MAT1856/APM466 Assignment 1

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Note: Statements of questions can be found in the assignment 1 document.

Fundamental Questions - 25 points

1.

   (a) It depends on the interest rate: When interest rate is low, it is more beneficial to issue bonds; When interest rate is high, it is more sensible to increase money supply.

   (b) One possible scenario where the yield curve of a bond might flatten is when the supply and demand for bonds reach an equilibrium. This can happen, for example, if the interest rates stabilize. In this case, the price of a bond will eventually stop varying, thereby flattening its yield curve.

   (c) Quantitative easing refers to how central banks drive interest rates up and down by buying/selling government issued bonds: Buying raises bond prices and lowers interest rates, while selling brings down bond prices and raises interest rates.[1]

2. The selected bonds are all issued by the government of Canada during 2018 and 2022. They are the following(ISIN+name as appeared in the website):1.CA135087J397(CDA 18/29);2.CA135087J546(CDA 2024);3.CA135087H490(CDA 2023);4.CA135087L930(CANADA 21/26);5.CA135087J967(CDA 19/24); 6.CA135087K528(CANADA 19/25);7.CA135087N266(CANADA 21/31);8.CA135087N670(CANADA 22/29).[2]

   All of them are rated as 'Aaa' by Moody's Rating. Their maturity dates range from 2023 to 2031 with two clusters centered at 2024 and 2029. It was originally anticipated that they would be comparable because they have nearly identical credit risk, similar coupon rates. However, the discrepancies in coupon rates of the remaining were still too high. This introduced unintended fluctuations to our regressions in part 2.

3. The covariance matrix of a collection of stochastic processes describes how the variances of the underlying processes correlate with each other.

   The eigenvectors identify the main directions of variation, and the eigenvalues indicate the degree of correlation with those directions compared to a uniform distribution. Finally, on the relative scale, the larger an eigenvalue, the more dominant is its direction of variations.

[1] Technically, the "easing" refers to lowering the interest rates by buying.
[2] Two data points were knocked out at the analysis stage due to inconsistent credit risk.

Empirical Questions - 75 points

Note: All the rates were first obtained as daily rates, and then converted into annual rates. The rates in plots are all annual.

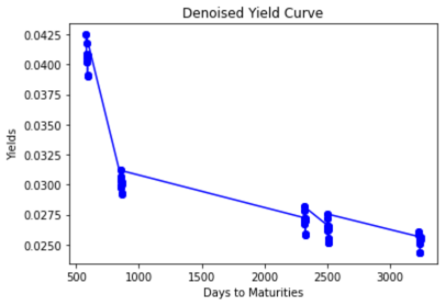
4.

(a) For yield to maturity(YTM), we use the bond-dependent rates with instantaneous compounding. This amounts to solving the following equation:

$$P = \sum_{i=1}^{n} N_i e^{-rT_i}, \tag{1}$$

Where r is the yield, T_i's denote the days remaining until payment of the i-th coupon, and P is the dirty price, which is the listed price plus $C_1 \times \frac{182.5-T_1}{182.5}$[3] Moreover, we combine the last coupon payment and the principal payment, whose payment dates are all < 10 days apart for bonds in our collection.

Two of the bonds in the basket have maturities greater than 5 years(6 and 7 years respectively). They were included to make the choice of 'Aaa' bonds consistent. We expand the range of maturities to account for these two exceptions. Furthermore, the time scale is measured in days to reflect its daily nature. The left portion of the yield curve is too noisy due to large deviations from the average coupon rate. The denoised version[4] is shown below.



Although still somewhat noisy, the downward trend is clearly shown. This is largely because of the recent increase of interest rate, which is now nearly twice as much as the average coupon rate of all bonds in the basket. As a result, the prices of these bonds necessarily drop, so are the yields. Longer term bonds have more coupons left, therefore they devalue more since the drop in price is proportional to the number of coupons.

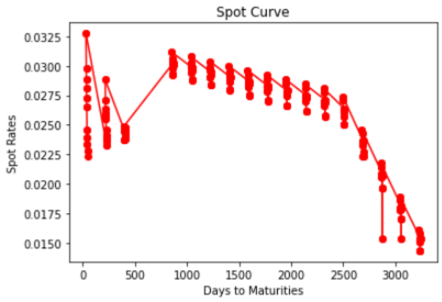
(b) Similar to 4a, we arrive at a system of equations, each equation is of the following form, where r_i's are the spot rates:

$$P = \sum_{i=1}^{n} N_i e^{-r_i T_i} \tag{2}$$

The unknowns are the spot rates r_i which are not necessarily the same at different scale. There are as many equations as the sampled bonds. To solve the system, we start with the bond with fewest coupon payments remaining and bootstrap up. Since there are more unknown spot rates than the number of equations after knocking out two bonds and including two long-term bonds, regularization assumptions need to be imposed to make them solvable. We assume:1.The spot

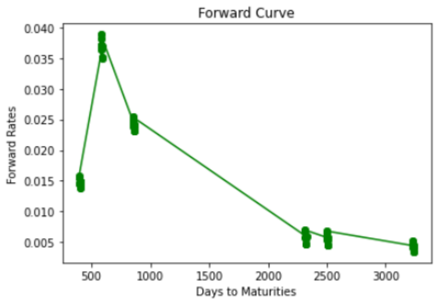
[3] This is the portion of the next coupon that should belong to the current bond holder.
[4] Admittedly this is still very noisy.

rates form arithmetic progression when there are gaps between half-years.[5];2.In case a spot rate r_i has multiple solutions, we take their average and continue bootstrapping. The spot curve is:



Similar to the noisy yield curve, the left portion is unstable, but the overall trend clearly indicates that the spot rate is decreasing and is lower for long term bonds. This is consistent with the information revealed by the yield curve.

