场论与凝聚态笔记

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目录

1 Invitation: The Cartoon of Confinement

Never see individual quarks.

For separatable particles, like electron charges, their potential is $V(r) \sim \frac{1}{r}$, thus $V(r) - V(r_0)$ is always bounded.

Two quarks forms a pion. They interacts through gluon, and forms a structure called gluon tube or string. The potential is $V(r) \sim r$ and the energy density per length is appropriately constant. To separate a quark pair, the energy inputed $V(r) - V(r_0)$ is unbounded.

Similar phenomenon appears in superconductor(type II). When electronic charge condensed, the interaction of magneticmonopole becomes $V(r) \sim r$. According to EM duality, when magneticmonopole condensed, analogy goes to its counterpart. (Perspective by t'Hooft, Polyakov and Manldstan.)

2 Path Integral for Single Particles

From the two-slit interference, we've known the picture of wave. Whilst, the view of particle could recover the result by computing the phase $\mathrm{e}^{\mathrm{i}S}$.

For single particle mechanic, we start from the Schrödinger equation

$$\mathrm{i}\hbar\partial_t |\psi(t)\rangle = H(p, x, t) |\psi(t)\rangle . \tag{2.1}$$

We have the time evolution operator

$$|\psi(t)\rangle = U(t, t_0) |\psi(t)\rangle , \tag{2.2}$$

which is unitary,

$$U^\dagger(t, t_0)U(t, t_0) = 1. \tag{2.3}$$

If H is time-dependent, we split the time interval into small slices, and we get the infinitesimal U operator as time-independent cases,

$$U(t, t_0) = \prod_{n=0}^{N-1} U(\overbrace{t_{n-1}, t_n}^{\delta t}), \tag{2.4}$$

where $t_n \equiv t + n\delta t$. Note that the product is time ordered.

Another perspective is from the Schrödinger equation, by finite differential,

$$|\psi(t+\delta t)\rangle = \left[1 - \frac{\mathrm{i}}{\hbar} H(p, x, t + \frac{\delta t}{2})\right] |\psi(t)\rangle \tag{2.5}$$

To the order of δt, we have

$$|\psi(t+\delta t)\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H(p,x,t+\frac{\delta t}{2})} |\psi(t)\rangle \tag{2.6}$$

Next, we put the time evolution operator in spacial basis, considering

$$\langle x'| U(t+\delta t, t) |x\rangle . \tag{2.7}$$

Suppose $H = \frac{p^2}{2m} + V(x)$ for simplicity, we obtain

$$\langle x'| \left[1 - \frac{\mathrm{i}\delta t}{\hbar} \left(\frac{p^2}{2m} + V(x, t + \frac{\delta t}{2})\right)\right] |x\rangle . \tag{2.8}$$

Make a substitution

$$V \to \frac{V(x', t + \frac{\delta t}{2}) + V(x, t + \frac{\delta t}{2})}{2} 1, \tag{2.9}$$

and insert a completeness relation of p in each time slice, we arrive at

$$\langle x'| U(t+\delta t, t) |x\rangle = \int \mathrm{d}p \, \frac{1}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}p(x'-x)}{\hbar}} \exp\left[-\frac{i\delta t}{\hbar} \frac{H(p, x', t + \frac{\delta t}{2}) + H(p, x, t + \frac{\delta t}{2})}{2}\right] . \tag{2.10}$$

written in a more compact form,

$$\langle x'| U(t+\delta t, t) |x\rangle \sim \int \mathrm{d}p \, \frac{1}{2\pi\hbar} \mathrm{e}^{\mathrm{i}\frac{p\delta x}{\hbar} - \mathrm{i}\frac{H\delta t}{\hbar}} . \tag{2.11}$$

The finite-time evolution operator,

$$U(t_N, t_0) = \prod_{n=0}^{N-1} U(t_{n-1}, t_n) \tag{2.12}$$

inserting an identity operator as x basis completeness relation, the element is

$$U(t_{n+2}, t_{n+1}) \underbrace{1}_{\int \mathrm{d}x_n |x_n\rangle\langle x_n|} U(t_{n+1}, t_n) \tag{2.13}$$

then we obtain

$$\begin{aligned}\langle x_N | U(t_N, t_0) | x_0 \rangle = &\left(\prod_{n=1}^{N-1} \int \mathrm{d}x_n \, \langle x_{n+1} | U(t_{n+1}, t_n) | x_n \rangle \right) \\ &\times \langle x_1 | U(t_1, t_0) | x_0 \rangle \end{aligned} \tag{2.14}$$

in full,

$$\begin{aligned}\langle x_N | U(t_N, t_0) | x_0 \rangle = &\left(\prod_{n=1}^{N-1} \int \frac{\mathrm{d}p_{n+\frac{1}{2}} \, \mathrm{d}x_n}{2\pi\hbar} \right) \int \frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar} \\ \times \exp &\left(\frac{\mathrm{i}}{\hbar} \sum_{n=0}^{N-1} \left[p_{n+\frac{1}{2}}(x_{n+1} - x_n) - \delta t \frac{H(p_{n+\frac{1}{2}}, x_{n+1}, t_{n+\frac{1}{2}}) + H(p_{n+\frac{1}{2}}, x_n, t_{n+\frac{1}{2}})}{2} \right] \right) \\ \sim &\left(\prod_{n=0}^{N-1} \int \frac{\mathrm{d}p_{n+\frac{1}{2}} \, \mathrm{d}x_n}{2\pi\hbar} \right) \int \frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int p \, \mathrm{d}x - H \, \mathrm{d}t} \end{aligned} \tag{2.15}$$

Insert a operator $\hat{B}(x)$ in between the path integral,

$$\begin{aligned}\langle x_N | U(t_N, t_m) \hat{B}(x) U(t_m, t_0) | x_0 \rangle = &\left(\prod_{n=1}^{N-1} \int \frac{\mathrm{d}p_{n+\frac{1}{2}} \mathrm{d}x_n}{2\pi\hbar} \right) \int \frac{\mathrm{d}p_{\frac{1}{2}}}{2\pi\hbar} B(x_m) \\ \times \exp \frac{\mathrm{i}}{\hbar} \sum_{n=0}^{N-1} &\left[p_{\frac{1}{2}}(x_{n-1} - x_n) - \delta t \frac{H(p_{n+\frac{1}{2}}, x_{n+1}, t_{n+\frac{1}{2}}) + H(p_{n+\frac{1}{2}}, x_n, t_{n+\frac{1}{2}})}{2} \right]. \end{aligned} \tag{2.16}$$

We simply need to add the value of the operator as a function of certain space coordinate in the expression of path integral.

