场论与凝聚态笔记

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目录

1 Invitation: The Cartoon of Confinement

Never see individual quarks.

For separatable particles, like electron charges, their potential is $V(r) \sim \frac{1}{r}$, thus $V(r) - V(r_0)$ is always bounded.

Two quarks forms a pion. They interacts through gluon, and forms a structure called gluon tube or string. The potential is $V(r) \sim r$ and the energy density per length is appropriately constant. To separate a quark pair, the energy inputed $V(r) - V(r_0)$ is unbounded.

Similar phenomenon appears in superconductor(type II). When electronic charge condensed, the interaction of magneticmonopole becomes $V(r) \sim r$. According to EM duality, when magneticmonopole condensed, analogy goes to its counterpart. (Perspective by t'Hooft, Polyakov and Manldstan.)

2 Path Integral for Single Particles

From the two-slit interference, we've known the picture of wave. Whilst, the view of particle could recover the result by computing the phase $\mathrm{e}^{\mathrm{i}S}$.

For single particle mechanic, we start from the Schrödinger equation

$$\mathrm{i}\hbar\partial_t \left|\psi(t)\right\rangle = H(p,x,t)\left|\psi(t)\right\rangle . \tag{2.1}$$

We have the time evolution operator

$$\left|\psi(t)\right\rangle = U(t,t_0)\left|\psi(t)\right\rangle , \tag{2.2}$$

which is unitary,

$$U^\dagger(t,t_0)U(t,t_0) = 1. \tag{2.3}$$

If H is time-dependent, we split the time interval into small slices, and we get the infinitesimal U operator as time-independent cases,

$$U(t,t_0) = \prod_{n=0}^{N-1} U(\overbrace{t_{n-1},t_n}^{\delta t}), \tag{2.4}$$

where $t_n \equiv t + n\delta t$. Note that the product is time ordered.

Another perspective is from the Schrödinger equation, by finite differential,

$$\left|\psi(t+\delta t)\right\rangle = \left[1 - \frac{\mathrm{i}}{\hbar}H(p,x,t+\frac{\delta t}{2})\right]\left|\psi(t)\right\rangle \tag{2.5}$$

To the order of δt, we have

$$\left|\psi(t+\delta t)\right\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(p,x,t+\frac{\delta t}{2})}\left|\psi(t)\right\rangle \tag{2.6}$$

Next, we put the time evolution operator in spacial basis, considering

$$\left\langle x'\right| U(t+\delta t,t)\left|x\right\rangle . \tag{2.7}$$

Suppose $H = \frac{p^2}{2m} + V(x)$ for simplicity, we obtain

$$\langle x'| \left[1 - \frac{\mathrm{i}\delta t}{\hbar}\left(\frac{p^2}{2m} + V(x, t + \frac{\delta t}{2})\right)\right] |x\rangle . \tag{2.8}$$

Make a substitution

$$V \to \frac{V(x', t + \frac{\delta t}{2}) + V(x, t + \frac{\delta t}{2})}{2} 1, \tag{2.9}$$

and insert a completeness relation of p in each time slice, we arrive at

$$\langle x'| U(t + \delta t, t) |x\rangle = \int \mathrm{d}p \, \frac{1}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}p(x'-x)}{\hbar}} \exp\left[-\frac{i\delta t}{\hbar} \frac{H(p, x', t + \frac{\delta t}{2}) + H(p, x, t + \frac{\delta t}{2})}{2}\right]. \tag{2.10}$$

written in a more compact form,

$$\langle x'| U(t + \delta t, t) |x\rangle \sim \int \mathrm{d}p \, \frac{1}{2\pi\hbar} \mathrm{e}^{\mathrm{i}\frac{p\delta x}{\hbar} - \mathrm{i}\frac{H\delta t}{\hbar}}. \tag{2.11}$$

