Q.1)

1a) If $u = \log\left(\frac{x}{y}\right) + \log\left(\frac{y}{x}\right)$, find $\frac{\partial u / \partial x}{\partial u / \partial y}$. (Chp: Partial Differentiation) (3)

Ans. $u = \log\left(\frac{x}{y}\right) + \log\left(\frac{y}{x}\right)$

$\therefore\ u = \log x - \log y + \log y - \log x$

$\therefore\ u = 0$

Partially Differentiating w.r.t. 'x', $\frac{\partial u}{\partial x} = 0$

Partially Differentiating 'u' w.r.t. 'y', $\frac{\partial u}{\partial y} = 0$

$\therefore\ \frac{\partial u / \partial x}{\partial u / \partial y} = \frac{0}{0}$, which is an indeterminate form.

1b) Find the value of tanh (log x) if $x=\sqrt{3}$. (Chp: Hyperbolic Functions) (3)

Ans. We know, $\tanh a = \dfrac{\sinh a}{\cosh a} = \dfrac{\left(e^{a}-e^{-a}\right)/\cancel{2}}{\left(e^{a}+e^{-a}\right)/\cancel{2}} \rightarrow (1)$

Put a = log x

i.e. Put $e^{a}=x$

And, $e^{-a}=\dfrac{1}{e^{a}}=\dfrac{1}{x}$

From (1), $\tanh(\log x)=\dfrac{x-\dfrac{1}{x}}{x+\dfrac{1}{x}}$

$\therefore\ \tanh(\log x)=\dfrac{\left(x^{2}-1\right)/\cancel{x}}{\left(x^{2}+1\right)/\cancel{x}}$

Put, $x=\sqrt{3}$

$\therefore\ \tanh(\log x)=\dfrac{\sqrt{3}^{2}-1}{\sqrt{3}^{2}+1}$

$\therefore\ \tanh(\log x)=\dfrac{3-1}{3+1}$

$\therefore\ \boxed{\tanh(\log x)=\dfrac{1}{2}}$

1c) Evaluate $\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{1}{\log(x-2)}\right]$. (Chp: Indeterminate Forms) (3)

Ans. Let $L=\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{1}{\log(x-2)}\right]$

$$=\lim_{x\to 3}\left[\frac{\log(x-2)-(x-3)}{(x-3)\log(x-2)}\right]\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 3}\left[\frac{\frac{1}{x-2}\cdot 1-(1-0)}{(x-3)\cdot\frac{1}{x-2}\cdot 1+\log(x-2)\cdot(1-0)}\right]\text{ (L'Hospital's Rule)}$$

$$=\lim_{x\to 3}\left[\frac{\frac{1}{x-2}-1}{\frac{x-3}{x-2}+\log(x-2)}\right]$$

$$=\lim_{x\to 3}\left[\frac{\frac{1-(x-2)}{\cancel{(x-2)}}}{\frac{x-3+(x-2)\log(x-2)}{\cancel{(x-2)}}}\right]$$

$$=\lim_{x\to 3}\left[\frac{3-x}{x-3+(x-2)\log(x-2)}\right]\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 3}\left[\frac{0-1}{1-0+\cancel{(x-2)}\cdot\frac{1}{\cancel{(x-2)}}\cdot 1+\log(x-2)\cdot 1}\right]\text{ (L'Hospital's Rule)}$$

$$=\lim_{x\to 3}\left[\frac{-1}{1+1+\log(x-2)}\right]$$

$$=\frac{-1}{2+\log(3-2)}$$

$$=\frac{-1}{2+0}$$

$$\therefore\ \lim_{x\to 3}\left[\frac{1}{x-3}-\frac{1}{\log(x-2)}\right]=\frac{-1}{2}$$

1d) If $u = r^2 \cos 2\theta$; $v = r^2 \sin 2\theta$ find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$. (Chp: Jacobian) (3)

Ans. Given, $u = r^2 \cos 2\theta$ and $v = r^2 \sin 2\theta \rightarrow (1)$

Partially differentiating w.r.t. 'r', we get

$$\therefore\ u_r = \frac{\partial u}{\partial r} = 2r\cos 2\theta \text{ and } v_r = \frac{\partial v}{\partial r} = 2r\sin 2\theta \rightarrow (2)$$

Partially differentiating (1) w.r.t. 'θ', we get

$$\therefore\ u_\theta = \frac{\partial u}{\partial \theta} = r^2 \cdot -\sin 2\theta \cdot 2 \text{ and } v_\theta = \frac{\partial v}{\partial \theta} = r^2 \cdot \cos 2\theta \cdot 2 \rightarrow (3)$$

We know $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix}$

$$= u_r v_\theta - v_r u_\theta$$

$$= (2r\cos 2\theta)(2r^2\cos 2\theta) - (2r\sin 2\theta)(-2r^2\sin 2\theta) \text{ (From 2 \& 3)}$$

$$= 4r^3\cos^2 2\theta + 4r^3\sin^2 2\theta$$

$$= 4r^3(\cos^2 2\theta + \sin^2 2\theta)$$

$$= 4r^3 \times 1$$

$$= 4r^3$$

$$\therefore\ \boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

1e) Express the matrix $A=\begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix}$ as the Sum of a Hermitian and a Skew-Hermitian matrix.

