Q.1)

1a) Prove that $\tanh^{-1}(\sin\theta)=\cosh^{-1}(\sec\theta)$. (Chp: Hyperbolic Functions) (3)

Ans. We know, $\sin^2\theta+\cos^2\theta=1 \rightarrow (1)$ and $\text{sech}^2 x=1-\tanh^2 x \rightarrow (2)$

Let $\tanh^{-1}(\sin\theta)=x \rightarrow (3)$

$\therefore\ \sin\theta=\tanh x$

On squaring, $\sin^2\theta=\tanh^2 x$

$\therefore\ \cancel{1}-\cos^2\theta=\cancel{1}-\text{sech}^2 x$ (From 1 & 2)

$\therefore\ \cos^2\theta=\text{sech}^2 x$

On taking square roots, $\cos\theta=\text{sech}\, x$

$\therefore\ \dfrac{1}{\cos\theta}=\dfrac{1}{\text{sech}\, x}$

$\therefore\ \sec\theta=\cosh x$

$\therefore\ x=\cosh^{-1}(\sec\theta) \rightarrow (4)$

Hence, from (3) & (4), $\boxed{\tanh^{-1}(\sin\theta)=\cosh^{-1}(\sec\theta)}$

1b) Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary. (Chp: Rank of Matrix) (3)

Ans. Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$\therefore$ Taking Transpose, $A^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

$\therefore$ Taking conjugate, $\left(\overline{A^T}\right) = A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Consider, $A^\theta A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$= \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1+(1+i)(1-i) & (1+i)-(1+i) \\ (1-i)-(1-i) & (1-i)(1+i)+1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $A^\theta A = I$

Similarly, we can prove $AA^\theta = I$

Hence, $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

1d) If $x=uv$, $y=\frac{u}{v}$. Prove that $JJ'=1$. (Chp: Jacobian) (3)

Ans. x = uv → (1)

$\therefore\ x_u=\frac{\partial x}{\partial u}=v$ and $x_v=\frac{\partial x}{\partial v}=u \rightarrow (2)$

And, $y=\frac{u}{v} \rightarrow (3)$

$\therefore\ y_u=\frac{\partial y}{\partial u}=\frac{1}{v}$ and $y_v=\frac{\partial y}{\partial v}=u\cdot\frac{-1}{v^2} \rightarrow (4)$

$\therefore\ J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$=x_u y_v - x_v y_u$

$=v\cdot\frac{-u}{v^2}-u\cdot\frac{1}{v}$ (From 2 & 4)

$=\frac{-u}{v}-\frac{u}{v}$

$=\frac{-2u}{v}$

$\therefore\ J=-2y \rightarrow (5)$

From (3), u = v y → (6)

Substituting ‘u’ in (1) we get, $x=(vy)v$

$\therefore\ \frac{x}{y}=v^2$

$\therefore\ v=\frac{\sqrt{x}}{\sqrt{y}}=x^{1/2}y^{-1/2} \rightarrow (7)$

$\therefore\ v_x=y^{-1/2}\cdot\frac{1}{2}x^{-1/2}$ and $v_y=x^{1/2}\cdot\frac{-1}{2}y^{-3/2} \rightarrow (8)$

From (6) and (7), $u=\left(x^{1/2}y^{-1/2}\right)y$

$\therefore\ u=x^{1/2}y^{1/2}$

$\therefore\ u_x=y^{1/2}\cdot\frac{1}{2}x^{-1/2}$ and $u_y=x^{1/2}\cdot\frac{1}{2}y^{-1/2} \rightarrow (9)$

$\therefore\ J'=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$=u_x v_y - u_y v_x$

$=\left(y^{1/2}\cdot\frac{1}{2}x^{-1/2}\right)\times\left(x^{1/2}\cdot\frac{-1}{2}y^{-3/2}\right)-$
$\left(x^{1/2}\cdot\frac{1}{2}y^{-1/2}\right)\times\left(y^{-1/2}\cdot\frac{1}{2}x^{-1/2}\right)$ (From 8 & 9)

$=\frac{-1}{4}x^{\frac{-1}{2}+\frac{1}{2}}\cdot y^{\frac{1}{2}-\frac{3}{2}}-\frac{-1}{4}x^{\frac{1}{2}-\frac{1}{2}}\cdot y^{\frac{-1}{2}-\frac{1}{2}}$

$=\frac{-1}{4}y^{-1}-\frac{1}{4}y^{-1}$

$=\frac{-2}{4}y^{-1}$

$J'=\frac{-1}{2y} \rightarrow (10)$

From (5) and (10), $J\cdot J'=-2\cancel{y}\cdot\frac{-1}{2\cancel{y}}$

$\therefore$ J . J’ = 1

1d) If $Z=\tan^{-1}\left(\frac{x}{y}\right)$ where x = 2t, $y=1-t^2$, prove that $\frac{dZ}{dt}=\frac{2}{1+t^2}$. (Chp: Partial Differentiation) (3)

Ans. Method I: Using Partial Differentiation

x = 2t, $y=1-t^2 \rightarrow$ (1)

Differentiating w.r.t. t,

$\frac{dx}{dt}=2$; $\frac{dy}{dt}=0-2t=-2t$; $\rightarrow$ (2)

