ASEN 5070
Statistical Orbit Determination I
Fall 2012

Professor Jeffrey S. Parker
Professor George H. Born

Lecture 26: Smoothing, Monte Carlo

Announcements

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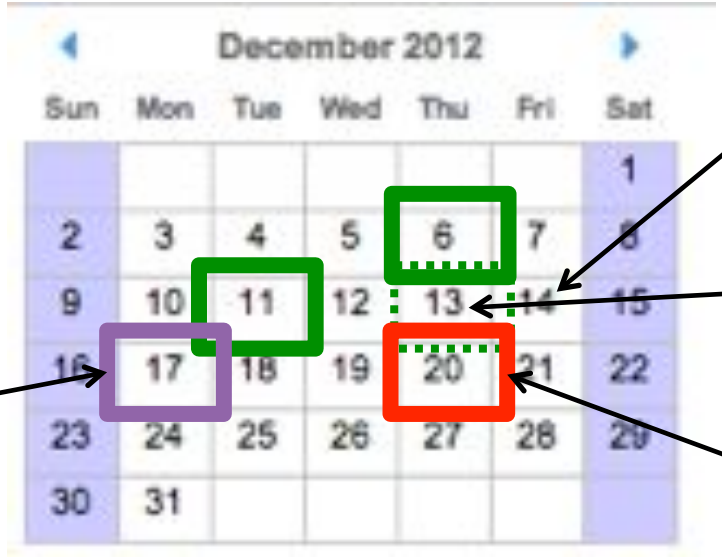
- HW 11 due.
- We are *still* catching up with grading. I guess we never caught up after HW2!
- Check your grades (for those graded anyway), especially quizzes.



Last Day of Classes

CCAR Conflict

Take-Home Exam Due

Final Project Due
All HW Due

Quiz 22 Review

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Question 1 (1 point)

Let's say we have a satellite is low Earth orbit at an altitude of about 600 km. We have an estimate of its state and an associated state estimate covariance matrix. Recall that we can represent the position components of the covariance matrix as a 3D probability ellipsoid.

At some moment in time, the satellite's state estimate probability ellipsoid is very close to a sphere of radius 100 meters. At that moment we take several precise laser-range observations of the satellite from a ground station directly beneath the satellite. The range observations have an error of about 1 meter. What would happen to the probability ellipsoid?

- It would remain the same.
- It would remain spherical and get smaller in radius.
- It would flatten out into a pancake: about 100 meters wide and very thin, with the thin axis pointing toward the ground station.
- It would become very narrow, like a straight banana, where the long axis remains around 100 meters and is pointing toward the ground; the other two axes are significantly smaller than 100 meters.

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Quiz 22 Review

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Question 2 (1 point)

Let's say we have the same scenario as in Problem 1, but we replace the satellite with a cloud of tiny particles. The cloud is roughly spherical and has a radius of about 100 meters.

If you propagate this cloud of particles forward in time for a day, such that the cloud orbits the Earth a dozen times or so before the center of the cloud returns to nearly the same spot above the Earth's surface, what would the cloud look like?

- It would look roughly spherical with a radius of about 100 meters.
- It would resemble a long, thin, curved banana, extending along the orbit's track +/- a bit in any direction.
- The particles would be randomly distributed anywhere around the orbit.
- The particles would be distributed around the entire Earth.

Quiz 22 Review

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Quiz 22 Review

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Question 3 (1 point)

The final exam may certainly cover any of the basic topics of Stat OD including the Batch, the Sequential, and the EKF; it may also cover anything having to do with numerical errors and process noise. Since it's take-home, it will probably include some computer programming. The word "covariance" will undoubtely show up :)

What topics in particular would you like us to review before the exam?

I'm writing the test now; I'll make sure to cover those topics again. As time permits, we'll cover everything else, but I'll try to satisfy the most number of requests ;)

Contents

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- Smoothing
- Monte Carlo
- Special Topics:
  - Consider Covariance, consider filter
  - Chandrayaan-1 Navigation
- TA Evaluations!

Smoothing

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- Smoothing is a method by which a state estimate (and optionally, the covariance) may be constructed using observations before and after the epoch.

- Step 1. Process all observations using a CKF with process noise compensation.
- Step 2. Start with the last observation processed and *smooth* back through the observations.

Smoothing

- On Tuesday we showed that if there is no process noise, smoothing ends up just mapping the final estimate and covariance back through time.

- Smoothing is good if you manipulate the covariance matrix during the sequential filter in any way.
  - Process noise

Smoothing visualization

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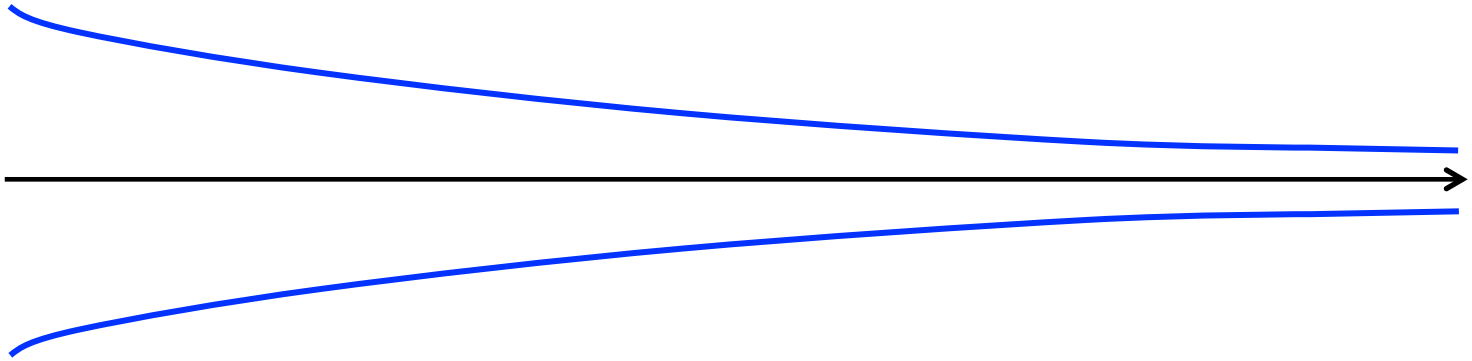
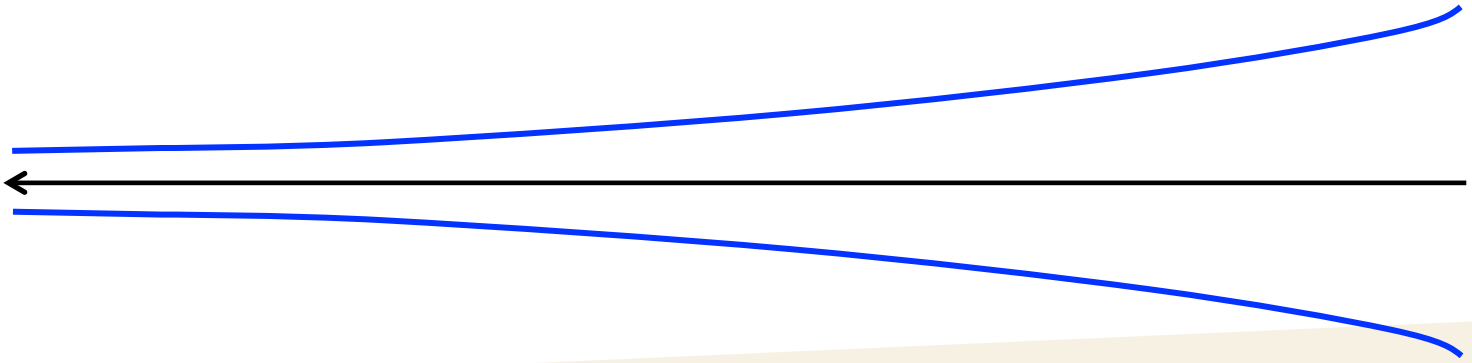
- Process observations forward in time:



