基于$\mathrm{P_N}$方法的中子输运方程近似及二阶导归并

周星光

学号：4122303028

zxg818@stu.xjtu.edu.cn

2023 年 4 月 27 日

笛卡尔坐标系下，裂变源项为各向同性的稳态多群中子输运方程可以写为：

$$\begin{aligned}\mathbf{\Omega}\cdot\nabla\psi_g\left(\mathbf{r},\mathbf{\Omega}\right)+\Sigma_{t,g}\left(\mathbf{r}\right)\psi_g\left(\mathbf{r},\mathbf{\Omega}\right)=\sum_{g'=1}^{G}\int_0^{4\pi}d\mathbf{\Omega}'\Sigma_{s,g'\to g}\left(\mathbf{r},\mathbf{\Omega}'\to\mathbf{\Omega}\right)\psi_{g'}\left(\mathbf{r},\mathbf{\Omega}'\right)\\+\frac{\chi_g}{4\pi k_{eff}}\sum_{g'=1}^{G}\nu\Sigma_{f,g'}\int_0^{4\pi}\psi_{g'}\left(\mathbf{r},\mathbf{\Omega}'\right)d\mathbf{\Omega}'\end{aligned}\tag{1}$$

散射源项按照球谐函数近似展开可以得到：

$$\begin{aligned}\mathbf{\Omega}\cdot\nabla\psi_g\left(\mathbf{r},\mathbf{\Omega}\right)+\Sigma_{t,g}\left(\mathbf{r}\right)\psi_g\left(\mathbf{r},\mathbf{\Omega}\right)=Q_{f,g}\left(\mathbf{r},\mathbf{\Omega}\right)\\+\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\sum_{g'=1}^{G}\Sigma_{n,g'\to g}\left[\sum_{m=0}^{n}a_{n,m}\widetilde{\psi}_{n,m,g'}P_n^m\left(\cos\theta\right)\cos m\varphi\right.\\\left.+\sum_{m=1}^{n}a_{n,m}\widetilde{\gamma}_{n,m,g}P_n^m\left(\cos\theta\right)\sin m\varphi\right]\end{aligned}\tag{2}$$

式中，$\widetilde{\psi}_{n,m}$与$\widetilde{\gamma}_{n,m}$为中子角通量密度的勒让德函数矩。为了方便起见，中子角通量密度也可以展开为下列形式：

$$\begin{aligned}\psi\left(\mathbf{r},\mathbf{\Omega}\right)&=\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\sum_{m=-n}^{n}a_{n,m}\widetilde{\phi}_{n,m}\left(\mathbf{r}\right)Y_{n,m}\left(\theta,\varphi\right)\\&=\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\left[\sum_{m=0}^{n}a_{n,m}\widetilde{\psi}_{n,m}\left(\mathbf{r}\right)P_n^m\left(\cos\theta\right)\cos m\varphi\right.\\&\left.+\sum_{m=1}^{n}a_{n,m}\widetilde{\gamma}_{n,m}\left(\mathbf{r}\right)P_n^m\left(\cos\theta\right)\sin m\varphi\right]\end{aligned}\tag{3}$$

而

$$
\begin{aligned}
Q_f(\mathbf{r},\boldsymbol{\Omega}) = \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\Bigg[&\sum_{m=0}^{n} a_{n,m}\widetilde{Q}_{f,n,m}^{c}(\mathbf{r})\,P_n^m(\cos\theta)\cos m\varphi \\
&+\sum_{m=1}^{n} a_{n,m}\widetilde{Q}_{f,n,m}^{s}(\mathbf{r})\,P_n^m(\cos\theta)\sin m\varphi\Bigg]
\end{aligned} \tag{4}
$$

这里，球谐函数定义为：

$$
Y_{n,m}(\boldsymbol{\Omega}) = \begin{pmatrix} Y_{n,m}^{c}(\boldsymbol{\Omega}) \\ Y_{n,m}^{s}(\boldsymbol{\Omega}) \end{pmatrix} = \begin{pmatrix} P_n^m(\cos\theta)\cos m\varphi, m=0,1,\cdots,n \\ P_n^m(\cos\theta)\sin m\varphi, m=1,\cdots,n \end{pmatrix}, n=0,1,\cdots \tag{5}
$$

