

The following table summarizes methods known to me.

Name	Original Recurrence	Sufficient Condition of Applicability	Original Complexity	Optimized Complexity	Links
Convex Hull Optimization1	$dp[i] = \min_{j < i} \{dp[j] + b[j] * a[i]\}$	$b[j] \geq b[j + 1]$ optionally $a[i] \leq a[i + 1]$	$O(n^2)$	$O(n)$	<a href="#">1</a> <a href="#">2</a> <a href="#">3</a> <a href="#">p1</a>
Convex Hull Optimization2	$dp[i][j] = \min_{k < j} \{dp[i - 1][k] + b[k] * a[j]\}$	$b[k] \geq b[k + 1]$ optionally $a[j] \leq a[j + 1]$	$O(kn^2)$	$O(kn)$	<a href="#">1</a> <a href="#">p1</a> <a href="#">p2</a>
Divide and Conquer Optimization	$dp[i][j] = \min_{k < j} \{dp[i - 1][k] + C[k][j]\}$	$A[i][j] \leq A[i][j + 1]$	$O(kn^2)$	$O(kn \log n)$	<a href="#">1</a> <a href="#">p1</a>
Knuth Optimization	$dp[i][j] = \min_{i < k < j} \{dp[i][k] + dp[k][j]\} + C[i][j]$	$A[i, j - 1] \leq A[i, j] \leq A[i + 1, j]$	$O(n^3)$	$O(n^2)$	<a href="#">1</a> <a href="#">2</a> <a href="#">p1</a>

Notes:

- $A[i][j]$  — the smallest  $k$  that gives optimal answer, for example in  $dp[i][j] = dp[i - 1][k] + C[k][j]$
- $C[i][j]$  — some given cost function
- We can generalize a bit in the following way:  $dp[i] = \min_{j < i} \{F[j] + b[j] * a[i]\}$ , where  $F[j]$  is computed from  $dp[j]$  in constant time.
- It looks like **Convex Hull Optimization2** is a special case of **Divide and Conquer Optimization**.
- It is claimed (in the references) that **Knuth Optimization** is applicable if  $C[i][j]$  satisfies the following 2 conditions:
  - **quadrangle inequality**:  $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$ ,  $a \leq b \leq c \leq d$
  - **monotonicity**:  $C[b][c] \leq C[a][d]$ ,  $a \leq b \leq c \leq d$
- It is claimed (in the references) that the recurrence  $dp[j] = \min_{i < j} \{dp[i] + C[i][j]\}$  can be solved in  $O(n \log n)$  (and even  $O(n)$ ) if  $C[i][j]$  satisfies **quadrangle inequality**. [YuukaKazami](#) [described](#) how to solve some case of this problem.