Master in Computer Vision Barcelona

M2 – Optimisation in Computer Vision
Part 3 – Segmentation

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1. Segmentation:
Problem Definition

Problem Definition

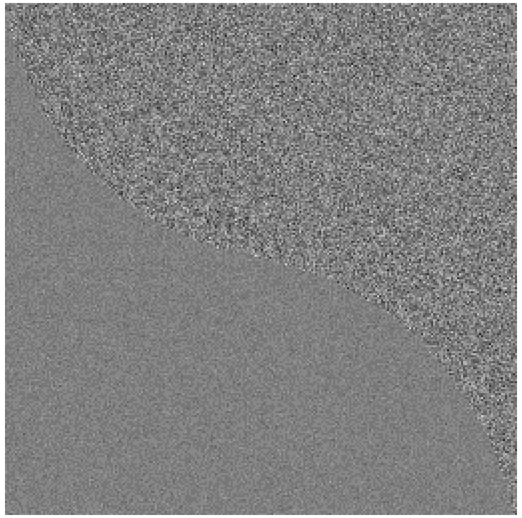
- Goal: Finding the different (meaningful) **regions / segments** of an image
- Result: **Image** that is a **map** of the different regions / segmentation
- Is it **well-defined** or **ill-defined**?

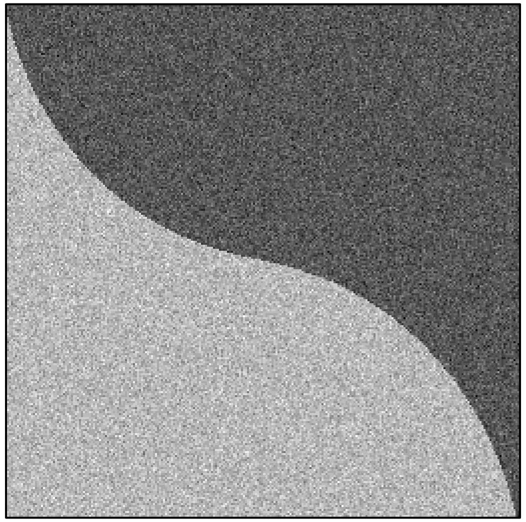
Well-Define Problems

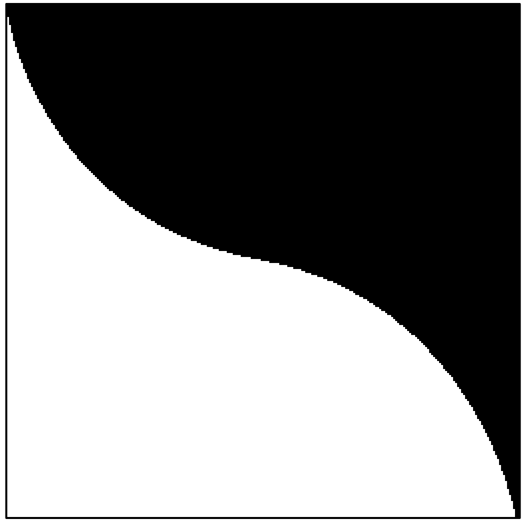
- **Well-posed problem**:
  1. A solution exists
  2. The solution is unique
  3. The solution's behaviour changes continuously with the initial conditions

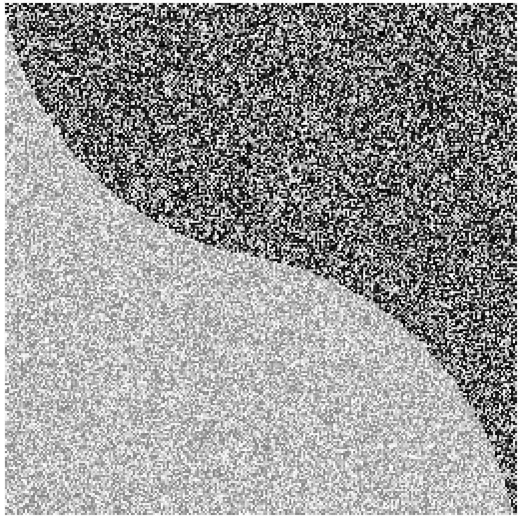
- In computer vision, we often deal with complex **ill-posed problems**.
- Analytical solutions are often not available.
- Optimisation can help find an acceptable solution to an ill-posed problem

