Module 4:Lecture notes

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Successive Differentiation

Definition
If a function is differentiated once and if derivative of a function differentiated again and again with respect to same independent variable then such process is known as **Successive Differentiation** of a function
i.e. If $y = f(x)$ is a differentiable function of x then

$$y_1 = \frac{dy}{dx}$$
$$\implies y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$
$$\implies y_3 = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Continuing this process we have

$$y_n = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$$

are called successive n^{th} derivatives of y and are denoted by $y_n, f^{(n)}(x), \frac{d^n}{dx^n}$ and the values of this derivatives at an arbitrary point a is denoted by $[y_n]_{(a)}, f^{(n)}(a), \left[\frac{d^ny}{dx^n}\right]_{x=a}$

n^{th} derivatives of Algebraic Functions

Case(1)
If $y = (ax+b)^m$ then

$$y_n = m(m-1)(m-2)...(m-n+1)a^n(ax+b)^{(m-n)} \qquad n < m$$

Proof
Let

$$y = (ax+b)^m$$
$$\implies y_1 = m(ax+b)^{(m-1)}a$$
$$\implies y_2 = m(m-1)(ax+b)^{(m-2)}a^2$$
$$\implies y_3 = m(m-1)(m-2)(ax+b)^{(m-3)}a^3$$

Continuing differentiation upto n^{th} derivatives, we have

$$y_n = m(m-1)(m-2).....(m-(n-1))(ax+b)^{(m-n)}a^n$$

$$y_n = m(m-1)(m-2).....(m-n+1)(ax+b)^{(m-n)}a^n....\mathbf{(1)}$$

This holds true for positive integer m and $n < m$ Multiplying and dividing result (1) by $(m-n)(m-n-1)...3.2.1$ we have

$$y_n = \frac{m!}{(m-n)!}(ax+b)^{(m-n)}a^n....\mathbf{(2)}$$

If $n = m$ then from above result

$$y_n = n!a^n....\mathbf{(3)}$$

This holds true for positive integer m and $n < m$ **Case(2)**
If $y = (ax+b)^{(-m)}$ then

$$y_n = \frac{(-1)^n(m+n-1)!a^n}{(m-1)!(ax+b)^{(m+n)}}$$

Proof
If m is negative real integer then let $m = -p$ where p is positive integer, then

$$y = \frac{1}{(ax+b)^m} \implies y = (ax+b)^{(-m)}$$

Then by(1)

$$y_n = (-m)(-m-1)(-m-2).....(-m-n+1)(ax+b)^{(-m-n)}a^n$$

$$y_n = (-1)^n m(m-1)(m-2).....(m+n-1)(ax+b)^{-(m+n)}a^n$$

$$y_n = \frac{(-1)^n(m+n-1)!(ax+b)^{-(m+n)}a^n}{(m-1)!}$$

(Multiplying and dividing by $(m-1)(m-2)...3.2.1$ and simplifying)
Hence for negative integer m i.e. for $y = \frac{1}{(ax+b)^m}$

$$y_n = \frac{(-1)^n(m+n-1)!a^n}{(m-1)!(ax+b)^{(m+n)}}$$

Case(3)
If $y = log_e(ax+b)$ then

$$y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

Proof
Let

$$\begin{aligned} y &= log_e(ax+b) \\ \implies y_1 &= \frac{1}{ax+b}a \\ \implies y_2 &= \frac{(-1)}{(ax+b)^2}a^2 \\ \implies y_3 &= \frac{(-1)(-2)}{(ax+b)^3}a^3 \end{aligned}$$

Continuing differentiation upto n^{th} derivatives, we have

$$y_n = \frac{(-1)(-2)...(-n+1)}{(ax+b)^n}a^n$$

$$y_n = \frac{(-1)^{(n-1)}(n-1)!a^n}{(ax+b)^n}....\mathbf{(1)}$$

Case(4)
If $y = a^{(bx+c)}$ then

$$y_n = b^n \left(log_e a\right)^n a^{bx+c}$$

Proof
Let

$$\begin{aligned} y &= a^{(bx+c)} \\ \implies y_1 &= a^{(bx+c)}log_e ab \\ \implies y_2 &= a^{(bx+c)}\left(log_e a\right)^2 b^2 \\ \implies y_3 &= a^{(bx+c)}\left(log_e a\right)^3 b^3 \end{aligned}$$

