Lecture notes

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Multiple Integration

Preface

- The integral of a function of two variables $f(x, y)$ over a region in plane and an integrals over a function of three variables $f(x, y, z)$ over a region in space are known as **Multiple Integrals**

- These integrals are defined as the limits of **Reimann Sums** , like single variable Integrals

- **Multiple Integrals** are used to calculates quantities that are vary over mostly two or three dimensions such as total mass, angular momentum of an object with varying desnsity and the volumes of a solid with generalized curved boundries.

Definite integrals

- The concept of definite integral
$$\int_a^b f(x)\,dx$$
is physically the **area under a curve** $y = f(x)$, (say), the x-axis and the two ordinates $x = a$ and $x = b$

- It is defined as the limit of the sum
$$f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_n)\Delta x_n$$
when $n \to \infty$ and each of the lengths Δx_i tends to zero

- Here Δx_i are n subdivisions into which the range of integration has been divided and x_i are the values of x lying respectively in the 1st, 2nd, ..., nth subintervals i.e. The Riemann sums for the integral of a single-variable function $f(x)$ are obtained by partitioning a finite interval into thin subintervals, multiplying the width of each subinterval by the value of f at a point inside that subinterval, and then adding together all the products.

- We extend this concept to define integral of a contionuous function of two variables $f(x, y)$ over a bounder region R in xy Plane.

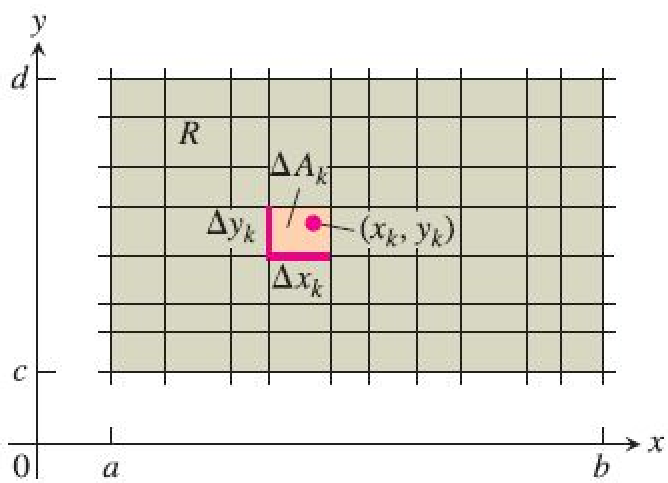
Double Integration

Let $z = f(x, y)$ be a two variable continuous function defined over a given region R

Case(I): Rectangular Region

$$R : a \leq x \leq b; c \leq y \leq d$$

- A rectangular region bounded by two vertical lines $x = a$ and $x = b$ and two horizontal lines $y = c$ and $y = d$



- Then dividing region into small subregions by drawing a lines parrallel to x and y axes we get rectagular subregion of area $\Delta A_i = \Delta x_i * \Delta y_i$ and considering a function value at point(x_i, y_i) along with the area ΔA_i consider a sum S_n

$$S_n = \sum_{i=0}^{n} f(x_i, y_i)\Delta A_i$$

item As $n \to \infty$ or $\Delta A_i \to 0$ the Double Integration over a given rectangular region is defined as

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i, y_i)\Delta A_i \\ &= \iint\limits_{R} f(x, y) dA \end{aligned}$$

- For a given rectegular region

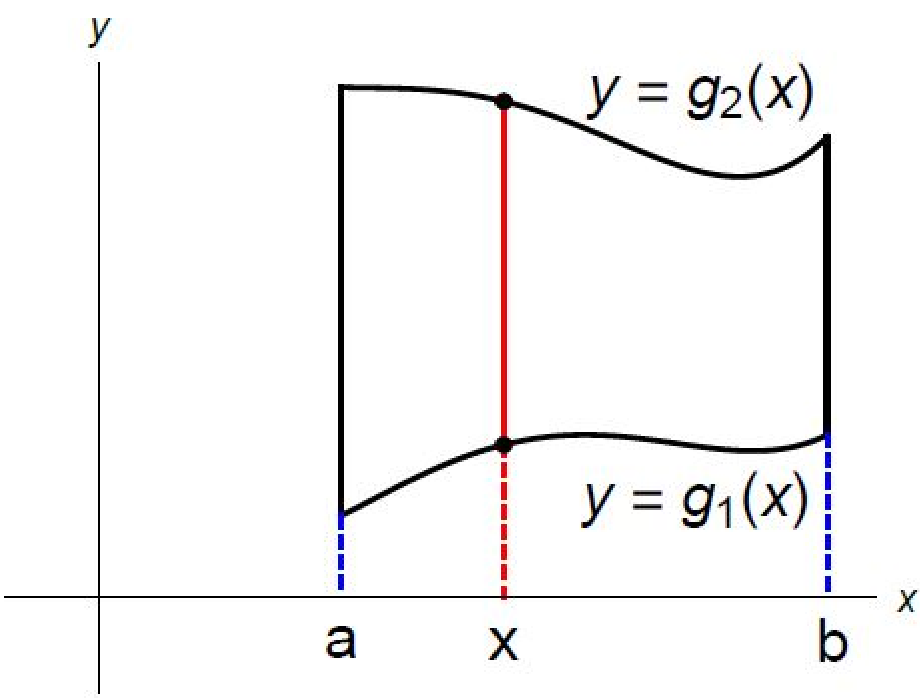
$$\begin{aligned} S &= \int\limits_{c}^{d} \int\limits_{a}^{b} f(x, y)\ dx\ dy \\ &= \int\limits_{a}^{b} \int\limits_{c}^{d} f(x, y)\ dy\ dx \end{aligned}$$

