SECTION 4

Chapter 13, 14 and 15

CHAPTER 13

Chance Error

- Chance/Probability
- Conditional Probability
- The Multiplication Rule
- Independence

Chance

Definition: the percentage of time it is expected to happen, when the basic process is done over and over again, independently and under the same conditions.

Properties:
Assume there is some event A. The chance that it is expected to happen is p. (Or we can actually say the the **probability** of event A will happen is p).
Notation: P(A) = p
(1). If event A is impossible: p = 0
(2). If event A is sure to happen: p = 1 (So, if chance is closer to 1, it will be more likely to happen)
(3). 0 <= p <= 1 or in another form, 0% <= p <= 100%
(4). The chance of the **opposite event** for A: 1 – p (Obviously, 0 <= 1 – p <= 1, it still statisfies the (3))

Example Questions in Page 226:

- A coin will be tossed 1,000 times. About how many heads are expected?
  N =1,000 P(Head) = 0.5 E(#of Head) = N × P(Head) = 1000 × 0.5 = 500
  Note: P(Head) means the probability that a head shows for a toss
- A die will be rolled 6,000 times. About how many aces are expected?

Conditional Probability

Definition: a measure of the probability of an event occurring, given that another event has already occurred.

Assume there is some event A, and some event B. The probability that the event A will happen given that the event B has already happened is p.

Notation: $P(A \mid B) = p$ (the vertical bar is read "given")

Example Questions in Page 227:

1. Two tickets are drawn at random without replacement from the box 1 2 3 4 .

(a) What is the chance that the second ticket is 4? $\frac{1}{4}$ (see Example 2 in the book)

(b) What is the chance that the second ticket is 4, given the first is 2?

$P(T_2 = 4 \mid T_1 = 2) = \frac{1}{3}$ The first is 2, since without replacement, only the boxes 1,3,4 are left, the chance of 4 is $\frac{1}{4-1}$

2. Repeat exercise 1, if the draws are made with replacement.

$P(T_2 = 4 \mid T_1 = 2) = \frac{1}{4}$ The first is 2, since with replacement, all the boxes 1,2,3,4 are left for the second draw

Notes:

With replacement: When we sample with replacement, the two sample values are independent.
Practically, this means that what we get on the first one doesn't affect what we get on the second.

Without replacement: When we sample without replacement, the two sample values aren't independent.
Practically, this means that what we got on the for the first one affects what we can get for the second one.

The Multiplication Rule

Definition: The chance that two things will both happen equals the chance that the first will happen, multiplied by the chance that the second will happen given the first has happened.

Assume there is some event A, and some event B.
The probability that both A and B will happen equals to
the probability of A will happen given B have already happened times the probability of event B will happen.
OR
the probability of B will happen given A have already happened times the probability of event A will happen.

Notation: $P(A \text{ and } B) = P(A \mid B) \times P(B) = P(B \mid A) \times P(A)$

$P(A \mid B) = \dfrac{P(AB)}{P(B)}$ And if there is another event C: $P(A \text{ and } B \text{ and } C) = P(A) \times P(B \mid A) \times P(C \mid A,B)$

Example Questions in Page 230:

1. A deck is shuffled and three cards are dealt.

(a) Find the chance that the first card is a king. $P(K_1) = \frac{4}{52} = \frac{1}{13}$

(b) Find the chance that the first card is a king, the second is a queen, and the third is a jack.

$P(K_1Q_2J_3) = P(K_1) \times P(Q_2 \mid K_1) \times P(J_3 \mid K_1, Q_2) = \frac{4}{52} \times \frac{4}{52-1} \times \frac{4}{52-2}$

Independence

Definition: Two things are independent if the chances for the second given the first are the same, no matter how the first one turns out. Otherwise, the two things are *dependent*.

Assume there is some event A, and some event B.
If A and B are independent, the probability of A will happen given B has already happened equals to the probability of A will happen.
OR
the probability of B will happen given A has already happened equals to the probability of B will happen.

Notation: $P(A \mid B) = P(A)$, $P(B \mid A) = P(A)$ (It can also be the rule to judge if A and B are independent)
$P(A \text{ and } B) = P(A \mid B) \times P(B) = P(A) \times P(B)$ (a special case for multiplication rule)
If there is another event C, independent with A, B, $P(ABC)=P(A) \times P(B) \times P(C)$

Example Questions in Page 230:

1. A die is rolled three times.

(a) Find the chance that the first roll is an ace. $P(A_1) = \frac{1}{6}$

(b) Find the chance that the first roll is an ace, the second roll is a deuce, and the third roll is a trey.