3 Observables in QM

- $$\langle \psi | A | \psi \rangle, \;\; A^\dagger = A. \tag{3.1}$$

- Probability of projection,

$$P = | \langle \phi | \psi \rangle |^2 = \langle \psi | \underbrace{| \phi \rangle \langle \phi |}_{\hat{A}} | \psi \rangle \tag{3.2}$$

- Observables after evaluation,

$$\langle \phi | \underbrace{\mathrm{e}^{\mathrm{i}Ht/\hbar} A \mathrm{e}^{-iHt/\hbar}}_{\hat{A}(t)} | \psi \rangle \tag{3.3}$$

- Projection after evaluation: scattering

$$P = |\langle\psi| \mathrm{e}^{-\mathrm{i}Ht/\hbar} |\psi\rangle|^2 = \langle\psi| \underbrace{\mathrm{e}^{\mathrm{i}Ht/\hbar} |\phi\rangle \langle\phi| \mathrm{e}^{-iHt/\hbar}}_{\hat{A}(t)} |\psi\rangle \tag{3.4}$$

- Retarded correlation

$$H(t) = H_0 + \underbrace{a(t)}_{\text{small}} B. \tag{3.5}$$

The contribution of $\langle\psi| U^\dagger(t,0)AU(t,0) |\psi\rangle$ to the first order correction in a is

$$-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t' a(t')\Big[\langle\psi| U_0^\dagger(t,0)AU_0(t,t')BU_0(t',0) - U_0^\dagger(t,0)BU_0(t,t')AU_0(t',0) |\psi\rangle\Big] \tag{3.6}$$

4 Path Integrals for Fields

There's a mattress with springs and massive balls connected. Denoting the offset of each ball as $\phi_{\vec{r}}$ the Hamiltonian is

$$H = \sum_{\vec{r}\text{ on lattice}} \left[\frac{p_{\vec{r}}^2}{2m} + V(\phi_{\vec{r}}) + \sum_{\hat{\tau}=1}^{d} \frac{k}{2}\left(\phi_{\vec{r}+\alpha\vec{\tau}} - \phi_{\vec{r}}\right)^2\right], \tag{4.1}$$

with a commutation relation

$$[\phi_{\vec{r}}, p_{\vec{r}'}] = \mathrm{i}\hbar\delta_{\vec{r},\vec{r}'}. \tag{4.2}$$

The interacting part of the Hamiltonian only involves pairs nearby.

In continuum limit, $H = \displaystyle\int \mathrm{d}^d\vec{r}\mathcal{H}(\vec{r})$, and we can write the Hamilton density as

$$\mathcal{H} = \frac{\pi(\vec{r})^2}{2\rho} + \mathcal{V}(\phi(\vec{r})) + \frac{\kappa}{2}\left[\partial_{\vec{r}}\phi(\vec{r})\right]^2, \tag{4.3}$$

where $\pi(\vec{r}) = \dfrac{p_{\vec{r}}}{\alpha^d}, \mathcal{V} = \dfrac{V}{\alpha^d}, \rho = \dfrac{m}{\alpha^d}, \kappa = \dfrac{k\alpha^2}{\alpha^d}$.

Path Integral At t_n, we use $\bigotimes_{\vec{r}} |\phi(\vec{r})\rangle$ basis, and $\bigotimes_{\vec{r}} |\pi(\vec{r})\rangle$ for $t_{n+\frac{1}{2}}$.

Analogously, we get

$$\langle\text{end}| U(t,0) |\text{start}\rangle = \left(\prod_{t_n,\vec{r}} \frac{\mathrm{d}p_{t_{n+\frac{1}{2}},\vec{r}}\,\mathrm{d}\phi_{t_n,\vec{r}}}{2\pi\hbar}\right)_{\text{suitable boundary condition}}$$

$$\times \exp\frac{\mathrm{i}}{\hbar}\sum_{t_n,\vec{r}}\left[p_{t_{n+\frac{1}{2}},\vec{r}}\left(\phi_{t_{n+1},\vec{r}} - \phi_{t_n,\vec{r}}\right) - \delta t\left(\frac{p^2_{n+\frac{1}{2},\vec{r}}}{2m} + \frac{k}{2}\sum_{\hat{\tau}=1}^{d}\left(\phi_{t_n,\vec{r}+\alpha\hat{\tau}} - \phi_{t_n,\vec{r}}\right)^2 + V(\phi_{t_n,\vec{r}})\right)\right]. \tag{4.4}$$

Integrate out p, we obtain

$$\begin{aligned}&\left(\prod_{t_n,\vec{r}}\sqrt{\frac{2\pi\hbar m}{\mathrm{i}\delta t}}\frac{\mathrm{d}\phi_{t_n,\vec{r}}}{2\pi\hbar}\right)_{\text{suitable boundary condition}}\\ &\times\exp\frac{i\delta}{\hbar}\sum_{t_n,\vec{r}}\left[\frac{m}{2}\left(\frac{\phi_{t_{n+1},\vec{r}}-\phi_{t_n,\vec{r}}}{\delta t}\right)-\frac{k\alpha^2}{2}\sum_{\hat{r}=1}^{d}\left(\frac{\phi_{t_n,\vec{r}+\alpha\hat{r}}-\phi_{t_n,\vec{r}}}{\alpha}\right)^2-V(\phi_{t_n,\vec{r}})\right],\end{aligned} \tag{4.5}$$

in which time and space stand in the same place, similar to relativistic K-G field.

5 Free Field Theory

For free fields, $V(\phi_{\vec{r}})=\frac{u}{2}\phi_{\vec{r}}^2$, it behaves just like coupled simple harmonic oscillators. To find the normal modes, we use Fourier transformation.

$$\phi_{\vec{r}}=\int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}}\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}\phi_{\vec{k}}, \tag{5.1}$$

likewise for $p_{\vec{r}}$.

$\phi_{\vec{r}}$ is real, thus $\phi_{\vec{r}}=\phi_{\vec{r}}^\dagger \implies \phi_{-\vec{k}}=\phi_{\vec{k}}^\dagger$, and the commutation relation is

$$\left[\phi_{\vec{k}},p_{\vec{k}'}\right]=\mathrm{i}\hbar(2\pi)^d\delta^d(\vec{k}+\vec{k}'). \tag{5.2}$$

The Hamiltonian becomes

$$H=\int_{-\frac{\pi}{\alpha}}^{\frac{\pi}{\alpha}}\frac{\mathrm{d}^d\vec{k}}{(2\pi)^d}\frac{1}{2}\left[\frac{p_{-\vec{k}}p_{\vec{k}}}{2m}+\left(\frac{k}{2}\sum_{\vec{r}=1}^{d}\left(2\sin\frac{\alpha k_i}{2}\right)^2+\frac{u}{2}\right)\phi_{-\vec{k}}\phi_{\vec{k}}\right], \tag{5.3}$$

from which we can directly figure out the frequency $\omega_{\vec{k}}=\omega_{-\vec{k}}=\sqrt{\frac{k\sum_{\vec{r}}\left(\sin\frac{\alpha k_i}{2}\right)^2+U}{m}}$

Next, we need to make a substitution, or Bogoliubov transformation:

$$\phi_{\vec{k}}^c=\frac{\phi_{\vec{k}}+\phi_{-\vec{k}}}{\sqrt{2}}, \tag{5.4}$$

$$\phi_{\vec{k}}^s=\mathrm{i}\frac{\phi_{\vec{k}}-\phi_{-\vec{k}}}{\sqrt{2}}. \tag{5.5}$$

the integration range becomes half of $\vec{k}$.