Given, $Z=\tan^{-1}\left(\frac{x}{y}\right)$

$$\therefore\ \frac{\partial Z}{\partial x}=\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$=\frac{y^2}{y^2+x^2}\cdot\frac{1}{y}$$

$$=\frac{y}{y^2+x^2}\rightarrow \text{(3) and,}$$

$$\frac{\partial Z}{\partial y}=\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$$=\frac{y^2}{y^2+x^2}\cdot x\cdot\frac{-1}{y^2}$$

$$=\frac{-x}{y^2+x^2}\rightarrow (4)$$

Now, Z → x, y → t

$$\therefore\ \frac{dZ}{dt}=\frac{\partial Z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial Z}{\partial y}\cdot\frac{dy}{dt}$$

$$=\frac{y}{y^2+x^2}\cdot 2+\frac{-x}{y^2+x^2}\cdot -2t \quad \text{(From 2, 3 and 4)}$$

$$=\frac{2}{y^2+x^2}(y+tx)$$

$$=\frac{2}{(1-t^2)^2+(2t)^2}\left[\left(1-t^2\right)+t(2t)\right] \quad \text{(From 1)}$$

$$=\frac{2}{1-2t^2+t^4+4t^2}\left[1-t^2+2t^2\right]$$

$$=\frac{2}{1+2t^2+t^4}\left(1+t^2\right)$$

$$=\frac{2}{(1+t^2)^{\cancel{2}}}\cdot\cancel{(1+t^2)}$$

$$\therefore\ \boxed{\frac{dZ}{dt}=\frac{2}{1+t^2}}$$

Method II: Without Partial Differentiation

$Z=\tan^{-1}\left(\frac{x}{y}\right)$

Put x = 2t, $y=1-t^2$

$$\therefore\ Z=\tan^{-1}\left(\frac{2t}{1-t^2}\right)$$

Put $t=\tan\theta$ or $\theta=\tan^{-1}t$

$$\therefore\ Z=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$=\tan^{-1}(\tan 2\theta)$$

$$=2\theta$$

$$=2\tan^{-1}t$$

Differentiating w.r.t. t, $\frac{dz}{dt}=2\cdot\frac{1}{1+t^2}$

$$\therefore\ \boxed{\frac{dZ}{dt}=\frac{2}{1+t^2}}$$

1e) Find the nth derivative of cos 5x · cos 3x · cos x. (Chp: Successive Differentiation) (4)

Ans. Let $y=\cos 5x\cdot\cos 3x\cdot\cos x\times\frac{2}{2}$

$$=\frac{1}{2}\cos 5x\left(\cos 4x+\cos 2x\right)\times\frac{2}{2}\quad\left\{\text{Using, } 2\cos A\cos B=\cos\left(A+B\right)+\cos\left(A-B\right)\right\}$$

$$=\frac{1}{4}\left(2\cos 5x\cos 4x+2\cos 5x\cos 2x\right)$$

$$=\frac{1}{4}\left(\cos 9x+\cos x+\cos 7x+\cos 3x\right)\quad\left\{\text{Using, } 2\cos A\cos B=\cos\left(A+B\right)+\cos\left(A-B\right)\right\}$$

We know, if $y=\cos\left(ax+b\right)$ then $y_n=a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$

Taking n^{th} order derivative

$$y_n=\frac{1}{4}\left[9^n\cos\left(9x+\frac{n\pi}{2}\right)+7^n\cos\left(7x+\frac{n\pi}{2}\right)+3^n\cos\left(3x+\frac{n\pi}{2}\right)+\cos\left(x+\frac{n\pi}{2}\right)\right]$$

1f) Evaluate $\lim\limits_{x \to 0} (x)^{1/1-x}$. (Chp: Indeterminate Forms) (4)

Ans. Let $L = \lim\limits_{x \to 0} (x)^{1/1-x}$

$$\therefore \log L = \lim_{x \to 0} \log (x)^{1/1-x}$$

$$= \lim_{x \to 0} \frac{1}{1-x} \log x$$

Put y = 1 – x

$\therefore$ x = 1 – y

As x $\to$ 0, y $\to$ 1

$$\therefore \log L = \lim_{y \to 1} \frac{1}{y} \log(1-y)$$

$$= \lim_{y \to 1} \frac{1}{y} \times \left(-y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots \right)$$

$$= \lim_{y \to 1} \left(-1 - \frac{y}{2} - \frac{y^2}{3} - \frac{y^3}{4} - \ldots \right)$$

$$= -\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \right)$$

$$\therefore \log L = \infty$$

$$\therefore L = e^{-\infty}$$

$$\therefore \boxed{\lim_{x \to 0} (x)^{1/1-x} = 0}$$

Q.2)

2a) Find all the value of $(1+i)^{1/3}$ & show that their continued product is (1 + i). (Chp: Complex - DMT) (6)

Ans. Let $z=(1+i)^{1/3}$

$\therefore\ z^3=1+i=x+iy$

$\therefore$ x = y = 1

$\therefore$ z^3 lies in first quadrant.

