Lecture notes

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Beta and Gamma Function

Preface

- **Gamma Function** also known as **Euler's Integral of the second kind** is an extention or generaliztion to **Factorial Function** to non integer values

- The function can be extended to negative non-integer real numbers and to complex numbers as long as the real part is greater than or equal to 1.

- **Beta Function** also known as **Euler's Integral of the first kind** is closely related to **Gamma Function** and to binomial Coefficients

- Gamma and Beta functions are useful for modeling situations involving continuous change, with important applications to calculus, differential equations, complex analysis, statistics, probability Theory, integral Transforms and so on.

- Certain kind of real definite integrals can be evaluated using **Beta and Gamma functions and their properties**

- Their use is also very prominent in the evaluation of Multiple Integration

Gamma Function

Definition

The definite integral $\int_0^\infty e^{-x}\ x^{n-1}\ dx$ is a function of $n > 0$ and denoted by the symbol $\Gamma(n)$. Thus

$$\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1}\ dx \quad (n > 0)$$

Properties of Gamma Function

Property 1: Reduction Formula

$$\Gamma(n+1) = n\ \Gamma(n)$$

Proof

By Definition of Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1}\ dx$$

replacing n by $n+1$

$$\Gamma(n+1) = \int_0^\infty e^{-x}\ x^{n}\ dx$$

Simplifying R.H.S using Integration by parts

$$\Gamma(n+1) = \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty nx^{n-1}\ (-e^{-x})\ dx$$

By L'Hospital Rule $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ Hence

$$\Gamma(n+1) = 0 + \int_0^\infty nx^{n-1}\ e^{-x}\ dx$$

$$\Gamma(n+1) = n\ \int_0^\infty x^{n-1}\ e^{-x}\ dx$$

$$\Gamma(n+1) = n\ \Gamma(n)$$

Note 1:If n is positive integer

$$\begin{aligned}\Gamma(n+1) &= n\ \Gamma(n)\\ &= n\ (n-1)\ \Gamma(n-1)\\ &= n\ (n-1)\ (n-2)\ \Gamma(n-2)\\ &= n\ (n-1)\ (n-2)\ ...\ 3.2.1\Gamma(1)\end{aligned}$$

By Definition of Gamma Function

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-x}\ x^0\ dx\\ &= \left[\left(-e^{-x}\right)\right]_0^\infty\\ &= \left[\left(-e^{-\infty}\right)+e^0\right]\\ \therefore \Gamma(1) &= 1\end{aligned}$$

Hence

$$\begin{aligned}\Gamma(n+1) &= n\ (n-1)\ (n-2)\ ...\ 3.2.1\Gamma(1)\\ &= n\ (n-1)\ (n-2)\ ...\ 3.2.1\\ \Gamma(n+1) &= n! \quad if\ n\ is\ positive\ integer\end{aligned}$$

Note 2:If n is zero Then

$$\Gamma(0) = \infty$$

$$\because \Gamma(n) = \frac{\Gamma(n+1)}{0}$$

$$\therefore \Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty$$

Property 2: Alternate Definition of Gamma Functions

(1) $\Gamma(n) = 2 \int_0^\infty e^{-x^2}\ x^{2n-2}\ x\ dx$

(2) $\Gamma(n) = a^n \int_0^\infty e^{-ax}\ x^{n-1}\ dx$

Proof
By Definition of Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-y}\ y^{n-1}\ dy\text{.........}\mathbf{(A)}$$

Now
(1) Let $y = x^2 \implies dy = 2\ x\ dx$
as $y = 0 \implies x = 0$ and
$y \to \infty \implies x \to \infty$
substituting in (A)

$$\begin{aligned}\Gamma(n) &= \int_0^\infty e^{-y}\ y^{n-1}\ dy \\ &= \int_0^\infty e^{-x^2}\ (x^2)^{n-1}\ 2\ x\ dx \\ &= 2\int_0^\infty e^{-x^2}\ x^{2n-2}\ x\ dx \\ \therefore \Gamma(n) &= 2\int_0^\infty e^{-x^2}\ x^{2n-1}\ dx\end{aligned}$$

