MITx 18.6501x
Fundamentals of Statistics

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7. Modes of Convergence

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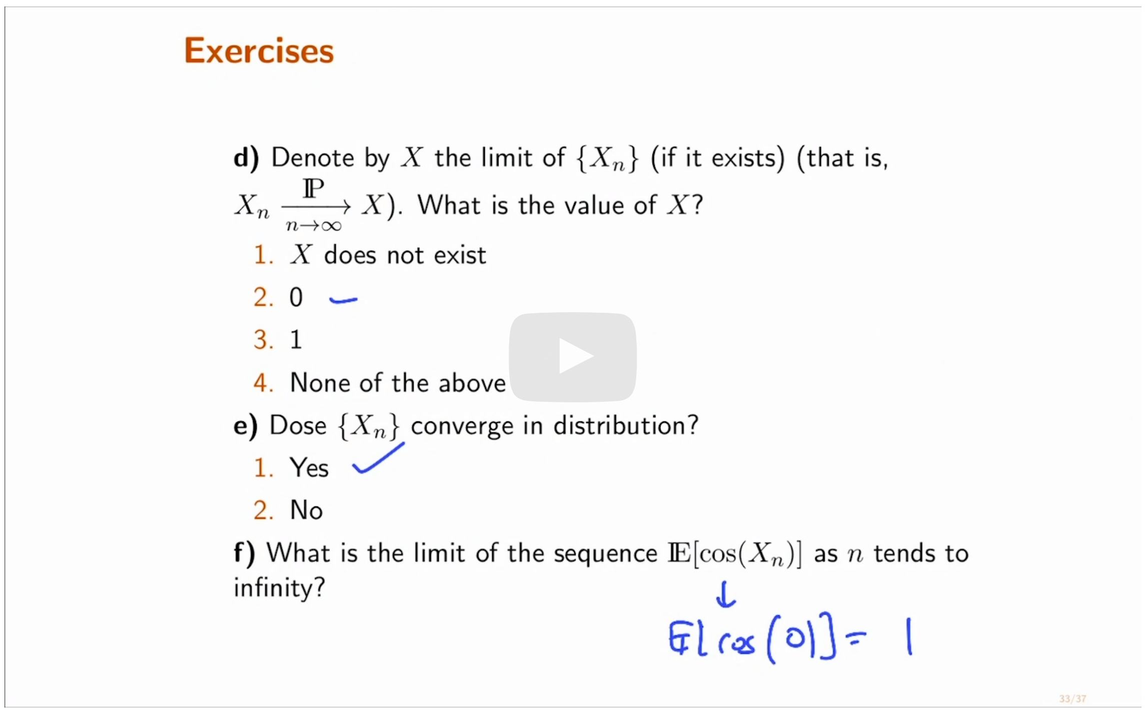
Exercises due May 25, 2021 19:59 EDT

Convergence almost surely, in probability, and in distribution



12:15 / 12:19 1.25x

Xn

takes two values-- cosine of 0 and cosine of 1,

one with probability 1 over n and one with probability

1 minus 1 over n.

So compute it, and that will give you

a little hint as to why you need those functions to be bounded.

If they were not bounded, then you

might not be able to conclude.

So it's important that things are bounded.

Video

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Note: We did not study modes of convergence in great detail in *6.431x Probability–the Science of Uncertainty and Data*. This is one of the most theoretical topic in this course. Basic understanding of their differences will be expected, but in the rest of this course, convergence will mostly be discussed in the context of the laws of large numbers and central limit theorem. **You will not be tested directly on modes of convergence in any exam**.

Equivalent definition of convergence in distribution for real r.v.s

Convergence in distribution is also known as **convergence in law** and **weak convergence** .