Continuing differentiation upto n^{th} derivatives, we have

$$y_n = a^{(bx+c)}\left(log_e a\right)^n b^n....\mathbf{(1)}$$

n^{th} derivatives of Trigonometric Functions

Case(1)
If $y = sin(ax + b)$ then

$$y_n = a^n sin\left(ax + b + \frac{n\pi}{2}\right)$$

Proof 1
Let

$$\begin{aligned}
y &= sin(ax + b) \\
\implies y_1 &= a\ cos(ax + b) \\
&= asin\left(ax + b + \frac{\pi}{2}\right) \\
\implies y_2 &= a^2\ cos\left(ax + b + \frac{\pi}{2}\right) \\
&= a^2 sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right) \\
&= a^2 sin\left(ax + b + \pi\right) \\
\implies y_3 &= a^3\ cos\left(ax + b + \frac{2\pi}{2}\right) \\
&= a^3 sin\left(ax + b + \frac{2\pi}{2} + \frac{\pi}{2}\right) \\
&= a^3 sin\left(ax + b + \frac{3\pi}{2}\right)
\end{aligned}$$

Continuing differentiation upto n^{th} derivatives, we have

$$y_n = \frac{d^n}{dx^n}\left[sin(ax + b)\right]$$

$$y_n = a^n sin\left(ax + b + \frac{n\pi}{2}\right) \text{....}\mathbf{(1)}$$

Case(2)
If $y = cos(ax + b)$ then

$$y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right)$$

Proof 2
Let

$$\begin{aligned}
&y = cos(ax + b)\\
\implies &y_1 = -a\ sin(ax + b)\\
&= acos\left(ax + b + \frac{\pi}{2}\right)\\
\implies &y_2 = -a^2\ sin\left(ax + b + \frac{\pi}{2}\right)\\
&= a^2cos\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)\\
&= a^2cos\left(ax + b + \pi\right)\\
\implies &y_3 = -a^3\ sin\left(ax + b + \frac{2\pi}{2}\right)\\
&= a^3cos\left(ax + b + \frac{2\pi}{2} + \frac{\pi}{2}\right)\\
&= a^3cos\left(ax + b + \frac{3\pi}{2}\right)
\end{aligned}$$

Continuing differentiation upto n^{th} derivatives, we have

$$y_n = \frac{d^n}{dx^n}\left[cos(ax + b)\right]$$

$$y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right) \text{....}\mathbf{(2)}$$

Special Cases

Case(1)
If $y = e^{ax}sin(bx + c)$ then

$$y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} sin\left[bx + c + ntan^{-1}\left(\frac{b}{a}\right)\right]$$

Proof

Let

$$\begin{aligned} y &= e^{ax} sin(bx+c) \\ \implies y_1 &= a\ e^{ax}\ sin(bx+c) + e^{ax}\ b\ cos(bx+c) \\ &= e^{ax}\left[a\ sin(bx+c) + b\ cos(bx+c)\right] .....\mathbf{(1)} \end{aligned}$$

Let $a = r\ cos\ \theta$ and $b = r\ sin\ \theta$
Then $r = (a^2+b^2)^{\frac{1}{2}}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$
Substituting in (1) , we have

$$\begin{aligned} y_1 &= e^{ax}\left[a\ sin(bx+c) + b\ cos(bx+c)\right] \\ &= e^{ax}\left[(r\ cos\ \theta)\ \ sin(bx+c) + (r\ sin\ \theta)\ cos(bx+c)\right] \\ &= r\ e^{ax}\left[sin(bx+c) cos\ \theta\ \ + cos(bx+c)\ sin\ \theta\right] \\ y_1 &= r\ e^{ax}\left[sin(bx+c+\theta)\right] \end{aligned}$$

Again differentiating in the same manner

$$\begin{aligned} y_1 &= r\ e^{ax}\left[sin(bx+c+\theta)\right] \\ \implies y_2 &= r\left[a\ e^{ax}\left(sin(bx+c+\theta)\right) + r\left(b\ e^{ax}\ cos(bx+c+\theta)\right)\right] \\ &= r\ e^{ax}\left[a\ sin(bx+c+\theta) + b\ cos(bx+c+\theta)\right] \\ &= r\ e^{ax}\left[r\ cos\ \theta\ sin(bx+c+\theta) + r\ sin\ \theta\ cos(bx+c+\theta)\right] \\ &= r^2\ e^{ax}\left[sin(bx+c+\theta) cos\ \theta + cos(bx+c+\theta) sin\ \theta\right] \\ &= r^2\ e^{ax}\left[sin(bx+c+\theta+\theta)\right] \\ y_2 &= r^2\ e^{ax}\left[sin(bx+c+2\theta)\right] \end{aligned}$$

