Lecture notes

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Differential Equation of First Order and First Degree

Def: Differential Equation

An equation involving a dependent variable, an independent variable and the differential coefficient of various orders is called **Differential Equation**

Def: Ordinary Differential Equation

A differential equation involving one and only one independent variable and one or more dependent variables and their derivatives with respect to independent variables is called **Ordinary Differential Equation(O.D.E.)**

Def: Partial Differential Equation

A differential equation involving two or more independent variable and one or more dependent variables and their derivatives with respect to independent variables are called **Partial Differential Equation(P.D.E.)**

Examples

- $\dfrac{dy}{dx} - y = 0$
  (O.D.E)
- $\sqrt{1 + \dfrac{dy}{dx}} = \dfrac{d^2y}{dx^2}$
  (O.D.E)
- $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$
  (P.D.E)
- $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$
  (P.D.E)

Def: Order of Differential Equation

The **order of a differential equation** is the order of the highest order derivative appears in the equation

Def: Degree of Differential Equation

The **degree of a differential equation** is the degree of the highest order derivative or differential coefficient when differential coefficients are free from radicals and fractions

Def: Solution of Differential Equation

The **solution or primitive of a differential equation** is any relation between dependent and independent variables free from derivatives and satisfying differential equation

Def: General Solution (G.S) of Differential Equation

The **general solution or complete Integral of a differential equation** is any relation between dependent and independent variables free from derivatives and satisfying differential equation and contains number of arbitrary constantsequal to the order of the differential equation.

Thus the general solution of differential equation of order$n$ must contains n arbitrary constants

Def: Particular Solution or Particular Integral of Differential Equation

The **particular solution of a differential equation** is obtained by assigning particular values to the arbitrary constants in G.S.of a differential equation.

Exact Differential Equation

Definition

The differential equation $M(x,y)dx+N(x,y)dy=0$ is said to be **Exact Differential Equation** in a region R of the xy-plane if there exists a function $\phi(x,y)$such that

$$\frac{\partial \phi}{\partial x} = M \ and \ \frac{\partial \phi}{\partial y} = N.....\mathbf{(1)}$$

for all (x,y) in R.

- Any function ϕ satisfying (1) , if exists then given Equation can be written as

$$M(x,y)dx + N(x,y)dy = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = d\phi = 0$$

- In this case Differential Equation is known as **Exact Differential Equation** and general solution is given by $\phi(x,y) = c$

<u>Condition for exactness</u>

Condition

Let M, N and their first partial derivatives M_y and N_x , be continuous in a region R of the xy-plane. Then the differential equation $M(x,y)dx + N(x,y)dy = 0$ is **exact** for all x, y in R if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

<u>General Solution</u>

- If given differential equation $M(x,y)dx+N(x,y)dy=0$ satiesfies the condition of exactness then its general solution is given by

$$\int M dx + \int_{xfree} N dy = c......\mathbf{(2)}$$

or

$$\int N dy + \int_{yfree} M dx = c......\mathbf{(3)}$$

Alternative Solution Method

Step 1 Check the condition of exactness

Step 2 Since $\frac{\partial \phi}{\partial x} = M$(or $\frac{\partial \phi}{\partial y} = N$) then

$$\phi(x,y) = \int M dx + h(y)...\textbf{(A)}$$

or

$$\phi(x,y) = \int N dy + g(x)...\textbf{(B)}$$

Step 3 Partially differentiating ϕ w.r.t. y and comparing with $\frac{\partial \phi}{\partial y} = N$ we get value of $h^{'}(y)$ (OR Partially differentiating ϕ w.r.t. x and comparing with $\frac{\partial \phi}{\partial x} = M$ we get value of $g^{'}(x)$)

Step 4 Integrating $h^{'}(y)$ w.r.t. y (OR $g^{'}(x)$ w.r.t. x) we get values of $h(y)$ (OR $g(x)$)

Step 5 Resubstituting values in (A) or (B) we get the requires Solution

Example(Exact)

Solve

$$2xe^y dx + (x^2e^y + cosy)dy = 0$$

Solution :Method 1

Given

$$2xe^y dx + (x^2e^y + cosy)dy = 0$$

Here $M = 2xe^y$ and $N = (x^2e^y + cosy)$
Now

$$\frac{\partial M}{\partial y} = 2xe^y$$

and

$$\frac{\partial N}{\partial x} = 2xe^y$$

Hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given Equation is exact.
$\therefore$ General Solution is given by

$$\int M dx + \int_{xfree} N dy = c$$
$$\int (2xe^y)\ dx + \int_{xfree} (x^2e^y + cosy)\ dy = c$$
$$\int (2xe^y)\ dx + \int cosy\ dy = c$$
$$x^2e^y + siny = c$$