Case(II): Non rectangular region (Type 1)

$$R : a \leq x \leq b; g_1(x) \leq y \leq g_2(x)$$

A non-rectangular region bounded by two vertical lines $x = a$ and $x = b$ and two curves $y = g_1(x)$ and $y = g_2(x)$ Then

$$S = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

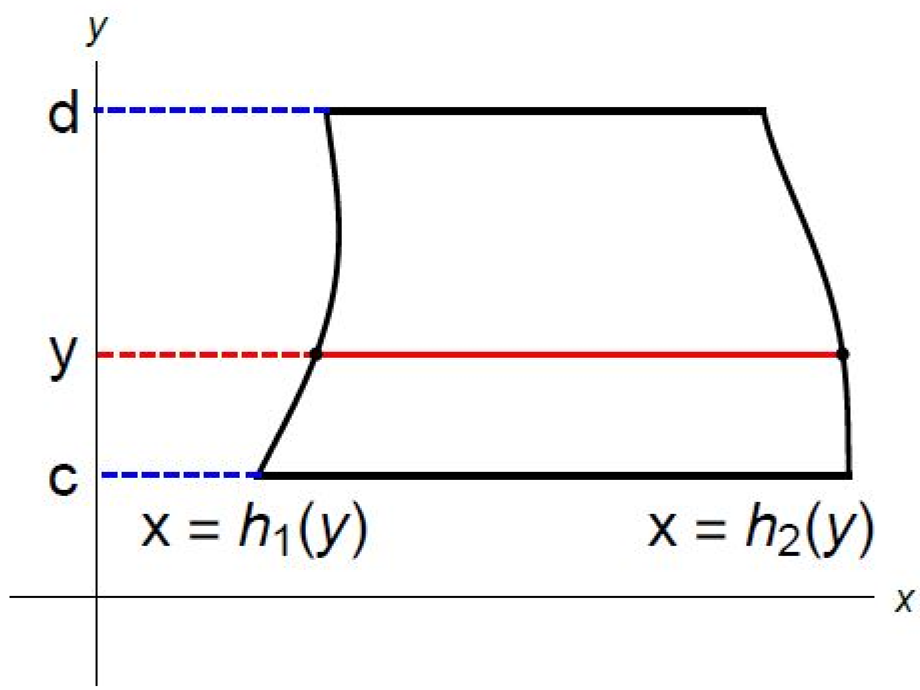


Case(III) Non rectangular region (Type 2)

$$R : c \leq y \leq d; h_1(y) \leq x \leq h_2(y)$$

A non-rectangular region bounded by two horizontal lines $y = c$ and $y = d$ and two curves $x = h_1(y)$ and $x = h_2(y)$ Then

$$S = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$



Limits of Integration using Double integration

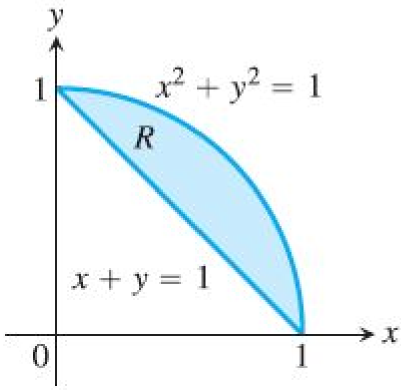
Decide limits of integration, to evaluate $f(x, y)$ over a region bounded by the curves $x^2 + y^2 = 1$ and $x + y = 1$.

Solution:

To sketch the region we need to fine intersecting points
For given curves $x^2 + y^2 = 1$ and line $x + y = 1$, intersecting points are

$$\begin{aligned} x + y = 1 \implies y &= 1 - x \\ \therefore x^2 + (1 - x)^2 &= 1 \\ \therefore 2x^2 - 2x &= 1 \\ \therefore x = 0; x &= 1 \\ y = 1; y &= 0 \end{aligned}$$

$\therefore$ intersection points are $(1, 0)$ and $(0, 1)$ and region is shown as

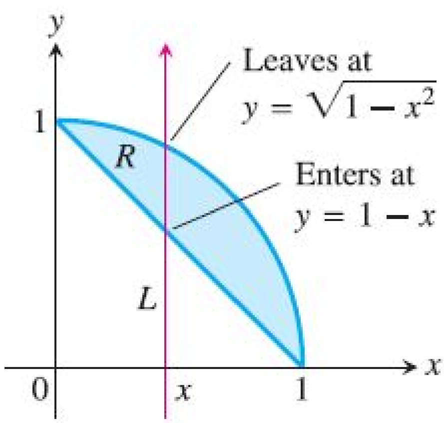


Considering vertical strip

for y limits of integration

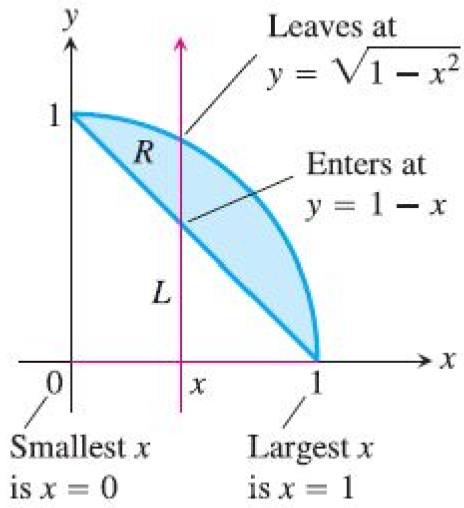
imagine a line parallel to y axis entering the region R at $y = 1 - x$ and leaving the region at $y = \sqrt{1 - x^2}$

these are the y limits of integration.



For x limits of integration

moving the strip in R from left to right x varies from $x = 0$ to $x = 1$. these are x limits of integrtaion.

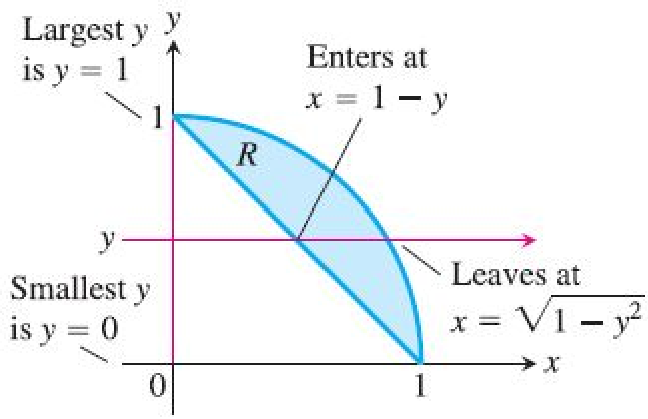


Hence considering vertical strip limits of integration are

$$1 - x \leq y \leq \sqrt{1 - x^2}$$
$$0 \leq x \leq 1$$

Considering Horizontal strip

For x limits of integration
imagine a line parallel to x axis entering the region R at $x = \sqrt{1 - y^2}$ and leaving the region at $x = 1 - y$. these are the x limits of integration
for y limits of integration
moving the strip in R from bottom to top y varies from $y = 0$ to $y = 1$. these are y limits of integrataion.