- If you were to process them backward in time (given everything needed to *do* that):



Smoothing visualization

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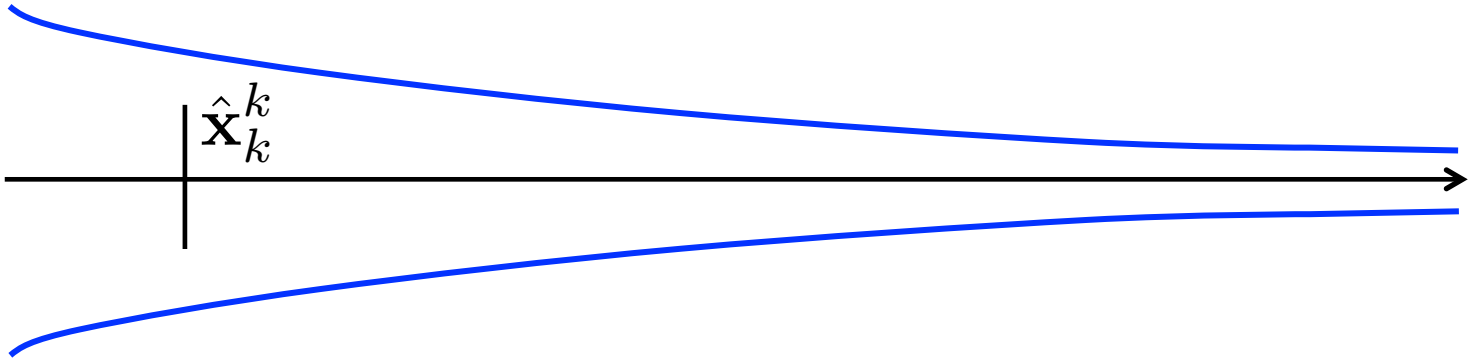
- Process observations forward in time:



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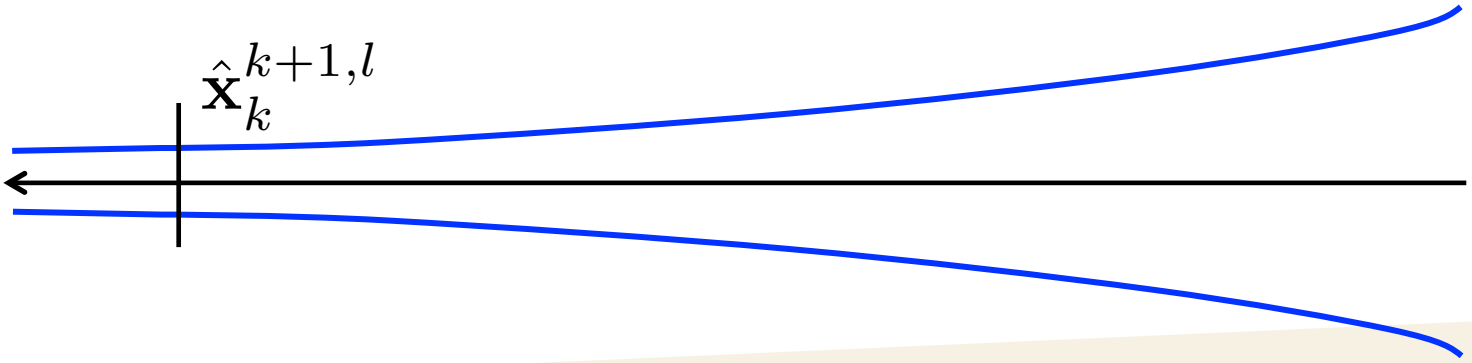

Smoothing visualization

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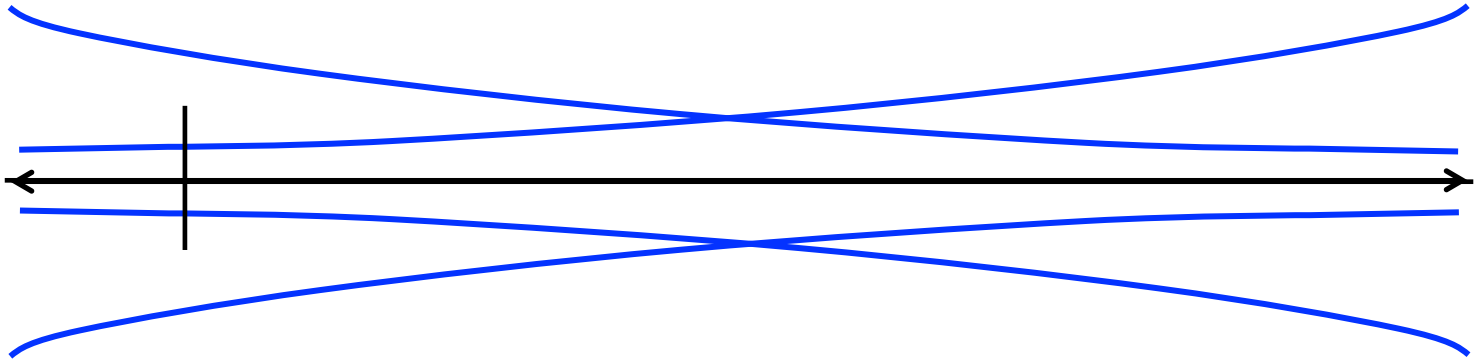
- Smoothing does not actually combine them, but you can think about it in order to conceptualize what smoothing does.



- Smoothing results in a much more consistent solution over time. And it results in an *optimal* estimate using all observations.