The finite-time evolution operator,

$$U(t_N, t_0) = \prod_{n=0}^{N-1} U(t_{n-1}, t_n) \tag{2.12}$$

inserting an identity operator as x basis completeness relation, the element is

$$U(t_{n+2}, t_{n+1}) \underbrace{1}_{\int \mathrm{d}x_n \, |x_n\rangle\langle x_n|} U(t_{n+1}, t_n) \tag{2.13}$$

then we obtain

$$\begin{aligned} \langle x_N| U(t_N, t_0) |x_0\rangle = &\left(\prod_{n=1}^{N-1} \int \mathrm{d}x_n \, \langle x_{n+1}| U(t_{n+1}, t_n) |x_n\rangle\right) \\ &\times \langle x_1| U(t_1, t_0) |x_0\rangle \end{aligned} \tag{2.14}$$

in full,

$$\begin{aligned} \langle x_N| U(t_N, t_0) |x_0\rangle &= \left(\prod_{n=1}^{N-1} \int \frac{\mathrm{d}p_{n+\frac{1}{2}} \, \mathrm{d}x_n}{2\pi\hbar}\right) \int \frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar} \\ \times \exp\left(\frac{\mathrm{i}}{\hbar} \sum_{n=0}^{N-1} \left[p_{n+\frac{1}{2}}(x_{n+1} - x_n) - \delta t \frac{H(p_{n+\frac{1}{2}}, x_{n+1}, t_{n+\frac{1}{2}}) + H(p_{n+\frac{1}{2}}, x_n, t_{n+\frac{1}{2}})}{2}\right]\right) \\ &\sim \left(\prod_{n=0}^{N-1} \int \frac{\mathrm{d}p_{n+\frac{1}{2}} \, \mathrm{d}x_n}{2\pi\hbar}\right) \int \frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int p \, \mathrm{d}x - H \, \mathrm{d}t} \end{aligned} \tag{2.15}$$

Insert a operator $\hat{B}(x)$ in between the path integral,

$$\begin{aligned}\langle x_N| U(t_N,t_m)\hat{B}(x)U(t_m,t_0)|x_0\rangle &= \left(\prod_{n=1}^{N-1}\int\frac{\mathrm{d}p_{n+\frac{1}{2}}\mathrm{d}x_n}{2\pi\hbar}\right)\int\frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar}B(x_m)\\ \times\exp\frac{\mathrm{i}}{\hbar}\sum_{n=0}^{N-1}&\left[p_{\frac{1}{2}}(x_{n-1}-x_n)-\delta t\frac{H(p_{n+\frac{1}{2}},x_{n+1},t_{n+\frac{1}{2}})+H(p_{n+\frac{1}{2}},x_n,t_{n+\frac{1}{2}})}{2}\right].\end{aligned} \tag{2.16}$$

We simply need to add the value of the operator as a function of certain space coordinate in the expression of path integral.

3 Observables in QM

- $$\langle\psi| A|\psi\rangle,\ A^\dagger = A. \tag{3.1}$$

- Probability of projection,

$$P = |\langle\phi|\psi\rangle|^2 = \langle\psi|\underbrace{|\phi\rangle\langle\phi|}_{\hat{A}}|\psi\rangle \tag{3.2}$$

- Observables after evaluation,

$$\langle\phi|\underbrace{\mathrm{e}^{\mathrm{i}Ht/\hbar}A\mathrm{e}^{-iHt/\hbar}}_{\hat{A}(t)}|\psi\rangle \tag{3.3}$$

- Projection after evaluation: scattering

$$P = |\langle\psi|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|\psi\rangle|^2 = \langle\psi|\underbrace{\mathrm{e}^{\mathrm{i}Ht/\hbar}|\phi\rangle\langle\phi|\mathrm{e}^{-iHt/\hbar}}_{\hat{A}(t)}|\psi\rangle \tag{3.4}$$