Hence considering Horizontal strip strip limits of integration are

$$1 - y \leq x \leq \sqrt{1 - y^2}$$
$$0 \leq y \leq 1$$

EXAMPLES(Evaluation)

1. $\int_0^1 \int_1^2 x\ y^2\ dy\ dx$

2. $\int_1^2 \int_y^{3y} x + y\ dx\ dy$

3. $\int_1^2 \int_2^3 e^{x+y}\ dy\ dx$

4. $\int_0^{\pi} \int_0^{cos(\theta)} r sin(\theta)\ dr\ d\theta$

5. $\int_1^{\infty} \int_0^1 e^{-y}\ y^x\ log\ y\ dx\ dy$

6. $\int_{-1}^1 \int_0^2 x + y\ dy\ dx$

7. $\int_1^2 \int_1^x \frac{x^2}{y^2}\ dy\ dx$

8. $\int_0^1 \int_{\frac{y^2}{2}}^{\sqrt{y}} dx\ dy$

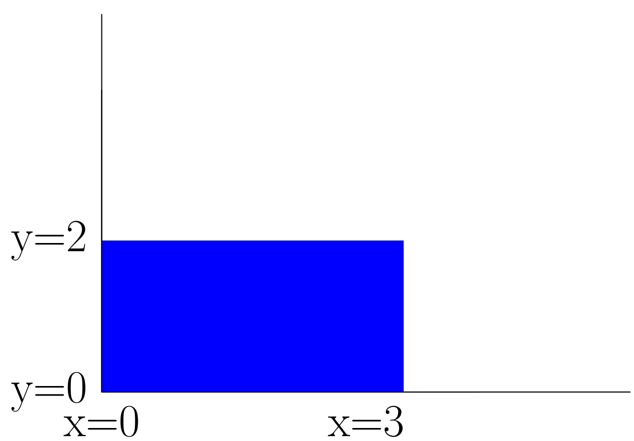
EXAMPLES(Sketch the region and Evaluate)

1. **Evaluate $f(x, y) = 4 - y^2$ over a region bounded by the lines $x = 3$, $x = 0$, $y = 2$ and $y = 0$.**

   Solution:

   Here region is bounded by two horizontal lines $x = 0$ and $x = 3$ and two verticallines $y = 0$ and $y = 2$

   $\therefore$ region is shown by shadded portion in the figure as follows

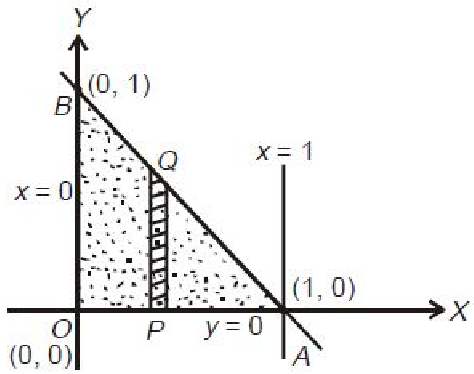
   

   Considering imaginary horizontal strip in the region limits of integration are $0 \leq x \leq 3$ and $0 \leq y \leq 2$

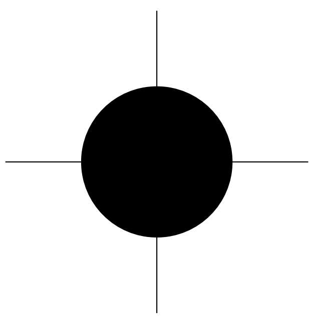
$\therefore$ required Integral is

$$I = \int_0^2 \int_0^3 (4 - \ y^2)\ dx\ dy$$
$$= \int_0^2 (4 - \ y^2)\ [x]_0^3\ dy$$
$$= \int_0^2 (4 - \ y^2)(3 - 0)\ dy$$
$$= 3 \int_0^2 (4 - \ y^2)\ dy$$
$$= 3\ [4y - \frac{y^3}{3}]_0^2$$
$$= 3\ [8 - \frac{8}{3}]$$
$$= 16 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots Ans$$

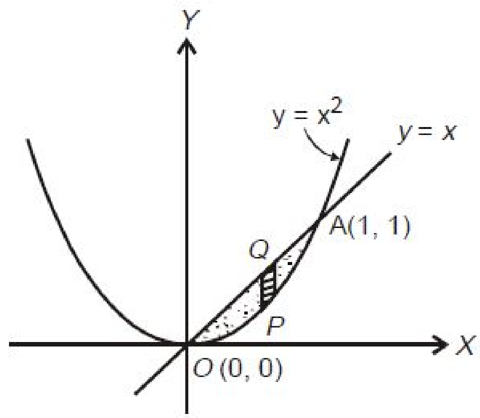
2. Evaluate $f(x, y) = e^{2x+3y}$ over a triangular region with vertices $(0, 0)$, $(1, 0)$and $(0, 1)$.
   Ans:$\frac{1}{6}[(2e + 1)(e - 1)^2]$



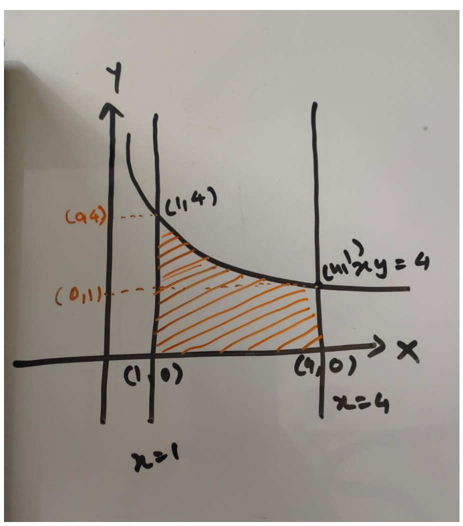
3. Evaluate $\iint xy\,dx\,dy$ over a region in first quadrant bounded by the circle $x^2 + y^2 = a^2$
Ans:$\frac{a^4}{8}$



4. Evaluate $\iint xy(x + y)\ dx\ dy$ over a region between the curves $y = x$ and $y = x^2$
Ans:$\frac{3}{56}$



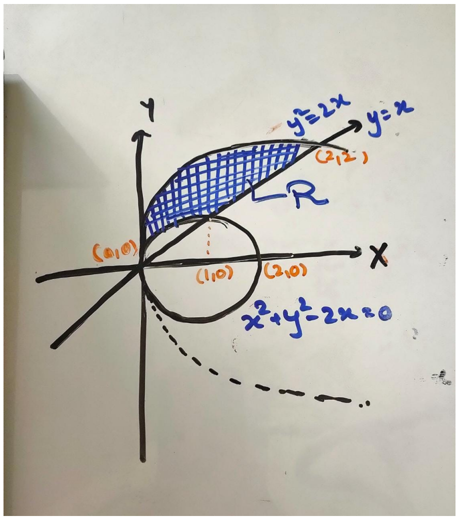
5. Evaluate $\iint xy(x+y)\,dx\,dy$ over a region bounded by the curves $xy = 4$,$x = 1$, $x = 4$ and $y = 0$
   Ans:40