Continuing differentiation upto n^{th} derivatives, we have

$$y_n = r^n\ e^{ax}\left[sin(bx+c+n\theta)\right]$$

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} sin\left[bx+c+ntan^{-1}\left(\frac{b}{a}\right)\right]$$

Case(2)
If $y = e^{ax}cos(bx+c)$ then

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} cos\left[bx + c + ntan^{-1}\left(\frac{b}{a}\right)\right]$$

Proof
Let

$$\begin{aligned} y &= e^{ax}cos(bx+c) \\ \Longrightarrow y_1 &= a\ e^{ax}\ cos(bx+c) - e^{ax}\ b\ sin(bx+c) \\ &= e^{ax}\left[a\ cos(bx+c) - b\ sin(bx+c)\right] .....\mathbf{(1)} \end{aligned}$$

Let $a = r\ cos\ \theta$ and $b = r\ sin\ \theta$
Then $r = (a^2+b^2)^{\frac{1}{2}}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$
Substituting in (1) , we have

$$\begin{aligned} y_1 &= e^{ax}\left[a\ cos(bx+c) - b\ sin(bx+c)\right] \\ &= e^{ax}\left[(r\ cos\ \theta)\ \ cos(bx+c) - (r\ sin\ \theta)\ sin(bx+c)\right] \\ &= r\ e^{ax}\left[cos(bx+c)cos\ \theta\ - sin(bx+c)\ sin\ \theta\right] \\ y_1 &= r\ e^{ax}\left[cos(bx+c+\theta)\right] \end{aligned}$$

Again differentiating in the same manner

$$\begin{aligned} y_1 &= r\ e^{ax}\left[cos(bx+c+\theta)\right] \\ \Longrightarrow y_2 &= r\left[a\ e^{ax}\left(cos(bx+c+\theta)\right) - r\left(b\ e^{ax}\ sin(bx+c+\theta)\right)\right] \\ &= r\ e^{ax}\left[a\ cos(bx+c+\theta) - b\ sin(bx+c+\theta)\right] \\ &= r\ e^{ax}\left[r\ cos\ \theta\ cos(bx+c+\theta) - r\ sin\ \theta\ sin(bx+c+\theta)\right] \\ &= r^2\ e^{ax}\left[cos(bx+c+\theta)cos\ \theta - sin(bx+c+\theta)sin\ \theta\right] \\ &= r^2\ e^{ax}\left[cos(bx+c+\theta+\theta)\right] \\ y_2 &= r^2\ e^{ax}\left[cos(bx+c+2\theta)\right] \end{aligned}$$

Continuing differentiation upto n^{th} derivatives, we have

$$y_n = r^n\ e^{ax}\left[cos(bx+c+n\theta)\right]$$

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax} cos\left[bx + c + ntan^{-1}\left(\frac{b}{a}\right)\right]$$

Examples

Example 1
Find n^{th} derivatives of following functions

$$y = \frac{x+3}{(x-1)(x+2)}$$

Solution
Given

$$y = \frac{x+3}{(x-1)(x+2)}$$
$$y = \frac{A}{(x-1)} + \frac{B}{(x+2)} \ldots\textbf{(1)}$$

Using Partial fraction method, let

$$x+3 = A(x+2) + B(x-1)$$

For $x = 1$; $4 = A(3) \implies A = \frac{4}{3}$
$x = -2$; $1 = B(-3) \implies B = \frac{-1}{3}$
Substituting in (1)

$$y = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$
$$y = \frac{\frac{4}{3}}{(x-1)} + \frac{\frac{-1}{3}}{(x+2)}$$

Taking n^{th} derivative

$$\frac{d^n}{dx^n}[y] = \frac{d^n}{dx^n}\left[\frac{\frac{4}{3}}{(x-1)}\right] + \frac{d^n}{dx^n}\left[\frac{\frac{-1}{3}}{(x+2)}\right]$$

$$= \frac{4}{3}\frac{d^n}{dx^n}\left[\frac{1}{(x-1)}\right] - \frac{1}{3}\frac{d^n}{dx^n}\left[\frac{1}{(x+2)}\right]$$

$$= \frac{4}{3}\left[\frac{(-1)^n\ n!}{(x-1)^{n+1}}\right] - \frac{1}{3}\left[\frac{(-1)^n\ n!}{(x+2)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n\ n!}{3}\left[\frac{4}{(x-1)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right]$$

