ASEN 5070
Statistical Orbit Determination I
Fall 2012

CCAR
Colorado Center for
Astrodynamics Research

Professor Jeffrey S. Parker
Professor George H. Born

Lecture 22: Householder, Information Filter

Announcements

- Homework 10 AND 11 announced today.

- HW 10 due the Thursday after break.
- HW 11 due the week after.

- Have a good Fall Break!

Quiz 18 Review

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Information

This is a fun quiz (wheee!) that considers Stat OD principles applied to different scenarios.

Question 1 (1 point)

The Cassini spacecraft recently flew past Titan. This isn't new: Cassini has done this nearly a hundred times! Each flyby has different objectives, including science and engineering objectives.

One of the objectives of the recent flyby was to characterize Titan's atmosphere. The nav team had a unique opportunity to contribute to the scientific investigation of Titan! Cassini was to fly through the upper layers of Titan's atmosphere. As you know from our standard equations of motion, passing through the atmosphere will impart a force of drag on the spacecraft. The NAV team collected precise tracking data during the flyby in order to observe this force of drag as it impacted Cassini's trajectory. After obtaining a precise solution for Cassini's trajectory, the NAV team was able to characterize the density profile of the upper atmosphere.

Of course, the science team probably wanted to fly as deep into Titan's atmosphere as possible. How deep could they go? Let's say hypothetically Cassini could not get any closer than 500 km to Titan without overheating due to the friction from the atmosphere (I made this number up!).

Let's say the NAV team is able to predict Cassini's closest approach with a 1-sigma value of 10 km.

Let's say the mission required at least a 99.99% chance that Cassini would survive the flyby. Thus, the NAV team has to design a flyby that has a 0.01% chance of flying closer than 500 km to Titan.

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What should be the aimpoint for the flyby? How close should the nominal, planned trajectory get to Titan at its periapse?

- 500 km
- 521 km
- 533 km
- 561 km

Quiz 18 Review

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Quiz 18 Review

Question 2 (1 point)

If Cassini's Nav team was able to estimate the periapse altitude to 1 km (1 sigma), could the aimpoint be lowered? (ignoring all effects of the flyby on the trajectory, etc.)

- Yes
- No

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- Yes
- No

Quiz 18 Review

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Question 3 (1 point)

The Cassini spacecraft has all sorts of sensors and instruments onboard. Which of the following would help the NAV team estimate the atmospheric density profile of Titan (some of course more than others).

- ☐ Attitude sensors (star trackers, gyros, etc) that measure Cassini's orientation in space, relative to Earth, Titan, Saturn, etc.
- ☐ Thermisters that measure Cassini's temperature at various places around the spacecraft.
- ☐ Accelerometers that measure Cassini's accelerations (typically caused by thruster firings).
- ☐ The wheel speed of the reaction control wheels inside of the spacecraft (let's assume that they are actively compensating for any torques on the spacecraft).

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HW#10

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- Due the Thursday after Fall Break

ASEN 5070 HW # 10

1. Program the Givens square root free algorithm (Eq. 5.4.70).

   Given the problem of HW # 8 i. e.,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \bar{P}_0 = \begin{bmatrix} 1/\delta^2 & 0 \\ 0 & 1/\delta^2 \end{bmatrix}$$

$$R = I,\ \delta = 1\text{x}10^{-2}, 1\text{x}10^{-4}, 1\text{x}10^{-6}, \ldots\ldots 1\text{x}10^{-16}$$

   a. For each value of δ solve for $\hat{X}$ and P_2. Compare your results for P_2 to the exact solution (Eq (4.7.24)) by plotting the trace difference as you did in HW # 8.

   b. For δ = $1\text{x}10^{-2}$, amaze your friends by demonstrating that $\sum_{i=1}^{2} e_i^2$ from your solution** agrees with the results obtained from Eq (5.4.33). See also Eq(5.6.21).

**Note $\sum_{i=1}^{2} e_i^2 = 6.51300624\text{x}10^{-3}$ for this problem

HW#11

- Due the Thursday after HW10

ASEN 5070 **HW #11**

1. Calculate $\Phi(18340,0)$ using the equation $\Phi(t_k,0)=\Phi(t_k,t_n)\Phi(t_n,0)$

2. Report the state deviation after one pass through the sequential filter.

3. Compare the result from question 2 to your batch results.

4. Using the convention Kalman filter covariance measurement update equation, plot the trace of the position covariance on a semilog y scale (at each measurement time).

5. Using the Joseph formulation of the covariance measurement update equation, plot the trace of the position covariance on a semilog y scale (at each measurement time).

6. Plot the position error ellipsoid for your batch state error covariance. (You may use the function plotEllipsoid.m)

7. Work problem 4-41. You can use this data file hw11.dat, where time is in the first column and the observations are in the second column.

Exam 2 Debrief

- I think anyone with questions has spoken with me, but if you have any questions about your grade on Exam 2, see me asap.
- Check the math and make sure you agree that all grading was fair (did we dock enough points!?).

- Quick Break

- Next up: Stuff.