6. Evaluate $\iint xy\,dx\,dy$ over a region bounded by the curves

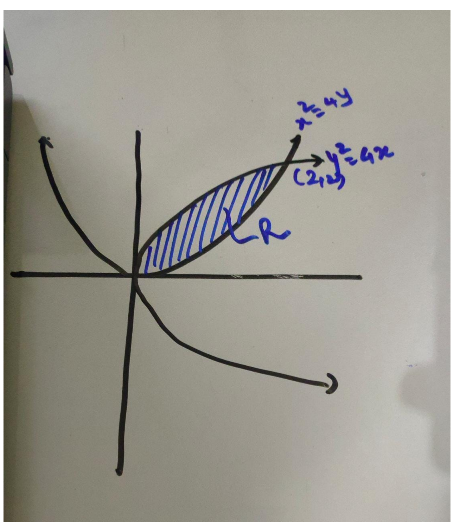
$x^2 + y^2 - 2x = 0$, $y^2 = 2x$ and $y = x$

Ans: $\frac{7}{12}$



7. Evaluate $\iint y \; dA$ over a region bounded by the curves $y^2 = 4x$ and $x^2 = 4y$

Ans:9.6

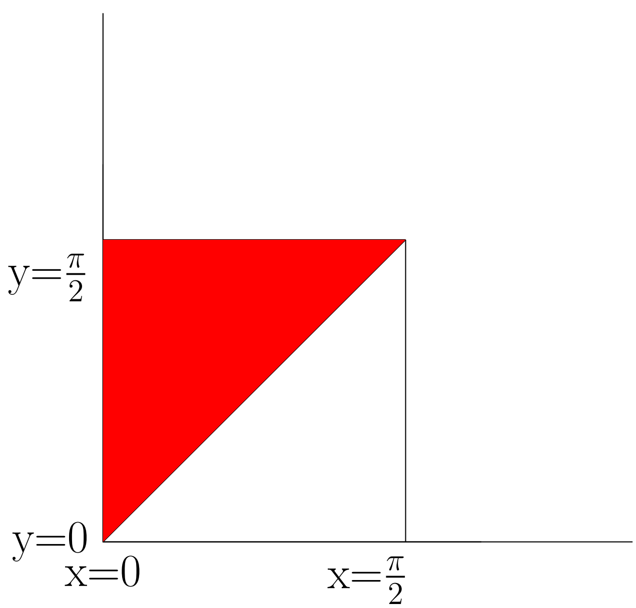


Change the order of Integration

Example: Change the order of the following Integral

$$I = \int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} f(x,y)\ dy\ dx$$

Solution:
Here given region of integration is bounded by lines $y = x$, $y = \frac{\pi}{2}$, $x = 0$ and $x = \frac{\pi}{2}$
Region is shown by shaded portion in the figure by considering vertical strip.



To change the order of integration we consider horizontal strip

in the same region.
$\therefore$ limits of integration are

$$0 \le x \le y$$

and

$$0 \le y \le \frac{\pi}{2}$$

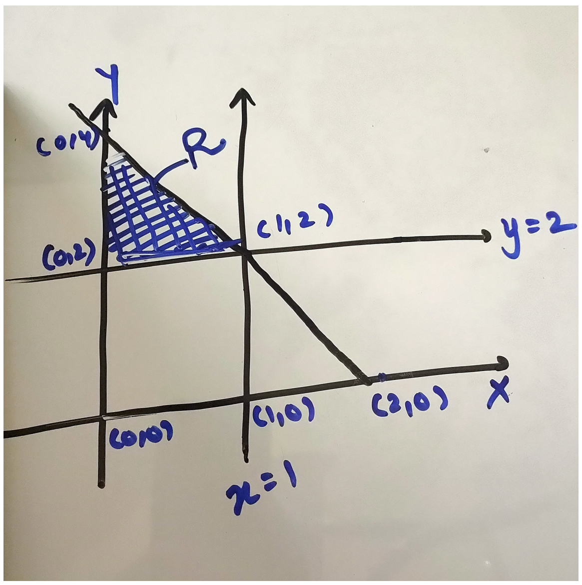
$\therefore$ required integral after changing order is

$$I = \int\limits_{0}^{\frac{\pi}{2}} \int\limits_{0}^{y} f(x, y)\; dx\; dy$$

EXAMPLES(Change the order and Evaluate)

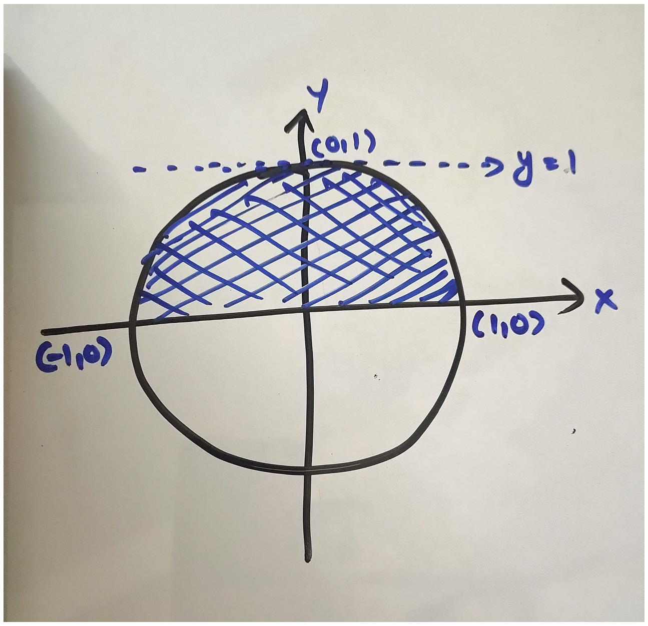
1. $I = \int_0^1 \int_2^{4-2x} dy\, dx$

   Ans: 1



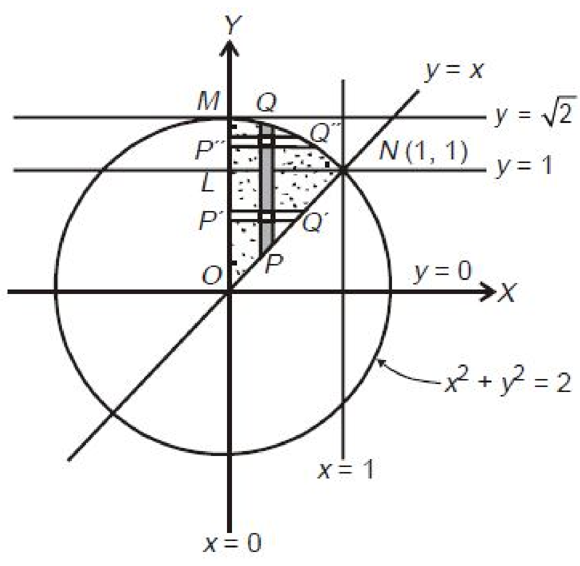
2. $I = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3\, y\, dx\, dy$