Problem Definition

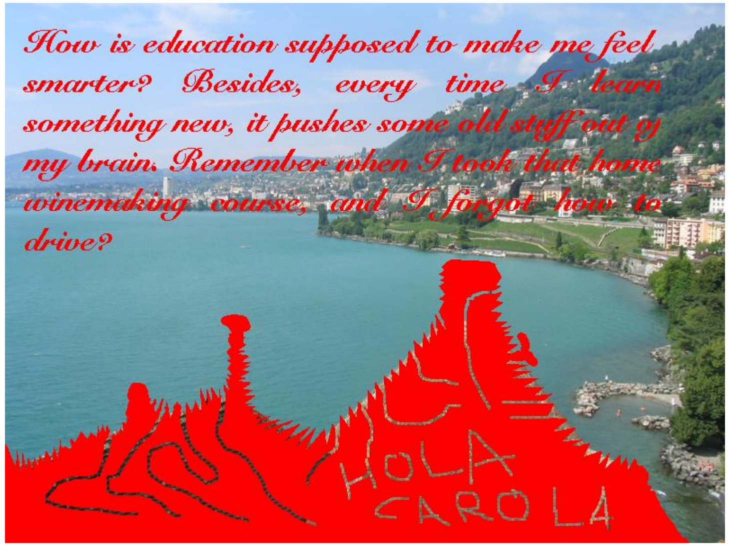








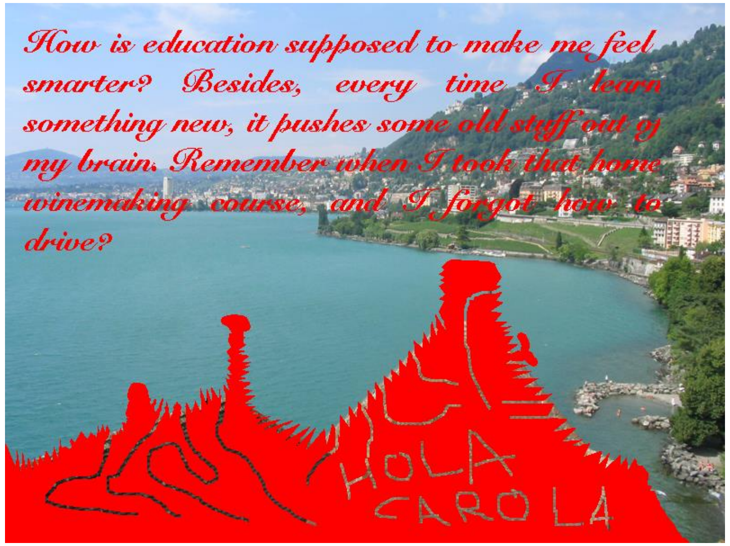
Problem Definition

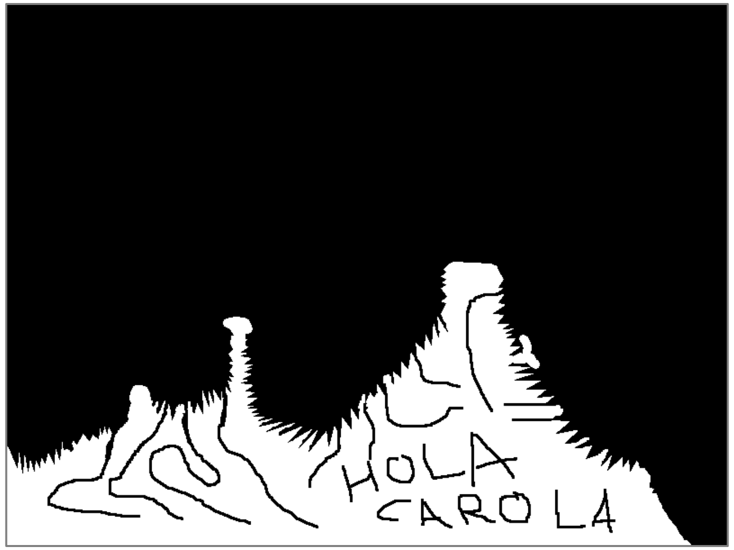


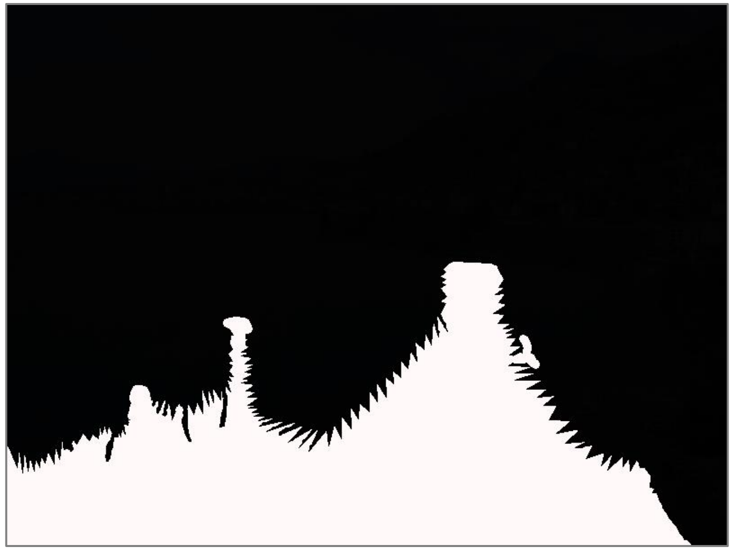
- Goal: Automatically segment the bottom region to perform inpainting
- **Criterion: Identify the red pixels**

Problem Definition

Criterion: Identify the red pixels







Criteria Definition

f

Chan–Vese
binary approximation

u

Criteria

- Criterion 1: The segmentation result is an image
- Criterion 2: The segmentation image is similar to the original image
- Criterion 3: The segmented regions are homogeneous
- Criterion 4: The segmented regions have smooth boundaries

Criterion 1

- Criterion 1: The segmentation result is an image: $f \rightarrow u$

Criterion 2

- Criterion 1: The segmentation result is an image: $f \rightarrow u$
- Criterion 2: The segmentation image is similar to the original image, i.e. minimise:

$$\int_\Omega \big(f(x) - u(x)\big)^2 \, dx$$

Criterion 3

- Criterion 1: The segmentation result is an image: $f \rightarrow u$
- Criterion 3: The segmented regions are homogeneous, i.e. minimise:

$$\int_{\Omega \setminus C} |\nabla u(x)|^2 \, dx$$

Criterion 4

- Criterion 1: The segmentation result is an image: $f \rightarrow u$
- Criterion 4: The segmented regions have smooth boundaries:

$$\arg\min_{u,C} \; \mu \, \mathrm{Length}(C)$$

Mumford-Shah Solution

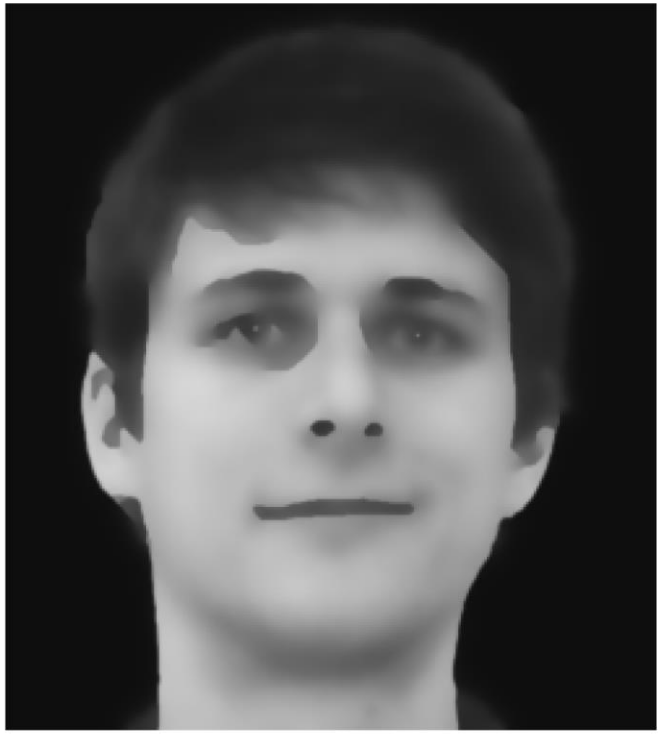
- Criterion 1: The segmentation result is an image
- Criterion 2: The segmentation image is similar to the original image
- Criterion 3: The segmented regions are homogeneous
- Criterion 4: The segmented regions have smooth boundaries

$$\arg\min_{u,C}\ \mu\,\mathrm{Length}(C) + \lambda \int_\Omega \big(f(x) - u(x)\big)^2\,dx + \int_{\Omega\setminus C} |\nabla u(x)|^2\,dx$$

Criteria Definition

f

Mumford–Shah
piecewise-smooth approximation



Criteria Definition

f

Chan–Vese
binary approximation

u

Chan-Vese

- Criterion 1: The segmentation result is an image
- Criterion 2: The segmentation image is similar to the original image
- Criterion 3: The segmented regions are homogeneous
- Criterion 4: The segmented regions have smooth boundaries
- **Criterion 5: The segmentation image has two regions**

Energy Functions

Mumford-Shah

$$\underset{u,C}{\arg\min}\ \mu\,\mathrm{Length}(C) + \lambda \int_{\Omega} \big(f(x) - u(x)\big)^2\,dx + \int_{\Omega\setminus C} |\nabla u(x)|^2\,dx$$

Chan-Vese

$$u(x) = \begin{cases} c_1 & \text{where } x \text{ is inside } C, \\ c_2 & \text{where } x \text{ is outside } C, \end{cases}$$

$$\begin{aligned} \underset{c_1,c_2,C}{\arg\min}\ & \mu\,\mathrm{Length}(C) + \nu\,\mathrm{Area}(\mathit{inside}(C)) \\ & + \lambda_1 \int_{\mathit{inside}(C)} |f(x) - c_1|^2\,dx + \lambda_2 \int_{\mathit{outside}(C)} |f(x) - c_2|^2\,dx. \end{aligned}$$

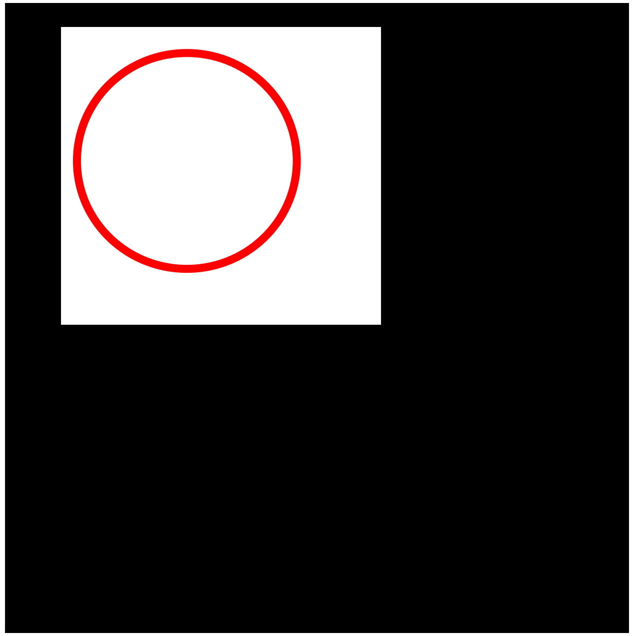
2. From Active Contours to Active Surfaces (Level Sets)

