CCAR
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ASEN 5070
Statistical Orbit determination I
Fall 2012

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Lecture 10: Batch Processors

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University of Colorado
Boulder

Announcements

- Homework 3 Graded
  - Comments included on D2L
  - Any questions, talk with us this week
- Homework 4 due Today
- Homework 5 due next week
- Bobby Braun will be a guest speaker next Thursday.
  - NASA Chief Technologist
  - Now at Georgia Tech.
- Exam on 10/11.
- I'll be out of the state 10/9 – 10/16; we'll have guest lecturers then.

Syllabus Review

- We're right on track with the syllabus.

- Topics today:
  - Least Squares
  - Weighted Least Squares
  - Minimum Norm
  - A Priori
  - Minimum Variance
  - The Batch Processor
  - Bayesian

Quiz Results

Question 1 (1 point)

The least squares solution minimizes the sum of the residuals.

- True
- False

$$\min\left(\sum_{i=1}^{N}\epsilon_i\right)?$$

Quiz Results

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- False

$$\min\left(\sum_{i=1}^{N}\epsilon_i\right)?$$

$$\min\left(\sum_{i=1}^{N}\epsilon_i^2\right)$$

Quiz Results

Question 2 (1 point)

What is the rank of the following H-matrix?

H =

[2 4 1 ;

6 12 3]

(If it's unclear, that's a matrix with two row and three columns. First row = [2, 4, 1]. 2nd row = [6, 12 3].)

- 0
- 1
- 2
- 3

$$H = \begin{bmatrix} 2 & 4 & 1 \\ 6 & 12 & 3 \end{bmatrix}$$

Quiz Results

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Question 2 (1 point)

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Quiz Results

Question 3 (1 point)

If we have perfect observations and no errors in the system, then we can map each observation to a state deviation vector using the relationship: y(t) = H*x(t_0) (notice there's no "epsilon" error).

True/False: If our state "X" has 7 parameters in it, we can deterministically solve for X using only 7 linearly independent observations.

- True
- False

Quiz Results

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- True
- False

$$y(t) = H\mathbf{x}(t_0)$$

$$y_1 = \tilde{H}\Phi_{10}\mathbf{x}(t_0)_{7\times 1}$$

$$y_2 = \tilde{H}\Phi_{20}\mathbf{x}(t_0)_{7\times 1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_7 = \tilde{H}\Phi_{70}\mathbf{x}(t_0)_{7\times 1}$$

Quiz Results

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Quiz Results

Question 4 (1 point)

We have a state vector "X" of 6 parameters that we're estimating. We have 10 range observations made at different times that each include an error. We're able to map each range observation to a particular epoch t_0 using the relationship y(t) = H*x(t_0) + e(t).

How many unknowns are present in this system?

- 6
- 10
- 16
- 60

Quiz Results

Question 4 (1 point)

We have a state vector "X" of 6 parameters that we're estimating. We have 10 range observations made at different times that each include an error. We're able to map each range observation to a particular epoch t_0 using the relationship y(t) = H*x(t_0) + e(t).

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- ○ 6
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- ○ 16
- ○ 60

$$y_1 = \tilde{H}\Phi_{1,0}\ \mathbf{x}(t_0)_{6\times 1} + \epsilon_1$$

$$y_2 = \tilde{H}\Phi_{2,0}\ \mathbf{x}(t_0)_{6\times 1} + \epsilon_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_{10} = \tilde{H}\Phi_{10,0}\ \mathbf{x}(t_0)_{6\times 1} + \epsilon_{10}$$

Quiz Results

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$$y_{10} = \tilde{H}\Phi_{10,0}\ \mathbf{x}(t_0)_{6\times 1} + \epsilon_{10}$$

Stat OD Filters

- Given:
  - Nominal state at some time $\mathbf{X}^*(t_0)$
  - Dynamical Models
  - Observations $\mathbf{y}(t)$
- Want:
  - Best estimate of state at some (other) time $\hat{\mathbf{X}}(t_i)$
  - Covariance of that state $\hat{P}(t_i)$
- Optional:
  - *a priori* information about state or covariance $\bar{\mathbf{X}}(t_0)$ $\bar{P}(t_0)$
  - Weighting matrix for observations
  - Statistical information about observations

Information

- The state has n parameters
  - n unknowns at any given time.
- There are l observations of any given type.
- There are p types of observations (range, range-rate, angles, etc)
- We have $p \; x \; l = m$ total equations.
- Three situations:
  - $n < m$: Least Squares
  - $n = m$: Deterministic
  - $n > m$: Minimum Norm

Least Squares

- The state deviation vector that minimizes the least-squares cost function:

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

$$\hat{\mathbf{X}}(t_k) = \mathbf{X}^*(t_k) + \hat{\mathbf{x}}(t_k)$$

- Additional Details:
  - $H^T H$ is called the *normal matrix*
  - If H is full rank, then this will be positive definite.
    - If it's not then we don't have a least squares estimate!

