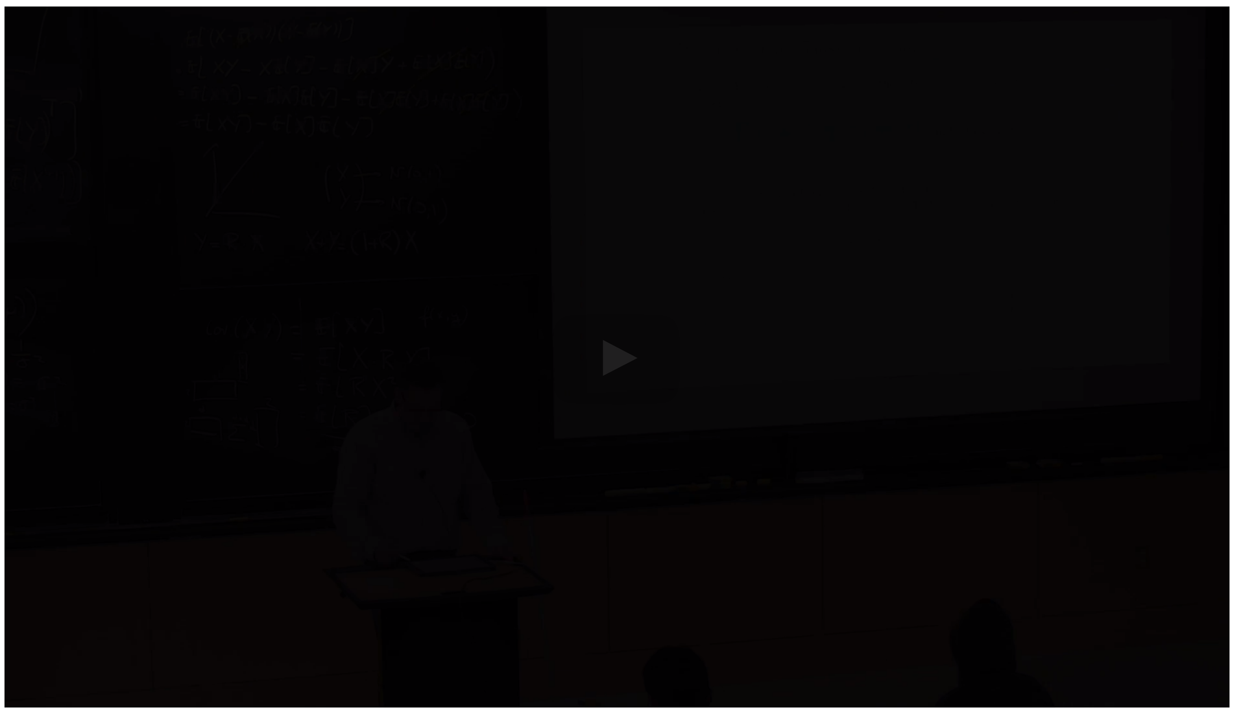
# 9. Multivariate Gaussian Distribution

Exercises due Jun 29, 2021 19:59 EDT

**Note:** Now is a good time to review Gaussian random variables from Lecture 2.

**Video Note:** In the slide of the video below, there is a typo in the formula of the pdf of the multivariate Gaussian distribution: the exponent $d$ in overall scaling factor should apply only to $2\pi$, rather than $2\pi\det\Sigma$. The correct version is in the note below the video. (The unannotated slides in the resource section have also been corrected).

### Multivariate Gaussian Distribution: Definition



multivariate PDF by heart.

What I'm really trying to say is if you give me

a covariance matrix and a vector of expectations--

the expectation vector, I guess--

then I actually can fully characterize a Gaussian PDF.

I have just an entire distribution.

Just like if you give me a expectation and a variance,

I have a Gaussian PDF in one dimension.

9:21 / 9:21 1.50x

**Video**
Download video file

**Transcripts**
Download SubRip (.srt) file
Download Text (.txt) file

**Multivariate Gaussian Random Variable**

A random vector $\mathbf{X} = \left(X^{(1)}, \ldots, X^{(d)}\right)^T$ is a **Gaussian vector** , or **multivariate Gaussian or normal variable** , if any linear combination of its components is a (univariate) Gaussian variable or a constant (a "Gaussian" variable with zero variance), i.e., if $\alpha^T \mathbf{X}$ is (univariate) Gaussian or constant for any constant non-zero vector $\alpha \in \mathbb{R}^d$.

The distribution of $\mathbf{X}$, the **$d$-dimensional Gaussian or normal distribution** , is completely specified by the vector mean $\mu = \mathbb{E}[\mathbf{X}] = \left(\mathbb{E}[X^{(1)}], \ldots, \mathbb{E}[X^{(d)}]\right)^T$ and the $d \times d$ covariance matrix $\Sigma$. If $\Sigma$ is invertible, then the pdf of $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1}(\mathbf{x}-\mu)}, \quad \mathbf{x} \in \mathbb{R}^d$$

where $\det(\Sigma)$ is the determinant of the $\Sigma$, which is positive when $\Sigma$ is invertible.