$\therefore$ Argument (θ) $=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{1}{1}\right|=\frac{\pi}{4}$

$\therefore$ Modulus $r=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2}$

$\therefore$ In polar form, $z^3=re^{-i\theta}$

$\therefore\ 1+i=\sqrt{2}\,e^{-i\pi/4}\rightarrow(1)$ (Principal Value)

$=\sqrt{2}\,e^{-i\left(\frac{\pi}{4}+2n\pi\right)}$ (General Value)

$=2^{1/2}e^{-i\left(\frac{\pi+8n\pi}{4}\right)}$

$\therefore\ z^3=2^{\frac{1}{2}}e^{-i(1+8n)\frac{\pi}{4}}$

$\therefore\ z=\left(2^{\frac{1}{2}}e^{-i(1+8n)\frac{\pi}{4}}\right)^{\frac{1}{3}}$

$\therefore\ z=2^{1/6}e^{-i(1+8n)\frac{\pi}{12}}$

Put n = 0, $z_1=2^{1/6}e^{-i\frac{\pi}{12}}=2^{1/6}\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)$

Put n = 1, $z_2=2^{1/6}e^{-i\frac{9\pi}{12}}=2^{1/6}e^{-i\frac{3\pi}{4}}$

$=2^{1/6}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$

$=2^{1/6}\left(\frac{-1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$

$=2^{1/6}\times\frac{-1}{\sqrt{2}}(1+i)$

$=2^{1/6}\times-2^{-1/2}(1+i)$

$=-2^{-1/3}(1+i)$

$\therefore\ z_2=2^{1/6}e^{-i\frac{3\pi}{4}}=-2^{-1/3}(1+i)$

Put n = 2, $z_3=2^{1/6}e^{-i\frac{17\pi}{12}}$

$=2^{1/6}e^{-i\left(2\pi-\frac{7\pi}{12}\right)}$

$=2^{1/6}e^{-i2\pi}e^{i\frac{7\pi}{12}}$

$=2^{1/6}\times1\times\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$

$\therefore\ z_3=2^{1/6}e^{-i\frac{17\pi}{12}}=2^{1/6}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$

$\therefore$ Continued Product of all the roots $=z_1\cdot z_2\cdot z_3$

$=2^{1/6}e^{-i\frac{\pi}{12}}.2^{1/6}e^{-i\frac{9\pi}{12}}.2^{1/6}e^{-i\frac{17\pi}{12}}$

$=\left(2^{1/6}\right)^3e^{-i\left(\frac{\pi}{12}+\frac{9\pi}{12}+\frac{17\pi}{12}\right)}$

$=2^{1/2}e^{-i\frac{27\pi}{12}}$

$=\sqrt{2}\,e^{-i\frac{9\pi}{4}}$

$=\sqrt{2}\,e^{-i\left(2\pi+\frac{\pi}{4}\right)}$

$=\sqrt{2}\,e^{-i\frac{\pi}{4}}$

$=1+i$ (From 1)

Hence, Continued Product of all the roots is (1 + i)

2b) Find non-singular matrices P & Q such that PAQ is in normal form where $A=\begin{bmatrix}2&-2&3\\3&-1&2\\1&2&-1\end{bmatrix}$.

(Chp: Rank of Matrix) (6)

Ans. Let $A_{3\times3}=I_{3\times3}\times A_{3\times3}\times I_{3\times3}$

$$\therefore \begin{bmatrix}2&-2&3\\3&-1&2\\1&2&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$R_3\leftrightarrow R_1$ $$\Rightarrow\begin{bmatrix}1&2&-1\\3&-1&2\\2&-2&3\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$R_2-3R_1;\ R_3-2R_1$ $$\Rightarrow\begin{bmatrix}1&2&-1\\0&-7&5\\0&-6&5\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$C_2-2C_1;\ C_3+C_1$ $$\Rightarrow\begin{bmatrix}1&0&0\\0&-7&5\\0&-6&5\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-2&1\\0&1&0\\0&0&1\end{bmatrix}$$

$-R_2+R_3$ $$\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&-6&5\end{bmatrix}=\begin{bmatrix}0&0&1\\1&-1&1\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-2&1\\0&1&0\\0&0&1\end{bmatrix}$$

R_3+6R_2 $$\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&5\end{bmatrix}=\begin{bmatrix}0&0&1\\1&-1&1\\7&-6&4\end{bmatrix}A\begin{bmatrix}1&-2&1\\0&1&0\\0&0&1\end{bmatrix}$$

$\frac{1}{5}C_3$ $$\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&1\\1&-1&1\\7&-6&4\end{bmatrix}A\begin{bmatrix}1&-2&1/5\\0&1&0\\0&0&1/5\end{bmatrix}$$

RHS is the required PAQ form and the LHS is the Normal form.

Here, $P=\begin{bmatrix}0&0&1\\1&-1&1\\7&-6&4\end{bmatrix}$ and $Q=\begin{bmatrix}1&-2&1/5\\0&1&0\\0&0&1/5\end{bmatrix}$.