(Chp: Rank of Matrix) (4)

Ans. $A=\begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix}$

$\therefore\ A'=\begin{bmatrix} 2+3i & -2i & 4 \\ 2 & 0 & 2+5i \\ 3i & 1+2i & -i \end{bmatrix}$

$A^{\theta}=\overline{A'}=\begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}$

Let $P=\frac{1}{2}\left(A+A^{\theta}\right)$

$=\frac{1}{2}\begin{bmatrix} 4 & 2+2i & 4+3i \\ 2-2i & 0 & 3-3i \\ 4-3i & 3+3i & 0 \end{bmatrix}$

$=\begin{bmatrix} 2 & 1+i & 2+1.5i \\ 1-i & 0 & 1.5-1.5i \\ 2-1.5i & 1.5+1.5i & 0 \end{bmatrix}$,

We observe, $p_{ij}=\overline{p_{ji}}$

$\therefore$ P is Hermitian.

Let $Q=\frac{1}{2}\left(A-A^{\theta}\right)$

$=\frac{1}{2}\begin{bmatrix} 6i & 2-2i & -4+3i \\ -2-2i & 0 & -1+7i \\ 4+3i & 1+7i & -2i \end{bmatrix}$

$=\begin{bmatrix} 3i & 1-i & -2+1.5i \\ -1-i & 0 & -0.5+3.5i \\ 2+1.5i & 0.5+3.5i & -i \end{bmatrix}$,

We observe, $q_{ij}=-\overline{q_{ji}}$

$\therefore$ Q is skew-Hermitian.

Now,

$$P+Q=\begin{bmatrix} 2 & 1+i & 2+1.5i \\ 1-i & 0 & 1.5-1.5i \\ 2-1.5i & 1.5+1.5i & 0 \end{bmatrix}+\begin{bmatrix} 3i & 1-i & -2+1.5i \\ -1-i & 0 & -0.5+3.5i \\ 2+1.5i & 0.5+3.5i & -i \end{bmatrix}$$

$\therefore\ P+Q=\begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix}$

$\therefore$ P + Q = A

Hence, A is expressed as a sum of a Hermitian matrix and a Skew-Hermitian matrix.

1f) Expand $\tan^{-1} x$ in powers of $\left(x-\frac{\pi}{4}\right)$. (Chp: Expansion) (4)

Ans. Let $f(x)=\tan^{-1} x$ and $a=\frac{\pi}{4}$

$\therefore\ f'(x)=\frac{1}{1+x^2}$

$=\left(1+x^2\right)^{-1}$

$\therefore\ f''(x)=-1\cdot\left(1+x^2\right)^{-2}\cdot 2x$

$=-2x\left(1+x^2\right)^{-2}$

$\therefore\ f'''(x)=-2\left[x\cdot-2\left(1+x^2\right)^{-3}\cdot 2x+\left(1+x^2\right)^{-2}\cdot 1\right]$

$=-2\left[\frac{-4x^2}{(1+x^2)^3}+\frac{1}{(1+x^2)^2}\right]$

$=-2\left[\frac{-4x^2+1+x^2}{(1+x^2)^3}\right]$

$=\frac{-2(-3x^2+1)}{(1+x^2)^3}$

$=\frac{2(3x^2-1)}{(1+x^2)^3}$

At $a=\frac{\pi}{4}$,

$f(a)=\tan^{-1}\frac{\pi}{4}$

$\therefore\ f'(a)=\frac{1}{1+(\pi/4)^2}=\frac{16}{16+\pi^2}$

$\therefore\ f''(a)=\frac{-2(\pi/4)}{\left[1+(\pi/4)^2\right]^2}$

$=\frac{-\pi}{2}\div\left[\frac{16+\pi^2}{16}\right]^2$

$=\frac{-\pi}{2}\times\frac{16^2}{(16+\pi^2)^2}$

$=\frac{-128\pi}{(16+\pi^2)^2}$

$\therefore\ f'''(a)=2\left[3\left(\frac{\pi}{4}\right)^2-1\right]\div\left[1+\left(\frac{\pi}{4}\right)^2\right]^3$

$=2\left[\frac{3\pi^2}{4^2}-1\right]\div\left[1+\frac{\pi^2}{4^2}\right]^3$

$=2\left[\frac{3\pi^2-16}{16}\right]\div\left[\frac{16+\pi^2}{16}\right]^3$

$=2\times\frac{3\pi^2-16}{16}\times\frac{16^3}{(16+\pi^2)^3}$

$=\frac{512(3\pi^2-16)}{(16+\pi^2)^3}$

$\therefore$ By Taylor Series, $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots$

$$\therefore\ \tan^{-1}x=\tan^{-1}\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\times\left(\frac{16}{16+\pi^2}\right)+\frac{1}{2}\left(x-\frac{\pi}{4}\right)^2\times\frac{-128\pi}{(16+\pi^2)^2}+\frac{1}{6}\left(x-\frac{\pi}{4}\right)^3\times\frac{512(3\pi^2-16)}{(16+\pi^2)^3}+\ldots$$

$$\therefore\ \tan^{-1}x=\tan^{-1}\frac{\pi}{4}+\frac{16}{16+\pi^2}\left(x-\frac{\pi}{4}\right)-\frac{64\pi}{(16+\pi^2)^2}\left(x-\frac{\pi}{4}\right)^2+\frac{256(3\pi^2-16)}{3(16+\pi^2)^3}\left(x-\frac{\pi}{4}\right)^3+\ldots$$

Q.2)

2a) Expand $\sin^7\theta$ in a series of sines of multiples of θ. (Chp: Complex - DMT) (6)

Ans. Let $e^{i\theta} = x$

$$\therefore\ e^{-i\theta} = \frac{1}{x}$$

$$\therefore\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(x - \frac{1}{x}\right) \rightarrow (1)$$

