11. Multivariate Delta Method

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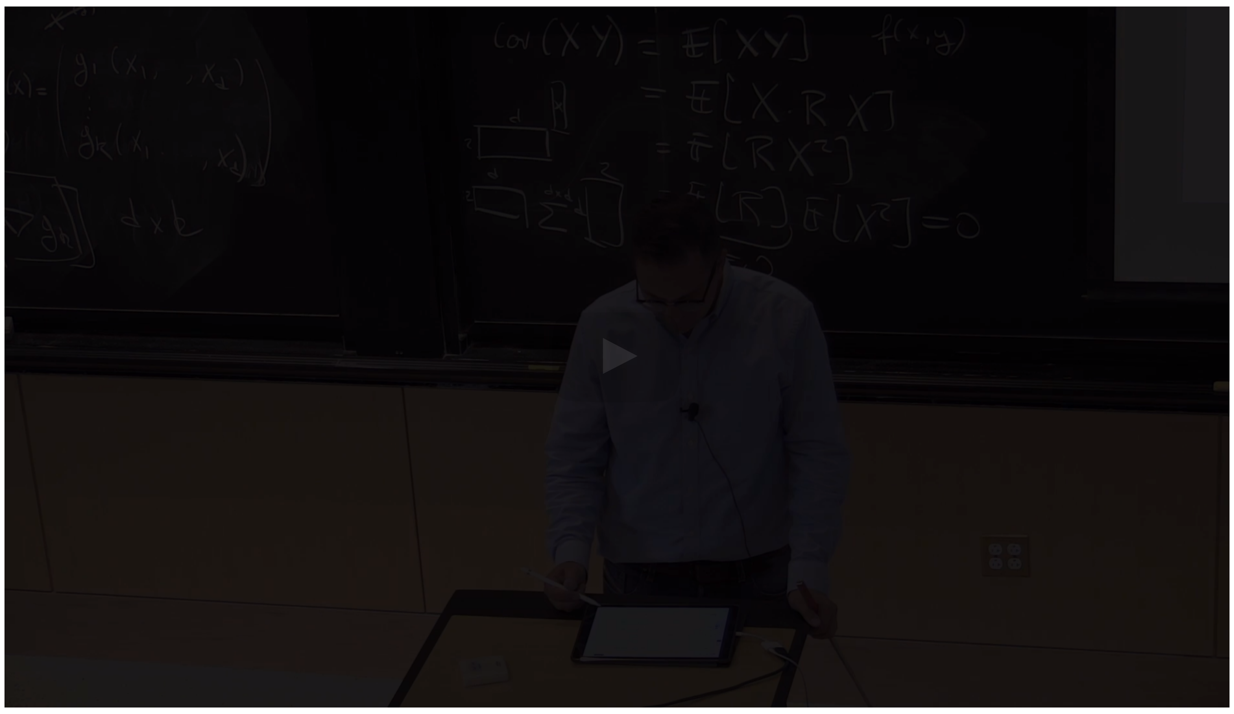
Exercises due Jun 29, 2021 19:59 EDT

Multivariate Delta Method



8:55 / 8:55 1.50x

columns,

you could just put them as rows.

And that would be fine, too.

You would just have to adjust, and not put the transpose here,

but put it here.

So it depends on how.

If you've seen that before, for example,

you might have used a different convention.

And don't let this deter you.

It's just really conventions at this point.

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Gradient Matrix of a Vector Function

4 points possible (graded)

Given a vector-valued function $f: \mathbb{R}^d \to \mathbb{R}^k$, the **gradient** or the **gradient matrix** of f, denoted by ∇f, is the $d \times k$ matrix

$$\nabla f = \begin{pmatrix} | & | & & | \\ \nabla f_1 & \nabla f_2 & \ldots & \nabla f_k \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_k}{\partial x_1} \\ \vdots & \ldots & \vdots \\ \frac{\partial f_1}{\partial x_d} & \ldots & \frac{\partial f_k}{\partial x_d} \end{pmatrix}.$$

This is also the transpose of what is known as the **Jacobian matrix** $\mathbf{J}_f$ of f.

Let $f(x, y, z) = \begin{pmatrix} x^2 + y^2 + z^2 \\ 2xy \\ y^3 + z^3 \\ z^4 \end{pmatrix}$.

How many rows does $\nabla f(x, y, z)$ have?

How many columns does $\nabla f(x, y, z)$ have?

What does column 2 represent in the gradient matrix?

- ◯ Derivative of $2xy$ with respect to x, y, z stacked as a column
- ◯ Derivative of the individual functions with respect to y stacked as a column

What is $\nabla f(x,y,z)_{3,2}$?

Submit You have used 0 of 2 attempts

General Statement of the Multivariate Delta Method

The multivariate delta method states that given

- a sequence of random vectors $(\mathbf{T}_n)_{n\geq 1}$ satisfying $\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right) \xrightarrow[n\to\infty]{(d)} \mathbf{T}$,
- a function $\mathbf{g}: \mathbb{R}^d \to \mathbb{R}^k$ that is continuously differentiable at $\vec{\theta}$,

then

$$\sqrt{n}\left(\mathbf{g}\left(\mathbf{T}_n\right) - \mathbf{g}\left(\vec{\theta}\right)\right) \xrightarrow[n\to\infty]{(d)} \nabla\mathbf{g}(\vec{\theta})^T\,\mathbf{T} \qquad \text{where } \nabla\mathbf{g} = \begin{pmatrix} | & | & \cdots & | \\ \nabla\mathbf{g}_1 & \nabla\mathbf{g}_2 & \cdots & \nabla\mathbf{g}_k \\ | & | & \cdots & | \end{pmatrix}.$$