Least Squares

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- Additional Details:
  - $P_k = \left(H^T H\right)^{-1}$ is related to the variance-covariance matrix
  - If it exists (i.e., if $H^T H$ is invertible), then P_k is symmetric and positive definite.

- The Minimum Norm Solution

Minimum Norm

For the least squares solution

$$\hat{\mathbf{x}} = (H^T H)^{-1} H^T \mathbf{y}$$

to exist $m \geq n$ and H be of rank n

Consider a case with $m \leq n$ and rank $H < n$

There are more unknowns than linearly independent observations

Minimum Norm

Option 1: specify any *n* – *m* of the *n* components of x and solve for remaining *m* components of x using observation equations with $\boldsymbol{\epsilon} = 0$

Result: an infinite number of solutions for $\hat{\boldsymbol{x}}$

Option 2: use the minimum norm criterion to uniquely determine $\hat{\boldsymbol{x}}$

Using the generally available nominal/initial guess for x the minimum norm criterion chooses x to minimize the sum of the squares of the difference between X and X* with the constraint that $\boldsymbol{\epsilon} = 0$

Minimum Norm

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Recall: $$\boldsymbol{x} = \boldsymbol{X} - \boldsymbol{X}^*$$

Want to minimize the sum of the squares of the difference given $\boldsymbol{\epsilon} = 0$

That is $\boldsymbol{y} = H\boldsymbol{x}$

Minimum Norm

Hence the performance index becomes:

$$J(\mathbf{x}, \boldsymbol{\lambda}) = 1/2\mathbf{x}^T\mathbf{x} + \boldsymbol{\lambda}^T(\mathbf{y} - H\mathbf{x})$$

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$$\frac{\partial J(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 = \mathbf{x} - H^T\lambda$$

$$\frac{\partial J(\mathbf{x}, \lambda)}{\partial \lambda} = 0 = \mathbf{y} - H\mathbf{x}$$

$$\hat{\mathbf{x}} = H^T(HH^T)^{-1}\mathbf{y}$$

Pseudo-Inverses

Apply when there are more unknowns than equations or more equations than unknowns

$$
\begin{aligned}
\hat{\mathbf{x}} &= (H^T H)^{-1} H^T \mathbf{y}, && \text{if } m > n \\
\hat{\mathbf{x}} &= H^{-1} \mathbf{y}, && \text{if } m = n \\
\hat{\mathbf{x}} &= H^T (H H^T)^{-1} \mathbf{y}, && \text{if } m < n.
\end{aligned}
$$

- Weighted Least Squares

Weighted Least Squares

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- If we have information about how to weigh different observations in the filter

$$
\begin{aligned}
\mathbf{y}_1 &= H_1 \mathbf{x}_k + \epsilon_1; && w_1 \\
\mathbf{y}_2 &= H_2 \mathbf{x}_k + \epsilon_2; && w_2 \\
&\vdots \quad \vdots \quad \vdots \\
\mathbf{y}_\ell &= H_\ell \mathbf{x}_k + \epsilon_\ell; && w_\ell
\end{aligned}
$$

$$
H_i = \widetilde{H}_i \Phi(t_i, t_k).
$$

Weighted Least Squares Solution

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$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_\ell \end{bmatrix}; \quad H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_\ell \end{bmatrix};$$

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_\ell \end{bmatrix}; \quad W = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & w_\ell \end{bmatrix}$$

Weighted Least Squares Solution

$$\mathbf{y} = H\mathbf{x}_k + \boldsymbol{\epsilon}; \quad W.$$

$$J(\mathbf{x}_k) = 1/2\boldsymbol{\epsilon}^T W \boldsymbol{\epsilon} = \sum_{i=1}^{\ell} 1/2\boldsymbol{\epsilon}_i^T w_i \boldsymbol{\epsilon}_i .$$

$$J(\mathbf{x}_k) = 1/2(\mathbf{y} - H\mathbf{x}_k)^T W(\mathbf{y} - H\mathbf{x}_k).$$

Weighted Least Squares Solution

$$J(\mathbf{x}_k) = 1/2(\mathbf{y} - H\mathbf{x}_k)^T W(\mathbf{y} - H\mathbf{x}_k).$$