- Retarded correlation

$$H(t) = H_0 + \underbrace{a(t)}_{\text{small}}B. \tag{3.5}$$

The contribution of $\langle\psi| U^\dagger(t,0)AU(t,0)|\psi\rangle$ to the first order correction in a is

$$-\frac{\mathrm{i}}{\hbar}\int\mathrm{d}t'a(t')\Big[\langle\psi| U_0^\dagger(t,0)AU_0(t,t')BU_0(t',0) - U_0^\dagger(t,0)BU_0(t,t')AU_0(t',0)|\psi\rangle\Big] \tag{3.6}$$

4 Path Integrals for Fields

There's a mattress with springs and massive balls connected. Denoting the offset of each ball as $\phi_{\vec{r}}$ the Hamiltonian is

$$H = \sum_{\vec{r}\text{ on lattice}} \left[\frac{p_{\vec{r}}^2}{2m} + V(\phi_{\vec{r}}) + \sum_{\hat{\tau}=1}^{d} \frac{k}{2} \left(\phi_{\vec{r}+\alpha\vec{\tau}} - \phi_{\vec{r}}\right)^2 \right], \tag{4.1}$$

with a commutation relation

$$[\phi_{\vec{r}}, p_{\vec{r}'}] = \mathrm{i}\hbar\delta_{\vec{r},\vec{r}'}. \tag{4.2}$$

The interacting part of the Hamiltonian only involves pairs nearby.

In continuum limit, $H = \displaystyle\int \mathrm{d}^d\vec{r}\mathcal{H}(\vec{r})$, and we can write the Hamilton density as

$$\mathcal{H} = \frac{\pi(\vec{r})^2}{2\rho} + \mathcal{V}(\phi(\vec{r})) + \frac{\kappa}{2}\left[\partial_{\vec{r}}\phi(\vec{r})\right]^2, \tag{4.3}$$

where $\pi(\vec{r}) = \dfrac{p_{\vec{r}}}{\alpha^d}, \mathcal{V} = \dfrac{V}{\alpha^d}, \rho = \dfrac{m}{\alpha^d}, \kappa = \dfrac{k\alpha^2}{\alpha^d}$.

Path Integral At t_n, we use $\bigotimes_{\vec{r}} |\phi(\vec{r})\rangle$ basis, and $\bigotimes_{\vec{r}} |\pi(\vec{r})\rangle$ for $t_{n+\frac{1}{2}}$.

Analogously, we get

$$\begin{aligned} \langle\text{end}|\, U(t,0)\, |\text{start}\rangle = &\left(\prod_{t_n,\vec{r}} \frac{\mathrm{d}p_{t_{n+\frac{1}{2}},\vec{r}}\, \mathrm{d}\phi_{t_n,\vec{r}}}{2\pi\hbar}\right)_{\text{suitable boundary condition}} \\ \times \exp\frac{\mathrm{i}}{\hbar}\sum_{t_n,\vec{r}} &\left[p_{t_{n+\frac{1}{2}},\vec{r}}\left(\phi_{t_{n+1},\vec{r}} - \phi_{t_n,\vec{r}}\right) - \delta t\left(\frac{p^2_{n+\frac{1}{2},\vec{r}}}{2m} + \frac{k}{2}\sum_{\hat{\tau}=1}^{d}\left(\phi_{t_n,\vec{r}+\alpha\hat{\tau}} - \phi_{t_n,\vec{r}}\right)^2 + V(\phi_{t_n,\vec{r}}) \right)\right]. \end{aligned} \tag{4.4}$$

Integrate out p, we obtain

$$\begin{aligned} &\left(\prod_{t_n,\vec{r}} \sqrt{\frac{2\pi\hbar m}{\mathrm{i}\delta t}} \frac{\mathrm{d}\phi_{t_n,\vec{r}}}{2\pi\hbar}\right)_{\text{suitable boundary condition}} \\ \times \exp\frac{i\delta}{\hbar}\sum_{t_n,\vec{r}} &\left[\frac{m}{2}\left(\frac{\phi_{t_{n+1},\vec{r}} - \phi_{t_n,\vec{r}}}{\delta t}\right) - \frac{k\alpha^2}{2}\sum_{\hat{\tau}=1}^{d}\left(\frac{\phi_{t_n,\vec{r}+\alpha\hat{\tau}} - \phi_{t_n,\vec{r}}}{\alpha}\right)^2 - V(\phi_{t_n,\vec{r}})\right], \end{aligned} \tag{4.5}$$

in which time and space stand in the same place, similar to relativistic K-G field.

