DECEMBER 2016 FIRST SEMESTER MATHEMATICS PAPER SOLUTION

Q.1)

1a) If $\cos\alpha\cosh\beta=\frac{x}{2}$, $\sin\alpha\sinh\beta=\frac{y}{2}$, prove that $\sec(\alpha-i\beta)+\sec(\alpha+i\beta)=\frac{4x}{x^2+y^2}$.

(Chp: Hyperbolic Functions) (3)

Ans. $\text{LHS}=\sec(\alpha-i\beta)+\sec(\alpha+i\beta)$

$$=\frac{1}{\cos(\alpha-i\beta)}+\frac{1}{\cos(\alpha+i\beta)}$$

$$=\frac{1}{\cos\alpha\cos i\beta+\sin\alpha\sin i\beta}+\frac{1}{\cos\alpha\cos i\beta-\sin\alpha\sin i\beta}$$

$$=\frac{1}{\cos\alpha\cosh\beta+i\sin\alpha\sinh\beta}+\frac{1}{\cos\alpha\cosh\beta-i\sin\alpha\sinh\beta}\quad\{\because \cos(ix)=\cosh x;\sin(ix)=i\sinh x\}$$

$$=\frac{1}{\frac{x}{2}+i\cdot\frac{y}{2}}+\frac{1}{\frac{x}{2}-i\cdot\frac{y}{2}}\quad\text{(Given)}$$

$$=\frac{2}{x+iy}+\frac{2}{x-iy}$$

$$=\frac{2x-\cancel{2iy}+2x+\cancel{2iy}}{(x+iy)(x-iy)}$$

$$=\frac{4x}{x^2-i^2y^2}$$

$$=\frac{4x}{x^2+y^2}$$

= RHS

Hence, $\boxed{\sec(\alpha-i\beta)+\sec(\alpha+i\beta)=\frac{4x}{x^2+y^2}}$

1b) If $z=\log\left(e^x+e^y\right)$, show that $rt-s^2=0$, where $r=\frac{\partial^2 z}{\partial x^2}$, $t=\frac{\partial^2 z}{\partial y^2}$, $s=\frac{\partial^2 z}{\partial x \partial y}$. (Chp: Partial Differentiation)

(3)

Ans. $z=\log\left(e^x+e^y\right)$

Differentiating partially w.r.t. x, $\frac{\partial z}{\partial x}=\frac{1}{e^x+e^y}\cdot\left(e^x+0\right)=\frac{e^x}{e^x+e^y}\rightarrow(1)$

Again, differentiating partially w.r.t. x, $\frac{\partial^2 z}{\partial x^2}=\frac{(e^x+e^y)\cdot e^x-e^x\cdot(e^x+0)}{(e^x+e^y)^2}$

$\therefore\ r=\frac{\partial^2 z}{\partial x^2}=\frac{\cancel{(e^x)^2}+e^y\cdot e^x-\cancel{(e^x)^2}}{(e^x+e^y)^2}$

$\therefore\ r=\frac{e^{x+y}}{(e^x+e^y)^2}\rightarrow(2)$

Similarly, we can prove, $t=\frac{\partial^2 z}{\partial y^2}=\frac{e^{x+y}}{(e^x+e^y)^2}\rightarrow(3)$

Now, differentiating (1) partially w.r.t. y, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=e^x\cdot\frac{-1}{(e^x+e^y)^2}\cdot(0+e^y)$

$\therefore\ \frac{\partial^2 z}{\partial x \partial y}=\frac{-e^x\cdot e^y}{(e^x+e^y)^2}$

$\therefore\ s=\frac{\partial^2 z}{\partial x \partial y}=\frac{-e^{x+y}}{(e^x+e^y)^2}\rightarrow(4)$

From (2), (3) and (4), $rt-s^2=\frac{e^{x+y}}{(e^x+e^y)^2}\times\frac{e^{x+y}}{(e^x+e^y)^2}-\left[\frac{-e^{x+y}}{(e^x+e^y)^2}\right]^2$

$\therefore\ rt-s^2=\frac{(e^{x+y})^2}{(e^x+e^y)^4}-\frac{(e^{x+y})^2}{(e^x+e^y)^4}$

$\therefore\ \boxed{rt-s^2=0}$

1c) If x = u v, $y = \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (Chp: Jacobian) (3)

Ans. Given, x = u v

$$\therefore \frac{\partial x}{\partial u} = v \cdot 1 \text{ and } \frac{\partial x}{\partial v} = u \cdot 1$$

$$\therefore x_u = \frac{\partial x}{\partial u} = v \text{ and } x_v = \frac{\partial x}{\partial v} = u \rightarrow (1)$$

Given, $y = \frac{u+v}{u-v}$

$$\therefore \frac{\partial y}{\partial u} = \frac{(u-v)\cdot(1+0)-(u+v)\cdot(1-0)}{(u-v)^2} \text{ and } \frac{\partial y}{\partial v} = \frac{(u-v)\cdot(0+1)-(u+v)\cdot(0-1)}{(u-v)^2}$$

$$\therefore \frac{\partial y}{\partial u} = \frac{\cancel{u}-v-\cancel{u}-v}{(u-v)^2} \text{ and } \frac{\partial y}{\partial v} = \frac{u-\cancel{v}+u+\cancel{v}}{(u-v)^2}$$

$$\therefore y_u = \frac{\partial y}{\partial u} = \frac{-2v}{(u-v)^2} \text{ and } y_v = \frac{\partial y}{\partial v} = \frac{2u}{(u-v)^2} \rightarrow (2)$$

Let, $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$$= x_u\, y_v - x_v\, y_u$$

$$= v \times \frac{2u}{(u-v)^2} - u \times \frac{-2v}{(u-v)^2} \text{ (From 1 \& 2)}$$

$$= \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2}$$

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{4uv}{(u-v)^2}$$

$$\therefore \boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{J} = \frac{(u-v)^2}{4uv}}$$

1d) If $y=2^x \sin^2 x \cos x$ find y_n. (Chp: Successive Differentiation) (3)

Ans. Consider, $\sin^2 x \cos x = \sin x \cdot \sin x \cos x \times \frac{2}{2}$

$$= \frac{1}{2} \sin x \cdot \sin 2x \times \frac{2}{2}$$

$$= \frac{1}{4}\left[\cos(x-2x)-\cos(x+2x)\right]$$

$$= \frac{1}{4}\left[\cos x - \cos 3x\right] \rightarrow (1)$$

Given, $y = 2^x \sin^2 x \cos x$

$$y = 2^x \times \frac{1}{4}\left[\cos x - \cos 3x\right] \text{ (From 1)}$$

$$\therefore\ y = \frac{1}{4} \times 2^x \cos x - \frac{1}{4} \times 2^x \cos 3x$$

