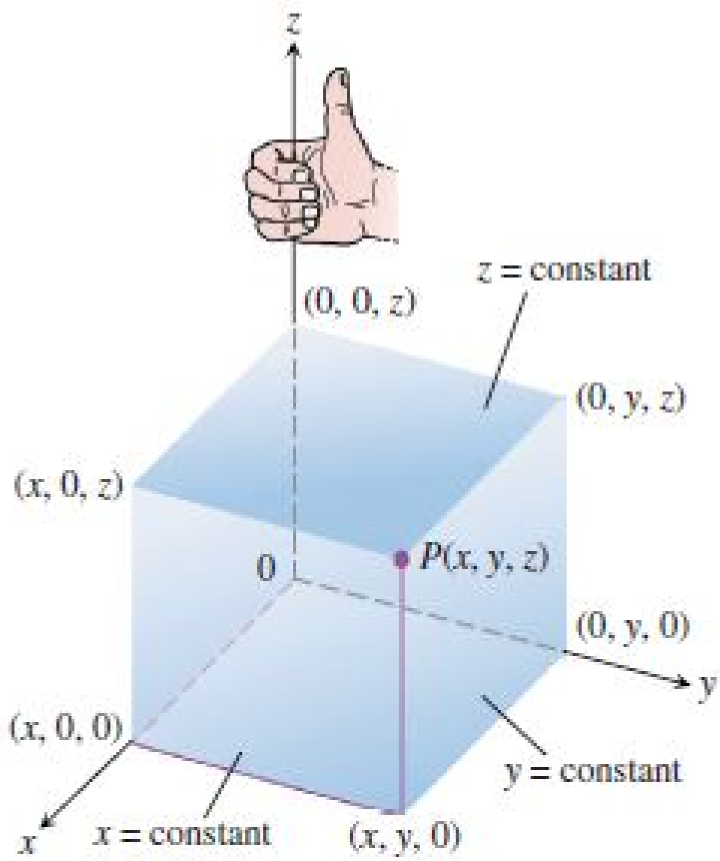
Module 5.2:Lecture notes:Triple Integration

By Dr. Keya Doshi

TRIPLE INTEGRATION

Three Dimensional coordinate system

To locate the point in a space we consider three mutually perpendicular cordinate axis as shown in a figure.



- The Cartesian coordinates (x, y, z) of a point P in space are the numbers at which the planes through P perpendicular to the axes cut the axes.

- Cartesian coordinates for spaceare also called rectangular coordinates because the axes that define them meet at right angles.

- Points on the x-axis have y- and z-coordinates equal to zero. That is, they have coordinates of the form $(x, 0, 0)$. Similarly, points on the y-axis have coordinates of the form $(0, y, 0)$, and points on the z-axis have coordinates of the form $(0, 0, z)$.

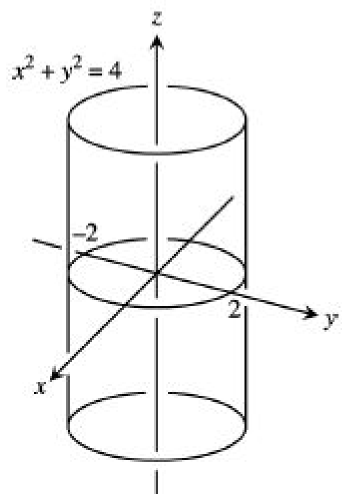
- The planes determined by the coordinates axes are the xy-plane, whose standard equation is $z = 0$, the yz-plane, whose standard equation is $x = 0$ and the xz-plane, whose standard equation is $y = 0$. They meet at the origin $(0, 0, 0)$

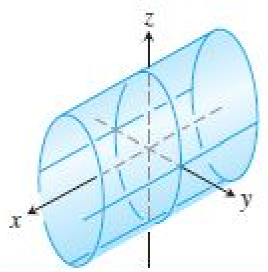
- The origin is also identified by simply 0 or sometimes the letter O.

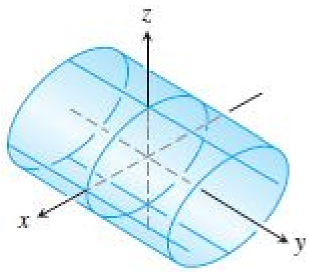
- The three coordinate planes $x = 0$, $y = 0$ and $z = 0$ divide the space into eight cells called **octants**. The octant in which the point coordinates are all positive is called the **first octant or Positive octant**; there is no conventional numbering for the other seven octants.

Some standard solids

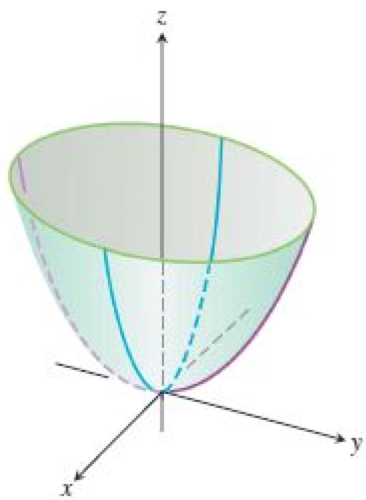
1. Right circular cylinder parallel to z axis, x axis and y axis respectively

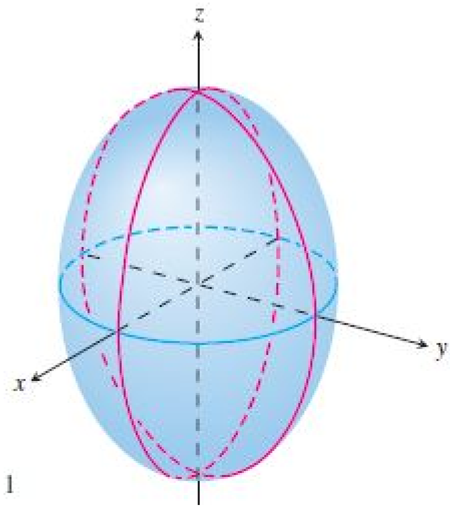




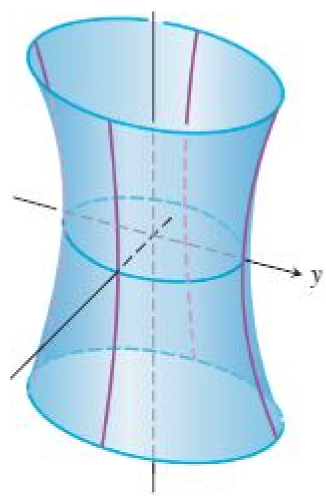


2. Paraboloids

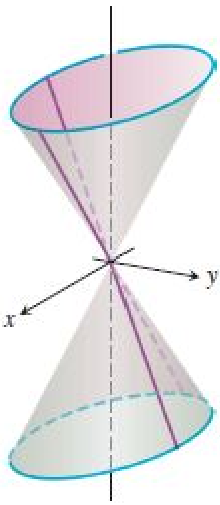




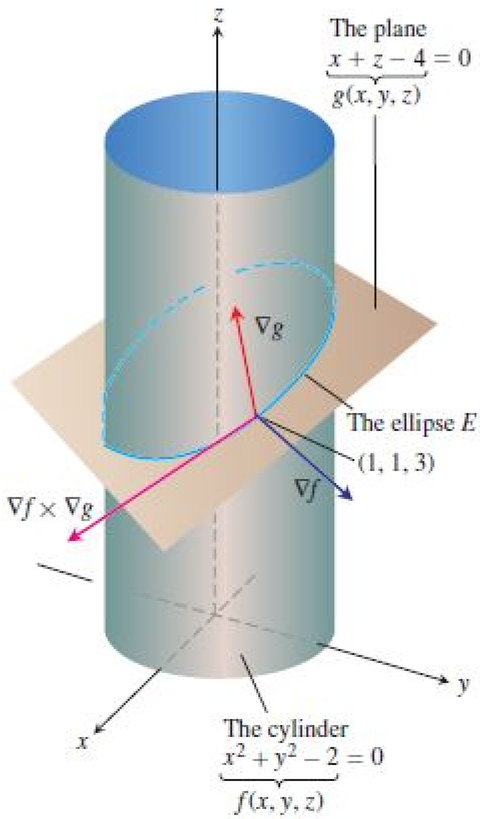
3. Ellipsoid [1]



