Linear Differential Equation

A Differential Equation in which the dependent variable and its derivatives appear in first degree ony and are not multiplied together is called Linear Differential Equation. The General Form of Linear Differential Equation is given by

$$\frac{dy}{dx} + Py = Q$$

Or

$$\frac{dx}{dy} + P'x = Q'$$

where P and Q are constants or functions of x only and P' and Q' are constants or functions of y only.

The General Solution of First form Linear Differential Equation is given by

$$y.(I.F.) = \int Q.(I.F)dx + c$$

where $I.F. = e^{\int P(x)dx}$ and

The General Solution of Second form Linear Differential Equation is given by

$$x.(I.F.) = \int Q'.(I.F)dy + c$$

where $I.F. = e^{\int P'(y)dy}$

Example

Solve: $xlog(x)\dfrac{dy}{dx} + y = 2log(x)$

Solution:

Given Equation is not in Linear Differential Equation Form. Re writing the Equation by dividing throughout with $xlog(x)$, we have

$$\frac{dy}{dx} + \frac{y}{xlog(x)} = \frac{2}{log(x)}$$

Which is the linear Differential Equation where

$$P = \frac{1}{xlog(x)}$$

and

$$Q = \frac{2}{x}$$

Now

$$\begin{aligned} I.F. &= e^{\int P(x)dx} \\ &= e^{\int \frac{1}{xlog(x)}dx} \\ &= e^{\int \frac{1/x}{log(x)}dx} \\ &= e^{\log(log(x))} \\ I.F. &= log(x) \end{aligned}$$

therefore General Solution is

$$y.(I.F.) = \int Q.(I.F)dx + c$$

$$y.(log(x)) = \int \frac{2}{log(x)}.\left[log(x)\right]dx + c$$

For R.H.S.
Let $log(x) = u \implies \frac{1}{x}dx = du$

$$y.(log(x)) = \int 2udu + c$$

$$y.(log(x)) = [log(x)]^2 + c$$

$$y = log(x) + \frac{c}{log(x)}$$

is the required solution

Non-Linear Differential Equation

Equations of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q.....\mathbf{(A)}$$

and

$$\frac{dy}{dx} + Py = Qy^n.....\mathbf{(B)}$$

where P and Q are constants or functions of x
are known as **Non-Linear differential equations** and are reducible to linear differential equation Form.

For D.E. of the form (1) we substitute $v = f(y)$ so that it reduced in Linear Differential equation in v and x which is solvable by L.D.E. solution method and then re substituting v we get the solution of original Equation.

For D.E. of the form (2) we divide throughout by y^n so that it reduced in form (1) then again follow the solution method of Form (1), we can obtain the solution of given D.E.

Equation (B) are known as Bernoulli's Equation
Example

Solve: (1) $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Solution:

$$\frac{dy}{dx} = e^{x-y}(e^x - e^y)$$

$$\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y)$$

$$e^y\frac{dy}{dx} = e^{2x} - e^x e^y$$

$$e^y \frac{dy}{dx} + e^x e^y = e^{2x}$$

Let $e^y = v$
$e^y \frac{dy}{dx} = \frac{dv}{dx}$
Substituting (2) in (1) , we have

$$\frac{dv}{dx} + e^x v = e^{2x}$$

which is L.D.E. in v and x
$I.F. = e^{\int P dx}$
$= e^{\int e^x dx}$
$= e^{e^x}$

Solution is,

$$v.(I.F.) = \int Q.(I.F) dx + c$$

$$v.(e^{e^x}) = \int e^{2x}.(e^{e^x}) dx + c$$

$$v.(e^{e^x}) = \int e^x.(e^{e^x}) e^x dx + c$$

Let $e^x = t$
$e^x dx = dt$

Hence

$$v.(e^t) = \int t.(e^t) dt + c$$

$$v.(e^t) = te^t - e^t + c$$

$$v.(e^{e^x}) = e^x e^{e^x} - e^{e^x} + c$$

$$v = e^x - 1 + ce^{-e^x}$$

$$e^y = e^x - 1 + ce^{-e^x}$$

which is require solution.