Smoothing

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- One major caveat to this.
  - If you use process noise or some other way to raise the covariance, the result is that the *optimal* estimate at any time really only pays attention to observations nearby.
  - While this is good, it also means smoothing doesn't always have a big effect.

- Smoothing shouldn't remove the white noise found on the signals.
  - It's not a "cleaning" function, it's a "use all the data for your estimate" function.

Smoothing

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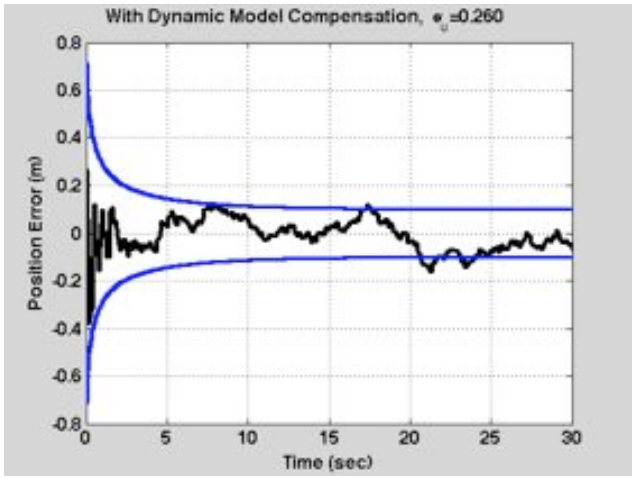
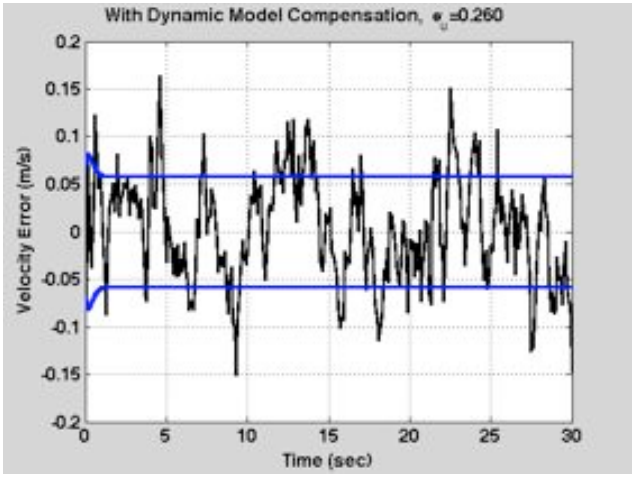
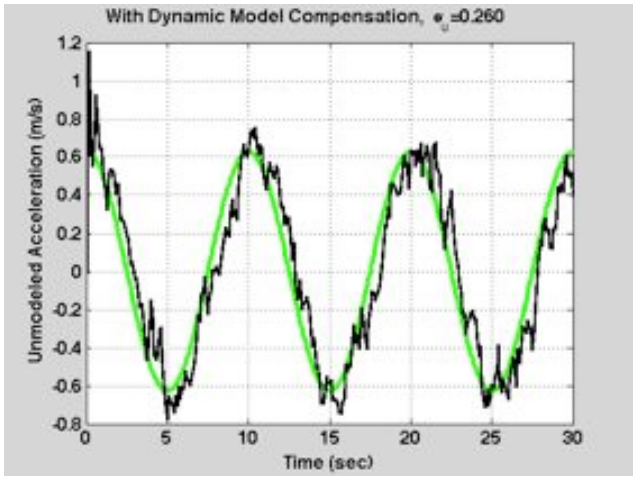
- Returning to the process noise / DMC example from earlier:
- Recall the particle on the x-axis moving at ~10 m/s with an unmodeled acceleration acting on it.







Smoothing

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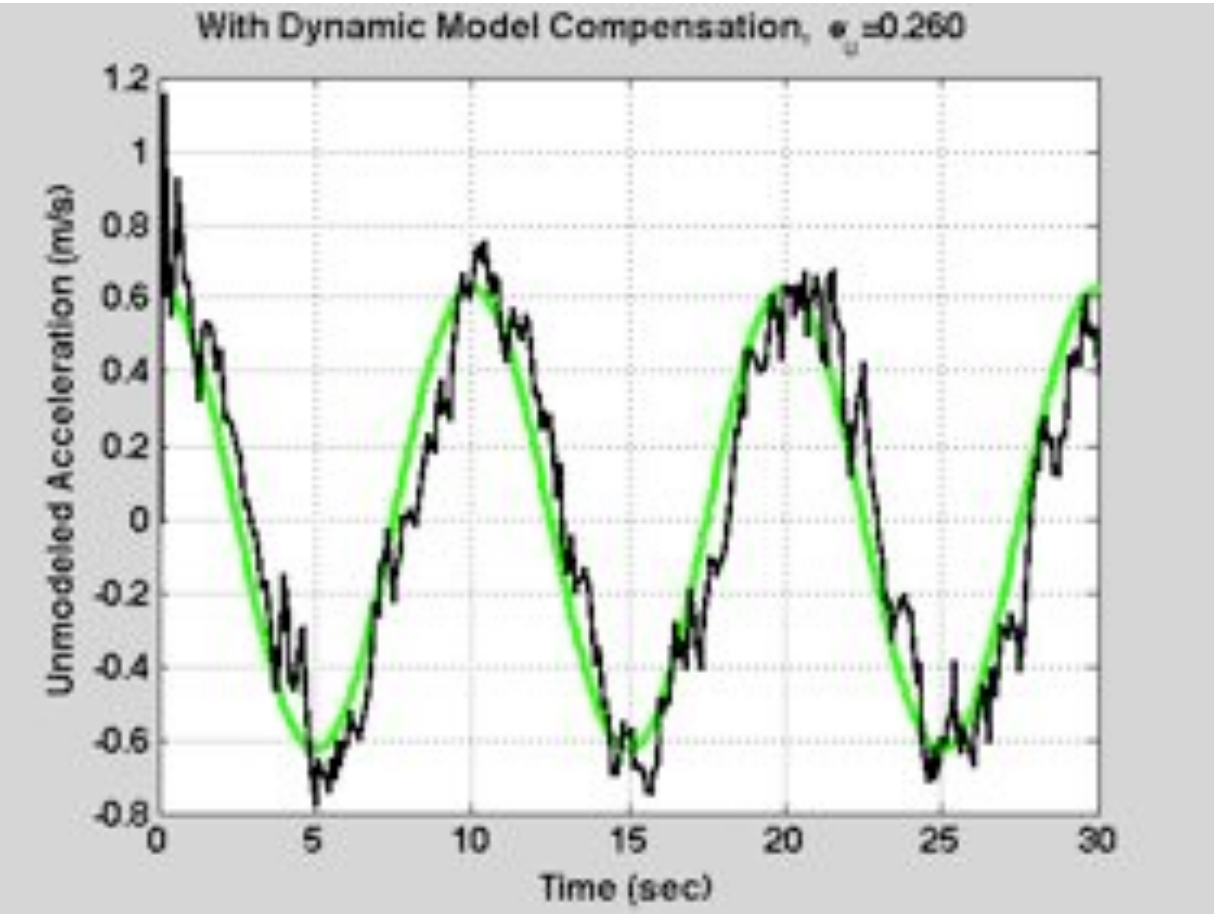
- Each acceleration estimate is based on previous observations.
- We'll demo the smoothing process and show it's results.