5.1 Ground State Wave Function and Entanglement

"You will gain a full comprehension and admire its subtleties in homework."

5.2 Energy Gap and Correlation

There's no gapless excitations in finite crystal constant lattice theory since the minimal wave number is $\sim \frac{\pi}{a}$

6 Where comes the i0^+: a Non-Perturbative Interpretation

Unitarity and locality plays an unreplaceable role in QFT and a lot of deep conclusion can be gained directly from these two properties, without the need of calculation of specific model.

In QFT, we compute something like

$$\langle\phi|\cdots|\psi\rangle\,,\ \text{or}\ \int_{\text{b.c.}}\mathcal{D}(\square)\exp\left(\cdots\right),\tag{6.1}$$

respectively in operator and path integral methodology, and b.c. stands for boundary condition.

In condensed matter, the experiment are done by disturbing a ground state, and the system would decay from excited state to ground state eventually, thus (near $T\sim 0$) what we usually compute is

$$\langle 0|\cdots|0\rangle\,.\tag{6.2}$$

And in particle phys., after performing LSZ reduction, any initial or final state would be turned into creation/annihilation operators acting on a ground state, thus, similarly, we compute

$$\langle\phi|\cdots|\psi\rangle\to\langle 0|\,O_\phi^\dagger\cdots O_\psi\,|0\rangle\,.\tag{6.3}$$

Furthermore, it seems that we need to get the wave function of ground state ,which would be extremely hard in any theory with interaction, however, we do not actually need to do so. The following of this section would explain it in detail.

A typically computation would be in the form of

$$\langle\phi|\,\mathrm{e}^{\mathrm{i}t_f H}\mathrm{e}^{\mathrm{i}t_i H}\,|\psi\rangle\,,\tag{6.4}$$

where the initial and final time are set at $-\infty$ and $+\infty$ for the reason that the time interval of a physical process would be greatly smaller that of an experiment.

Operator A commonly used trick is to make a substitution of $\mathrm{e}^{-\mathrm{i}\Delta t H} \to \mathrm{e}^{-i\Delta t H\left(1-\mathrm{i}0^+\right)}$ in time evolution. [1] Let's check its property in two cases: finite time and infinite time. When Δt is finite, there's nothing abnormal. When $\Delta t \to +\infty$, a factor of exponentially suppress appears, yet except for states whose $H = 0$. This term would effectively become $|0\rangle\langle 0|$ or $\sum_i |0_i\rangle\langle 0_i|$ when $H|0\rangle = 0$.

If, unfortunately, the experiment time isn't literally infinite, the 0^+ have to be

$$0^+ \sim \frac{1}{\text{experiment time}} \quad \text{(or relaxation time } t\text{)}. \tag{6.5}$$

In Green function, this projection to ground state is included by substitution of

$$\begin{aligned} \omega &\to \left(1+\mathrm{i}0^+\right)\omega \quad \text{(time ordered)} \\ \omega &\to \omega + \mathrm{i}0^+ \quad \text{(retarded)} \end{aligned}. \tag{6.6}$$

Path Integral After a substitution of $H\delta t \to H\left(1-\mathrm{i}0^+\right)\delta t$, we obtain

$$\int_{\text{No b.c.}} \mathcal{D}p\,\mathcal{D}x\,\exp\left(\frac{\mathrm{i}}{\hbar}\int \left[p\,\mathrm{d}x - H\left(1-\mathrm{i}0^+\right)\,\mathrm{d}t\right]\right). \tag{6.7}$$

Wick Rotation In the process given above, we have *rotated* the trajectory of t from the real axis to another line through the original point with a small angle 0^+. A generalization of this trick is to choose a integral contour along the imaginary axis, which is called *Wick rotation*. It can be shown that, in principle, if we know all observables along the $z = -\mathrm{i}\tau$ axis, all observables on real time can be recovered to the spirit of analytical continuation. Under

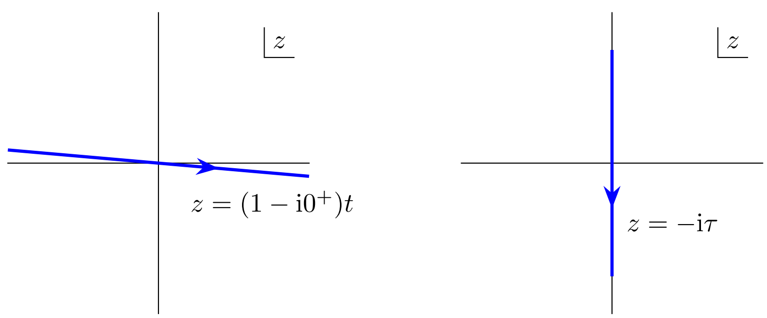


Fig 6.1. Wick rotation

this, the expression of path integral becomes

$$\int \mathcal{D}\pi\,\mathcal{D}\phi\,\exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}\tau\,\mathrm{d}^3 r\left(\pi\mathrm{i}\partial_t\phi - \mathcal{H}\right)\right]. \tag{6.8}$$

[1] It is no longer unitary, but that's what we want.

This formalism is also called *Euclidean field theory* or *imaginary time* path integral.

The name "Euclidean" field theory comes from the fact that after completing the integration on π of a free field, we would find the term on the exponential becomes

$$\mathcal{L}_{\text{Euclidean}} \sim (\partial_t \phi)^2 + (\partial_{\vec{r}} \phi)^2 + \phi^2, \tag{6.9}$$

instead of the Lagrangian density in Minkovski spacetime, whose form is typically

$$\mathcal{L}_{\text{Minkovski}} = (\partial \phi)^2 - m^2 \phi^2 = (\partial_t \phi)^2 - (\partial_{\vec{r}} \phi)^2 - m^2 \phi^2. \tag{6.10}$$

The iconic minus signs from Minkovski metric were eliminated after Wick rotation as if it morphed into a field living in Euclidean space.

7 Dualities of QFT and Stat. Mech.

7.1 Partition Function as Path Integral

In statistic mechanics, we compute the partition functions of fields,

$$\mathcal{Z} = \sum_s \mathrm{e}^{-\beta E_s} = \sum_s \mathrm{e}^{-\beta \int \mathrm{d}\vec{r}\, \mathcal{H}}, \tag{7.1}$$

which is easy to compute in classical statistic mechanics where $\hbar = 0$, i.e., all operators commute.