Active Contours

- Criteria 1: Smooth contour
- Criteria 2: Strong edges in the image

$$J_1(C) = \alpha \int_0^1 |C'(s)|^2 \, ds + \beta \int_0^1 |C''(s)| \, ds - \lambda \int_0^1 |\nabla u_0(C(s))|^2 \, ds.$$

$$g(|\nabla u_0(x, y)|) = \frac{1}{1 + |\nabla G_\sigma(x, y) * u_0(x, y)|^p}$$



Active Contours

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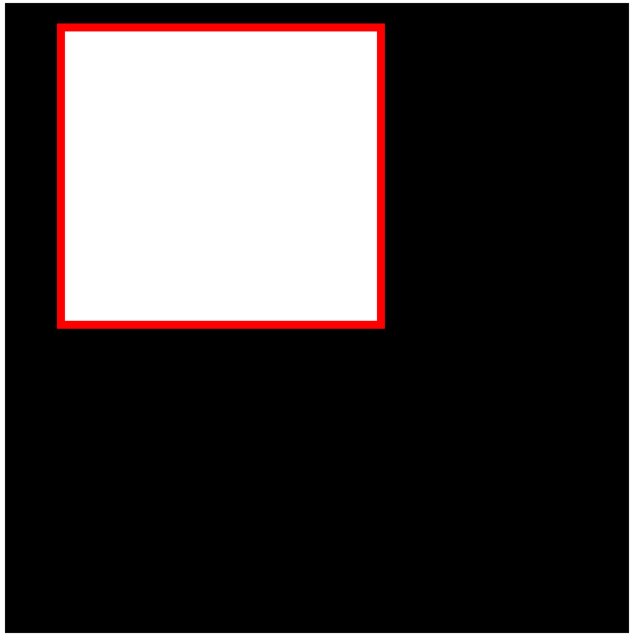
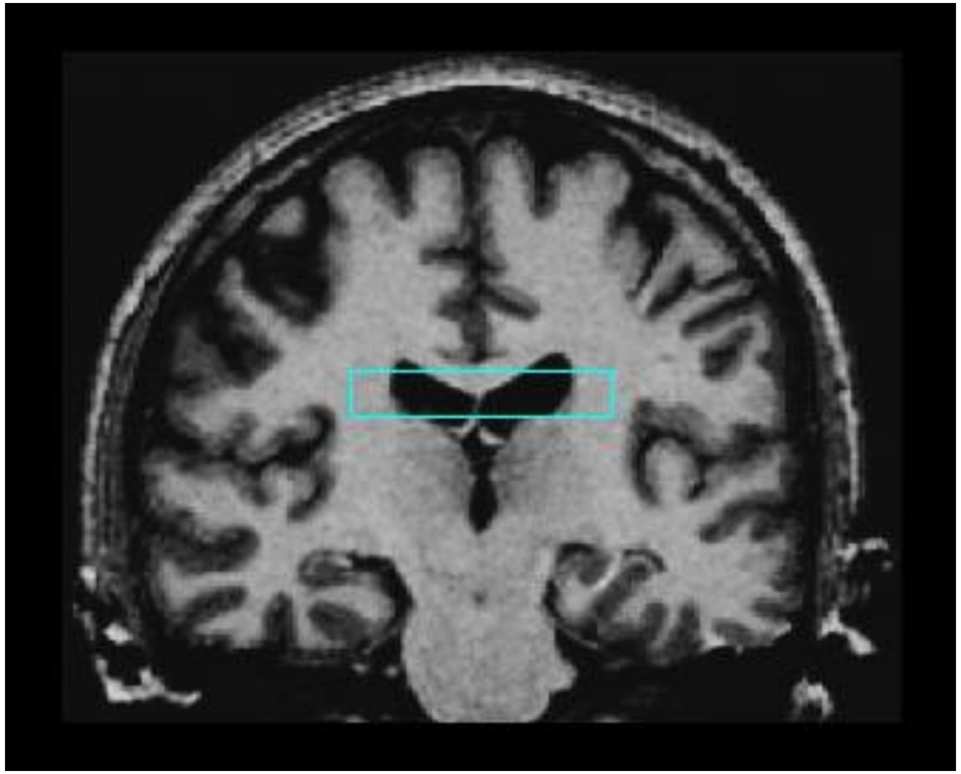
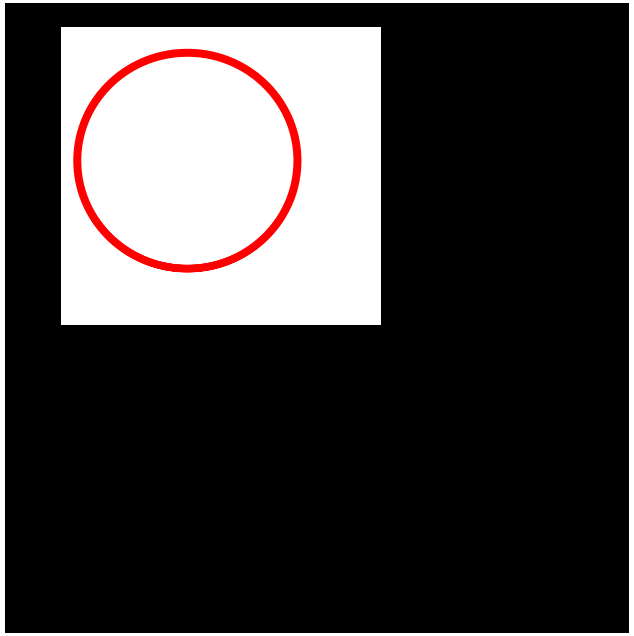
$$g(|\nabla u_0(x,\, y)|) = \frac{1}{1 + |\nabla G_\sigma(x, y) * u_0(x, y)|^p}$$