Smoothing

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- Say there are 100 observations

Y_{97}	Y_{98}	Y_{99}	Y_{100}
$\hat{\mathbf{x}}_{97}^{97}$	$\hat{\mathbf{x}}_{98}^{98}$	$\hat{\mathbf{x}}_{99}^{99}$	$\hat{\mathbf{x}}_{100}^{100}$
P_{97}^{97}	P_{98}^{98}	P_{99}^{99}	P_{100}^{100}

- We want to construct new estimates using all data, i.e., $\hat{\mathbf{x}}_{97}^{100}$ $\hat{\mathbf{x}}_{98}^{100}$ $\hat{\mathbf{x}}_{99}^{100}$

Smoothing

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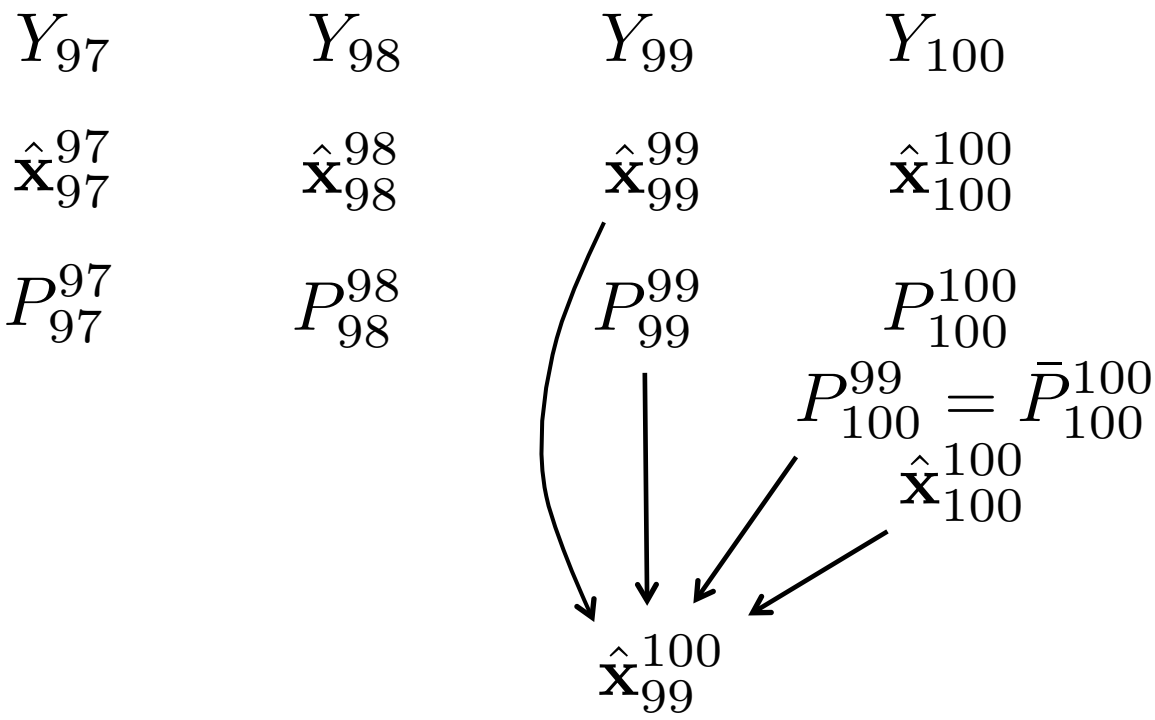
- Say there are 100 observations



Smoothing

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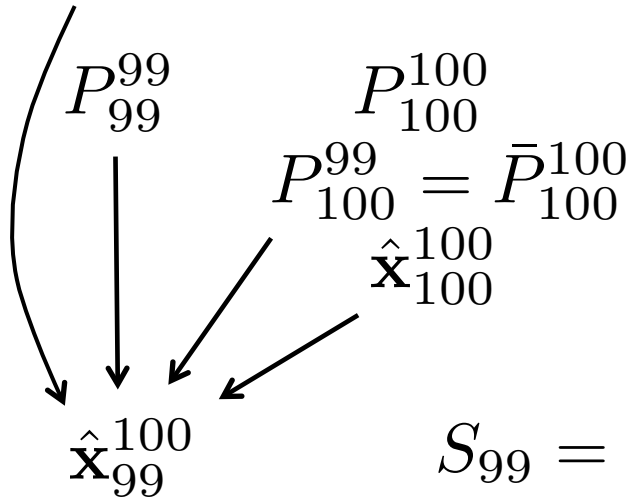
- Say there are 100 observations

$$Y_{97} \quad Y_{98} \quad Y_{99} \quad Y_{100}$$

$$\hat{\mathbf{x}}_{97}^{97} \quad \hat{\mathbf{x}}_{98}^{98} \quad \hat{\mathbf{x}}_{99}^{99} \quad \hat{\mathbf{x}}_{100}^{100}$$

$$P_{97}^{97} \quad P_{98}^{98} \quad P_{99}^{99} \quad P_{100}^{100}$$



$$S_{99} = P_{99}^{99}\Phi^T(t_{100}, t_{99})(P_{100}^{99})^{-1}$$

$$\hat{\mathbf{x}}_{99}^{100} = \hat{\mathbf{x}}_{99}^{99} + S_{99}(\hat{\mathbf{x}}_{100}^{100} - \Phi(t_{100}, t_{99})\hat{\mathbf{x}}_{99}^{99})$$