Example 2
If $y = x\ log\left(\frac{x-1}{x+1}\right)$ then prove that

$$y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

Solution
Given

$$y = x\ log\left(\frac{x-1}{x+1}\right)$$

$$\Longrightarrow y = x\ log\,(x-1) - x\ log\,(x+1)$$

$$\Longrightarrow y_1 = log\,(x-1) - log\,(x+1) + \frac{x}{x-1} - \frac{x}{x+1}$$

$$\Longrightarrow y_1 = log\,(x-1) - log\,(x+1) + 1 + \frac{1}{x-1} - 1 + \frac{1}{x+1}$$

$$\Longrightarrow y_1 = log\,(x-1) - log\,(x+1) + \frac{1}{x-1} + \frac{1}{x+1}$$

Taking n^{th} derivative of y, i.e. taking $(n-1)^{th}$ derivative of y_1

$$\frac{d^n}{dx^n}[y] = \frac{d^{n-1}}{dx^{n-1}}\left[log\,(x-1) - log\,(x+1) + \frac{1}{x-1} + \frac{1}{x+1}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}[log\,(x-1)] - \frac{d^{n-1}}{dx^{n-1}}[log\,(x+1)] +$$

$$\frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{x-1}\right] + \frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{x+1}\right]$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^{n}} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^{n}}$$

$$y_n = (-1)^{n-2}(n-2)!\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} + \frac{-n+1}{(x-1)^n} + \frac{-n+1}{(x+1)^n}\right]$$

$$y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

Example 3
If

$$y = sin^2x\; cos^3x$$

then find y_n **Solution**

Given

$$
\begin{aligned}
& y = sin^2x\ cos^3x \\
\implies & y = sin^2x\ cos^2x\ \ cosx \\
\implies & y = \frac{1}{4}(2sin^2x\ cos^2x)^2\ cosx \\
\implies & y = \frac{1}{4}(sin\ 2x)^2\ cosx \\
\implies & y = \frac{1}{8}(2sin^2\ 2x)\ cosx \\
\implies & y = \frac{1}{8}(1 - cos\ 4x)\ cosx \\
\implies & y = \frac{1}{8}(cosx - cos\ 4x\ cosx) \\
\implies & y = \frac{1}{16}(2cosx - 2\ cos\ 4x\ cosx) \\
\implies & y = \frac{1}{16}(2cosx - cos\ 5x - cos3x)
\end{aligned}
$$

Taking n^{th} derivative of y

$$y_n = \frac{1}{16}\left[2\ cos\left(x + \frac{n\pi}{2}\right) - 5^n\ cos\left(5x + \frac{n\pi}{2}\right) - 3^n\ cos\left(3x + \frac{n\pi}{2}\right)\right]$$

$$y_n = \frac{1}{8}\ cos\left(x + \frac{n\pi}{2}\right) - \frac{5^n}{16}\ cos\left(5x + \frac{n\pi}{2}\right) - \frac{3^n}{16}\ cos\left(3x + \frac{n\pi}{2}\right)$$

Liebnitz theorem

Statement
If u and v are two functions of x with u_n and v_n be their n^{th} derivatives, then n^{th} derivative of their product uv is given by

$$(uv)_n =^n C_0\ u_n\ v +^n C_1\ u_{n-1}\ v_1 + ... +^n C_r\ u_{n-r}\ v_r + ... +^n C_n\ u\ v_n$$

NOTE:
1)$(uv)_n = (vu)_n$
2) Function whose n^{th} derivative is known is considered as u

Example(Type 1)
Find n^{th} derivative of
1)$x\ sin\ 3x$ 2)$x^2\ e^{ax}$

Solution
(1) Given $y = x\ sin\ 3x$
Let $u = sin\ 3x$ and $v = x$
By Liebnitz Rule

$$(uv)_n =^n C_0\ u_n\ v +^n C_1\ u_{n-1}\ v_1 + ... +^n C_r\ u_{n-r}\ v_r + ... +^n C_n\ u\ v_n....\mathbf{(1)}$$