(2) Let $y = ax \implies dy = a\ dx$
as $y = 0 \implies x = 0$ and
$y \to \infty \implies x \to \infty$

substituting in (A)

$$\begin{aligned}\Gamma\left(n\right) &= \int_0^\infty e^{-y}\ y^{n-1}\ dy \\ &= \int_0^\infty e^{-ax}\ (ax)^{n-1}\ a\ dx \\ &= a^{n-1+1}\ \int_0^\infty e^{-ax}\ x^{n-1}\ dx \\ \therefore \Gamma\left(n\right) &= a^n\ \int_0^\infty e^{-ax}\ x^{n-1}\ dx\end{aligned}$$

Solved Examples

Evaluate

(1) $\displaystyle\int_0^\infty e^{-y^{\frac{1}{n}}}\ dy$

(2) $\displaystyle\int_0^1 \left[log\left(\frac{1}{y}\right)\right]^{\frac{1}{n}}\ dy$

Solution
(1) Given

$$I = \int_0^\infty e^{-y^{\frac{1}{n}}}\ dy.......\textbf{(A)}$$

Let $y^{\frac{1}{n}} = x$
$\implies y = x^n$
$\implies dy = n\ x^{n-1}\ dx$

as $y = 0 \implies x = 0$ and
$y \to \infty \implies x \to \infty$

substituting in (A)

$$\begin{aligned} I &= \int_0^\infty e^{-y^{\frac{1}{n}}} \ dy \\ &= \int_0^\infty e^{-x} n \ x^{n-1} \ dx \\ &= n \ \int_0^\infty e^{-x} x^{n-1} \ dx \\ &= n \ \Gamma(n) \end{aligned}$$

$$\therefore I = \Gamma(n+1)$$

(2) Given

$$I = \int_0^1 \left[log \left(\frac{1}{y} \right) \right]^{\frac{1}{n}} \ dy.......\textbf{(A)}$$

Let $log\left(\frac{1}{y}\right) = x$
$\implies \frac{1}{y} = e^x$
$\implies y = e^{-x}$
$\implies dy = - \ e^{-x} \ dx$

as $y = 0 \implies x \to \infty$ and
$y = 1 \implies x = 0$

substituting in (A)

$$
\begin{aligned}
I &= \int_0^1 \left[log \left(\frac{1}{y} \right) \right]^{\frac{1}{n}} \ dy \\
&= \int_\infty^0 \ x^{\frac{1}{n}} \ (- \ e^{-x} \ )dx \\
&= \int_0^\infty \ e^{-x} x^{\frac{1}{n}} \ dx \\
&= \int_0^\infty \ e^{-x} x^{(\frac{1}{n}+1-1)} \ dx \\
&= n \ \Gamma \left(\frac{1}{n} + 1 \right)
\end{aligned}
$$

$$
\therefore I = \frac{1}{n} \ \Gamma \left(\frac{1}{n} \right)
$$

Beta Function

Definition

The definite integral $\int_0^1 x^{m-1}\ (1-x)^{n-1}\ dx$ is a function of $m > 0, n > 0$ and denoted by the symbol $\beta\,(m, n)$. Thus

$$\beta\,(m,n) = \int_0^1 x^{m-1}\ (1-x)^{n-1}\ dx \ \ (m > 0, n > 0)$$

Properties of Beta Function

Property 1:

$$\beta(m,n) = \beta(n,m)$$

Proof
By Definition of Beta Function

$$\beta\,(m,n) = \int_0^1 y^{m-1}\ (1-y)^{n-1}\ dy.....\mathbf{(1)}$$

Let $y = 1 - x$
$\implies dy = -dx$

as $y = 0 \implies x = 1$ and
$y = 1 \implies x = 0$
substituting in (A)

$$\begin{aligned}\beta\,(m,n) &= \int_0^1 y^{m-1}\ (1-y)^{n-1}\ dy \\ &= \int_1^0 (1-x)^{m-1}\ [1-(1-x)]^{n-1}\ (-dx) \\ &= \int_0^1 x^{n-1}(1-x)^{m-1}\ \ dx\end{aligned}$$

$$\therefore \beta\,(m,n) = \beta\,(n,m)$$

Property 2:

$$\beta(m,n) = 2 \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta \;\; cos^{2n-1}\theta \; d\theta$$