Three rolls are independent. $P(A_1D_2T_3) = P(A_1) \times P(D_2) \times P(T_3) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$

CHAPTER 14

More about Chance

- Listing the Ways (Sample Space)
- The Addition Rule
- Mutually Exclusive and Independent
- Multiplication Rule VS Addition Rule

Listing the Ways

Listing all the possible ways is sometimes very helpful for use to figure the chance.

Example Questions in Page 240-241:

1. Two draws are made at random with replacement from the box 1,2,3,4,5. What is the chance that the sum of the two draws turns out to equal 6?

First draw: 5 ways; Second draw: 5 ways. Total: $5^2 = 25$ ways

The ways of 2 draws having sum of 6: {15, 24, 33, 42, 51} = 5 ways

$$P(\text{sum} = 6) = \frac{\#\ of\ ways\ having\ sum\ of\ 6}{\#\ of\ total\ ways} = \frac{5}{25} = 0.2$$

2. A pair of dice is thrown 1,000 times. What total should appear most often? What totals should appear least often?

Dice one: 6 ways; Dice two: 6 ways. Total: $6^2 = 36$ different combinations

The ways of 2 dices having sum of 2: {11} = 1, $P(\text{sum} = 2) = \frac{\#\ of\ ways\ having\ sum\ of\ 2}{\#\ of\ total\ ways} = \frac{1}{36}$

Sum = 3: {12, 21} = 2
Sum = 4: {13, 22, 31} = 3
Sum = 5: {14, 23, 32, 41} = 4
Sum = 6: {15, 24, 33, 42, 51} = 5
Sum = 7: {16, 25, 34, 43, 52, 61} = 6

Sum = 8: {26, 35, 44, 53, 62} = 5
Sum = 9: {36, 45, 54, 63} = 4
Sum = 10: {46, 55, 64} = 3
Sum = 11: {56, 65} = 2
Sum = 12: {66} = 1

The Addition Rule

Definition: to calculate the chance that at least one of two specified things will happen: either the first happens, or the second, or both. Check if they are *mutually exclusive*. If they are, add the chances.

Notation: P(A or B) = P(A) + P(B) – P(AB) ≠ P(A) + P(B) if they are not mutually exclusive
When A and B are mutually exclusive, P (AB) = 0, so P(A or B) = P(A) + P(B)
What will happen if accidentally took 2 events as mutually exclusive when they are actually not?

Example Questions in Page 243:

4. Two dice will be rolled. The chance that the first one lands ⚀ is 1/6. The chance that the second one lands ⚁ is 1/6. True or false: the chance that the first one lands ⚀ or the second one lands ⚁ equals 1/6 + 1/6. Explain briefly.

The events that the first one lands 1 and the second one lands 2 are not mutually exclusive. But they are independent. Why?

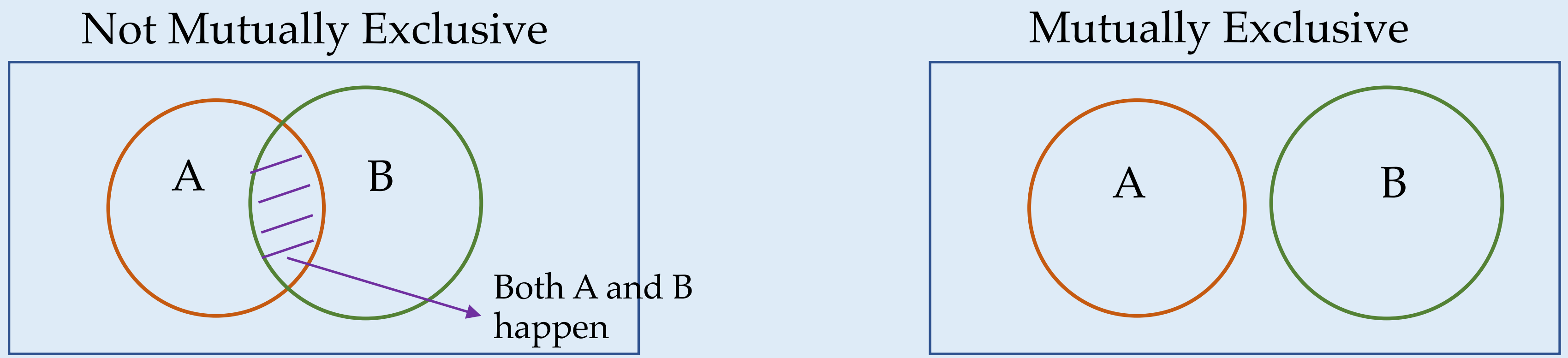
P(first = 1) $= \frac{1}{6}$ P(second = 2) $= \frac{1}{6}$ P(first = 1 and second = 2) = $\frac{1}{6} \times \frac{1}{6}$

So P(first = 1 or second = 2) = P(first = 1) + P(second = 2) - P(first = 1 and second = 2) $= \frac{1}{6} + \frac{1}{6} - \frac{1}{36}$

Mutually Exclusive and Independent

Mutually Exclusive: if the occurrence of one prevents the other from happening.