   Ans: 2



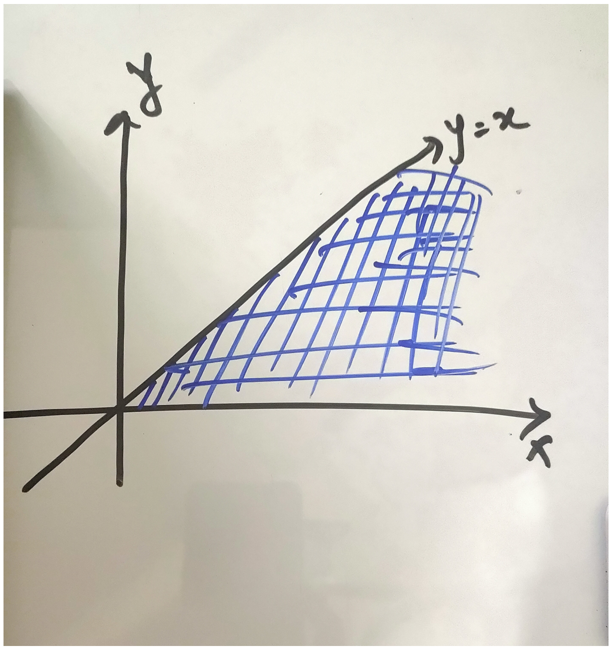
3. $I = \int_{0}^{1} \int_{x}^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

Ans: $\frac{1}{2}(\sqrt{2}-1)$



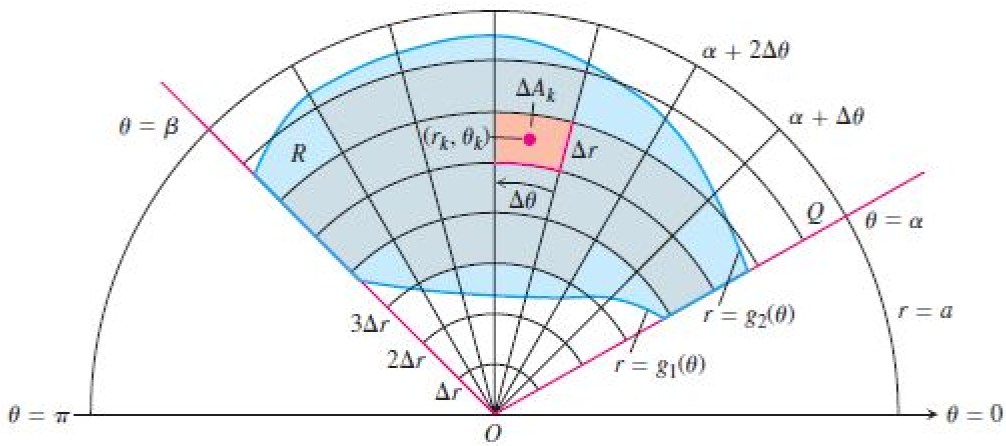
4. $I = \int\limits_0^{\infty} \int\limits_0^{x} x \; e^{-\frac{x^2}{y}} \; dy \; dx$

Ans: $\frac{1}{2}$



Double Integration in polar coordinates

In polar plane consider a region bounded by $r = g_1(\theta)$, $r = g_2(\theta)$, $\theta_1 = \alpha$ and $\theta_2 = \beta$ then double integration in polar plane to evaluate a function $f(r, \theta)$ is defined as



$$I = \iint f(r, \theta)\; dA$$

$$= \int\limits_{\theta_1 = \alpha}^{\theta_2 = \beta} \int\limits_{r_1 = g_1(\theta)}^{r_2 = g_2(\theta)} f(r, \theta)\; dr\; d\theta$$

Example:Evaluation in Polar coordinates

1. $\int_{0}^{\frac{\pi}{2}} \int_{0}^{acos(\theta)} r^2 \; dr \; d\theta$

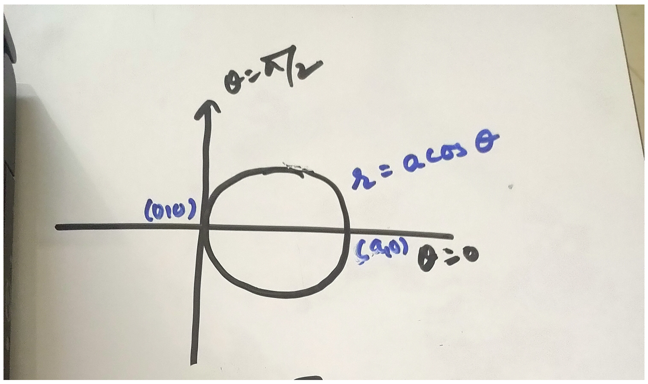
   Ans: $\frac{2a^3}{9}$

2. $\int_{0}^{\frac{\pi}{4}} \int_{0}^{\sqrt{cos(2\theta)}} \frac{r}{(1+r^2)^2} \; dr \; d\theta$

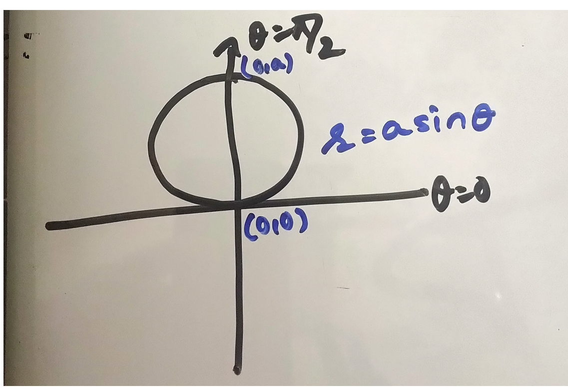
   Ans: $\frac{\pi - 2}{8}$

List of standard polar curves

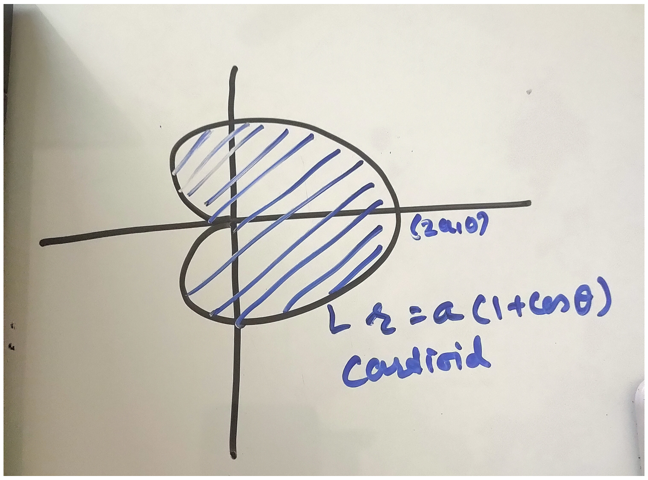
1. A circle $r = acos(\theta)$
   symmetry about the line $\theta = 0$