Similarly, $x^n + \dfrac{1}{x^n} = 2i\sin n\theta \rightarrow (2)$

Now, $\sin^7\theta = \dfrac{1}{2^7 i^7}\left(x - \dfrac{1}{x}\right)^7$ (From 1)

$$= \frac{1}{-128i}\left(1x^7 - 7x^6\cdot\frac{1}{x} + 21x^5\cdot\frac{1}{x^2} - 35x^4\cdot\frac{1}{x^3} + 35x^3\cdot\frac{1}{x^4} - 21x^2\cdot\frac{1}{x^5} + 7x\cdot\frac{1}{x^6} - \frac{1}{x^7}\right)$$

$$= \frac{1}{-128i}\left(x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}\right)$$

$$= \frac{1}{-128i}\left[\left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)\right]$$

$$= \frac{1}{-128i}\left[2i\sin 7\theta - 7\times 2i\sin 5\theta + 21\times 2i\sin 3\theta - 35\times 2i\sin\theta\right] \text{ (From 2)}$$

$$= \frac{1}{-128i}\times 2i\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$$

$$= \frac{1}{-64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$$

Hence, $\sin^7\theta = \dfrac{-1}{64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$

2b) If $y=\sin^2 x\cos^3 x$, find y_n. (Chp: Successive Differentiation) (6)

Ans. $\sin^2 x\cos^3 x=\left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)^2\times\left(\dfrac{e^{ix}+e^{-ix}}{2}\right)^3$

$$=\frac{\left(e^{ix}-e^{-ix}\right)^2\left(e^{ix}+e^{-ix}\right)^2\left(e^{ix}+e^{-ix}\right)}{2^2 i^2 2^3}$$

$$=\frac{-1}{2^5}\cdot\left[\left(e^{ix}\right)^2-\left(e^{-ix}\right)^2\right]^2\cdot\left(e^{ix}+e^{-ix}\right)$$

$$=\frac{-1}{2^5}\left(e^{2ix}-e^{-2ix}\right)^2\cdot\left(e^{ix}+e^{-ix}\right)$$

$$=\frac{-1}{32}\left[\left(e^{2ix}\right)^2-2\left(e^{2ix}\right)\left(e^{-2ix}\right)+\left(e^{-2ix}\right)^2\right]\cdot\left(e^{ix}+e^{-ix}\right)$$

$$=\frac{-1}{32}\left[e^{4ix}-2+e^{-4ix}\right]\cdot\left(e^{ix}+e^{-ix}\right)$$

$$=\frac{-1}{32}\left\{e^{5ix}-2e^{ix}+e^{-3ix}+e^{3ix}-2e^{-ix}+e^{-5ix}\right\}$$

$$=\frac{-1}{32}\left[\left(e^{5ix}+e^{-5ix}\right)+\left(e^{3ix}+e^{-3ix}\right)-2\left(e^{ix}+e^{-ix}\right)\right]$$

$$=\frac{-1}{32}\left[2\cos 5x+2\cos 3x-2\cdot 2\cos x\right]$$

Taking nth derivative and using, If $y=a\cos(ax+b)$ then $y_n=a^n\cos\left(ax+b+\dfrac{n\pi}{2}\right)$

$$\therefore\ y_n=\frac{-1}{32}\times 2\left[5^n\cos\left(5x+\frac{n\pi}{2}\right)+3^n\cos\left(3x+\frac{n\pi}{2}\right)+2\times 1^n\cos\left(1x+\frac{n\pi}{2}\right)\right]$$

$$\therefore\ y_n=\frac{-1}{16}\left[5^n\cos\left(5x+\frac{n\pi}{2}\right)+3^n\cos\left(3x+\frac{n\pi}{2}\right)+2\cos\left(x+\frac{n\pi}{2}\right)\right]$$

3b) Find the stationary values of x^3+y^3-3axy, a > 0. (6)

Ans. Let $f(x,y)=x^3+y^3-3axy$

$f_x = 3x^2+0-3ay$

$r=f_{xx}=6x \rightarrow (1)$

$f_y=3y^2-3ax$

$t=f_{yy}=6y \rightarrow (2)$

$s=f_{xy}=-3a \rightarrow (3)$

Put $f_x=0$ and $f_y=0$

$\therefore\ 3x^2-3ay=0$

$\therefore\ 3x^2=3ay$

$\therefore\ y=\dfrac{x^2}{a} \rightarrow (4)$

And, $3y^2-3ax=0$

$\therefore\ y^2=ax$

$\therefore\ \left(\dfrac{x^2}{a}\right)^2=ax$ (From 4)

$\therefore\ x^4=a^3x$

$\therefore\ x^4-a^3x=0$

$\therefore\ x(x^3-a^3)=0$

$\therefore\ x=0$ or $x^3-a^3=0$

$\therefore\ x=0$ or $x=a \rightarrow (5)$

From (4) and (5),

When x = 0, y = 0

When x = a, $y=\dfrac{a^2}{a}=a$

$\therefore$ Stationary points are (0, 0) & (a, a)

(i) At (0, 0)

From (1), r = 0

$\therefore$ Maximum or minimum cannot be determined.

(ii) At (a, a)

From (1), r = 6 a > 0

From (2), t = 6 a > 0

From (3), s = –3 a

$\therefore\ rt-s^2=6a\times 6a-3^2a^2=27a^2>0$

$\therefore$ f has minimum at (a, a)

Minimum value of

$f(x,y)=a^3+a^3-3a(a)(a)=-a^3$

Q.3)

3a) Compute the real root of $x\log_{10}x-1.2=0$ correct to three places of decimals using Newton-Raphson method.

(Chp: Transcendental equations) (6)

Ans. Let $f(x)=x\log_{10}x-1.2$

When x = 2, $f(2)=2\log_{10}2-1.2=-0.5979$

When x = 3, $f(3)=3\log_{10}3-1.2=0.2314$

$\therefore$ Root of f (x) lies between 2 and 3.

