RECENT PROGRESS OF THE RESEARCH ON NAMBU-GOLDSTONE PHENOMENA

Lunch seminar on 19th December by Yoshihiro Michishita

Outline

① Introduction

② Symmetry spontaneously breaking and NG mode

③ Classical (and quantum action) in open system

④ “Noether charge” and NG mode in open system

⑤ Summary

① Introduction

Nambu-Goldstone theorem (under the Lorentz symmetry)

- Lagrangian has Global continuous symmetries
- Noether charge is locally conserved

⬇ If the ground state doesn't have the symmetry

In the ground state, the gapless mode (NG mode) appears.

N_{BS} corresponds to N_{NG}.

Eg)

	type-A	type-B
Closed	Pendulum,phonon in SF	Spin wave in FM
Open	Flock of bird	Spin wave in FM joined with something ?

① Introduction

The history of the study on NG modes

・NG modes in relativistic system … Nambu - Lasinio (1961)
Goldstone - Cimento (1961)
Goldstone-Salam-Weinberg (1962)

→ Higgs phenomena … Brout, Englert, Guralnik, Hagan, Higgs, Kibble (1964~66)

→ NG modes in non-relativistic system

・Nielsen - Chadha (1976) found type-1 and type-2 modes and

give the inequality $N_{BS} - N_{GS} \leq \mathrm{rank} < [Q_a, Q_b] > /2$

・Watanabe-Murayama (2012) and Hidaka (2013) show this inequality saturates

→ NG modes in non-relativistic open system
Minami - Hidaka (2018)

① Introduction

Symmetry, generator, and the Noether current

Unitary transformation (global) $\hat{U}_Q(\epsilon) = \exp(i\epsilon\hat{Q})$ (Q … generator)

Under this symmetry transformation $\phi \to \phi' = U_Q^{-1}(\epsilon)\phi U_Q(\epsilon) = \phi - i\epsilon[\hat{Q}, \phi]$

$$\delta\phi \equiv -i\epsilon[\hat{Q}, \phi] = \epsilon G$$

"The action has the symmetry" means $\delta S[\phi, \partial\phi] = U_Q^{-1}(\epsilon) S U_Q(\epsilon) - S = 0$

The Lagrangian can change by $\delta\mathscr{L}[\phi, \partial\phi] = \epsilon\partial_\mu(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}G) = \epsilon\partial_\mu X^\mu$

Then we can define the local conserved current by $\partial_\mu j^\mu = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}G - X^\mu\right) = 0$

$$Q' = \int d^3x j^0(x) = Q \qquad dQ'/dt = 0$$

(Q' … Noether charge)

② Symmetry spontaneously breaking and NG mode

- Symmetry spontaneously breaking

Density matrix in the ground state ρ_{GS} (=|GS><GS|) can changes under the symmetry transformation $\delta\rho_{GS} = -i\epsilon[Q, \rho_{GS}]$

If the symmetry is broken in the ground state, some operator ϕ get the expectation value $\mathrm{Tr}[\phi\delta\rho_{GS}] = \epsilon\mathrm{Tr}\big[[iQ, \phi]\rho_{GS}\big] = \epsilon < [iQ, \phi] > \neq 0$

- Emergence of zero modes

Energy density $\mathscr{H}[\{\phi_n\}] = \dfrac{1}{2}\sum_n (\dot{\phi}_n{}^2 + (\nabla\phi_n)^2) + V(\{\phi_n\})$

has the global symmetry described by $\phi \to R\phi,\ \ R = \exp[i\epsilon^a T_a],\ \ T_a = [Q_a, \sim]$

$$V(R\phi) - V(\phi) = \epsilon^a \frac{\partial V}{\partial \phi_n}(iT_a\phi)_n = 0 \quad \leftrightarrow \quad \frac{\partial^2 V}{\partial\phi_m\partial\phi_n}(iT_a\phi)_n + \frac{\partial V}{\partial \phi_n}(iT_a)_{nm} = 0$$

➡ $\mathscr{M}_{mn} < [iQ_a, \phi_n] > = 0 \qquad \left(\mathscr{M}_{mn} = \dfrac{\partial^2 V}{\partial\phi_m\partial\phi_n}\Big|_{\phi=\phi_G} \right)$

➡ Mass Matrix $\mathscr{M}_{mn}$ has the zero modes as many as the linear independent $< [iQ_a, \phi_n] >$ vector.