4. Hyperboloid of one sheet

5. cone 

6. intersection of cylinder $x^2 + y^2 = 2$ and plane $x + z = 4$



Triple Integration Concept

Let $F(x, y, z)$ be a continuous function defined over a closeed and bounded surface surface D
Then dividing a surface D into small surfaces or rectangular boxes by the planes parrallel to coordinate axes we get rectagular boxes or surfaces of volume $\Delta v_i = \Delta x_i * \Delta y_i * \Delta z_i$ and considering a function value at point(x_i, y_i, z_i) along with the

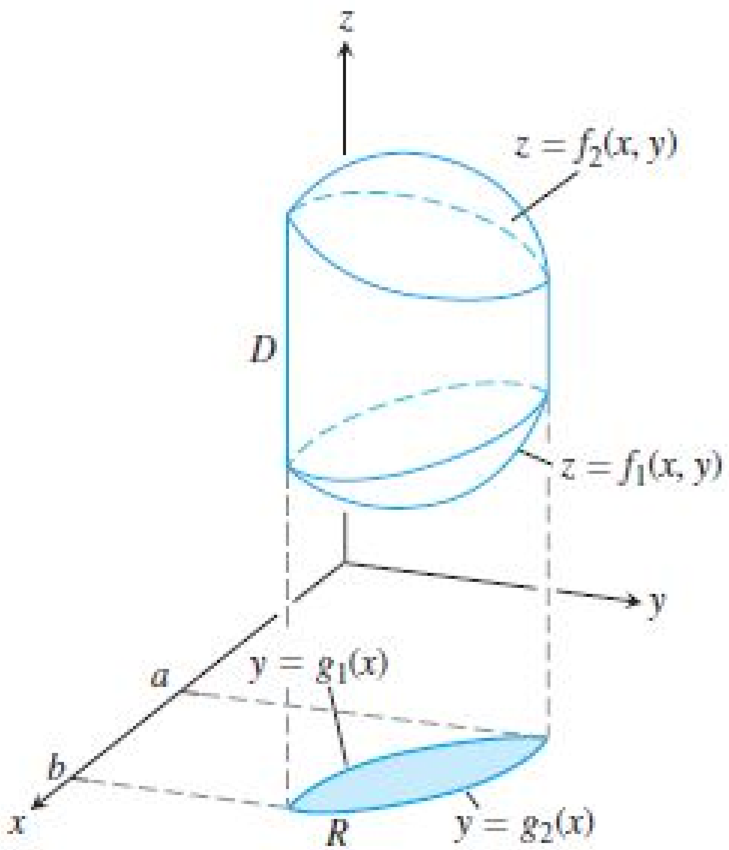
volume Δv_i , we consider a sum S_n

$$S_n = \sum_{i=0}^{n} f(x_i, y_i, z_i)\Delta v_i$$

As $n \to \infty$ or $\Delta v_i \to 0$ the Double Integration over a given rectangular region is defined as

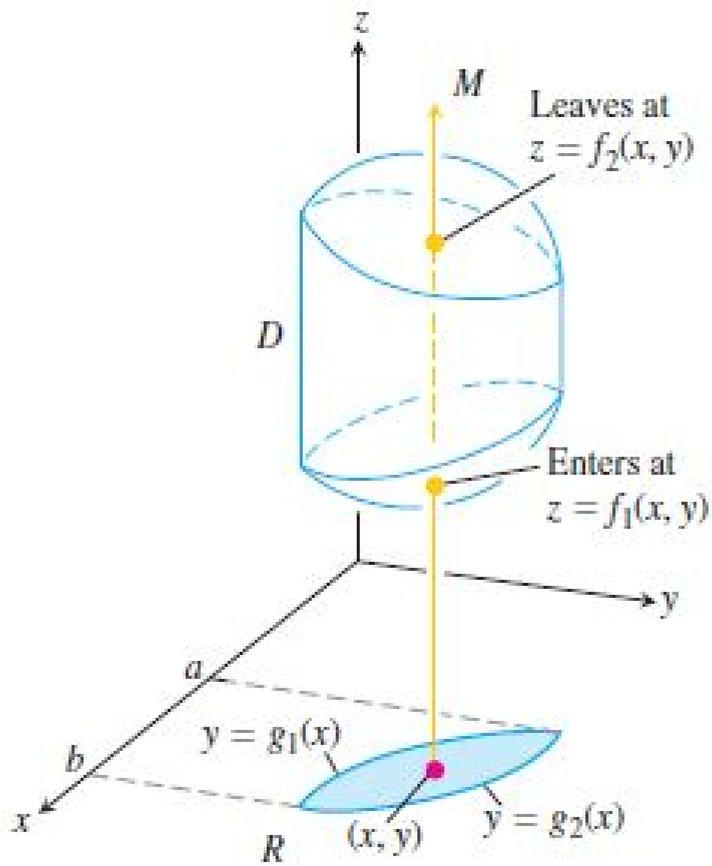
$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i, y_i, z_i)\Delta v_i \\ &= \iiint_D f(x, y, z)dv \end{aligned}$$

Consider a solid region D along with its vertical projection on xy plane given by region R as shown in the figure.

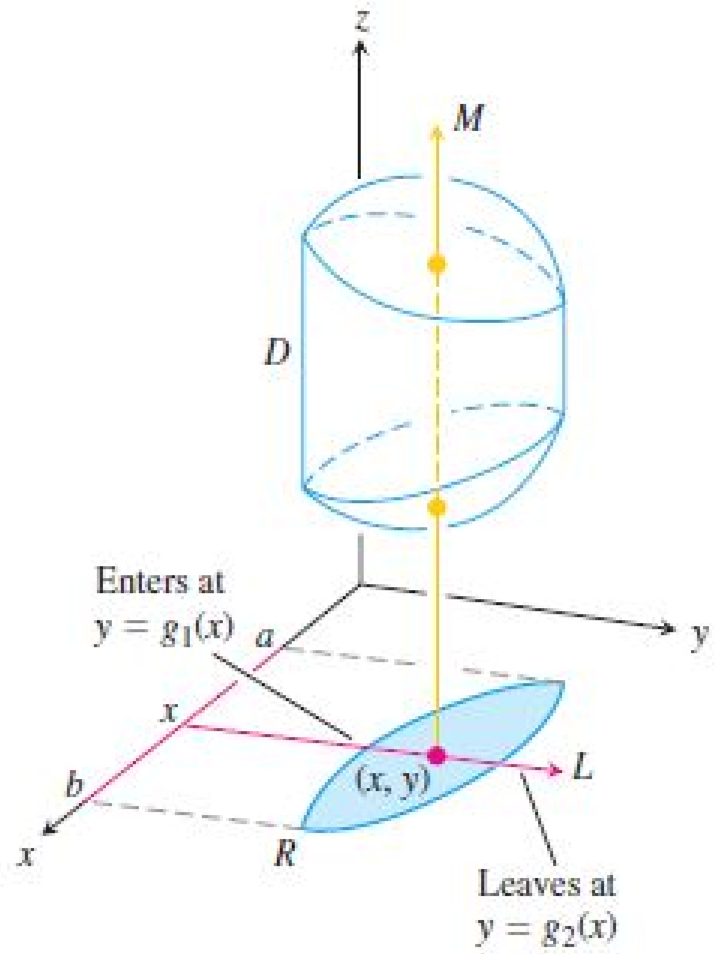


A solid region D in the space bounded below by two surfaces $z = f_1(x, y)$ and above by $z = f_2(x, y)$

To consider limits, consider an imaginary strip from a point(x, y) in a vertical Projection R parallel to z axis entering the solid D at $z = f_1(x, y)$ and leaving the surface at $z = f_2(x, y)$.These are z limits of integration.