Let initial value $x_0=3$

Now, $f'(x)=\cancel{x}\cdot\dfrac{1}{\cancel{x}\log 10}+\log_{10}x\cdot 1-0$

$=\dfrac{1}{\log 10}+\dfrac{\log x}{\log 10}$

$=\dfrac{1+\log x}{\log 10}$

By Newton-Raphson's Method $x_{n+1}=x_n-\dfrac{f(x)}{f'(x)}$

$=x_n-\dfrac{x_n\log_{10}x_n-1.2}{(1+\log x_n)/\log 10}$

$=x_n-\dfrac{(x_n\log_{10}x_n-1.2)\log 10}{(1+\log x_n)}$

$=x_n-\dfrac{x_n\times\dfrac{\log x_n}{\cancel{\log 10}}\times\cancel{\log 10}-1.2\log 10}{(1+\log x_n)}$

$=\dfrac{x_n(1+\log x_n)-(x_n\log x_n-1.2\log 10)}{(1+\log x_n)}$

$=\dfrac{x_n+\cancel{x_n\log x_n}-\cancel{x_n\log x_n}+1.2\log 10}{(1+\log x_n)}$

$\therefore\ x_{n+1}=\dfrac{x_n+1.2\log 10}{1+\log x_n}\rightarrow(1)$

Iteration 1: Put n = 0 and $x_0=3$ in (1)

$\therefore\ x_1=\dfrac{x_0+1.2\log 10}{1+\log x_0}=\dfrac{3+1.2\log 10}{1+\log 3}=2.7461$

Iteration 2: Put n = 1 and $x_1=2.7461$ in (1)

$\therefore\ x_2=\dfrac{x_1+1.2\log 10}{1+\log x_1}=\dfrac{2.7461+1.2\log 10}{1+\log 2.7461}=2.7406$

Iteration 3: Put n = 2 and $x_2=2.7406$ in (1)

$\therefore\ x_3=\dfrac{x_2+1.2\log 10}{1+\log x_2}=\dfrac{2.7406+1.2\log 10}{1+\log 2.7406}=2.7406$

Hence, Root of $x\log_{10}x-1.2=0$ is 2.7406

3b) Show that the system of equations: 2x – 2y + z = λx, 2x – 3y + 2z = λy; –x + 2y = λz can possess a non-trivial solution only if λ = 1, λ = –3. Obtain the general solution in each case. (Chp: Linear Equations) (6)

Ans. $2x - 2y + z = \lambda x$

$\therefore\ 2x - \lambda x - 2y + z = 0$

$\therefore\ (2 - \lambda)\, x - 2y + z = 0 \rightarrow (1)$

Similarly solving, we get

$2x - (3 + \lambda)\, y + 2z = 0; \rightarrow (2)$ and

$-x + 2y - \lambda z = 0$

Writing the equations in the matrix form,

$$\begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (3)$$

which is of the type AX = 0.

Part I:

For non - trivial solution, | A | = 0

$$\therefore \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

On expanding,

$$(2-\lambda)\left[-\lambda(-3-\lambda)-4\right]+2\left[-2\lambda+2\right]+1\left[4+1(-3-\lambda)\right]=0$$

$$\therefore\ (2-\lambda)\left(3\lambda+\lambda^2-4\right)-4\lambda+4+\left(4-3-\lambda\right)=0$$

$$\therefore 6\lambda+2\lambda^2-8-3\lambda^2-\lambda^3+\cancel{4\lambda}-\cancel{4\lambda}+4+1-\lambda=0$$

$$\therefore\ -\lambda^3-\lambda^2+5\lambda-3=0$$

$$\therefore\ 0=\lambda^3+\lambda^2-5\lambda+3$$

On solving, λ = 1, 1, –3

Case I: Put λ = 1 in (3)

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 + R_1;\ R_2 - 2R_1 \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On expansion, $1x - 2y + 1z = 0 \rightarrow (4)$

Number of unknowns = n = 3

rank of A (r) = number of non-zero rows = 1

n – r = 3 – 1 = 2

We have to assume two unknowns.

Let x = t (≠ 0); y = s (≠ 0)

From (4), 1t – 2s + 1z = 0

$\therefore\ z = 2s - t$

Hence, the solution is x = t; y = s; z = 2s – t (Infinite Solutions)

Case II: When λ = –3

From (1), 5x – 2y + 1z = 0 and,

From (2), 2x – 0y + 2z = 0

By Crammer's Rule, $\dfrac{x}{\begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} 5 & -2 \\ 2 & 0 \end{vmatrix}}$

$$\therefore\ \frac{x}{-4} = \frac{-y}{8} = \frac{z}{4}$$

$$\therefore\ \frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = t\,(\neq 0)\ \text{(let)}$$

$\therefore$ x = t; y = 2t; z = –t (Infinite Solutions)

Part II:

For trivial solution, | A | ≠ 0

$\therefore$ k ≠ 1, –3

Hence, the solution is x = y = z = 0

3c) If $\tan(\alpha + i\beta) = \cos\theta + i\sin\theta$, prove that $\alpha = \frac{n\pi}{2} + \frac{\pi}{4}$ and $\beta = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$. (Chp: Hyperbolic Functions)

OR Separate into real and imaginary parts $\tan^{-1}\left(e^{i\theta}\right)$.