Smoothing

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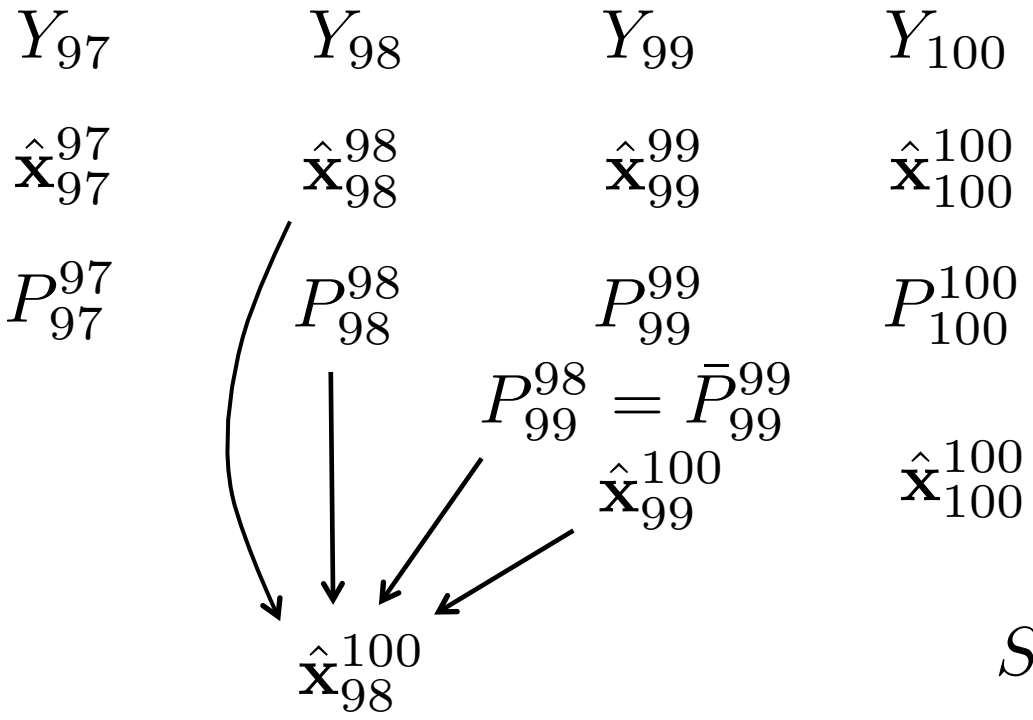
- Say there are 100 observations



$$S_{98} = P_{98}^{98}\Phi^T(t_{99}, t_{98})(P_{99}^{98})^{-1}$$

$$\hat{\mathbf{x}}_{98}^{100} = \hat{\mathbf{x}}_{98}^{98} + S_{98}(\hat{\mathbf{x}}_{100}^{99} - \Phi(t_{99}, t_{98})\hat{\mathbf{x}}_{98}^{98})$$

Smoothing

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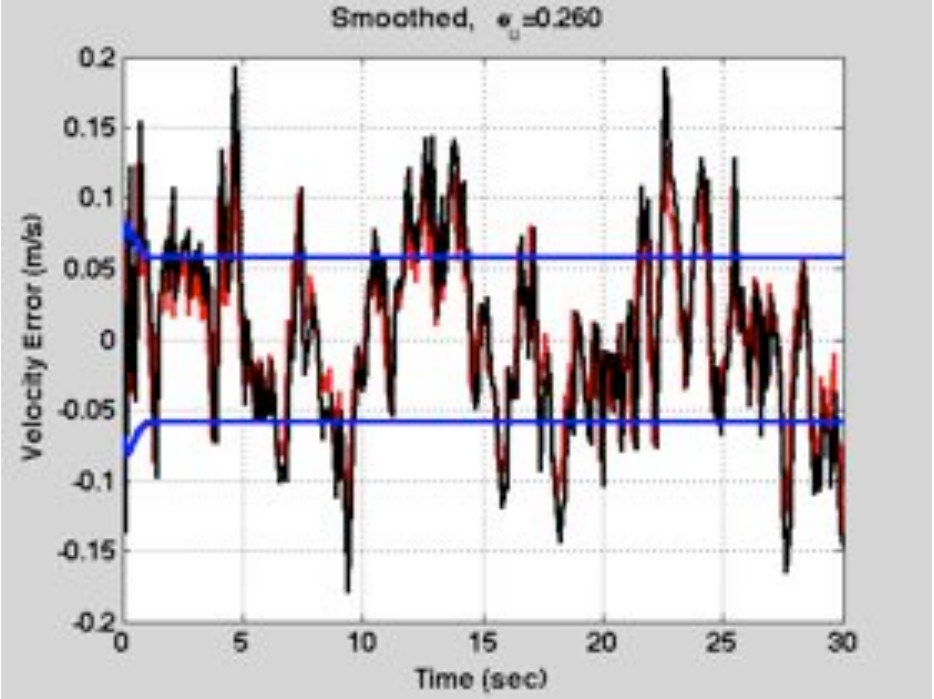
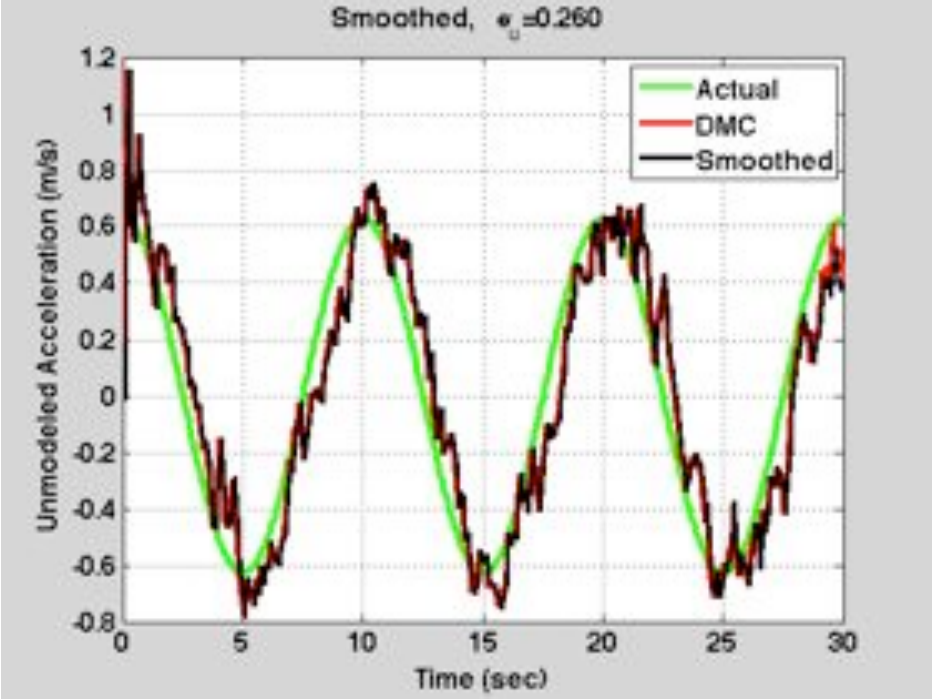
- When applied to the example problem:





Dropped the RMS, but not by much – a few percent.