  - Orthogonal Filters
    - Givens review
    - Householder
  - Prediction Residual
  - Future: Process Noise, Smoothing

Orthogonal Filters

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- Reformulate Cost Function using Q

$$J(\mathbf{x}) = \left\| Q W^{\frac{1}{2}} (H\mathbf{x} - \mathbf{y}) \right\|^2$$
$$= \left(H\mathbf{x} - \mathbf{y}\right)^T W^{\frac{1}{2}} Q^T Q W^{\frac{1}{2}} \left(H\mathbf{x} - \mathbf{y}\right)$$

Select Q such that

$$Q W^{\frac{1}{2}} H = \begin{bmatrix} R \\ O \end{bmatrix} \text{ and define } Q W^{\frac{1}{2}} \mathbf{y} \equiv \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix}$$

- Q = Orthogonal Matrix
- Givens, Householder, many methods

Orthogonal Filters

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Select Q such that

$$QW^{\frac{1}{2}}H = \begin{bmatrix} R \\ O \end{bmatrix} \text{ and define } QW^{\frac{1}{2}}\mathbf{y} \equiv \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix} \qquad (5.3.6)$$

where

R is a $n \times n$ upper-triangular matrix of rank n

O is a $(m-n) \times n$ null matrix

$\mathbf{b}$ is a $n \times 1$ column vector

$\mathbf{e}$ is a $(m-n) \times 1$ column vector.

Orthogonal Filters

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- When you do reformulate J…

$$J(\mathbf{x}) = \left\| QW^{\frac{1}{2}}(H\mathbf{x} - \mathbf{y}) \right\|^2$$
$$= \left(H\mathbf{x} - \mathbf{y}\right)^T W^{\frac{1}{2}} Q^T Q W^{\frac{1}{2}} \left(H\mathbf{x} - \mathbf{y}\right)$$

Select Q such that

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- Then we simply solve for x-hat:

$$R\hat{\mathbf{x}} = \mathbf{b}$$

Orthogonal Filters

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- The challenge is building Q to construct

$$QW^{\frac{1}{2}}H = \begin{bmatrix} R \\ O \end{bmatrix}$$

given Q, we can build b easily enough

$$QW^{\frac{1}{2}}\mathbf{y} \equiv \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix}$$

$$R\hat{\mathbf{x}} = \mathbf{b}$$

Givens

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The procedure described in the previous section is direct and for implementation requires only that a convenient procedure for computing Q be obtained. One such procedure can be developed based on the Givens plane rotation (Givens, 1958). Let $\mathbf{x}$ be a 2×1 vector having components $\mathbf{x}^T = [x_1 \;\; x_2]$ and let G be a 2×2 orthogonal matrix associated with the plane rotation through the angle θ. Then select G such that

$$G\mathbf{x} = \mathbf{x}' = \begin{pmatrix} x_1' \\ 0 \end{pmatrix}. \tag{5.4.1}$$

Givens

To this end, consider the transformation

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5.4.2)$$

or

$$\begin{aligned} x_1' &= \quad \cos\theta x_1 + \sin\theta x_2 \\ x_2' &= -\sin\theta x_1 + \cos\theta x_2. \end{aligned} \quad (5.4.3)$$

Equations (5.4.3) represent a system of two equations in three unknowns; that is, x_1', x_2', and θ. The Givens rotation is defined by selecting the rotation θ such that $x_2' = 0$. That is, let

$$x_1' = \quad \cos\theta x_1 + \sin\theta x_2 \quad (5.4.4)$$

$$0 = -\sin\theta x_1 + \cos\theta x_2. \quad (5.4.5)$$

Givens

Consider the application of the transformation

$$G(\theta) = \begin{bmatrix} \cos\theta \; \sin\theta \\ -\sin\theta \; \cos\theta \end{bmatrix} \tag{5.4.8}$$

to two general row vectors $\mathbf{h}_i$ and $\mathbf{h}_k$; for example,

$$G \begin{bmatrix} h_{ii} \; h_{ii+1} \; \ldots \, h_{in} \\ h_{ki} \; h_{ki+1} \ldots \, h_{kn} \end{bmatrix} = \begin{bmatrix} h'_{ii} \; h'_{ii+1} \; \ldots \, h'_{in} \\ 0 \quad h'_{ki+1} \ldots \, h'_{kn} \end{bmatrix}. \tag{5.4.9}$$

Givens

$$
\begin{bmatrix}
1 & & & & & & & & \\
& 1 & & & & & & & \\
& & C^{4,3} & S^{4,3} & & & & & \\
& & -S^{4,3} & C^{4,3} & & & & & \\
& & & & 1 & & & & \\
& & & & & 1 & & & \\
& & & & & & 1 & & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\
0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\
0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\
0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\
0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\
0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & h_{m3} & \cdots & h_{mn} & y_m
\end{bmatrix}
=
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\
0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\
0 & 0 & h'_{33} & \cdots & h'_{3n} & y'_3 \\
0 & 0 & 0 & \cdots & h'_{4n} & y'_4 \\
0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\
0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & h_{m3} & \cdots & h_{mn} & y_m
\end{bmatrix}
$$

Givens

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$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & C^{4,3} & S^{4,3} & & & & & \\ & & -S^{4,3} & C^{4,3} & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\ 0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\ 0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\ 0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\ 0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\ 0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & h_{m3} & \cdots & h_{mn} & y_m \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\ 0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\ 0 & 0 & h'_{33} & \cdots & h'_{3n} & y'_3 \\ 0 & 0 & 0 & \cdots & h'_{4n} & y'_4 \\ 0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\ 0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & h_{m3} & \cdots & h_{mn} & y_m \end{bmatrix}$$

$$QW^{\frac{1}{2}}H = \begin{bmatrix} R \\ O \end{bmatrix} \text{ and define } QW^{\frac{1}{2}}\mathbf{y} \equiv \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix} \qquad R\hat{\mathbf{x}} = \mathbf{b}$$

Givens

- As mentioned last week, Givens is essentially a “smart” row-reduction technique.
- Normal row-reduction does not require that the operating matrix be orthogonal.
- Givens assures that it is.

Accuracy Comparison for Batch and Givens – Finite Precision Computer

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Consider

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}$$

Notice that a vector of Observations is not needed. Why?

