Moment Generating Function of Gaussian Distribution

Contents

Theorem

Let $X \sim N\left(\mu, \sigma^2\right)$ for some $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_{>0}$, where N is the Gaussian distribution.

Then the moment generating function M_X of X is given by:

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Proof

From the definition of the Gaussian distribution, X has probability density function:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

From the definition of a moment generating function:

$$M_X(t) = \mathsf{E}\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x) \, \mathrm{d}x$$

So:

$$\begin{aligned}
M_X(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(tx - \frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x \\
&= \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\left(\sqrt{2}\sigma u + \mu\right) t - u^2\right) \mathrm{d}u && \text{substituting } u = \frac{x-\mu}{\sqrt{2}\sigma} \\
&= \frac{\exp \mu t}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(u^2 - \sqrt{2}\sigma u t\right)\right) \mathrm{d}u \\
&= \frac{\exp \mu t}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(u - \frac{\sqrt{2}}{2}\sigma t\right)^2 + \frac{1}{2}\sigma^2 t^2\right) \mathrm{d}u \\
&= \frac{\exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-v^2\right) \mathrm{d}v && \text{substituting } v = u - \frac{\sqrt{2}}{2}\sigma t
\end{aligned}$$

$$= \frac{\sqrt{\pi} \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)}{\sqrt{\pi}} \qquad \text{Gaussian Integral}$$

$$= \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

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Examples

First Moment

The first moment generating function of X is given by:

$$M_X{}'(t) = \left(\mu + \sigma^2 t\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Second Moment

The second moment generating function of X is given by:

$$M_X{}''(t) = \left(\sigma^2 + \left(\mu + \sigma^2 t\right)^2\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Third Moment

The third moment generating function of X is given by:

$$M_X{}'''(t) = \left(3\sigma^2 \left(\mu + \sigma^2 t\right) + \left(\mu + \sigma^2 t\right)^3\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Fourth Moment

The fourth moment generating function of X is given by:

$$M_X{}^{(4)}(t) = \left(3\sigma^4 + 6\sigma^2\left(\mu + \sigma^2 t\right)^2 + \left(\mu + \sigma^2 t\right)^4\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$