Simultaneously in xy plane from the point (x, y) consider a strip parallel to y axis entering a region R at $y = g_1(x)$ and leaving at $y = g_2(x)$ these are y limits of integration
and on x axis strip varies from $x = a$ to $x = b$.these are x limits of integration



Then for a given solid region triple integral is defined as

$$S = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\ dz\ dy\ dx$$

EXAMPLES(Evaluation)

1. $\int_0^1 \int_{-1}^2 \int_1^3 x + y^3 + z^3 \, dx \, dy \, dz$

Solution:

$$\begin{aligned}
I &= \int_0^1 \int_{-1}^2 \int_1^3 x + y^3 + z^3 \, dx \, dy \, dz \\
&= \int_0^1 \int_{-1}^2 \left[\frac{x^2}{2} + x\ y^3 + x\ z^3\right]_1^3 dy \, dz \\
&= \int_0^1 \int_{-1}^2 \left[4 + 2\ y^3 + 2\ z^3\right] dy \, dz \\
&= \int_0^1 \left[4y + \frac{y^4}{2} + 2y\ z^3\right]_{-1}^2 dz \\
&= \int_0^1 \left[12 + \frac{15}{2} + 6z^3\right] dz \\
&= \left[12z + \frac{15}{2}z + \frac{3\ z^4}{2}\right]_0^1 \\
&= \frac{42}{2} \\
&= 21.........Ans
\end{aligned}$$

2. $\int\limits_{1}^{e} \int\limits_{1}^{e} \int\limits_{1}^{e} \frac{dx\ dy\ dz}{xyz}$

Ans:1

3. $\int\limits_{0}^{1} \int\limits_{0}^{\sqrt{1-x^2}} \int\limits_{0}^{\sqrt{1-x^2-y^2}} \frac{dz\ dy\ dx}{\sqrt{1-x^2-y^2-z^2}}$

Ans:$\frac{\pi^2}{8}$

4. $\int\limits_{1}^{e} \int\limits_{1}^{logy} \int\limits_{1}^{e^x} logz\ \ dz\ dx\ dy$

Ans:$\frac{1}{4}\left[e^2 - 8e + 13\right]$

5. $\int\limits_{0}^{a} \int\limits_{0}^{\sqrt{a^2-x^2}} \int\limits_{0}^{\sqrt{a^2-x^2-y^2}} xyz\ dz\ dy\ dx$

Ans:$\frac{a^6}{48}$

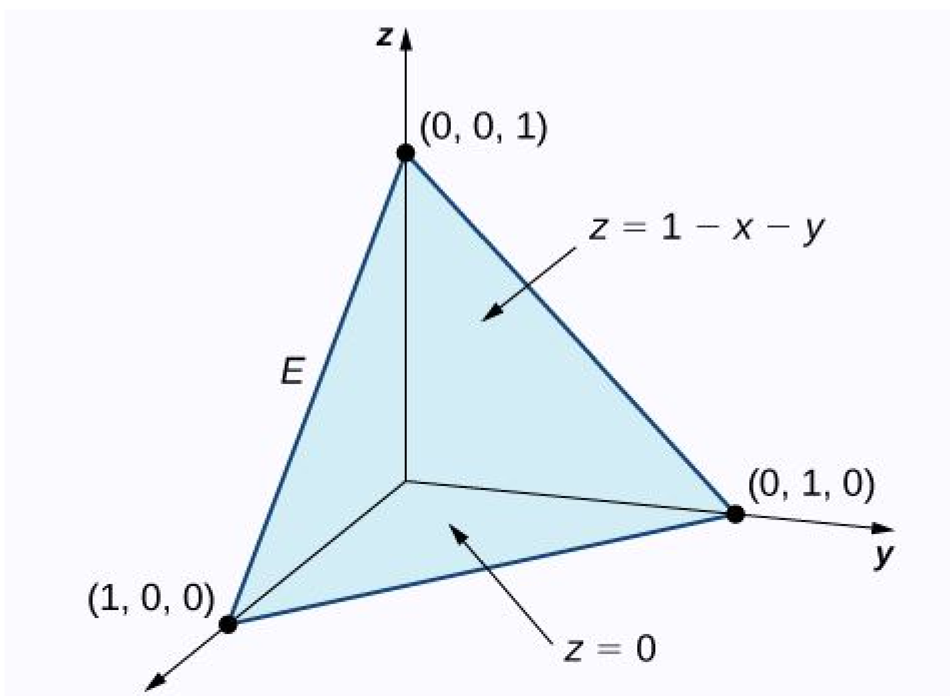
6. $\int\limits_{0}^{2} \int\limits_{0}^{z} \int\limits_{0}^{y} xyz\ dx\ dy\ dz$

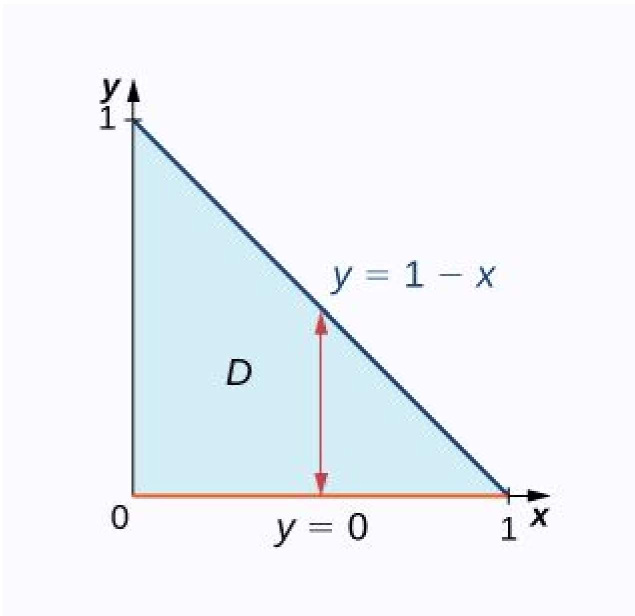
Ans:$\frac{4}{3}$

EXAMPLES(Sketch the region and Evaluation)

1. Evaluate $\iiint\limits_D \frac{dz\ dy\ dx}{(1+x+y+z)^3}$ where D is a solid region bounded by the lines $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

   Solution:
   Here region is bounded by coordinate axes and a plane $x + y + z = 1$.These planes intersect oneanother in three slanted planes $x+y = 1$, $y+z = 1$ and $x+z = 1$ respectively. $\therefore$ region is shown by shadded portion in the figure as follows





Considering imaginary strip parallel to z axis in the solid and correspondingly a strip parallel to y axis in the region in xy plane, limits of integration are $0 \le x \le 1$,$0 \ \le y \le 1 - x$ and $0 \ \le z \le 1 - x - y$

$\therefore$ required Integral is

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\ dy\ dx}{(1+x+y+z)^3}$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{-1}{2(1+x+y+z)^2}\right]_0^{1-x-y} dy\ dx$$

$$= \int_0^1 \int_0^{1-x} \frac{-1}{2}\left[\frac{1}{4} - \frac{-1}{(1+x+y)^2}\right] dy\ dx$$

$$= \int_0^1 \frac{-1}{2}\left[\frac{y}{4} + \frac{1}{(1+x+y)}\right]_1^{1-x} dx$$

$$= \int_0^1 \frac{-1}{2}\left[\frac{1}{4}(1-x) + \left(1 - \frac{1}{1+x}\right)\right] dx$$

$$= \int_0^1 \frac{-1}{2}\left[\frac{5}{4} - \frac{x}{4} - \frac{1}{1+x}\right] dx$$

$$= \frac{-1}{2}\left[\frac{5}{4}x - \frac{x^2}{8} - log(1+x)\right]_0^1$$

$$= \frac{-1}{2}\left[\frac{5}{4} - \frac{1}{8} - log(2)\right]$$

$$= \frac{1}{2}log(2) - \frac{9}{16} ....Ans$$

2. Evaluate $\iiint\limits_D dz\ dy\ dx$ where D is enclosed by surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

Solution:

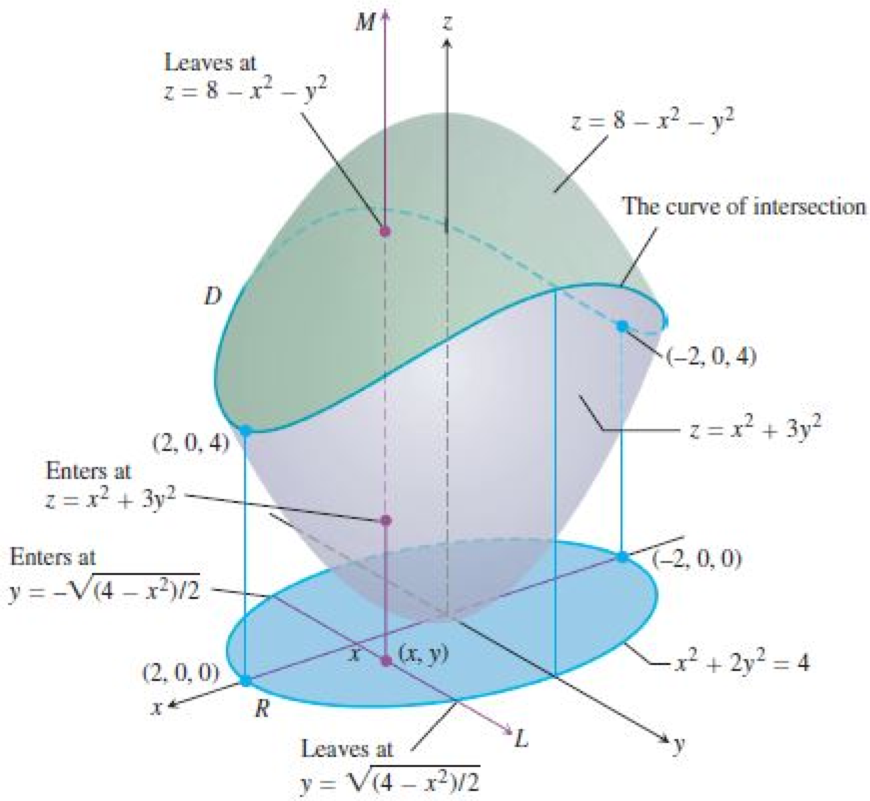
Here to evaluate the integral we first find the intersecting curve of two surfaces whose verical projection will give our region in xy palne.

Now

$x^2 + 3y^2 = 8 - x^2 - y^2 \implies x^2 + 2y^2 = 4$ which is an elliptic cylinder.

$\therefore$ our region in xy plane will be an ellipse with same equation $x^2 + 2y^2 = 4$

The solid region is shown as follows:



for z limits of integration consider imaginary strip parallel to z axis in the solid , strip enters at $z = x^2+3y^2$and leaves at $z = 8-x^2-y^2$ For y limits of integrationncorrespondingly consider a strip parallel to y axis in the ellipse $x^2 + 2y^2 = 4$ in xy plane enters at $y = -\sqrt{\frac{4-x^2}{2}}$ and leaves at $y = \sqrt{\frac{4-x^2}{2}}$ and correspondingly x limits varies from $x = -2$ to $x = 2$ hence limits of integration are

$$-2 \leq x \leq 2$$

,

$$-\sqrt{\frac{4-x^2}{2}} \;\leq y \leq \sqrt{\frac{4-x^2}{2}}$$

and

$$x^2+3y^2 \;\leq z \leq 8-x^2-y^2$$

$\therefore$ required Integral is

$$I = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\ dy\ dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \left(8 - 2x^2 - 4y^2\right)\ dy\ dx$$

$$= \int_{-2}^{2} \left[(8-2x^2)y - \frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\ dx$$

$$= \int_{-2}^{2} \left[2(8-2x^2)\left(\frac{4-x^2}{2}\right) - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}}\right]\ dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}}\right]\ dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} \left(4-x^2\right)^{\frac{3}{2}}\ dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} \left(4-x^2\right)^{\frac{3}{2}}\ dx$$

$$= \frac{4\sqrt{2}}{3} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(4 - 4sin^2u\right)^{\frac{3}{2}}\ 2\ cosu\ du$$

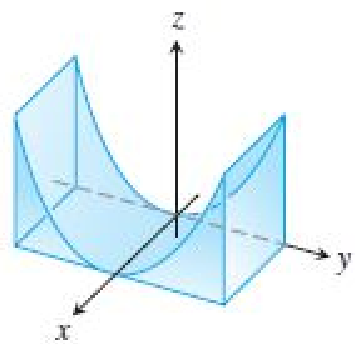
$$= \frac{64\sqrt{2}}{3} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos^4 u \; du$$

$$= \frac{64\sqrt{2}}{3} \; \beta\left(\frac{5}{2}, \frac{1}{2}\right)$$

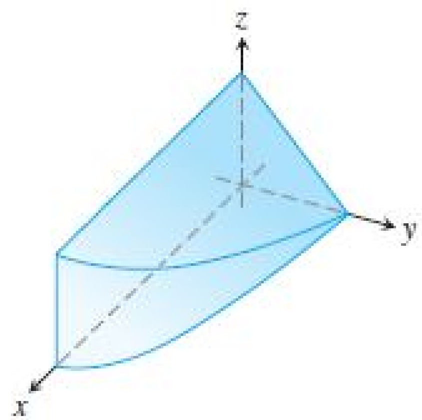
$$. = \frac{64\sqrt{2}}{3} \; \frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= 8\sqrt{2}\pi........Ans$$

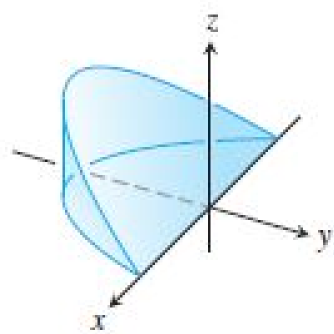
3. Evaluate $\iiint\limits_D \; dz \; dy \; dx$ where D is a solid region bounded by the cylinder $z = y^2$and in xy plane by the planes $x = 0$, $x = 1$, $y = 1$,$y = -1$
   Ans:$\frac{2}{3}$



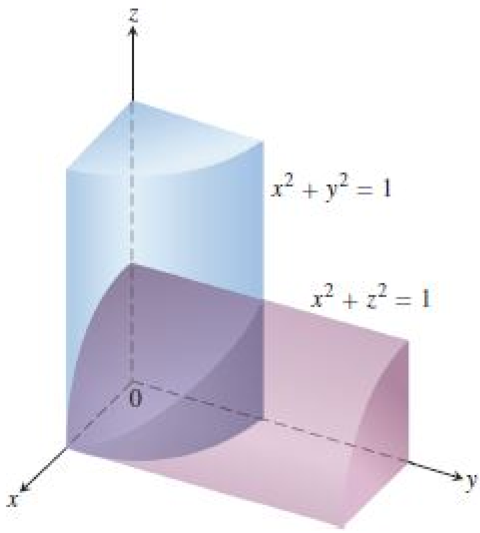
4. Evaluate $\iiint\limits_D \; dz \; dy \; dx$ where D is a solid region in first octant bounded by coordinate plane, the plane $y + z = 2$ and a cylinder $x = 4 - y^2$
   Ans:$\frac{20}{3}$



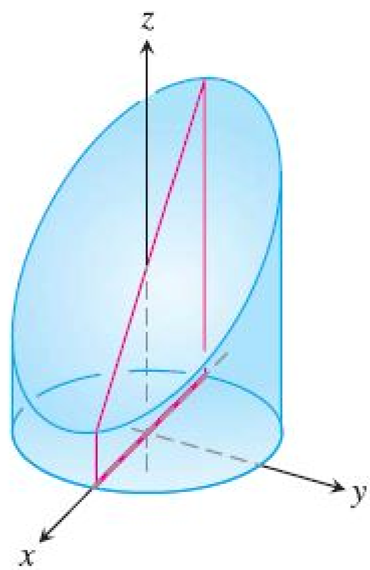
5. Evaluate $\iiint\limits_D \, dz\, dy\, dx$ where D is a wedge cut from the cylinder$x^2+y^2=1$ by the planes $z=-y$ and $z=1$
Ans:$\frac{2}{3}$