Machine precision is such that

$$1 \pm \epsilon^2 = 1$$

The normal matrix is given by

$$H^T H = \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon+\varepsilon^2 \end{bmatrix}; \quad \text{exact solution}$$

our computer will drop the ε^2 and

$$H^T H = \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon \end{bmatrix}$$

To order ε $\left|H^T H\right| = 0$, hence it is singular

Accuracy Comparison for Batch and Givens – Finite Precision Computer

Consequently, the Batch Processor will fail to yield a solution. Note that this Illustrates the problem with forming H^TH, i.e. numerical problems are amplified.

The Cholesky decomposition yields:

$$R = \begin{bmatrix} \sqrt{3} & \dfrac{3-\varepsilon}{\sqrt{3}} \\ 0 & 0 \end{bmatrix}$$

R is singular and will not yield a solution for $\hat{x}$.

Accuracy Comparison for Batch and Givens – Finite Precision Computer

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Use the Givens transformation to determine *R*

1st zero element (2,1) of *H*

$$\begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}$$

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$$x_1 = 1 \qquad x_2 = 1$$

$$S_\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{\sqrt{2}} \qquad C_\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{\sqrt{2}}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}_{3\times 2} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix}$$

Note that the magnitude of the columns of [H y] are unchanged.

Accuracy Comparison for Batch and Givens – Finite Precision Computer

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$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}_{3\times 2} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix}$$

Next zero element (3,1)

$$x_1 = \frac{2}{\sqrt{2}}, \quad x_2 = 1$$

$$S_\theta = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}, \quad C_\theta = \frac{2/\sqrt{2}}{\sqrt{3}} = \frac{2}{\sqrt{6}}$$

$$\begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix} = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & 0 \\ 0 & -\sqrt{2}\varepsilon/\sqrt{3} \end{bmatrix}$$

Accuracy Comparison for Batch and Givens – Finite Precision Computer

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$$\begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix} = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & 0 \\ 0 & -\sqrt{2}\varepsilon/\sqrt{3} \end{bmatrix}$$

Next zero element (3,2) $x_1 = 0\,,\; x_2 = -\dfrac{\sqrt{2}}{\sqrt{3}}\varepsilon \rightarrow \quad S_\theta = -1,\, C_\theta = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & 0 \\ 0 & -\sqrt{2}\varepsilon/\sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & \sqrt{2}\varepsilon/\sqrt{3} \\ 0 & 0 \end{bmatrix}$$

Accuracy Comparison for Batch and Givens – Finite Precision Computer

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The Givens transformations yield

$$R = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix}$$

Which will yield a valid solution for $\hat{x}$

In fact

$$R^T R = \begin{bmatrix} \sqrt{3} & 0 \\ (3-\varepsilon)/\sqrt{3} & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix} \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon+\varepsilon^2 \end{bmatrix}$$

Accuracy Comparison for Batch and Givens – Finite Precision Computer

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Which is the exact solution result for $H^T H$. Hence, for this example the orthogonal transformations would yield the correct solution. However, the estimation error covariance matrix would be incorrect because our computer would drop the ε^2 term.

Givens

$$R^T R = \begin{bmatrix} \sqrt{3} & 0 \\ (3-\varepsilon)/\sqrt{3} & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix} \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon+\varepsilon^2 \end{bmatrix}$$

Perfect

$$H^T H = \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon+\varepsilon^2 \end{bmatrix}$$

Other Orthogonal Transformations

- Givens used rotations to null values until R became upper-triangular
- Householder uses reflections to accomplish the same goal

Householder

- Rather than applying a rotation that effects two rows and nulls one element, apply an orthogonal reflection and null an entire column in one operation!

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\ 0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\ 0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\ 0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\ 0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\ 0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & h_{m3} & \cdots & h_{mn} & y_m \end{bmatrix} \qquad \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\ 0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\ 0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\ 0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\ 0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\ 0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & h_{m3} & \cdots & h_{mn} & y_m \end{bmatrix}$$

Householder

- Effect of the reflection.
  - Which elements of the matrix will be impacted?

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\ 0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\ 0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\ 0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\ 0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\ 0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & h_{m3} & \cdots & h_{mn} & y_m \end{bmatrix}$$

Householder

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What do you expect h_{33} to be equal to?

Householder Example

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5.6.7 THE HOUSEHOLDER TRANSFORMATION

The matrix we wish to transform is given by Eq. (5.6.25). In terms of $\overline{R}$ and $\overline{\mathbf{b}}$ the matrix is given by

$$\begin{bmatrix} \overline{R} & \overline{\mathbf{b}} \\ H & \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.1 & 0.2 \\ 1 & -2 & -1.1 \\ 2 & -1 & 1.2 \\ 1 & 1 & 1.8 \end{bmatrix}. \tag{5.6.29}$$

Householder Example

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The Householder transformation algorithm given by Eq. (5.5.29) nulls the elements of each column below the main diagonal. The first transformation yields

$$\begin{bmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.1 & 0.2 \\ 1 & -2 & -1.1 \\ 2 & -1 & 1.2 \\ 1 & 1 & 1.8 \end{bmatrix} \longrightarrow \begin{bmatrix} -2.4515 & 1.2237 & -1.2727 \\ 0 & 0.1 & 0.2 \\ 0 & -1.5204 & -1.6772 \\ 0 & -0.0408 & 0.0457 \\ 0 & 1.4796 & 1.2228 \end{bmatrix}$$

Householder Example

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The second transformation results in

$$\begin{bmatrix} -2.4515 & 1.2237 & -1.2727 \\ 0 & 0.1 & 0.2 \\ 0 & -1.5204 & -1.6772 \\ 0 & -0.0408 & 0.0457 \\ 0 & 1.4796 & 1.2228 \end{bmatrix} \longrightarrow \begin{bmatrix} -2.4515 & 1.2237 & -1.2727 \\ 0 & -2.1243 & -2.0607 \\ 0 & 0 & -0.1319 \\ 0 & 0 & 0.0871 \\ 0 & 0 & -0.2810 \end{bmatrix}$$

Don't forget the goal: $R\hat{x} = b$

Householder

Several points should be noted:

1. The Householder values of R and b are identical to the Givens results. Hence, the solution for $\hat{\mathbf{x}}$ and P will be identical.