OR Separate into real and imaginary parts $\tan^{-1}(\cos\theta + i\sin\theta)$. (8)

Ans. Let $\alpha + i\beta = \tan^{-1}\left(e^{i\theta}\right)$ **OR**

$$\alpha + i\beta = \tan^{-1}(\cos\theta + i\sin\theta)$$

$$\tan(\alpha + i\beta) = \cos\theta + i\sin\theta = e^{i\theta} \quad \rightarrow (1)$$

On taking conjugates, $\tan(\alpha - i\beta) = e^{-i\theta} \quad \rightarrow (2)$

Now,

$$\tan(2\alpha) = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$$

$$= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta)\tan(\alpha - i\beta)}$$

$$= \frac{e^{i\theta} + e^{-i\theta}}{1 - e^{i\theta}e^{-i\theta}} \text{ (From 1 \& 2)}$$

$$= \frac{2\cos\theta}{1-1} \quad \left\{\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right\}$$

$$\therefore \tan(2\alpha) = \infty$$

$$\therefore 2\alpha = n\pi + \frac{\pi}{2}$$

$$\therefore \boxed{\alpha = \frac{n\pi}{2} + \frac{\pi}{4}}$$

Similarly,

$$\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$$

$$= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta)\tan(\alpha - i\beta)}$$

$$\therefore \tan(2i\beta) = \frac{e^{i\theta} - e^{-i\theta}}{1 + e^{i\theta}e^{-i\theta}} \text{ (From 1 \& 2)}$$

$$\therefore \cancel{i}\tanh(2\beta) = \frac{\cancel{2}\cancel{i}\sin\theta}{\cancel{1+1}} \quad \left\{\because \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right\}$$

$$\therefore \tanh(2\beta) = \sin\theta$$

$$\therefore \beta = \frac{1}{2}\tanh^{-1}(\sin\theta)$$

$$= \frac{1}{2} \times \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) \quad \left\{\because \tanh^{-1}x = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right\}$$

$$= \frac{1}{2}\log\left\{\frac{[\cos(\theta/2) + \sin(\theta/2)]^2}{[\cos(\theta/2) - \sin(\theta/2)]^2}\right\}^{1/2}$$

$$= \frac{1}{2}\log\left\{\frac{\cos(\theta/2) + \sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)}\right\}$$

$$= \frac{1}{2}\log\left\{\frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)}\right\} \text{ \{Dividing N \& D by cos(θ/2)\}}$$

$$= \frac{1}{2}\log\left\{\frac{\tan(\pi/4) + \tan(\theta/2)}{1 - \tan(\pi/4)\tan(\theta/2)}\right\}$$

$$\therefore \boxed{\beta = \frac{1}{2}\log\left[\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]}$$

Q.4)

4a) Using the encoding matrix as $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, encode and decode the message MOVE. (Chp: Coding) (6)

Ans. We use following numerical values of each alphabet for coding

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	*
15	16	17	18	19	20	21	22	23	24	25	26	27

Step 1:

Message: MOVE

As per the above table, the numerical values of each alphabet in the message are

M	O	V	E
13	15	22	5

Step 2:

Writing the above numerical values column-wise in a 2-row matrix we get, $A = \begin{bmatrix} 13 & 22 \\ 15 & 5 \end{bmatrix}$

Encoding matrix $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow (1)$

Now, $EA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 13 & 22 \\ 15 & 5 \end{bmatrix}$

$\therefore \; EA = \begin{bmatrix} 28 & 27 \\ 15 & 5 \end{bmatrix} \rightarrow (2)$

Writing the numbers in EA matrix column wise gives the encoded message.

$\therefore$ Encoded Message = 28 15 27 5

This encoded message is transmitted.

Step 3:

Assume there is no corruption of data, the message at the receiving end is 28 15 27 5

This message is decoded

We know, if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $P^{-1} = \frac{1}{|P|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

From (1), | E | = 1 – 0 = 1 → (3)

$\therefore \; E^{-1} = \frac{1}{|E|} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ Decoding matrix $E^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (From 3) → (4)

From (2) & (4), $E^{-1}(EA) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 28 & 27 \\ 15 & 5 \end{bmatrix}$

$\therefore \; A = \begin{bmatrix} 13 & 22 \\ 15 & 5 \end{bmatrix}$

Step 4:

Considering the numbers column-wise we get,

13 15 22 5

Reconverting each of the above numbers into corresponding alphabet,

Decoded Message = MOVE

4b) If $u = f\left(e^{x-y}, e^{y-z}, e^{z-x}\right)$, then prove that, $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$. (Chp: Partial Differentiation) (6)

Ans. Let $v = e^{y-z}$; $w = e^{z-x}$; $t = e^{x-y}$

$$\therefore \frac{\partial v}{\partial x}=0;\ \frac{\partial v}{\partial y}=e^{y-z}\cdot 1=v;\ \frac{\partial v}{\partial z}=e^{y-z}\cdot -1=-v$$

Similarly,

$$\frac{\partial w}{\partial x}=e^{z-x}\cdot -1=-w;\ \frac{\partial w}{\partial y}=0;\ \frac{\partial w}{\partial z}=e^{z-x}\cdot 1=w$$

And,

$$\frac{\partial t}{\partial x}=e^{x-y}\cdot 1=t;\ \frac{\partial t}{\partial y}=e^{x-y}\cdot -1=-t;\ \frac{\partial t}{\partial z}=0$$

