8. The One-Dimensional Delta Method

Exercises due Jun 8, 2021 19:59 EDT Completed

Applying Linear Functions to a Random Sequence

3/3 points (graded)

Let $(Z_n)_{n\geq 1}$ be a sequence of random variables such that

$$\sqrt{n}\,(Z_n - \theta) \xrightarrow[n\to\infty]{(d)} Z$$

for some $\theta \in \mathbb{R}$ and some random variable Z.

Let $g(x) = 5x$ and define another sequence by $Y_n = g(Z_n)$.

The sequence $\sqrt{n}\,(Y_n - g(\theta))$ converges. In terms of Z, what random variable does it converge to?

$$\sqrt{n}\,(Y_n - g(\theta)) \xrightarrow[n\to\infty]{(d)} \quad Y.$$

(Answer in terms of Z)

$Y =$

What theorem did we invoke to compute Y?
(There can be more than 1 acceptable answers.)

- (selected) Laws of large number
- Central Limit theorem
- Slutsky theorem
- Continuous mapping theorem

If Var $(Z) = \sigma^2$, what is Var (Y)? This is the asymptotic variance of $(Y_n)_{n\geq 1}$.
(Answer in terms of σ^2.)

Var $(Y) =$

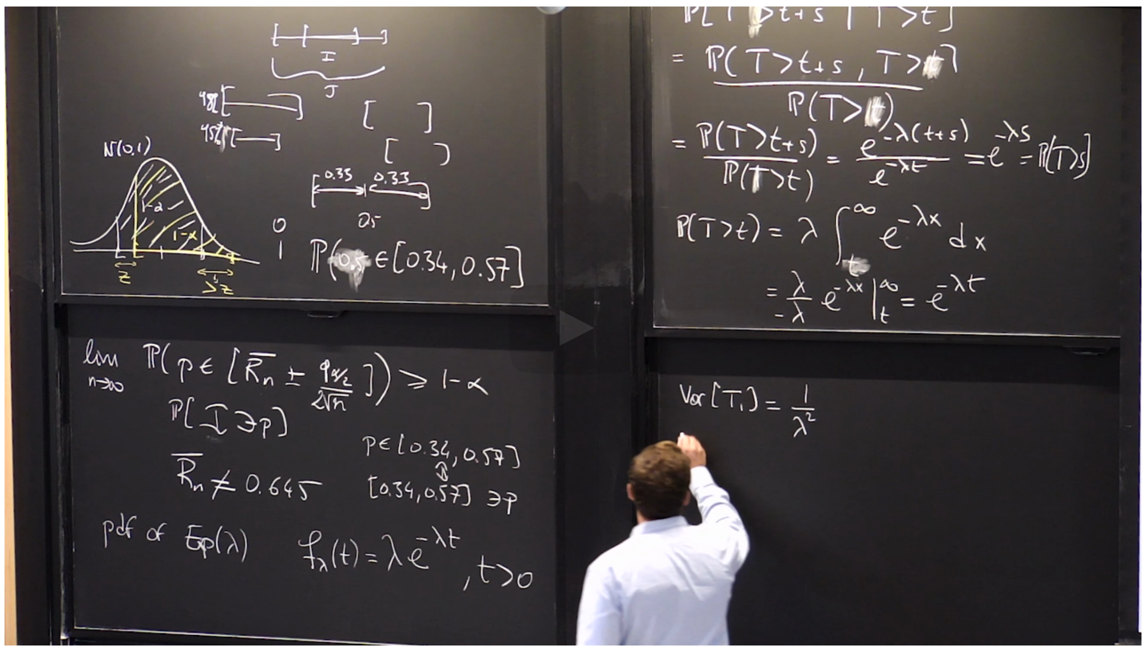
STANDARD NOTATION

Submit You have used 1 of 2 attempts

Video note: In the video below, there is an important misprint at roughly 1:26, which will be corrected in the video on the next page. The Central limit theorem applied to $\overline{T}_n$ should read

$$\sqrt{n}\left(\bar{T}_n - \frac{1}{\lambda}\right) \xrightarrow[n\to\infty]{} \mathcal{N}\left(0, \frac{1}{\lambda^2}\right).$$

the Delta Method



0:00 / 0:00 1.50x

The problem is that this is not something of the form estimator

of lambda minus lambda.