2c) Find the maximum & minimum values of $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$. (Chp: Maxima and Minima)(8)

Ans. S1: Let $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$

$f_x=3x^2+3y^2-30x+72$

Let $r=f_{xx}=6x-30 \rightarrow (1)$

$f_y=6xy-30y$

Let $t=f_{yy}=6x-30 \rightarrow (2)$

Let $s=f_{xy}=6y \rightarrow (3)$

S2: Put $f_x=0$ and $f_y=0$

$\therefore\ 3x^2+3y^2-30x+72=0$

$\therefore\ x^2+y^2-10x+24=0 \rightarrow (4)$ (Dividing by 3)

And, $6xy-30y=0$

$\therefore\ 6y(x-5)=0$

Substituting y = 0 in (4), $x^2-10x+24=0$

$\therefore$ x = 4 or x = 6

Substituting x = 5 in (4), $5^2+y^2-10\times5+24=0$

$\therefore\ y^2-1=0$

$\therefore$ y = 1 or y = –1

$\therefore$ Stationary Points are (4, 0); (6, 0); (5, 1); (5, –1)

S3:

(i) At (4, 0)

From (1), r = 6 (4) – 30 = –6 < 0

From (2), t = 6 (4) – 30 = –6 < 0

From (3), s = 6 (0) = 0

$\therefore\ rt-s^2$ = (–6) (–6) – 0 = 36 > 0

$\therefore$ *f* has maximum at (4, 0)

$\therefore\ f_{\max}=(4)^3+0-15(4)^2-0+72(4)$

$\therefore$ Maximum value $f_{\max}=112$

(ii) At (6, 0)

From (1), r = 6 (6) – 30 = 6 > 0

From (2), t = 6 (6) – 30 = 6 > 0

From (3), s = 6 (0) = 0

$\therefore\ rt-s^2$ = (6) (6) – 0 = 36 > 0

$\therefore$ *f* has minimum at (6, 0)

$\therefore\ f_{\min}=(6)^3+0-15(6)^2-0+72(6)$

Minimum value $f_{\min}=108$

(iii) At (5, ± 1)

From (1), r = 6 (5) – 30 = 0

From (2), t = 6 (5) – 30 = 0

From (3), s = 6 (± 1) = ± 6

$\therefore\ rt-s^2=(0)(0)-(\pm1)^2=-1$

$\therefore$ Maximum or minimum cannot be found at (5, ± 1)

Q.3)

3a) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial z} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$. (Chp: Partial Differentiation) (6)

Ans. (Question is wrong) Correct question is

Show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Let $v = \frac{y-x}{xy}$ and $w = \frac{z-x}{xz}$

$\therefore\ v = \frac{y}{xy} - \frac{x}{xy} = \frac{1}{x} - \frac{1}{y}$ and $w = \frac{z}{xz} - \frac{x}{xz} = \frac{1}{x} - \frac{1}{z}$

$\therefore\ \frac{\partial v}{\partial x} = \frac{-1}{x^2}$; $\frac{\partial v}{\partial y} = \frac{1}{y^2}$; $\frac{\partial v}{\partial z} = 0 \rightarrow (1)$ and

$\frac{\partial w}{\partial x} = \frac{-1}{x^2}$; $\frac{\partial w}{\partial y} = 0$; $\frac{\partial w}{\partial z} = \frac{1}{z^2}$; $\rightarrow (2)$

Now, u $\rightarrow$ v, w $\rightarrow$ x, y, z

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\times\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\times\frac{\partial w}{\partial x}$$

$$= u_v \times \frac{-1}{x^2} + u_w \times \frac{-1}{x^2} \quad \text{(From 1 and 2)}$$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{-u_v - u_w}{x^2}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\times\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\times\frac{\partial w}{\partial y}$

$$= u_v \times \frac{1}{y^2} + u_w \times 0 \quad \text{(From 1 and 2)}$$

$$\therefore\ \frac{\partial u}{\partial y} = \frac{u_v}{y^2}$$

And, $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\times\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\times\frac{\partial w}{\partial z}$

$$= u_v \times 0 + u_w \times \frac{1}{z^2} \quad \text{(From 1 and 2)}$$

$$\therefore\ \frac{\partial u}{\partial z} = \frac{u_w}{z^2}$$

$$\therefore\ x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} =$$

$$= x^2\left(\frac{-u_v - u_w}{x^2}\right) + y^2\left(\frac{u_v}{y^2}\right) + z^2\left(\frac{u_w}{z^2}\right)$$

$$= -u_v - u_w + u_v + u_w$$

$$= 0$$

Hence, $\boxed{x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0}$

3b) Using encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, encode & decode the message 'MUMBAI'. (Chp: Coding) (6)

Ans. We use following numerical values of each alphabet for coding

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	*
15	16	17	18	19	20	21	22	23	24	25	26	27

Step 1:

Message: MUMBAI

As per the above table, the numerical values of each alphabet in the message are

M	U	M	B	A	I
13	21	13	2	1	9

Step 2:

Writing the above numerical values column-wise in a 2 row matrix we get, $A = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$

Encoding matrix $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow (1)$

Now, $EA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$

$\therefore EA = \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix} \rightarrow (2)$

Writing the numbers in EA matrix column wise gives the encoded message.

$\therefore$ Encoded Message = 34 21 15 2 10 9

This encoded message is transmitted.