(c) For every fixed day t out of the total 11 days of observations, let T_1 be the shortest time to a maturity date, and T_i be any other time from a maturity date, we compute the rate by the formula $r(T_1, T_i) = -\frac{\log P(t,T_i) - \log P(t,T_1)}{T_i - T_1}$. To make the plot, we place these rates above T_i's. We have:



The forward curve exhibits the same trend, except for the beginning left portion. This is because, when interest rate rises, borrowing costs more, while the expected future value of a longer term bound with lower coupon rate goes down.

5. We use Numpy array notations for vectors and matrices. The covariance matrix of daily log returns of yields is [[-0.13404029, -0.98174428, -0.02837439, -0.11314385, -0.05022794, -0.04538579, -0.00469264], [-0.31595755, 0.05416954, -0.6119569 , 0.30807249, -0.63612598, 0.09246733, -0.12095499], [-0.4594921 , 0.05931507, -0.20575222, 0.08819205, 0.28182332, -0.40181938, 0.70310292], [-0.34115042, -0.05960404, -0.05780229, 0.50941872, 0.58435781, 0.44512785, -0.27857201], [-0.47866888, 0.12413184, 0.00895969, -0.33870695, 0.11925623, -0.50130799, -0.61248099], [-0.37048869, 0.0088592 , 0.75850356, 0.36000891, -0.38932116, -0.04492458, 0.06431486], [-0.4339374 , 0.10337175, 0.0601718 , -0.61698924, -0.07829075, 0.61356694, 0.18472255]].

   The covariance matrix of daily log returns of forward rate is [[0.00104532, 0.00079852, 0.00065254, 0.00325328, 0.00299495, 0.00299495], [0.00079852, 0.00077371, 0.00062951, 0.00318051, 0.00303533, 0.00303533], [0.00065254, 0.00062951, 0.00054435, 0.00248355, 0.00237918, 0.00237918], [0.00325328, 0.00318051, 0.00248355, 0.01362964, 0.01297419, 0.01297419], [0.00299495, 0.00303533, 0.00237918, 0.01297419, 0.01245594, 0.01245594], [0.00299495, 0.00303533, 0.00237918, 0.01297419, 0.01245594, 0.01245594]].

[5]For example, if the first bond only has r_1 term, and the second bond has r_1,r_2,r_3 terms, we impose $r_2 = \frac{r_1+r_3}{2}$. This can happen when no bonds in the basket mature in a year.

6. The eigenvalues for covariance matrix of daily log returns of the yields is [2.61287321e-03, 6.46846219e-04, 2.73355309e-04, 5.14406130e-05, 2.79635457e-05, 1.30111940e-06, 3.70477361e-07]. The corresponding eigenvectors are [[-0.13404029, -0.98174428, -0.02837439, -0.11314385, -0.05022794, -0.04538579, -0.00469264], [-0.31595755, 0.05416954, -0.6119569 , 0.30807249, -0.63612598, 0.09246733, -0.12095499], [-0.4594921 , 0.05931507, -0.20575222, 0.08819205, 0.28182332, -0.40181938, 0.70310292], [-0.34115042, -0.05960404, -0.05780229, 0.50941872, 0.58435781, 0.44512785, -0.27857201], [-0.47866888, 0.12413184, 0.00895969, -0.33870695, 0.11925623, -0.50130799, -0.61248099], [-0.37048869, 0.0088592 , 0.75850356, 0.36000891, -0.38932116, -0.04492458, 0.06431486], [-0.4339374 , 0.10337175, 0.0601718 , -0.61698924, -0.07829075, 0.61356694, 0.18472255]].

   The eigenvalues for covariance matrix of daily log returns of forward rates are [4.04173590e-02, 3.60018669e-04, 9.97350289e-05, 2.46820024e-05, 3.11836954e-06, 8.65771732e-19]. The corresponding eigenvectors are [[1.36814473e-01, -8.86025237e-01, 1.33657407e-01, -4.20457003e-01, 3.99084738e-02, -4.36171349e-14], [1.35899194e-01, -1.69735005e-01, -3.67410910e-01, 2.01431755e-01, -8.81564466e-01, -1.52820968e-13], [1.06688536e-01, -2.39431819e-01, -7.76967928e-01, 3.36246141e-01, 4.63194703e-01, 1.27850617e-13], [5.79872216e-01, -1.48865114e-01, 4.55171855e-01, 6.54475704e-01, 7.78943282e-02, 9.37792723e-14], [5.54607510e-01, 2.30933854e-01, -1.34692877e-01, -3.47304795e-01, 1.78121659e-02, -7.07106781e-01], [5.54607510e-01, 2.30933854e-01, -1.34692877e-01, -3.47304795e-01, 1.78121659e-02, 7.07106781e-01]].

   The first eigenvectors represent the direction with the largest concentration of variations of the given family of random variables. Its corresponding eigenvalues measure how spread out these variations are relative to this direction.

   In our data, both first eigenvalues are small compared to $\frac{1}{7}$, the predicted variance if the variations follow the uniform distribution on the spheres. This indicates strong signal in the first directions of both covariance matrices.

References and GitHub Link to Code

1. Link to my GitHub repository for the course. It contains codes, original data, up-sized plots. The naming of folders and files in the repository are consistent with the assignment.
2. All data were collected from Business Insider's website.