Now, $u \to v, w, t \to x, y, z$

$$\therefore \frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial v}\cdot 0+\frac{\partial u}{\partial w}\cdot -w+\frac{\partial u}{\partial t}\cdot t$$

$$=-w\frac{\partial u}{\partial w}+t\cdot\frac{\partial u}{\partial t}$$

$$\therefore \frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$=\frac{\partial u}{\partial v}\cdot v+\frac{\partial u}{\partial w}\cdot 0+\frac{\partial u}{\partial t}\cdot -t$$

$$=v\frac{\partial u}{\partial v}-t\frac{\partial u}{\partial t}$$

$$\therefore \frac{\partial u}{\partial z}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$=\frac{\partial u}{\partial v}\cdot -v+\frac{\partial u}{\partial w}\cdot w+\frac{\partial u}{\partial t}\cdot 0$$

$$=-v\frac{\partial u}{\partial v}+w\frac{\partial u}{\partial w}$$

On adding, $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=$

$$=\left(-w\frac{\partial u}{\partial w}+t\cdot\frac{\partial u}{\partial t}\right)+\left(v\frac{\partial u}{\partial v}-t\frac{\partial u}{\partial t}\right)+\left(-v\frac{\partial u}{\partial v}+w\frac{\partial u}{\partial w}\right)$$

$$=0$$

$$\therefore \boxed{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0}$$

4c) If y = a cos (log x) + b sin (log x), then show that $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$.

(Chp: Successive Differentiation) (8)

Ans. y = a cos (log x) + b sin (log x) → (1)

Differentiating w.r.t. 'x', $y_1 = a \cdot -\sin(\log x) \cdot \frac{1}{x} + b \cdot \cos(\log x) \cdot \frac{1}{x}$

Multiplying by 'x', $x y_1 = -a \sin(\log x) + b \cos(\log x)$

Again, differentiating w.r.t. 'x', $x \cdot \frac{d}{dx} y_1 + y_1 \cdot \frac{d}{dx} x = -a \cdot \cos(\log x) \cdot \frac{d}{dx}(\log x) + b \cdot -\sin(\log x) \cdot \frac{d}{dx}(\log x)$

$\therefore \; x y_2 + y_1 \cdot 1 = -a \cdot \cos(\log x) \cdot \frac{1}{x} + b \cdot -\sin(\log x) \cdot \frac{1}{x}$

Multiplying by 'x', $x^2 y_2 + x y_1 = -[a \sin(\log x) + b \cos(\log x)]$

$\therefore \; x^2 y_2 + x y_1 = -y$ (From 1)

$\therefore \; x^2 y_2 + x y_1 + y = 0$

Taking nth derivative by applying Leibnitz Theorem,

$$\left[x^2 \cdot y_{n+2} + n \cdot 2x \cdot y_{n+1} + \frac{n(n-1)}{\cancel{2}} \cdot \cancel{2}(1) \cdot y_n\right] + \left[x\, y_{n+1} + n \cdot 1 \cdot y_n\right] + y_n = 0$$

$\therefore \; x^2\, y_{n+2} + 2nx\, y_{n+1} + (n^2 - n) y_n + x\, y_{n+1} + n y_n + y_n = 0$

$\therefore \; x^2\, y_{n+2} + (2nx + x) y_{n+1} + (n^2 - \cancel{n} + \cancel{n} + 1) y_n = 0$

$\therefore \; \boxed{x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0}$

Hence proved.

Q.2)

2a) Show that the roots of $x^5 = 1$ can be written as 1, α, α^2, α^3, α^4.

Hence show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$. (Chp: Complex - DMT) (6)

Ans. $x^5 = 1$

$\therefore\ x^5 = e^{i0} = e^{i(0+2n\pi)}$

$\therefore\ x = e^{i2n\pi/5}$

Put n = 0, $x_1 = e^{i0} = 1$

Put n = 1, $x_2 = e^{i2\pi/5} = \alpha$ (let)

Put n = 2, $x_3 = e^{i4\pi/5} = \left(e^{i2\pi/5}\right)^2 = \alpha^2$

Put n = 3, $x_4 = e^{i6\pi/5} = \left(e^{i2\pi/5}\right)^3 = \alpha^3$

Put n = 4, $x_5 = e^{i8\pi/5} = \left(e^{i2\pi/5}\right)^4 = \alpha^4$

Hence, the roots of $x^5 = 1$ can be written as 1, α, α^2, α^3, α^4.

$\because x_1, x_2, x_3, x_4$ and x_5 are roots of $x^5 - 1 = 0$,

$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5) = x^5 - 1$

$\therefore\ \cancel{(x-1)}(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \cancel{(x-1)}(x^4+x^3+x^2+x+1)$

Put x = 1,

$\therefore\ (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 1^4+1^3+1^2+1+1$

$\therefore\ (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$

5b) If $\theta = t^n e^{-r^2/4t}$, find n which will make at $\dfrac{\partial\theta}{\partial t} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)$. (Chp: Partial Differentiation) (6)

Ans. $\theta = t^n e^{-r^2/4t}$

Taking logarithm on both sides, we get,

$$\log\theta = \log\left[t^n \cdot e^{-r^2/4t}\right]$$

$$\therefore\ \log\theta = \log t^n + \log e^{-r^2/4t}$$

$$\therefore\ \log\theta = n\log t - \frac{r^2}{4t}\log e$$

$$\therefore\ \log\theta = n\log t - \frac{r^2}{4t} \rightarrow (1)$$

Partially Differentiating w.r.t. 't',

$$\frac{1}{\theta}\frac{\partial\theta}{\partial t} = n\cdot\frac{1}{t} - \frac{r^2}{4}\cdot\frac{-1}{t^2}$$

$$\therefore\ \frac{\partial\theta}{\partial t} = \theta\left(\frac{n}{t} + \frac{r^2}{4t^2}\right) \rightarrow (2)$$

Partially Differentiating (1) w.r.t. 'r',

$$\frac{1}{\theta}\frac{\partial\theta}{\partial r} = 0 - \frac{1}{4t}\cdot 2r$$

$$\therefore\ \frac{\partial\theta}{\partial r} = \frac{-r\theta}{2t} \rightarrow (3)$$