Smoothing

$$\hat{\mathbf{x}}_k^\ell = \hat{\mathbf{x}}_k^k + S_k\left(\hat{\mathbf{x}}_{k+1}^\ell - \Phi(t_{k+1}, t_k)\hat{\mathbf{x}}_k^k\right) \tag{4.15.9}$$

where

$$\begin{aligned} S_k &= P_k^k \Phi^T(t_{k+1}, t_k)[\Phi(t_{k+1}, t_k) P_k^k \Phi^T(t_{k+1}, t_k) \\ &\quad + \Gamma(t_{k+1}, t_k) Q_k \Gamma^T(t_{k+1}, t_k)]^{-1} \\ &= P_k^k \Phi^T(t_{k+1}, t_k)(P_{k+1}^k)^{-1}. \end{aligned} \tag{4.15.10}$$

Eq. (4.15.9) is the smoothing algorithm. Computation goes backward in index k, with $\hat{\mathbf{x}}_\ell^\ell$, the filter solution, as initial conditions. Note that the filter solutions for $\hat{\mathbf{x}}_k^k$, P_k^k, $\Phi(t_{k+1}, t_k)$, and $\Gamma(t_{k+1}, t_k)$ are required and should be stored in the filtering process. The time update of the covariance matrix, P_{k+1}^k, may be stored or recomputed.

Smoothing

The equation for propagating the smoothed covariance is derived next (Jazwinski, 1970; Rausch *et al.*, 1965) . It can easily be shown from Eq. (4.15.9) that $\hat{\mathbf{x}}_k^\ell$ is unbiased; hence, the smoothed covariance is defined by

$$P_k^\ell = E\left[(\hat{\mathbf{x}}_k^\ell - \mathbf{x}_k)(\hat{\mathbf{x}}_k^\ell - \mathbf{x}_k)^T\right] . \qquad (4.15.11)$$

The equation for the smoothed covariance is given by

$$P_k^\ell = P_k^k + S_k\,(P_{k+1}^\ell - P_{k+1}^k)S_k^T . \qquad (4.15.24)$$

Smoothing Computational Algorithm

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Given (from the filtering algorithm)

$$\hat{\mathbf{x}}_\ell^\ell,\ \hat{\mathbf{x}}_{\ell-1}^{\ell-1},\ P_\ell^{\ell-1},\ P_{\ell-1}^{\ell-1},\ \Phi(t_\ell, t_{\ell-1});$$

set $k = \ell - 1$

$$S_{\ell-1} = P_{\ell-1}^{\ell-1}\,\Phi^T(t_\ell, t_{\ell-1})\,(P_\ell^{\ell-1})^{-1} \qquad (4.15.29)$$

$$\hat{\mathbf{x}}_{\ell-1}^\ell = \hat{\mathbf{x}}_{\ell-1}^{\ell-1} + S_{\ell-1}(\hat{\mathbf{x}}_\ell^\ell - \Phi(t_\ell, t_{\ell-1})\,\hat{\mathbf{x}}_{\ell-1}^{\ell-1}).$$

Smoothing Computational Algorithm

Given (from the filtering algorithm and the previous step of the smoothing algorithm)

$$\hat{\mathbf{x}}_{\ell-2}^{\ell-2},\quad P_{\ell-1}^{\ell-2},\quad P_{\ell-2}^{\ell-2},\quad \hat{\mathbf{x}}_{\ell-1}^{\ell},\quad \Phi(t_{\ell-1}, t_{\ell-2});$$

set $k = \ell - 2$, and compute

$$S_{\ell-2} = P_{\ell-2}^{\ell-2}\, \Phi^T\,(t_{\ell-1}, t_{\ell-2})\,(P_{\ell-1}^{\ell-2})^{-1} \qquad (4.15.30)$$

$$\hat{\mathbf{x}}_{\ell-2}^{\ell} = \hat{\mathbf{x}}_{\ell-2}^{\ell-2} + S_{\ell-2}\,(\hat{\mathbf{x}}_{\ell-1}^{\ell} - \Phi\,(t_{\ell-1}, t_{\ell-2})\,\hat{\mathbf{x}}_{\ell-2}^{\ell-2})$$

$$\vdots$$

and so on.

Smoothing

- If we suppose that there is no process noise (Q=0), then the smoothing algorithm reduces to the CKF mapping relationships:

$$S_k = \Phi^{-1}(t_{k+1}, t_k)$$
$$\hat{\mathbf{x}}_k^l = \Phi(t_k, t_l)\hat{\mathbf{x}}_l^l$$
$$P_k^l = \Phi(t_k, t_l)P_l^l\Phi^T(t_k, t_l)$$

A better example: 4-41 and 4-42

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(41) Generate 1000 equally spaced observations of one cycle of a sine wave with amplitude 1 and period 10. Add Gaussian random noise with zero mean and variance = 0.25. Set up a sequential estimation procedure to estimate the amplitude of the sine wave as a function of time using the noisy raw data. Model the sine wave as a Gauss-Markov process as given by Eq. (4.9.60),

$$\eta_{i+1} = m_{i+1}\eta_i + \Gamma_{i+1}u_i$$

where

$$u_i = N(0,1)$$

$$m_{i+1} = e^{-\beta(t_{i+1}-t_i)}$$

$$\Gamma_{i+1} = \sqrt{\frac{\sigma^2}{2\beta}(1-m_{i+1}^2)}$$

$$\beta = \frac{1}{\tau}$$

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and τ is the time constant. The sequential algorithm is given by

1. $\bar{\eta}_i = \Phi(t_i, t_{i-1})\hat{\eta}_{i-1} \quad (i = 1, 2 \ldots 1000)$

$\Phi(t_i, t_{i-1}) = m_i = e^{-\beta(t_i - t_{i-1})}$

$\overline{P}_i = \Phi(t_i, t_{i-1}) P_{i-1} \Phi^T(t_i, t_{i-1}) + \Gamma_i Q_{i-1} \Gamma_i^T$

Note that P, Φ, Q, and Γ are scalars

$Y_i = \eta_i$, thus $\widetilde{H}_i = 1$, assume $R_i = 1, Q_i = 1, \bar{\eta}_0 = 0, \overline{P}_0 = 1$

$$K_i = \overline{P}_i \widetilde{H}_i^T (\widetilde{H}_i \overline{P}_i \widetilde{H}_i^T + R_i)^{-1} = \frac{\overline{P}_i}{\overline{P}_i + 1}$$

$\hat{\eta}_i = \bar{\eta}_i + K_i(Y_i - \widetilde{H}_i \bar{\eta}_i) = \bar{\eta}_i + K_i(Y_i - \bar{\eta}_i)$, ($Y_i$ is the observation data)

$P_i = (I - K_i \widetilde{H}_i)\overline{P}_i = K_i$

Next i

Plot your observations, the truth data, and $\hat{\eta}$ versus time. You will need to guess initial values for σ and β.