Common Application

In the lecture and in most applications, $\mathbf{T}_n = \overline{\mathbf{X}}_n$ where $\overline{\mathbf{X}}_n$ is the sample average of $\mathbf{X}_1, \ldots, \mathbf{X}_n \overset{iid}{\sim} \mathbf{X}$, and $\vec{\theta} = \mathbb{E}\left[\mathbf{X}\right]$. The (multivariate) CLT then gives $\mathbf{T} \sim \mathcal{N}\left(\mathbf{0}, \Sigma_{\mathbf{X}}\right)$ where $\Sigma_{\mathbf{X}}$ is the covariance of $\mathbf{X}$. In this case, we have

$$\sqrt{n}\left(\mathbf{g}\left(\mathbf{T}_n\right) - \mathbf{g}\left(\vec{\theta}\right)\right) \xrightarrow[n\to\infty]{(d)} \nabla\mathbf{g}(\vec{\theta})^T\mathbf{T} \sim \mathcal{N}\left(0, \nabla\mathbf{g}(\vec{\theta})^T\Sigma_{\mathbf{X}}\nabla\mathbf{g}\left(\vec{\theta}\right)\right) \qquad \left(\mathbf{T} \sim \mathcal{N}\left(\mathbf{0}, \Sigma_{\mathbf{X}}\right)\right).$$

(Optional) Proof of Multivariate Delta Method

As in the univariate case, the main idea of the proof of the multivariate delta method is to apply the first order multivariate Taylor theorem (i.e. linear approximation with a remainder term), and then use (multivariate) Slutsky's, and the continuous mapping theorem to establish the required convergence.

Slutsky's theorem and the continuous mapping theorems in higher dimensions are straightforward generalizations of these same theorems in one dimension, i.e. where applicable, scalar random variables are replaced with random vectors.

Proof:
Let $(\mathbf{T}_n)_{n\geq 1}$ be a sequence of random vectors in $\mathbb{R}^d$ such that

$$\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right) \xrightarrow[n\to\infty]{(d)} \mathbf{T},$$

for some $\vec{\theta} \in \mathbb{R}^d$.

Let $\mathbf{g}: \mathbb{R}^d \to \mathbb{R}^k$ be continuously differentiable at $\vec{\theta}$. Then, for any vector $\mathbf{t} \in \mathbb{R}^d$, the first order multivariate Taylor's expansion at $\vec{\theta}$ gives

$$\mathbf{g}(\mathbf{t}) = \mathbf{g}(\vec{\theta}) + \nabla \mathbf{g}(\vec{\theta})^T \left(\mathbf{t} - \vec{\theta}\right) + \left\|\mathbf{t} - \vec{\theta}\right\| \mathbf{u}(\mathbf{t})$$

where $\mathbf{u}(\mathbf{t}) \to \mathbf{0}$ as $\mathbf{t} \to \vec{\theta}$.

Extend the above equation by replacing $\mathbf{t}$ with a random vector $\mathbf{T}_n$, rearrange and multiply both sides by $\sqrt{n}$:

$$\sqrt{n}\left(\mathbf{g}(\mathbf{T}_n) - \mathbf{g}(\vec{\theta})\right) \quad = \quad \nabla \mathbf{g}(\vec{\theta})^T \left(\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right) + \left\|\sqrt{n}(\mathbf{T}_n - \vec{\theta})\right\| \mathbf{u}(\mathbf{T}_n).$$

Let us look at convergence of each term on the right as $n \to \infty$. We will apply the multivariate version of continuous mapping theorem and Slutsky's theorem multiple times to our ingredient:

$$\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right) \xrightarrow[n\to\infty]{(d)} \mathbf{T},$$

which also implies

$$\left(\mathbf{T}_n - \vec{\theta}\right) \xrightarrow[n\to\infty]{(d)/(p)} \mathbf{0}.$$

The first term $\nabla \mathbf{g}(\vec{\theta})^T \left(\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right)$ is a continuous function of $\left(\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right)$, hence

$$\nabla \mathbf{g}(\vec{\theta})^T \left(\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right) \quad \xrightarrow[n\to\infty]{(d)} \left(\nabla \mathbf{g}(\vec{\theta})\right)^T \mathbf{T} \qquad \text{by continuous mapping theorem.}$$

For the second term, the first factor $\left\|\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right\|$ is again a continuous function of $\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)$, and therefore

$$\left\|\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right\| \xrightarrow[n\to\infty]{(d)} \|\mathbf{T}\| \qquad \text{by continuous mapping theorem.}$$

The second factor in the second term is a continuous function of $\mathbf{T}_n$ near θ. Hence

$$\mathbf{u}(\mathbf{T}_n) \xrightarrow[n\to\infty]{(d)/(p)} \mathbf{u}(\vec{\theta}) \;=\; \mathbf{0} \qquad \text{by continuous mapping theorem.}$$

By (multivariate) Slutsky theorem, the entire second term converges to $\mathbf{0}$:

$$\left\|\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right\| \mathbf{u}(\mathbf{T}_n) \xrightarrow[n\to\infty]{(d)/\mathbf{P}} \|\mathbf{T}\|(\mathbf{0}) \;=\; \mathbf{0}.$$

Finally, applying the (multivariate) Slutsky theorem to the sum of the two terms gives:

$$\nabla \mathbf{g}(\vec{\theta})^T \left(\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right) + \left\|\sqrt{n}\left(\mathbf{T}_n - \vec{\theta}\right)\right\| \mathbf{u}(\mathbf{T}_n) \xrightarrow[n\to\infty]{(d)} \nabla \mathbf{g}(\vec{\theta})^T \mathbf{T} + \mathbf{0} \;=\; \nabla \mathbf{g}(\vec{\theta})^T \mathbf{T}.$$

This establishes the multivariate delta method.

Discussion

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