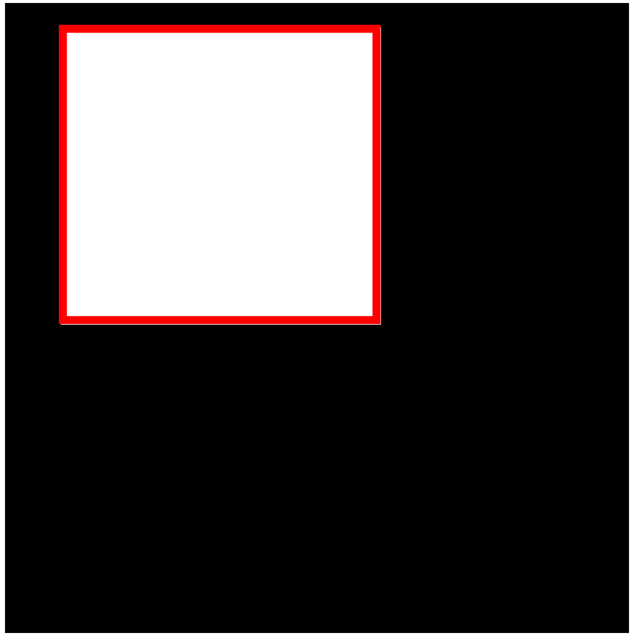


Examples

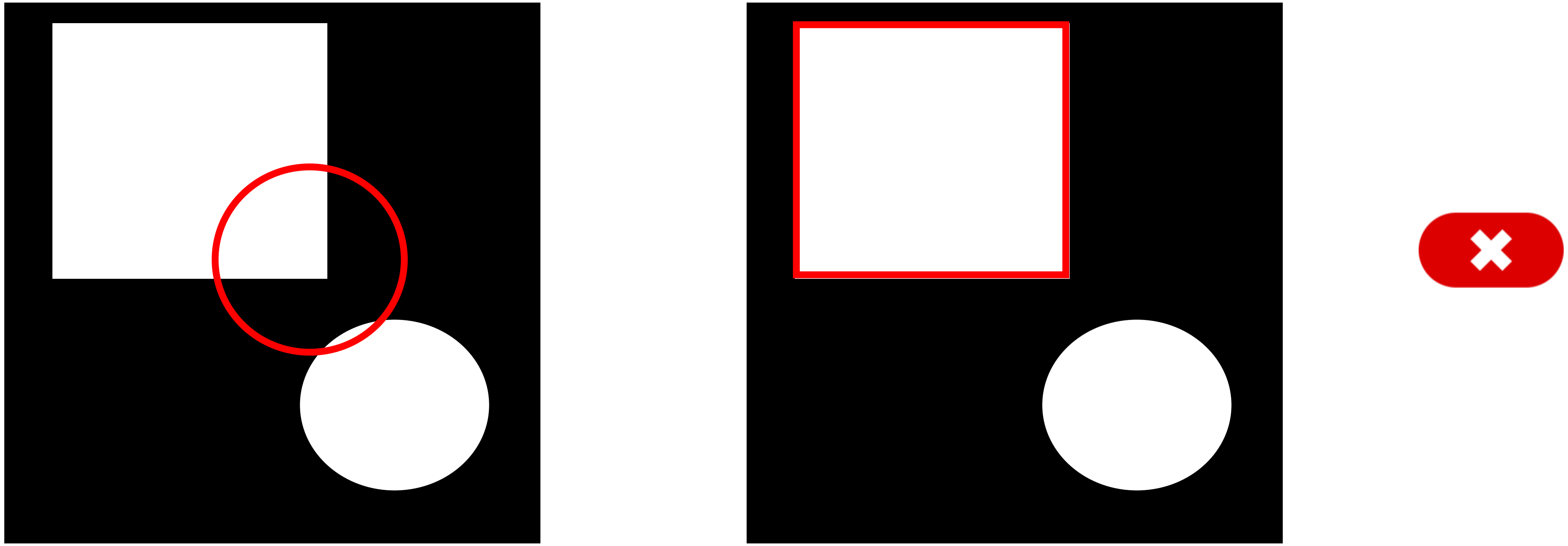


Limitation

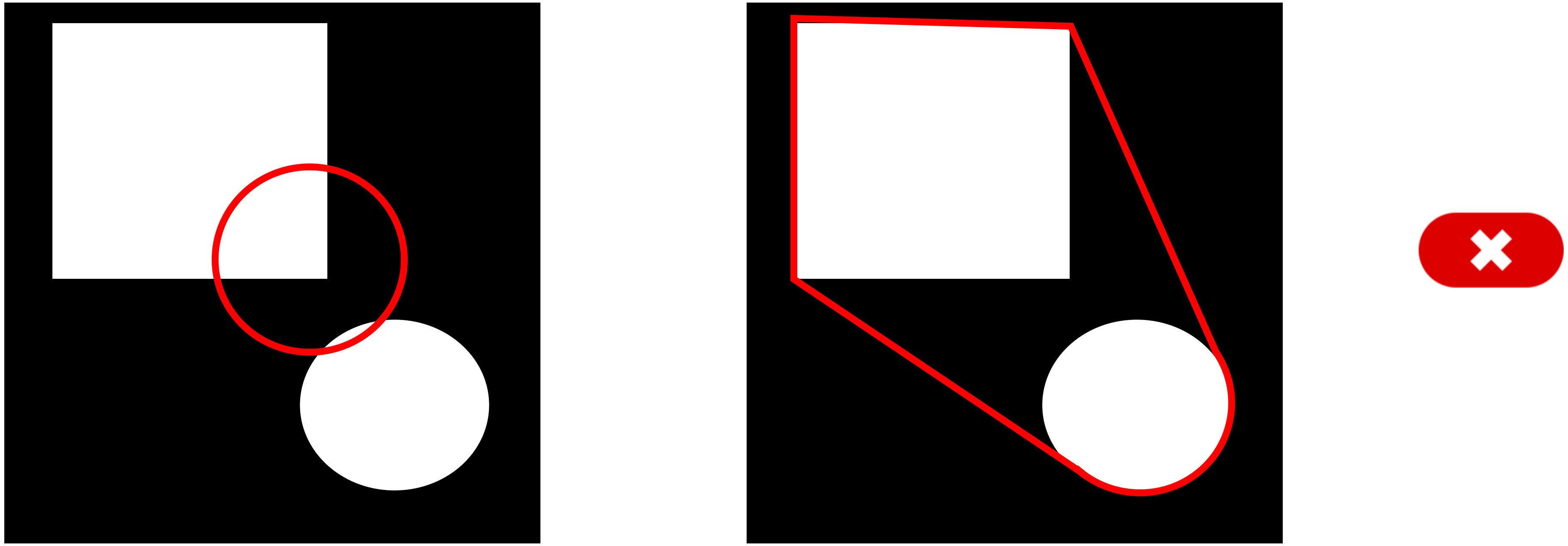




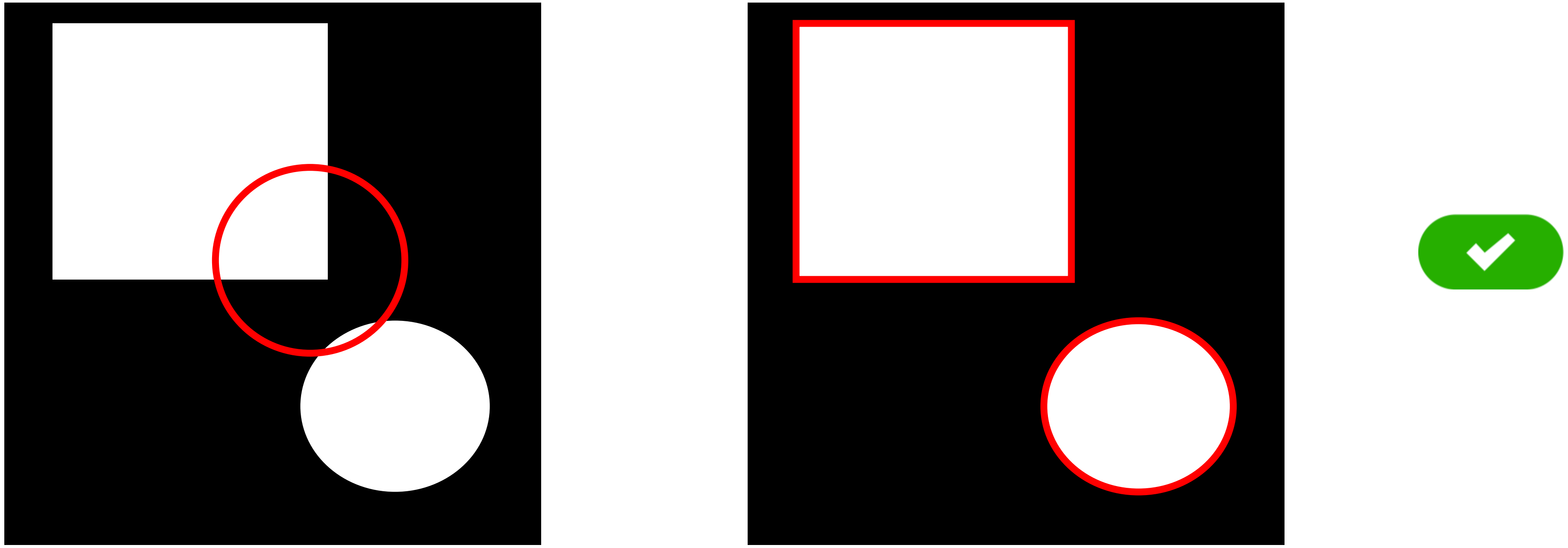
Limitation



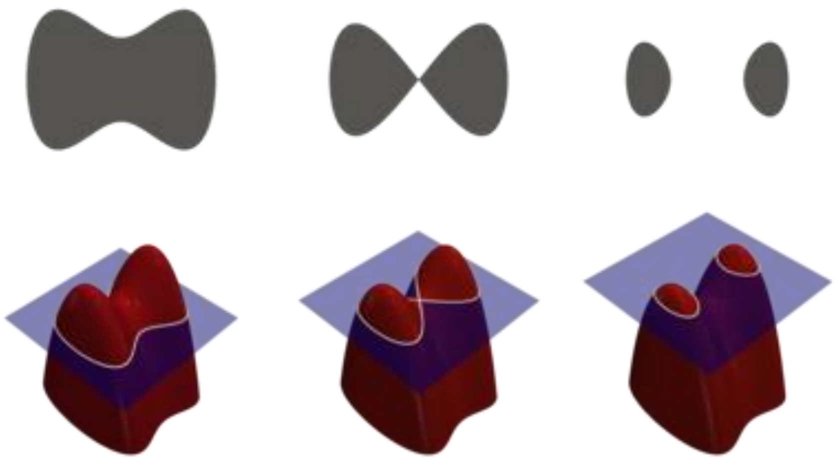
Limitation

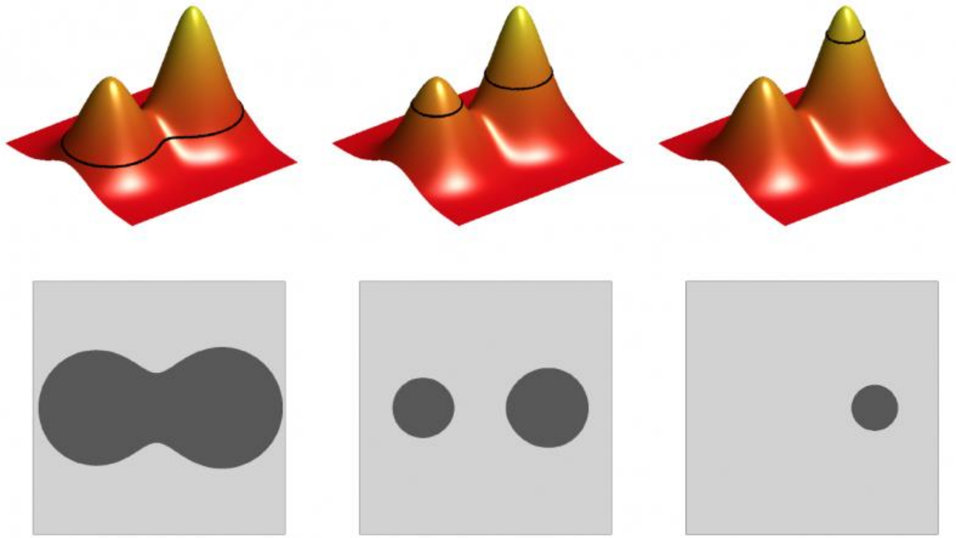


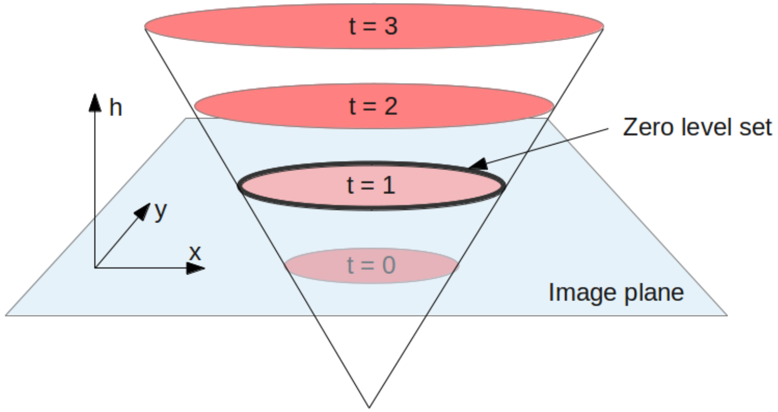
Limitation



Solution: Active Surfaces