For a sequence $(T_n)_{n\geq 1}$ of random variables that take values in $\mathbb{R}$, the definition of convergence in distribution given in lecture is equivalent to the definition we have learned in the course *6.431x: Probability–the Science of Uncertainty and Data*. That is, the following two notions are equivalent:

1. For all continuous and bounded function f,

$$T_n \xrightarrow[n\longrightarrow\infty]{(d)} T \quad \text{iff} \quad \mathbb{E}\left[f\left(T_n\right)\right] \xrightarrow[n\longrightarrow\infty]{} \mathbb{E}\left[f\left(T\right)\right]$$

2. For all $x \in \mathbb{R}$ at which the cdf of T is continuous,

$$T_n \xrightarrow[n\to\infty]{(d)} T \quad \text{iff} \quad \mathbf{P}[T_n \le x] \xrightarrow[n\to\infty]{} \mathbf{P}[T \le x]$$

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Convergence in probability and in distribution 1

1.5/2 points (graded)

Let $(T_n)_{n\ge1} = T_1, T_2, \ldots$ be a sequence of r.v.s such that

$$T_n \sim \text{Unif}\left(5 - \frac{1}{2n}, 5 + \frac{1}{2n}\right).$$

Given an arbitrary fixed number $0 < \delta < 1,$ find the smallest number N (in terms of δ) such that $\mathbf{P}(\{|T_n - 5| > \delta\}) = 0$ whenever $n > N$.

$N =$

Does $(T_n)_{n\ge1}$ converge in probability to a constant? If so, what is the limiting value? Enter **DNE** if $(\{X_n\})$ does not converge in probability.

$(T_n)_{n\ge1} \xrightarrow{\mathbf{P}}$

Does $(T_n)_{n\ge1}$ converge in distribution?

- [ ] Yes
- [x] No

Let $F_n(t)$ be the cdf of T_n and $F(t)$ be the cdf of the constant limit. For which values of t does $\lim_{n\to\infty} F_n(t) = F(t)$? (Choose all that apply.)

- [ ] $t < 5$
- [ ] $t = 5$
- [ ] $t > 5$

STANDARD NOTATION

Submit You have used 2 of 3 attempts

✱ Partially correct (1.5/2 points)

Convergence in probability and in distribution 2

Convergence in probability and in distribution 2

4 points possible (graded)
Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $Y_n \sim$ Unif $(0,1)$.

Let

$$M_n = \max(Y_1, Y_2, \ldots, Y_n).$$

For any fixed number $0 < \delta < 1$, find $\mathbf{P}(|M_n - 1| > \delta\})$. (Type **delta** for δ.)

$\mathbf{P}(|M_n - 1| > \delta\}) =$

Does the sequence $(M_n)_{n\geq 1}$ converge in probability to a constant? If yes, enter the value of the constant limit; if no, enter **DNE**.

$(M_n)_{n\geq 1} \xrightarrow{\mathbf{P}}$

Find the CDF $F_{M_n}(x)$ for $0 \leq x \leq 1$.

$F_{M_n}(x) = P(M_n \leq x) =$

Does $(M_n)_{n\geq 1}$ converge in distribution?

- Yes
- No

STANDARD NOTATION

Submit You have used 0 of 3 attempts

Expectations and convergence in probability

3 points possible (graded)
Let $(T_n)_{n\geq 1}$ be a sequence of r.v.s such that for each n, T_n takes only two possible values 0 and 2^n with the following probabilities:

$$\mathbf{P}(T_n = 0) = 1 - \frac{1}{n}$$
$$\mathbf{P}\left(T_n = 2^n\right) = \frac{1}{n}.$$

Does the sequence $(T_n)_{n\geq 1}$ converge in probability to a constant? If so, enter the limiting value; if not, enter **DNE**.

$T_n \xrightarrow{\mathbf{P}}$

Compute $\mathbb{E}[T_n]$ in terms of n.

$\mathbb{E}[T_n] =$

Does the sequence of expectations $\mathbb{E}[T_n]$ converge? If so, enter the limiting value; if not, enter **DNE**.

$\lim_{n\to\infty} \mathbb{E}[T_n] =$

STANDARD NOTATION

Submit You have used 0 of 3 attempts

Convergence almost surely (a.s) is also known as **convergence with probability 1 (w.p.1)** and **strong convergence** . We will not discuss this type of convergence much beyond this lecture.