其中$a_{n,m}$有以下关系：

$$
a_{n,m} = \frac{(n-m)!}{(n+m)!}\cdot\frac{2}{1+\delta_{0,m}} \tag{6}
$$

P_n^m为伴随勒让德函数，其有正交性，可以表示为：

$$
\int_0^{\pi} P_n^m(\cos\theta)\,P_{n'}^m(\cos\theta)\sin\theta d\theta = \frac{(n+m)!}{(n-m)!}\cdot\frac{2}{2n+1}\delta_{n,n'} \tag{7}
$$

此外，三角级数也是正交函数，有正交关系：

$$
\begin{aligned}
&\int_0^{2\pi}\cos m\varphi\cdot\cos m'\varphi d\varphi = \begin{cases} 0, & m\neq m' \\ \pi, & m=m'\neq 0 \\ 2\pi, & m=m'=0 \end{cases} \\
&\int_0^{2\pi}\sin m\varphi\cdot\sin m'\varphi d\varphi = \begin{cases} 0, & m\neq m' \\ \pi, & m=m' \end{cases}
\end{aligned} \tag{8}
$$

对式(3)可以利用级数的正交性得到中子角通量密度的展开矩。对(3)两边同时乘以$\cos m'\varphi$，并对φ进行0到2π的积分，可以得到：

$$
\int_0^{2\pi}\psi(\mathbf{r},\boldsymbol{\Omega})\cos m'\varphi d\varphi = \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}a_{n,m}\widetilde{\psi}_{n,m}(\mathbf{r})\,P_n^m(\cos\theta)\begin{pmatrix}\pi \\ 2\pi\end{pmatrix} \tag{9}
$$

对式(9)两边同时乘以$P_{n'}^m(\cos\theta)\sin\theta$，并对$\theta$进行0到$\pi$的积分，可以得到：

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi}\psi(\mathbf{r},\boldsymbol{\Omega})P_n^m(\cos\theta)\cos m\varphi\sin\theta d\theta d\varphi \\
&\qquad = \frac{2n+1}{4\pi}a_{n,m}\widetilde{\psi}_{n,m}(\mathbf{r})\frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}\begin{pmatrix}\pi \\ 2\pi\end{pmatrix}
\end{aligned} \tag{10}
$$

将式(6)代入(10)中，并同时考虑$n=0,m=0$；$n\neq 0,m=0$；$n\neq 0,m\neq 0$三种

情况，均有：

$$\widetilde{\psi}_{n,m}(\mathbf{r})=\int_0^{2\pi}\int_0^{\pi}\psi(\mathbf{r},\mathbf{\Omega})P_n^m(\cos\theta)\cos m\varphi\sin\theta d\theta d\varphi, n=0,\cdots;m=0,\cdots,n \tag{11}$$

此即为勒让德函数展开矩的第一个矩，另外一个矩也可以通过上述过程，利用级数的正交性得到：

$$\widetilde{\gamma}_{n,m}(\mathbf{r})=\int_0^{2\pi}\int_0^{\pi}\psi(\mathbf{r},\mathbf{\Omega})P_n^m(\cos\theta)\sin m\varphi\sin\theta d\theta d\varphi, n=1,\cdots;m=1,\cdots,n \tag{12}$$

对于球谐函数$Y_{n,m}(\mathbf{\Omega})$，可以得到下列一些递推关系式：

$$\begin{aligned}
\mu Y_{n,m}(\theta,\varphi)&=\frac{1}{2n+1}\left[(n-m+1)Y_{n+1,m}+(n+m)Y_{n-1,m}\right]\\
\eta Y_{n,m}(\theta,\varphi)&=\frac{1}{2(2n+1)}\{[Y_{n+1,m+1}-Y_{n-1,m+1}](1+\delta_{m,0})\\
&+[-(n-m+2)(n-m+1)Y_{n+1,m-1}\\
&+(n+m)(n+m-1)Y_{n-1,m-1}]\}\\
\xi Y_{n,m}^{c,(s)}(\mathbf{\Omega})&=\frac{1,(-1)}{2(2n+1)}\left[\left(Y_{n+1,m+1}^{s,(c)}-Y_{n-1,m+1}^{s,(c)}\right)(1+\delta_{m,0})\right.\\
&+(n-m+2)(n-m+1)Y_{n+1,m-1}^{s,(c)}\\
&\left.-(n+m)(n+m-1)Y_{n-1,m-1}^{s,(c)}\right]
\end{aligned} \tag{13}$$