Using Formula,

If $y = k^x \cos(bx+c)$ then $y_n = r^n k^x \cos(bx+c+n\phi)$ where $r=\sqrt{(\log k)^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{\log k}\right)$

$$\therefore\ y_n = \frac{1}{4} \times r_1^{\,n}\, 2^x \cos(x+n\phi_1) - \frac{1}{4} \times r_2^{\,n}\, 2^x \cos(3x+n\phi_2)$$

$$\therefore\ y_n = \frac{2^x}{4}\left[r_1^{\,n} \cos(x+n\phi_1) - r_2^{\,n} \cos(3x+n\phi_2)\right]$$

where, $r_1 = \sqrt{(\log 2)^2+1^2} = \sqrt{(\log 2)^2+1}$ and $\phi_1 = \tan^{-1}\left(\frac{1}{\log 2}\right)$

$r_2 = \sqrt{(\log 2)^2+3^2} = \sqrt{(\log 2)^2+9}$ and $\phi_2 = \tan^{-1}\left(\frac{3}{\log 2}\right)$

1e) Express the matrix $A=\begin{bmatrix}1 & 0 & 5 & 3\\ -2 & 1 & 6 & 1\\ 3 & 2 & 7 & 1\\ 4 & -4 & 2 & 0\end{bmatrix}$ as the sum of symmetric and skew–symmetric matrices.

(Chp: Rank of Matrix) (4)

Ans. $A=\begin{bmatrix}1 & 0 & 5 & 3\\ -2 & 1 & 6 & 1\\ 3 & 2 & 7 & 1\\ 4 & -4 & 2 & 0\end{bmatrix}$ and $A'=\begin{bmatrix}1 & -2 & 3 & 4\\ 0 & 1 & 2 & -4\\ 5 & 6 & 7 & 2\\ 3 & 1 & 1 & 0\end{bmatrix}$

Let $P=\frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}2 & -2 & 8 & 7\\ -2 & 2 & 8 & -3\\ 8 & 8 & 14 & 3\\ 7 & -3 & 3 & 0\end{bmatrix}=\begin{bmatrix}1 & -1 & 4 & 3.5\\ -1 & 1 & 4 & -1.5\\ 4 & 4 & 7 & 1.5\\ 3.5 & -1.5 & 1.5 & 0\end{bmatrix}$

We observe, $p_{ij}=p_{ji}$

$\therefore$ P is Symmetric.

Let $Q=\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}0 & 2 & 2 & -1\\ -2 & 0 & 4 & 5\\ -2 & -4 & 0 & -1\\ 1 & -5 & 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1 & 1 & -0.5\\ -1 & 0 & 2 & 2.5\\ -1 & -2 & 0 & -0.5\\ 0.5 & -2.5 & 0.5 & 0\end{bmatrix}$

We observe, $q_{ij}=-q_{ji}$

$\therefore$ Q is Skew-Symmetric.

Now, $P+Q=\begin{bmatrix}1 & -1 & 4 & 3.5\\ -1 & 1 & 4 & -1.5\\ 4 & 4 & 7 & 1.5\\ 3.5 & -1.5 & 1.5 & 0\end{bmatrix}+\begin{bmatrix}0 & 1 & 1 & -0.5\\ -1 & 0 & 2 & 2.5\\ -1 & -2 & 0 & -0.5\\ 0.5 & -2.5 & 0.5 & 0\end{bmatrix}$

$=\begin{bmatrix}1 & 0 & 5 & 3\\ -2 & 1 & 6 & 1\\ 3 & 2 & 7 & 1\\ 4 & -4 & 2 & 0\end{bmatrix}$

= A.

Hence, A is expressed as a sum of symmetric and skew-symmetric matrix.

1f) Evaluate $\lim_{x\to 0}\dfrac{e^{2x}-(1+x)^2}{x\log(1+x)}$. (Chp: Indeterminate Forms) (4)

Ans. Let $L=\lim_{x\to 0}\dfrac{e^{2x}-(1+x)^2}{x\log(1+x)}$

$$=\lim_{x\to 0}\frac{\left(1+\frac{2x}{1!}+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\frac{2^4x^4}{4!}+\ldots\right)-\left(1+2x+x^2\right)}{x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)}$$

$$\left\{\because e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots;\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right\}$$

$$=\lim_{x\to 0}\frac{\cancel{1}+\cancel{2x}+\frac{2x^2}{2}+\frac{8x^3}{6}+\frac{16x^4}{24}+\ldots-\left(\cancel{1}+\cancel{2x}+x^2\right)}{x^2-\frac{x^3}{2}+\frac{x^4}{3}-\ldots}$$

$$=\lim_{x\to 0}\frac{\cancel{x^2}\left(1+\frac{4x}{3}+\frac{2x^2}{3}+\ldots\right)}{\cancel{x^2}\left(1-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)}$$

$$=\frac{1+0+0+\ldots}{1-0+0-\ldots}$$

$=1$

Hence, $\boxed{\lim_{x\to 0}\dfrac{e^{2x}-(1+x)^2}{x\log(1+x)}=1}$

Q.2)

2a) Show that the roots of $x^5 = 1$ can be written as 1, α, α^2, α^3, α^4.
Hence show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$. (Chp: Complex - DMT) (6)

Ans. $x^5 = 1$

$\therefore\ x^5 = e^{i0} = e^{i(0+2n\pi)}$

$\therefore\ x = e^{i2n\pi/5}$

Put n = 0, $x_1 = e^{i0} = 1$

Put n = 1, $x_2 = e^{i2\pi/5} = \alpha$ (let)

Put n = 2, $x_3 = e^{i4\pi/5} = \left(e^{i2\pi/5}\right)^2 = \alpha^2$

Put n = 3, $x_4 = e^{i6\pi/5} = \left(e^{i2\pi/5}\right)^3 = \alpha^3$

Put n = 4, $x_5 = e^{i8\pi/5} = \left(e^{i2\pi/5}\right)^4 = \alpha^4$

Hence, the roots of $x^5 = 1$ can be written as 1, α, α^2, α^3, α^4.

$\because x_1, x_2, x_3, x_4$ and x_5 are roots of $x^5 - 1 = 0$,

$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5) = x^5 - 1$

$\therefore\ \cancel{(x-1)}(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \cancel{(x-1)}(x^4+x^3+x^2+x+1)$

Put x = 1,

$\therefore\ (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 1^4+1^3+1^2+1+1$

$\therefore\ \boxed{(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5}$

2b) Reduce the following matrix to its normal form and hence find its rank $A=\begin{bmatrix}3&-2&0&1\\0&2&2&7\\1&-2&-3&2\\0&1&2&1\end{bmatrix}$.