6. Evaluate $\iiint\limits_D \, dz\ dy\ dx$ where D is the region common to the interior of the cylinders$x^2+y^2=1$ and $x^2+z^2=1$
Ans:$\frac{16}{3}$



7. Evaluate $\iiint\limits_D dz\ dy\ dx$ where D is the region cut from the cylinder $x^2 + y^2 = 4$ planes $z = 0$ and $x + z = 3$
Ans:12π

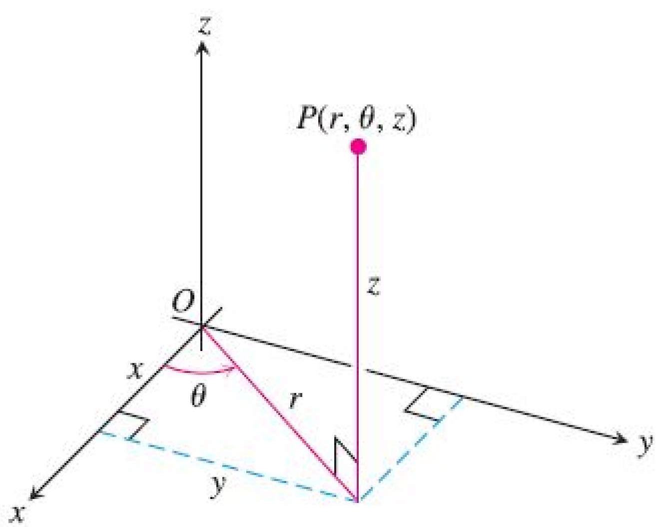


Triple Integrals in cylindrical and spherical polar coordinates

When calculation involves a solid region in the shape of cylinder, cone or spere we can often simplify our work by using cylindrical or spherical polar coordinates.

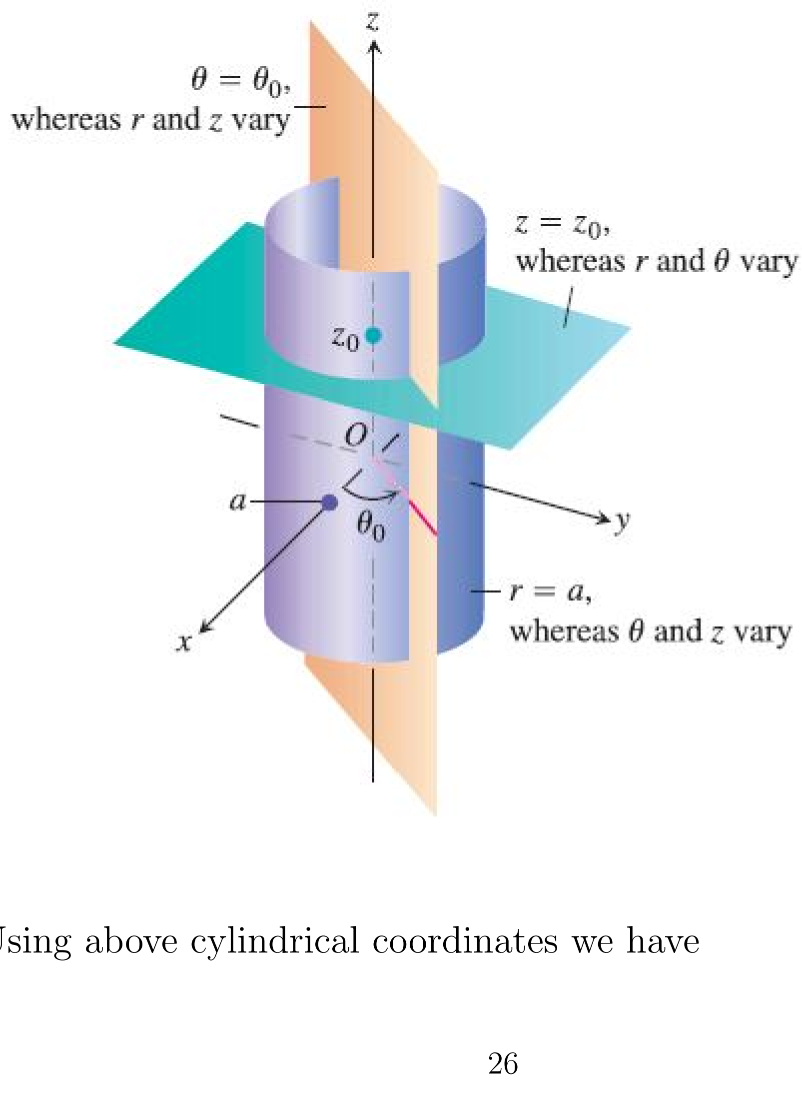
Triple Integrals in cylindrical polar coordinates

For a solid region in space,we obtain its cylindrical polar coordinates by combinding polar coordinates in xy plane with usual z axis. This assigns to every point in the space one or more coordinate triples of the form (r, θ, z) as shown in the figure.



Cylindrical coordinates represent a point P in a space by ordered triplets (r, θ, z) in which r and θ are the polar coordinates of vertical projection of P in xy plane and z is the rectangular vertical coordinate. The rectangular coordinates(x, y, z) and cylindrical coordinates (r, θ, z) are related by usual equations
$x = rcos\ \theta$

$y = r sin\ \theta$
$z = z$
$\therefore x^2 + y^2 = r^2$ and $tan\ \theta = \frac{y}{x}$
In cylindrical coordinates the equation $x^2 + y^2 = r^2$ not just describe the circle in xy plane but a cylinder about z axis whose base is in xy plane.In the following figure the equation of z axis is $r = 0$; the equation $\theta = \theta_0$ represents a plane which contains z axis and makes an angle θ_0 with positive x axis and in rectangular coordinates plane $z = z_0$ describes a plane perpendicular to z axis.



Using above cylindrical coordinates we have

$dx\ dy\ dz\ = dz\ J\left(\frac{x,y,z}{r,\theta,z}\right)\ dr\ d\theta$ where Jacobian of (x, y, z) with respect to (r, θ, z) is given by

$$J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta & 0 \\ sin\theta & rcos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$\therefore$ In cylindrical polar coordinates triple integral over a region D for a function f is defined as

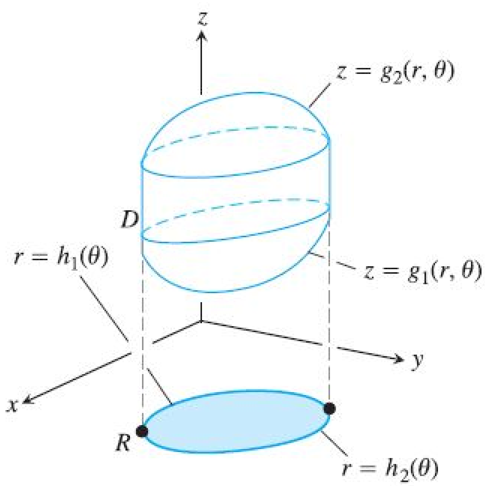
$$\begin{aligned} I &= \iiint\limits_D f\ dv \\ &= \iiint\limits_D f\ dz\ r\ dr\ d\theta \end{aligned}$$

Or we can also define triple integral in cylindrical polar cordinates over a solid region D for a function f by partitoning D into small cylindrical wedges of volume $\Delta v = \Delta z\ r\ \Delta r\ \Delta\theta$ in the same manner as in cartesian.