$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 =$$

Weighted Least Squares Solution

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$$J(\mathbf{x}_k) = 1/2\underbrace{(\mathbf{y} - H\mathbf{x}_k)^T W (\mathbf{y} - H\mathbf{x}_k)}_{1/2\left[A^T B\right]}.$$

$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 = \quad 1/2\left[A^T \frac{\partial B}{\partial X} + B^T \frac{\partial A}{\partial X}\right]$$

Weighted Least Squares Solution

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$$J(\mathbf{x}_k) = 1/2\underbrace{(\mathbf{y} - H\mathbf{x}_k)^T W(\mathbf{y} - H\mathbf{x}_k)}_{1/2\left[A^T B\right]}.$$

$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 = \quad 1/2\left[A^T \frac{\partial B}{\partial X} + B^T \frac{\partial A}{\partial X}\right]$$

$$A^T = (\mathbf{y} - H\mathbf{x}_k)^T$$

$$\frac{\partial B}{\partial X} = W(-H) = -WH$$

$$B^T = (W(\mathbf{y} - H\mathbf{x}_k))^T = (\mathbf{y} - H\mathbf{x}_k)^T W^T$$

$$\frac{\partial A}{\partial X} = -H$$

Weighted Least Squares Solution

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$$\begin{aligned}
\frac{\partial J}{\partial \mathbf{x}_k} = 0 &= 1/2\left[(\mathbf{y} - H\mathbf{x}_k)^T(-WH) + (\mathbf{y} - H\mathbf{x}_k)^T W^T(-H)\right] \\
&= 1/2\left[-(\mathbf{y} - H\mathbf{x}_k)^T WH - (\mathbf{y} - H\mathbf{x}_k)^T WH\right] \\
&= -(\mathbf{y} - H\mathbf{x}_k)^T WH
\end{aligned}$$

Weighted Least Squares Solution

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$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 = 1/2\left[(\mathbf{y} - H\mathbf{x}_k)^T(-WH) + (\mathbf{y} - H\mathbf{x}_k)^T W^T(-H)\right]$$

$$= 1/2\left[-(\mathbf{y} - H\mathbf{x}_k)^T WH - (\mathbf{y} - H\mathbf{x}_k)^T WH\right]$$

$$= -(\mathbf{y} - H\mathbf{x}_k)^T WH$$

$$0 = (\mathbf{y} - H\mathbf{x}_k)^T WH$$

$$= \left[(\mathbf{y} - H\mathbf{x}_k)^T WH\right]^T$$

$$= H^T W(\mathbf{y} - H\mathbf{x}_k)$$

$$= H^T W\mathbf{y} - H^T WH\mathbf{x}_k$$

Weighted Least Squares Solution

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$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 = -(y - H\mathbf{x}_k)^T W H = -H^T W(\mathbf{y} - H\mathbf{x}_k)$$

$$(H^T W H)\mathbf{x}_k = H^T W \mathbf{y}.$$

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}.$$

Least Squares

- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

Least Squares

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$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

$$P_k = \left(H^T H\right)^{-1}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

$$P_k = \left(H^T W H\right)^{-1}$$

Note: These P matrices are not necessarily the covariance matrices. They ARE if W is properly selected.

- Weighted Least Squares with *a priori* information

Least Squares with *a priori* Information

If an apriori value is available for x_k (call it $\bar{x}_k$) and an associated symmetric weighting matrix $\bar{w}_k$, the weighted least squares estimate of $\hat{x}_k$ can be obtained.

Least Squares with *a priori* Information

Given

$$y = Hx_k + \varepsilon$$

$$\bar{x}_k = x_k + \eta_k$$

Where η_k is the error in $\bar{x}_k$ and its influence on $\hat{x}_k$ is reflected in the weighting matrix $\bar{W}_k$

and $[y]_{m\times 1}$, $[\bar{x}_k]_{n\times 1}$

Choose $\hat{x}_k$ to minimize the performance index

$$J(x_k) = \frac{1}{2}\varepsilon^T w \varepsilon + \frac{1}{2}\eta_k \bar{w}_k \eta_k^{\ T}$$

