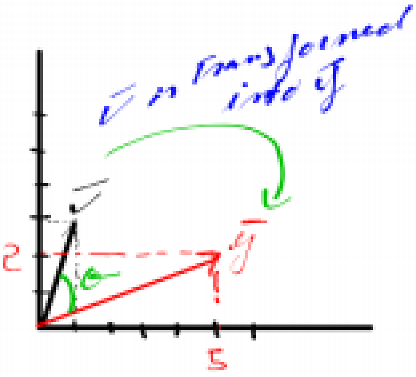
SINGULAR VALUE DECOMPOSITION : SVD

Let's start seeing what a matrix (M) multiplication does to a vector ($\bar{v}$)

$$\bar{v} = \begin{pmatrix}1\\3\end{pmatrix} \qquad M = \begin{pmatrix}2 & 1\\-1 & 1\end{pmatrix}$$

$$\bar{y} = M \times \bar{v} = \begin{pmatrix}2 & 1\\-1 & 1\end{pmatrix} * \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$$



A matrix multiplication can be seen as a Rotation and stretching. This is the idea beneath The Singular Value Decomposition

Let's aply this Idea of Rotation and stretching to an ortho-normal basis composed by vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n$

Let's A representing Rotations and stretchin

$$\bar{v}_1, \bar{v}_2, \bar{v}_3, \dots, \bar{v}_n \xRightarrow{A} \bar{y}_1, \bar{y}_2, \dots, \bar{y}_n$$

We can transform into unitary vectors just multiplying by some stretching factor

$$\begin{tikzcd} \sigma_1 \bar{u}_1 \arrow[d] & \sigma_2 u_2 \arrow[d] & \sigma_3 \bar{u}_3 \arrow[d] & \dots & \sigma_n \bar{u}_n \arrow[d] \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & & \bar{y}_n \end{tikzcd}$$

i.e. $A\bar{v}_1 = \sigma_1 \bar{u}_1$

$A\bar{v}_2 = \sigma_2 \bar{u}_2$

$\vdots$

$A\bar{v}_n = \sigma_n \bar{u}_n$

Let me write this in a compact (matrix way)

$$\begin{bmatrix} A \end{bmatrix}_{m\times n} \times \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}_{n\times n}}_{V} = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}_{m\times n}}_{\hat{U}} \times \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots\ \sigma_n \end{bmatrix}_{n\times n}}_{\hat{\Sigma}}$$

The elements in matrices $\hat{U}$ and $\hat{\Sigma}$ are sorted such as $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, then what we have is the following

$$A \times V = \hat{U} \times \hat{\Sigma}$$

notice that

V: orthonormal basis matrix

$\hat{U}$: orthonormal basis matrix

$\hat{\Sigma}$: stretching factors

big deal:
$$V^{-1} = V^{T}$$
$$\hat{U}^{-1} = \hat{U}^{T}$$

The inverse of V and $\hat{U}$ are they transposes. Lets do something:

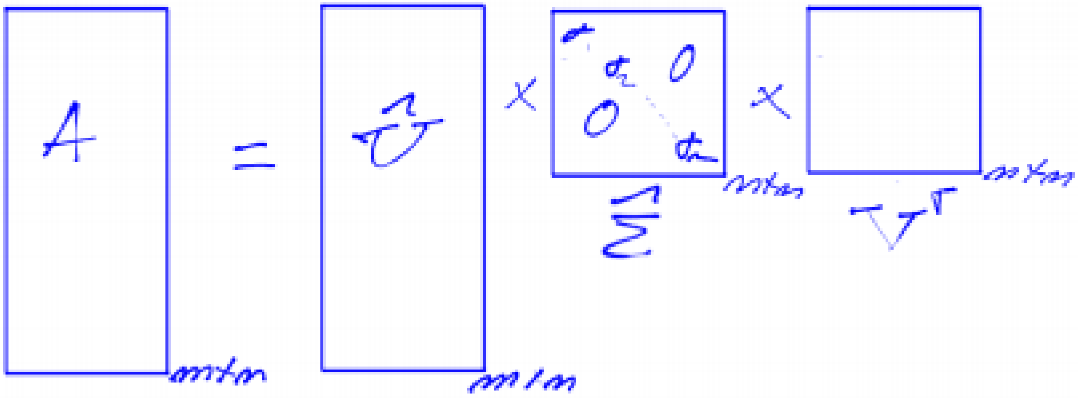
$$A \times V = \hat{U} \times \hat{\Sigma}$$
$$A \times V \times V^{-1} = \hat{U} \times \hat{\Sigma} \times V^{-1}$$
$$A = \hat{U} \times \hat{\Sigma} \times V^{-1}$$
$$\boxed{A = \hat{U} \times \hat{\Sigma} \times V^{T}}$$