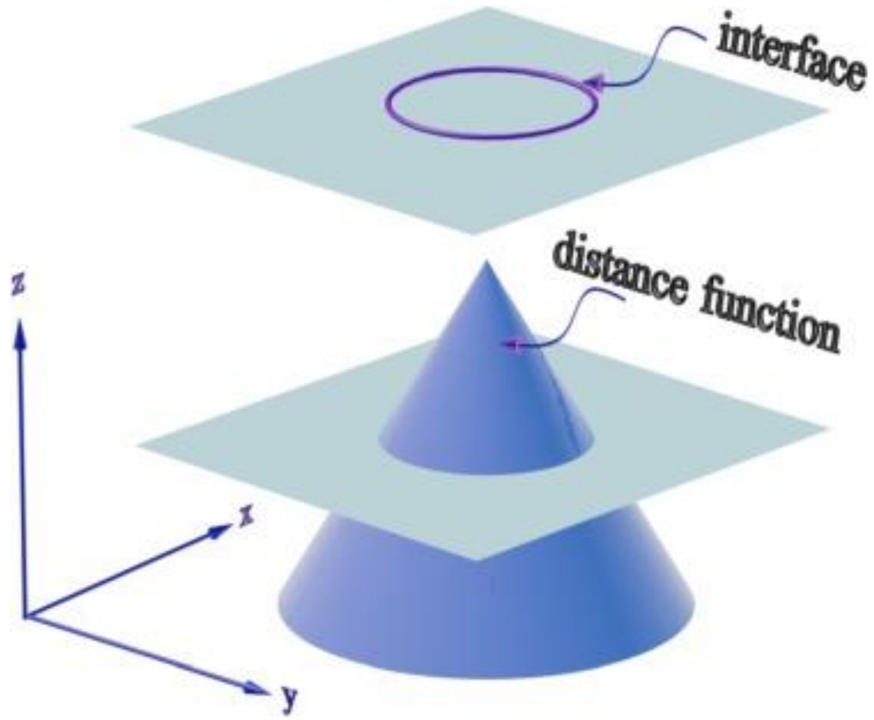


Active Surfaces / Level Sets

Initialisation with active contours:
Circle

Initialisation with active surfaces:
Cone

$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$



Active Surfaces / Level Sets

Initialisation with active contours:
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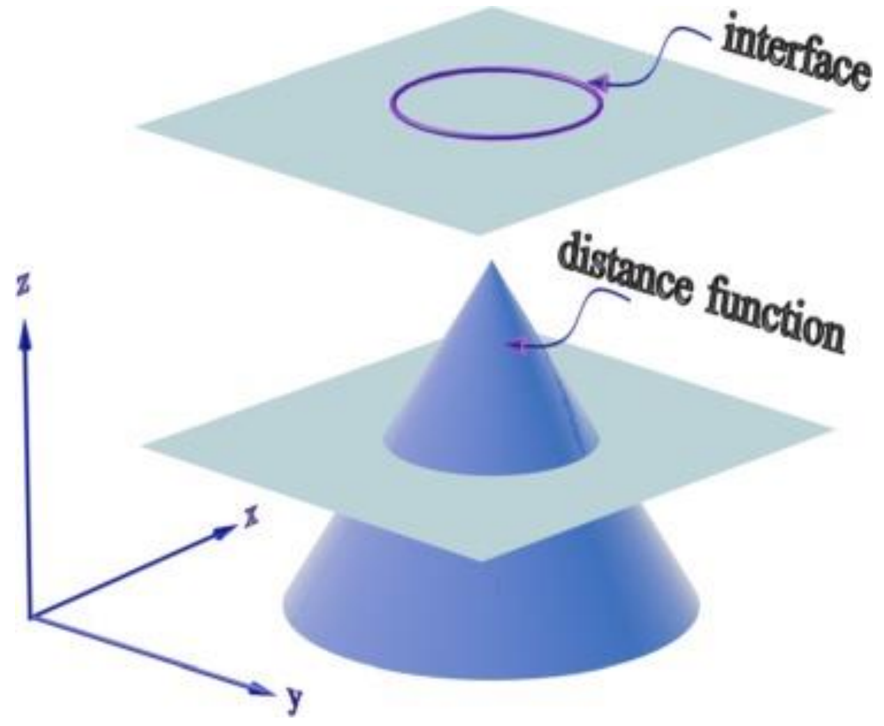
Cone

