Zagier's Rankin-Selberg Method

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Abstract

This is a note on a generalization of Rankin-Selberg method, proved by Zagier and Gupta, which is applicable to a class of automorphic functions with respect to a general congruence subgroup which are not of rapid decay. Some preliminary results are reviewed for completeness of the presentation.

Contents

1 Introduction:Convolution of L-functions

We begin by recalling how the original Rankin-Selberg transformation is to "convolute" L-functions.

Let $\Gamma = \Gamma(1)$, the full modular group. Denote by $E(z, s)$ its Eisenstein series, given by the formula $\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} I^s(\gamma z)$, where $I^s(z) = y^s$, and Γ_∞ is the subgroup of Γ fixing ∞.

Suppose $F(z)$ is a continuous Γ-automorphic function, then $F(z)$ admits a Fourier expansion:

$$F(z) = \sum_{n\in\mathbb{Z}} a_n(y)e^{2\pi nxi} \qquad \text{(eq1)}$$

Moreover, if $F(z)$ decays rapidly at ∞,there is the following "unfolding trick" which expresses the integral of it against $E(z, s)$ over $\Gamma\backslash\mathbb{H}^2$ (with respect to the measure $d\mu = \frac{dxdy}{y^2}$ induced by the standard hyperbolic metric) in terms of $a_0(y)$:

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}^2} F(z)E(z,s)d\mu &= \int_{\Gamma\backslash\mathbb{H}^2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} F(\gamma z)I^s(\gamma z)d\mu \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}^2} F(z)I^s(z)d\mu = \int_{\mathbb{R}_{>0}} y^{s-2}\int_0^1 F(x+yi)dxdy \\ &= \int_{\mathbb{R}_{>0}} y^{s-2}a_0(y)dy\end{aligned} \tag{eq2}$$

The last integral in the above equations defines a function $R(F,s)$ in s,which is called the Rankin-Selberg transformation of F. The rapid decay condition is needed to justify the first two equalities.

We can complete $R(F,s)$ to $R^*(F,s)=\zeta^*(2s)R(F,s)$,where $\zeta^*(s)$ is the completed zeta function $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. Two immediate consequences of the integral identity are the meromorphic continuation of $R^*(F,s)$(hence $R(F,s)$) with at most simple poles at $s=0$ and $s=1$,and a functional equation of $R^*(F,s)$ with respect to the transformation $s\rightarrow 1-s$. They essentially follow from the basic properties of Eisenstein series([1],p66,Theorem1.6.1).

Given two cusp forms of weight k, which we denote by f and g for concreteness. $\phi(z)=f(z)g(z)y^k$ is an automorphic function of rapid decay, in which case eq2 is true.

Let $\{A(n)\}_{n\geq 0}$ and $\{B(n)\}_{n\geq 0}$ be the Fourier coefficients of f and g respectively.

It can be shown that([1],p71)

$$\phi_0(y)=\sum_{n=1}^{\infty} A(n)\bar{B}(n)e^{-4\pi ny}y^k \tag{eq3}$$

Furthermore, if $g(z)$ is an Hecke eigenform, then

$$\phi_0(y)=\sum_{n=1}^{\infty} A(n)B(n)e^{-4\pi ny}y^k \tag{eq4}$$

The properties of Rankin-Selberg transformation then imply that the "convolution" of $L(s,f)$ and $L(s,g)$,$L(s,f\times g)=\sum_{n\geq 1}\frac{A(n)B(n)}{n^s}$, has meromorphic continuation and satisfies a functional equation. This is the so-called Rankin-Selberg method,which was first introduced to study Ramanujan's τ function([5]).

Two obvious questions can be asked:1.What happens when the automorphic function does not decay rapidly? 2.What happens if $\Gamma(1)$ is replaced by a general congruence subgroup?

These very natural questions were addressed by the works of Zagier and Gupta([6],[2]). A generalization of Rankin-Selberg transformation, now called “Zagier’s Rankin-Selberg transformation” was proposed to handle a more general class of automorphic functions, called “renormalizable automorphic functions”, with respect to an arbitrary congruence subgroup.

2 Results and Proofs by Zagier and Gupta

2.1 Results by Zagier

$R(F,s)$ might fail to converge if F does not decay rapidly. Zagier proposed to consider the following class of automorphic functions, in which case the convergence issue can be remedied:

Definition (Renormalizable Functions)**.** A continuous Γ-automorphic function $F(z)$ is said to be renormalizatible if there is a function $\psi(y) = \sum_{i=1}^{l<\infty} \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i}(y)$, with $c_i, \alpha_i \in \mathbb{C}$ and $n_i \in \mathbb{Z}_{\geq 0}$, such that $F(z) - \psi(y)$ decays rapidly in y uniformly.