Now $u = sin\ 3x \implies u_n = 3^n\ sin(3x + \dfrac{n\pi}{2})$
and $v = x \implies v_1 = 1\ and\ v_n = 0\ \ n \geq 2$
Substituting in (1), we have

$$(x\ sin\ 3x)_n = x\ 3^n\ sin(3x + \frac{n\pi}{2}) + n\ 1\ 3^{n-1}\ sin(3x + \frac{(n-1)\pi}{2})$$
$$(x\ sin\ 3x)_n = 3^{n-1}\ \left[3x\ sin(3x + \frac{n\pi}{2}) + n\ sin(3x + \frac{(n-1)\pi}{2})\right]$$

(2) Given $y = x^2\ e^{ax}$
Let $u = e^{ax}$ and $v = x^2$

By Liebnitz Rule

$$(uv)_n =^n C_0\ u_n\ v +^n C_1\ u_{n-1}\ v_1 + ... +^n C_r\ u_{n-r}\ v_r + ... +^n C_n\ u\ v_n$$

$$(x^2\ e^{ax})_n = x^2\ a^n\ e^{ax} + n\ 2x\ a^{n-1}\ e^{ax} + \frac{n(n-1)}{2!}\ 2\ a^{n-2}\ e^{ax}$$

$$(x^2\ e^{ax})_n = a^n\ e^{ax}\ \left[x^2 + \frac{2n}{a}x + \frac{n(n-1)}{a^2}\right]$$

Example(Type 2)
If $y = \frac{log\ x}{x}$ Prove that

$$y_5 = \frac{5!}{x^6}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - log\ x\right]$$

Solution
Given $y = \dfrac{log\ x}{x} = \dfrac{1}{x}\ log\ x$
Let $u = \dfrac{1}{x}$ and $v = log\ x$
Now

$$u = \frac{1}{x}$$
$$\implies u_n = \frac{(-1)^n\ n!}{x^{n+1}}$$

and

$$v = log\ x$$
$$\implies v_n = \frac{(-1)^{n-1}\ (n-1)!}{x^n}$$

By Liebnitz Rule

$$(uv)_n =^n C_0\ u_n\ v +^n C_1 u_{n-1}v_1 + ... +^n C_r\ u_{n-r}v_r + ... +^n C_n\ uv_n$$

$$\begin{aligned}\left(\frac{1}{x}\ log\ x\right)_n &= \left(\frac{(-1)^n n!}{x^{n+1}}\right)\ log\ x + n\left(\frac{(-1)^{n-1}(n-1)!}{x^n}\right)\frac{1}{x} + \frac{n(n-1)}{2!}\left(\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\right)\ \frac{-1}{x^2} \\ &+ \quad \frac{1}{x}\ \left(\frac{(-1)^{n-1}(n-1)!}{x^n}\right)\end{aligned}$$

$$y_n = \left(\frac{1}{x}\ log\ x\right)_n = \frac{(-1)^n n!}{x^{n+1}}\left[log\ x - \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)\right]$$

For $n = 5$

$$y_5 = \frac{-5!}{x^6}\left[log\ x - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}\right)\right]$$

$$y_5 = \frac{5!}{x^6}\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}\right) - log\ x\right]$$

Example(Type 3)

If $log\ y = tan^{-1}x$ Prove that

$$(1 + x^2)y_{n+2} - [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$$

Solution

Given

$$log\ y = tan^{-1}x$$
$$\implies y = e^{tan^{-1}x}$$
$$\implies y_1 = e^{tan^{-1}x}\ \left(\frac{1}{1+x^2}\right)$$
$$\implies (1+x^2)y_1 = e^{tan^{-1}x} = y$$

again differentiating

$$(1+x^2)y_2 + 2x\ y_1 = y_1$$

$$(1+x^2)y_2 + (2x-1)\ y_1 = 0....\mathbf{(1)}$$

Differentiating (1) n^{th} time using Liebnitz Rule

$$\left[(1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}2y_n\right] + [(2x-1)y_{n+1} + n(2)y_n] = 0$$

$$\implies (1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + (n^2-n+2n)y_n = 0$$

$$\implies (1+x^2)y_{n+2} + [2(n+1)-1]y_{n+1} + n(n+1)y_n = 0$$

Hence Proved

Example(Type 3)

If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ Prove that

$$(x^2-1)y_{n+2} + 2(n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Solution