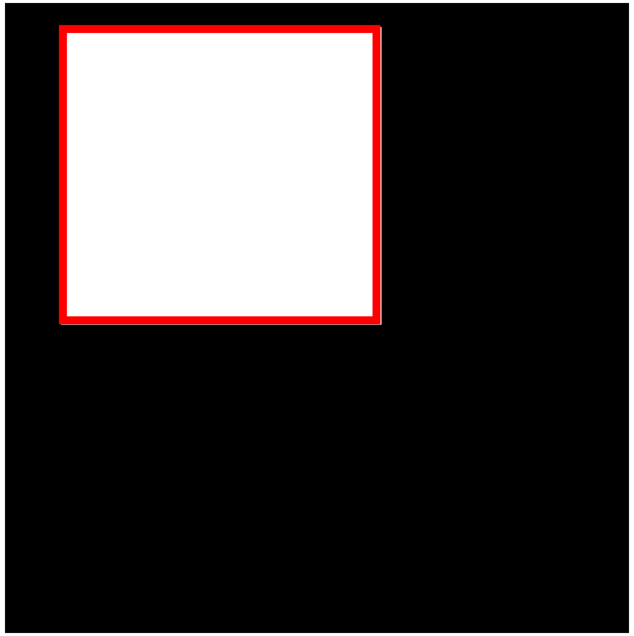
$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$



3. Solving Chan-Vese

Energy Function

$$\underset{c_1,c_2,C}{\arg\min}\;\; \mu\,\mathrm{Length}(C) + \nu\,\mathrm{Area}(inside(C)) + \lambda_1 \int_{inside(C)} |f(x) - c_1|^2\,dx + \lambda_2 \int_{outside(C)} |f(x) - c_2|^2\,dx.$$



$$u(x) = \begin{cases} c_1 & \text{where } x \text{ is inside } C, \\ c_2 & \text{where } x \text{ is outside } C, \end{cases}$$

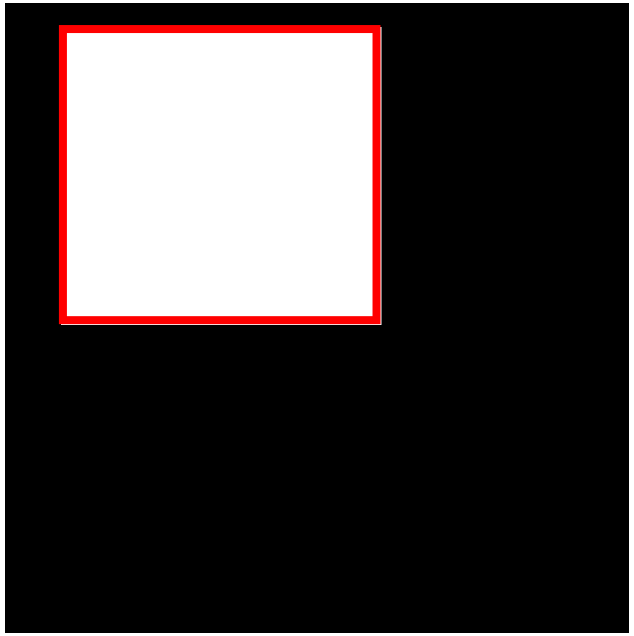
$\mathrm{c}_1 = 255$

$\mathrm{c}_2 = 0$

C **is the red contour**

Energy Function

$$\underset{c_1, c_2, C}{\arg\min} \;\; \mu \,\mathrm{Length}(C) + \nu \,\mathrm{Area}(inside(C)) + \lambda_1 \int_{inside(C)} |f(x) - c_1|^2 \, dx + \lambda_2 \int_{outside(C)} |f(x) - c_2|^2 \, dx.$$



$$u(x) = \begin{cases} c_1 & \text{where } x \text{ is inside } C, \\ c_2 & \text{where } x \text{ is outside } C, \end{cases}$$

$$\mathrm{c}_1 = 255$$

$$\mathrm{c}_2 = 0$$

Φ is a surface that intersects the image at the position of the red contour

Energy Function

$$\underset{c_1,c_2,C}{\arg\min} \quad \mu\,\mathrm{Length}(C) + \nu\,\mathrm{Area}(\mathit{inside}(C)) + \lambda_1 \int_{inside(C)} |f(x) - c_1|^2\,dx + \lambda_2 \int_{outside(C)} |f(x) - c_2|^2\,dx.$$

$$u(x) = \begin{cases} c_1 & \text{where } x \text{ is inside } C, \\ c_2 & \text{where } x \text{ is outside } C, \end{cases}$$

$$\boxed{C = \{x \in \Omega : \varphi(x) = 0\}}$$

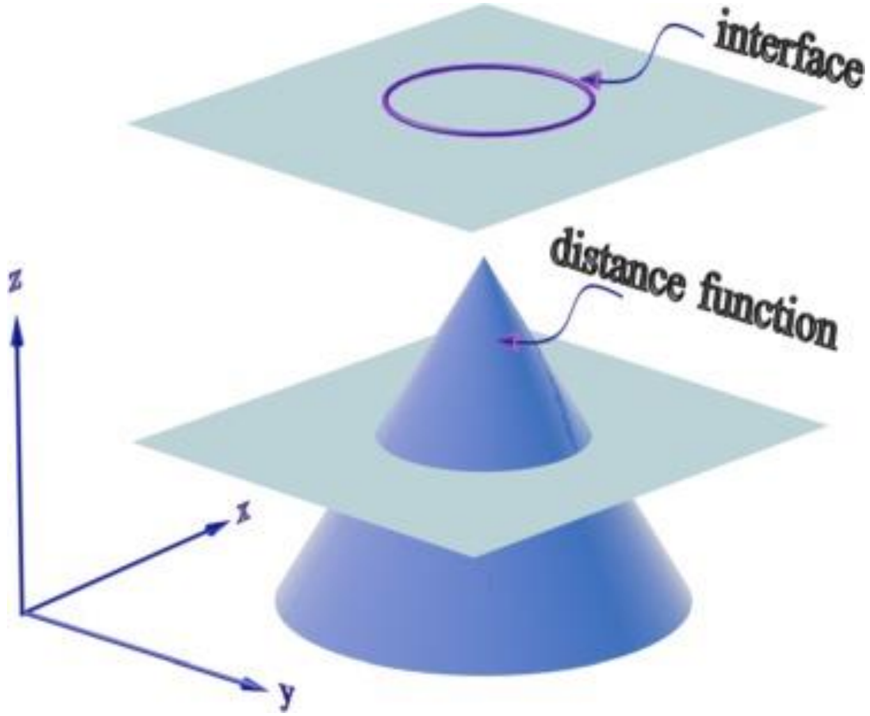
$$\underset{c_1,c_2,\varphi}{\arg\min} \quad \mu \int_\Omega \delta(\varphi(x))|\nabla\varphi(x)|\,dx + \nu \int_\Omega H(\varphi(x))\,dx + \lambda_1 \int_\Omega |f(x) - c_1|^2 H(\varphi(x))\,dx + \lambda_2 \int_\Omega |f(x) - c_2|^2 (1 - H(\varphi(x)))\,dx$$

Term 2

$$\mathrm{Area}(inside(C))$$

$$\int_{\Omega} H\big(\varphi(x)\big)\, dx$$

$$H(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0, \end{cases} \quad \delta(t) = \frac{d}{dt} H(t).$$