However, in quantum cases, the calculation of

$$\mathcal{Z} = \operatorname{Tr}\left(\mathrm{e}^{-\beta H}\right) = \operatorname{Tr}\left(\mathrm{e}^{-\beta \int \mathrm{d}^d \vec{r}\, \mathcal{H}}\right) \tag{7.2}$$

becomes impossible, because energy eigenstates are non-local and the eigenvalues of Hamiltonian are hard to find. To solve this, we consider the $\mathrm{e}^{-\beta H}$ as a product of a series of $\mathrm{e}^{-\beta\epsilon}$. By analogy, we insert a completeness relation of momenta and field operators in between each of them. Whereby, we get

$$\mathcal{Z} = \prod_{\vec{r},t} \left(\frac{1}{2\pi\hbar} \int \mathrm{d}\pi_{\tau+\frac{\delta\tau}{2},\vec{r}}\, \mathrm{d}\phi_{\tau,\vec{r}} \right) \exp\left[\frac{1}{\hbar} \int_0^{\beta\hbar} \mathrm{d}\tau\, \mathrm{d}^d r\ \left(\pi \mathrm{i} \partial_\tau \phi - \mathcal{H}(\pi, \phi)\right) \right] \tag{7.3}$$

This expression is local [2] in space"time", with the cost of introducing an additional dimension on $\mathbb{S}^1$ (note that calculating trace implies that initial state and final state are the same thus this dimension is rolled up.) **We have clearly verified that a d-dim space statistic field theory is close related to a QFT in $(d+1)$-dim spacetime.**

[2] It is of full necessity to indicate that a theory with locality means it is defined locally and does not have to live in $\mathbb{R}^d \times \mathbb{S}^1$ (or $\mathbb{R}^d \times \mathbb{R}^1$).

- When $\beta \to \infty$, it becomes the Euclidean QFT.
- When $\hbar \to 0$,

$$\frac{1}{\hbar}\int^{\hbar\beta} \mathrm{d}\tau\ (\square) \to \beta\frac{\partial}{\partial\tau}\int \mathrm{d}\tau\ (\square) \to \beta\,(\square)\,, \tag{7.4}$$

hence it goes back to classical case.

Additional compatibilities can be intuited directly from the form of path integral mopped up the momentum π, listed as follows:

$$\begin{gathered} d\text{-space quantum Stat. Mech.} \\ \Updownarrow \\ (d+1)\text{-spacetime QFT} \\ \Updownarrow \\ (d+1)\text{-space classical Stat. Mech.} \end{gathered}$$

The statistic field theory integrated out momenta is

$$\mathcal{Z} = \int \mathcal{D}\phi\ \exp\left[-\int \underbrace{\mathrm{d}\tau\,\mathrm{d}^d r}_{\mathrm{d}^{d+1}r \text{ in } \Sigma\times\mathbb{S}^1} \left(\frac{(\partial_\tau\phi)^2}{2} + \frac{(\partial_{\vec{r}}\phi)^2}{2} + V(\phi)\right)\right] \tag{7.5}$$

which can be rewritten as

$$\mathcal{Z} = \int \mathcal{D}\phi\ \exp\left[-\int \mathrm{d}^{d+1} r \left(\frac{(\partial_\tau\phi)^2}{2} + \frac{(\partial_{\vec{r}}\phi)^2}{2} + V(\phi)\right)\right]. \tag{7.6}$$

It can be reinterpreted as classical stat. mech. in $(d+1)$-space, whose, however, $\tilde{\beta}$ has no physical reflections.

There might be a puzzle through the derivation above,

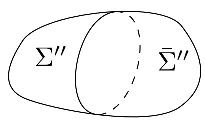
> *Before integrating out* π*, the* exp *term contains* $\pi \mathrm{i} \partial_\tau \phi$. *This match seems presentation-depending. How to figure out this mapping in a presentation independent way?*

7.2 General Development

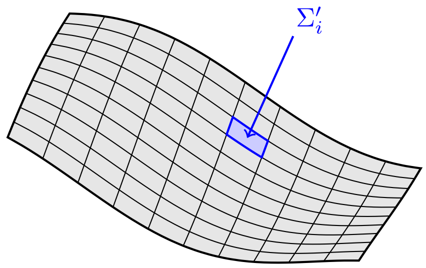
We' ve seen that a quantum field theory in $d+1$-dim spacetime can be reinterpreted as a d-dim space statistic field theory by regarding the time dimension as a inverse temperature, but it is not so obvious when we did not mop up the integration of conjugate momenta π. We need to find a way in whole, to figure out in what situation a quantum field theory can be seen as a statistic field theory. *"Just as the time when we heard about the formalism of Lagrangian mechanic, all cumbersome processes of force analysis disappeared."*

The answer is by locality and unitarity.

Let us first compare the behaviors between QFT and stat. mech. [3]

<table>
<tr><th>Real time</th><th>Imaginary time</th></tr>
<tr><td>unitarity</td><td>reflection positivity</td></tr>
<tr><td>$\left(\mathrm{e}^{-\mathrm{i}H\Delta t}\right)^* = \mathrm{e}^{-\mathrm{i}H(-\Delta t)}$
$\Downarrow$
$\mathcal{Z}^*_{\mathbb{R}\times\Sigma} = \mathcal{Z}_{\underset{\substack{\uparrow \\ \text{orientation inversed}}}{\bar{\mathbb{R}}}\times\Sigma}$</td><td>also true because the only
imaginary part is $\mathrm{e}^{-\int \mathrm{d}\tau\, \pi \mathrm{i}\partial_\tau\phi}$
$\Longrightarrow \mathcal{Z}^*_{\Sigma'} = \mathcal{Z}_{\bar{\Sigma}'}$</td></tr>
<tr><td colspan="2">The partition function of a locally defined theory can be splited into two parts, each on half of the base manifold. For $\Sigma =$  , the degree of freedom is the arbitrary boundary condition on $\partial\Sigma''$. By treating $\mathcal{Z}_{\Sigma''}$ and $\mathcal{Z}_{\bar{\Sigma}''}$ as functionals of $\phi|_{\partial\Sigma''}$. we have
$$\mathcal{Z}_{\Sigma'} = \int \mathcal{D}\phi|_{\partial\Sigma''}\, \mathcal{Z}_{\Sigma''}\mathcal{Z}_{\bar{\Sigma}''} = \int \mathcal{D}\phi|_{\partial\Sigma''}\, \left(\mathcal{Z}_{\Sigma''}\right)^2 \geq 0$$</td></tr>
<tr><td>time reversal invariant
$\Downarrow$
$\begin{cases} \mathcal{Z}_{\Sigma'\times\bar{\mathbb{R}}} = \mathcal{Z}_{\Sigma'\times\mathbb{R}} \\ \mathcal{Z}^*_{\Sigma'} = \mathcal{Z}_{\bar{\Sigma}'} \end{cases}$
$\Downarrow$
$\mathcal{Z}$ is always real.</td><td>also
(useful in *topological insulator*)</td></tr>
</table>

[3]The properties mentioned in the table are only valid with an implicit assumption that the base manifold of the field is orientable.



$$\Sigma' = \bigcup_i \Sigma'_i$$

Fig 7.1. Splited base manifold

We have came to the conclusion that a theory with locality (of course with Hermitian Hamiltonian whilst) is always accompanied by a real and positive partition function $\mathcal{Z}$.