Step 3:

Assume there is no corruption of data, the message at the receiving end is 34 21 15 2 10 9

This message is decoded

We know, if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $P^{-1} = \frac{1}{|P|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

From (1), | E | = 1 – 0 = 1 → (3)

$\therefore E^{-1} = \frac{1}{|E|}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ Decoding matrix $E^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (From 3) → (4)

From (2) & (4), $E^{-1}(EA) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$

Step 4:

Considering the numbers column-wise we get,

13 21 13 2 1 9 27

Reconverting each of the above numbers into corresponding alphabet,

Decoded Message = MUMBAI

3c) Prove that $\log\left[\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)\right]=i\tan^{-1}(\sinh x)$. (Chp: Log of Complex Numbers) (8)

Ans. LHS = $\log\left[\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)\right]$

$$=\log\left[\frac{\tan(\pi/4)+\tan(ix/2)}{1-\tan(\pi/4)\tan(ix/2)}\right]$$

$$=\log\left[\frac{1+i\tanh(x/2)}{1-i\tanh(x/2)}\right]$$

$$=\log\left[1+i\tanh\frac{x}{2}\right]-\log\left[1-i\tanh\frac{x}{2}\right]$$

$$=\left\{\cancel{\frac{1}{2}\log\left(1^2+\tanh^2\frac{x}{2}\right)}+i\tan^{-1}\left(\frac{\tanh(x/2)}{1}\right)\right\}-\left\{\cancel{\frac{1}{2}\log\left(1^2+\tanh^2\frac{x}{2}\right)}+i\tan^{-1}\left(\frac{-\tanh(x/2)}{1}\right)\right\}$$

$\left\{\because \text{ For Principal Values, } \log(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left|\frac{y}{x}\right|\right\}$

$$=i\tan^{-1}\left(\tanh\frac{x}{2}\right)-i\tan^{-1}\left(-\tanh\frac{x}{2}\right)$$

$$=i\left[\tan^{-1}\left(\tanh\frac{x}{2}\right)+\tan^{-1}\left(\tanh\frac{x}{2}\right)\right] \quad \left\{\because \tan^{-1}a+\tan^{-1}b=\tan^{-1}\left(\frac{a+b}{1-ab}\right)\right\}$$

$$=i\tan^{-1}\left[\frac{\tanh(x/2)+\tanh(x/2)}{1-\tanh(x/2)\tanh(x/2)}\right]$$

$$=i\tan^{-1}\left[\frac{2\tanh(x/2)}{1-\tanh^2(x/2)}\right] \quad \left\{\because \text{Sinh } 2A=\frac{2\tanh A}{1-\tanh^2 A}\right\}$$

$$=i\tan^{-1}(\sinh x)$$

= RHS

Hence, $\boxed{\log\left[\tan\left(\frac{\pi}{4}+\frac{ix}{2}\right)\right]=i\tan^{-1}(\sinh x)}$

Q.4)

4a) Obtain tan 5θ in terms of tan θ and show that $1-10\tan^2\frac{\pi}{10}+5\tan^4\frac{\pi}{10}=0$. (Chp: Complex - DMT) (6)

Ans. By De Moivre's Theorem, $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

Let c = cos θ and s = sin θ

$\therefore\ \cos 5\theta + i\sin 5\theta = (c+is)^5$

$= c^5 + 5c^4\cdot is + 10c^3\cdot i^2 s^2 + 10c^2\cdot i^3 s^3 + 5c\cdot i^4 s^4 + i^5 s^5$

$= c^5 + i5c^4 s - 10c^3 s^2 - i10c^2 s^3 + 5cs^4 + is^5$

$\therefore\ \cos 5\theta + i\sin 5\theta = \left(c^5 - 10c^3 s^2 + 5cs^4\right) + i\left(5c^4 s - 10c^2 s^3 + s^5\right)$

Comparing real and Imaginary parts, $\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$ & $\sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5 \rightarrow (1)$

Now, $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$\therefore\ \tan 5\theta = \dfrac{5c^4 s - 10c^2 s^3 + s^5}{c^5 - 10c^3 s^2 + 5cs^4}$ (From 1)

$\therefore\ \tan 5\theta = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$

Dividing N and D by $\cos^5\theta$, $\boxed{\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}}$

Put $\theta = \dfrac{\pi}{10}$,

$\therefore\ \tan\left(\dfrac{5\pi}{10}\right) = \dfrac{5\tan(\pi/10) - 10\tan^3(\pi/10) + \tan^5(\pi/10)}{1 - 10\tan^2(\pi/10) + 5\tan^4(\pi/10)}$

$\therefore\ \dfrac{1}{0} = \dfrac{5\tan(\pi/10) - 10\tan^3(\pi/10) + \tan^5(\pi/10)}{1 - 10\tan^2(\pi/10) + 5\tan^4(\pi/10)}$ $\left\{\because \tan\left(\dfrac{5\pi}{10}\right) = \tan\dfrac{\pi}{2} = \text{N.D.} = \dfrac{1}{0}\right\}$

$\therefore$ On cross multiplication, $\boxed{1 - 10\tan^2\dfrac{\pi}{10} + 5\tan^4\dfrac{\pi}{10} = 0}$

4b) If $y = e^{\tan^{-1}x}$, prove that $\left(1+x^2\right)y_{n+2} + \left[2(n+1)x-1\right]y_{n+2} + n(n+1)y_n = 0$. (Chp: Successive Differentiation)(6)