将上述中子角通量密度的球谐函数展开式代入中子输运方程，可以得到：

$$\begin{aligned}
\mu\frac{\partial\psi_g(\mathbf{r},\mathbf{\Omega})}{\partial x}&=\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\sum_{m=0}^{n}\mu a_{n,m}Y_{n,m}(\theta,\varphi)\frac{\partial\widetilde{\phi}_{n,m,g}(\mathbf{r})}{\partial x}\\
&=\sum_{n=0}^{\infty}\frac{1}{4\pi}\sum_{m=0}^{n}a_{n,m}\left[(n-m+1)Y_{n+1,m}+(n+m)Y_{n-1,m}\right]\frac{\partial\widetilde{\phi}_{n,m,g}(\mathbf{r})}{\partial x}
\end{aligned} \tag{14}$$

$$\begin{aligned}
\eta\frac{\partial\psi_g(\mathbf{r},\mathbf{\Omega})}{\partial y}&=\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\sum_{m=0}^{n}\eta a_{n,m}Y_{n,m}(\theta,\varphi)\frac{\partial\widetilde{\phi}_{n,m,g}(\mathbf{r})}{\partial y}\\
&=\sum_{n=0}^{\infty}\frac{1}{8\pi}\sum_{m=0}^{n}a_{n,m}\{[Y_{n+1,m+1}-Y_{n-1,m+1}](1+\delta_{m,0})\\
&+[-(n+m+2)(n-m+1)Y_{n+1,m-1}\\
&+(n+m)(n+m-1)Y_{n-1,m-1}]\}\frac{\partial\widetilde{\phi}_{n,m,g}(\mathbf{r})}{\partial y}
\end{aligned} \tag{15}$$

$$
\begin{aligned}
\xi\frac{\partial\psi_g\left(\mathbf{r},\mathbf{\Omega}\right)}{\partial z} &= \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\sum_{m=0}^{n}\xi a_{n,m}Y_{n,m}\left(\theta,\varphi\right)\frac{\partial\widetilde{\phi}_{n,m,g}\left(\mathbf{r}\right)}{\partial z} \\
&= \sum_{n=0}^{\infty}\frac{1}{4\pi}\sum_{m=0}^{n}\frac{1,(-1)}{2}\left[\left(Y^{s,(c)}_{n+1,m+1}-Y^{s,(c)}_{n-1,m+1}\right)\left(1+\delta_{m,0}\right)\right. \\
&+\left(n-m+2\right)\left(n-m+1\right)Y^{s,(c)}_{n+1,m-1} \\
&\left.-\left(n+m\right)\left(n+m-1\right)Y^{s,(c)}_{n-1,m-1}\right]\frac{\partial\widetilde{\phi}_{n,m,g}\left(\mathbf{r}\right)}{\partial z}
\end{aligned}
\tag{16}
$$