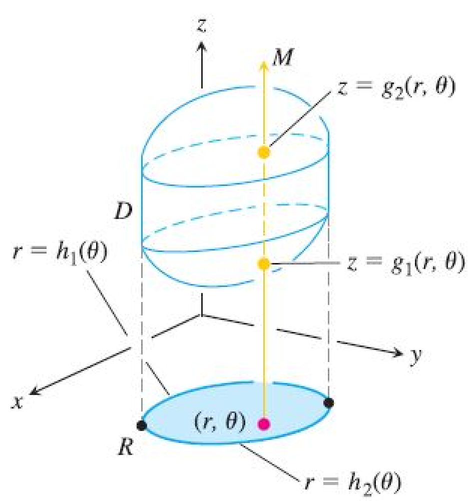
Limits in Cylindrical Polar Coordinates

To evaluate $\iiint\limits_D f(r, \theta, z)\ dv$ in cylindrical coordinates

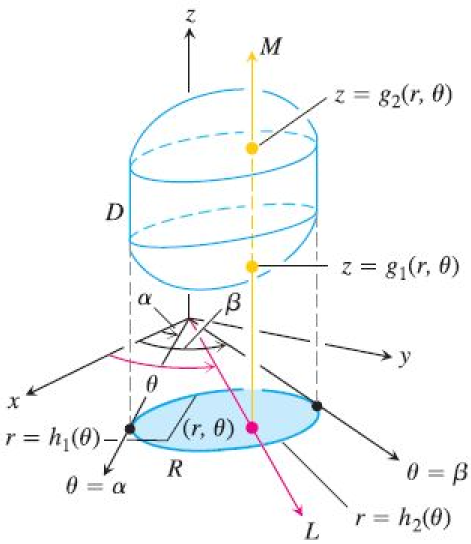
First sketch the region D along with its projection R in xy plane. label the surfaces and curves that bounds D and R



for z limits draw a line through typical point (r, θ) in region R parallel to z axis enteres the region D at $z = g_1(r, \theta)$ and leaves the region D at $z = g_2(r, \theta)$



For r limits a ray through (r, θ) in R from the origin enters the region R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$



For θ limits the ray through (r, θ) making an angle with positive x axis runs from $\theta = \alpha$ to $\theta = \beta$

$\therefore$ required integral is

$$I = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z) dz \ r \ dr \ d\theta$$

EXAMPLES:(Type I)

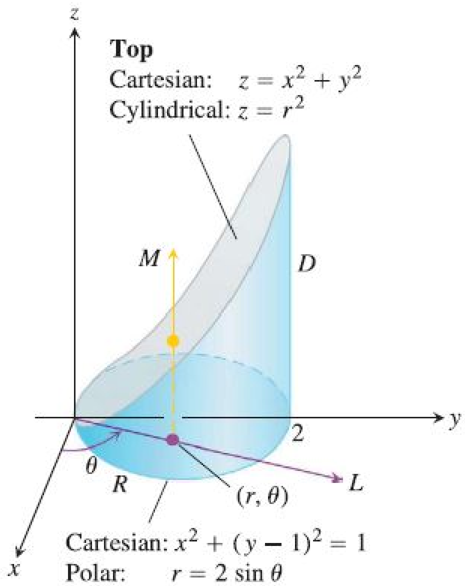
1. $\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz\ r\ dr\ d\theta$

2. $\int_0^{2\pi} \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[r^2 sin^2(\theta) d + z^2\right] dz\ r\ dr\ d\theta$

EXAMPLES:(Type II)

1. Find limits in cylindrical coordinates for intrgration a function $f(r, \theta, z)$ over a region D bounded below by plane $z = 0$ laterally by circular cylinder $x^2 + (y-1)^2 = 1$ and above by paraboloid $z = x^2 + y^2$

   Solution:
   Here base of the circular cylinder is a region in xy plane which is also a vertical projection of solid in xy plane.



Now considering cylindrical coordinates $x = rcos\ \theta$, $y = rsin\ \theta$ and $z = z$ equation of cylinder in polar coordinates is

$$\begin{aligned}
&x^2 + (y-1)^2 = 1 \\
&\implies x^2 + y^2 - 2y + 1 = 1 \\
&\implies r^2 = 2rsin\ \theta \\
&\implies r = 2sin\ \theta
\end{aligned}$$

To find limits of integration

for z limits a line through typical point (r, θ) in region R parallel to z axis enteres the region D at $z = 0$ and leaves the region D at $z = x^2 + y^2 = r^2$

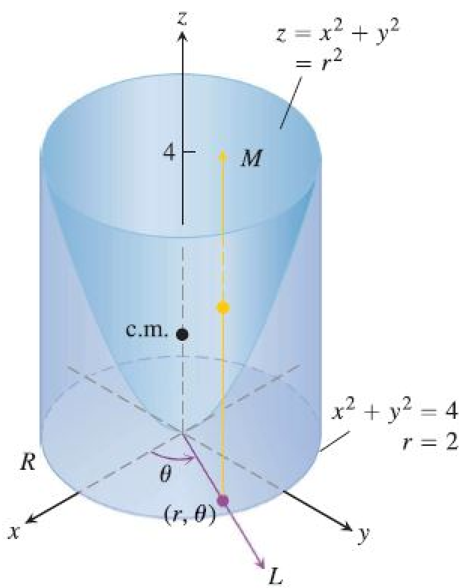
For r limits a ray through (r, θ) from the origin enters the region r at $r = 0$ and leaves at $r = 2sin\ \theta$
For θ limits the ray through (r, θ) making an angle with positive x axis runs from $\theta = 0$ to $\theta = \pi$

$\therefore$ required integral is

$$I = \int_0^{\pi} \int_0^{2sin\ \theta} \int_0^{r^2} f(r, \theta, z) dz\ r\ dr\ d\theta$$

2. Evaluate $\iiint_D dv$ over a region D enclosed by cylinder $x^2 + y^2 = 4$, bounded above by paraboloid $z = x^2 + y^2$ and below by the xy plane.

   Solution:
   The solid region$D$ along with its projection R is shown in the figure.



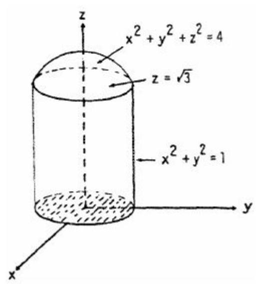
Now considering cylindrical coordinates $x = rcos\ \theta$, $y = rsin\ \theta$ and $z = z$ equation of cylinder in polar coordinates is

$$
\begin{aligned}
&x^2 + y^2 = 4 \\
&\implies x^2 + y^2 = 4 \\
&\implies r^2 = 4 \\
&\implies r = 2
\end{aligned}
$$

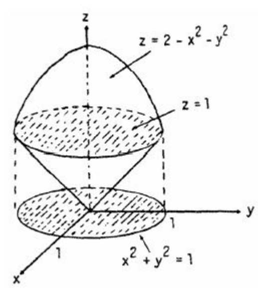
and equation of the paraboloid$z = x^2 + y^2$ is $z = r^2$
$\therefore$ Limits of integration are
$0 \leq \theta \leq 2\pi$; $0 \leq r \leq 2$ and $0 \leq z \leq r^2$
$\therefore$ required integral is

$$
\begin{aligned}
I &= \int_0^{2\pi}\int_0^{2}\int_0^{r^2} dz\ r\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^{2} [z]_0^{r^2}\ r\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^{2} r^3\ dr\ d\theta \\
&= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2\ d\theta \\
&= \int_0^{2\pi} 4d\theta \\
&= 4(2\pi - 0) \\
&= 8\pi......Ans
\end{aligned}
$$

3. Evaluate $\iiint_D dv$ over a region D bounded above by the sphere $x^2+y^2+z^2=4$, on the sides by the cylinder $x^2+y^2=1$, bounded below by xy plane.