<u>Solution :Method 2</u>

Given

$$2xe^y dx + (x^2e^y + cosy)dy = 0$$

Here $M = 2xe^y$ and $N = (x^2e^y + cosy)$
Now

$$\frac{\partial M}{\partial y} = 2xe^y$$

and

$$\frac{\partial N}{\partial x} = 2xe^y$$

Hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given Equation is exact. Since equation is exact there exists a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M = 2xe^y......\mathbf{(1)}$$
$$\frac{\partial \phi}{\partial y} = N = x^2e^y + cosy......\mathbf{(2)}$$

Integrating (1) w.r.t y ,keeping x constant we have

$$\phi(x, y) = x^2e^y + h(y)......\mathbf{(3)}$$

Now partially differentiating (3) w.r.t y

$$\frac{\partial\phi}{\partial y} = x^2e^y + \frac{dh}{dy}......\mathbf{(4)}$$

comparing (4) with (2)

$$\frac{dh}{dy} = cosy$$

Integrating we get

$$h(y) = siny$$

Substituting in (3)

$$\phi(x, y) = x^2e^y + siny$$

Hence general solution of given differential equation is given by

$$x^2e^y + siny = c$$

Non-Exact Differential Equation

Definition: Integrating Factor
The function $k(x, y)$ is said to be an **integrating factor (I.F.)** of the equation $M(x, y)dx + N(x, y)dy = 0$, if it is possible to obtain a function $u(x, y)$ such that

$$k\,(M(x, y)dx + N(x, y)dy) = 0 = du$$

i.e.Integrating factor is a multplying factor by which non-exact equation can be reduced as exact equation.

Rules to find Integrating Factors

Cases

If the equation $M(x, y)dx + N(x, y)dy = 0$ is not exact then we can obtain suiatble integrating factors from following cases to reduce it as an exact equation

(1) If $Mx + Ny \neq 0$ and given differential equation is homogeneous then

$$I.F. = \frac{1}{Mx + Ny}$$

(2) If $Mx - Ny \neq 0$ and given differential equation is of the form $f_1(xy)y\ dx + f_2(xy)x\ dy = 0$ then

$$I.F. = \frac{1}{Mx - Ny}$$

(3) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then

$$I.F. = e^{\int f(x)dx}$$

(4) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$ then

$$I.F. = e^{\int g(y)dy}$$

Example(Non-exact)

Solve

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$$

Solution

Given

$$(y^4+2y)dx+(xy^3+2y^4-4x)dydy=0$$

Here $M=(y^4+2y)$ and $N=(xy^3+2y^4-4x)$

Now

$$\frac{\partial M}{\partial y}=4y^3+2$$

and

$$\frac{\partial N}{\partial x}=y^3-4$$

Hence

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$\therefore$ Given Equation is not exact. To reduce it into exact equation we need to find suitable I.F.

Now

$$\begin{aligned}\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}&=4y^3+2-y^3+4\\&=3y^3+6\\ \therefore\ \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}&=\frac{3y^3+6}{-y^4-2y}\\&=\frac{3(y^3+2)}{-y(y^3+2)}\\&=-\frac{3}{y}=g(y)\end{aligned}$$

$\therefore I.F=e^{\int g(y)dy}=e^{\int -\frac{3}{y}dy}=e^{-3\int logy}=\dfrac{1}{y^3}$ Hence by definition of Integrating factor, required Exact equation is given by

$$\frac{1}{y^3}\left[(y^4+2y)dx+(xy^3+2y^4-4x)dy\right]=0$$

$$\implies\ \left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)=0.....\mathbf{(1)}$$

Hence General Solution (G.S.) of exact equation(1) is given by

$$\left(y+\frac{2}{y^2}\right)\int dx+2\int ydy=c$$
$$\implies \left(y+\frac{2}{y^2}\right)x+y^2=c\text{.....}\textbf{(Ans)}$$

Inspection Method

Some differential Equations can simply be solved by observation into direct derivative of some functions

- $xdy+ydx=d(xy)$
- $\dfrac{xdy+ydx}{xy}=d(log(xy))$
- $\dfrac{xdy-ydx}{x^2}=d\left(\dfrac{y}{x}\right)$
- $\dfrac{xdy-ydx}{x^2+y^2}=d\left[tan^{-1}\dfrac{y}{x}\right]$
- $\dfrac{xdy-ydx}{xy}=d\left[log\left(\dfrac{y}{x}\right)\right]$
- $\dfrac{ydx-xdy}{y^2}=d\left(\dfrac{x}{y}\right)$
- $\dfrac{ydx-xdy}{x^2+y^2}=d\left[tan^{-1}\dfrac{x}{y}\right]$
- $\dfrac{ydx-xdy}{xy}=d\left[log\left(\dfrac{x}{y}\right)\right]$

