iteration 4:

Put $x_3 = 2.64;\ y_3 = 1.68;\ z_3 = -1.128$

$\therefore\ x_4 = 0.2(10 + y_3 - z_3) = 0.2(10 + 1.68 + 1.128)$

$= 2.5616$

$\therefore\ y_4 = 0.5(6 - x_3) = 0.5(6 - 2.64) = 1.68$

$\therefore\ z_4 = -0.2(1 + x_3 + y_3) = -0.2(1 + 2.64 + 1.68)$

$= -1.0640$

Iteration 5:

Put $x_4 = 2.5616;\ y_4 = 1.68;\ z_4 = -1.0640$

$\therefore\ x_5 = 0.2(10 + y_4 - z_4) = 0.2(10 + 1.68 + 1.064)$

$= 2.5488$

$\therefore\ y_5 = 0.5(6 - x_4) = 0.5(6 - 2.5616) = 1.7192$

$\therefore\ z_5 = -0.2(1 + x_4 + y_4) = -0.2(1 + 2.5616 + 1.68)$

$= -1.0483$

Iteration 6:

Put $x_5 = 2.5488;\ y_5 = 1.7192;\ z_5 = -1.0483$

$\therefore\ x_6 = 0.2(10 + y_5 - z_5) = 0.2(10 + 1.7192 + 1.0483)$

$= 2.5535$

$\therefore\ y_6 = 0.5(6 - x_5) = 0.5(6 - 2.5488) = 1.7256$

$\therefore\ z_6 = -0.2(1 + x_5 + y_5) = -0.2(1 + 2.5488 + 1.7192)$

$= -1.0536$

Iteration 7:

Put $x_6 = 2.5535;\ y_6 = 1.7256;\ z_6 = -1.0536$

$\therefore\ x_7 = 0.2(10 + y_6 - z_6) = 0.2(10 + 1.7256 + 1.0536)$

$= 2.5558$

$\therefore\ y_7 = 0.5(6 - x_6) = 0.5(6 - 2.5535) = 1.7232$

$\therefore\ z_7 = -0.2(1 + x_6 + y_6) = -0.2(1 + 2.5535 + 1.7256)$

$= -1.0558$

Iteration 8:

Put $x_7 = 2.5558;\ y_7 = 1.7232;\ z_7 = -1.0558$

$\therefore\ x_8 = 0.2(10 + y_7 - z_7) = 0.2(10 + 1.7232 + 1.0558)$

$= 2.5558$

$\therefore\ y_8 = 0.5(6 - x_7) = 0.5(6 - 2.5558) = 1.7221$

$\therefore\ z_8 = -0.2(1 + x_7 + y_7) = -0.2(1 + 2.5558 + 1.7232)$

$= -1.0558$

Hence, by Gauss Jacobi Iteration Method, the solution is
x = 2.5558, y = 1.7221, z = –1.0558

Q.6)

6a) Prove that $\cos^{-1}\left[\tanh(\log x)\right]=\pi-2\left(x-\frac{x^3}{3}+\frac{x^5}{5}\ldots\right)$. (Chp: Expansion) (6)

Ans. We know, $\tanh\theta=\frac{\sinh\theta}{\cosh\theta}$

$$=\frac{\left(e^{\theta}-e^{-\theta}\right)/\cancel{2}}{\left(e^{\theta}+e^{-\theta}\right)/\cancel{2}}$$

$$\therefore\ \tanh\theta=\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$$

Put $\theta=\log x$, $\rightarrow(1)$

$$\therefore\ \tanh(\log x)=\frac{e^{\log x}-e^{-\log x}}{e^{\log x}+e^{-\log x}}$$

$$=\frac{e^{\log x}-e^{\log x^{-1}}}{e^{\log x}+e^{\log x^{-1}}}$$

$$=\frac{x-x^{-1}}{x+x^{-1}}$$

$$=\frac{\cancel{x}\left(1-x^{-2}\right)}{\cancel{x}\left(1+x^{-2}\right)}\rightarrow(2)$$

Let $y=\cos^{-1}\left[\tanh(\log x)\right]$

$$=\cos^{-1}\left[\frac{1-x^{-2}}{1+x^{-2}}\right]\ \text{(From 2)}$$

$$=\cos^{-1}\left[\frac{1-\left(x^{-1}\right)^2}{1+\left(x^{-1}\right)^2}\right]$$

Put $x^{-1}=\tan\theta$

$$\therefore\ y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$=\cos^{-1}(\cos 2\theta)$$

$$=2\theta$$

$$=2\tan^{-1}\left(\frac{1}{x}\right)\ \text{(From 1)}$$

$$=2\cot^{-1}x$$

$$=2\left(\frac{\pi}{2}-\tan^{-1}x\right)$$

$$=\pi-2\tan^{-1}x$$

$$=\pi-2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)$$

$$\left\{\because \tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right\}$$

Hence, $\cos^{-1}\left[\tanh(\log x)\right]=\pi-2\left(x-\frac{x^3}{3}+\frac{x^5}{5}\ldots\right)$

6b) If $y=e^{2x}\sin\frac{x}{2}\cos\frac{x}{2}\sin 3x$. Find y_n . (Chp: Successive Differentiation) (6)

Ans. $y=e^{2x}\sin\frac{x}{2}\cos\frac{x}{2}\sin 3x\times\frac{2}{2}$

$=\frac{1}{2}e^{2x}\left[\sin\cancel{2}\left(\frac{x}{\cancel{2}}\right)\right]\sin 3x \quad \{\because 2\sin A\cos A=\sin 2A\}$