② Symmetry spontaneously breaking and NG mode

- Linear independence of $< [iQ_a, \phi_n] >$

$$\sum_k < [iQ_a, \phi_k] > < [iQ_b, \phi_k] > = 0$$

$$\Rightarrow \sum_k \Big[< [iQ_a, \phi_k] > , < [iQ_b, \phi_k] > \Big] = 0$$

$$\Rightarrow \quad < [iQ_a, Q_b] > = 0$$

$[iQ_a, Q_b(x)] = Q(x)$ is not Lorentz invariant

$\Rightarrow$ When the system is Lorentz invariant, $< [iQ_a, Q_b] >$ = 0
And $N_{BS} = N_{NG}$ is satisfied !

② Symmetry spontaneously breaking and NG mode

- Counting rules in Lorentz non-invariant system

When $< [iQ_a, Q_b] > \neq 0$, N_{BS} increase by 2
about $< [iQ_a, Q_b] > \neq 0$ and $< [iQ_b, Q_a] > \neq 0$

However, $< [iQ_a, Q_b] > \neq 0$ and $< [iQ_b, Q_a] > \neq 0$ is the same mode.
We call the NG mode "type-B", which is described by the order parameter
$< [iQ_a, Q_b] > \neq 0$

$$N_{BS} = N_A + 2N_B \qquad N_{BS} - N_{GS} = N_B = \frac{1}{2}\mathrm{rank}(< [\mathrm{iQ_a}, \mathrm{Q_b}] >)$$

- Dispersion of type-B NG modes

When $< [iQ_a, Q_b] > = < Q > \neq 0$, the term $C_{ab}(Q_b\dot{Q}_a - Q_a\dot{Q}_b)$ emerges
In the effective action in the ground state.

From the Euler-Lagrange equation, the NG modes has the dispersion
$\omega \sim k^2$

② Symmetry spontaneously breaking and NG mode

・Exercise (Heisenberg model, $Q_a = \{s_x, s_y, s_z\}$)

$$\mathscr{H} = -J\sum_i \overrightarrow{S_i}\cdot\overrightarrow{S_{i+1}} = -\frac{J}{2}\int\frac{d^3x}{a^3}(\nabla\overrightarrow{S})^2$$

$$\Rightarrow \quad S = \int\frac{d^4x}{a^3}\left(\bar{S}(1-\cos\theta)\dot{\phi} + \frac{J}{2}(\nabla\overrightarrow{S})^2\right)$$

($<\theta+\delta\theta, \phi+\delta\phi\,|\,\theta,\phi>/\delta t = \bar{S}(1-\cos\theta)\dot{\phi}$ emerges due to the path integral.
$\bar{S}$ is the magnetization of the unit cell)

・Ferromagnet ($J > 0$)
When we choose the GS as $(S_x, S_y, S_z) = (0,0,S)$,
type-B NG mode appears because $<[iQ_x, Q_y]> = -<S_z> \neq 0$

・Anti-ferromagnet ($J < 0$)
When we choose the GS as $(S_x^A, S_y^A, S_z^A) = -(S_x^B, S_y^B, S_z^B) = (0,0,S)$,
type-A NG mode appears because $<[iQ_x, Q_y]_{A+B}> = 0$

③ "Noether charge" and NG modes in open system

- Strategy

(1) In non-equilibrium system, there is no conventional action
and the conventional charge is not conserved.

(2) We employ the Keldysh action because we can derive the Langevin eq.
and Fokker-Plank eq. by using the stationary-point solution from it.
Moreover, the new 'Noether charge' can be defined.

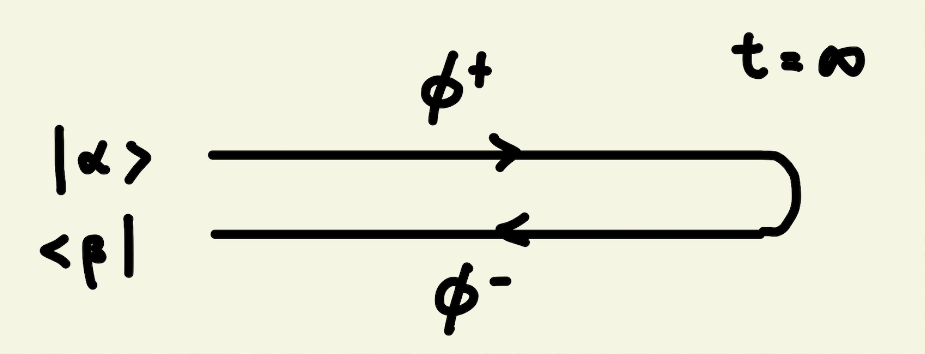
(3) FP eq. can be described by the Hamiltonian and the state vector
because it is the stochastic process. In this formulation, we can verify
the new 'Noether charge' is conserved in the dynamics of the stationary
-point solution of the MSR action.