Proof

By Definition of Beta Function

$$\beta\,(m,n) = \int_0^1 y^{m-1}\;(1-y)^{n-1}\;dy.....\mathbf{(1)}$$

Let $y = sin^2\theta$
$\implies dy = -2\; sin\theta \;\cos\theta\; d\theta$

as $y = 0 \implies \theta = 0$ and
$y = 1 \implies \theta = \frac{\pi}{2}$
substituting in (A)

$$\begin{aligned}\beta\,(m,n) &= \int_0^1 y^{m-1}\;(1-y)^{n-1}\;dy\\ &= \int_0^{\frac{\pi}{2}} (sin^2\theta)^{m-1}\;[1-sin^2\theta]^{n-1}\;(2\;sin\theta\;\cos\theta\;d\theta)\\ &= \int_0^{\frac{\pi}{2}} (sin\theta)^{2m-2}\;(cos\theta)^{2n-2}\;(2\;sin\theta\;\cos\theta\;d\theta)\\ &= 2\int_0^{\frac{\pi}{2}} (sin\theta)^{2m-1}\;(cos\theta)^{2n-1}\;d\theta)\\ \therefore \beta\,(m,n) &= 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\;\;cos^{2n-1}\theta\;d\theta\end{aligned}$$

Note: Let $2m-1 = p$ and $2n-1-q$
$\therefore m = \dfrac{p+1}{2}\;\;,\;\; n = \dfrac{q+1}{2}$ then

$$\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} sin^p\;\theta\;\;cos^q\;\theta\;d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}} sin^p\;\theta\;\;cos^q\;\theta\;d\theta = \frac{1}{2}\;\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

Property 2: Alternate Definition of Beta Function

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Proof
By Definition of beta Function

$$\beta\,(m,n) = \int_0^1 y^{m-1}\,(1-y)^{n-1}\, dy.....\mathbf{(1)}$$

Let $y = \dfrac{x}{1+x} \implies dy = \dfrac{(1+x)-x}{(1+x)^2}dx = \dfrac{dx}{(1+x)^2}$
Also $y = \dfrac{x}{1+x}$
$\implies y(1+x) = x$
$\implies y = x(1-y)$
$\implies x = \dfrac{y}{1-y}$

as $y = 0 \implies x = 0$ and
$y = 1 \implies x \to \infty$
substituting in (A)

$$\begin{aligned}\beta\,(m,n) &= \int_0^1 y^{m-1}\,(1-y)^{n-1}\, dy \\ &= \int_0^\infty \left(\frac{x}{1+x}\right)^{m-1} \left[1-\frac{x}{1+x}\right]^{n-1} \frac{dx}{(1+x)^2} \\ &= \int_0^\infty \left(\frac{x}{1+x}\right)^{m-1} \left(\frac{1}{1+x}\right)^{n-1} \frac{dx}{(1+x)^2} \\ &= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-1+n-1+2}}\, dx\end{aligned}$$

$$\therefore \beta\,(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

Note: Another possible substitution for this sum is $y = \dfrac{1}{1+x}$

Property 3: Relation between beta and Gamma function

$$\beta(m,n) = \frac{\Gamma(m)\ \ \Gamma(n)}{\Gamma(m+n)}$$

Proof
By alternate definition of Gamma function, we have

$$\Gamma(n) = 2\int_0^\infty\ e^{-y^2}\ y^{2n-1}\ dy......\mathbf{(1)}$$

and

$$\Gamma(m) = 2\int_0^\infty\ e^{-x^2}\ x^{2m-1}\ dx......\mathbf{(2)}$$

By (1) and (2)

$$\begin{aligned}\Gamma(m)\ \Gamma(n) &= \left\{2\int_0^\infty\ e^{-x^2}\ x^{2m-1}\ dx\right\}\ \left\{2\int_0^\infty\ e^{-y^2}\ y^{2n-1}\ dy\right\} \\ &= 4\ \int_0^\infty\ \int_0^\infty\ e^{-(x^2+y^2)}\ x^{2m-1}\ y^{2n-1}\ dx\ dy.......\mathbf{(3)}\end{aligned}$$

For solving above integral we will change the coordinates from cartesian to polar by substituting
$x = r\ cos\ \theta$ and $y = r\ sin\ \theta$
This gives
$r^2 = x^2 + y^2$ and $dx\ dy = r\ dr\ d\theta$

For limits of integration
since in cartesian plane x and y varies from 0 to ∞
The region of integration is first quadrant only
Hence in polar plane with same region r varies from 0 to ∞ and θ varioes from 0 to $\frac{\pi}{2}$