2. Although the individual values of e_i differ, both algorithms yield identical values of $e^2 = 0.1039$. This also agrees with the Cholesky result.

3. The Euclidean norm of each column is preserved by an orthogonal transformation.

4. The square root free Givens algorithm as derived here operates on the matrix row by row, whereas the Householder algorithm transforms column by column. The Givens algorithm can be modified to operate column by column (see Section 5.4.2).

5. The orthogonal transformations do not require the formation of $H^T H$. Hence, they will generally be more accurate than Cholesky decomposition.

Status

- Talked about numerical issues.

- If a matrix is poorly conditioned (and P will be, especially for the final project):
  - Square Root Formulations
    - Cholesky
    - Potter
  - Square Root via Orthogonal Transformations
    - Givens
    - Householder

- These help the P matrix retain positive definiteness.

Status

- Also spoke about filter saturation and divergence.

- One way to avoid divergence is to edit the data.
  - Very important if you expect any *bad* data points.
  - “Bad” = waaaay out of expected statistics.

The Prediction Residual

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4.7.4 THE PREDICTION RESIDUAL

It is of interest to examine the variance of the predicted residuals, which are sometimes referred to as the *innovation*, or new information, which comes from each measurement. The *predicted residual*, or innovation, is the observation residual based on the *a priori* or predicted state, $\bar{\mathbf{x}}_k$, at the observation time, t_k, and is defined as

$$\beta_k = \mathbf{y}_k - \widetilde{H}_k \bar{\mathbf{x}}_k. \tag{4.7.33}$$

As noted previously,

$$\bar{\mathbf{x}}_k = \mathbf{x}_k + \eta_k$$
$$\mathbf{y}_k = \widetilde{H}_k \mathbf{x}_k + \epsilon_k$$

The Prediction Residual

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4.7.4 The Prediction Residual

It is of interest to examine the variance of the predicted residuals, which are sometimes referred to as the *innovation*, or new information, which comes from each measurement. The *predicted residual*, or innovation, is the observation residual based on the *a priori* or predicted state , $\bar{\mathbf{x}}_k$, at the observation time, t_k, and is defined as

$$\beta_k = \mathbf{y}_k - \widetilde{H}_k \bar{\mathbf{x}}_k. \tag{4.7.33}$$

As noted previously,

$$\begin{aligned} \bar{\mathbf{x}}_k &= \mathbf{x}_k + \eta_k \\ \mathbf{y}_k &= \widetilde{H}_k \mathbf{x}_k + \epsilon_k \end{aligned}$$

$$\begin{aligned} \beta_k &= \tilde{H}_k \mathbf{x}_k + \epsilon_k - \tilde{H}_k \bar{\mathbf{x}}_k \\ &= \tilde{H}(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \epsilon_k \end{aligned}$$

The Prediction Residual

where $\mathbf{x}_k$ is the true value of the state deviation vector and η_k is the error in $\bar{\mathbf{x}}_k$. Also,

$$E[\eta_k] = 0, \; E[\eta_k \eta_k^T] = \overline{P}_k$$

and

$$\begin{aligned} E[\epsilon_k] &= 0, \; E[\epsilon_k \epsilon_k^T] = R_k \\ E[\epsilon_k \eta_k^T] &= 0. \end{aligned}$$

From these conditions it follows that β_k has mean

$$\begin{aligned} E[\beta_k] \equiv \overline{\beta}_k &= E(\widetilde{H}_k \mathbf{x}_k + \epsilon_k - \widetilde{H}_k \bar{\mathbf{x}}_k) \\ &= E(\epsilon_k - \widetilde{H} \eta_k) = 0 \end{aligned}$$

The Prediction Residual

and variance-covariance

$$
\begin{aligned}
P_{\beta_k} &= E[(\beta_k - \overline{\beta}_k)(\beta_k - \overline{\beta}_k)^T] = E[\beta_k \beta_k^T] \\
&= E[(\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k)(\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k)^T] \\
&= E[(\epsilon_k - \widetilde{H}_k \eta_k)(\epsilon_k - \widetilde{H}_k \eta_k)^T] \\
P_{\beta_k} &= R_k + \widetilde{H}_k \overline{P}_k \widetilde{H}_k^T .
\end{aligned} \tag{4.7.34}
$$

Hence, for a large prediction residual variance-covariance, the Kalman gain

$$K_k = \overline{P}_k \widetilde{H}_k^T P_{\beta_k}^{-1} \tag{4.7.35}$$

will be small, and the observation will have little influence on the estimate of the state. Also, large values of the prediction residual relative to the prediction residual standard deviation may be an indication of bad tracking data and hence may be used to edit data from the solution.

This would be especially important in the case of the EKF

The End

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- Homework 10 AND 11 announced today.
- HW 10 due the Thursday after break.
- HW 11 due the week after.
- Have a good Fall Break!