(Chp: Rank of Matrix) (6)

Ans. $A=\begin{bmatrix}3&-2&0&1\\0&2&2&7\\1&-2&-3&2\\0&1&2&1\end{bmatrix}$

$R_1 \leftrightarrow R_3 \qquad \Rightarrow \begin{bmatrix}1&-2&-3&2\\0&2&2&7\\3&-2&0&1\\0&1&2&1\end{bmatrix}$

$R_3-3R_1 \qquad \Rightarrow \begin{bmatrix}1&-2&-3&2\\0&2&2&7\\0&4&9&-5\\0&1&2&1\end{bmatrix}$

$C_2+2C_1; C_3+3C_1; C_4-2C_1$

$\Rightarrow \begin{bmatrix}1&0&0&0\\0&2&2&7\\0&4&9&-5\\0&1&2&1\end{bmatrix}$

$R_2-R_4 \qquad \Rightarrow \begin{bmatrix}1&0&0&0\\0&1&0&6\\0&4&9&-5\\0&1&2&1\end{bmatrix}$

$R_3-4R_2; R_4-R_2 \Rightarrow \begin{bmatrix}1&0&0&0\\0&1&0&6\\0&0&9&-29\\0&0&2&-5\end{bmatrix}$

$C_4-6C_2 \qquad \Rightarrow \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&9&-29\\0&0&2&-5\end{bmatrix}$

$R_3-4R_4 \qquad \Rightarrow \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-9\\0&0&2&-5\end{bmatrix}$

$R_4-2R_3 \qquad \Rightarrow \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-9\\0&0&0&13\end{bmatrix}$

$C_4+9C_3 \qquad \Rightarrow \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&13\end{bmatrix}$

$\frac{1}{13}R_4 \qquad \Rightarrow \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$, which is the Normal Form.

Hence, Rank of A = Number of non-zero rows = 4

2c) Solve the following system of equations by Gauss – Seidal Iterative Method up to 4 iterations (8)
$4x - 2y - z = 40$; $x - 6y + 2z = -28$; $x - 2y + 12z = -86$. (Chp: Linear algebraic equations)

Ans The three equations are

$4x - 2y - z = 40$;
$x - 6y + 2z = -28$;
$x - 2y + 12z = -86$;

From 1^{st} equation, $4x = 40 + 2y + z$

$\therefore\ x = \frac{1}{4}(40+2y+z) = 4^{-1}(40+2y+z)$

Similarly, from 2^{nd} equation,

$y = 6^{-1}(28+x+2z)$

And, from 3^{rd} equation, $z = 12^{-1}(2y-86-x)$

Iteration 1:

Put $y_0 = z_0 = 0$;

$\therefore\ x_1 = 4^{-1}(40+2y_0+z_0)$

$= 4^{-1}(40+0+0)$

$= 10$

Put $x_1 = 10$; $z_0 = 0$;

$\therefore\ y_1 = 6^{-1}(28+x_1+2z_0)$

$= 6^{-1}(28+10-0)$

$= 6.3333$

Put $x_1 = 10$; $y_1 = 6.3333$;

$\therefore\ z_1 = 12^{-1}(2y_1-86-x_1)$

$= 12^{-1}(2\times 6.3333-86-10)$

$= -6.9444$

Iteration 2:

Put $y_1 = 6.3333$; $z_1 = -6.9444$;

$\therefore\ x_2 = 4^{-1}(40+2y_1+z_1)$

$= 4^{-1}(40+2\times 6.3333-6.9444)$

$= 11.4306$

Put $x_2 = 11.4306$; $z_1 = -6.9444$;

$\therefore\ y_2 = 6^{-1}(28+x_2+2z_1)$

$= 6^{-1}(28+11.4306-2\times 6.9444)$

$= 4.2570$

Put $x_2 = 11.4306$; $y_2 = 4.2570$;

$\therefore\ z_2 = 12^{-1}(2y_2-86-x_2)$

$= 12^{-1}(2\times 4.2570-86-11.4306)$

$= -7.4097$

Iteration 3:

Put $y_2 = 4.2570$; $z_2 = -7.4097$;

$\therefore\ x_3 = 4^{-1}(40+2y_2+z_2)$

$= 4^{-1}(40+2\times 4.2570-7.4097)$

$= 10.2761$

Put $x_3 = 10.2761$; $z_2 = -7.4097$;

$\therefore\ y_3 = 6^{-1}(28+x_3+2z_2)$

$= 6^{-1}(28+10.2761-2\times 7.4097)$

$= 3.9095$

Put $x_3 = 10.2761$; $y_3 = 3.9095$;

$\therefore\ z_3 = 12^{-1}(2y_3-86-x_3)$

$= 12^{-1}(2\times 3.9095-86-10.2761)$

$= -7.3714$

Iteration 4:

Put $y_3 = 3.9095$; $z_3 = -7.3714$;

$\therefore\ x_4 = 4^{-1}(40+2y_3+z_3)$

$= 4^{-1}(40+2\times 3.9098-7.3714)$

$= 10.1121$

Put $x_4 = 10.1121$; $z_3 = -7.3714$;

$\therefore\ y_4 = 6^{-1}(28+x_4+2z_3)$

$= 6^{-1}(28+10.1121-2\times 7.3714)$

$= 3.8949$

Put $x_4 = 10.1121$; $y_4 = 3.8949$;

$\therefore\ z_4 = 12^{-1}(2y_3-86-x_3)$

$= 12^{-1}(2\times 3.8949-86-10.1121)$

$= -7.3602$

Hence, by Gauss-Seidal method, solution of given set of equations is
$x = 10.1121$, $y = 3.8949$, $z = -7.3602$.

Q.3)

3a) Investigate for what values of λ and μ the system of equations: $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = \mu$ has (i) no solution, (ii) a unique solution, (iii) an infinite no. of solutions. (Chp: Linear Equations) (6)

Ans. Writing the equations in the matrix form, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

$R_3 - R_2$; $R_2 - R_1$; $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \mu - 10 \end{bmatrix}$

Augmented Matrix [A | B] $= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$

Number of unknowns = n = 3

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when $\mu \neq 10$ and $\lambda = 3$.

We then have, rank of A (r_A) = 2 and rank of [A | B] (r_{AB}) = 3

Case II: Unique Solution

For which, $r_A = r_{AB} = n$

This is only possible, when $\lambda \neq 3$ and μ has any value.

We then have, rank of A (r_A) = rank of [A | B] (r_{AB}) = 3

Case III: Infinite Solution

For which, $r_A = r_{AB} < 3$ (i.e. < 3)

This is only possible, when $\mu = 10$ and $\lambda = 3$.