Hence result (3) can be written as

$$\begin{aligned}
\Gamma(m)\ \Gamma(n) &= 4\ \int_0^{\infty}\ \int_0^{\infty}\ e^{-(x^2+y^2)}\ x^{2m-1}\ y^{2n-1}\ dx\ dy \\
&= 4\ \int_0^{\infty}\ \int_0^{\frac{\pi}{2}}\ e^{-(r^2)}\ (r\ cos\theta)^{2m-1}\ (r\ sin\ \theta)^{2n-1}\ r\ dr\ d\theta \\
&= 4\ \int_0^{\infty}\ \int_0^{\frac{\pi}{2}}\ e^{-(r^2)}\ r^{2m+2n-2+1}\ (cos\theta)^{2m-1}\ (sin\ \theta)^{2n-1}\ dr\ d\theta \\
&= \left\{2\int_0^{\infty}\ e^{-r^2}\ r^{2(m+n)-1}\ dr\right\}\ \left\{2\int_0^{\frac{\pi}{2}}\ (cos\theta)^{2m-1}\ (sin\ \theta)^{2n-1}\ d\theta\right\} \\
&= \Gamma(m+n)\ \beta(m,n)
\end{aligned}$$

Hence

$$\beta(m,n) = \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}$$

Note:
Substituting $m = n = \frac{1}{2}$ in above relation, we have

$$\beta(m,n) = \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \text{.......}\mathbf{(1)}$$

Now

$$\beta(m,n) = 2\ \int_0^{\frac{\pi}{2}}\ sin^{2m-1}\theta\ cos^{2n-1}\theta\ d\theta$$

Substituting $m = n = \frac{1}{2}$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \int_{0}^{\frac{\pi}{2}} sin^{1-1}\theta \ cos^{1-1}\theta \ d\theta$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \int_{0}^{\frac{\pi}{2}} d\theta$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \ [\theta]_{0}^{\frac{\pi}{2}}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi ....\mathbf{(2)}$$

From (1) and (2)

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Property 4: Duplication Formula

$$\Gamma(m) \ \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi} \ \ \Gamma(2m)}{2^{2m-1}}$$

Proof

By relaton between beta and Gamma function, we have

$$\beta(m, n) = \frac{\Gamma(m) \ \Gamma(n)}{\Gamma(m+n)} ....\mathbf{(1)}$$

and by alternate definition of beta

$$\beta(m, n) = 2 \int_{0}^{\frac{\pi}{2}} sin^{2m-1}\theta \ cos^{2n-1}\theta \ d\theta ....\mathbf{(2)}$$

Replacing $'n'$ by $'m'$ in (1)

$$\beta(m, m) = \frac{\Gamma(m) \ \Gamma(m)}{\Gamma(m+m)}$$

$$\beta(m, m) = \frac{[\Gamma(m)]^2}{\Gamma(2m)} ....\mathbf{(3)}$$

Also replacing $'n'$ by $'m'$ in (2)

$$\beta(m,m) = 2 \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta \ cos^{2m-1}\theta \ d\theta$$
$$= 2 \int_0^{\frac{\pi}{2}} \frac{1}{2^{2m-1}} \left[2 \ sin\theta \ cos\theta\right]^{2m-1} \ d\theta$$
$$= \frac{2}{2^{2m-1}} \int 0^{\frac{\pi}{2}} \left[sin2\theta\right]^{2m-1} \ d\theta....\mathbf{(4)}$$

in (4) let $2\theta = \phi \implies 2 \ d\theta = d\phi$
As $\theta = 0 \implies \phi = 0$
$\theta = \frac{\pi}{2} \implies \phi = \pi$
Hence by (4)

$$\beta(m,m) = \frac{2}{2^{2m-1}} \int 0^{\pi} \left[sin\phi\right]^{2m-1} \ \frac{d\phi}{2}$$
$$= \frac{2}{2^{2m-1}} \int 0^{\pi} \left[sin\phi\right]^{2m-1} \ \frac{d\phi}{2}$$
$$= \frac{1}{2^{2m-1}} \int 0^{\pi} \left[sin\phi\right]^{2m-1} \ d\phi$$
$$= \frac{2}{2^{2m-1}} \int 0^{\frac{\pi}{2}} \left[sin\phi\right]^{2m-1} \ d\phi....\mathbf{(5)}$$