5.4.1 *A PRIORI* INFORMATION AND INITIALIZATION

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The formulation given earlier does not specifically address the question of *a priori* information. Assume *a priori* information, $\overline{\mathrm{x}}$ and $\overline{P}$, are available. The procedure is initialized by writing the *a priori* information in the form of a data equation; that is, in the form of $\mathrm{y} = H\mathrm{x} + \epsilon$. This is accomplished by writing

$$\overline{\mathrm{x}} = \mathrm{x} + \eta \tag{5.4.15}$$

where x is the true value and η is the error in $\overline{\mathrm{x}}$. We assume that

$$E[\eta] = 0,\ E[\eta\eta^T] = \overline{P}. \tag{5.4.16}$$

Compute $\overline{S}$, the upper triangular square root of $\overline{P}$,

$$\overline{P} = \overline{S}\,\overline{S}^T. \tag{5.4.17}$$

5.4.1 A PRIORI INFORMATION AND INITIALIZATION

If $\overline{P}$ is not diagonal, the Cholesky decomposition may be used to accomplish this. Next compute $\overline{R}$, the square root of the *a priori* information matrix, $\overline{\Lambda}$,

$$\overline{\Lambda} = \overline{P}^{-1} = \overline{S}^{-T}\overline{S}^{-1} = \overline{R}^T\overline{R} \tag{5.4.18}$$

hence,

$$\overline{R} = \overline{S}^{-1}. \tag{5.4.19}$$

Multiplying Eq. (5.4.15) by $\overline{R}$ yields

$$\overline{R}\overline{\mathrm{x}} = \overline{R}\mathrm{x} + \overline{R}\eta. \tag{5.4.20}$$

Define

$$\overline{b} = \overline{R}\overline{\mathrm{x}}, \; \overline{\eta} = \overline{R}\eta, \tag{5.4.21}$$

5.4.1 A Priori Information and Initialization

then

$$\bar{b} = \bar{R}x + \bar{\eta} \tag{5.4.22}$$

where $\bar{\eta} \sim (O, I)$. Note that Eq. (5.4.22) expresses the *a priori* information in the form of the data equation, $y = Hx + \epsilon$. Hence, the equations we wish to solve for $\hat{x}$, using orthogonal transformations, are

$$\begin{aligned} \bar{b} &= \bar{R}x + \bar{\eta} \\ y &= Hx + \epsilon \end{aligned} \tag{5.4.23}$$

where x is an n vector and y is an m vector.

5.4.1 A PRIORI INFORMATION AND INITIALIZATION

The least squares solution for x in Eq. (5.4.23) is found by minimizing the performance index (we assume that ϵ has been prewhitened so that $\epsilon \sim (O, I)$; if not, replace ϵ with $W^{1/2}\epsilon$ in J)

$$\begin{aligned} J &= \|\overline{\eta}\|^2 + \|\epsilon\|^2 \\ &= \|\overline{R}\mathbf{x} - \overline{\mathbf{b}}\|^2 + \|H\mathbf{x} - \mathbf{y}\|^2 \\ &= \left\| \begin{bmatrix} \overline{R} \\ H \end{bmatrix} \mathbf{x} - \begin{bmatrix} \overline{\mathbf{b}} \\ \mathbf{y} \end{bmatrix} \right\|^2 . \end{aligned} \qquad (5.4.24)$$

5.4.1 *A Priori* Information and Initialization

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After multiplying by an orthogonal transformation, Q, Eq. (5.4.24) may be written as

$$J = \left\{ \begin{bmatrix} \overline{R} \\ H \end{bmatrix} \mathbf{x} - \begin{bmatrix} \overline{\mathbf{b}} \\ \mathbf{y} \end{bmatrix} \right\}^T Q^T Q \left\{ \begin{bmatrix} \overline{R} \\ H \end{bmatrix} \mathbf{x} - \begin{bmatrix} \overline{\mathbf{b}} \\ \mathbf{y} \end{bmatrix} \right\}. \tag{5.4.25}$$

Choose Q so that

$$Q \begin{bmatrix} \overline{R} \\ H \end{bmatrix} = \begin{bmatrix} R \\ O \end{bmatrix}, \text{ and define } Q \begin{bmatrix} \overline{\mathbf{b}} \\ \mathbf{y} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix} \tag{5.4.26}$$

where R is upper triangular. Eq. (5.4.24) can now be written as

$$J = \left\| \begin{bmatrix} R \\ O \end{bmatrix} \mathbf{x} - \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix} \right\|^2 \tag{5.4.27}$$

or

$$J = \| R\mathbf{x} - \mathbf{b} \|^2 + \| \mathbf{e} \|^2 \tag{5.4.28}$$

5.4.1 A PRIORI INFORMATION AND INITIALIZATION

as noted before. The minimum value of J is found by choosing $\hat{\mathbf{x}}$ so that

$$R\hat{\mathbf{x}} - \mathbf{b} = 0. \tag{5.4.29}$$

The vector $\hat{\mathbf{x}}$ is obtained by the backward substitution described by Eq. (5.2.8), where z and r are replaced by b and R, respectively. Observe that $\hat{\mathbf{x}}$ usually would be determined after processing all observations. However, intermediate values of $\hat{\mathbf{x}}$ could be determined at any point in the process.