利用勒让德多项式与球谐函数的正交关系，可以得到和前面一维相似的一组方程，即：

$$
\begin{aligned}
&(n+m)\frac{\partial\widetilde{\psi}_{n-1,m,g}}{\partial x}+\left(n-m+1\right)\frac{\partial\widetilde{\psi}_{n+1,m,g}}{\partial x} \\
&\quad+\frac{1}{2}\left[\left(1+\delta_{m,0}\right)\left(n+m\right)\left(n+m-1\right)\left(\frac{\partial\widetilde{\psi}_{n-1,m-1,g}}{\partial y}-\frac{\partial\widetilde{\gamma}_{n-1,m-1,g}}{\partial z}\right)\right. \\
&\quad-\left(1+\delta_{m,0}\right)\left(n-m+2\right)\left(n-m+1\right)\left(\frac{\partial\widetilde{\psi}_{n+1,m-1,g}}{\partial y}-\frac{\partial\widetilde{\gamma}_{n+1,m-1,g}}{\partial z}\right) \\
&\quad-\left(1+\delta_{m,0}\right)\left(\frac{\partial\widetilde{\psi}_{n-1,m+1,g}}{\partial y}+\frac{\partial\widetilde{\gamma}_{n-1,m+1,g}}{\partial z}\right) \\
&\quad\left.+\left(1+\delta_{m,0}\right)\left(\frac{\partial\widetilde{\psi}_{n+1,m+1,g}}{\partial y}+\frac{\partial\widetilde{\gamma}_{n+1,m+1,g}}{\partial z}\right)\right] \\
&\quad+\left(2n+1\right)\Sigma_{t,g}\widetilde{\psi}_{n,m,g}=\left(2n+1\right)\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\psi}_{n,m,g'}+\left(2n+1\right)Q^{c}_{n,m,g} \\
&\quad n=0,1,\cdots,N;\quad m=0,1,\cdots,n
\end{aligned}
\tag{17}
$$

和

$$
\begin{aligned}
&(n+m)\frac{\partial\widetilde{\gamma}_{n-1,m,g}}{\partial x}+(n-m+1)\frac{\partial\widetilde{\gamma}_{n+1,m,g}}{\partial x}\\
&\quad+\frac{1}{2}\left[(n+m)(n+m-1)\left(\frac{\partial\widetilde{\gamma}_{n-1,m-1,g}}{\partial y}+\frac{\partial\widetilde{\psi}_{n-1,m-1,g}}{\partial z}\right)\right.\\
&\quad-(n-m+2)(n-m+1)\left(\frac{\partial\widetilde{\gamma}_{n+1,m-1,g}}{\partial y}+\frac{\partial\widetilde{\psi}_{n+1,m-1,g}}{\partial z}\right)\\
&\quad\left.-\left(\frac{\partial\widetilde{\gamma}_{n-1,m+1,g}}{\partial y}-\frac{\partial\widetilde{\psi}_{n-1,m+1,g}}{\partial z}\right)+\left(\frac{\partial\widetilde{\gamma}_{n+1,m+1,g}}{\partial y}-\frac{\partial\widetilde{\psi}_{n+1,m+1,g}}{\partial z}\right)\right]\\
&\quad+(2n+1)\Sigma_{t,g}\widetilde{\gamma}_{n,m,g}=(2n+1)\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\gamma}_{n,m,g'}+(2n+1)Q^{s}_{n,m,g}\\
&\quad n=1,\cdots,N;\quad m=1,\cdots,n
\end{aligned}
\tag{18}
$$

式中

$$
Q^{c,(s)}_{n,m,g}=\int_0^{4\pi}Q_g(\mathbf{r},\mathbf{\Omega})\,Y^{c,(s)}_{c,m}(\mathbf{\Omega})\,d\mathbf{\Omega}
\tag{19}
$$

接下来首先回到式(11)和(12)讨论其物理意义，实际上中子角通量密度的球谐函数展开矩可以重新写为立体角的4π积分的形式：

$$
\widetilde{\psi}_{n,m}(\mathbf{r})=\int_0^{4\pi}\psi(\mathbf{r},\mathbf{\Omega})\,P^m_n(\cos\theta)\cos m\varphi d\mathbf{\Omega},n=0,\cdots;m=0,\cdots,n
\tag{20}
$$

$$
\widetilde{\gamma}_{n,m}(\mathbf{r})=\int_0^{4\pi}\psi(\mathbf{r},\mathbf{\Omega})\,P^m_n(\cos\theta)\sin m\varphi d\mathbf{\Omega},n=1,\cdots;m=1,\cdots,n
\tag{21}
$$