In result (2) with $n = \frac{1}{2}$

$$\beta(m, \frac{1}{2}) = 2 \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta \ \theta \ d\theta...\mathbf{(6)}$$

By (5) and (6)

$$\beta(m,m) = \frac{\beta(m, \frac{1}{2})}{2^{2m-1}}....\mathbf{(7)}$$

By (3) and (7)

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{\beta(m, \frac{1}{2})}{2^{2m-1}} = \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+\frac{1}{2}) \ 2^{2m-1}}$$

Hence

$$\Gamma(m) \ \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi} \ \ \Gamma(2m)}{2^{2m-1}}$$

Important Substitutions

Gamma Function

Types of integrals	Substitution
$\int_0^\infty e^{-ax^n}\ dx\ or\ \int_0^\infty x^m\ e^{-ax^n}\ dx$	$ax^n = t$
$\int_0^1 (logx)^n\ dx\ or\ \int_0^1 x^m\ (logx)^n\ dx$	$logx = -t$
$\int_0^a \frac{x^a}{a^x}\ dx$	$a^x = e^t$
$\int_0^\infty a^{-kx^2}\ dx$	$a^{-kx^2} = e^{-t}$

Beta Function

Types of integrals	**Substitution**
$\int_0^a x^m\ (a-x)^n\ dx$	$x = at$
$\int_0^1 (x)^m (1-x^n)^p\ dx$	$x^n = t$
$\int_0^1 \left(1-\sqrt[n]{x}\right)^m\ dx$	$\sqrt[n]{x} = t$
$\int_a^b (x-a)^m\ (b-x)^n\ dx$	$(x-a) = (b-a)t$

Practice examples

Evaluate

(1) $\int_0^9 x^{\frac{3}{2}}\ (9-x)^{\frac{1}{2}}\ dx$ **Ans:** $\frac{729\ \pi}{16}$

(2) $\int_0^\infty e^{-k^2x^2}\ dx$ **Ans:** $\frac{\sqrt{\pi}}{2k}$

(3) $\int_0^\infty x^2\ e^{-x^4}\ dx\ *\ \int_0^\infty e^{-x^4}\ dx$ **Ans:** $\frac{1}{4}\ \Gamma\left(\frac{1}{4}\right)\ \Gamma\left(\frac{3}{4}\right)$

(4) $\int_0^2 x\sqrt[3]{8-x^3}\ dx$ **Ans:** $\frac{8}{9}\ \Gamma\left(\frac{1}{3}\right)\ \Gamma\left(\frac{2}{3}\right)$

(5) $\int_0^1 (log\ x)^5\ dx$ **Ans:** -120

(6) $\int_0^1 \sqrt{1-\sqrt{x}}\ dx\ *\ \int_0^{\frac{1}{2}} \sqrt{2y-4y^2}\ dy$ **Ans:** $\frac{\pi}{30}$

(7) $\int_0^1 \frac{dx}{\sqrt{x\ log\frac{1}{x}}}$ **Ans:**$\sqrt{2\pi}$

(8) $\int_0^{2a} x^2\ \sqrt{2ax-x^2}\ dx$ **Ans:**$\frac{5\ a^4\ \pi}{8}$

(9) $\int_0^{\infty} \frac{x^4}{4^x}$ **Ans:**$\frac{24}{(log\ 4)^5}$

(10) $\int_3^7 \sqrt[4]{(x-3)(7-x)}\ dx$ **Ans:**$\frac{2}{3}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 \frac{1}{\sqrt{\pi}}$

(11) $\int_0^{\infty} 3^{-4x^2}\ dx$ **Ans:**$\frac{\sqrt{\pi}}{4\ \sqrt{log\ 3}}$

(12) $\int_0^1$

$fracx^7\sqrt{1-x^2}\ dx$ **Ans:**$\frac{32}{245}$

(13) $\int_0^{\infty} x^{n-1}\ cos\ (ax)\ dx$ **Ans:**$\frac{\Gamma(n)}{a^n} cos\left(\frac{n\pi}{2}\right)$

(14) $\int_0^1 \frac{x^2(1-x^4)}{\sqrt{1-x^2}}\ dx$ **Ans:**$\frac{3\pi}{32}$

(15) $\int_0^{\infty} \frac{dx}{(1+x)^{\frac{7}{2}}} dx$ **Ans:**$\frac{8}{15}$