10. Multivariate Central Limit Theorem

Exercises due Jun 29, 2021 19:59 EDT

Note: The following exercise will be presented in the video that follows. We encourage you to attempt it before watching the video.

Vector Version of the Central Limit Theorem

1/1 point (graded)

Let $\mathbf{X}$ be a random vector of dimension $d \times 1$ and let μ and Σ be its mean and covariance. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d. copies of $\mathbf{X}$. Let $\overline{\mathbf{X}}_n \triangleq \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$.

Based on your knowledge of the central limit theorem for a single random variable, select from the following the correct shift and scale factor for $\overline{\mathbf{X}}_n$ so that $\overline{\mathbf{X}}_n$ could potentially converge to the Gaussian random vector $\mathcal{N}\left(0, I_{d\times d}\right)$.

- ( ) $\sqrt{d} \cdot \Sigma^{-\frac{1}{2}}\left(\overline{\mathbf{X}}_n - \mu\right)$
- ( ) $\sqrt{d} \cdot \Sigma^{-1}\left(\overline{\mathbf{X}}_n - \mu\right)$
- ( ) $\sqrt{n} \cdot \Sigma^{-1}\left(\overline{\mathbf{X}}_n - \mu\right)$
- ( ) $\sqrt{n} \cdot \Sigma^{-\frac{1}{2}}\left(\overline{\mathbf{X}}_n - \mu\right)$
- (●) None of the above

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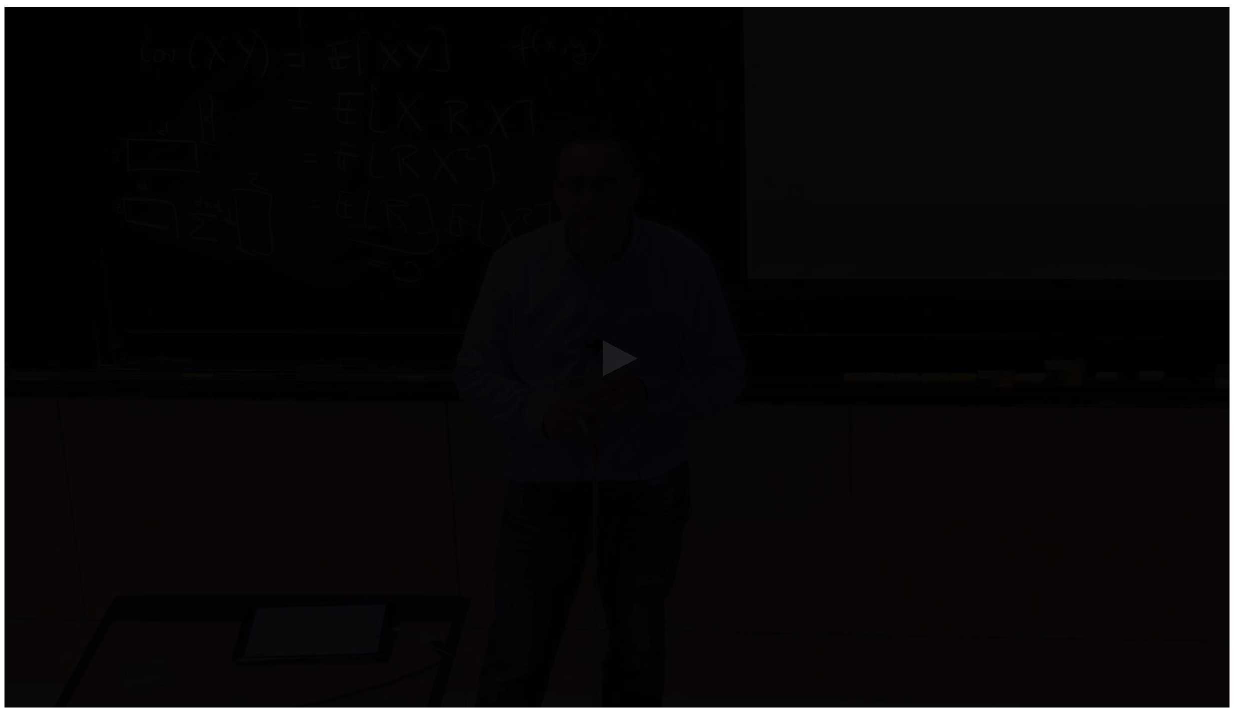
You have used 1 of 3 attempts

✔ Correct (1/1 point)

Multivariate Central Limit Theorem



5:46 / 5:46 1.50x

page for this.