以一维问题中的求解经验来看，我们求出$n=1$，$m=1$和$m=0$的各个矩来分析其物理意义。

以x轴为极轴，当$n=1,m=0$时

$$
\begin{aligned}
\widetilde{\psi}_{1,0}(\mathbf{r})&=\int_0^{2\pi}\int_0^{\pi}\psi(\mathbf{r},\mathbf{\Omega})\cos\theta\sin\theta d\theta d\varphi\\
&=\int_0^{4\pi}\mathbf{e}_x\cdot\mathbf{\Omega}\psi(\mathbf{r},\mathbf{\Omega})\,d\mathbf{\Omega}=\int_0^{4\pi}\mathbf{e}_x\cdot\mathbf{J}(\mathbf{r},\mathbf{\Omega})\,d\mathbf{\Omega}\\
&=\int_0^{4\pi}J_x(\mathbf{r},\mathbf{\Omega})\,d\mathbf{\Omega}=J_x(\mathbf{r})
\end{aligned}
\tag{22}
$$

当$n=1,m=1$时

$$\begin{aligned}
\widetilde{\psi}_{1,1}(\mathbf{r}) &= \int_0^{2\pi}\int_0^{\pi}\psi(\mathbf{r},\mathbf{\Omega})(-\sin\theta)\cos\varphi\sin\theta d\theta d\varphi \\
&= -\int_0^{4\pi}\mathbf{e}_y\cdot\mathbf{\Omega}\psi(\mathbf{r},\mathbf{\Omega})d\mathbf{\Omega} = -\int_0^{4\pi}\mathbf{e}_y\cdot\mathbf{J}(\mathbf{r},\mathbf{\Omega})d\mathbf{\Omega} \\
&= -\int_0^{4\pi}J_y(\mathbf{r},\mathbf{\Omega})d\mathbf{\Omega} = -J_y(\mathbf{r}) \\
\widetilde{\gamma}_{1,1}(\mathbf{r}) &= \int_0^{2\pi}\int_0^{\pi}\psi(\mathbf{r},\mathbf{\Omega})(-\sin\theta)\sin\varphi\sin\theta d\theta d\varphi \\
&= -\int_0^{4\pi}\mathbf{e}_z\cdot\mathbf{\Omega}\psi(\mathbf{r},\mathbf{\Omega})d\mathbf{\Omega} = -\int_0^{4\pi}\mathbf{e}_z\cdot\mathbf{J}(\mathbf{r},\mathbf{\Omega})d\mathbf{\Omega} \\
&= -\int_0^{4\pi}J_z(\mathbf{r},\mathbf{\Omega})d\mathbf{\Omega} = -J_z(\mathbf{r})
\end{aligned} \tag{23}$$

可以看到，$\widetilde{\psi}_{1,0}(\mathbf{r})$，$-\widetilde{\psi}_{1,1}(\mathbf{r})$和$-\widetilde{\gamma}_{1,1}(\mathbf{r})$分别为$x$方向的中子流密度，$y$方向的中子流密度和$z$方向的中子流密度。

考虑$n=0$与$m=0$的情况，可以得到：

$$\widetilde{\psi}_{0,0}(\mathbf{r}) = \int_0^{2\pi}\int_0^{\pi}\psi(\mathbf{r},\mathbf{\Omega})\sin\theta d\theta d\varphi = \int_0^{4\pi}\psi(\mathbf{r},\mathbf{\Omega})d\mathbf{\Omega} = \phi(\mathbf{r}) \tag{24}$$

故$\widetilde{\psi}_{0,0}(\mathbf{r})$的物理意义可以视为中子标通量密度。针对式(17)与(18)，考虑P1近似，三维情况下可以得到$(N+1)^2=4$个方程。

当$n=0,m=0$时：

$$\begin{aligned}
&\frac{\partial\widetilde{\psi}_{1,0,g}}{\partial x} - 2\left(\frac{\partial\widetilde{\psi}_{1,-1,g}}{\partial y} - \frac{\partial\widetilde{\gamma}_{1,-1,g}}{\partial z}\right) - \left(\frac{\partial\widetilde{\psi}_{-1,1,g}}{\partial y} + \frac{\partial\widetilde{\gamma}_{-1,1,g}}{\partial z}\right) \\
&\qquad + \left(\frac{\partial\widetilde{\psi}_{1,1,g}}{\partial y} + \frac{\partial\widetilde{\gamma}_{1,1,g}}{\partial z}\right) + \Sigma_{t,g}\widetilde{\psi}_{0,0,g} = \sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\psi}_{0,0,g'} + Q_{0,0,g}^{c}
\end{aligned} \tag{25}$$