The minimum value of J is given by substituting $\hat{\mathbf{x}}$ into Eq. (5.4.24):

$$J = \|\mathbf{e}\|^2 = \sum_{i=1}^{m} e_i^2 = \|\overline{R}\hat{\mathbf{x}} - \overline{\mathbf{b}}\|^2 + \sum_{i=1}^{m} (H_i\hat{\mathbf{x}} - y_i)^2 . \tag{5.4.30}$$

Note that the first term on the right-hand side of Eq. (5.4.30) corresponds to the norm of the error in the *a priori* value for $\mathbf{x}$ multiplied by the square root of the inverse of the *a priori* covariance matrix,

$$\|\overline{R}\hat{\mathbf{x}} - \overline{\mathbf{b}}\|^2 = \|\overline{R}(\hat{\mathbf{x}} - \overline{\mathbf{x}})\|^2 \tag{5.4.31}$$

which also can be expressed as

$$\|\overline{R}(\hat{\mathrm{x}} - \overline{\mathrm{x}})\|^2 = (\hat{\mathrm{x}} - \overline{\mathrm{x}})^T \overline{P}^{-1} (\hat{\mathrm{x}} - \overline{\mathrm{x}}). \tag{5.4.32}$$

From Eqs. (5.4.30) and (5.4.31) it is seen that

$$\sum_{i=1}^{m} e_i^2 = \|\overline{R}(\hat{\mathrm{x}} - \overline{\mathrm{x}})\|^2 + \sum_{i=1}^{m} (H_i \hat{\mathrm{x}} - y_i)^2$$

$$\begin{aligned} \sum_{i=1}^{m} e_i^2 &= \|\overline{R}\hat{\eta}\|^2 + \sum_{i=1}^{m} \hat{\epsilon}_i^2 \\ &= \hat{\eta}^T \overline{R}^T \overline{R} \hat{\eta} + \sum_{i=1}^{m} \hat{\epsilon}_i^2, \end{aligned} \tag{5.4.33}$$

where

$$\hat{\eta} = \hat{\mathrm{x}} - \overline{\mathrm{x}}, \quad \hat{\epsilon}_i = y_i - H_i \hat{\mathrm{x}}. \tag{5.4.34}$$

5.4.1 A PRIORI INFORMATION AND INITIALIZATION

Consequently, $\epsilon_i \neq \hat{\epsilon}_i$; that is, the elements of the error vector $[e]_{m\times 1}$ contain a contribution from errors in the *a priori* information as well as the observation residuals. The RMS of the observation residuals, $\hat{\epsilon}_i$, is given by

$$\mathrm{RMS} = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\hat{\epsilon}_i^2} \tag{5.4.35}$$

and from Eq. (5.4.33)

$$\sum_{i=1}^{m}\hat{\epsilon}_i^2 = \sum_{i=1}^{m}e_i^2 - \hat{\eta}^T\overline{R}^T\overline{R}\hat{\eta}. \tag{5.4.36}$$

5.4.1 *A Priori* Information and Initialization

If the procedure is initialized with n observations in place of *a priori* information, $\overline{\mathbf{x}}$ and $\overline{P}$, Eq. (5.4.33) becomes

$$\sum_{i=1}^{m-n} \epsilon_i^2 = \sum_{i=1}^{m} \hat{\epsilon}_i^2 \tag{5.4.37}$$

and again $\epsilon_i \neq \hat{\epsilon}_i$ because the first n observations serve the same function as *a priori* values of $\overline{\mathbf{x}}$ and $\overline{P}$. Here we have assumed that the weighting matrix is the identity matrix. If not, $(H_i\hat{\mathbf{x}} - y_i)^2$ in Eq. (5.4.30) and subsequent equations should be replaced by $W_i(H_i\hat{\mathbf{x}} - y_i)^2$.

5.4.1 A PRIORI INFORMATION AND INITIALIZATION

In summary, given *a priori* information $\overline{R}$ and $\overline{b}$ and observations

$$y_i = H_i \mathrm{x} + \epsilon_i, \quad i = 1, \ldots, m,$$

the matrix we wish to reduce to upper triangular form is

$$\begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{1} \end{array}$$

$$\left[\begin{array}{cc} \overline{R} & \overline{b} \\ H_1 & y_1 \\ H_2 & y_2 \\ \vdots & \vdots \\ H_m & y_m \end{array}\right] \begin{array}{l} \}\, n \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} m \end{array} \tag{5.4.38}$$

where $\overline{R}$ is upper triangular.

Addition of New Data

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After application of this algorithm, the $(n+m) \times (n+1)$ matrix will appear as

$$
Q \begin{bmatrix} \overline{R} & \vdots & \overline{\mathrm{b}} \\ H & \vdots & \mathrm{y} \end{bmatrix} = \begin{array}{c} \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{1} \end{array} \\ \begin{bmatrix} R & \mathrm{b} \\ O & \mathrm{e} \end{bmatrix} \end{array} \begin{array}{l} \}n \\ \}m \end{array} \tag{5.4.41}
$$

which is the required form for solution of the least squares estimation problem as

Addition of New Data

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Once the array has been reduced to the form given by Eq. (5.4.41), subsequent observations can be included by considering the following array:

$$\begin{bmatrix} R & \mathbf{b} \\ H_{m+1} & y_{m+1} \\ 0 & e^2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} & b_1 \\ 0 & R_{22} & \cdots & R_{2n} & b_2 \\ 0 & 0 & \cdots & R_{3n} & b_3 \\ \vdots & & & & \\ 0 & 0 & \cdots & R_{nn} & b_n \\ H_{m+1,1} & H_{m+1,2} & \cdots & H_{m+1,n} & y_{m+1} \\ 0 & 0 & \cdots & 0 & e^2 \end{bmatrix} \tag{5.4.42}$$

where

$$e^2 = \sum_{k=1}^{m} e_k^2 = \text{Sum}.$$

Addition of New Data

Then by application of a Givens rotation to rows 1 and $n+1$, $H_{m+1,1}$ can be nulled. Successive applications moving down the main diagonal can be used to null the remaining $n-1$ elements of the $n+1^{st}$ row and reduce the array to upper triangular form:

$$\begin{bmatrix} R' & \mathbf{b}' \\ 0 & e_{m+1} \\ 0 & e^2 \end{bmatrix}.$$

Next e^2 is replaced by $e^2 + e_{m+1}^2$ and the procedure is repeated with the next observation, and so on. It is also obvious that a group of m' observations could be included by replacing the array (H_{m+1}, y_{m+1}) with an array in which H_{m+1} has dimension $(m' \times n)$ and $\mathbf{y}_{m+1}$ has dimension m'. The Givens rotation would be used as before to reduce the augmented array to upper triangular form.