5 Free Field Theory

For free fields, $V(\phi_{\vec{r}}) = \frac{u}{2}\phi_{\vec{r}}^2$, it behaves just like coupled simple harmonic oscillators. To find the normal modes, we use Fourier transformation.

$$\phi_{\vec{r}} = \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\phi_{\vec{k}}, \tag{5.1}$$

likewise for $p_{\vec{r}}$.

$\phi_{\vec{r}}$ is real, thus $\phi_{\vec{r}} = \phi_{\vec{r}}^\dagger \implies \phi_{-\vec{k}} = \phi_{\vec{k}}^\dagger$, and the commutation relation is

$$\left[\phi_{\vec{k}}, p_{\vec{k}'}\right] = \mathrm{i}\hbar(2\pi)^d\delta^d(\vec{k}+\vec{k}'). \tag{5.2}$$

The Hamiltonian becomes

$$H = \int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}} \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\frac{1}{2}\left[\frac{p_{-\vec{k}}p_{\vec{k}}}{2m} + \left(\frac{k}{2}\sum_{\vec{r}=1}^{d}\left(2\sin\frac{\alpha k_i}{2}\right)^2 + \frac{u}{2}\right)\phi_{-\vec{k}}\phi_{\vec{k}}\right], \tag{5.3}$$

from which we can directly figure out the frequency $\omega_{\vec{k}} = \omega_{-\vec{k}} = \sqrt{\frac{k\sum_{\vec{r}}\left(\sin\frac{\alpha k_i}{2}\right)^2 + U}{m}}$

Next, we need to make a substitution, or Bogoliubov transformation:

$$\phi_{\vec{k}}^c = \frac{\phi_{\vec{k}} + \phi_{-\vec{k}}}{\sqrt{2}}, \tag{5.4}$$

$$\phi_{\vec{k}}^s = \mathrm{i}\frac{\phi_{\vec{k}} - \phi_{-\vec{k}}}{\sqrt{2}}. \tag{5.5}$$

the integration range becomes half of $\vec{k}$.

5.1 Ground State Wave Function and Entanglement

"You will gain a full comprehension and admire its subtleties in homework."

5.2 Energy Gap and Correlation

There's no gapless excitations in finite crystal constant lattice theory since the minimal wave number is $\sim \frac{\pi}{a}$

6 Where comes the i0$^+$: a Non-Perturbative Interpretation

Unitarity and locality plays an unreplaceable role in QFT and a lot of deep conclusion can be gained directly from these two properties, without the need of calculation of specific model.

In QFT, we compute something like

$$\langle\phi|\cdots|\psi\rangle\,, \text{ or } \int_{\text{b.c.}}\mathcal{D}(\square)\exp\left(\cdots\right), \tag{6.1}$$

respectively in operator and path integral methodology, and b.c. stands for boundary condition.

In condensed matter, the experiment are done by disturbing a ground state, and the system would decay from excited state to ground state eventually, thus (near $T\sim 0$) what we usually compute is

$$\langle 0|\cdots|0\rangle\,. \tag{6.2}$$

And in particle phys., after performing LSZ reduction, any initial or final state would be turned into creation/annihilation operators acting on a ground state, thus, similarly, we compute

$$\langle\phi|\cdots|\psi\rangle \to \langle 0|\,O^\dagger_\phi\cdots O_\psi\,|0\rangle\,. \tag{6.3}$$

Furthermore, it seems that we need to get the wave function of ground state ,which would be extremely hard in any theory with interaction, however, we do not actually need to do so. The following of this section would explain it in detail.