In the following it will be convinced that *if and only if* the generative functional of a Euclidean field theory is real and positive, it has a *classical stat. mech. interpretation*.

Sufficiency can be easily verified, as we have

$$\mathcal{Z} = \int \mathcal{D}\phi \, \exp\left(-\beta \int \mathrm{d}^{d+1}r' \, \mathcal{H}(\phi)\right), \tag{7.7}$$

no imaginary part appears in the expression of $\mathcal{Z}$. [4]

Proof of necessity is carried out by utilizing locality, as path integral can be performed in a certain subset on spacetime.[5] Splitting the base manifold into small parts, as shown in Fig7.1, the path integral on each sub-manifold is fully determined by the b.c. on each $\partial\Sigma'_i$. Hence the path integral on Σ' can be expressed as

$$\mathcal{Z}_{\Sigma'} = \int (\text{b.c. of all boundary}) \prod_i \mathcal{Z}_{\Sigma'_i \ (\text{b.c.})}. \tag{7.8}$$

Given that locality and unitarity guarantee that all $\mathcal{Z}_{\Sigma'_i}$ are *real and positive*, such $\mathcal{Z}_{\Sigma'_i}$ is undoubtedly doable to be turned into the form of $\mathrm{e}^{-\tilde{\beta}(\cdots)}$, thus there always exists an effective Hamiltonian density and corresponding inverse temperature to match this QFT with a stat. mech. interpretation.

[4] The integration of $\mathcal{D}\phi$ can be approximated by summation on a large amount of field configurations, which turns out to be Mont Carlo method for stat. mech.

[5] but there's no locality in momentum space.

8 SSB: What is Broken? Why no Tunneling?

A name of ambiguity was designated to refer as the physical picture behind SSB, resulting a widespread misconstruction among juvenile learners, that "*a potential with minimal value at non-zero field strength will lead to SSB as vacuum state is static and tends to minimize its Hamiltonian, so it has to pick a state.*" But such explanation will be no longer compatible if one try to replicate this in a statistic field theory and he would eventually figure out that $\langle s \rangle = 0$. What on earth is broken!

To answer the question of what is breaking, we need to investigate a simplest case, that is, a 0-dim field. The picture of a 0-dim field is generally characterized by Fig8.1. Directly from

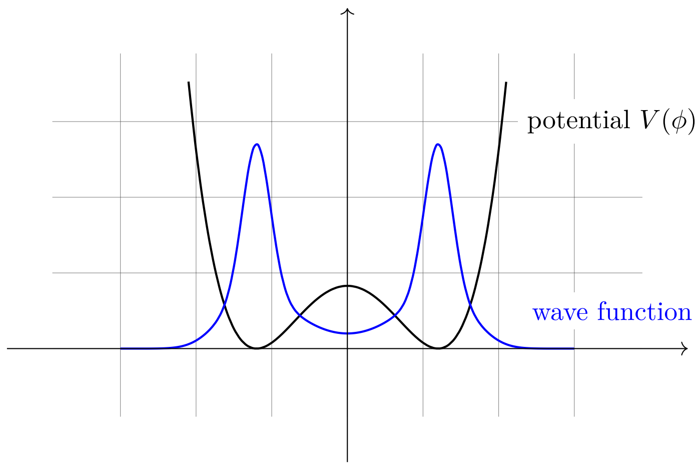


Fig 8.1. 1-dim field in a double-minimal potential

the figure, one can discover that the vacuum expectation value of ϕ is 0, even though the field is trapped in a potential with two valleys. The key is that as long as the shape of potential is not so sharp, a non-zero tunneling amplitude will gradually expunge any symmetry-broken state as time evolutes. We need to get a approach to form a *naturally* infinite potential barrier, i.e. at least one parameter of the system should be literally infinite.

In a system with infinite degrees of freedom, we describe it by a Hamiltonian density, thus

$$V(\phi = 0) \propto \text{ volume of space.} \tag{8.1}$$

Hence, a spontaneous symmetry broken only appears under *thermodynamic limit*, and the dimension of base manifold should be greater than 2 in stat. mech., which would be discussed later.

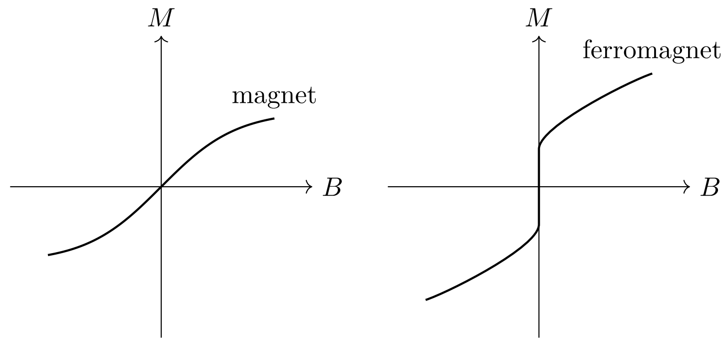


Fig 8.2. Magnetization curve of two phases

In the following we need to develop a well-defined language of SSB, since the ambiguity of describing SSB as a shifted vacuum expectation field strength will keep bothering us as long as we are about to investigate phenomena of SSB in stat. mech.

Taking ferromagnet as an example, a double-minimum potential would lead to degeneracy of ground state. To avoid vagueness in physical picture, we introduce a background magnetic field B, and consider the behavior of magnet strength in such case. There will be a susceptibility divergence in presence of SSB, equivalently infinitesimal change of B leads to a finite change of M.(Fig 8.2)

By definition, we have

$$M = \frac{\partial F}{\partial B} \tag{8.2}$$

thus, a discontinuity of M is corresponded to a non smoothness of shape F.

The "microscopic free energy"[6] will be modified to

$$\tilde{F} = \tilde{F} - B\tilde{M} \cdot \mathrm{Vol}. \tag{8.3}$$

and we have

$$\mathcal{Z}\left(B\right) = \mathrm{e}^{-\frac{F(B)}{T}} = \int_{-\infty}^{\infty} \mathrm{d}\tilde{M}\,\mathrm{e}^{-\frac{1}{T}\left(\tilde{F} - B\tilde{M}\cdot\mathrm{Vol}\right)}, \tag{8.4}$$

which is dominated by the double well in $\tilde{F}$. Meanwhile, we have the relation

$$M = \frac{1}{\mathrm{Vol}}\frac{\partial F}{\partial B} = \frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial B} = \left\langle \tilde{M} \right\rangle \tag{8.5}$$

and

$$\chi = \frac{\partial M}{\partial B} = \frac{1}{\mathcal{Z}}\frac{\partial^2 \mathcal{Z}}{\partial B^2} - \frac{1}{\mathcal{Z}^2}\frac{\partial \mathcal{Z}}{\partial B}\frac{\partial \mathcal{Z}}{\partial B} = \left\langle M^2 \right\rangle - \left\langle M \right\rangle^2 \tag{8.6}$$

[6]The Mellin transformation of F of variable β.