(4) Check the NG mode

③ Classical (and quantum)action in open system

- Brief review of the Keldysh formalism

von-Neumann equation … $\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)]$



Because the order of time changes at $t = \infty$,
The fields become double $\phi \to (\phi^+, \phi^-)$

Expectation value …

$$< \mathcal{O}(t) > = \mathrm{Tr}[\mathcal{O}\rho(t)] = \mathrm{Tr}[\mathcal{U}_{\infty,t}\mathcal{O}\mathcal{U}_{t,-\infty}\rho(-\infty)\mathcal{U}_{-\infty,\infty}]/\mathrm{Tr}[\rho(-\infty)]$$
$$= \frac{i}{2}\frac{\delta Z[V]}{\delta V(t)}|_{V=0}$$

Generating function : $Z[V] \equiv \mathrm{Tr}\left[\mathcal{U}_C[V]\rho(-\infty)\right]/\mathrm{Tr}[\rho(-\infty)]$

➡ We can define the action and the calculate the expectation value by the functional formalism.

③ Classical (and quantum)action in open system

- Before moving to the main topic, we should do some exercise.

  Here, I want to show the relationship among the Langevin eq. , Keldysh action, and the Martin-Siggia-Rose method.

- Exercise for the Keldysh formalism (free particle in the potential)

Action … $S[X] = \int_C dt\Big[\frac{1}{2}\dot{X}^2 - V(X)\Big] = \int_{-\infty}^{\infty} dt\Big[-2X^q\ddot{X}^{cl} - V(X^{cl}+X^q) + V(X^{cl}-X^q)\Big]$

$$(X^{cl}(t) = \frac{1}{2}[X^+(t) + X^-(t)]; \quad X^q(t) = \frac{1}{2}[X^+(t) - X^-(t)])$$

We can derive the classical EOM : $\frac{\delta S[X]}{\delta X^q}|_{X^q=0} = \ddot{X}^{cl} - \partial_X V(X^{cl}) = 0$

∗ $X^{cl}(t)$ describes the classical dynamics because it represent the dynamics of diagonal elements such as $\sum_X P(X,t)|X><X|$

③ Classical (and quantum)action in open system

- Exercise for the Keldysh formalism (Caldeira-Leggett model)

Let's consider the particle coupling with the dissipative Ohmic bath

$$S_P[X] = \int_{-\infty}^{\infty} dt \Big[-2X^q \ddot{X}^{cl} - V(X^{cl} + X^q) + V(X^{cl} - X^q) \Big]$$

$$S_B[\phi_s] = \frac{1}{2} \sum_s \int_{-\infty}^{\infty} dt \overrightarrow{\phi}_s^T D_s^{-1} \overrightarrow{\phi}_s; \qquad S_{int}[X, \phi_s] = \sum_s g_s \int_{-\infty}^{\infty} dt \overrightarrow{X}^T \sigma^x \overrightarrow{\phi}_s .$$

After integrating out the bath, we get the dissipative action,

$$S_{diss}[X] = \frac{1}{2} \sum_s \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt \overrightarrow{X}^T \mathscr{D}_s^{-1} \overrightarrow{X} \qquad \Big(\mathscr{D}^{-1} = -\sigma^x \Big[\sum_s g_s^2 D_s(t-t') \Big] \sigma^x; \qquad \mathscr{D}^{R(A)-1} = \mp 2\gamma \delta(t-t') \partial_{t'} \Big)$$

$$S[X] \simeq \int_{-\infty}^{\infty} dt \Big[-2X^q \big(\ddot{X}^{cl} + \gamma \dot{X}^{cl} + \partial_X V(X^{cl}) \big) \Big] + 2i\gamma \int_{-\infty}^{\infty} dt \Big[2T(X^q(t))^2 + \sim \Big]$$

By using $\exp\Big[-4\gamma T \int dt (X^q(t))^2 \Big] = \int \mathscr{D}[\xi(t)] \ \exp\Big[\frac{\xi^2(t)}{4\gamma T} - 2i\xi(t) X^q(t) \Big]$

