2. Linear Independence and Rank

Exercises due Aug 24, 2021 19:59 EDT

Note: Below are exercises from homework 0 that cover the ideas of linear independence, dimensions, and rank, which will be used in this lecture. These exercises were optional in homework 0, but is graded in this unit.

Linear Independence

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_n$ such that

1. not all c_i's are zero, i.e. there is i such that $c_i \neq 0$;

2. $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$

Vectors that are **not** linearly dependent are said to be **linearly independent** . In other words, vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if the only scalars $c_1, \ldots, c_n$ such that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$ are $c_1 = \cdots = c_n = 0$, i.e.

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0 \quad \Longrightarrow \quad c_i = 0 \qquad \text{for all } i \qquad (\text{linear independence})\,.$$

Two non-zero vectors $\mathbf{v}_1,\ \mathbf{v}_2$ are linear dependent if and only if $\mathbf{v}_1 = c\mathbf{v}_2$, i.e. if one is a scalar multiple of the other.

Examples:

1. $\begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are linear dependent.

2. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are linear independent.

3. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are linear independent.

4. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are linear dependent, because

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or written in a more symmetric form:

$$1\begin{pmatrix} 2 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

5. $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ are linearly dependent, because

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0.$$

Span and dimension

The collection of non-zero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ determines a subspace of $\mathbb{R}^m$. This subspace **subspace of** $\mathbb{R}^m$, also known as the **span** of the vector $\mathbf{v}_1, \dots, \mathbf{v}_n$, is the set of all linear combinations $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ over different choices of $c_1, \dots, c_n \in \mathbb{R}$. denoted by

$$\langle \mathbf{v}_1, \dots, \mathbf{v}_n \rangle = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n\} \qquad (\text{span of } \mathbf{v}_1, \dots, \mathbf{v}_n).$$

The **dimension** of this subspace $\langle \mathbf{v}_1, \dots, \mathbf{v}_n \rangle$ is the size of the **largest possible, linearly independent** sub-collection of the (non-zero) vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Referring to the examples above:

1. $\left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$, that is, either vector spans the entire subspace. Hence, this is a 1-dimensional subspace of $\mathbb{R}^2$.

2. $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \mathbb{R}^2$, and is 2-dimensional.

3. $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle = \mathbb{R}^2$, and is 2-dimensional.

4. $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = \mathbb{R}^2$, and again is 2-dimensional. That is, any 2 of the 3 given vectors span all of $\mathbb{R}^2$.

5. $$\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right.$$
   That is, any two of the first three vectors along with the fourth vector span the subspace; hence. this is a 3-dimensional subspace of $\mathbb{R}^5$.

Rank

The **column space** and **row space** of matrix is the subspace spanned by its columns and its rows respectively. It is a fact from linear algebra that the dimension of the column space of a matrix $\mathbf{M}$ is equal to the dimension of its row space (try to show it by row-reduction). This dimension is the **rank** of the matrix, and denoted $\mathbf{rank}\,(\mathbf{M})$. Note that $\mathbf{rank}\,(\mathbf{M}) = \mathbf{rank}\,(\mathbf{M}^T)$.

Examples

Refer to the examples above. For each example, define a matrix $\mathbf{M}$ whose columns are the given vectors.

1. $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}$ has column rank 1 because the column space $\left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$ is 1-dimensional. Check that the row space, $\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \right\rangle$, spanned by the rows of the matrix, is also 1-dimensional.

2. $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is of rank 2 since the column space $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ is 2-dimensional. The row space and column space are both R^2.

3. $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ is of rank 2 since the column space $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$ is 2-dimensional. The row space and column space of this matrix is equal.

4. $\mathbf{M} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$ is of rank 2, since the dimension of the column space $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle$ is $\mathbf{2}$. The row space $\left\langle \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$ is a subspace of $\mathbb{R}^3$ of dimension $\mathbf{2}$.

5. $\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 4 \\ 1 & 1 & 0 & 5 \end{pmatrix}$ has rank 3. We have seen that tts column space is a 3-dimensional subspace of $\mathbb{R}^5$. This means its row space is also 3-dimensional; it is a subspace of $\mathbb{R}^4$.

Row and Column Rank

2 points possible (graded)

Suppose $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. The rows of the matrix, $(1, 3)$ and $(2, 6)$, span a subspace of dimension

. This is the **row rank** of $\mathbf{A}$.

The columns of the matrix, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ span a subspace of dimension

. This is the **column rank** of $\mathbf{A}$.

We will be using these ideas throughout this lecture, where we will work with larger, possibly rectangular matrices.

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The rank of a matrix

3 points possible (graded)

In general, row rank is always equal to the column rank, so we simply refer to this common value as the **rank** of a matrix.

What is the largest possible rank of a 2×2 matrix?

What is the largest possible rank of a 5×2 matrix?

In general, what is the largest possible rank of an $m \times n$ matrix?

- m
- n
- $\min(m, n)$
- $\max(m, n)$
- None of the above

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Examples of rank

5 points possible (graded)

What is the rank of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

What is the rank of $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$?

What is the rank of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$?

What is the rank of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$?

What is the rank of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$?

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The rank of a matrix continued

2 points possible (graded)

This question is meant to serve as an answer to the following: *If you sum two rank-1 matrices, do you get a rank-2 matrix? What about products? More generally, what rank is the sum of a rank-r_1 and a rank-r_2 matrix?"*

Let $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Observe that all four of these matrices are rank 1.

There are many ways to determine rank. Here is one useful fact that you could use for this problem:

> **"Every rank-1 matrix can be written as an outer product. Conversely, every outer product $\mathbf{u}\mathbf{v}^T$ is a rank-1 matrix."**

For example, $\mathbf{A} = \mathbf{u}\mathbf{v}^T$, $\mathbf{B} = \mathbf{v}\mathbf{v}^T$, $\mathbf{C} = \mathbf{w}\mathbf{w}^T$ and $\mathbf{D} = \mathbf{x}\mathbf{x}^T$, where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Which combination of these matrices has rank 2? Choose all that apply.

- [ ] $\mathbf{A} + \mathbf{A}$
- [ ] $\mathbf{A} + \mathbf{B}$

- [ ] $\mathbf{A}+\mathbf{C}$
- [ ] $\mathbf{AB}$
- [ ] $\mathbf{AC}$
- [ ] $\mathbf{BD}$

Which combination of these matrices has rank 1? Choose all that apply.

- [ ] $\mathbf{A}+\mathbf{A}$
- [ ] $\mathbf{A}+\mathbf{B}$
- [ ] $\mathbf{A}+\mathbf{C}$
- [ ] $\mathbf{AB}$
- [ ] $\mathbf{AC}$
- [ ] $\mathbf{BD}$

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Invertibility of a matrix

1 point possible (graded)

An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ has full rank, i.e. $\mathbf{rank}\,(\mathbf{A}) = n$.

Which of the following matrices are invertible? Choose all that apply.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$
$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

- [ ] $\mathbf{A}$
- [ ] $\mathbf{B}$
- [ ] $\mathbf{C}$
- [ ] $\mathbf{D}$

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