A typically computation would be in the form of

$$\langle\phi|\,\mathrm{e}^{\mathrm{i}t_f H}\mathrm{e}^{\mathrm{i}t_i H}\,|\psi\rangle\,, \tag{6.4}$$

where the initial and final time are set at $-\infty$ and $+\infty$ for the reason that the time interval of a physical process would be greatly smaller that of an experiment.

Operator A commonly used trick is to make a substitution of $\mathrm{e}^{-\mathrm{i}\Delta t H}\to\mathrm{e}^{-i\Delta t H\left(1-\mathrm{i}0^+\right)}$ in time evolution. [1] Let's check its property in two cases: finite time and infinite time. When Δt is finite, there's nothing abnormal. When $\Delta t\to+\infty$, a factor of exponentially suppress

[1]It is no longer unitary, but that's what we want.

appears, yet except for states whose $H = 0$. This term would effectively become $|0\rangle\langle 0|$ or $\sum_i |0_i\rangle\langle 0_i|$ when $H|0\rangle = 0$.

If, unfortunately, the experiment time isn't literally infinite, the 0^+ have to be

$$0^+ \sim \frac{1}{\text{experiment time}} \quad (\text{or relaxation time } t). \tag{6.5}$$

In Green function, this projection to ground state is included by substitution of

$$\begin{aligned} \omega &\to \left(1 + \mathrm{i}0^+\right)\omega \quad (\text{time ordered}) \\ \omega &\to \omega + \mathrm{i}0^+ \quad (\text{retarded}) \end{aligned}. \tag{6.6}$$

Path Integral After a substitution of $H\delta t \to H\left(1 - \mathrm{i}0^+\right)\delta t$, we obtain

$$\int_{\text{No b.c.}} \mathcal{D}p\,\mathcal{D}x\,\exp\left(\frac{\mathrm{i}}{\hbar}\int\left[p\,\mathrm{d}x - H\left(1 - \mathrm{i}0^+\right)\,\mathrm{d}t\right]\right). \tag{6.7}$$

Wick Rotation In the process given above, we have *rotated* the trajectory of t from the real axis to another line through the original point with a small angle 0^+. A generalization of this trick is to choose a integral contour along the imaginary axis, which is called *Wick rotation*. It can be shown that, in principle, if we know all observables along the $z = -\mathrm{i}\tau$ axis, all observables on real time can be recovered to the spirit of analytical continuation. Under

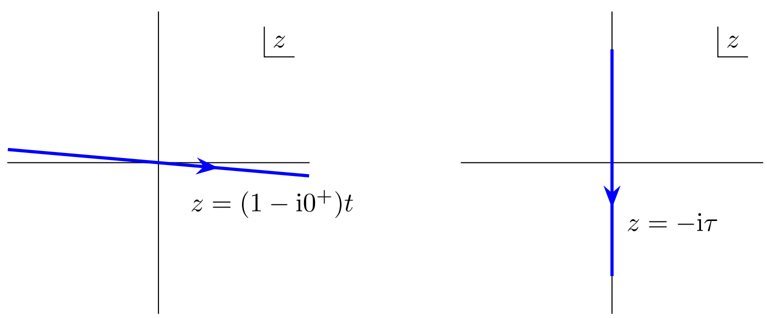


Fig 1. Wick rotation

this, the expression of path integral becomes

$$\int \mathcal{D}\pi\,\mathcal{D}\phi\,\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}\tau\,\mathrm{d}^3r\,(\pi\mathrm{i}\partial_t\phi - \mathcal{H})\right]. \tag{6.8}$$

This formalism is also called *Euclidean field theory* or *imaginary time* path integral.