Langevin equation

$$< \mathscr{O}[X^{cl}] > = \int \mathscr{D}[X^{cl}, X^q] \mathscr{O}[X^{cl}] e^{iS[X]} = \int \mathscr{D}[\xi] e^{-\frac{1}{4\gamma T} \int dt \xi^2} \int \mathscr{D}[X^{cl}] \mathscr{O}[X^{cl}] \delta\Big(\ddot{X}^{cl} + \gamma \dot{X}^{cl} + \partial_X V(X^{cl}) - \xi \Big)$$

③ Classical (and quantum)action in open system

When we consider the over damped limit ($\gamma \gg \omega$) and the space dependent noise, Langevin equation changes to the following form (Fokker-Plank equation),

$$\dot{X} = A(X) + b(X)\xi(t)$$

Then the partition function changes and do the invert of the procedure in before slide.

$$Z = \int \mathscr{D}[\xi] e^{-\frac{1}{4\gamma T}\int dt \xi^2} \int \mathscr{D}[X^{cl}] \mathscr{J}[X^{cl}] \delta\Big(\dot{X}^{cl} + \partial_X V(X^{cl}) - b(X^{cl})\xi\Big)$$

$$= \int \mathscr{D}[X^{cl}, X^q] \exp\Big[\int dt \big\{-2iX^q(\partial_t^R X^{cl} - A(X^{cl})) - 4(X^q)^2 D(X^{cl})\big\}\Big] \qquad (D(X) = b(X)^2)$$

By the stationary path approximation, we can derive

$$\dot{X}^{cl} = A(X^{cl}) + 4iX^q D(X^{cl}); \quad i\dot{X}^q = -iX^q \partial_X A(X^{cl}) + 2X^q \partial_X D(X^{cl})$$

For convenience, we set $X^q(t) = P(t)/2i$ and then we can construct the Hamiltonian-like Equation such as,

$$H(P, X) = PA(X) + P^2 D(X); \quad \dot{X} = \partial_P H(P, X); \quad \dot{P} = -\partial_X H(P, X)\,.$$

Then, we can rewrite the action as, $iS[X, P] = -\int dt \big[P\dot{X} - H(P, X)\big]$,

And stationary path appr. gives us $H(P, X) = \mathrm{const}$. along the trajectory.

③ Classical (and quantum)action in open system

Dynamics of the density matrix

⇕

Keldysh formalism

⇓ Perturbation about quantum op.

MSR formalism

⇓ Stationary path app.

Langevin equation

→ Fokker-Plank equation

④"Noether charge" and NG modes in open system

- Symmetry and the conserved quantity of the Langevin equation

$$\dot{\vec{x}} = \vec{u}(t); \quad \dot{\vec{u}} = -\gamma \vec{u}(t) + \vec{\xi}(t);$$

System has the rotational symmetry and the conventional conserved quantity is $\vec{L}_R = \vec{x} \times \vec{u}$. However in this case, it is not conserved $\frac{d}{dt}\vec{L}_R = \vec{x} \times (-\gamma \vec{u} + \vec{\xi}) \neq 0$.

We map to the Fokker-Plank equation in the Hamiltonian-like formalism,

$$\partial_t P(\vec{v}, t) = -H_{FP} P(\vec{v}, t) = \left(\gamma T \frac{\partial^2}{\partial \vec{v}^2} + \gamma \frac{\partial}{\partial \vec{v}}\right) P(\vec{v}, t); \quad P(\vec{v}, t) \equiv < \delta(\vec{u}(t) - \vec{v}) >$$

This Hamiltonian also has the rotational symmetry and the corresponding conserved quantity $\vec{L}_A = -i \vec{v} \times \frac{\partial}{\partial \vec{v}} \left(\neq \vec{L}_R \right)$. This is conserved because $[H_{FP}, \vec{L}_A] = 0$.

④"Noether charge" and NG modes in open system

- Noether charge in MSR action

Remember the MSR action $iS[X,P] = -\int dt\big[P\dot{X} - H(P,X)\big]$ in the previous section,

Now we set $\overrightarrow{X} = (\overrightarrow{\phi}_R, \overrightarrow{\pi}_R)$, $\overrightarrow{P} = (\overrightarrow{\phi}_A, \overrightarrow{\pi}_A)$ and get the following action.

$$iS = -\int d^4x\Big\{ i\pi_A^a(\partial_t\phi_R^a - \pi_R^a) - i\phi_A^a\big[\partial_t\pi_R^a + \gamma\pi_R^a + (-\nabla^2 + m^2 + u^2\phi_R^{b2})\phi_R^a\big] - \frac{A}{2}\phi_A^{a2}\Big\}$$

(Here we employ the ϕ^4-model like potential up to the second order of quantum op.)