Probability review: the (Strong) Law of Large Numbers

1 point possible (graded)

A digital signal receiver decodes bits of incoming signal as 0s or 1 and makes an error in decoding a bit with probability 10^{-4}.

Assuming decoding success is independent for different bits, as the receiver receives more and more signals, what is the fraction of erroneously decoded bits?

Fraction of errors:

Submit You have used 0 of 3 attempts

(Optional theoretical material) Distinguishing different types of convergences

In the following examples, the explicit definition of a random variable as a function on a probability space to $\mathbb{R}$, rather than just its distribution, will be needed to establish the type of convergence.

Convergence in distribution but NOT in probability

Let $X_1, X_2, \ldots, X_i, \ldots$ be a sequence of random variables.

For i odd, $X_i = f_{\text{odd}}(x)$ where $f_{\text{odd}}(x) = x$ in $[0,1]$.
For i even, $X_i = f_{\text{even}}(x)$ where $f_{\text{even}} = 1 - x$ in $[0,1]$.

Then for all i, $X_i \sim \text{Unif}(0,1)$. Since the distribution of all X_i is the same, the sequence converges in distribution to $\text{Unif}(0,1)$.

However, $\{X_i\}$ does not converge in probability: there is no random variable X (i.e. no function from $[0,1]$ to $\mathbb{R}$) such that $\mathbf{P}(|X_i - X| > \epsilon) \to 0$.

Convergence in probability but NOT almost surely

As discussed in the lecture, a sequence $X_n \sim \text{Ber}(1/n)$ converges in probability to 0. However, depending on how the random variables are defined as functions on the underlying probability space, (and different random variables can have the same distribution), the sequence can converge almost surely or not.

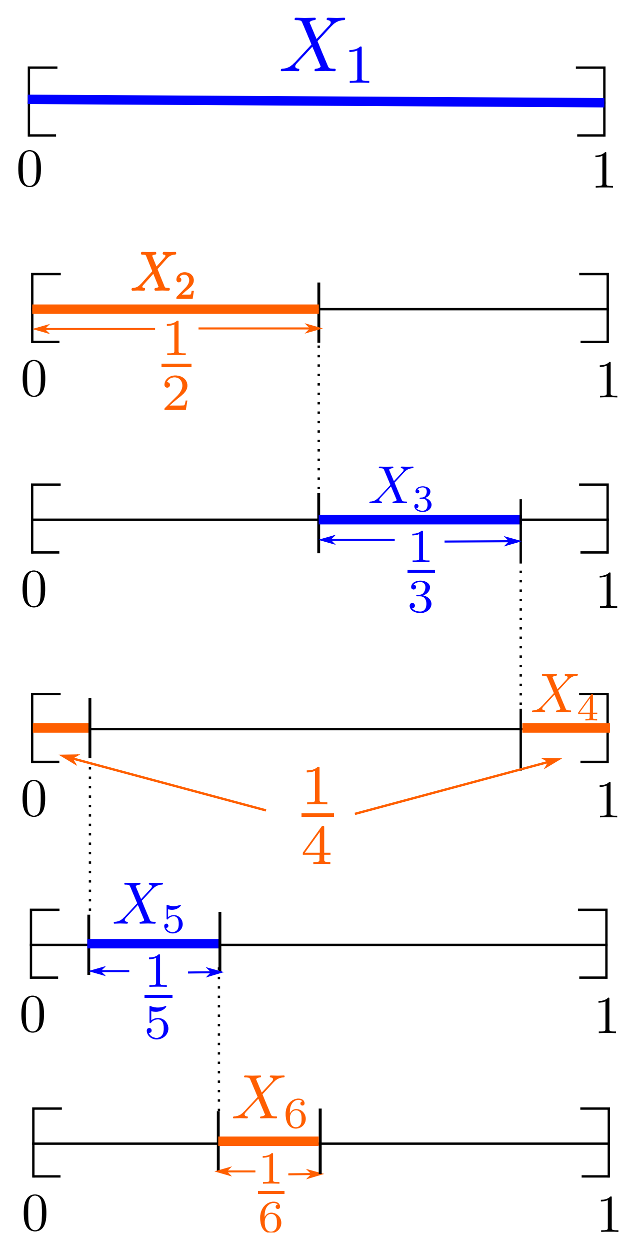
Examples:

1. $\{X_n\}$ converges almost surely: Define $X_n : [0,1] \longrightarrow \mathbb{R}$ by

$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \in [0, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

   Then for each $w \in (0,1]$, $X_n(\omega) \longrightarrow 0$; hence $\mathbf{P}(\{\omega : X_n(\omega) \longrightarrow 0\}) = 1$, i.e. $X_n \xrightarrow{a.s.} 0$.