(2) $\frac{dy}{dx} = x^3y^3 - xy$

Solution:

$$\frac{dy}{dx} = x^3y^3 - xy$$

$$\frac{dy}{dx} + xy = x^3y^3$$

which is Bernoulli's equation
Dividing throughot by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}x = x^3$$

substituting $y^2 = v$ we get

$$\frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{y^3}\frac{dy}{dx} = -[\frac{1}{2}\frac{dv}{dx}]$$

Hence

$$-[\frac{1}{2}\frac{dv}{dx}] + vx = x^3$$

Multiplying throughout by (-2), we get

$$\frac{dv}{dx} + (-2)xv = -2x^3$$

Which is linear equation in x and v where $P = -2x$ and $q = -2x^3$
$I.F. = e^{\int Pdx}$
$= e^{\int -2xdx}$
$= e^{-x^2}$

Solution is,

$$v.(I.F.) = \int Q.(I.F)dx + c$$

$$v.(e^{-x^2}) = \int (-2x^3).(e^{-x^2})dx + c$$

$$v.(e^{-x^2}) = \int (-x^2).(e^{-x^2})2xdx + c$$

Let $-x^2 = t$
$2xdx = -dt$

Hence

$$v.(e^t) = \int t.(e^t)(-dt) + c$$

Integrating by parts we get

$$v.(e^t) = -[te^t - \int (e^t)dt] + c$$

$$v.(e^t) = (e^t)(1-t) + c$$

Re substituting v and t require solution is

$$\frac{1}{y^2} = (1+x^2) + ce^{x^2}$$

Practice Examples

Solve:

(1) $y\frac{dy}{dx} + \frac{4x}{3} - \frac{y^2}{3x} = 0$

(2) $xy(1+xy^2).\frac{dy}{dx} = 1$

(3) $\frac{dy}{dx} + xsin2y = x^3cos^2y$

(4) $\frac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^2$

(5) $\frac{y^2}{x^2} = \frac{2}{3}e^{x^{-3}} + c$

(6) $\frac{dy}{dx} + \frac{ylogy}{x-logy} = 0$

(7) $\frac{dy}{dx} + \frac{y}{x}logy = \frac{y}{x^2}(logy)^2$

(8) $\frac{dy}{dx} + \frac{1-2x}{x^2}.y = 1$

(9) $x.\frac{dy}{dx} + 2.y = log(x)$

(10) $(x+2y^3).\frac{dy}{dx} = y$

(11) $\frac{dy}{dx}coshx = 2cosh^2xsinhx - ysinhx$

(12) $x(x-1)\frac{dy}{dx} - (x-2)y = (x^3).(2x-1)$

(13) $(1+y^2)\frac{dx}{dy} = tan^{-1}y - x$

(14) $sin2x\frac{dy}{dx} = y + tanx$

(15) $(1+siny)\frac{dx}{dy} = 2ycosy - x(secy + tany)$

(16) $(1+x+xy^2)dy + (y+y^3)dx = 0$

(17) $(1-x^2)\frac{dy}{dx}+2xy=x.\sqrt{1-x^2}$

Answers

(1) $y^2+2x^2=cx^{\frac{2}{3}}$

(2)$-\frac{1}{x}=(y^2-2)+ce^{\frac{y^2}{2}}$

(3) $tany=\frac{1}{2}(x^2-1)+ce^{-x^2}$

(4) $\frac{1}{y-x}=-(x^2+2)+ce^{x^2}2$

(5) $y=x^3+\frac{cx^2}{x-1}$

(6) $xlogy-\frac{1}{2}(logy)^2=c$

(7) $\frac{1}{xlogy}=\frac{1}{2x^2}+c$

(8) $y=x^2+ce^{1/x}.x^2$

(9)$y=\frac{2log(x)-1}{4}+\frac{c}{x^2}$

(10) $x=y^3+cy$

(11) $y=\frac{2(coshx)^2}{3}+csechx$

(12) $y=x^3+\frac{cx^2}{x-1}$

(13) $x = tan^{-1}y - 1 + ce^{tan^{-1}y}$

(14) $y = tanx - 1 + c\sqrt{tanx}$

(15) $x(1 + siny) = y^2cosy + ccosy$

(16) $xy + tan^{-1}y = c$

(17) $2tan^{-1}y = x^2 - 1 + ce^{(x^2)}$