We then have, rank of A (r_A) = rank of [A | B] (r_{AB}) = 2

Hence,

No Solution	$\mu \neq 10, \lambda = 3$
Unique Solution	μ = any value, $\lambda \neq 3$
Infinite Solution	$\mu = 10, \lambda = 3$

3b) If $u = x^2 + y^2 + z^2$, where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$. Prove that $\frac{du}{dt} = 4e^{2t}$.(Chp: Partial Differentiation)(6)

Ans. Method I:

$x = e^t$, $y = e^t \sin t$, $z = e^t \cos t \rightarrow (1)$

Differentiating w.r.t. t, $\frac{dx}{dt} = e^t$; $\frac{dy}{dt} = e^t \cdot \cos t + \sin t \cdot e^t$; $\frac{dz}{dt} = e^t \cdot -\sin t + \cos t \cdot e^t$; $\rightarrow (2)$

Given, $u = x^2 + y^2 + z^2$

$\therefore \frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = 2y$; $\frac{\partial u}{\partial z} = 2z$; $\rightarrow (3)$

Now, u $\rightarrow$ x, y, z $\rightarrow$ t

$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$

$= 2x \cdot e^t + 2y \cdot \left(e^t \cdot \cos t + \sin t \cdot e^t\right) + 2z \cdot \left(e^t \cdot -\sin t + \cos t \cdot e^t\right)$ (From 2 and 3)

$= 2e^t \cdot e^t + 2e^t \sin t \cdot e^t \left(\cos t + \sin t\right) + 2e^t \cos t \cdot e^t \left(-\sin t + \cos t\right)$ (From 1)

$= 2e^t \cdot e^t \left[1 + \sin t\left(\cos t + \sin t\right) + \cos t\left(-\sin t + \cos t\right)\right]$

$= 2e^{2t}\left[1 + \cancel{\sin t \cos t} + \sin^2 t - \cancel{\sin t \cos t} + \cos^2 t\right]$

$= 2e^{2t}[1+1]$

$\therefore \boxed{\frac{du}{dt} = 4e^{2t}}$

Method II:

$u = x^2 + y^2 + z^2$

Put $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

$\therefore u = \left(e^t\right)^2 + \left(e^t \sin t\right)^2 + \left(e^t \cos t\right)^2$

$= e^{2t} + e^{2t}\sin^2 t + e^{2t}\cos^2 t$

$= e^{2t}\left(1 + \sin^2 t + \cos^2 t\right)$

$= e^{2t}(1+1)$

$\therefore u = 2e^{2t}$

Differentiating w.r.t. t, $\frac{du}{dt} = 2e^{2t} \cdot 2$

$\therefore \boxed{\frac{du}{dt} = 4e^{2t}}$

3c) Show that $\sin\left(e^x - 1\right) = x + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots$. (Chp: Expansion) (4)

Ans. $\text{LHS} = \sin\left(e^x - 1\right)$

$$= \sin\left(\cancel{1} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots - \cancel{1}\right) \quad \left\{\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right\}$$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) - \frac{1}{3!}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)^3 + \frac{1}{5!}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)^5 + \ldots \quad \left\{\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right\}$$

$$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots - \frac{1}{6}\left(x^3 + 3x^2 \cdot \frac{x^2}{2} + \ldots\right) + \frac{1}{120}\left(x^5 + \ldots\right) + \ldots$$

$$= x + \frac{x^2}{2} + \cancel{\frac{x^3}{6}} - \cancel{\frac{x^3}{6}} + \frac{x^4}{24} - \frac{3x^4}{12} + \ldots$$

$$\therefore \boxed{\sin\left(e^x - 1\right) = x + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots}$$

3d) Expand $2x^3+7x^2+x-6$ in powers of x – 2. (Chp: Expansion) (4)

Ans. Let $f(x)=2x^3+7x^2+x-6$

$\therefore\ f'(x)=6x^2+14x+1$

$\therefore\ f''(x)=12x+14$

$\therefore\ f'''(x)=12$

Let a = 2,

$\therefore\ f(a)=f(2)=2(2)^3+7(2)^2+(2)-6=40$

$\therefore\ f'(a)=f'(2)=6(2)^2+14(2)+1=53$

$\therefore\ f''(a)=f''(2)=12(2)+14=38$

$\therefore\ f'''(a)=f'''(2)=12$

By Taylor Series, $f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$

$\therefore\ f(x)=40+(x-2)\cdot 53+\frac{1}{2}(x-2)^2\cdot 38+\frac{1}{6}(x-2)^3\cdot 12$

$\therefore\ \boxed{2x^3+7x^2+x-6=40+53(x-2)+19(x-2)^2+2(x-2)^3}$

Q.4)

4a) If x = u + v + w, y = uv + vw + uw, z = uvw and ϕ is a function of x, y and z.

Prove that $x\frac{\partial\phi}{\partial x}+2y\frac{\partial\phi}{\partial y}+3z\frac{\partial\phi}{\partial z}=u\frac{\partial\phi}{\partial u}+v\frac{\partial\phi}{\partial v}+w\frac{\partial\phi}{\partial w}$. (Chp: Partial Differentiation) (6)

Ans. Given, x = u + v + w $\quad\therefore\ \frac{\partial x}{\partial u}=1;\ \frac{\partial x}{\partial v}=1$ and $\frac{\partial x}{\partial w}=1\ \rightarrow(1)$

Given, y = uv + vw + uw

$\therefore\ \frac{\partial y}{\partial u}=v\cdot1+0+w\cdot1=v+w;\ \frac{\partial y}{\partial v}=u\cdot1+w\cdot1+0=u+w$ and $\frac{\partial y}{\partial w}=0+v\cdot1+u\cdot1=v+u\ \rightarrow(2)$

Given, z = uvw $\quad\therefore\ \frac{\partial z}{\partial u}=vw;\ \frac{\partial z}{\partial v}=uw$ and $\frac{\partial z}{\partial w}=uv\ \rightarrow(3)$

Now, $\phi\rightarrow$ x, y, z $\rightarrow$ u, v, w

$$\therefore\ \frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\cdot\frac{dx}{\partial u}+\frac{\partial\phi}{\partial y}\cdot\frac{dy}{\partial u}+\frac{\partial\phi}{\partial z}\cdot\frac{dz}{\partial u}$$

$$\therefore\ \frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\cdot1+\frac{\partial\phi}{\partial y}(v+w)+\frac{\partial\phi}{\partial z}\cdot vw\quad\text{(From 1, 2 and 3)}\rightarrow(4)$$