当$n=1,m=0$时：

$$\begin{aligned}
&\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial x} + 2\frac{\partial\widetilde{\psi}_{2,0,g}}{\partial x} - 6\left(\frac{\partial\widetilde{\psi}_{2,-1,g}}{\partial y} - \frac{\partial\widetilde{\gamma}_{2,-1,g}}{\partial z}\right) - \left(\frac{\partial\widetilde{\psi}_{0,1,g}}{\partial y} + \frac{\partial\widetilde{\gamma}_{0,1,g}}{\partial z}\right) \\
&\qquad + \left(\frac{\partial\widetilde{\psi}_{2,1,g}}{\partial y} + \frac{\partial\widetilde{\gamma}_{2,1,g}}{\partial z}\right) + 3\Sigma_{t,g}\widetilde{\psi}_{1,0,g} = 3\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\psi}_{1,0,g'}
\end{aligned} \tag{26}$$

当$n=1,m=1$时：

$$
\begin{aligned}
&2\frac{\partial\widetilde{\psi}_{0,1,g}}{\partial x}+\frac{\partial\widetilde{\psi}_{2,1,g}}{\partial x}+\left(\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial y}-\frac{\partial\widetilde{\gamma}_{0,0,g}}{\partial z}\right)-2\left(\frac{\partial\widetilde{\psi}_{2,0,g}}{\partial y}-\frac{\partial\widetilde{\gamma}_{2,0,g}}{\partial z}\right)\\
&\quad-\frac{1}{2}\left(\frac{\partial\widetilde{\psi}_{0,2,g}}{\partial y}+\frac{\partial\widetilde{\gamma}_{0,2,g}}{\partial z}\right)+\frac{1}{2}\left(\frac{\partial\widetilde{\psi}_{2,2,g}}{\partial y}+\frac{\partial\widetilde{\gamma}_{2,2,g}}{\partial z}\right)+3\Sigma_{t,g}\widetilde{\psi}_{1,1,g}\\
&\quad=3\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\psi}_{1,1,g'}
\end{aligned}
\tag{27}
$$

当$n=1,m=1$时：

$$
\begin{aligned}
&2\frac{\partial\widetilde{\gamma}_{0,1,g}}{\partial x}+\frac{\partial\widetilde{\gamma}_{2,1,g}}{\partial x}+\left(\frac{\partial\widetilde{\gamma}_{0,0,g}}{\partial y}+\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial z}\right)-\left(\frac{\partial\widetilde{\gamma}_{2,0,g}}{\partial y}+\frac{\partial\widetilde{\psi}_{2,0,g}}{\partial z}\right)\\
&\quad-\frac{1}{2}\left(\frac{\partial\widetilde{\gamma}_{0,2,g}}{\partial y}-\frac{\partial\widetilde{\psi}_{0,2,g}}{\partial z}\right)+\frac{1}{2}\left(\frac{\partial\widetilde{\gamma}_{2,2,g}}{\partial y}-\frac{\partial\widetilde{\psi}_{2,2,g}}{\partial z}\right)+3\Sigma_{t,g}\widetilde{\gamma}_{1,1,g}\\
&\quad=3\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\gamma}_{1,1,g'}+3Q_{1,1,g}^{s}
\end{aligned}
\tag{28}
$$

对于式(25)至(28)，略去高阶矩和不满足运算规则的矩，并且由式(21)可以得知$\widetilde{\gamma}_{0,0}=0$，则上式可以化简为：

$$
\frac{\partial\widetilde{\psi}_{1,0,g}}{\partial x}+\frac{\partial\widetilde{\psi}_{1,1,g}}{\partial y}+\frac{\partial\widetilde{\gamma}_{1,1,g}}{\partial z}+\Sigma_{t,g}\widetilde{\psi}_{0,0,g}=\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\psi}_{0,0,g'}+Q_{0,0,g}^{c}
\tag{29}
$$