Example: Vann Diagram:



Independence: if the occurrence of one does not change the chances for the other.

Example: Roll 2 dies

Multiplication Rule VS Addition Rule

Multiplication Rule: find the chance that two things both happen.

Addition Rule: find the chance that at least one of two things happens.

Example Questions in Page 247:

3. A deck of cards is shuffled. True or false, and explain briefly:
   (a) The chance that the top card is the jack of clubs equals $1/52$.
   (b) The chance that the bottom card is the jack of diamonds equals $1/52$.
   (c) The chance that the top card is the jack of clubs or the bottom card is the jack of diamonds equals $2/52$.
   (d) The chance that the top card is the jack of clubs or the bottom card is the jack of clubs equals $2/52$.
   (e) The chance that the top card is the jack of clubs and the bottom card is the jack of diamonds equals $1/52 \times 1/52$.
   (f) The chance that the top card is the jack of clubs and the bottom card is the jack of clubs equals $1/52 \times 1/52$.

(a) True
(b) True
(c) False. Not mutually exclusive
(d) True. Mutually exclusive.
(e) False. Not independent
(f) False. 0

And when it's easier to calculate the chance of opposite, we want to use 1-P(opposite)

Conclusion

- Question Type
  - Find the chance that two things both happen → Multiplication Rule
    - Independent → Multiply the chances
    - Not independent → Multiply with conditional probability
  - Find the chance that at least one of two things happens → Addition Rule
    - Mutually exclusive → Sum the chances
    - Not mutually exclusive → Cannot just sum the chances

CHAPTER 15

Binomial Distribution

- Factorial
- Binomial Coefficient and Distribution

Factorial

The exclamation mark after the number is read as ‘factorial’.

$N! = N \times (N-1) \times \ldots \times 2 \times 1$, N is integer, N >= 0

$N! = N \times (N-1)\,!$

Eg: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$6! = 6 \times 5! = 6 \times 120 = 720$

$1! = 1$

NOTE:

- $0! = 1$
- No factorial definition for negative integer

Where do we need to use factorial?
<u>Combinations and Permutations</u>

Exercises:
How many ways for arranging 10 people to sit in 10 chairs ? $10!$
How many ways to shuffle a deck of cards? $52!$
How many ways to arrange 2 red balls and 4 blue balls? $\frac{6!}{2! \times 4!} \Rightarrow \binom{6}{2}$

Binomial Coefficient and Distribution

- In many applied problems, we are interested in the probability that an event will occur exactly k times out of n:

Binomial Formular:

$$P(X = k) = \underbrace{\boxed{\frac{n!}{k!\,(n-k)!}}}_{\textit{Binomial Coefficient}} p^k (1-p)^{n-k}$$

n: the number of trials
k: the number of times the event is to occur
p: the probability that the event will occur on any particular trial

Binomial coefficient can also be writtern as $\binom{n}{k}$, read as 'n choose k'

Binomial Coefficient and Distribution

★ Binomial coefficient: the number of ways picking k unordered outcomes from n possibilities.

Binomial coefficient properties:

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

- $\binom{n}{k} = \binom{n}{n-k}$ symmetry rule. (k <= n)
- $\binom{0}{0} = 1$
- $\binom{n}{0} = \binom{n}{1} = 1$
- $\binom{n}{1} = \binom{n}{n-1} = n$

Example:
How many ways to arrange 2 red balls and 4 blue balls?
⇔ How many ways to have 2 red balls in 6 balls?
⇔ How many ways to have 4 blue balls in 6 balls?

Binomial Coefficient and Distribution

★ Review addition rule: the event contains m possible ways, and the possibility of the event should be the summation of the possibilities for each way when they are mutually exclusive.

Exercise:
The possibility of get 2 heads in 4 tosses of a coin?

{HHTT}, {HTHT}, {HTTH}, {THHT}, {THTH}, {TTHH}. 6 ways

P(Head) $= \frac{1}{2}$, P(Tail) $= \frac{1}{2}$

Total of ways: use binomial coefficient, $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\times3\times2\times1}{2\times1\times2\times1} = 6$

For each way, the possibility is: $(\frac{1}{2})^2(\frac{1}{2})^2$

Apply for addition rule: $6\times (\frac{1}{2})^2(\frac{1}{2})^2 \Leftrightarrow$ Binomial formular $\binom{4}{2} (\frac{1}{2})^2(\frac{1}{2})^2$ when $n = 4$, $k = 2$, $p = \frac{1}{2}$