Similarly, $\frac{\partial\phi}{\partial v}=\frac{\partial\phi}{\partial x}\cdot\frac{dx}{\partial v}+\frac{\partial\phi}{\partial y}\cdot\frac{dy}{\partial v}+\frac{\partial\phi}{\partial z}\cdot\frac{dz}{\partial v}$

$$\therefore\ \frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\cdot1+\frac{\partial\phi}{\partial y}(u+w)+\frac{\partial\phi}{\partial z}\cdot uw\quad\text{(From 1, 2 and 3)}\rightarrow(5)$$

Similarly, $\frac{\partial\phi}{\partial w}=\frac{\partial\phi}{\partial x}\cdot\frac{dx}{\partial w}+\frac{\partial\phi}{\partial y}\cdot\frac{dy}{\partial w}+\frac{\partial\phi}{\partial z}\cdot\frac{dz}{\partial w}$

$$\therefore\ \frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\cdot1+\frac{\partial\phi}{\partial y}(u+v)+\frac{\partial\phi}{\partial z}\cdot uv\quad\text{(From 1, 2 and 3)}\rightarrow(6)$$

RHS $=u\frac{\partial\phi}{\partial u}+v\frac{\partial\phi}{\partial v}+w\frac{\partial\phi}{\partial w}$

$$=u\left[\frac{\partial\phi}{\partial x}+(v+w)\frac{\partial\phi}{\partial y}+vw\frac{\partial\phi}{\partial z}\right]+v\left[\frac{\partial\phi}{\partial x}+(u+w)\frac{\partial\phi}{\partial y}+uw\frac{\partial\phi}{\partial z}\right]+w\left[\frac{\partial\phi}{\partial x}+(u+v)\frac{\partial\phi}{\partial y}+uv\frac{\partial\phi}{\partial z}\right]\quad\text{(From 4, 5 and 6)}$$

$$=u\frac{\partial\phi}{\partial x}+u(v+w)\frac{\partial\phi}{\partial y}+uvw\frac{\partial\phi}{\partial z}+v\frac{\partial\phi}{\partial x}+v(u+w)\frac{\partial\phi}{\partial y}+uvw\frac{\partial\phi}{\partial z}+w\frac{\partial\phi}{\partial x}+w(u+v)\frac{\partial\phi}{\partial y}+uvw\frac{\partial\phi}{\partial z}$$

$$=\frac{\partial\phi}{\partial x}(u+v+w)+\frac{\partial\phi}{\partial y}\left[u(v+w)+v(u+w)+w(u+v)\right]+3uvw\frac{\partial\phi}{\partial z}$$

$$=\frac{\partial\phi}{\partial x}(u+v+w)+\frac{\partial\phi}{\partial y}\left[uv+uw+vu+vw+wu+wv\right]+3uvw\frac{\partial\phi}{\partial z}$$

$$=\frac{\partial\phi}{\partial x}(u+v+w)+\frac{\partial\phi}{\partial y}\times2\left[uv+vw+uw\right]+3uvw\frac{\partial\phi}{\partial z}$$

$=x\frac{\partial\phi}{\partial x}+2y\frac{\partial\phi}{\partial y}+3z\frac{\partial\phi}{\partial z}$ (Given)

= LHS

4b) If tan (θ + i ϕ) = tanα + i secα. Prove that : (i) $e^{2\phi} = \cot\frac{\alpha}{2}$; (ii) $2\theta = n\pi + \frac{\pi}{2} + \alpha$. (Chp: Homogenous Functions) (6)

Ans. $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha \rightarrow (1)$

Taking conjugates,

$\tan(\theta - i\phi) = \tan\alpha - i\sec\alpha \rightarrow (2)$

Now, $\tan[(\theta+i\phi)+(\theta-i\phi)] = \dfrac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$

$\therefore \tan\left[\theta + \cancel{i\phi} + \theta - \cancel{i\phi}\right] = \dfrac{(\tan\alpha + \cancel{i\sec\alpha}) + (\tan\alpha - \cancel{i\sec\alpha})}{1-(\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$ (From 1 & 2)

$\therefore \tan 2\theta = \dfrac{2\tan\alpha}{1-(\tan^2\alpha - i^2\sec^2\alpha)}$

$\therefore \tan 2\theta = \dfrac{2\tan\alpha}{1-(\tan^2\alpha + \sec^2\alpha)}$

$\therefore \tan 2\theta = \dfrac{2\tan\alpha}{\cancel{1}-(\tan^2\alpha + \cancel{1} + \tan^2\alpha)}$

$\therefore \tan 2\theta = \dfrac{\cancel{2}\ \cancel{\tan\alpha}}{-\cancel{2}\tan^{\cancel{2}}\alpha}$

$\therefore \tan 2\theta = -\cot\alpha$

$\therefore \tan 2\theta = \tan\left(\frac{\pi}{2} + \alpha\right)$

$\therefore \boxed{2\theta = n\pi + \frac{\pi}{2} + \alpha}$

Similarly, $\tan[(\theta+i\phi)-(\theta-i\phi)] = \dfrac{\tan(\theta+i\phi)-\tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$

$\therefore \tan\left[\cancel{\theta} + i\phi - \cancel{\theta} + i\phi\right] = \dfrac{(\cancel{\tan\alpha} + i\sec\alpha) - (\cancel{\tan\alpha} - i\sec\alpha)}{1+(\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$ (From 1 & 2)

$\therefore \tan 2i\phi = \dfrac{2i\sec\alpha}{1+(\tan^2\alpha - i^2\sec^2\alpha)}$

$\therefore \cancel{i}\tanh 2\phi = \dfrac{2\cancel{i}\sec\alpha}{1+\tan^2\alpha + \sec^2\alpha}$

$\therefore \tanh 2\phi = \dfrac{2\sec\alpha}{\sec^2\alpha + \sec^2\alpha}$

$\therefore \tanh 2\phi = \dfrac{\cancel{2}\ \cancel{\sec\alpha}}{\cancel{2}\sec^{\cancel{2}}\alpha}$

$\therefore \tanh 2\phi = \cos\alpha$

$\therefore 2\phi = \tanh^{-1}(\cos\alpha)$

$\therefore 2\phi = \frac{1}{2}\log\left|\dfrac{1+\cos\alpha}{1-\cos\alpha}\right|$

$\therefore 2\phi = \log\left|\dfrac{\cancel{2}\cos^2\alpha/2}{\cancel{2}\sin^2\alpha/2}\right|^{1/2}$

$\therefore 2\phi = \log\left|\dfrac{\cos\alpha/2}{\sin\alpha/2}\right|$

$\therefore 2\phi = \log\left|\cot\frac{\alpha}{2}\right|$

$\therefore \boxed{e^{2\phi} = \cot\frac{\alpha}{2}}$

Hence Proved.