$=\frac{1}{2}e^{2x}\sin x\sin 3x\times\frac{2}{2}$

$=\frac{1}{4}e^{2x}\left[\cos(3x-x)-\cos(3x+x)\right] \quad \{\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)\}$

$\therefore\ y=\frac{1}{4}\left[e^{2x}\cos 2x-e^{2x}\cos 4x\right]$

Taking n[th] order derivative, $y_n=\frac{1}{4}\left\{\frac{d^n}{dx^n}\left(e^{2x}\cos 2x\right)-\frac{d^n}{dx^n}\left(e^{2x}\cos 4x\right)\right\}\rightarrow(1)$

We know, If $y=e^{ax}\cos(bx+c)$, $y_n=r^ne^{ax}\cos(bx+c+n\phi)\ \rightarrow(2)$

Here, a = 2, c = 0, $b_1=2$ and $b_2=4$

$\therefore\ r_1=\sqrt{a^2+b_1^{\ 2}}=\sqrt{2^2+2^2}=\sqrt{8}=8^{1/2}$ and $r_2=\sqrt{a^2+b_2^{\ 2}}=\sqrt{2^2+4^2}=\sqrt{20}=20^{1/2}\ \rightarrow(3)$

And, $\phi_1=\tan^{-1}\frac{b_1}{a}=\tan^{-1}\frac{2}{2}=\tan^{-1}1=\frac{\pi}{4}$ & $\phi_2=\tan^{-1}\frac{b_2}{a}=\tan^{-1}\frac{2}{4}=\tan^{-1}\frac{1}{2}\ \rightarrow(4)$

$\therefore$ From (1), (2), (3) and (4),

$y_n=\frac{1}{4}\left\{\left(8^{1/2}\right)^ne^{2x}\cos\left(2x+0+n\phi_1\right)-\left(20^{1/2}\right)^ne^{2x}\cos\left(4x+0+n\phi_2\right)\right\}$

$\therefore\ y_n=\frac{1}{4}e^{2x}\left[8^{n/2}\cos\left(2x+\frac{n\pi}{4}\right)-20^{n/2}\cos\left(4x+n\phi_2\right)\right]$, where $\phi_2=\tan^{-1}\frac{1}{2}$

6c) Evaluate $\lim\limits_{x \to 0} (\cot x)^{\sin x}$. (Chp: Indeterminate Forms) (4)

Ans. Let $L = \lim\limits_{x \to 0} (\cot x)^{\sin x}$

$$\therefore \log L = \log\left\{\lim_{x \to 0} (\cot x)^{\sin x}\right\}$$

$$= \lim_{x \to 0} \left\{\log(\cot x)^{\sin x}\right\}$$

$$= \lim_{x \to 0} \sin x \cdot \log(\cot x)$$

$$= \lim_{x \to 0} \frac{\log(\cot x)}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{\cot x} \cdot -\operatorname{cosec}^{\cancel{2}} x}{-\cancel{\operatorname{cosec}} x \cot x} \quad \text{(L' Hospital's Rule)}$$

$$= \lim_{x \to 0} \tan x \cdot \frac{1}{\sin x} \cdot \tan x$$

$$= \lim_{x \to 0} \tan x \cdot \frac{1}{\cancel{\sin x}} \times \frac{\cancel{\sin x}}{\cos x}$$

$$= \tan 0 \times \frac{1}{\cos 0}$$

$$\therefore \log L = 0$$

$$\therefore L = e^0$$

$$\therefore \boxed{\lim_{x \to 0} (\cot x)^{\sin x} = 1}$$

6d) Prove that $\log\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right]=2i\tan^{-1}(\cot x\tanh y)$. (Chp: Log of Complex Numbers) (4)

Ans. Consider, $\log[\sin(x+iy)]=\log[\sin x\cos(iy)+\cos x\sin(iy)]$

$$\therefore\ \log[\sin(x+iy)]=\log[\sin x\cosh y+i\cos x\sinh y]\quad\{\because \cos(ix)=\cosh x;\sin(ix)=i\sinh x;\}$$

$$\therefore\ \log[\sin(x+iy)]=\frac{1}{2}\log\left[\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right]+i\tan^{-1}\left|\frac{\cos x\sinh y}{\sin x\cosh y}\right|$$

$$\left\{\because \log(x+iy)=\frac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\left|\frac{y}{x}\right|\right\}$$

$$\therefore\ \log[\sin(x+iy)]=\frac{1}{2}\log\left[\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right]+i\tan^{-1}|\cot x\tanh y|\ \rightarrow(1)$$

Taking Conjugates, $\log[\sin(x-iy)]=\dfrac{1}{2}\log\left[\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right]-i\tan^{-1}|\cot x\tanh y|\ \rightarrow(2)$

Now, $\log\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right]=\log[\sin(x+iy)]-\log[\sin(x-iy)]$

$$=\left\{\cancel{\frac{1}{2}\log\left[\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right]}+i\tan^{-1}|\cot x\tanh y|\right\}-$$

$$\left\{\cancel{\frac{1}{2}\log\left[\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\right]}-i\tan^{-1}|\cot x\tanh y|\right\}\ \text{(From 1 \& 2)}$$

$$=2i\tan^{-1}(\cot x\tanh y)$$

$$\therefore\ \boxed{\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right]=2i\tan^{-1}(\cot x\tanh y)}$$