Multiplying by 'r^2', $r^2\dfrac{\partial\theta}{\partial r} = \dfrac{-r^3\theta}{2t}$

Again, Partially Differentiating (1) w.r.t. 'r',

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{-1}{2t}\left[r^3\frac{\partial\theta}{\partial r} + \theta\cdot 3r^2\right]$$

$$\therefore\ \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{-1}{2t}\left[r^3\cdot\frac{-r\theta}{2t} + \theta\cdot 3r^2\right] \text{ (From 3)}$$

$$\therefore\ \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{-1}{2t}\times\theta\cdot r^2\left[\frac{-r^2}{2t} + 3\right]$$

$$\therefore\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \theta\left[\frac{r^2}{4t^2} - \frac{3}{2t}\right] \rightarrow (4)$$

Given, $\dfrac{\partial\theta}{\partial t} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right)$

$$\therefore\ \cancel{\theta}\left(\frac{n}{t} + \frac{r^2}{4t^2}\right) = \cancel{\theta}\left[\frac{r^2}{4t^2} - \frac{3}{2t}\right] \text{ (From 2 \& 4)}$$

$$\therefore\ \frac{n}{t} + \cancel{\frac{r^2}{4t^2}} = \frac{-3/2}{t} + \cancel{\frac{r^2}{4t^2}}$$

Comparing both sides, $\boxed{n = \dfrac{-3}{2}}$

5c) Find the root (correct to three places of decimals) of $x^3-4x-9=0$ lying between 2 and 3 by using Regula-Falsi method. (Chp: Transcendental equations) (8)

Ans. Let $f(x)=x^3-4x-9 \rightarrow (1)$

Let a = 1.3 and b = 1.4

$\therefore\ f(a)=f(2)=(2)^3-4(2)-9=-9<0$ and $f(b)=f(3)=(3)^3-4(3)-9=6>0$

$\therefore$ Root of f (x) lies between 2 and 3

By Regula Falsi Method $x=\dfrac{a\,f(b)-b\,f(a)}{f(b)-f(a)} \rightarrow (2)$

Method I:

Iteration	a	b	f(a)	f(b)	x	f(x)
1)	2	3	–9.0000	6.0000	2.6000	–1.8240
2)	2.6	3	–1.8240	6.0000	2.6933	–0.2369
3)	2.6933	3	–0.2369	6.0000	2.7049	–0.0292
4)	2.7049	3	–0.0292	6.0000	2.7063	–0.0041
5)	2.7063	3	–0.0041	6.0000	2.7065	

Method II:

Iteration I:

Let a = 2, b = 3, f(a) = –9 and f(b) = 6

$\therefore$ From (2), $x_1=\dfrac{2(6)-3(-9)}{(6)-(-9)}=2.6$

$\therefore$ From (1),

$f(x_1)=f(2.6)=(2.6)^3-4(2.6)-9=-1.824<0$

Iteration II:

Let a = 2.6, b = 3, f(a) = –1.8240 and f(b) = 6

From (2), $x_2=\dfrac{2.6(6)-3(-1.8240)}{(6)-(-1.8240)}=2.6933$

$\therefore$ From (1),

$f(x_2)=f(2.6933)=(2.6933)^3-4(2.6933)-9$

$=-0.2369<0$

Iteration III:

Let a = 2.6933, b = 3, f(a) = –0.2372 and f(b) = 6

From (2), $x_3=\dfrac{2.6933(6)-3(-0.2372)}{(6)-(-0.2372)}=2.7049$

$\therefore$ From (1),

$f(x_3)=f(2.7049)=(2.7049)^3-4(2.7049)-9$

$=-0.0292<0$

Iteration IV:

Let a = 2.7049, b = 3, f(a) = –0.0289 and f(b) = 6

From (2), $x_4=\dfrac{2.7049(6)-3(-0.0289)}{(6)-(-0.0289)}=2.7063$

$\therefore$ From (1),

$f(x_4)=f(2.7063)=(2.7063)^3-4(2.7063)-9$

$=-0.0041<0$

Iteration V:

Let a = 2.7063, b = 3, f(a) = –0.0035 and f(b) = 6

From (2), $x_5=\dfrac{2.7063(6)-3(-0.0035)}{(6)-(-0.0035)}=2.7065$

Hence, by Regula Falsi Method, Root of the equation $x^3-4x-9=0$ is 2.7065

6b) Find non-singular matrices P & Q such that $A=\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}$ is reduced to normal form. Also find its rank.

(Chp: Rank of Matrix) (6)

Ans. Let $A_{3\times4}=I_{3\times3}\times A_{3\times4}\times I_{4\times4}$

$$\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_2-2R_1;R_3-R_1$ $\Rightarrow$ $$\begin{bmatrix}1&2&3&2\\0&-1&-1&-3\\0&1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_2-2C_1;C_3-3C_1;C_4-2C_1$ $\Rightarrow$ $$\begin{bmatrix}1&0&0&0\\0&-1&-1&-3\\0&1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_3+R_2;-R_2$ $\Rightarrow$ $$\begin{bmatrix}1&0&0&0\\0&1&1&3\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_3-C_2;C_4-2C_2$ $\Rightarrow$ $$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}\rightarrow(1)$$

RHS is the required PAQ form.