Givens Operational Procedure

Assume we have a set of observations we wish to process with apriori information via Givens Orthogonal transformations.

Given observation y_i, $i = 1 \ldots m$ where y_i is a scalar (i.e. we plan to process the observations one at a time).

$$y_i = H_i x_0 + \varepsilon_i \qquad i = 1 \ldots m$$

Assume: $\varepsilon_i \approx (0, 1)$

$$[x]_{3\times1}, \quad [y_i]_{1\times1}$$

then $[H_i]_{1\times3}, \ [\varepsilon_i]_{3\times1}$

apriori $[\bar{x}_0]_{3\times1}, \ [\bar{P}_0]_{3\times3}$

Givens Operational Procedure

Procedure

1. Write the *a priori* in the form of a data equation i.e.

$$\bar{x} = x + \eta \qquad \text{i.e. looks like } \; y = Hx + \varepsilon$$

where

$$E[\eta] = 0, \quad E\left[\eta\eta^T\right] = \bar{P}_0$$

Compute the square root of the information matrix

$$\bar{\Lambda} = \bar{P}_0^{-1} = \bar{R}^T R$$

Multiply $\bar{x}$ by $\bar{R}$

$$\bar{R}\bar{x} = \bar{R}x + \bar{R}\eta$$

Define

$$\bar{R}\bar{x} \equiv \bar{b} \quad , \quad \bar{R}\eta \equiv \bar{\eta}$$

Givens Operational Procedure

Now $\bar{\eta} \approx (0, 1)$ and

$$\bar{b} = \bar{R}x + \bar{\eta}$$

The system now is

$$[y]_{m\times 1} = [H]_{m\times n}[x]_{n\times 1} + [\varepsilon]_{m\times 1}$$

$$[\bar{b}]_{n\times 1} = [\bar{R}]_{n\times n}[x]_{n\times 1} + [\bar{\eta}]_{n\times 1}$$

and $\hat{x}$ is chosen to minimize the performance index

$$J = \|\bar{\eta}\|^2 + \|\varepsilon\|^2$$

which results in Eqs (5.4.28) and (5.4.29)

Givens Operational Procedure

Given $\bar{P}_0$, $\bar{x}_0$ and $[y]_{m\times 1}$

1. compute $\bar{R}$ from (Use Cholesky if $\bar{P}_0$ is not diagonal)

$$\bar{R}^T\bar{R} = \bar{P}_0^{-1}$$ where $\bar{R}$ is upper triangular

compute $\bar{b} = \bar{R}\bar{x}_0$

2. Process the 1st of m observations by applying a series of orthogonal transformations. Q_i , S is a storage area for summing squares of residuals.

$$[Q_i]_{4\times 4}\begin{bmatrix} \bar{R} & \bar{b} \\ H_1 & y_1 \end{bmatrix}_{4\times 4}$$

$$S$$

we assumed that $[X]_{3\times 1}$, then $[\bar{R}]_{3\times 3}$ upper triangular, $[\bar{b}]_{3\times 1}$, S is a scalar storage area

set S=0, $[H_1]_{1\times 3} = \tilde{H}_1\Phi(t_1,t_0)$

Givens Operational Procedure

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$$\underset{i=1,3}{Q_i}\begin{array}{r}\begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} & \bar{R}_{13} & \bar{b}_1 \\ 0 & \bar{R}_{22} & \bar{R}_{23} & \bar{b}_2 \\ 0 & 0 & \bar{R}_{33} & \bar{b}_3 \\ H_{11} & H_{12} & H_{13} & y_1 \end{bmatrix} \\ S\quad\end{array} = \begin{array}{r}\begin{bmatrix} R_{11} & R_{12} & R_{13} & b_1 \\ 0 & R_{22} & R_{23} & b_2 \\ 0 & 0 & R_{33} & b_3 \\ 0 & 0 & 0 & e_1 \end{bmatrix} \\ S\quad\end{array}$$

set $S = e_1^2$

3. Process 2nd obs

$$\underset{i=1,2,3}{Q_i}\begin{array}{r}\begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} & \bar{R}_{13} & \bar{b}_1 \\ 0 & \bar{R}_{22} & \bar{R}_{23} & \bar{b}_2 \\ 0 & 0 & \bar{R}_{33} & \bar{b}_3 \\ H_{21} & H_{22} & H_{23} & y_2 \end{bmatrix} \\ S\quad\end{array} = \begin{array}{r}\begin{bmatrix} R_{11} & R_{12} & R_{13} & b_1 \\ 0 & R_{22} & R_{23} & b_2 \\ 0 & 0 & R_{33} & b_3 \\ 0 & 0 & 0 & e_2 \end{bmatrix} \\ S\quad\end{array}$$