$$
\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial x}+3\Sigma_{t,g}\widetilde{\psi}_{1,0,g}=3\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\psi}_{1,0,g'}
\tag{30}
$$

$$
\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial y}+3\Sigma_{t,g}\widetilde{\psi}_{1,1,g}=3\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\psi}_{1,1,g'}
\tag{31}
$$

$$
\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial z}+3\Sigma_{t,g}\widetilde{\gamma}_{1,1,g}=3\sum_{g'=1}^{G}\Sigma_{s,g'\to g}\widetilde{\gamma}_{1,1,g'}+3Q_{1,1,g}^{s}
\tag{32}
$$

定义散射源项算子$\mathcal{S}$

$$
\mathcal{S}=\sum_{g'=1}^{G}\Sigma_{s,g'\to g}
\tag{33}
$$

依据之前分析的中子角通量密度各阶展开矩的物理意义，可以将式(30)至(32)代

入式(29)中以得到二阶导项的扩散方程形式，有：

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\left(\frac{1}{3\Sigma_{t,g}}\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{1}{3\Sigma_{t,g}}\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{1}{3\Sigma_{t,g}}\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial z}\right)\\
&\quad+\Sigma_{t,g}\widetilde{\psi}_{0,0,g}+\frac{\partial}{\partial x}\left(\frac{\mathcal{S}\widetilde{\psi}_{1,0,g'}}{\Sigma_{t,g}}\right)+\frac{\partial}{\partial y}\left(\frac{\mathcal{S}\widetilde{\psi}_{1,1,g'}}{\Sigma_{t,g}}\right)+\frac{\partial}{\partial z}\left(\frac{\mathcal{S}\widetilde{\gamma}_{1,1,g'}+Q^{s}_{1,1,g}}{\Sigma_{t,g}}\right)\\
&\quad=\mathcal{S}\widetilde{\psi}_{0,0,g'}+Q_{0,0,g}
\end{aligned}
\tag{34}
$$

令$D_g=\frac{1}{3\Sigma_{t,g}}$，则式(34)进一步化简为：

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\left(D_g\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial x}\right)-\frac{\partial}{\partial y}\left(D_g\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial y}\right)-\frac{\partial}{\partial z}\left(D_g\frac{\partial\widetilde{\psi}_{0,0,g}}{\partial z}\right)\\
&\quad+\Sigma_{t,g}\widetilde{\psi}_{0,0,g}+\frac{\partial}{\partial x}\left(\frac{\mathcal{S}\widetilde{\psi}_{1,0,g'}}{\Sigma_{t,g}}\right)+\frac{\partial}{\partial y}\left(\frac{\mathcal{S}\widetilde{\psi}_{1,1,g'}}{\Sigma_{t,g}}\right)+\frac{\partial}{\partial z}\left(\frac{\mathcal{S}\widetilde{\gamma}_{1,1,g'}+Q^{s}_{1,1,g}}{\Sigma_{t,g}}\right)\\
&\quad=\mathcal{S}\widetilde{\psi}_{0,0,g'}+Q_{0,0,g}
\end{aligned}
\tag{35}
$$

由式(24)可以得到三维情况下中子输运方程球谐展开的P1近似方程：

$$
-\nabla\cdot\left(D_g\nabla\phi_g\left(\mathbf{r}\right)\right)+\Sigma_{t,g}\phi_g\left(\mathbf{r}\right)=\mathcal{S}\phi_{g'}\left(\mathbf{r}\right)+Q_{0,0,g}-\nabla\cdot\boldsymbol{\mathcal{S}} \tag{36}
$$

$$
\boldsymbol{\mathcal{S}}=\left(\frac{\mathcal{S}\widetilde{\psi}_{1,0,g'}}{\Sigma_{t,g}}\mathbf{e}_i,\frac{\mathcal{S}\widetilde{\psi}_{1,1,g'}}{\Sigma_{t,g}}\mathbf{e}_j,\frac{\mathcal{S}\widetilde{\gamma}_{1,1,g'}+Q^{s}_{1,1,g}}{\Sigma_{t,g}}\mathbf{e}_k\right) \tag{37}
$$