4. Evaluate $\iiint_D dv$ over a region D bounded above by the paraboloid $z = 2 - x^2 - y^2$ and bounded below by the cone $z^2 = x^2 + y^2$



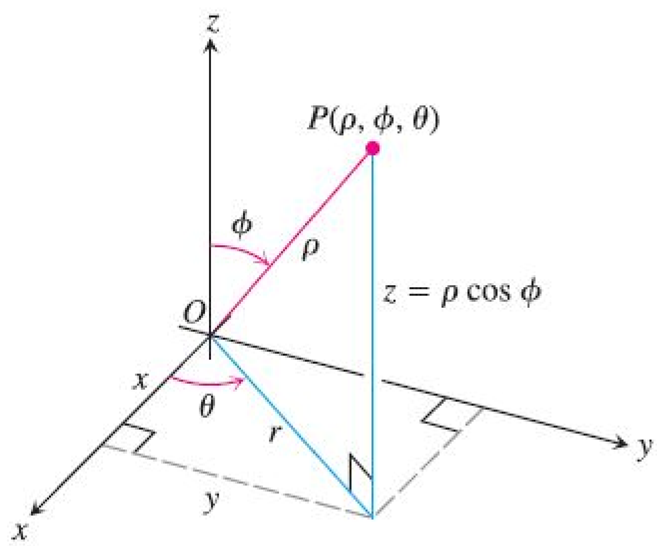
5. Evaluate $\iiint_D xyz\ dv$ over a region D bounded by planes $x = 0$, $y = 0$, $z = 0$, $z = 1$ and a cylinder $x^2 + y^2 = 1$

6. Evaluate $\iiint_D z^2\ dv$ where d is the surface common to sphere$x^2 + y^2 + z^2 = a^2$ and a cylinder $x^2 + y^2 = ax$

7. Evaluate $\iiint_D x^2\ dv$ over a region D bounded above by by the cone $z^2 = 4x^2 + 4y^2$, on the sides by the cylinder $x^2 + y^2 = 1$ and below by the plane $z = 0$

Triple Integrals in Spherical polar coordinates

For a solid region in space,spherical coordinates locate the point in space with two angles and one distance as shown in the figure.



Spherical coordinates represents a point P in space by ordered triplets(ρ, ϕ, θ) where ρ ,is a distance of point P from the origin $(\rho \geq 0)$, ϕ , is the angle that ray OP makes with positive z axis $(0 \leq \phi \leq \pi)$ and θ is the angle from cylindrical coordinates.
The rectangular coordinates(x, y, z) and spherical coordinates (ρ, ϕ, θ) are related by usual equations
$x = \rho \ \ sin\ \phi\ cos\ \theta$
$y = \rho \ \ sin\ \phi\ sin\ \theta$
$z = \rho\ cos\ \phi$
$\therefore$ Using above spherical coordinates we have
$dx\ dy\ dz\ = J\left(\frac{x,y,z}{\rho,\phi,\theta}\right) d\ \rho\ d\ \phi\ d\theta$ where Jacobian of (x, y, z) with respect to (ρ, ϕ, θ) is given by

$$J\left(\frac{x,y,z}{\rho,\phi,\theta}\right)=\begin{vmatrix}x_\rho & x_\phi & x_\theta\\ y_\rho & y_\phi & y_\theta\\ z_\rho & z_\phi & z_\theta\end{vmatrix}=\begin{vmatrix}sin\phi\ cos\theta & -r\ sin\phi\ sin\theta & \rho\ cos\phi\ cos\theta\\ sin\phi\ sin\theta & r\ sin\phi cos\theta & \rho\ cos\phi\ cos\theta\\ cos\phi & 0 & -\rho\ sin\phi\end{vmatrix}=\rho^2 sin\phi$$

$\therefore$ In spherical polar coordinates triple integral over a region D for a function f is defined as

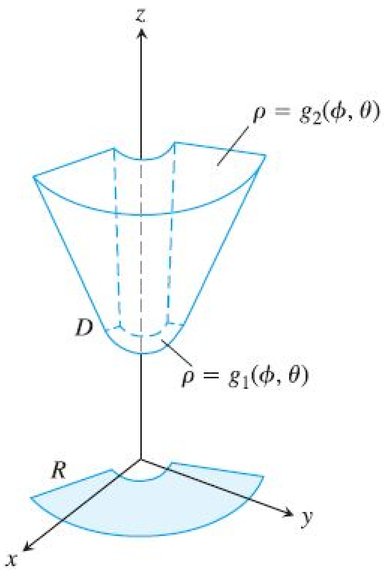
$$\begin{aligned}I &= \iiint\limits_D f\ dv\\ &= \iiint\limits_D f\ \rho^2 sin\phi\ d\rho\ d\phi\ d\theta\end{aligned}$$

Or we can also define triple integral in spherical polar cordinates over a solid region D for a function f by partitoning D into small spherical wedges of volume $\Delta v=\rho^2 sin\phi\ \Delta\rho\ \Delta\phi\ \Delta\theta$ in the same manner as in cartesian.

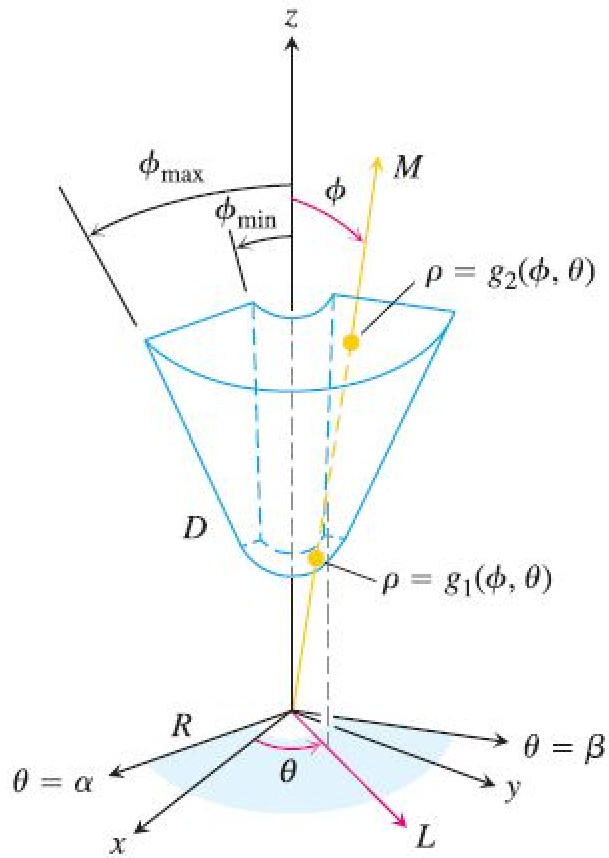
Limits in Spherical coordinates

To evaluate $\iiint\limits_d f(\rho,\phi,\theta)\ dv$ in spherical coordinates

First sketch the region D along with its projection in xy plane. label the surfaces that bound D



for ρ limits draw a ray through D from the origin making an angle ϕ with positive z axis and also a ray in the Projection in xy plane making an agle θ with positive x axis.The ray through D enters at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$



For ϕ limits , for any θ,an angle ϕ that ray through D makes with z axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$

For θ limits the ray in projection in xy runs from $\theta = \alpha$ to $\theta = \beta$
$\therefore$ required integral is

$$I = \int\limits_{\alpha}^{\beta} \int\limits_{\phi_{min}}^{\phi_{max}} \int\limits_{g_1(\phi,\theta)}^{g_2(\phi,\theta)} f(\rho\ ,\phi\ ,\theta)\ \rho^2 sin\phi\ d\rho\ d\phi\ d\theta$$

EXAMPLES:(Type I)

1. Find spherical coordinate equation for
   1) $x^2 + y^2 + (z-1)^2 = 1$
   2) $z = \sqrt{x^2 + y^2}$

   Solution:
   1) Considering spherical polar coordinates
   $x = \rho \;\; sin\ \phi\ cos\ \theta$
   $y = \rho \;\; sin\ \phi\ sin\ \theta$
   $z = \rho\ cos\ \phi$
   the equation of sphere is given by

$$
\begin{aligned}
& x^2 + y^2 + (z-1)^2 = 1 \\
& \implies \rho^2 sin^2\phi\ cos^2\theta + \rho^2 sin^2\phi\ sin^2\theta + (\rho\ cos\phi - 1)^2 = 1 \\
& \implies \rho^2 sin^2\phi \left(cos^2\theta + sin^2\theta\right) + \rho^2 cos^2\phi - 2\rho\ cos\phi + 1 = 1 \\
& \implies \rho^2 sin^2\phi + \rho^2 cos^2\phi - 2\rho\ cos\phi + 1 = 1 \\
& \implies \rho^2 \left(sin^2\phi + cos^2\phi\right) = 2\rho\ cos\phi \\
& \implies \rho^2 = 2\rho\ cos\phi \\
& \implies \rho = 2\ cos\phi
\end{aligned}
$$