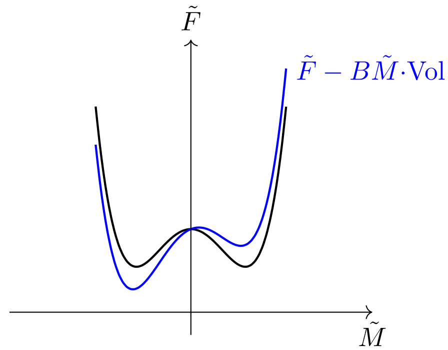


Fig 8.3. free energy in external field

It can be seen that there's some relation between long range correlation and spontaneous symmetry broken.

8.1 Long Range Correlation and SSB in Ising Model

The Hamiltonian of this model is

$$\begin{aligned} H_{\text{classical}} &= \sum_{\text{link } l = \langle v\, v' \rangle} (-J)\sigma_v^z \sigma_{v'}^z \\ H_{\text{quantum}} &= \sum_{\text{vertex } v} (-h)\sigma_v^x + H_{\text{classical}} \end{aligned} \tag{8.7}$$

in which h and $-h$ are equivalent for the $\mathbb{Z}_2$ global symmetry. The corresponding flip operator is

$$\prod v\sigma_v^x. \quad \text{(one for each part of the space)} \tag{8.8}$$

Note that we doesn't need to flip all spins as our basis spacetime may consist separate parts which do not interact with each other. The $\mathbb{Z}_2$ symmetry broken term in this model would be $-\sum_v B\sigma_v^z$, and our order parameter is σ_v^z.

- H_{quantum} in $d+1$ spacetime can b mapped to $H_{\text{classical}}$ in $d' = d+1$ space, but anisotropic $\tilde{J}_d \neq \tilde{J}_\tau$.

Now let us investigate this system through its partition function

$$\mathcal{Z} = \left(\prod v \sum_{x=\pm 1} \right) \mathrm{e}^{-\frac{E}{T}} \tag{8.9}$$

where $E = \sum_l (-J) s_v s_{v'} - \sum_v B_v s_v$.

Derivations of $\mathcal{Z}$'s logarithm gives statistic observables

$$M = \frac{1}{\mathrm{Vol}}\frac{\partial F}{\partial B} = \frac{\left(\prod_v \sum_{s_v=\pm 1}\right) s_{v_0} \mathrm{e}^{-\frac{E}{T}}}{\mathcal{Z}} = \langle s_{u_0} \rangle \tag{8.10}$$

and

$$\begin{aligned}\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} &= \frac{1}{T}\sum_{v_1} \left(\langle s_{v_0} s_{v_1}\rangle - \langle s_{v_0}\rangle\langle s_{v_1}\rangle\right)_{B=0} \\ &= \frac{1}{T}\sum_{v_1} \langle s_{v_0} s_{v_1}\rangle\end{aligned} \tag{8.11}$$

The relation above gives two cases:

- if $\langle s_{v_0} s_{v_1}\rangle \sim \mathrm{e}^{-\# r}$, χ is finite, and no SSB, which corresponds to high T and $d \leq 1$.
- if $\langle s_{v_0} s_{v_1}\rangle \sim$ constant, χ is infinite, SSB, which corresponds to low T and $d \geq 2$.

8.2 Expansion of $\mathcal{Z}$ in Low and High T

An useful observation is given as

$$\mathrm{e}^{\frac{J}{T} s_v s_{v'}} = \mathrm{e}^{\frac{J}{T}} \left(\delta_{s_v, s_{v'}} + \mathrm{e}^{-\frac{2J}{T}} \delta_{s_v, -s_{v'}}\right) = \cosh\frac{J}{T}\left(1 + \tanh\frac{J}{T} s_v s_{v'}\right) \tag{8.12}$$

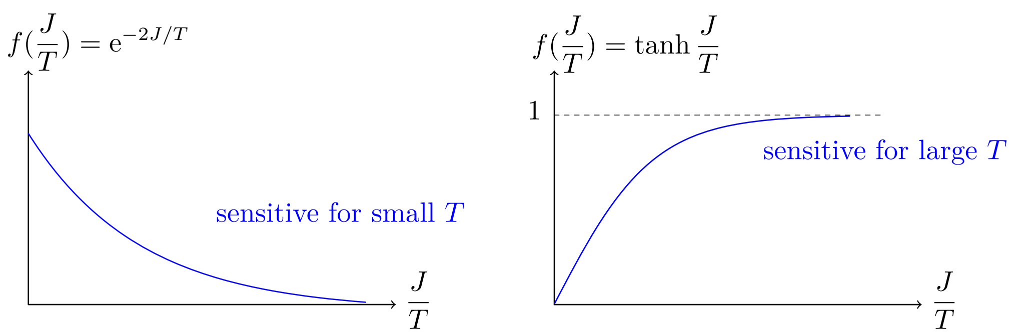


Fig 8.4. Expansion of ising

The overall factor only contributes a constant shift of Hamiltonian density, what we focus now would be the latter term of each expression.

8.2.1 Low T Expansion

We can obtain from above that

$$\mathcal{Z} = \left(\prod_v \sum_{s_v=\pm 1}\right)\left(\prod_{l=\langle v,v'\rangle} \mathrm{e}^{\frac{J}{T}}\left(\delta_{s_v,s_{v'}} + \mathrm{e}^{-\frac{2J}{T}}\delta_{s_v,-s_{v'}}\right)\right). \tag{8.13}$$

Expanding the $\prod$ in partition function, we will be expected to arrive at a huge product of $\delta_{s_v,s_{v'}}$ and $\delta_{s_v,-s_{v'}}$. After summing over all possible field configuration, we can match each field config with a diagram, by coloring each link where $\delta_{s_v,-s_{v'}} = 1$.

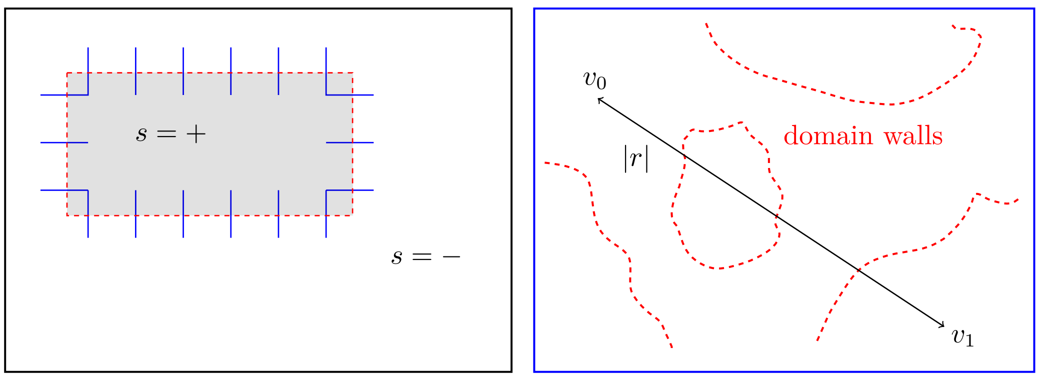


Fig 8.5. Domain wall

$$\mathcal{Z} = \sum_{\text{domain config}} \left(\mathrm{e}^{-2J/T}\right)^{\#\text{ of links on domain wall}} \underbrace{\left(\mathrm{e}^{J/T}\right)^{\#\text{ of links}}}_{\text{constant}} \tag{8.14}$$