4c) Find the root of the equation $x^4+x^3-7x^2-x+5=0$ which lies between 2 and 2.1 Correct to three places of decimal using Regula Falsi Method. (Chp: Linear algebraic equations) (8)

Ans. Let $f(x)=x^4+x^3-7x^2-x+5 \rightarrow (1)$

Let a = 2 and b = 2.1

$\therefore\ f(a)=f(2)=(2)^4+(2)^3-7(2)^2-(2)+5=-1 < 0$ and

$f(b)=f(2.1)=(2.1)^4+(2.1)^3-7(2.1)^2-(2.1)+5=0.7391 > 0$

$\therefore$ Root of f(x) lies between 2 and 2.1

By Regula Falsi Method $x=\dfrac{a\,f(b)-b\,f(a)}{f(b)-f(a)} \rightarrow (2)$

Method I:

Iteration	a	b	f(a)	f(b)	x	f(x)
1)	2	2.1	–1	0.7391	2.0575	–0.0597
2)	2.0575	2.1	–0.0597	0.7391	2.0607	–0.0027
3)	2.0607	2.1	–0.0027	0.7391	2.0608	

Method II:

Iteration I:

Let a = 2, b = 2.1, f(a) = –1 and f(b) = 0.7391

$\therefore$ From (2), $x_1=\dfrac{2(0.7391)-2.1(-1)}{(0.7391)-(-1)}=2.0575$

$\therefore$ From (1), $f(x_1)=f(2.0575)=(2.0575)^4+(2.0575)^3-7(2.0575)^2-(2.0575)+5=-0.0597 < 0$

Iteration II:

Let a = 2.0575, b = 2.1, f(a) = –0.0597 and f(b) = 0.7391

From (2), $x_2=\dfrac{2.0575(0.7391)-2.1(-0.0597)}{(0.7391)-(-0.0597)}=2.0607$

$\therefore$ From (1), $f(x_2)=f(2.0607)=(2.0607)^4+(2.0607)^3-7(2.0607)^2-(2.0607)+5=-0.0027 < 0$

Iteration III:

Let a = 2.0607, b = 2.1, f(a) = –0.0027 and f(b) = 0.7391

From (2), $x_3=\dfrac{2.0607(0.7391)-2.1(-0.0027)}{(0.7391)-(-0.0027)}=2.0608$

Hence, by Regula Falsi Method, Root of the equation $x^4+x^3-7x^2-x+5=0$ is $2.0607 \approx 2.061$

Q.5)

5a) If $y=\left(x+\sqrt{x^2-1}\right)^m$, Prove that $\left(x^2-1\right)y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-m^2\right)y_n=0$.

(Chp: Successive Differentiation) (6)

Ans. $y=\left(x+\sqrt{x^2-1}\right)^m \quad \to (1)$

Differentiate w.r.t. x, $y_1=m\left(x+\sqrt{x^2-1}\right)^{m-1}\cdot\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right)$

$=m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{x}{\sqrt{x^2-1}}\right)$

$=m\left(x+\sqrt{x^2-1}\right)^{m-1}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}$

$\therefore\ y_1=\frac{m}{\sqrt{x^2-1}}\cdot\left(x+\sqrt{x^2-1}\right)^{m-1+1}$

$\therefore\ y_1\sqrt{x^2-1}=my$ (From 1)

$\therefore$ On squaring, $y_1{}^2\left(x^2-1\right)=m^2y^2$

Again, differentiating w.r.t. x, $y_1{}^2\cdot 2x+\left(x^2-1\right)\cdot 2y_1y_2=m^2\cdot 2yy_1$

$\therefore$ Dividing by $2y_1$, $y_1x+\left(x^2-1\right)y_2=m^2y$

$\therefore\ \left(x^2-1\right)y_2+xy_1-m^2y=0$

Applying Leibnitz theorem, $\left[\left(x^2-1\right)y_{n+2}+n\cdot(2x)y_{n+1}+\frac{n(n-1)}{\cancel{2}!}\cdot(\cancel{2})y_n\right]+\left[x\cdot y_{n+1}+n\cdot 1\cdot y_n\right]-m^2y_n=0$

$\therefore\ \left(x^2-1\right)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n-m^2y_n=0$

$\therefore\ \left(x^2-1\right)y_{n+2}+xy_{n+1}(2n+1)+\left(n^2-\cancel{n}+\cancel{n}-m^2\right)y_n=0$

$\therefore\ \boxed{\left(x^2-1\right)y_{n+2}+(2n+1)xy_{n+1}+\left(n^2-m^2\right)y_n=0}$

5b) Using the encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, encode and decode the message I*LOVE*MUMBAI*. (Chp: Coding) (6)

Ans. We use following numerical values of each alphabet for coding

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	*
15	16	17	18	19	20	21	22	23	24	25	26	27

Step 1:
Message: I*LOVE*MUMBAI*
As per the above table, the numerical values of each alphabet in the message are

I	*	L	O	V	E	*	M	U	M	B	A	I	*
9	27	12	15	22	5	27	13	21	13	2	1	9	27

Step 2:

Writing the above numerical values column-wise in a 2 row matrix we get, $A = \begin{bmatrix} 9 & 12 & 22 & 27 & 21 & 2 & 9 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$

Encoding matrix $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \to (1)$

Now, $EA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 9 & 12 & 22 & 27 & 21 & 2 & 9 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$

$\therefore\ EA = \begin{bmatrix} 36 & 27 & 27 & 40 & 34 & 3 & 36 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix} \to (2)$

Writing the numbers in EA matrix column wise gives the encoded message.
$\therefore$ Encoded Message = 36 27 27 15 27 5 40 13 34 13 3 1 36 27
This encoded message is transmitted.

Step 3:
Assume there is no corruption of data, the message at the receiving end is
36 27 27 15 27 5 40 13 34 13 3 1 36 27
This message is decoded

We know, if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $P^{-1} = \frac{1}{|P|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

From (1), | E | = 1 – 0 = 1 → (3)

$\therefore\ E^{-1} = \frac{1}{|E|}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ Decoding matrix $E^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (From 3) → (4)

From (2) & (4), $E^{-1}(EA) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 36 & 27 & 27 & 40 & 34 & 3 & 36 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$

$\therefore\ A = \begin{bmatrix} 9 & 12 & 22 & 27 & 21 & 2 & 9 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$

Step 4:
Considering the numbers column-wise we get, 9 27 12 15 22 5 27 13 21 13 2 1 9 27

Reconverting each of the above numbers into corresponding alphabet,
Decoded Message = I*LOVE*MUMBAI*

5c) Considering only principal values separate into real and imaginary parts $i^{\log(1+i)}$.

(Chp: Log of Complex Numbers) (4)

Ans. We know, $\log(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{y}{x}\right) \rightarrow (1)$

We consider only principal values of the roots.