Here, $P=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}$ and $Q=\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}$

Rank of A = Number of non-zero rows on LHS of (1) = 2

6b) Find the principle value of $(1+i)^{(1-i)}$. (Chp: Log of Complex Numbers) (6)

Ans. Let $a=(1+i)^{(1-i)}$

Taking log on both sides, $\log a=\log(1+i)^{(1-i)}$

$\therefore\ \log a=(1-i)\log(1+1i)$

We know, $\log(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}$ (Principal Form)

$$\therefore\ \log a=(1-i)\times\left[\frac{1}{2}\log(1^2+1^2)+i\tan^{-1}\left(\frac{1}{1}\right)\right]$$

$$=(1-i)\times\left[\frac{1}{2}\log 2+i\times\frac{\pi}{4}\right]$$

$$=(1-i)\times\left[\log 2^{1/2}+i\frac{\pi}{4}\right]$$

$$=\log\sqrt{2}+i\frac{\pi}{4}-i\log\sqrt{2}+i^2\times\frac{\pi}{4}$$

$$\therefore\ \log a=\log\sqrt{2}-\frac{\pi}{4}+i\left(\frac{\pi}{4}-\log\sqrt{2}\right)$$

$$\therefore\ a=e^{\left[\log\sqrt{2}-\frac{\pi}{4}+i\left(\frac{\pi}{4}-\log\sqrt{2}\right)\right]}$$

$$\therefore\ a=e^{\left(\log\sqrt{2}-\frac{\pi}{4}\right)}\cdot e^{i\left(\frac{\pi}{4}-\log\sqrt{2}\right)}$$

$$\therefore\ a=e^{\log\sqrt{2}}\cdot e^{-\pi/4}\cdot e^{i\left(\frac{\pi}{4}-\log\sqrt{2}\right)}$$

$\therefore\ (1+i)^{(1-i)}=\sqrt{2}\,e^{-\pi/4}\cdot\left[\cos\left(\frac{\pi}{4}-\log\sqrt{2}\right)+i\sin\left(\frac{\pi}{4}-\log\sqrt{2}\right)\right]$, is the principle value.

6c) Solve the following equations by Gauss-Seidel method

27 x + 6 y – z = 85; 6 x + 15 y + 2 z = 72; x + y + 54 z = 110. (Take three iterations).

(Chp: Linear algebraic equations) (8)

Ans From 1st equation, 27 x = 85 – 6 y + z

$\therefore\ x = \frac{1}{27}(85 - 6y - z) = 27^{-1}(85 - 6y - z)$

Similarly,

6 x + 15 y + 2 z = 72 gives $y = 15^{-1}(72 - 6x - 2z)$ &

x + y + 54 z = 110 gives $z = 54^{-1}(110 - x - y)$

Iteration 1:

Put $y_0 = 0;\ z_0 = 0$

$\therefore\ x_1 = 27^{-1}(85 - 6y_0 - z_0)$

$= 27^{-1}(85 - 0 - 0)$

= 3.1481

Put $x_1 = 3.1481;\ z_0 = 0$

$\therefore\ y_1 = 15^{-1}(72 - 6x_1 - 2z_0)$

$= 15^{-1}(72 - 6 \times 3.1481 - 0)$

$= 15^{-1}(72 - 6 \times 3.1481 - 0)$

= 3.5407

Put $x_1 = 3.1481;\ y_1 = 3.5407$

$\therefore\ z_1 = 54^{-1}(110 - x_1 - y_1)$

$= 54^{-1}(110 - 3.1481 - 3.5407)$

= 1.9132

Iteration 2:

Put $y_1 = 3.5407;\ z_1 = 1.9132$

$\therefore\ x_2 = 27^{-1}(85 - 6y_1 - z_1)$

$= 27^{-1}(85 - 6 \times 3.5407 - 1.9132)$

= 2.4322

Put $x_2 = 2.4322;\ z_1 = 1.9132$

$\therefore\ y_2 = 15^{-1}(72 - 6x_2 - 2z_1)$

$= 15^{-1}(72 - 6 \times 2.4322 - 1.9132)$

= 3.5720

Put $x_2 = 2.4322;\ y_2 = 3.5720$

$\therefore\ z_2 = 54^{-1}(110 - x_2 - y_2)$

$= 54^{-1}(110 - 2.4322 - 3.5720)$

= 1.9258

Iteration 3:

Put $y_2 = 3.5720;\ z_2 = 1.9258$

$\therefore\ x_3 = 27^{-1}(85 - 6y_2 - z_2)$

$= 27^{-1}(85 - 6 \times 3.5720 - 1.9258)$

= 2.4257

Put $x_3 = 2.4257;\ z_2 = 1.9258$

$\therefore\ y_3 = 15^{-1}(72 - 6x_3 - 2z_2)$

$= 15^{-1}(72 - 6 \times 2.4257 - 1.9258)$

= 3.5729

Put $x_3 = 2.4257;\ y_3 = 3.5729$

$\therefore\ z_3 = 54^{-1}(110 - x_3 - y_3)$

$= 54^{-1}(110 - 2.4257 - 3.5729)$

= 1.9260

Iteration 4:

Put $y_3 = 3.5729;\ z_3 = 1.9260$

$\therefore\ x_4 = 27^{-1}(85 - 6y_3 - z_3)$

$= 27^{-1}(85 - 6 \times 3.5729 - 1.9260)$

= 2.4255

Put $x_4 = 2.4255;\ z_3 = 1.9260$

$\therefore\ y_4 = 15^{-1}(72 - 6x_4 - 2z_3)$

$= 15^{-1}(72 - 6 \times 2.4255 - 1.9260)$

= 3.5730

Put $x_4 = 2.4255;\ y_4 = 3.5730$

$\therefore\ z_4 = 54^{-1}(110 - x_4 - y_4)$

$= 54^{-1}(110 - 2.4255 - 3.5730)$

= 1.9260

Hence, by Gauss-Seidal method, solution of given set of equations is x = 2.4255, y = 3.5730, z = 1.9260.