Each place between v_0 and v_1 is a possible position for domain wall to appear, thus we could write

$$\#\text{ of possibilities} \sim \Big(\mathcal{O}(1)\Big)^{|r|}. \tag{8.15}$$

Given that entropy is logarithm of number of microstates, we discover that a new domain wall gives out a shift of free energy satisfying

$$\mathrm{e}^{-\frac{\Delta F}{T}} \sim \left(\mathrm{e}^{-2J/T}\right)^{l}\Big(\mathcal{O}(1)\Big)^{|r|} \tag{8.16}$$

In long-range correlation, both l and r scale with system size[7], thus $\mathrm{e}^{-\Delta F/T}$ will be

[7] For 1-dim, $l = 1$, no SSB

extremely small, revealing that it is unlikely to form domain wall in large scale, thus

$$\langle s_{v_0} s_{v_1} \rangle \sim \text{constant} \tag{8.17}$$

8.2.2 High T Expansion

Re-write the partition function as

$$\mathcal{Z} = \left(\prod_v \sum_{s_v = \pm 1} \right) \left(\prod_{l = \langle v, v' \rangle} \cosh \frac{J}{T} \left(1 + \tanh \frac{J}{T} s_v s_{v'} \right) \right). \tag{8.18}$$

Similarly, expanding the product in the second parenthesis, $\mathcal{Z}$ now is a polynomial of $s_v s_{v'}$ to be summed over all config. Now we denote each $s_v s_{v'}$ as a colored link.

After $\sum_{s_v=\pm 1}$, all monomials containing odd power of s_v become zero. The remaining terms can be regarded as summation of all closed loops,

$$\mathcal{Z} = \sum_{\text{closed loops}} \left(\tanh \frac{J}{T} \right)^{|\text{length of loop}|} \cdot 2^{|\#\text{ of vertices}|} \cdot \left(\cosh \frac{J}{T} \right)^{|\#\text{ of links}|} \tag{8.19}$$

Now we insert $s_{v_0} s_{v_1}$ into the path integral to obtain $\langle s_{v_0} s_{v_1} \rangle$. The discussion above is still compatible, however, the only difference is at the vertices of v_0 and v_1, where the original monomial with odd power of s_{v_0}, s_{v_1} will remain.

Now it's easy to see that in high temperature, the partition function is dominated by the shortest line between v_0 and v_1. Any extra deformation of the line would contribute a free energy of

$$\mathrm{e}^{-\frac{\Delta F}{T}} \sim \mathcal{O}(1) \cdot \left(\tanh \frac{J}{T} \right)^{\Delta l} \ll 1 \tag{8.20}$$

when T is high, any deformation is unlikely, thus

$$\langle s_{v_0} s_{v_1} \rangle \sim \left(\tanh \frac{J}{T} \right)^{r} \tag{8.21}$$

9 Classical $U(1)$ Theory

Assigning a $U(1)$ valued field on each vertex, and we can write the partition function as

$$\mathcal{Z} = \left(\prod_{\text{vertex}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_v}{2\pi} \right) \prod_{\text{link}} W \left(\mathrm{e}^{\mathrm{i}\mathrm{d}\theta_l} \right), \tag{9.1}$$

where W is a statistical weight periodic in θ, and other symbolic conventions are used as in lattice exterior derivative.

An realization of this model is separated superconductors with Joseph coupling, as shown in Fig. 9.1.

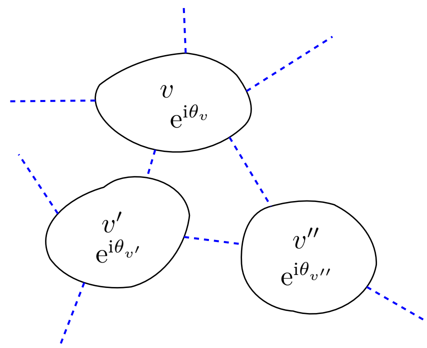


Fig 9.1. Superconductor of Joseph Interaction

This system acquires a $U(1)$ global symmetry[8], invariant under transformation of $\mathrm{e}^{\mathrm{i}\alpha}$, in which α satisfies $\mathrm{d}\alpha_l = 0$. Note that α might be different in uncoupled areas. In classical statistical mechanic this model is called *linear sigma model*, and in condensed matter, called *XY model*.

The θ is 2π periodic, thus its spectrum is a series of integer, inferring that the conjugate momentum of θ is integer-quantized — the property of angular momentum. A usual choice for $W(\theta)$ is

$$W = \exp\left(\frac{\cos \mathrm{d}\theta - 1}{T}\right) \tag{9.2}$$

Under low temperature, we can expand the weight as (to the spirit of renormalization group)

$$\exp\left(\frac{\cos \mathrm{d}\theta_l - 1}{T}\right) \sim \exp\left(-\frac{\mathrm{d}\theta_l^2}{2T}\right) \tag{9.3}$$

which lost the 2π periodicity.

[8]Also a $\mathbb{Z}_2$ symmetry, i.e., reflection for rotor or charge conjugation for superconductor.

10 Adding Topology Structures to XY Model

In classical statistic mechanics, XY model is widely proposed to describe the interactions between superconductors, whose weight reads

$$\mathcal{Z}=\left(\prod_{\text{vertex}}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_l}{2\pi}\right)\prod_{\text{link}}W\left(\mathrm{e}^{\mathrm{id}\theta_l}\right), \tag{10.1}$$

where W is the Boltzmann weight of the system, restricted by the $U(1)$ symmetry of θ.

11 Mixed Anomaly between $U(1)$ and $\tilde{U}(1)$

In XY model, the expression of weight is

$$\text{weight}=\frac{1}{T}\sum_l\left(\cos(\mathrm{d}\theta_l)-1\right). \tag{11.1}$$

There exists a $U(1)$ global symmetry,

$$\theta_v\to\theta_v+\alpha_v,\quad \alpha\in U(1),\quad \mathrm{d}\alpha=0\mod 2\pi. \tag{11.2}$$

We have known the order parameter of $U(1)$ global symmetry is $\mathrm{e}^{\mathrm{i}\theta_v}$, which **transforms with the global $U(1)$ symmetry transformation**. We can construct invariant observables with order parameters, to determent the correlation scale or other properties of the system. Non-trivial observables should be $\left\langle\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta_{v'}}\right\rangle$ or more generally,

$$\left\langle\mathrm{e}^{\mathrm{i}\sum_v B_v\theta_v}\right\rangle=\begin{cases}0, & B_v\neq\nabla W_l\\ \text{non zero}, & B_v=\nabla W\ (=\mathrm{d}^*W)\end{cases} \tag{11.3}$$

for the later case, we can obtain the following equation by lattice integration by part. If $B_v=\mathrm{d}^*\left(W_l\right)$,

$$\left\langle\mathrm{e}^{\mathrm{i}\sum_v B_v\theta_v}\right\rangle=\left\langle\mathrm{e}^{\mathrm{i}\sum_v(\mathrm{d}^*\theta_l)W_l}\right\rangle \tag{11.4}$$

which is an invariant under $U(1)$.