Ans. $y = e^{\tan^{-1}x} \rightarrow (1)$

Differentiating w.r.t. x, $\dfrac{dy}{dx} = e^{\tan^{-1}x} \cdot \dfrac{d}{dx}\tan^{-1}x$

$$\therefore \frac{dy}{dx} = y \cdot \frac{1}{1+x^2} \text{ (From 1)}$$

$$\therefore \left(1+x^2\right)\frac{dy}{dx} = y$$

Again, differentiating w.r.t. x, $\left(1+x^2\right)\cdot\dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] + \dfrac{dy}{dx}\cdot\dfrac{d}{dx}\left(1+x^2\right) = \dfrac{d}{dx}y$

$$\therefore \left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(0+2x) = \frac{dy}{dx}$$

$$\therefore \left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\therefore \left(1+x^2\right)y_2 + (2x-1)y_1 = 0$$

Using Leibnitz theorem the nth order derivative is,

$$\therefore \left\{\left(1+x^2\right)\cdot y_{n+2} + n\cdot 2x\cdot y_{n+1} + \frac{n(n-1)}{\cancel{2}}\cdot \cancel{2}\cdot y_n\right\} + \left\{(2x-1)\cdot y_{n+1} + n\cdot(2-0)\cdot y_n\right\} = 0$$

$$\therefore \left(1+x^2\right)y_{n+2} + 2nxy_{n+1} + \left(n^2-n\right)y_n + (2x-1)y_{n+1} + 2ny_n = 0$$

$$\therefore \left(1+x^2\right)y_{n+2} + \left[2nx+2x-1\right]y_{n+1} + \left(n^2-n+2n\right)y_n = 0$$

$$\therefore \boxed{\left(1+x^2\right)y_{n+2} + \left[2(n+1)x-1\right]y_{n+1} + n(n+1)y_n = 0}$$

Hence, proved.

4c) Express $\left(2x^3+3x^2-8x+7\right)$ in terms of (x – 2) using Taylor's theorem. (Chp: Expansion) (4)

Ans. Let $f(x)=2x^3+3x^2-8x+7$

$\therefore\ f'(x)=6x^2+6x-8$

$\therefore\ f''(x)=12x+6$

$\therefore\ f'''(x)=12$

Let a = 2

$\therefore\ f(a)=f(2)=2(2)^3+3(2)^2-8(2)+7=19$

$\therefore\ f'(a)=f'(2)=6(2)^2+6(2)-8=28$

$\therefore\ f''(a)=f''(2)=12(2)+6=30$

$\therefore\ f'''(a)=f'''(2)=12$

By Taylor Series, $f(x)=f(a)+\dfrac{(x-a)}{1!}f'(a)+\dfrac{(x-a)^2}{2!}f''(a)+\dfrac{(x-a)^3}{3!}f'''(a)+\dots$

$\therefore\ f(x)=19+(x-2)\cdot 28+\dfrac{1}{2}(x-2)^2\cdot 30+\dfrac{1}{6}(x-2)^3\cdot 12$

$\therefore\ 2x^3+3x^2-8x+7=19+28(x-2)+15(x-2)^2+2(x-2)^3$

4d) Prove that $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ (Chp: Expansion) (4)

Ans. Part I:

Let $y = \tan^{-1}x$

Differentiating w.r.t. x, $\frac{dy}{dx} = \frac{1}{1+x^2}$

$= (1+x^2)^{-1}$

$= 1 - x^2 + (x^2)^2 - (x^2)^3 + \ldots$

$\therefore \frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + \ldots$

$\therefore dy = (1 - x^2 + x^4 - x^6 + \ldots)dx$

$\therefore$ On Integration, $y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

Part II:

From (1), $\frac{dy}{dx} = \frac{1}{1+x^2}$

$= 1 - x^2 + x^4 - x^6 + \ldots$

Multiplying throughout by 2x, $\frac{2x}{1+x^2} = 2(x - x^3 + x^5 - x^7 + \ldots)$

On Integration, $\log(1+x^2) = 2\left[\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \ldots\right]$

$\therefore \log(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$

Q.5)

5a) If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, prove that $\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$. (Chp: Partial Differentiation) (6)

Ans. $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Partially differentiate w.r.t. x, $\frac{\partial z}{\partial x} = \left[x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right) \cdot 2x\right] - \left[y^2 \cdot \frac{1}{1+(x/y)^2} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)\right]$

$$= \cancel{x^2} \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot y \cdot \frac{-1}{\cancel{x^2}} + 2x \tan^{-1}\left(\frac{y}{x}\right) - y^{\cancel{2}} \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{\cancel{y}}$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2}{x^2+y^2} \cdot y - y \cdot \frac{y^2}{y^2+x^2}$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - y \cdot \left[\frac{x^2}{x^2+y^2} + \frac{y^2}{y^2+x^2}\right]$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - y \cdot \frac{\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}$$

$$\therefore \frac{\partial z}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y$$

Now, Partially differentiating w.r.t. y, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x \cdot \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) - 1$

$$\frac{\partial^2 z}{\partial y \partial x} = 2\cancel{x} \cdot \frac{x^2}{x^2+y^2} \cdot \frac{1}{\cancel{x}} - 1$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{2x^2 - \left(x^2+y^2\right)}{x^2+y^2}$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}}$$