Let $a+ib=i^{\log(1+i)}$

$\therefore\ \log(a+ib)=\log i^{\log(1+i)}$

$=\log(1+i)\cdot\log i$

$=\log(1+1i)\cdot\log(0+1i)$

$=\left[\frac{1}{2}\log(1^2+1^2)+i\tan^{-1}\left(\frac{1}{1}\right)\right]\times\left[\frac{1}{2}\log(0^2+1^2)+i\tan^{-1}\left(\frac{1}{0}\right)\right]$ (From 1)

$=\left[\frac{1}{2}\log 2+i\frac{\pi}{4}\right]\cdot\left[\frac{1}{2}\log 1+i\frac{\pi}{2}\right]$

$=\left[\frac{1}{2}\log 2+i\frac{\pi}{4}\right]\cdot\left[0+i\frac{\pi}{2}\right]$

$=i\frac{\pi}{4}\log 2+i^2\frac{\pi^2}{8}$

$\therefore\ \log(a+ib)=\frac{-\pi^2}{8}+i\cdot\frac{\pi}{4}\log 2$

$\therefore\ a+ib=e^{\frac{-\pi^2}{8}+i\cdot\frac{\pi}{4}\log 2}$

$\therefore\ a+ib=e^{-\pi^2/8}\,e^{i(\pi\log 2)/4}$

$\therefore\ a+ib=e^{-\pi^2/8}\left[\cos\left(\frac{\pi\log 2}{4}\right)+i\sin\left(\frac{\pi\log 2}{4}\right)\right]$

$\therefore\ a+ib=e^{-\pi^2/8}\cos\left(\frac{\pi\log 2}{4}\right)+i\,e^{-\pi^2/8}\sin\left(\frac{\pi\log 2}{4}\right)$

Comparing Real and Imaginary Parts, Real Part $=e^{-\pi^2/8}\cos\left(\frac{\pi\log 2}{4}\right)$ and Imaginary Part $=e^{-\pi^2/8}\sin\left(\frac{\pi\log 2}{4}\right)$

5d) Show that: $i\log\left(\dfrac{x-i}{x+i}\right)=\pi-2\tan^{-1}x$. (Chp: Log of Complex Numbers) (4)

Ans. We know, $\log(x+iy)=\dfrac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\left(\dfrac{y}{x}\right) \rightarrow (1)$

$$\text{LHS}=i\log\left(\frac{x-i}{x+i}\right)$$

$$=i\left[\log(x-i)-\log(x+i)\right]$$

$$=i\left\{\left[\cancel{\frac{1}{2}\log\left(x^2+1^2\right)}+i\tan^{-1}\left(\frac{-1}{x}\right)\right]-\left[\cancel{\frac{1}{2}\log\left(x^2+1^2\right)}+i\tan^{-1}\left(\frac{1}{x}\right)\right]\right\} \text{ (From 1)}$$

$$=i\left\{-i\tan^{-1}\frac{1}{x}-i\tan^{-1}\frac{1}{x}\right\}$$

$$=i\left\{-2i\tan^{-1}\frac{1}{x}\right\}$$

$$=-2i^2\cot^{-1}x \quad \left\{\because \cot^{-1}x=\tan^{-1}\frac{1}{x}\right\}$$

$$=2\left[\frac{\pi}{2}-\tan^{-1}x\right] \quad \left\{\because \cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x\right\}$$

$$=\pi-2\tan^{-1}x$$

= RHS

Hence, $i\log\left(\dfrac{x-i}{x+i}\right)=\pi-2\tan^{-1}x$

Q.6)

6a) Using De Moivre's theorem prove that $\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)$. (Chp: Complex - DMT) (6)

Ans. Let $x = \cos\theta + i\sin\theta$

$\therefore\ x^n = (\cos\theta + i\sin\theta)^n$

$\therefore\ x^n = \cos n\theta + i\sin n\theta \rightarrow (1)$ (De Moivre's theorem)

Put n = −1 in (1), $x^{-1} = \cos(-\theta) + i\sin(-\theta)$

$\therefore\ \frac{1}{x} = \cos\theta - i\sin\theta$

Replacing n by −n in (1), $x^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$\therefore\ \frac{1}{x^n} = \cos n\theta - i\sin n\theta \rightarrow (2)$

Adding (1) and (2), $x^n + \frac{1}{x^n} = (\cos n\theta + \cancel{i\sin n\theta}) + (\cos n\theta - \cancel{i\sin n\theta})$

$\therefore\ \cos n\theta = \frac{1}{2}\left(x^n + \frac{1}{x^n}\right) \rightarrow (3)$

For n = 1, $\cos\theta = \frac{1}{2}\left(x + \frac{1}{x}\right) \rightarrow (4)$

Subtracting (2) from (1), $x^n - \frac{1}{x^n} = (\cancel{\cos n\theta} + i\sin n\theta) - (\cancel{\cos n\theta} - i\sin n\theta)$

$\therefore\ \sin n\theta = \frac{1}{2i}\left(x^n - \frac{1}{x^n}\right) \rightarrow (5)$

For n = 1, $\sin\theta = \frac{1}{2i}\left(x - \frac{1}{x}\right) \rightarrow (6)$

LHS $= \cos^6\theta - \sin^6\theta$

$= \left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^6 - \left[\frac{1}{2i}\left(x - \frac{1}{x}\right)\right]^6$ (From 4 & 6)

$= \frac{1}{2^6}\left[x^6 + 6x^5\cdot\frac{1}{x} + 15x^4\cdot\frac{1}{x^2} + 20x^3\cdot\frac{1}{x^3} + 15x^2\cdot\frac{1}{x^4} + 6x\cdot\frac{1}{x^5} + \frac{1}{x^6}\right] -$

$\frac{1}{2^6 i^6}\left[x^6 - 6x^5\cdot\frac{1}{x} + 15x^4\cdot\frac{1}{x^2} - 20x^3\cdot\frac{1}{x^3} + 15x^2\cdot\frac{1}{x^4} - 6x\cdot\frac{1}{x^5} + \frac{1}{x^6}\right]$

$= \frac{1}{64}\left[x^6 + \cancel{6x^4} + 15x^2 + \cancel{20} + \frac{15}{x^2} + \frac{\cancel{6}}{\cancel{x^4}} + \frac{1}{x^6}\right] + \frac{1}{64}\left[x^6 - \cancel{6x^4} + 15x^2 - \cancel{20} + \frac{15}{x^2} - \frac{\cancel{6}}{\cancel{x^4}} + \frac{1}{x^6}\right]$

$= \frac{1}{64}\times 2\left[x^6 + \frac{1}{x^6} + 15\left(x^2 + \frac{1}{x^2}\right)\right]$

$= \frac{1}{32}[2\cos 6\theta + 15\times 2\cos 2\theta]$ (From 3 & 5)

$= \frac{1}{16}[\cos 6\theta + 15\cos 2\theta]$ = RHS

6b) If $u=\sin^{-1}\left(\dfrac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\tan u}{144}\left(\tan^2 u+13\right)$.