11.1 XY Model in $U(1)$ Background

Coupling to a $U(1)$ background field, the weight of XY model becomes

$$\text{weight}=\frac{1}{T}\sum_l\left[\cos\left(\mathrm{d}\theta_l-qA_l\right)-1\right]. \tag{11.5}$$

In this case, $U(1)$ global symmetry manifests as $U(1)$ gauge invariance, as we require A_l transforms with θ_v

$$\begin{cases} \theta_v \to \theta_v + \alpha_v, \quad & \alpha \in U(1) \\ A_l \to A_l + \mathrm{d}\alpha_l \end{cases} \tag{11.6}$$

where now α_v is *arbitrary*, without restriction of $\mathrm{d}\alpha_l = 0 \mod 2\pi$.

Note that the transformation $A_l \to A_l + \mathrm{d}\alpha_l$ gives

$$\mathcal{Z}[A] = \mathcal{Z}[A + \mathrm{d}\alpha], \tag{11.7}$$

which means different backgrounds (with the difference of a total derivation) are equivalent.

This A_l seems like a electromagnetic field, let us check its current behaviour. Take a Fourier transformation (or Poisson resummation), in current representation we get

$$\prod lI_{jl}\left(\frac{1}{T}\right)\mathrm{e}^{\mathrm{i}j_l(\mathrm{d}\theta_l + qA_l)} \tag{11.8}$$

we hope that A_l should be 2π periodic, thus q must be a integer.[9]

The operator correlation transforms as

$$\left\langle \mathrm{e}^{\mathrm{i}\theta_v}\mathrm{e}^{-\mathrm{i}\theta_{v'}} \right\rangle_A = \mathrm{e}^{\mathrm{i}(\alpha_v - \alpha_{v'})}\left\langle \mathrm{e}^{\mathrm{i}\theta_v}\mathrm{e}^{-\mathrm{i}\theta_{v'}} \right\rangle_{A+\mathrm{d}\alpha} \tag{11.9}$$

11.2 Villain Model in $U(1)$ Background

In Villain model, we have made the substitution,

$$\cos(\mathrm{d}\theta) - 1 \to \frac{1}{2}\mathrm{d}\theta^2 \tag{11.10}$$

thus it has no natural 2π periodicity. We add this manually, by introducing a dynamical degree of freedom m to be summed in path integral. The weight of Villain model is

$$-\frac{1}{2T}\sum_l \big(\underbrace{\mathrm{d}\theta + 2\pi m}_{\gamma_l} - A\big)_l^2. \tag{11.11}$$

m transforms with θ and A_l,

$$\begin{cases} A_l \to A_l + \mathrm{d}\alpha_l + 2\pi k_l, \\ \theta_v \to \theta_v + \alpha_l + 2\pi n_v, & \alpha_v \in (-\pi, \pi],\ k_l, n_v \in \mathbb{Z} \\ m_l \to m_l + k_l - \mathrm{d}n_l, \end{cases} \tag{11.12}$$

[9]We can also couple the θ_v to a series of A_l's, in this case, $\frac{q}{q'}$ must be a rational number.

In this case, the vortex previous is no longer well-defined, as

$$v_p \equiv \frac{\mathrm{d}\gamma_p}{2\pi} = \mathrm{d}m_p \to \mathrm{d}m_p + \mathrm{d}k_p, \tag{11.13}$$

v_p isn't a invariant now.

We need to find a well-defined vortex. A simple choice would be

$$\bar{v}_p \equiv \frac{\mathrm{d}\left(\gamma - A\right)}{2\pi}, \tag{11.14}$$

which is an invariant under gauge transformation, but it is *no longer an integer*.

Recalling that we have defined a vortex operator V_p under the limit of $u \to +\infty$, which is coupled with θ_v as

$$-\frac{u}{2}\sum_p \left(v_p - V_p\right)^2 \xrightarrow[u\to+\infty]{\text{H-S trans.}} \mathrm{i}\sum_p \tilde{\theta}_p \left(v_p - V_p\right). \tag{11.15}$$

This can be viewed as a Lagrangian constrain multiplier to restrict the vortex number equal to V_p.[10] If we let $V_p \to V_p + \mathrm{d}k_p$, the theory is compatible with the transformation. However, A_l and V_p cannot be viewed as independent, i.e., they together form $U(1)$ background. The $U(1)$ field is described by

$$\left(\mathrm{d}A - 2\pi V\right)_p \in \mathbb{R}. \tag{11.16}$$

instead of the previous one $\mathrm{d}A_l \in U\left(1\right)$. This $\mathrm{d}A - 2\pi V$ can be regarded as a Villainized version of electromagnetic field, which indicates that *a 2π flux cannot be distinguished with a vortex.*

Both two choice are OK for finite u, but interesting things happen when $u \to \infty$ We make a Hubbard-Stratonovich transformation and get the presentation of $\tilde{\theta}$.

1. $\mathrm{i}\sum_p \tilde{\theta}_p \left(\mathrm{d}m - \frac{\mathrm{d}A}{2\pi}\right)_p$, $\tilde{\theta}_p$ losses 2π periodicity. There's no global $\tilde{U}(1)$ symmetry and $\tilde{\theta}_p$ lives in $\mathbb{R}$ instead.

2. $\mathrm{i}\sum_p \tilde{\theta}_p \left(\mathrm{d}m - V\right)_p$, V_p is part of $U(1)$ background, and $\tilde{\theta}$ is actually 2π periodic.

   But let us recall how we identified this $\tilde{U}(1)$ symmetry. By lattice integration by part, we convert $\mathrm{i}\sum_l \tilde{\theta}_l \mathrm{d}m_l$ to $\mathrm{i}\sum_l \tilde{\mathrm{d}}\tilde{\theta}_{l*} m_l$. We want to be able to consider all possible $U(1)$ background, but some V_p will explicitly break $\tilde{U}(1)$ when there exists no $L_p \in \mathbb{Z}$ such $V_p = \mathrm{d}L_p$, leading to the result that the δ function in $\mathcal{Z}$ will never be met, thus

$$\mathcal{Z} = 0 \quad \text{for certain backgrounds } V_p. \tag{11.17}$$

[10] Or get the same result by completing the integration getting a δ function.

This aberrant phenomenon is called *mixed anomaly*.

In conclusion, when coupling with *a non-trivial* $U(1)$ *background*, a system with $U(1)$ and $\tilde{U}(1)$ symmetry suffers from $\tilde{U}(1)$ explicitly broken.

In traditional QFT, people believe that anomaly is due to non-invariance of path integral measurement, which is, however, presentation dependent and doesn't reveal the true picture.