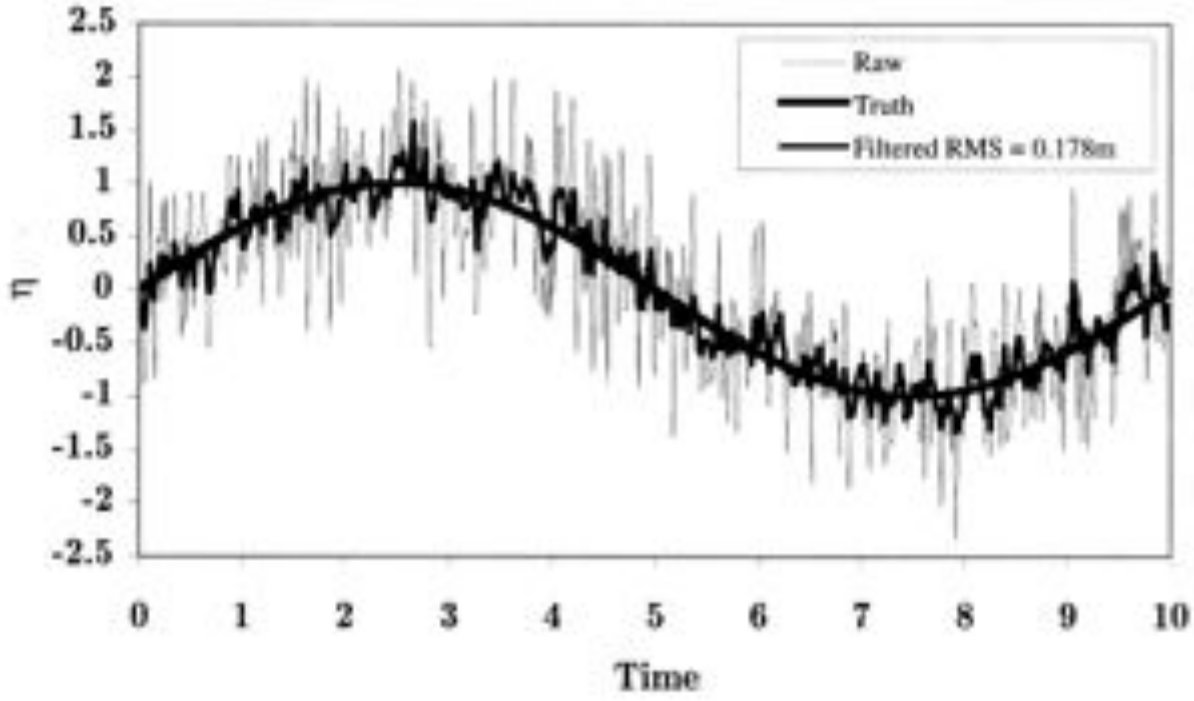

Figure 4.19.1: Process noise/sine wave recovery showing truth, raw data (truth plus noise) and the filtered solution. $\dot{\eta}_0 = 0, \sigma = 2.49, \beta = .045$.

where

$$\mathrm{RMS} = \left\{ \sum_{i=1}^{N} \frac{(T_i - \hat{\eta}_i)^2}{N} \right\}^{1/2}$$

Given (from the filtering algorithm)

$$\hat{\mathbf{x}}_{\ell}^{\ell},\ \hat{\mathbf{x}}_{\ell-1}^{\ell-1},\ P_{\ell}^{\ell-1},\ P_{\ell-1}^{\ell-1},\ \Phi(t_{\ell}, t_{\ell-1});$$

set $k = \ell - 1$

$$S_{\ell-1} = P_{\ell-1}^{\ell-1}\,\Phi^{T}(t_{\ell}, t_{\ell-1})\,(P_{\ell}^{\ell-1})^{-1} \tag{4.15.29}$$

$$\hat{\mathbf{x}}_{\ell-1}^{\ell} = \hat{\mathbf{x}}_{\ell-1}^{\ell-1} + S_{\ell-1}(\hat{\mathbf{x}}_{\ell}^{\ell} - \Phi(t_{\ell}, t_{\ell-1})\,\hat{\mathbf{x}}_{\ell-1}^{\ell-1}).$$

Given (from the filtering algorithm and the previous step of the smoothing algorithm)

$$\hat{\mathbf{x}}_{\ell-2}^{\ell-2},\ P_{\ell-1}^{\ell-2},\ P_{\ell-2}^{\ell-2},\ \hat{\mathbf{x}}_{\ell-1}^{\ell},\ \Phi(t_{\ell-1}, t_{\ell-2});$$

set $k = \ell - 2$, and compute

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$$\hat{\mathbf{x}}_{\ell-2}^{\ell} = \hat{\mathbf{x}}_{\ell-2}^{\ell-2} + S_{\ell-2}\,(\hat{\mathbf{x}}_{\ell-1}^{\ell} - \Phi\,(t_{\ell-1}, t_{\ell-2})\,\hat{\mathbf{x}}_{\ell-2}^{\ell-2})$$

$$\vdots$$

and so on.