It is easy to see that the function ψ in the definition is unique.

The modification of Zagier is very simple: do the Rankin-Selberg transformation after throwing away the parts which do not decay rapidly.

Definition (Zagier’s Rankin-Selberg Transformation)**.** If $F(z)$ is renormalizable, Zagier’s Rankin-Selberg transformation $\tilde{R}(F,s)$is defined to be the integral
$\int_{\mathbb{R}_{>0}} y^{s-2}(a_0(y) - \psi(y))dy,$
where $a_0(y)$ is the constant term of the Fourier expansion of $F(z)$ and $\psi(y)$ is the function in the definition of renormalizable functions.

$\tilde{R}(F,s)$ clearly converges. Moreover, it can be shown to have similar properties as the usual Rankin-Selberg transformation. In fact, they coincide when $F(z)$ decays rapidly.

Theorem (Zagier,[6],p 419)**.** *$\tilde{R^*}(z,s) = \zeta^*(2s)\tilde{R}(z,s)$ satisfies the same functional equation under $s \to 1-s$, and has at most poles at $s = 0, 1, \alpha_i$,and $\frac{\rho}{2}$’s, where α_i’s are those in the definition of $\psi(y)$ and ρ’s are the zeros of the Riemann Zeta function.*

The proof requires careful analysis of truncated domains and constant term of the Eisenstein series. We will omit the technical details but provide the relevant references. We will indicate the ideas and ingredients are used to prove Zagier’s theorem.

Fact (An Expression for Truncated Domain([6],p420,eq20))**.** Let $\mathcal{D} = \{z \in \mathbb{H}^2 : |z| \geq 1, |x| \leq \frac{1}{2}\}$ be the standard fundamental domain for Γ. For $T > 0$, define the truncated domain $\mathcal{D}_T := \{z \in \mathbb{H}^2 : |z| \geq 1, |x| \leq \frac{1}{2}, y \leq T\}$.

The set $\mathfrak{D}_T = \bigcup_{\gamma\in\Gamma} \gamma\mathcal{D}_T = \{z \in \mathbb{H}^2 : I(\gamma z) \leq T\}$, since $\mathcal{D}$ is a fundamental domain and the elements in Γ do not increase the imaginary parts of elements in $\mathbb{H}^2$.

$\mathfrak{D}_T$ can be re-expressed as $\{z \in \mathbb{H}^2 : I(z) \leq T\} - \bigcup\limits_{\substack{c\geq 1 \\ (a,c)=1}} \bigcup\limits_{a\in\mathbb{Z}} S_{a/c}$,

where $S_{a/c}$ is the disc of radius $\frac{1}{2c^2}T$ which is tangent to the x-axis at $\frac{a}{c}$.

In addition to the application of the Rankin-Selberg method to the truncated domain, some critical inputs from the classical theory of Eisensetein series are needed([6],eq 23,24):

$$\int_0^1 E(x+yi,s)dx = y^s + \frac{\zeta^*(2s-1)}{\zeta^*(2s)}y^{1-s},$$

e(y,s)=$\int_0^1 E^*(x+yi,s)dx = \begin{cases} \zeta^*(2s)y^s + \zeta^*(2s-1)y^{1-s} & s \neq 0, \frac{1}{2}, 1 \\ y^{\frac{1}{2}}\log y + (\gamma - \log 4\pi)y^{\frac{1}{2}} & s = \frac{1}{2}, \gamma \text{ is Euler's gamma constant.} \end{cases}$

e(y,s)=e(y,1-s)

Another important fact is that $E^*(z,s) - e(y,s)$ is entire in s and of rapid decay in y.

In the course of integrating over the truncated domain, an auxiliary function $h_T(s)$ is introduced. This is the Mellin transform of $\psi(y)$ cut off at T.

After carefully analyzing the contribution of the integral of F against E from each components of the truncated domain, the following expression([6],eq27) for $\tilde{R}(F,s)$ is obtained:

$$\tilde{R}(F,s) = \int_{\mathcal{D}_T} F(z)E^*(z,s)d\mu + \int_{\mathcal{D}-\mathcal{D}_T} (F(z)E^*(z,s) - \phi(y)e(y,s))d\mu$$

-$\zeta^*(2s)h_T(s) - \zeta^*(2s-1)h_T(1-s)$

It is then obvious from the right hand side that $\tilde{R}(F,s)$ has all the analytic properties it was claimed to have.