This action in invariant under the O(N) transformation and the corresponding Noether charge is ,

$$\delta_\alpha\phi^a_{R(A)}/\epsilon = [T^\alpha]^a_b\phi^b_{R(A)}; \quad \delta_\alpha\pi^a_{R(A)}/\epsilon = [T^\alpha]^a_b\pi^b_{R(A)}$$

$$Q_A^\alpha = -\int d^3x\Big[\frac{\delta S}{\delta(\dot{X})}\frac{\delta X}{\epsilon}\Big] = -\int d^3x\Big[\pi_A^a i[T^\alpha]^a_b\phi_R^b + \pi_R^a i[T^\alpha]^a_b\phi_A^b\Big]$$

This action is also invariant under another O(N) transformation

$$\delta_\alpha P^a/\epsilon = [T^\alpha]^a_b P^b; \quad \delta_\alpha X^a = 0$$

$$Q_R^\alpha = -\int d^3x\Big[\frac{\delta S}{\delta(\dot{P})}\frac{\delta P}{\epsilon}\Big] = -\int d^3x\Big[\pi_R^a i[T^\alpha]^a_b\phi_R^b\Big]$$

Doubling Symmetries occur
Due to the Keldysh formalism

④"Noether charge" and NG modes in open system

・Symmetry spontaneously breaking in stationary path

The effective potential of the MSR action is written by,

$$V_{eff} = im^2\phi_A^a\phi_R^a + \frac{A}{2}\phi_A^{a2} + iu^2\phi_R^{b2}\phi_A^a\phi_R^a$$

The stationary solution is given by

$$\delta V_{eff}/\delta\phi_R^a = 0; \quad \delta V_{eff}/\delta\phi_A^a = 0$$

$$<\phi_R^a>^2 = \frac{-m^2}{3u^2}; \quad <\phi_A^a> = i\frac{-2m^2}{3A}<\phi_R^a>$$

Hereafter, we choose the GS as $<\phi_R^a> = \frac{\phi_0}{\sqrt{3}}\delta_1^a$

And, in this case, the order parameter $-<[iQ_A^\alpha, \phi_R^a]> = i[T^\alpha]_1^a\phi_0$ becomes finite and then, the O(N) symmetry is spontanepusly broken into O(N-1) symmetry and ϕ_R^b for b={2,..,N} become GS boson.

④"Noether charge" and NG modes in open system

- NG modes in open system

Around the stationary state we chose before, we can parametrize the fields as,

$$\phi_R^a(x) = (\phi_0/\sqrt{3} + \sigma_R(x), \xi_R^b(x)), \quad \phi_A^a = (\sigma_A(x), \xi_A^b(x))$$

Using this, we derive the effective action as,

$$iS = \int d^4x \left[-\frac{1}{2}(\sigma_R \quad \sigma_A) \begin{pmatrix} 0 & iD_{\sigma,A}^{-1} \\ iD_{\sigma,R}^{-1} & A \end{pmatrix} \begin{pmatrix} \sigma_R \\ \sigma_A \end{pmatrix} - \frac{1}{2}(\chi_R^b \quad \chi_A^b) \begin{pmatrix} 0 & iD_{\chi,A}^{-1} \\ iD_{\chi,R}^{-1} & A \end{pmatrix} \begin{pmatrix} \chi_R^b \\ \chi_A^b \end{pmatrix} - V_{\text{int}} \right],$$

$$V_{int} = iu^2 \left[\sigma_A (3\phi_0 \sigma_R^2 + \phi_0 \chi_R^{b2} + \sigma_R \chi_R^{b2} + \sigma_R^3) + \chi_A^b \chi_R^b (2\phi_0 \sigma_R + \chi_R^{b2} + \sigma_R^2) \right]$$

Here, χ^b are the massless modes and its dispersion can be determined by

$$D_{\chi,R}^{-1}(\omega, k) = -\omega^2 - i\gamma\omega + k^2 = 0; \qquad \omega(k) = \frac{i}{2}\gamma \pm \frac{i}{2}\sqrt{\gamma^2 - 4k^2}$$