2. $\{X_n\}$ does NOT converge almost surely: As above, each random variable X_n is a function $X_n : [0,1] \longrightarrow \mathbb{R}$. In this example, for each X_n, we specify a subinterval of $[0,1]$ of length $1/n$ where X_n takes value 1 and outside which X_n takes value 0 by the figure below:



In the figure above,

$$\begin{aligned} X_1(\omega) &= 1 \quad \text{for all } \omega \in [0,1]; \\ X_2(\omega) &= 1 \quad \text{for all } \omega \in [0,1/2]; \\ X_3(\omega) &= 1 \quad \text{for all } \omega \in [1/2, 1/2+1/3] \\ X_4(\omega) &= 1 \quad \text{for all } \omega \in [1/2+1/3, 1] \cup [0, 1/4-(1-(1/2+1/3))] \end{aligned}$$

and so on. The subinterval(s) $\{\omega : X_n(\omega) = 1\}$ is of total length $\frac{1}{n}$, lies immediately to the right of the subinterval $\{\omega : X_{n-1}(\omega) = 1\}$, is truncated at $\omega = 1$ with the "rest" of the length $\frac{1}{n}$ interval "cycled" back to the right of $\omega = 0$.

Because $\sum_{n=1}^{\infty} \frac{1}{n} \to \infty$ but the interval $[0,1]$ has finite length, this "cycling process" will continue, and each number in $[0,1]$ will lie in a subinterval $\{\omega : X_n(\omega) = 1\}$ for infinitely many n's. Hence, $\{\omega : X_n(\omega) \longrightarrow 0\} = \emptyset$, and consequently $\mathbf{P}(\{\omega : X_n(\omega) \longrightarrow 0\}) = 0$.

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Discussion

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- Thank you for the additional material — 1
  I wanted to see the example the lecturer, as he said, would give during the office hours, and here the staff gave it to us as additional ...
- Feedback on filming the videos when professor points at projector screen — 2
  There are times when the professor is pointing at something on the projector screen, but the video is switched to the recording of th...
- Here's a great resource I found online about the topic of this video and exercises — 1
  If you want to understand the difference between Almost Sure Convergence and Convergence in Probability, here's the [link][1] wher...
- Convergence in probability and in distribution 2 — 4
  Hi, For The First Question Answer Should Be Numeric Value Or In Terms of delta? If Mn Is Always (0, 1) Then Mn - 1< 0 But Delta >0 S...
- ? Convergence in distribution — 2
  Lecture and lecture note (slide 30) explains "Convergence in distribution implies convergence of probabilities in the limit has a densit...
- ? What does 'cdf of constant limit' mean? — 4
  Referring to Exercise1:Q4
- Convergence in probability and in distribution 1: Last part clarification — 4
  If this converges in probability then it converges in distribution. If this converges in distribution then it is the CDF of a constant. In my...
- Do we need to know the sample space to apply the definition of almost-surely convergence? — 2
  Around 8:50 minutes into the video, there is an example: let {X1, X2, . . . , Xn} be a sequence of r.v. such that Xn are Ber(1/n). The vid...
- ? about the bounds on epsilon in problem 3 Expectations and convergence in probability — 4
- Optional theoretical material — 2
- ? Exercise 1: Question 1 — 4
  without giving too much away, we know that $\mathbf{P}(|T_n-5|>\delta)=0$ whenever $x<\delta$. I got the correct answer, but I am not confident I got the a...
- Why including a.s./P and (d), when one is sufficient? — 3
- Strong and Weak LNN — 1

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