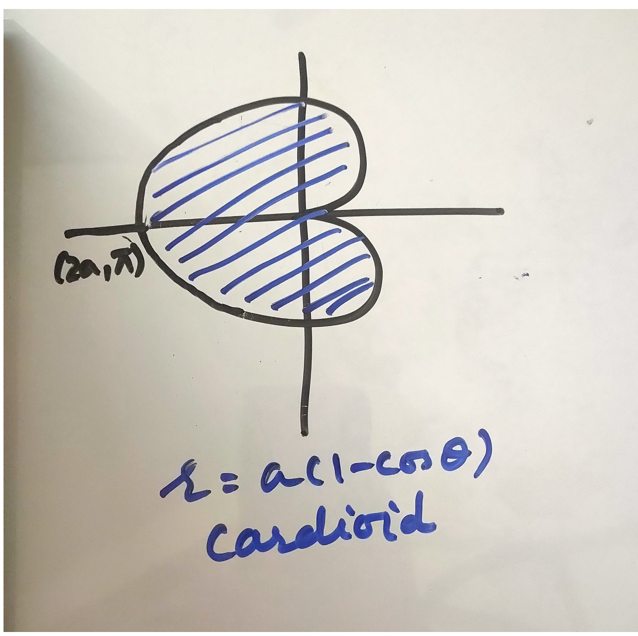
2. A circle $r = asin(\theta)$
   symmetry about the line $\theta = \frac{\pi}{2}$



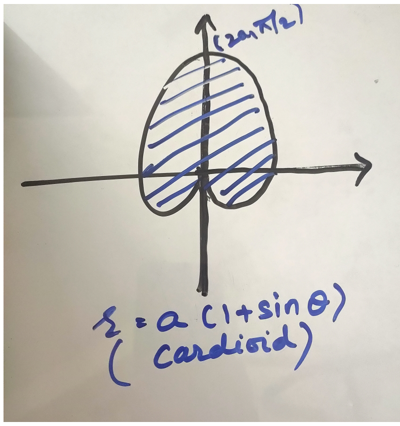
3. A cardioid $r = a(1 + cos(\theta))$
symmetry about the line $\theta = 0$



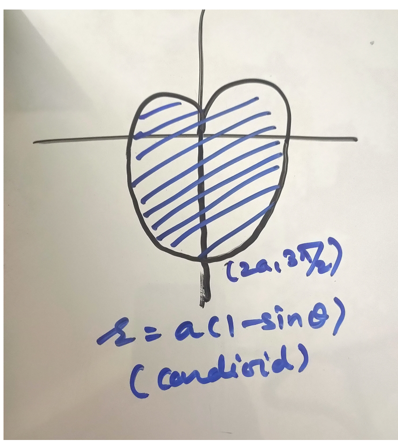
4. A cardioid $r = a(1 - cos(\theta))$
symmetry about the line $\theta = 0$



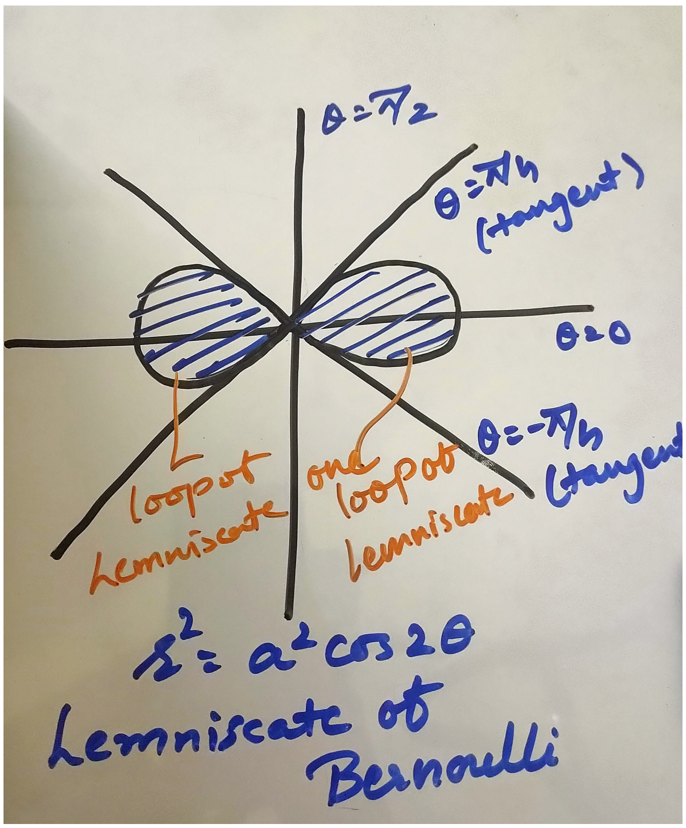
5. A cardioid $r = a(1 + sin(\theta))$
   symmetry about the line $\theta = \frac{\pi}{2}$



6. A cardioid $r = a(1 - sin(\theta))$
   symmetry about the line $\theta = \frac{\pi}{2}$



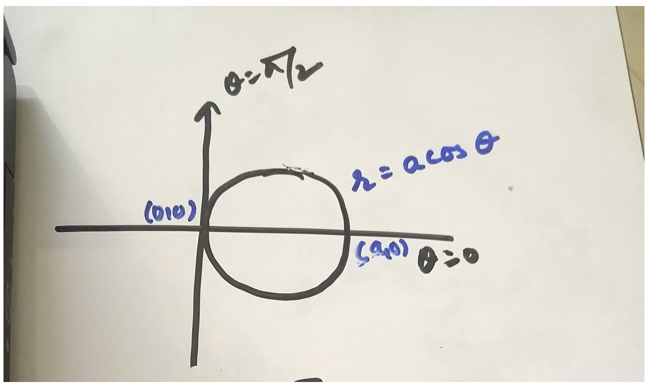
7. A lemniscate of Bernoulli $r^2 = a^2 cos(2\theta)$
symmetry about the line $\theta = \frac{\pi}{2}$ and $\theta = 0$
tangents at $\theta = \pm\frac{\pi}{4}$



Example: Evaluate

1. Evaluate $\iint r\sqrt{(a^2-r^2)}\,dr\,d\theta$ where region is upperhallf of the circle $r = acos\theta$

   Solution:

   The region is shown by shadded portion in the figure. To decide limits of integration considering a ray passing from the pole into the rgion we have $0 \le r \le acos\theta$ and $0 \le \theta \le \frac{\pi}{2}$

$\therefore$ required Integral

$$I = \int_0^{\frac{\pi}{2}} \int_0^{acos\theta} r\,\sqrt{a^2 - r^2}\; dr\; d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_{a^2}^{a^2 sin^2\theta} \frac{-dt}{2}\sqrt{t}\; dt\; d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{2}\left[\frac{2t^{\frac{3}{2}}}{3}\right]_{a^2}^{a^2 sin^2\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{3}\left[a^3 sin^3\theta - a^3\right]\; d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{3}a^3\left[sin^3\theta - 1\right]\; d\theta$$

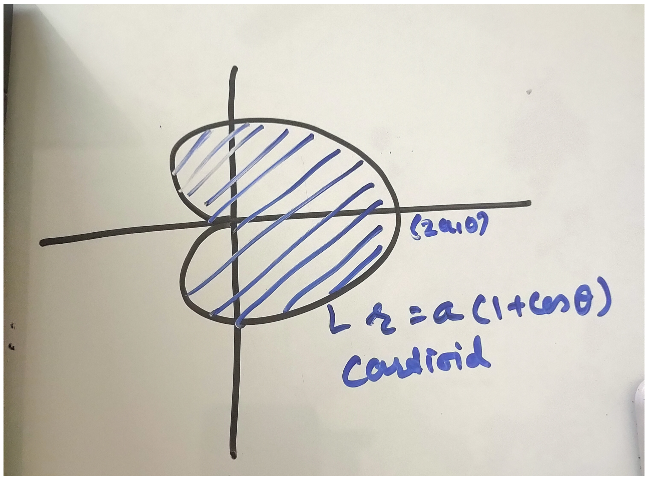
$$= -\frac{1}{3}a^3 \int_0^{\frac{\pi}{2}} \left[sin^3\theta - 1\right]\; d\theta$$

$$= -\frac{1}{3}a^3\left[\frac{1}{2}\beta\left(2, \frac{1}{2}\right) - \frac{\pi}{2}\right]$$

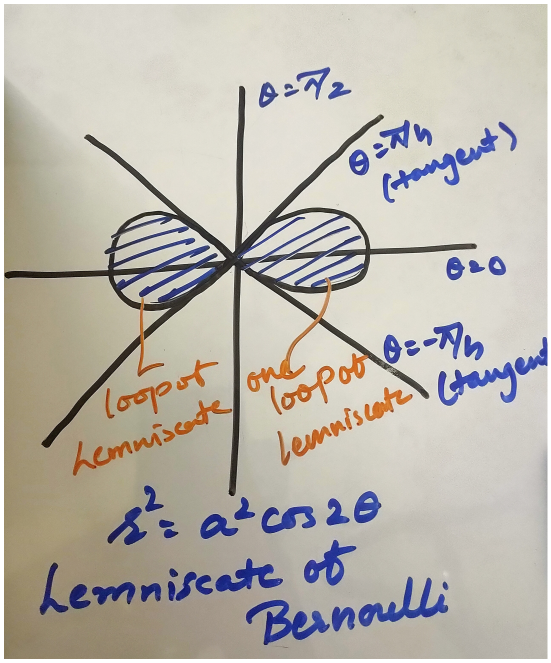
$$= -\frac{1}{3}a^3\left[\frac{1}{2}\left(\frac{\Gamma 2\Gamma\left(1/2\right)}{\Gamma\left(5/2\right)}\right) - \frac{\pi}{2}\right]$$

$$= -\frac{1}{3}a^3\left[\frac{2}{3} - \frac{\pi}{2}\right] \text{..........................} Ans$$

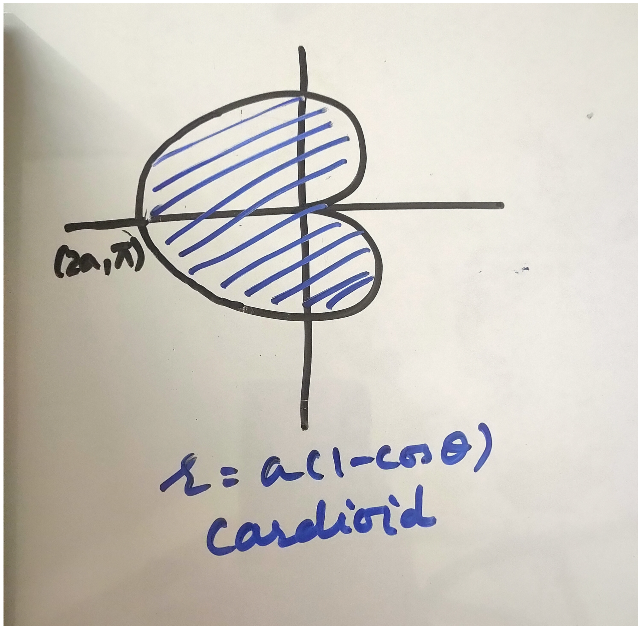
2. Evaluate $\iint r^2 \ sin\theta \ dr \ d\theta$ over a cardioid $r = a(1 + cos\theta)$ above the initial line.
   Ans:$\frac{4a^3}{3}$



3. Evaluate $\iint \frac{r}{\sqrt{r^2+a^2}} \ dr \ d\theta$ over a Lemniscate of Bernoulli $r^2 = a^2cos(2\theta)$
   Ans:$4a - a\pi$



4. Evaluate $\iint r^2 \, sin\theta \, dr \, d\theta$ over a cardioid $r = a(1 - cos\theta)$ above the initial line.
Ans:$\frac{4a^3}{3}$


(2a, π)
r = a(1-cos θ)
Cardioid

Evaluation of Double Integration by changing from Cartesian to Polar Coordinates

For a given point (x, y) in Cartesian plane consider a corresponding point (r, θ) in polar plane by taking polar coordinates $x = rcos\theta$ and $y = rsin\theta$, then change of coordinates is related to the Jacobian of (x, y) with respect to (r, θ) given by

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

$\therefore dx\ dy = J\left(\frac{x,y}{r,\theta}\right)\ dr\ d\theta = r\ dr\ d\theta$

then

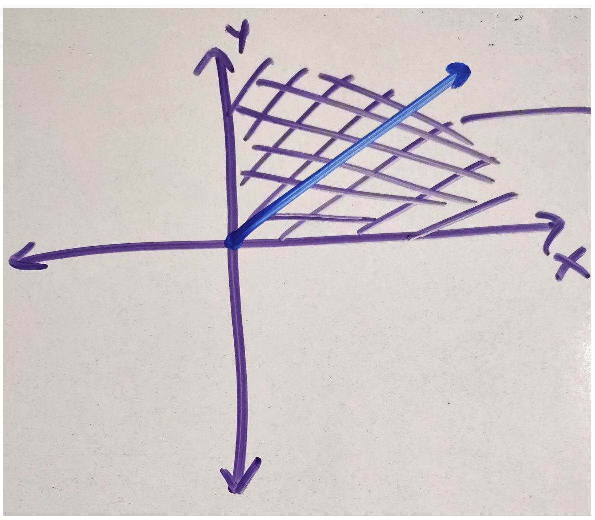
$$\begin{aligned} I &= \int\int F(x, y)\ dx\ dy \\ &= \int\int F(rcos\theta, rsin\theta)\ r\ dr\ d\theta \end{aligned}$$

where in polar plane point(r, θ) varies as point (x, y) varies in cartesian plane.