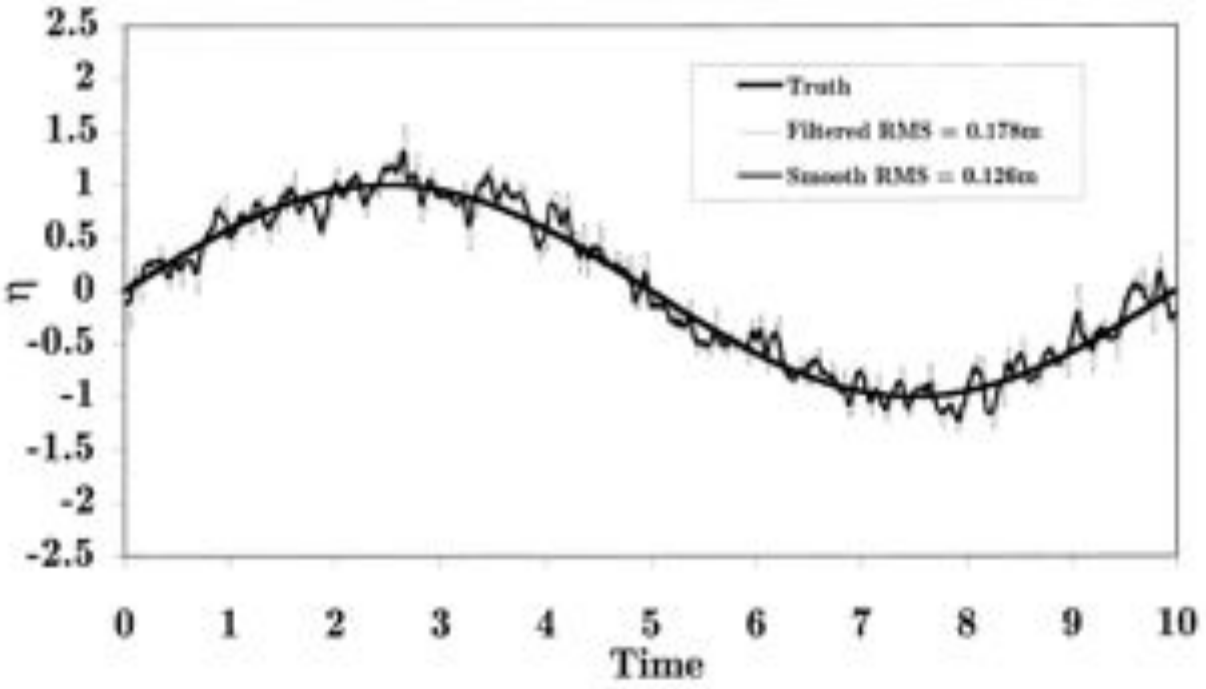

Figure 4.19.2: Process noise/sine wave recovery showing the truth, the filtered, and the smoothed solution. $\bar{r}_{\beta 0} = 0, \sigma = 2.49, \beta = .045$.

Monte Carlo

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- New topic!

- Let's say you want to do a Monte Carlo analysis to determine the costs of having uncertainty in your trajectory.
  - Downstream maneuvers
  - Pointing accuracy
  - Other statistics

- We need a way to take the state estimate's covariance matrix and sample that correctly.

Monte Carlo

Assume we have a state vector $\mathbf{X}$ with associated error covariance $P = E\left[\mathbf{x}\,\mathbf{x}^T\right]$ where $\mathbf{x}$ is a vector of zero mean error realizations of the state vector $\mathbf{X}$. Hence

$$P = E\left[\mathbf{x}\,\mathbf{x}^T\right]$$

- factor P into

$$P = S^T S$$

where S is upper triangular and can be computed via Cholesky decomposition or orthogonal transformations. Note that S is not unique.

- Pre-multiply P by S^{-T} and post-multiply by S^{-1}

$$S^{-T} P S^{-1} = S^{-T} S^T S S^{-1} = I$$

- using

$$P = E\left[\mathbf{x}\mathbf{x}^T\right]$$
$$S^{-T} P S^{-1} = E\left[S^{-T}\mathbf{x}\mathbf{x}^T S^{-1}\right] = I$$

let $\mathbf{e} \equiv S^{-T}\mathbf{x}$

so $E\left[\mathbf{e}\mathbf{e}^T\right] = I$

- Therefore **e** can be realized as an $N(O, I)$ vector of random numbers, and **x** calculated from

$$\mathbf{x} = S^T \mathbf{e}$$

- Therefore **x** is a realization of errors of the vector **X** for which P is the error covariance.

Implementation procedure using Matlab

- Given an *n*-vector **X** and *P*, compute *S*

$$S = \text{chol}(P)$$

- Generate an n-vector of Gaussian random numbers with $N(O, I)$

$$\mathbf{e} = \text{randn}(n, 1)$$

- If desired, an n-vector of Gaussian random number, **b**, with mean *M* and variance σ^2 can be computed from

$$\mathbf{b} = M + \text{sqrt}(\sigma^2) * \text{randn}(n, 1)$$

- Compute a realization of error in **x** from

$$\mathbf{x} = S^T \mathbf{e}$$

- Generate a new realization of **X**

$$\mathbf{X}_{new} = \mathbf{X} + \mathbf{x}$$

- $\mathbf{X}_{new}$ will have P as its error covariance

- Another realization may be computed by generating a new vector of random numbers, **e**.

- Unless you specify the seed, Matlab will generate a different random vector each time $\text{randn}(n,1)$ is used

- We could also use $A = \sqrt{P}$ in place of S

$$P = A\,A$$

$$A^{-1} P\, A^{-1} = I$$

- Let

$$\mathbf{e} = A^{-1}\mathbf{x}\,,\ \mathbf{e}\ \text{is}\ N(O, I)$$

then

$$\mathbf{x} = A\,\mathbf{e}$$

- Note that $A\,\mathbf{e} \neq S^T\mathbf{e}$

- Hence this will be a different realization of $\mathbf{x}$ given the same random vector, $\mathbf{e}$.
- However, it can be shown that

$$S = Q\,A$$

where Q is an orthogonal transformation matrix.

- Therefore,

$$\mathbf{x}_1 = S^T\mathbf{e} = A\,Q^T\mathbf{e}$$
$$\mathbf{x}_2 = A\,\mathbf{e}$$

and $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same Euclidean norm.

Consider Covariance

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- Our final project's state:
  - R, V, mu, J2, CD, S1, S2, S3
- We know we have mismodeled dynamics.
  - SNC, DMC
- In theory we could estimate an nxn gravity field.
  - Adds huge complexity.
  - Adds sensitivity.
  - Filter could diverge.
- Another option: we could *consider* parameters whose values are known to be unknown.
  - We could *consider* the J3 term, knowing something about its variance.
  - Real missions (GRAIL ☺) often consider parameters whose values are not known perfectly, but whose values are not estimated.
    - Earth's mass, planetary positions, station coordinates, etc.

Announcements

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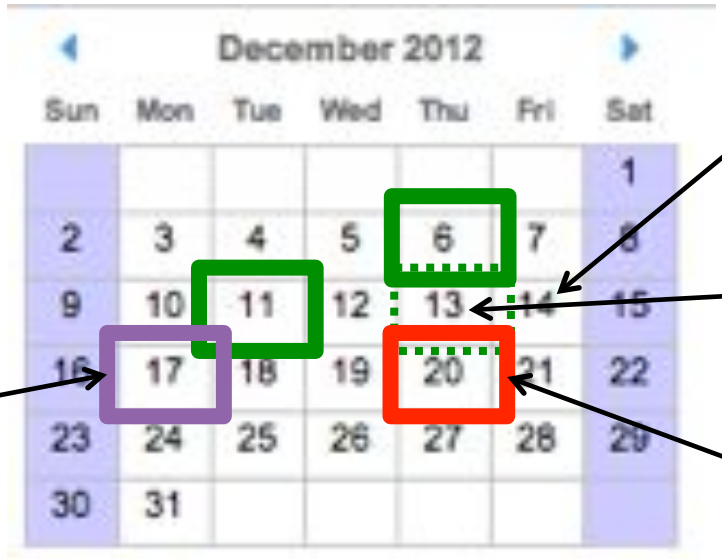

Last Day of Classes

CCAR Conflict

Take-Home Exam Due

Final Project Due
All HW Due