Least Squares with *a priori* Information

Writing $J\left(x_k\right)$ explicitly in terms of x_k

$$J\left(x_k\right)=\frac{1}{2}\left(y-Hx_k\right)^T w\left(y-Hx_k\right)+\frac{1}{2}\left(\bar{x}_k-x_k\right)^T \bar{w}_k\left(\bar{x}_k-x_k\right)$$

$$\frac{\partial J\left(\mathbf{x}_k\right)}{\partial \mathbf{x}_k}=0$$

$$\frac{\partial J\left(\mathbf{x}_k\right)}{\partial \mathbf{x}_k}=-\left(\mathbf{y}-\mathbf{H}\mathbf{x}_k\right)^T \mathbf{w}\mathbf{H}-\left(\bar{\mathbf{x}}_k-\mathbf{x}_k\right)^T \mathbf{w}_k=0$$

$$=-\mathbf{y}^T\mathbf{w}\mathbf{H}+\mathbf{x}_k{}^T\mathbf{H}^T\mathbf{w}\mathbf{H}-\bar{\mathbf{x}}_k{}^T\mathbf{w}_k+\mathbf{x}_k{}^T\mathbf{w}_k=0$$

Least Squares with *a priori* Information

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Solving for x_k yields $\hat{x}_k$

$$\mathbf{x}_k^T\left(H^T wH + \mathbf{w}_k\right) = y^T wH + \bar{\mathbf{x}}_k^T \mathbf{w}_k$$

$$\hat{\mathbf{x}}_k^T = \left(y^T wH + \bar{\mathbf{x}}_k^T \mathbf{w}_k\right)\left(H^T wH + \mathbf{w}_k\right)^{-1}$$

$$\hat{x}_k = \left(H^T wH + w_k\right)^{-1}\left(H^T wy + w_k \bar{x}_k\right)$$

Least Squares with *a priori* Information

Note that $\left(H^T w H + w_k\right)^{-1}$ is symmetric

also

$$\frac{\partial^2 J\left(x_k\right)}{\partial x_k^2} = H^T w H + w_k$$

which will be positive definite if *H* and/or w_k is full rank. Hence, $\hat{x}_k$ minimizes $J\left(x_k\right)$.

Summary so far

- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

- Least Squares with *a priori*

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

- Minimum Variance Estimate

Minimum Variance

- We can use the Minimum Variance Estimator to generate the best estimate of the state, given any statistical information about the observations.
  - Varying standard deviations
  - Correlations of observations over time
  - Observation drift
  - Etc.

Minimum Variance

- Given: System of state-propagation equations and observation state equations:

$$\mathbf{x}_i = \Phi(t_i, t_k)\mathbf{x}_k$$

$$\mathbf{y}_i = \tilde{H}_i \mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

- Find: The linear, unbiased, minimum variance estimate, $\hat{\mathbf{x}}_k$, of the state $\mathbf{x}_k$.

Minimum Variance

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- Find: The linear, unbiased, minimum variance estimate, $\hat{\mathbf{x}}_k$, of the state $\mathbf{x}_k$.

- Linear:

Minimum Variance

- Given: System of state-propagation equations and observation state equations:

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$$\mathbf{y}_i = \tilde{H}_i\mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

- Find: The linear, unbiased, minimum variance estimate, $\hat{\mathbf{x}}_k$, of the state $\mathbf{x}_k$.

- Unbiased:

Minimum Variance

- Given: System of state-propagation equations and observation state equations:

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$$\mathbf{y}_i = \tilde{H}_i\mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

- Find: The linear, unbiased, minimum variance estimate, $\hat{\mathbf{x}}_k$, of the state $\mathbf{x}_k$.

- Min Variance Estimate:

- (Jump from the derivation to 4.4.17 – see book for details)

Minimum Variance

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$$\mathbf{x}_i = \Phi(t_i, t_k)\mathbf{x}_k$$

$$\mathbf{y}_i = \tilde{H}_i\mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

- Find: The linear, unbiased, minimum variance estimate, $\hat{\mathbf{x}}_k$, of the state $\mathbf{x}_k$.

- Best Estimate:

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1}\mathbf{y}$$

Summary so far

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- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

- Least Squares with *a priori*

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

- Min Variance

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

$$\hat{\mathbf{x}}_k = P_k H^T R^{-1} \mathbf{y}$$

Propagating the estimate and covariance

$$\bar{\mathbf{x}}_k = \Phi(t_k, t_j)\hat{\mathbf{x}}$$

$$\bar{P}_k \equiv E\left[(\bar{\mathbf{x}}_k - \mathbf{x}_k)(\bar{\mathbf{x}}_k - \mathbf{x}_k)^T\right]$$

$$\bar{P}_k = \Phi(t_k, t_j) P_j \Phi^T(t_k, t_j)$$

More Options

- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

- Least Squares with *a priori*

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

- Min Variance

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

- Min Variance with *a priori*

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

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