EXAMPLES(Evaluate by changing into polar coordinates)

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing into polar coordinates.

   Solution:
   Here from a given data x and y varies from 0 to ∞.
   $0 \le x \le \infty$ and $0 \le y \le \infty$
   $\therefore$ in Cartesian plane region is the first quadrant as shown in the figure.

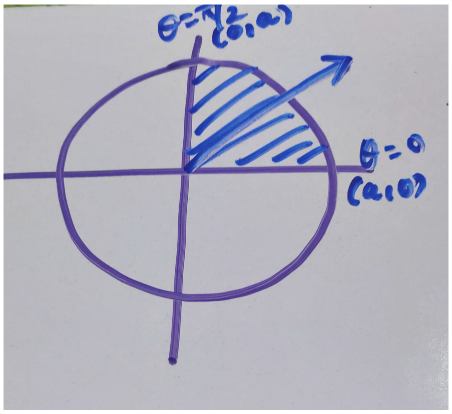


To change into polar coordinates consider $x = rcos\theta$ and $y = rsin\theta$
$\therefore$ we have $x^2 + y^2 = r^2$ and $dx\ dy = r\ dr\ d\theta$
For the limits consider a ray from the pole entering the region, we have
$0 \le r \le \infty$ and $0 \le \theta \le \frac{\pi}{2}$
$\therefore$ Required integral is

$$
\begin{aligned}
I &= \int\limits_0^\infty \int\limits_0^\infty e^{-(x^2+y^2)}\ dx\ dy \\
&= \int\limits_0^{\frac{\pi}{2}} \int\limits_0^\infty e^{-r^2}\ r\ dr\ d\theta \\
&= \int\limits_0^{\frac{\pi}{2}} \int\limits_0^{-\infty} \frac{e^t}{-2}\ dt\ d\theta \\
&= \frac{1}{2}\ \int\limits_0^{\frac{\pi}{2}} \int\limits_{-\infty}^{0} e^t\ dt\ d\theta \\
&= \frac{1}{2}\ \int\limits_0^{\frac{\pi}{2}} \left[e^t\right]_{-\infty}^{0}\ d\theta \\
&= \frac{1}{2}\ \int\limits_0^{\frac{\pi}{2}}\ d\theta \\
&= \frac{1}{2}\ [\theta]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}\ \frac{\pi}{2} \\
&= \frac{\pi}{4}..............Ans
\end{aligned}
$$

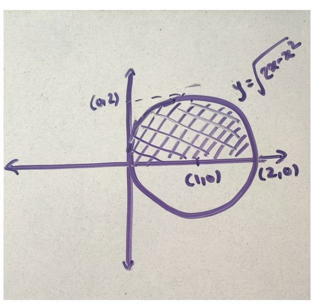
1. Evaluate $\int\limits_0^a \int\limits_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$ by changing into polar coordinates.

   Ans: $\frac{a^4}{4}$



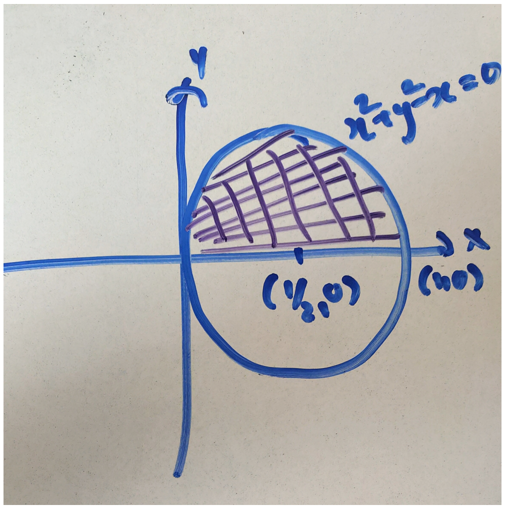
2. Evaluate $\int\limits_0^2 \int\limits_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$ by changing into polar coordinates.

   Ans: $\frac{4}{3}$



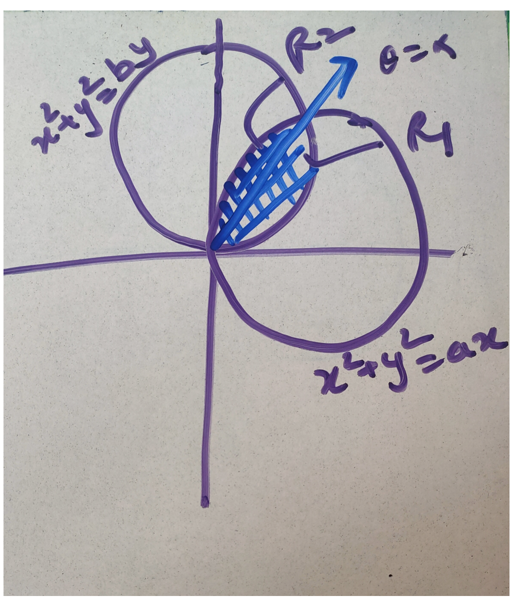
3. Evaluate $\iint \frac{4xy}{x^2+y^2}\, e^{-x^2-y^2}\, dx\, dy$ by changing into polar coordinates, over a circle $x^2+y^2-x=0$ in first quadrant

   Ans: $\frac{1}{e}$



4. Evaluate $\iint \frac{(x^2+y^2)^2}{x^2\ y^2}\ dx\ dy$ by changing into polar coordinates, where region is an area common to circles$x^2+y^2=ax$ and $x^2+y^2=by$; where $a, b > 0$

   Ans:ab



5. Evaluate $\iint \frac{x^2\ y^2}{x^2+y^2}\,dx\,dy$ where region is an annular region between the circles$x^2+y^2=a^2$ and $x^2+y^2=b^2$; where $a>b>0$
Ans:$\frac{(a^4-b^4)\pi}{16}$