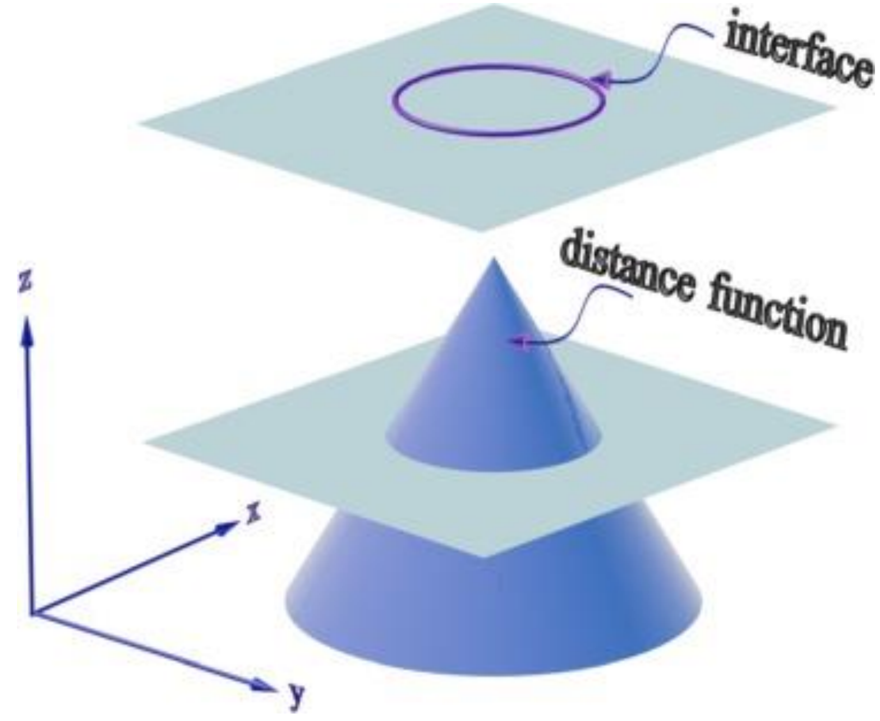
$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$

Term 1

$$\text{Length}(C)$$

$$\int_\Omega |\nabla H(\varphi(x))|\, dx = \int_\Omega \delta(\varphi(x))|\nabla \varphi(x)|\, dx.$$

$$H(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0, \end{cases} \quad \delta(t) = \frac{d}{dt} H(t).$$



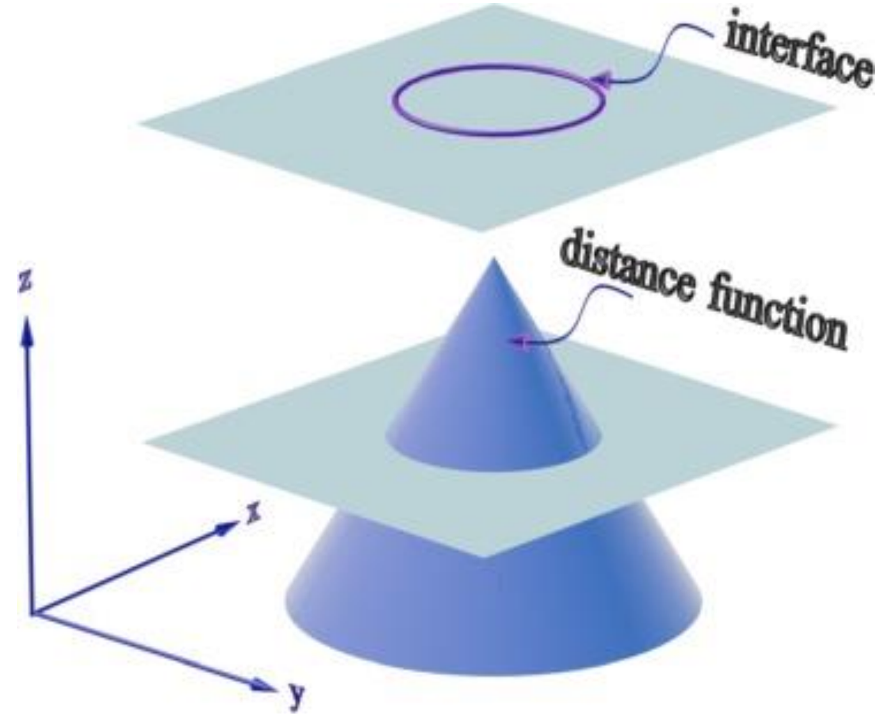
$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$

Term 3

$$\int_{inside(C)} |f(x) - c_1|^2 \, dx$$

$$\int_{\Omega} |f(x) - c_1|^2 H\big(\varphi(x)\big) \, dx$$

$$H(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0, \end{cases} \quad \delta(t) = \frac{d}{dt} H(t).$$



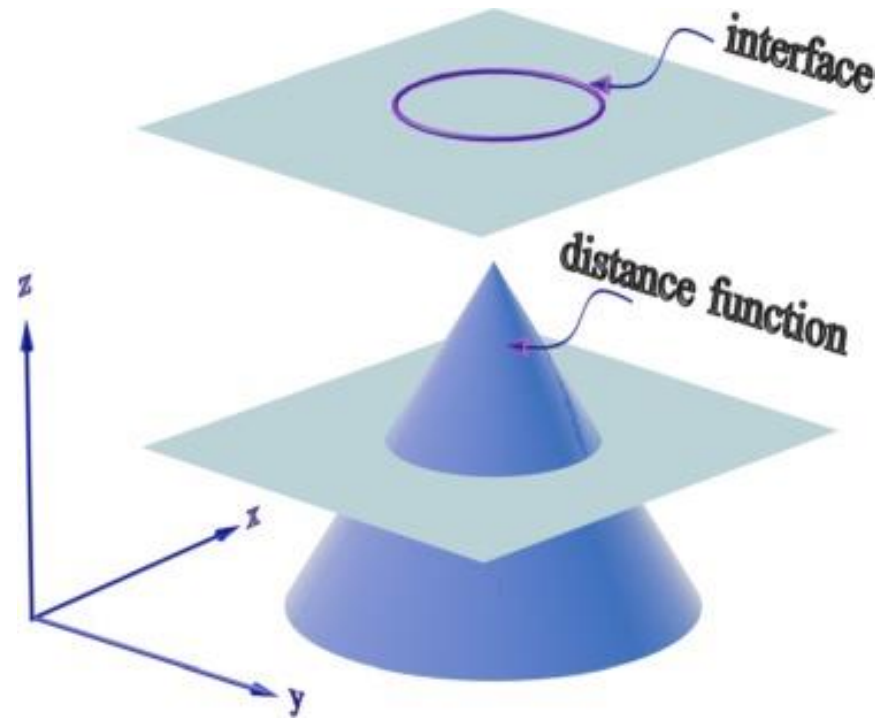
$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$

Term 4

$$\int_{outside(C)} |f(x) - c_2|^2 \, dx$$

$$\int_{\Omega} |f(x) - c_2|^2 \big(1 - H\big(\varphi(x)\big)\big) \, dx$$

$$H(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0, \end{cases} \quad \delta(t) = \frac{d}{dt} H(t).$$