Otherwise, this is just linear algebra.

So now that I have this notion of taking 1

over square root of sigma squared, which is just

multiplying by this matrix, then I actually

can talk about having something which converges

to a standard Gaussian.

Are there any questions?

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(Optional) Multivariate Convergence in Distribution and Proof of Multivariate CLT

Convergence in Distribution in Higher Dimensions

Convergence in distribution of a random vector is **not implied** by convergence in distribution of each of its components.

A sequence $\mathbf{T}_1, \mathbf{T}_2, \ldots$ of random vectors in $\mathbb{R}^d$ **converges in distribution** to a random vector $\mathbf{T}$ if

$$\mathbf{v}^T\mathbf{T}_n \xrightarrow[(d)]{n\to\infty} \mathbf{v}^T\mathbf{T} \qquad \text{for all } \mathbf{v}\in\mathbb{R}^d \qquad (\text{multivariate convergence in distribution})\,.$$

That is, the vector sequence $(\mathbf{T}_n)_{n\geq 1}$ converges in distribution only if its dot product $\mathbf{v}^T\mathbf{T}_n$ with **any** constant vector $\mathbf{v}$, which is a scalar random variable, converges in distribution (or equivalently, if the projection of the vector sequence onto **any** line converges in distribution.)

Univariate CLT Implies Multivariate CLT

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \overset{i.i.d.}{\sim} \mathbf{X}$ be random vectors in $\mathbb{R}^d$ with (vector) mean $\mathbb{E}[\mathbf{X}] = \mu_{\mathbf{X}}$ and covariance matrix $\Sigma_{\mathbf{X}}$.

Let $\mathbf{v}\in\mathbb{R}^d$ and define $Y_i = \mathbf{v}^T\mathbf{X}_i$. Then

- Y_i is a scalar random variable;
- Its mean and variance are $\mathbb{E}[Y_i] = \mathbf{v}^T\mathbb{E}[\mathbf{X}_i]$ and $\sigma^2_{Y_i} = \mathbf{v}^T\Sigma_{\mathbf{X}_i}\mathbf{v}$ (you can check that the variance is indeed a scalar).

Hence Y_i satisfies the univariate CLT:

$$\sqrt{n}\left(\overline{Y_n} - \mathbf{v}^T\mu_{\mathbf{X}}\right) \xrightarrow[(d)]{n\to\infty} \mathcal{N}\left(0, \mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}\right)$$

On the other hand, consider a multivariate Gaussian variable $\mathbf{Z}\sim\mathcal{N}(\mathbf{0},\Sigma_{\mathbf{X}})$. For any constant vector $\mathbf{v}\in\mathbb{R}^d$, $\mathbf{v}^T\mathbf{Z}$ is a univariate Gaussian with variance $\mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}$. Hence, $\mathbf{v}^T\mathbf{Z}\sim\mathcal{N}\left(0,\mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}\right)$, which is the distribution on the right hand side above. Therefore, $\overline{\mathbf{X}}_n$ converges in distribution:

$$\begin{aligned}
\sqrt{n}\left(\mathbf{v}^T\overline{\mathbf{X}}_n - \mathbf{v}^T\mu_{\mathbf{X}}\right) = \sqrt{n}\left(\overline{Y_n} - \mathbf{v}^T\mu_{\mathbf{X}}\right) &\xrightarrow[(d)]{n\to\infty} \mathcal{N}\left(0,\mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}\right) = \mathbf{v}^T\mathcal{N}\left(0,\Sigma_{\mathbf{X}}\right)\\
\Longleftrightarrow \qquad \sqrt{n}\left(\overline{\mathbf{X}}_n - \mu_{\mathbf{X}}\right) &\xrightarrow[(d)]{n\to\infty} \mathcal{N}\left(0,\Sigma_{\mathbf{X}}\right).
\end{aligned}$$

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