The name "Euclidean" field theory comes from the fact that after completing the integration on π of a free field, we would find the term on the exponential becomes

$$\mathcal{L}_{\text{Euclidean}} \sim (\partial_t \phi)^2 + (\partial_{\vec{r}} \phi)^2 + \phi^2, \tag{6.9}$$

instead of the Lagrangian density in Minkovski spacetime, whose form is typically

$$\mathcal{L}_{\text{Minkovski}} = (\partial \phi)^2 - m^2 \phi^2 = (\partial_t \phi)^2 - (\partial_{\vec{r}} \phi)^2 - m^2 \phi^2. \tag{6.10}$$

The iconic minus signs from Minkovski metric were eliminated after Wick rotation as if it morphed into a field living in Euclidean space.

7 Dualities of QFT and Stat. Mech.

7.1 Partition Function as Path Integral

In statistic mechanics, we compute the partition functions of fields,

$$\mathcal{Z} = \sum_s \mathrm{e}^{-\beta E_s} = \sum_s \mathrm{e}^{-\beta \int \mathrm{d}\vec{r}\, \mathcal{H}}, \tag{7.1}$$

which is easy to compute in classical statistic mechanics where $\hbar = 0$, i.e., all operators commute.

However, in quantum cases, the calculation of

$$\mathcal{Z} = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \mathrm{Tr}\left(\mathrm{e}^{-\beta \int \mathrm{d}^d \vec{r}\, \mathcal{H}}\right) \tag{7.2}$$

becomes impossible, because energy eigenstates are non-local and the eigenvalues of Hamiltonian are hard to find. To solve this, we consider the $\mathrm{e}^{-\beta H}$ as a product of a series of $\mathrm{e}^{-\beta \epsilon}$. By analogy, we insert a completeness relation of momenta and field operators in between each of them. Whereby, we get

$$\mathcal{Z} = \prod_{\vec{r},t} \left(\frac{1}{2\pi\hbar} \int \mathrm{d}\pi_{\tau+\frac{\delta\tau}{2},\vec{r}}\, \mathrm{d}\phi_{\tau,\vec{r}} \right) \exp\left[\frac{1}{\hbar} \int_0^{\beta\hbar} \mathrm{d}\tau\, \mathrm{d}^d r\ \left(\pi \mathrm{i} \partial_\tau \phi - \mathcal{H}(\pi, \phi)\right) \right] \tag{7.3}$$

This expression is local [2] in space"time", with the cost of introducing an additional dimension on $\mathbb{S}^1$ (note that calculating trace implies that initial state and final state are the same thus this dimension is rolled up.) **We have clearly verified that a d-dim space statistic field theory is close related to a QFT in $(d+1)$-dim spacetime.**

[2] It is of full necessity to indicate that a theory with locality means it is defined locally and does not have to live in $\mathbb{R}^d \times \mathbb{S}^1$ (or $\mathbb{R}^d \times \mathbb{R}^1$).

- When $\beta \to \infty$, it becomes the Euclidean QFT.
- When $\hbar \to 0$,
$$\frac{1}{\hbar}\int^{\hbar\beta} \mathrm{d}\tau\ (\Box) \to \beta\frac{\partial}{\partial\tau}\int \mathrm{d}\tau\ (\Box) \to \beta\,(\Box)\,, \tag{7.4}$$
hence it goes back to classical case.

Additional compatibilities can be intuited directly from the form of path integral mopped up the momentum π, listed as follows:

$$\begin{gathered} d\text{-space quantum Stat. Mech.} \\ \Updownarrow \\ (d+1)\text{-spacetime QFT} \\ \Updownarrow \\ (d+1)\text{-space classical Stat. Mech.} \end{gathered}$$

The statistic field theory integrated out momenta is

$$\mathcal{Z} = \int \mathcal{D}\phi\ \exp\left[-\int \underbrace{\mathrm{d}\tau\,\mathrm{d}^d r}_{\mathrm{d}^{d+1}r\text{ in }\Sigma\times\mathbb{S}^1}\left(\frac{(\partial_\tau\phi)^2}{2} + \frac{(\partial_{\vec{r}}\phi)^2}{2} + V(\phi)\right)\right] \tag{7.5}$$

which can be rewritten as

$$\mathcal{Z} = \int \mathcal{D}\phi\ \exp\left[-\int \mathrm{d}^{d+1}r\left(\frac{(\partial_\tau\phi)^2}{2} + \frac{(\partial_{\vec{r}}\phi)^2}{2} + V(\phi)\right)\right]. \tag{7.6}$$

It can be reinterpreted as classical stat. mech. in $(d+1)$-space, whose, however, $\tilde{\beta}$ has no physical reflections.