$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$

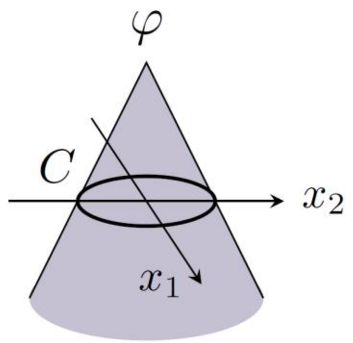
Resolution

$$c_1 = \frac{\int_\Omega f(x) H\big(\varphi(x)\big)\,dx}{\int_\Omega H\big(\varphi(x)\big)\,dx},$$
$$c_2 = \frac{\int_\Omega f(x)\big(1 - H\big(\varphi(x)\big)\big)\,dx}{\int_\Omega \big(1 - H\big(\varphi(x)\big)\big)\,dx}.$$

$$\begin{aligned}
\varphi_{i,j}^{n+1} \leftarrow \Big[&\varphi_{i,j}^n + dt\,\delta_\epsilon(\varphi_{i,j}^n)\big(A_{i,j}\varphi_{i+1,j}^n + A_{i-1,j}\varphi_{i-1,j}^{n+1} + B_{i,j}\varphi_{i,j+1}^n + B_{i,j-1}\varphi_{i,j-1}^{n+1} \\
&- \nu - \lambda_1(f_{i,j} - c_1)^2 + \lambda_2(f_{i,j} - c_2)^2\big)\Big] \\
&\Big/ \Big[1 + dt\,\delta_\epsilon(\varphi_{i,j})(A_{i,j} + A_{i-1,j} + B_{i,j} + B_{i,j-1})\Big].
\end{aligned}$$

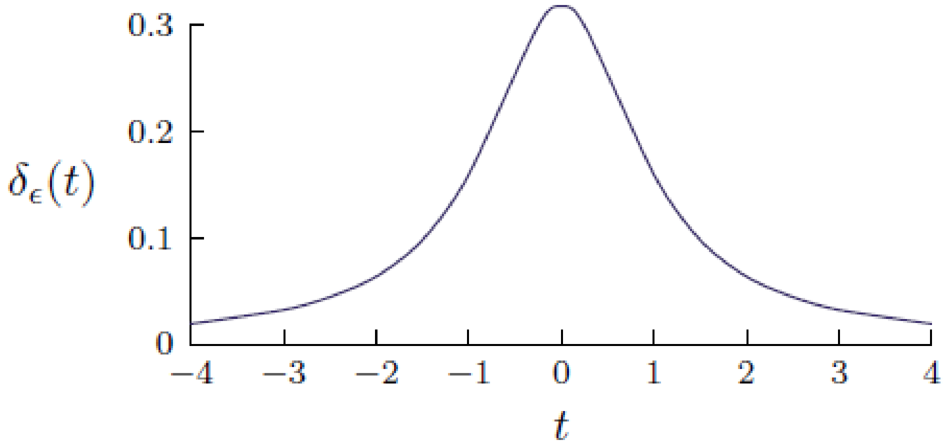
$$A_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^+ \varphi_{i,j})^2 + (\nabla_y^0 \varphi_{i,j})^2}}, \quad B_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^0 \varphi_{i,j})^2 + (\nabla_y^+ \varphi_{i,j})^2}},$$

Resolution



$$H(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0, \end{cases} \quad \delta(t) = \frac{d}{dt} H(t).$$

$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$



$$H_\epsilon(t) = \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\left(\frac{t}{\epsilon}\right)\right)$$

$$\delta_\epsilon(t) := \frac{d}{dt} H_\epsilon(t) = \frac{\epsilon}{\pi(\epsilon^2 + t^2)}$$

$$\varphi(x) = \sin(\tfrac{\pi}{5} x_1) \sin(\tfrac{\pi}{5} y)$$

4. Coding

Initial φ

$$\varphi(x) = r - \sqrt{x_1^2 + x_2^2}$$

```
[X, Y]=meshgrid(1:nj, 1:ni);

%%Initial phi
%phi_0=(-sqrt( ( X-round(ni/2)).^2 + (Y-round(nj/2)).^2)+50);

%%% This initialization allows a faster convergence for phantom 18
%phi_0=(-sqrt( ( X-round(ni/2)).^2 + (Y-round(nj/4)).^2)+50);
```

C1 & C2

$$c_1 = \frac{\int_\Omega f(x) H\big(\varphi(x)\big)\,dx}{\int_\Omega H\big(\varphi(x)\big)\,dx},$$

$$c_2 = \frac{\int_\Omega f(x)\big(1 - H\big(\varphi(x)\big)\big)\,dx}{\int_\Omega \big(1 - H\big(\varphi(x)\big)\big)\,dx}.$$

```
%Fixed phi, Minimization w.r.t c1 and c2 (constant estimation)
   c1 = ??; %TODO 1: Line to complete
   c2 = ??; %TODO 2: Line to complete
```

Dirac

$$\delta_\epsilon(t) := \frac{d}{dt}H_\epsilon(t) = \frac{\epsilon}{\pi(\epsilon^2 + t^2)}$$

```
function y = sol_diracReg( x, epsilon )
%  Dirac function of x
%    sol_diracReg( x, epsilon ) Computes the derivative of the
heaviside
%    function of x with respect to x. Regularized based on epsilon.

y = ??; %TODO 19: Line to complete
```

φ Derivates

$$A_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^+ \varphi_{i,j})^2 + (\nabla_y^0 \varphi_{i,j})^2}}, \quad B_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^0 \varphi_{i,j})^2 + (\nabla_y^+ \varphi_{i,j})^2}},$$

```
%derivatives estimation
%i direction, forward finite differences
phi_iFwd  = ??; %TODO 7: Line to complete (using DiBwd, DiFwd, DjBwd, DjFwd)
phi_iBwd  = ??; %TODO 8: Line to complete

%j direction, forward finitie differences
phi_jFwd  = ??; %TODO 9: Line to complete
phi_jBwd  = ??; %TODO 10: Line to complete

%centered finite diferences
phi_icent   = ??; %TODO 11: Line to complete
phi_jcent   = ??; %TODO 12: Line to complete
```

A & B

$$A_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^+ \varphi_{i,j})^2 + (\nabla_y^0 \varphi_{i,j})^2}}, \quad B_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^0 \varphi_{i,j})^2 + (\nabla_y^+ \varphi_{i,j})^2}},$$

```
%A and B estimation (A y B from the Pascal Getreuer's IPOL paper "Chan
    %Vese segmentation
    A = ??; %TODO 13: Line to complete
    B = ??; %TODO 14: Line to complete
```

φ(n+1)

$$\varphi_{i,j}^{n+1} \leftarrow \Big[\varphi_{i,j}^{n} + dt\,\delta_\epsilon(\varphi_{i,j}^{n})\big(A_{i,j}\varphi_{i+1,j}^{n} + A_{i-1,j}\varphi_{i-1,j}^{n+1} + B_{i,j}\varphi_{i,j+1}^{n} + B_{i,j-1}\varphi_{i,j-1}^{n+1}$$
$$- \nu - \lambda_1(f_{i,j} - c_1)^2 + \lambda_2(f_{i,j} - c_2)^2\big)\Big]$$
$$/ \ \big[1 + dt\,\delta_\epsilon(\varphi_{i,j})(A_{i,j} + A_{i-1,j} + B_{i,j} + B_{i,j-1})\big].$$

```
%%Equation 22, for inner points
    phi(??) = ??; %TODO 15: Line to complete
```

Initialisations

$$\underset{c_1,c_2,\varphi}{\arg\min}\ \mu\int_\Omega \delta(\varphi(x))|\nabla\varphi(x)|\,dx + \nu\int_\Omega H(\varphi(x))\,dx$$
$$+\lambda_1\int_\Omega |f(x)-c_1|^2 H(\varphi(x))\,dx + \lambda_2\int_\Omega |f(x)-c_2|^2(1-H(\varphi(x)))\,dx$$

```
%Length and area parameters
%phantom18 mu=0.2 mu=0.5
%hola carola
mu=1
nu=0;

%%Parameters
%lambda1=1;
%lambda2=1;
lambda1=10^-3; %Hola carola problem
lambda2=10^-3; %Hola carola problem
```

```
epHeaviside=1;
eta=0.01;
%eta=1
tol=0.1;
%dt=(10^-2)/mu;
dt=(10^-1)/mu;
iterMax=100000
%reIni=0; %Try both of them
%reIni=500;
reIni=100;
[X, Y]=meshgrid(1:nj, 1:ni);
```