   2)Considering spherical polar coordinates
   $x = \rho \;\; sin\ \phi\ cos\ \theta$
   $y = \rho \;\; sin\ \phi\ sin\ \theta$
   $z = \rho\ cos\ \phi$
   the equation of cone is given by

$$
\begin{aligned}
& z = \sqrt{x^2+y^2} \\
& \implies \rho\ cos\phi = \sqrt{\rho^2 sin^2\phi\ cos^2\theta + \rho^2 sin^2\phi\ sin^2\theta} \\
& \implies \rho\ cos\phi = \sqrt{\rho^2 sin^2\phi\ \ (cos^2\theta + sin^2\theta)} \\
& \implies \rho\ cos\phi = \sqrt{\rho^2 sin^2\phi} \\
& \implies \rho\ cos\phi = \rho\ sin\phi \\
& \implies tan\phi = 1 \\
& \implies \phi = \frac{\pi}{4}
\end{aligned}
$$

EXAMPLES:(Type II)

1. $\int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{4}} \int\limits_0^{2} \left[\rho^2 cos\ \phi\ \rho^2\ sin(\phi)\right]\ d\rho\ d\phi\ d\theta$

2. $\int\limits_0^{2\pi} \int\limits_0^{\frac{\pi}{3}} \int\limits_{sec\phi}^{2} \left[3\ \rho^2\ sin(\phi)\ d\right]\ d\rho\ d\phi\ d\theta$

EXAMPLES:(Type III)

1. Using spherical polar coordinates evaluate $\iiint\ xyz\ (x^2+y^2+z^2)\ dx\ dy\ dz$ over a first octant of the sphere $x^2+y^2+z^2=a^2$

   Solution:
   Considering spherical polar coordinates
   $x = \rho\ \ sin\ \phi\ cos\ \theta$
   $y = \rho\ \ sin\ \phi\ sin\ \theta$

$z = \rho \ cos \ \phi$
the equation of sphere is given by

$$
\begin{aligned}
& x^2 + y^2 + z^2 = a^2 \\
& \implies \rho^2 sin^2\phi \ cos^2\theta + \rho^2 sin^2\phi \ sin^2\theta + \rho^2 cos\phi = a^2 \\
& \implies \rho^2 sin^2\phi \left(cos^2\theta + sin^2\theta\right) + \rho^2 cos^2\phi = a^2 \\
& \implies \rho^2 sin^2\phi + \rho^2 cos^2\phi = a^2 \\
& \implies \rho^2 \left(sin^2\phi + cos^2\phi\right) = a^2 \\
& \implies \rho^2 = a^2 \\
& \implies \rho = a
\end{aligned}
$$

Given region of integration is first octant only. limits of integration are
$0 \ \leq \rho \ \leq a$
$0 \ \leq \phi \ \leq \frac{\pi}{2}$
$0 \ \leq \theta \ \leq \frac{\pi}{2}$

$\therefore$ required integral is

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{a} (\rho^3 \ sin^2\phi \ cos \ \phi \ cos \ \theta \ sin\theta)(\rho^2) \ \rho^2 \ sin\phi \ d\rho \ d\phi \ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{a} (\rho^7 \ sin^3\phi \ cos \ \phi \ cos \ \theta \ sin\theta) \ d\rho \ d\phi \ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^8}{8}\right]_0^a \ sin^3\phi \ cos \ \phi \ cos \ \theta \ sin\theta \ d\phi \ d\theta$$

$$= \frac{a^8}{8} \int_0^{\frac{\pi}{2}} cos \ \theta \ sin\theta \ d\theta \int_0^{\frac{\pi}{2}} sin^3\phi \ cos \ \phi \ d\phi$$

$$= \frac{a^8}{8} \ \frac{1}{2} \ \beta(1,1) \ \frac{1}{2} \ \beta(2,1)$$

$$= \frac{a^8}{8} \ \frac{1}{2} \ \frac{\Gamma1 \ \Gamma1}{\Gamma2} \ \frac{1}{2} \ \frac{\Gamma2 \ \Gamma1}{\Gamma3}$$

$$= \frac{a^8}{64} ....Ans$$

2. Using spherical polar coordinates evaluate $\iiint \frac{dx \ dy \ dz}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ over a solid bounded by spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ where $b > a > 0$

3. Using spherical polar coordinates evaluate $\iiint \ dx \ dy \ dz$ over a

region D which is a sphere $x^2 + y^2 + z^2 = a^2$

4. Evaluate $\iiint_D \, dx \, dy \, dz$ where D is region in first octant bounded by coordinate planes, a plane $y + z = 2$ and a cylinder $x = 4 - y^2$

5. Evaluate $\iiint_D \, dx \, dy \, dz$ where D is region of a sphere $x^2+y^2+z^2 = a^2$ cut by the cone $z^2 = x^2 + y^2$

6. Evaluate $\iiint_D \, dx \, dy \, dz$ where D is region enclosed by the cone $z^2 = x^2 + y^2$ and parabola $z = x^2 + y^2$

7. Evaluate $\iiint_D \, dv$ where D is region bounded above by the cone $z^2 = x^2 + y^2$ and below by the sphere $\rho = 2 \; cos \; \phi$

8. Evaluate $\iiint_D \, dv$ where D is region bounded above by the cone $\phi = \frac{\pi}{3}$, on the sides by the sphere $\rho = 2$ and below by xy plane

SUMMERY

(1) Conversion from Cartesian to polar using double integration

Steps:

1. Draw the region in cartesian plane (For morethan two curves find intersectiong points to sketch the region).

2. Find equation of each bounded curve of the region in polar plane using polar coordinates $x = r\ cos\ \theta$ and $y = r\ sin\ \theta$ and substitute $dx\ dy = r\ dr\ d\theta$

3. Consider a ray from the pole(origin)entering and leaving the region, to decide r limits of integration

4. Move the ray in counterclockwise rotation in the region to decide θ limits of integration.

(2) Conversion from Cartesian to Cylindrical polar using Triple integration

Steps:

1. Draw the solid region in space along with its vertical projection in opposite plane.(Find intersecting points to trace the region)

2. Find equation of each bounded curve of the region using cylindrical polar coordinates $x = r\ cos\ \theta$, $y = r\ sin\ \theta$ and $z = z$ and

substitute $dx\ dy\ dz = dz\ r\ dr\ d\theta$

3. Consider a line from the the point in the region in opposite plane, parallel to z axis entering and leaving the solid Region to decide z limits of integration

4. Consider a ray from the pole(origin)entering and leaving the region in opposite plane, to decide r limits of integration

5. Move the ray in counterclockwise rotation in the region to decide θ limits of integration.

(3) Conversion from Cartesian to Spherical polar using Triple integration

Steps:

1. Draw the solid region in space along with its vertical projection in opposite plane.(Find intersecting points to trace the region)

2. Find equation of each bounded curve of the region using spherical polar coordinates $x = \rho\ sin\ \phi\ cos\ \theta$, $y = \rho\ sin\ \phi\ sin\ \theta$ and $z = \rho\ cos\ \phi$ and substitute $dx\ dy\ dz = \rho^2\ sin\ \phi\ d\rho\ d\phi\ d\theta$

3. Consider a ray from the the origin in the region making an angle ϕ with positive z axis entering and leaving the solid Region to decide ρ limits of integration

4. Above ray from the pole(origin) making and angle ϕ with positive z axis runs from ϕ_{min} to ϕ_{max}

5. Move the ray in counterclockwise rotation in the region in xy plane to decide θ limits of integration.