There might be a puzzle through the derivation above,

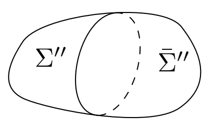
> *Before integrating out* π*, the* exp *term contains* $\pi \mathrm{i}\partial_\tau\phi$*. This match seems presentation-depending. How to figure out this mapping in a presentation independent way?*

7.2 General Development

We' ve seen that a quantum field theory in $d+1$-dim spacetime can be reinterpreted as a d-dim space statistic field theory by regarding the time dimension as a inverse temperature, but it is not so obvious when we did not mop up the integration of conjugate momenta π. We need to find a way in whole, to figure out in what situation a quantum field theory can be seen as a statistic field theory. *"Just as the time when we heard about the formalism of Lagrangian mechanic, all cumbersome processes of force analysis disappeared."*

The answer is by locality and unitarity.

Let us first compare the behaviors between QFT and stat. mech. [3]

<table>
<tr><th>Real time</th><th>Imaginary time</th></tr>
<tr><td>unitarity</td><td>reflection positivity</td></tr>
<tr><td>$\left(\mathrm{e}^{-\mathrm{i}H\Delta t}\right)^* = \mathrm{e}^{-\mathrm{i}H(-\Delta t)}$
$\Downarrow$
$\mathcal{Z}^*_{\mathbb{R}\times\Sigma} = \mathcal{Z}_{\underset{\text{orientation inversed}}{\underset{\uparrow}{\bar{\mathbb{R}}}}\times\Sigma}$</td><td>also true because the only imaginary part is $\mathrm{e}^{-\int \mathrm{d}\tau\, \pi \mathrm{i}\partial\tau\phi}$
$\Longrightarrow\ \mathcal{Z}^*_{\Sigma'} = \mathcal{Z}_{\bar{\Sigma}'}$</td></tr>
<tr><td colspan="2">The partition function of a locally defined theory can be splited into two parts, each on half of the base manifold. For $\Sigma =$  , the degree of freedom is the arbitrary boundary condition on $\partial\Sigma''$. By treating $\mathcal{Z}_{\Sigma''}$ and $\mathcal{Z}_{\bar{\Sigma}''}$ as functionals of $\phi|_{\partial\Sigma''}$. we have
$$\mathcal{Z}_{\Sigma'} = \int \mathcal{D}\phi|_{\partial\Sigma''}\, \mathcal{Z}_{\Sigma''}\mathcal{Z}_{\bar{\Sigma}''} = \int \mathcal{D}\phi|_{\partial\Sigma''}\, \left(\mathcal{Z}_{\Sigma''}\right)^2 \geq 0$$</td></tr>
<tr><td>time reversal invariant
$\Downarrow$
$\begin{cases} \mathcal{Z}_{\Sigma'\times\bar{\mathbb{R}}} = \mathcal{Z}_{\Sigma'\times\mathbb{R}} \\ \mathcal{Z}_{\Sigma'} = \mathcal{Z}_{\bar{\Sigma}'} \end{cases}$
$\Downarrow$
$\mathcal{Z}$ is always real.</td><td>also
(useful in *topological insulator*)</td></tr>
</table>

[3]The properties mentioned in the table are only valid with an implicit assumption that the base manifold on which the field lives is orientable.

We have came to the conclusion that a theory with locality (of course with Hermitian Hamiltonian whilst) is always accompanied by a real and positive partition function $\mathcal{Z}$.