For small k,

$$\omega(k) \sim \frac{i}{2\gamma}k^2, -i\gamma + \frac{i}{\gamma}k^2$$

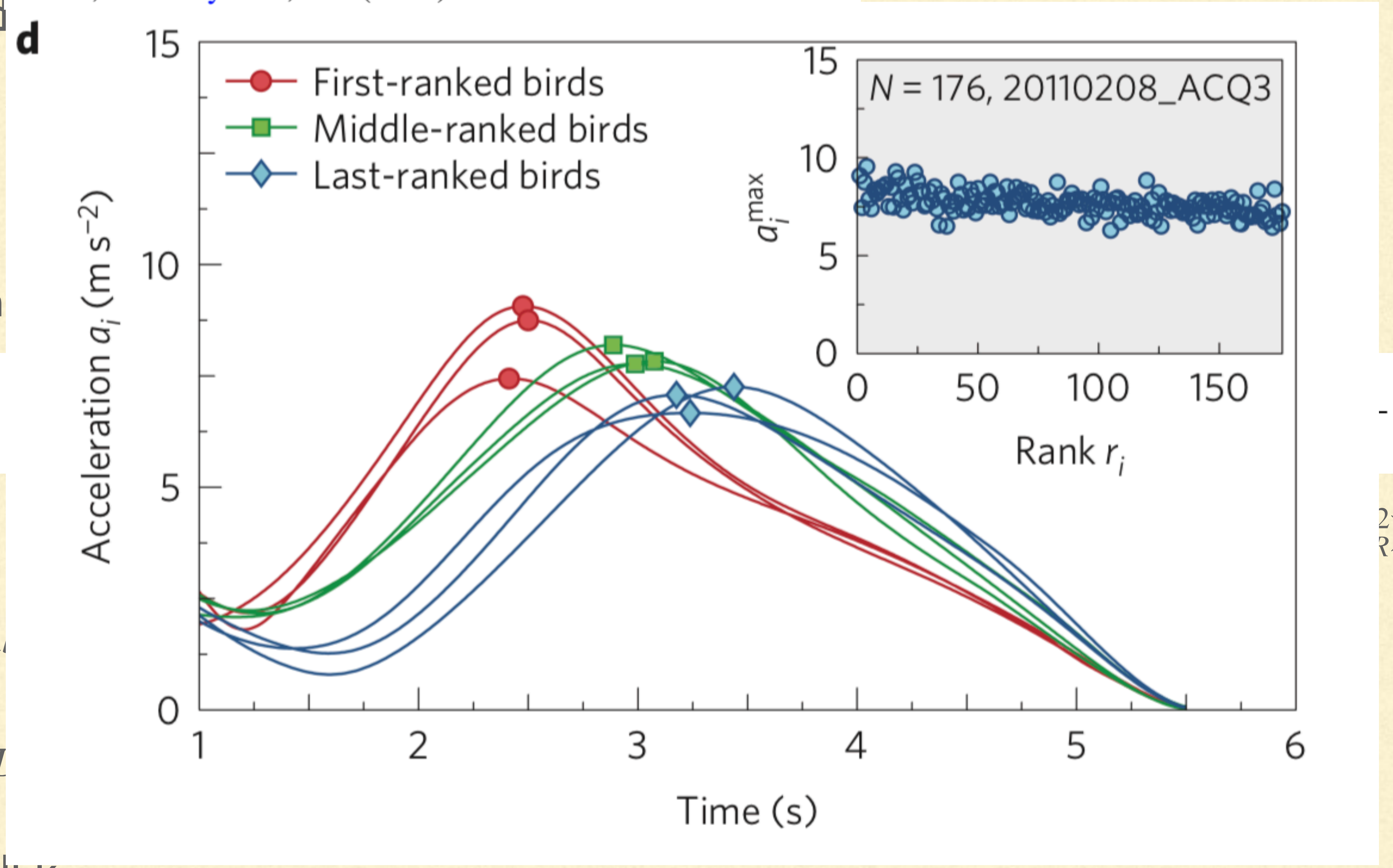
Diffusive NG mode!

④"Noeth... open system

A. Attanasi, A. Cavagna, L. Del Castello, I. Giardina, T. S. Grigera, A. Jelić, S. Melillo, L. Parisi, O. Pohl, E. Shen, and M. Viale, Information transfer and behavioural inertia in starling flocks, Nat. Phys. **10**, 691 (2014).

- NG mod...

Around



Using th...

$$iS = \int \ldots - V_{\mathrm{int}} \Big],$$

Here, χ...

For small k,

$$\omega(k) \sim \frac{i}{2\gamma}k^2, -i\gamma + \frac{i}{\gamma}k^2$$

$\boxed{\text{Diffusive NG mode!}}$