- $dx + dy = d(x + y)$

- $\dfrac{dx + dy}{x + y} = d\left[log(x + y)\right]$

- $\dfrac{ye^x dx - e^x dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$

Example(By Inspection)
Example 1
Solve

$$(x + y)^2 \left(x\frac{dy}{dx} + y\right) = xy\left(1 + \frac{dy}{dx}\right)$$

Solution
Given

$$\begin{aligned}
(x + y)^2 \left(x\frac{dy}{dx} + y\right) &= xy\left(1 + \frac{dy}{dx}\right) \\
\implies (x + y)^2 \left(\frac{xdy + ydx}{dx}\right) &= xy\left(\frac{dx + dy}{dx}\right) \\
\implies \left(\frac{xdy + ydx}{xy}\right) &= \left(\frac{dx + dy}{(x + y)^2}\right)
\end{aligned}$$

Integrating on both sides

$$\int \frac{xdy + ydx}{xy} = \int \frac{dx + dy}{(x + y)^2}$$

$$log(xy) = \frac{-1}{x + y} + c$$

Hence G.S. is

$$log(xy) + \frac{1}{x + y} = c$$

Practice Examples

Solve

(1) $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

(2) $e^{x+y}\left(x\frac{dy}{dx} + y\right) = e^{xy}\left(1 + \frac{dy}{dx}\right)$

(3) $x\frac{dy}{dx} = x^2y^2 - y$

Answers

(1) $\frac{x}{y} + e^{x^3} = c$

(2) $e^{-xy} = e^{-(x+y)} + c$

(3) $-\frac{1}{xy} = x + c$

Exercise
Solve

(1) $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

(2) $(x^4e^x - 2mxy^2)dx + (2mx^2y)dy = 0$

(3) $(xy^2 - 3y)dx + (3x^2y - 3x)dy = 0$

(4) $\left[y\left(1 + \frac{1}{x}\right) + cosy\right]dx + [x + logx - xsiny]\,dy = 0$

(5) $(2xlogx - xy)dy + (2y)dx = 0$

(6) $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

(7) $(x + y - 2)dx + (x - y - 4)dy = 0$

(8) $\left(\frac{y^2}{(y-x)^2} - \frac{1}{x}\right)\ dx + \left(\frac{1}{y} - \frac{x^2}{(x-y)^2}\right)\ dy = 0$

(9) $\frac{dy}{dx} = \frac{tany - 2xy - y}{x^2 - xtan^2y + sec^2y}$

(10) $[sin(xy) + xycos(xy) + 2x]dx + [x^2cos(xy) + 2y]dy = 0$

(11) $\left(\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right)\ dx - \frac{x}{(x-y)^2}\ dy = 0$

(12) $(cosx - xsinx + y^2)\ dx + (2xy)\ dy = 0$

(13) $(y^2 - x)\ dx + 2ydy = 0$

(14) $(xsec^2y - x^2cosy)dy = (tany - 3x^4)dx$

(15) $[xy - 2y^2]dx - [x^2 - 3xy]dy = 0$

(16) $(x^2y^3 + 2y)dx + (2x - 2x^3y^2)dy$

(17) $(2x^2y^2 + e^x y)dx = (e^x + y^3)dy$

Answers

(1) $e^{xy^2} + x^4 - y^3 = c$ (Exact)

(2) $e^x + \dfrac{my^2}{x^2} = c$ (I.F.=$\frac{1}{x^4}$)

(3) $log(xy^3) + \dfrac{3}{xy} = c$ (I.F.=$\frac{-1}{2x^2y^2}$)

(4) $y(x + logx) + xcosy = c$(Exact)

(5) $2ylogx - \dfrac{y^2}{2} = c$ (I.F.=$\frac{1}{x}$)

(6) $x^3y^2 + \dfrac{x^2}{y} = c$
(I.F.=$\frac{1}{y^2}$)

(7) $x^2 - y^2 + 2xy - 4x - 8y = c_1$ *where* $c_1 = 2c$

(8) $\dfrac{y^2}{y-x} + log\left(\dfrac{y}{x}\right) = c$

(9) $xtany - x^2y - xy - tany = c$

(10) $xsin(xy) + x^2 + y^2 = c$

(11) $\dfrac{2y}{x-y} + sin^{-1}x = c_1 \; where \; c_1 = -2c$

(12) $xy^2 + xcosx = c$

(13) $y^2e^x - xe^x + e^x = c$

(14) $\dfrac{tanx}{y} + x^3 - siny = C_1 \; where \; c_1 = -c$

(15) $\dfrac{x}{y} + log\left(\dfrac{x^2}{y^3}\right) = c$

(16) $log\left(\dfrac{x}{y^2}\right) - \dfrac{1}{x^2y^2} = c$

(17) $\dfrac{2x^3}{3} + \dfrac{e^x}{y} - \dfrac{y^2}{2} = c$