5b) Investigate for what values of λ & μ the equation, 2x + 3y + 5z = 9; 7x + 3y – 2z = 8; 2x + 3y + λz = μ have (i) no solution (ii) a unique solution (iii) an infinite no. of solutions. (Chp: Linear Equations) (6)

Ans. 2x + 3y + 5z = 9

7x + 3y – 2z = 8

2x + 3y + λz = μ

Writing the equations in the matrix form, $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

$R_3 - R_1 \quad \Rightarrow \quad \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu - 9 \end{bmatrix}$

Augmented Matrix [A | B] = $\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$

Number of unknowns = n = 3

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when μ ≠ 9 and λ = 5.

We then have, rank of A (r_A) = 2 and rank of [A | B] = 3

Case II: Unique Solution

For which, $r_A = r_{AB} = n$

This is only possible, when λ ≠ 5 and μ has any value.

We then have, rank of A (r_A) = rank of [A | B] = 3

Case III: Infinite Solution

For which, $r_A = r_{AB} < n$ (i.e. < 3)

This is only possible, when μ = 9 and λ = 5.

We then have, rank of A (r_A) = rank of [A | B] = 2

Hence,

No Solution	μ ≠ 9, λ = 5
Unique Solution	μ = any value, λ ≠ 5
Infinite Solution	μ = 9, λ = 5

5c) Using Newton Raphson method, find approximate root of $x^3 - 2x - 5 = 0$ (correct to three places of decimals). (Chp: Transcendental equations) (8)

Ans. Let $f(x) = x^3 - 2x - 5$

$\therefore\ f'(x) = 3x^2 - 2\ f'(x)$

When x = –3, $f(-3) = (-3)^3 - 2(-3) - 5 = -16$

When x = –2, $f(-2) = (-2)^3 - 2(-2) - 5 = 1$

$\therefore$ Root of f (x) lies between –3 and –2.

Let initial value $x_0 = -2$

By Newton-Raphson's Method $x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$

$$= x_n - \frac{x_n^3 - 2x_n + 5}{3x_n^2 - 2}$$

$$= \frac{x_n(3x_n^2 - 2) - (x_n^3 - 2x_n + 5)}{3x_n^2 - 2}$$

$$= \frac{3x_n^3 - \cancel{2x_n} - x_n^3 + \cancel{2x_n} - 5}{3x_n^2 - 2}$$

$$\therefore\ x_{n+1} = \frac{2x_n^3 - 5}{3x_n^2 - 2} \rightarrow (1)$$

Iteration 1: Put n = 0 in (1)

$$\therefore\ x_1 = \frac{2x_0^3 - 5}{3x_0^2 - 2} = \frac{2(-2)^3 - 5}{3(-2)^2 - 2} = -2.1$$

Iteration 2: Put n = 1 in (1)

$$\therefore\ x_2 = \frac{2x_1^3 - 5}{3x_1^2 - 2} = \frac{2(-2.1)^3 - 5}{3(-2.1)^2 - 2} = -2.0946$$

Iteration 3: Put n = 2 in (1)

$$\therefore\ x_3 = \frac{2x_2^3 - 5}{3x_2^2 - 2} = \frac{2(-2.0946)^3 - 5}{3(-2.0946)^2 - 2} = -2.0946$$

Hence, Root of $x^3 - 2x - 5 = 0$ is –2.0946

Q.6)

6a) Find tanh x, if 5sinh x – cosh x = 5. (Chp: Hyperbolic Functions) (6)

Ans. 5sinh x – cosh x = 5

$$\therefore\ 5\left(\frac{e^x-e^{-x}}{2}\right)-\left(\frac{e^x+e^{-x}}{2}\right)=5\quad \left\{\because \sinh x=\frac{e^x-e^{-x}}{2}\ \&\cosh x=\frac{e^x+e^{-x}}{2}\right\}$$

$$\therefore\ \frac{5}{2}\left(e^x-\frac{1}{e^x}\right)-\frac{1}{2}\left(e^x+\frac{1}{e^x}\right)=5$$

$$\therefore\ \frac{5}{2}e^x-\frac{5}{2e^x}-\frac{1}{2}e^x-\frac{1}{2e^x}=5$$

$$\therefore\ 2e^x-\frac{3}{e^x}=5$$

Multiplying by e^x, $2\left(e^x\right)^2-3=5e^x$

$$\therefore\ 2\left(e^x\right)^2-5e^x-3=0$$

$$\therefore\ 2\left(e^x\right)^2-6e^x+e^x-3=0$$

$$\therefore\ 2e^x\left(e^x-3\right)+1\left(e^x-3\right)=0$$

$$\therefore\ \left(2e^x+1\right)\left(e^x-3\right)=0$$

$$\therefore\ e^x=\frac{-1}{2}\ \text{or}\ e^x=3$$

But e^x cannot be negative.

$$\therefore\ e^x=3$$

$$\therefore\ e^{-x}=\frac{1}{e^x}=\frac{1}{3}$$

$$\therefore\ \tanh x=\frac{\sinh x}{\cosh x}$$

$$=\frac{\left(e^x-e^{-x}\right)/\cancel{2}}{\left(e^x+e^{-x}\right)/\cancel{2}}$$

$$=\frac{3-\frac{1}{3}}{3+\frac{1}{3}}$$

$$=\frac{8/3}{10/3}$$

$$\therefore\ \boxed{\tanh x=\frac{4}{5}}$$

6b) If $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that (i) $xu_x+yu_y=\frac{1}{2}\tan u$ (ii) $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{-\sin u\cos 2u}{4\cos^3 u}$.