If $\mu = \mathbf{0}$ and $\Sigma$ is the identity matrix, then $\mathbf{X}$ is called a **standard normal random vector** .

Note that when the covariant matrix $\Sigma$ is diagonal, the pdf factors into pdfs of univariate Gaussians, and hence the components are independent.

## Linear Transformation of a Multivariate Gaussian Random Vector

1 point possible (graded)

Consider the $2$-dimensional Gaussian $\mathbf{X} = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}$ with covariance matrix $\Sigma_X = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and mean $\mu_{\mathbf{X}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Consider the vector $\alpha = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so that $Y = \alpha^T \mathbf{X}$ is a $1$-dimensional Gaussian.

What is the variance $\textsf{Var}\,(Y)$ of $Y$?

$\textsf{Var}\,(Y) =$

Submit You have used 0 of 3 attempts

---

## Singular Covariance Matrices

1 point possible (graded)

Consider again a $2$-dimensional Gaussian $\mathbf{X} = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}$. But instead, $\Sigma_X$ is $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $\alpha = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, what is the variance $\textsf{Var}\,(Y)$ of $Y = \alpha^T \mathbf{X}$?

$\textsf{Var}\,(Y) =$

This result tells us that the Gaussian $\left(X^{(1)}, X^{(2)}\right)^T$ is actually a one-dimensional Gaussian, orthogonal to the direction of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Submit You have used 0 of 3 attempts

---

**(Optional) Diagonalization of the Covariance Matrix**

Let $\Sigma$ be a covariance matrix of size $d \times d$. Note that its entries are all real numbers with diagonal elements being non-negative. $\Sigma$ has the following properties:

- $\Sigma$ is symmetric. That is, $\Sigma = \Sigma^T$.
- $\Sigma$ is diagonalizable to a diagonal matrix $D$ via a transformation $D = U\Sigma U^T$, where $U$ is an orthogonal matrix (recall that a square matrix $A$ is orthogonal if $AA^T = A^T A = I$, where $I$ is the identity matrix). This implies that $\Sigma = U^T D U$.
- Moreover, $\Sigma$ is positive semidefinite. That is, the diagonal matrix $D$ has diagonal entries that are all non-negative.
- $\Sigma$ has a unique positive semidefinite square root matrix. That is, there exists a positive semi-definite matrix $\Sigma^{\frac{1}{2}}$ that is unique such that $\Sigma^{\frac{1}{2}} \cdot \Sigma^{\frac{1}{2}} = \Sigma$.
- If $\Sigma$ is of size $d \times d$, then it has $d$ orthonormal eigenvectors (even if there are repeated eigenvalues). Furthermore, if $U$ is a matrix with rows corresponding to the orthonormal eigenvectors, then the diagonal matrix $D = U\Sigma U^T$ contains the eigenvalues of $\Sigma$ along its diagonal. Therefore, diagonalization of a symmetric matrix involves finding its eigenvalues and the orthonormal eigenvectors.
- If $\Sigma$ is positive definite, i.e. the diagonal matrix $D = U\Sigma U^T$ has diagonal entries that are all strictly positive, then it is invertible and the inverse $\Sigma^{-1}$ satisfies the following: $\Sigma^{-\frac{1}{2}} \cdot \Sigma^{-\frac{1}{2}} = \Sigma^{-1}$, where $\Sigma^{-\frac{1}{2}}$ is the inverse of the square root of $\Sigma$.

Hide

---

## (Optional) Gaussian Random Vectors I

0 points possible (ungraded)

Recall from an earlier part of this lecture that the covariance between two random variables being 0 does not necessarily imply that the random variables are independent. However, this is true if the random variables are multivariate Gaussian.

Let $\mathbf{X}$ be a Gaussian random vector with mean $\mu$ and covariance $\Sigma$. Assume that $\Sigma$ is positive definite. Determine if the following statement is true or false.

“There exists a vector $B$ and a matrix $A$ such that $A(\mathbf{X}+B)$ is a Gaussian random vector whose components are independent and each of mean $\mathbf{0}$".

- True
- False

*Hint:* Refer to the note above on diagonalization of the covariance matrix.

Submit

You have used 0 of 1 attempt

## Discussion

Hide Discussion

**Topic:** Unit 3 Methods of Estimation:Lecture 10: Consistency of MLE, Covariance Matrices, and Multivariate Statistics / 9. Multivariate Gaussian Distribution

**Add a Post**

Show all posts | by recent activity

| | | |
|---|---|---|
| ? | these<br>on 3:19, what are these guys referring to? Maybe in the lecture, professor is pointing to 'these', but here I have no idea what these is referring to.... | 1 |
| ? | If covariance matrix is diagonal, components are independent? | 4 |
| | Singular Covariance Matrices<br>Although I got the answer right, I still don't understand the last line of the section. "This result tells us that the Gaussian (X(1),X(2))T is actually a ... | 2 |
| ? | Is there an orthogonal matrix for all Sigmas? | 4 |

< Previous

Next >

## Legal

Terms of Service & Honor Code
Privacy Policy
Accessibility Policy
Trademark Policy
Sitemap

## Connect

Blog
Contact Us
Help Center
Media Kit
Donate

GET IT ON Google play
Download on the App Store