Last Step of 3 by 3 Solution Idea

The last step of the implemented solution for the 3×3 cube comes after everything is solved except for the orientation of the corners of the *Down* face. When this step is reached, each corner of the *Down* face is already in its desired location but may be in the wrong orientation. This step rotates the corners in place into their correct orientation.

The typical simple algorithm to solve this step is to rotate each time 2 corners which share a row. Each sub-step like that requires 16 moves! For this reason it would be helpful to find the shortest combination of sub-steps which solves the cube. And in addition to be able to determine whether a give cube state is solvable before applying any move.

Definition: For the 4 corners of the *Down* face, we define a value based of the *Down*-colored sticker location (See Table 1). For example if the *Down*-colored location of the *Front-Left* corner is in the *Left* face, the *Down*-colored location of the *Left-Back* corner is in the *Back* face, of the *Back-Right* corner is in the *Right* face and of the *Right-Front* corner is on the *Down* face, then the values of the corners will be 2, 1, 0, 2 accordingly. (See Table 2).

Note that the *Right* and *Left* faces are swapped because the cube is being held upside down.

Table 1: Down Corner Values

			Back			
		1		1		
	2	0		0	2	
Right			Down			Left
	2	0		0	2	
		1		1		
			Front			

Table 2: Down Corner Values Example

			Back			
				x		
	x					
Right			Down			Left
		x			x	
			Front			

The “x”-s mark where the *Down*-colored location is in each ocrner.

Definition: Let Fm, Rm, Bm, Lm be the moves required for the sub-step if it were applied on the *Front*, *Right*, *Back* or *Left* faces accordingly.
For reference: $Lm = \langle L, D, D, L', D', L, D', L', R', D', D', R, D, R', D, R \rangle$.
For each sub-step moves $X = \langle m_1, \ldots, m_n \rangle$, let X^{-1} be the moves of X in opposite order and in opposite direction, I.E. $X^{-1} = \left\langle m_n^{-1}, \ldots, m_1^{-1} \right\rangle$.

By inspection and symmetry, we can deduce the effects of the sub-steps on the corner orientation values. Each sub-step X affects exactly 2 corners values, the 2 corners which has a side in X's face.
Here is how each sub-step affects its 2 corner values:

Table 3: Down Corner Values with Sub-Step Effects

(Results are $\mod 3$)

			Bm: $+2$ Bm^{-1}: $+1$			
		1	Back	1		
	2	0		0	2	
Rm: $+1$ Rm^{-1}: $+2$	Right		Down		Left	Lm: $+1$ Lm^{-1}: $+2$
	2	0		0	2	
		1	Front	1		
			Fm: $+2$, Fm^{-1}:$+1$			

Let's redefine our problem with the new terms: "For a given $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ corner values, is there a combination of adding 1 or 2 to two non-opposite corners in such way that the end result would be $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \mod 3$? And if so, what is the shortest combination that allows that?"

Solution: Let $\alpha, \beta, \gamma, \delta$ the added values to the left, right, top, and bottom pairs accordingly. I.E. after the addition, the end result would be $\left(\begin{smallmatrix} x+\alpha+\gamma & y+\beta+\gamma \\ z+\alpha+\delta & w+\beta+\delta \end{smallmatrix}\right)$. Hence, we need to solve the following linear equations: $\begin{cases} \text{I} & x+\alpha+\gamma=0 \\ \text{II} & y+\beta+\gamma=0 \\ \text{III} & z+\alpha+\delta=0 \\ \text{IV} & w+\beta+\delta=0 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -x \\ 0 & 1 & 1 & 0 & -y \\ 1 & 0 & 0 & 1 & -z \\ 0 & 1 & 0 & 1 & -w \end{array}\right) \xRightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -x \\ 0 & 1 & 1 & 0 & -y \\ 0 & 0 & -1 & 1 & x-z \\ 0 & 0 & -1 & 1 & y-w \end{array}\right) \xRightarrow[R_3 \to -R_3]{R_4 \to R_4 - R_3}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -x \\ 0 & 1 & 1 & 0 & -y \\ 0 & 0 & 1 & -1 & z-x \\ 0 & 0 & 0 & 0 & y+z-x-w \end{array}\right) \xRightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -z \\ 0 & 1 & 0 & 1 & x-y-z \\ 0 & 0 & 1 & -1 & z-x \\ 0 & 0 & 0 & 0 & y+z-x-w \end{array}\right)$$

$$
\text{Sols} = \begin{cases} \left\{ \begin{pmatrix} -t-z \\ -t-w \\ t+z-x \\ t \end{pmatrix} \mid t \in \mathbb{Z}_3 \right\} & \text{if } x+w \equiv y+z \mod 3 \\ \emptyset & \text{else} \end{cases}
$$

Hence, such combination exists iff $x+w \equiv y+z \mod 3$ I.E. iff the sum of the *Back-Right* and *Front-Left* corner values is equal to the sum of the *Left-Back* and the *Right-Front* corner values modulo 3. When exists, each solution for $\alpha, \beta, \gamma, \delta$ can be translated to a combination of sub-steps which solves the cube. Furthermore, because we can add 1 or 2 (mod 3) to any two non-opposite corners using only 1 sub-step, the shortest combination of sub-steps would be the solution with the most occurrences of 0.

For example, for the cube state as shown in Table 2, $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. Hence there are 3 solutions:

$$
\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}
$$

which are being translated into the sub-step combinations

$$
\left\{Lm, Bm^{-1}\right\}, \left\{Rm^{-1}, Bm, Fm^{-1}\right\}, \left\{Rm, Lm^{-1}, Fm\right\}
$$

accordingly. And the shortest combination of sub-steps is indeed the one represented by the vector with the most 0 occurrences.