(Chp: Homogenous Functions) (6)

Ans. $u(x,y)=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$\therefore\ \sin u(x,y)=\frac{x+y}{\sqrt{x}+\sqrt{y}}\rightarrow(1)$

$\therefore\ \sin u(X,Y)=\frac{X+Y}{\sqrt{X}+\sqrt{Y}}$

Now, Put X = x t, Y = y t

$\therefore\ \sin u(X,Y)=\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}$

$=\frac{t(x+y)}{\sqrt{t}\left(\sqrt{x}+\sqrt{y}\right)}$

$=\sqrt{t}\sin u(x,y)$

$\sin u(X,Y)=t^{1/2}\sin u(x,y)$ (From 1)

$\therefore$ sin u is homogenous function of degree $(n)=\frac{1}{2}$.

Let $f(u)=\sin u$

$\therefore\ f'(u)=\cos u$

Let $g(u)=n\frac{f(u)}{f'(u)}$

$=\frac{1}{2}\cdot\frac{\sin u}{\cos u}$

$=\frac{\tan u}{2}$

$\therefore\ g'(u)=\frac{1}{2}\sec^2 u$

Using Euler's Theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g(u)$

$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\tan u}{2}$

$\therefore\ xu_x+yu_y=\frac{1}{2}\tan u$

Using Corollary to Euler's Theorem,

$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\cdot\left[g'(u)-1\right]$

$\therefore\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{2}\cdot\left[\frac{1}{2}\sec^2 u-1\right]$

$=\frac{\sin u}{2\cos u}\cdot\left[\frac{1}{2\cos^2 u}-1\right]$

$=\frac{\sin u}{2\cos u}\cdot\left[\frac{1-2\cos^2 u}{2\cos^2 u}\right]$

$=\frac{\sin u}{4\cos^3 u}\cdot-\left(2\cos^2 u-1\right)$

$\therefore\ x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\frac{-\sin u\cos 2u}{4\cos^3 u}$

6c) Solve the following system or equation by Gauss-Seidel method:
20 x + y – 2z = 17; 3x + 20y – z = – 18; 2x – 3y + 20z = 25. (Chp: Linear Equations) (8)

Ans From 1st equation, 20 x = 17 + 2z – y

$\therefore\ x = \frac{1}{20}(17+2z-y) = 20^{-1}(17+2z-y)$

Similarly,

$y = 20^{-1}(-18+z-3x)$ and $z = 20^{-1}(25-2x+3y)$

Iteration 1:

Put $y_0 = z_0 = 0$

$\therefore\ x_1 = 20^{-1}(17+2z_0-y_0)$

$= 20^{-1}(17+0-0)$

= 0.85

Put $x_1 = 0.85;\ z_0 = 0;$

$\therefore\ y_1 = 20^{-1}(-18+z_0-3x_1)$

$= 20^{-1}[-18+0-3(0.85)]$

= – 1.0275

Put $x_1 = 0.85;\ y_1 = -1.0275;$

$\therefore\ z_1 = 20^{-1}(25-2x_1+3y_1)$

$= 20^{-1}[25-2(0.85)+3(-1.0275)]$

= 1.01088

Iteration 2:

Put $y_1 = -1.0275;\ z_1 = 1.01088;$

$\therefore\ x_2 = 20^{-1}(17+2z_1-y_1)$

$= 20^{-1}[17+2(1.01088)-(-1.0275)]$

= 1.0025

Put $x_2 = 1.0025;\ z_1 = 1.01088;$

$\therefore\ y_2 = 20^{-1}(-18+z_1-3x_2)$

$= 20^{-1}[-18+(1.01088)-3(1.0025)]$

= – 0.9998

Put $x_2 = 1.0025;\ y_2 = -0.9998;$

$\therefore\ z_2 = 20^{-1}(25-2x_2+3y_2)$

$= 20^{-1}[25-2(1.0025)+3(-0.9998)]$

= 0.9998

Iteration 3:

Put $y_2 = -0.9998;\ z_2 = 0.9998;$

$\therefore\ x_3 = 20^{-1}(17+2z_2-y_2)$

$= 20^{-1}[17+2(0.9998)-(-0.9998)]$

= 1

Put $x_3 = 1;\ z_2 = 0.9998;$

$\therefore\ y_3 = 20^{-1}(-18+z_2-3x_3)$

$= 20^{-1}[-18+(0.9998)-3(1)]$

= –1

Put $x_3 = 1;\ y_3 = -1;$

$\therefore\ z_3 = 20^{-1}(25-2x_3+3y_3)$

$= 20^{-1}[25-2(1)+3(-1)]$

= 1

Hence, by Gauss Seidal Method, solution of given set of equations is x = 1, y = –1, z = 1