Singular Value Decomposition

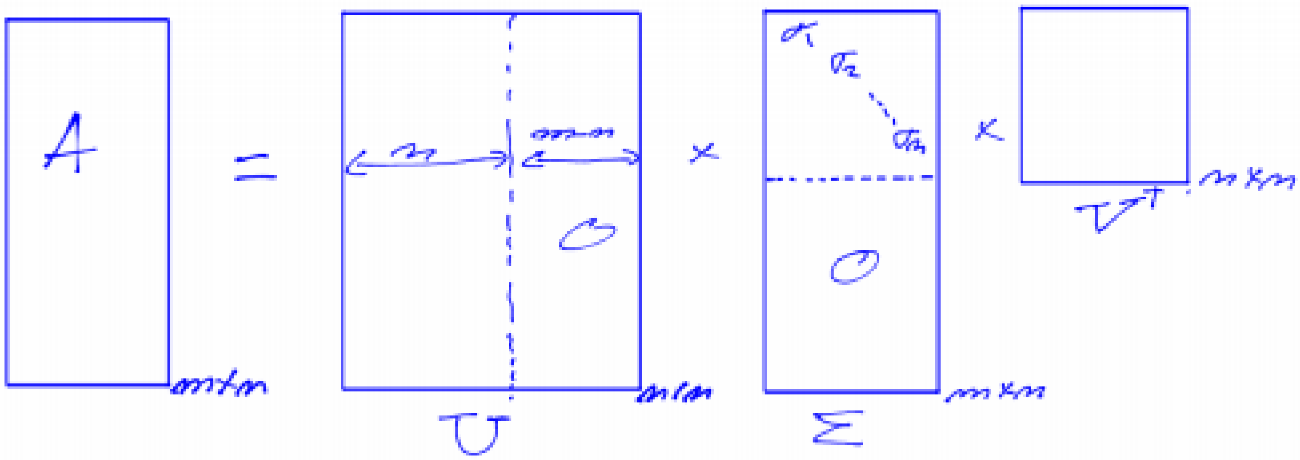
(Reduced)

Usually expands $\hat{U}$ and $\hat{\Sigma}$ with zeros

Suppose: A matrix of size $m \times n$



expands the matrices with zeros



Theorem: SVD

Every matrix $A \in \mathbb{C}^{m \times n}$ has a Singular Value Decomposition such as

$$A = U \Sigma V^T \Longrightarrow$$

Columns of U are vectors $\bar{u}_i$
Columns of V are $\bar{v}_i$
Σ diagonal matrix composed by σ_i
U orthonormal matrix
V orthonormal matrix

- Singular values $\{\sigma_i\}$ are uniquely determined, and if A square, σ_i distinct
- $\{\bar{u}_i\}$ and $\{\bar{v}_i\}$ are also unique

HOW TO DETERMINE THE SVD OF A

$\boxed{\text{IV}}$

Let's Play a little bit

$$A = U\Sigma V^T \quad \begin{cases} A: \text{known} \\ U, \Sigma, V \text{ unknown} \end{cases}$$

multiply on the left by A^T

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T)$$

Remember $(AB)^T = B^TA$

$$A^TA = V\Sigma U^T(U\Sigma V^T)$$

$$A^TA = V\Sigma \underbrace{U^TU}_{I}\Sigma V^T$$

(because $U^T = U^{-1}$)

$$A^TA = V\Sigma^2V^T$$

multiply on the right by V

$$A^TAV = V\Sigma^2\underbrace{V^TV}_{I} \Longrightarrow \underbrace{A^TA}_{}V = V\Sigma^2$$

Rename this matrix as : B

$$BV = V\Sigma^2 \Longrightarrow$$

EIGEN VALUE POBLEM!!
EIGEN VECTOR

$$\begin{cases} B\bar{v}_1 = \sigma_1^2\bar{v}_1 \\ B\bar{v}_2 = \sigma_2^2\bar{v}_2 \\ \vdots \\ B\bar{v}_n = \sigma_n^2\bar{v}_n \end{cases}$$

Remember the eigenvalue/eigenvector problem:

$$A\bar{v} = l\bar{v} \Longrightarrow l_j = \sigma_j^2 \Longrightarrow \sigma_j = \sqrt{l_j}$$

this is not the same A as before

For finding vectors $\bar{v}_1, \dots, \bar{v}_n$ (i.e. V) and Singular values $\sigma_1, \dots, \sigma_n$ (i.e. Σ) just:

- Find the eigenvalues and eigenvectors of A^TA
- The Singular values are the Square root of the eigenvalues.

$\boxed{V}$

Let work in a similar way for finding matrix U

$$A = U \Sigma V^T$$

$$AA^T = U \Sigma V^T (U \Sigma V^T)^T$$

$$AA^T = (U \Sigma V^T)(V \Sigma U^T)$$

$$AA^T = (U \Sigma \underbrace{V^T V}_{I} \Sigma U^T)$$

$$AA^T = (U \Sigma^2 U^T)$$

$$AA^T U = U \Sigma^2 \underbrace{\cancel{U^T U}}_{I}$$

$$AA^T U = U \Sigma^2$$ ⇒ Eigenvalue Problem

So: For finding U, Σ, V you have to solve two different eigenvalue problems:

- Eigenvalues and eigenvectors of A^TA give you V and Σ
- Eigenvalues and eigenvectors of AA^T give you U and Σ

APLICATIONS

-PSEUDO-INVERSE

What if you cannot compute the Inverse of a Matrix and you still need one?

We Define the pseudo-Inverse as:

$$A^{+} = V \Sigma^{+} U^{T}$$

where $\Sigma^{+} = \begin{cases} s_{ii}^{-1} & \text{if } s_{ii} \neq 0 \\ 0 & \text{if } s_{ii} = 0 \end{cases}$

The pseudo-inverse can be used for Solving $A\bar{x} = \bar{b}$

$$A\bar{x} = \bar{b}$$

$$\bar{x} = V \Sigma^{+} U^{T} \bar{b}$$

Notice that if A is not square, this is equivalent to solve the normal equation

$$X = (A^{t}A)^{-1} A^{t}\bar{b}$$