Right?

What I would want to see is something

that looks like square root of n one

over T n bar, which is actually my lambda hat,

minus lambda converges to some Gaussian

as n goes to infinity in distribution.

Maybe zero and some sigma squared here.

Right?

That's what I want to see.

Because once I know how to do this,

then I can start unpacking my confidence intervals

of the form lambda hat plus or minus

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(Optional) Proof of the Delta Method

For simplicity, we will prove only the case when g is continuously differentiable everywhere in $\mathbb{R}$, ie. g and g' exist and are continuous everywhere. Let μ be arbitrary. The mean value theorem (which is the zeroth order statement of Taylor's theorem) states that for any $z > \mu$,

$$g(z) = g(\mu) + g'(c_z)(z-\mu) \qquad \text{for some } c_z \in (\mu, z)$$

Note that c_z is a function of z. This works also for the case $z < \mu$. The two cases together give the statement that for any z:

$$g(z) = g(\mu) + g'(c_z)(z-\mu) \qquad \text{for some } c_z \text{ such that } |c_z - \mu| < |z-\mu|.$$

For each z, we can make a choice of c_z that makes the above statement true: we now think of c as being a function of z (but we will continue to write c_z to denote $c(z)$). This implies that for a random variable Z,

$$g(Z) - g(\mu) = g'(c_Z)(Z-\mu) \qquad \text{for some } c \text{ such that } |c_Z - \mu| < |Z-\mu|.$$

Now, given an arbitrary sequence $(Z_n)_{n\geq 1}$ and for any μ, the above statement is true for each random variable Z_n in the sequence:

$$g(Z_n) - g(\mu) = g'(c_{Z_n})(Z_n-\mu) \qquad \text{for some } c \text{ such that } |c_{Z_n} - \mu| < |Z_n-\mu|.$$

We return to the statistical context. Let $X_1, X_2, \ldots, X_n \overset{\text{i.i.d}}{\sim} X$, and let $Z_n = \overline{X}_n$ and $\mu = \mathbb{E}[X]$. Plugging these into the equation above and multiplying by $\sqrt{n}$, we have

$$\sqrt{n}\left(g(\overline{X}_n) - g(\mu)\right) \quad = \quad g'\left(c_{\overline{X}_n}\right)\left(\sqrt{n}(\overline{X}_n - \mu)\right) \qquad \text{where } \left|c_{\overline{X}_n} - \mu\right| < \left|\overline{X}_n - \mu\right|$$

We deal with the two factors on the right hand side separately. By CLT, we know the second factor is asymptotically normal:

$$\left(\sqrt{n}\,(\overline{X}_n - \mu)\right) \xrightarrow[n\to\infty]{(d)} N\left(0, \sigma^2\right).$$

For the second factor, observe that since $\left|c_{\overline{X}_n} - \mu\right| < \left|\overline{X}_n - \mu\right|$, we have for any $\epsilon > 0$,

$$\mathbf{P}\left(\left|c_{\overline{X}_n} - \mu\right| > \epsilon\right) \leq \mathbf{P}\left(\left|\overline{X}_n - \mu\right| > \epsilon\right).$$

Together with the fact that $\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mu$, this implies

$$c_{\overline{X}_n} \xrightarrow[n\to\infty]{\mathbf{P}} \mu.$$

Since g' is continuous, by the continous mapping theorem,

$$g'\left(c_{\overline{X}_n}\right) \xrightarrow[n\to\infty]{\mathbf{P}} g'(\mu).$$

Finally, by Slutsky Theorem,

$$\sqrt{n}\left(g\left(\overline{X}_n\right) - g(\mu)\right) \xrightarrow[n\to\infty]{(d)} N\left(0, \left(g'(\mu)\right)^2 \sigma^2\right).$$

Remark: Notice that g is only needed to be continuously differentiable close to μ.

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- Taylor series expansion for functions with non continuous second derivative ~6:30 in the video 3
- Is the purpose of delta method to get a "centered" interval? 3
  From n^0.5*(barT_n - 1/lambda), we can get a confidence interval for 1/lambda say [a, b] without delta method. Once we have that w...

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