Given

$$y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$$
$$\therefore y^{\frac{2}{m}} + 1 = 2xy^{\frac{1}{m}}$$
$$\therefore \left(y^{\frac{1}{m}}\right)^2 - 2xy^{\frac{1}{m}} + 1 = 0$$
$$\therefore y^{\frac{1}{m}} = \frac{-(-2x) \pm \sqrt{4x^2 - 4}}{2}$$
$$= \frac{(2x) \pm 2\sqrt{x^2 - 1}}{2}$$
$$\therefore y^{\frac{1}{m}} = x \pm \sqrt{x^2 - 1}$$
$$\therefore y = (x \pm \sqrt{x^2 - 1})^m$$

Considering

$$y = (x + \sqrt{x^2 - 1})^m$$
$$\implies y_1 = m(x + \sqrt{x^2 - 1})^{m-1} \left[1 + \frac{2x}{2\sqrt{x^2 - 1}}\right]$$
$$\implies y_1 = m(x + \sqrt{x^2 - 1})^{m-1} \left[\frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}\right]$$
$$\implies \sqrt{x^2 - 1}\ y_1 = m(x + \sqrt{x^2 - 1})^m$$
$$\implies \sqrt{x^2 - 1}\ y_1 = m\ y$$
$$\implies (x^2 - 1)\ y_1^2 = m^2\ y^2$$

Differentiating w.r.t x

$$(x^2 - 1)\ 2y_1\ y_2 + (2x)y_1^2 = 2\ m^2\ y\ y_1$$
$$(x^2 - 1)\ y_2 + xy_1 - \ m^2\ y = 0....\mathbf{(1)}$$

Differentiating (1) n^{th} time using Liebnitz Rule

$$\left[(x^2 - 1)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}2y_n\right] + [xy_{n+1} + ny_n] - m^2\ y_n = 0$$
$$\implies (x^2 - 1)y_{n+2} + (2nx + x)y_{n+1} + (n^2 - n + n - m^2)y_n = 0$$
$$\implies (x^2 - 1)y_{n+2} + [2(n+1)x]y_{n+1} + (n^2 - m^2)y_n = 0$$

Hence Proved

Practice Examples

(A)Find n^{th} derivatives of following functions

1) $y = e^x$

2) $y = a^x$

(B)Find n^{th} derivatives of following functions

1) $y = \dfrac{8x}{x^3 - 2x^2 - 4x + 8}$

2) $y = \dfrac{x^3}{x^2 - 1}$

3) $y = \dfrac{1}{1 + x + x^2 + x^3}$

4) $y = \dfrac{x^3}{(x+1)(x-2)}$

5) $y = \dfrac{2x+3}{(x-1)(x-2)}$

(C)Find n^{th} derivatives of following functions

1) $y = sin\ 2x\ sin\ 3x\ \ cos\ 4x$

2) $y = 2^x\ sin^3x\ cos^2x$

3) $y = x\ log(1 - x)$

4) $y = e^{2x} cosx\ sin^2\ 2x$

5) $y = e^{2x} cos\frac{x}{2} sin\frac{x}{2} sin\ 3x$

6) $y = 2^x cos\ 9x$

(D)If $y = sin\ rx + cos\ rx$ **then prove that**

$$y_n = r^n \left[1 + (-1)^n sin\ 2rx\right]^{\frac{1}{2}}$$

Also find $y_8(\pi)$ where $r = \frac{1}{4}$

(E)If $y = tan^{-1}\left(\frac{1+x}{1-x}\right)$ **then prove that**

$$y_n = (-1)^{n-1}(n-1)!\ sin^n\theta\ sin\ n\theta$$

where $\theta = tan^{-1}\left(\frac{1}{x}\right)$

(F)If $y = \frac{x}{x^2+a^2}$ **then prove that**

$$y_n = \frac{(-1)^n n!}{a^{n+1}} sin^{n+1}\theta\ cos(n+1)\theta$$

(G)Using Leibnitz's Theorem Find n^{th} **derivatives of following functions**

1)$y = x^3\ cos\ x$

2)$y = x^2\ e^x\ cos\ x$

3)$y = x\ log(x+1)$

(H)Using Leibnitz's Theorem, Prove the following results for given functions

1)$y = x^n \; log \; x$ then

$$y_{n+1} = \frac{n!}{x}$$

2)$y = sin[log(x^2 + 2x + 1)]$ then

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$

3)$y = cos^{-1}x$ then

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$$

4)$cos^{-1}\left(\frac{y}{b}\right) = log\left(\frac{x}{n}\right)^n$ then

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$$

5)$y = (sin^{-1}x)^2$ then

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$$

Hence find value of $y_n(0)$