2.2 Results by Gupta

It was pointed out by Zagier that similar analysis can be applied to general congruence subgroup to obtain analogous result, the only difference is that the unfolding has to be done around each cusp([6],p420,remark2). This was explicitly carried out by Gupta [2]. Some preliminaries are needed to state the results in a more compact form which resembles the case for the full modular group.

Suppose $\Gamma \leq \Gamma(1)$ is a congruence subgroup. For concreteness, assume Γ has h cusps, label them by $\{\kappa_1, ..., \kappa_h\}$, where $\kappa_1 = \infty$ in accordance with Gupta's notations. Once again, denote by Γ_∞ the subgroup of Γ fixing ∞, which is now a subgroup of the translation group, and similarly use Γ_i to denote the subgroup fixing κ_i.

For every cusp κ_i, there is at least one $\alpha_i \in SL(2, \mathbb{Q})$[1] sending ∞ to κ_i, such that $\alpha_i^{-1}\Gamma_i\alpha_i = \Gamma_\infty$. Therefore, if $F(z)$ is continuous and Γ-automorphic,$F_i(z) = F(\alpha_i z)$ has an integer period m for some $m \in \mathbb{N}$, hence it has a Fourier expansion:

$$F_i(z) = \sum_{n\in\mathbb{Z}} a_{i,n}(y)e^{\frac{2\pi nxi}{m}} \tag{6}$$

It makes sense to discuss renormalizability just as in the case of automorphic functions for the full-modular group, because the definition did not make explicit reference to automorphy. Moreover, it is easy to see that the definition of F_i is independent of the choice of the transformation α_i.

A continuous Γ-automorphic function is said to be renormalizable if F_i is renormalizable when i runs over the index of all the cusps.

Eisenstein series at the cusp κ_i is defined to be $E_i(z, s) = \sum_{\delta\in\Gamma_i\backslash\Gamma} I^s(\alpha_i^{-1}\delta z)$.

Form a column vector using these Eisenstein series,

$$\text{E(z,s)} = \begin{bmatrix} E_1(z, s) \\ E_2(z, s) \\ \vdots \\ E_h(z, s) \end{bmatrix} \tag{7}$$

The role of $\zeta^*(2s)$ is taken over by a matrix $\Phi(s)$,called "the scattering matrix"([3],part2), or the "constant term matrix"([4],p16).

We have a functional equation analogous to the case of full modular group:

$$E(z, s) = \Phi(s)E(z, 1 - s) \tag{8}$$

[1] Actually, it can be done using an element in $SL(2, \mathbb{Z})$.

$$\Phi(s)\Phi(1-s) = I_{h\times h} \tag{9}$$

Zagier's Rankin-Selberg transformation $\tilde{R}(F,s)$ is now vector valued, whose entries are $\tilde{R}_i(F,s) = \int_{\mathbb{R}_{>0}} y^{s-2}(a_{i,0}(y) - \psi_i(y))dy$.

$\tilde{R}(F,s)$ has similar analytic properties as in the case for the full modular group:

Theorem (Gupta [2],Main Theorem)**.** *Suppose*

$$\psi_i(y) = \sum_{j=1}^{l<\infty} \frac{c_{i,j}}{n_{i,j}!} y^{\alpha_{i,j}} \log^{n_{i,j}}(y),$$

where $c_{i,j}, n_{i,j}$ *and* $\alpha_{i,j}$ *are as in p2,then* $\tilde{R}_i$ *can be meromorphically continued, with possible poles at* $s = 0, 1, \alpha_{i,j}$*,and* $\frac{\rho}{2}$*'s.*

Moreover, $\tilde{R}(F,s)$ *satisfies the functional equation*

$$\tilde{R}(F,s) = \Phi(s)\tilde{R}(F,1-s).$$

Gupta's analysis is almost identical to those given by Zagier, except it is done one cusp at a time.

After arriving at the analogue of Zagier's eq 27, which is numbered equation 9 in [2], the functional equation follows from the properties of the scattering matrix and how it interacts with the Eisenstein series associated to Γ.

3 Applications: Computing Volumes of Fundamental Domains

Zagier's Rankin-Selberg transformation gives an alternative method to compute the volume of fundamental domain.

Apply the transformation to the constant function 1, Zagier's equation 27 becomes,after taking residue at $s = 1$,

$$0 = -\zeta^*(2) + \frac{1}{2T} + \frac{1}{2}\mathrm{Vol}(\mathcal{D}_T)$$

Letting $T \to +\infty$ gives $\mathrm{Vol}(\mathcal{D}) = \frac{\pi}{3}$. In general, the volume of a fundamental domain of a congruence subgroup Γ of $\Gamma(1)$ is $[\Gamma(1) : \Gamma]\frac{\pi}{3}$.

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