(Chp: Homogenous Functions) (6)

Ans. $u(x,y)=\sin^{-1}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$

$$\therefore\ \sin u(x,y)=\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}\rightarrow(1)$$

$$\therefore\ \sin u(X,Y)=\left(\frac{X^{1/3}+Y^{1/3}}{X^{1/2}-Y^{1/2}}\right)^{1/2}$$

Now, Put X = x t, Y = y t

$$\therefore\ \sin u(X,Y)=\left[\frac{(xt)^{1/3}+(yt)^{1/3}}{(xt)^{1/2}-(yt)^{1/2}}\right]^{1/2}$$

$$=\left[\frac{t^{1/3}\left(x^{1/3}+y^{1/3}\right)}{t^{1/2}\left(x^{1/2}-y^{1/2}\right)}\right]^{1/2}$$

$$=\left[t^{\frac{1}{3}-\frac{1}{2}}\right]^{1/2}\left[\frac{\left(x^{1/3}+y^{1/3}\right)}{\left(x^{1/2}-y^{1/2}\right)}\right]^{1/2}$$

$$=\left[t^{-1/6}\right]^{1/2}\left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right]^{1/2}$$

$\sin u(X,Y)=t^{-1/12}\sin u(x,y)$ (From 1)

$\therefore$ sin u is homogenous function of degree $(n)=\dfrac{-1}{12}$.

Let $f(u)=\sin u$

$\therefore\ f'(u)=\cos u$

Let $g(u)=n\dfrac{f(u)}{f'(u)}$

$$=\frac{-1}{12}\cdot\frac{\sin u}{\cos u}$$

$$=\frac{-\tan u}{12}$$

$$\therefore\ g'(u)=\frac{-1}{12}\sec^2 u$$

Using formula,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\cdot\left[g'(u)-1\right]$$

$$\therefore\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{-\tan u}{12}\cdot\left[\frac{-1}{12}\sec^2 u-1\right]$$

$$=\frac{\tan u}{12}\cdot\left[\frac{1}{12}\left(1+\tan^2 u\right)+1\right]$$

$$=\frac{\tan u}{12}\cdot\frac{1}{12}\left[1+\tan^2 u+12\right]$$

$$\therefore\ \boxed{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}\left(\tan^2 u+13\right)}$$

6c) Discuss the maxima and minima of $f(x,y)=x^3y^2(1-x-y)$. (Chp: Maxima and Minima) (8)

Ans. Let $f(x,y)=x^3y^2(1-x-y)$

$f(x,y)=x^3y^2-x^4y^2-x^3y^3 \rightarrow (1)$

$\therefore f_x=3x^2y^2-4x^3y^2-3x^2y^3$

$\therefore r=f_{xx}=6xy^2-12x^2y^2-6xy^3 \rightarrow (2)$

$\therefore f_y=2x^3y-2x^4y-3x^3y^2$

$\therefore t=f_{yy}=2x^3-2x^4-6x^3y \rightarrow (3)$

$\therefore s=f_{xy}=6x^2y-8x^3y-9x^2y^2 \rightarrow (4)$

Put $f_x=0$ and $f_y=0$

$\therefore 3x^2y^2-4x^3y^2-3x^2y^3=0$

$\therefore x^2y^2(3-4x-3y)=0$

$\therefore$ x = 0 or y = 0 or 3 – 4 x – 3 y = 0

$\therefore$ x = 0 or y = 0 or 4 x + 3 y = 3 $\rightarrow (5)$

And, $2x^3y-2x^4y-3x^3y^2=0$

$\therefore x^3y(2-2x-3y)=0$

$\therefore$ x = 0 or y = 0 or 2 – 2 x – 3 y = 0

$\therefore$ x = 0 or y = 0 or 2 x + 3 y = 2 $\rightarrow (6)$

Put x = 0 in (5), 0 + 3 y = 3 $\therefore$ y = 1

Put x = 0 in (6), 0 + 3 y = 2 $\therefore$ y = 2 / 3

Put y = 0 in (5), 4 x + 0 = 3 $\therefore$ x = 0.75

Put y = 0 in (6), 2 x + 0 = 2 $\therefore$ x = 1

Solving (5) & (6) simultaneously, we get,
x = 1 / 2 and y = 1 / 3

$\therefore$ Stationary Points are

(0, 1); $\left(0,\frac{2}{3}\right)$; (0.75, 0); (1, 0); $\left(\frac{1}{2},\frac{1}{3}\right)$

(i) At (0, 1) and $\left(0,\frac{2}{3}\right)$;

From (2), r = 0 – 0 – 0 = 0
From (3), t = 0 – 0 – 0 = 0
From (4), s = 0 – 0 – 0 = 0

$\therefore rt-s^2=(0)(0)-(0)^2=0$

$\therefore$ Maxima or minima cannot be found.

(ii) At (0.75, 0);
From (2), r = 0 – 0 – 0 = 0

From (3), $t=2(0.75)^3-2(0.75)^4-0=\frac{27}{128}$

From (4), s = 0 – 0 – 0 = 0

$\therefore rt-s^2=(0)\left(\frac{27}{128}\right)-(0)^2=0$

$\therefore$ Maxima or minima cannot be found at (0.75, 0).

(iii) At (1, 0)
From (2), r = 0 – 0 – 0 = 0

From (3), $t=2(1)^3-2(1)^4-0=0$

From (4), s = 0 – 0 – 0 = 0

$\therefore rt-s^2=(0)(0)-(0)^2=0$

$\therefore$ Maxima or minima cannot be found at (0.75, 0).

(iv) At $\left(\frac{1}{2},\frac{1}{3}\right)$

From (2),

$$r=6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2-12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2-6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3=\frac{-1}{9}<0$$

From (3), $t=2\left(\frac{1}{2}\right)^3-2\left(\frac{1}{2}\right)^4-6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)=\frac{-1}{8}<0$

From (4),

$$s=6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)-8\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)-9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2=\frac{-1}{12}<0$$

$\therefore rt-s^2=\left(\frac{-1}{9}\right)\left(\frac{-1}{8}\right)-\left(\frac{-1}{12}\right)^2=\frac{1}{144}>0$

$\therefore f$ has maximum at $\left(\frac{1}{2},\frac{1}{3}\right)$

$\therefore$ From (1), $f(x,y)=\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{432}$

$\therefore$ Maximum value $f(x,y)=\frac{1}{432}$