$S = S + e_2^2$

Givens Operational Procedure

4. Using this procedure, process all *m* observations

The three Q matrices for transforming each observation are given by:

$$Q_1 = \begin{bmatrix} C_\theta & 0 & 0 & S_\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -S_\theta & 0 & 0 & C_\theta \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_\theta & 0 & S_\theta \\ 0 & 0 & 1 & 0 \\ 0 & -S_\theta & 0 & C_\theta \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C_\theta & S_\theta \\ 0 & 0 & -S_\theta & C_\theta \end{bmatrix}$$

where

Q_1 zeros $[\ ]_{4,1}$ element

Q_2 zeros $[\ ]_{4,2}$ element

Q_3 zeros $[\ ]_{4,3}$ element

Givens Operational Procedure

Also, for the 1st observation, y_1, the values of θ for the three matrices Q_1 , Q_2 , Q_3 are:

$$\theta_1 = a\tan 2\left(H_{11}, \bar{R}_{11}\right) \quad \theta_2 = a\tan 2\left(H'_{12}, \bar{R}_{22}\right) \quad \theta_3 = a\tan 2\left(H''_{13}, \bar{R}_{33}\right)$$

where H'_{12} and H''_{13} represent the transformed elements of *H*. To process Subsequent observations replace the values of H_{ij} with those corresponding to the observation.

5. After processing *m* observations we have

$$\underset{S}{\begin{bmatrix} R & b \\ \phi & e_m \end{bmatrix}}$$

let $S = S + e_m^2$

Givens Operational Procedure

Then

$$R\hat{x} = b$$

$\hat{x}$ is obtained from a backward substitution (see Eq 5.2.8)

The scalar *S* is given by

$$S = \sum_{i=1}^{m} \left(y_i - H_i \hat{x} \right)^2 + \left(\bar{x} - \hat{x} \right)^T \bar{P}_0^{-1} \left(x - \hat{x} \right)$$

The RMS of observation residuals is generally defined as

$$RMS = \left[\frac{\sum_{i=1}^{m} \left(y_i - H_i \hat{x} \right)^2}{m} \right]^{1/2}$$

Givens Operational Procedure

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Hence,

$$RMS = \left[\frac{S - (\bar{x} - \hat{x})^T \bar{P}_0^{-1} (x - \hat{x})}{m} \right]^{1/2}$$

also

$$P = \left(R^T R\right)^{-1} = R^{-1} R^{-T} = S S^T$$

The elements of *S* are obtained from $SR = I$ and are given by Eq (5.2.9)
Note that R is independent of *y* , i.e. $P = \left(\sum H^T H + \bar{P}_0^{-1}\right)^{-1} = \left(R^T R\right)^{-1}$
If there are different types of observation, e.g. range and range rate, keep separate values of *S* for each data type so that an RMS can be computed for each data type. Note that there will be some contribution from $\bar{P}_0$ in each RMS but there is no way to separate this effect. Also, notice that if the weighting matrix is not the identity matrix, then

$$S = \sum_{i=1}^{m} W_i \left(y_i - H_i \hat{x}_i\right)^2 + (\bar{x} - \hat{x})^T \bar{P}_0^{-1} (x - \hat{x})$$

Givens Algorithm

For purposes of the computational algorithm we will write Eq. (5.4.38) as

$$\begin{array}{cc} \overbrace{\qquad}^{n} & \overbrace{\quad}^{1} \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \bar{R} & \bar{\mathbf{b}} \\ H & \mathbf{y} \end{array}\right]} \end{array} \begin{array}{l} \}n \\ \}m \end{array} = \begin{array}{c} \overbrace{\qquad}^{n+1} \\ \left[\begin{array}{c} \tilde{R} \\ \tilde{H} \end{array}\right] \end{array} \begin{array}{l} \}n \\ \}m. \end{array} \qquad (5.4.39)$$

Lowercase r and h in the following algorithm represent the elements of $\tilde{R}$ and $\tilde{H}$, respectively, in Eq. (5.4.39).

The algorithm using the Givens rotation for reducing the $(m+n) \times (n+1)$ matrix of Eq. (5.4.39) to upper triangular form can be expressed as follows:

Givens Algorithm

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Sum = 0.

1. Do $k = 1, ..., m$

2. Do $i = 1, \ldots, n$

If $(h_{ki} = 0)$ Go to 2

$$
\begin{aligned}
r'_{ii} &= \sqrt{r_{ii}^2 + h_{ki}^2} \\
S_{ik} &= h_{ki}/r'_{ii} \\
C_{ik} &= r_{ii}/r'_{ii} \\
h_{ki} &= 0 \\
r_{ii} &= r'_{ii}
\end{aligned}
$$

Givens Algorithm

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3. Do $j = i + 1, \ldots, n + 1$

$$
\begin{aligned}
r'_{ij} &= C_{ik} r_{ij} + S_{ik} h_{kj} \\
h_{kj} &= -S_{ik} r_{ij} + C_{ik} h_{kj} \\
r_{ij} &= r'_{ij}
\end{aligned}
\tag{5.4.40}
$$

Next j
Next i
$e_k = h_{kj}$
Sum = Sum $+ e_k^2$
Next k

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After application of this algorithm, the $(n+m) \times (n+1)$ matrix will appear as

$$Q \begin{bmatrix} \overline{R} & \vdots & \overline{\mathbf{b}} \\ H & \vdots & \mathbf{y} \end{bmatrix} = \begin{array}{c} \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{1} \end{array} \\ \begin{bmatrix} R & \mathbf{b} \\ O & \mathbf{e} \end{bmatrix} \end{array} \begin{array}{l} \}n \\ \}m \end{array} \tag{5.4.41}$$

which is the required form for solution of the least squares estimation problem as given by Eq. (5.3.6). Note that $r_{i,n+1}(i = 1, \ldots, n)$ and $h_{k,n+1}(k = 1, \ldots, m)$ given by the algorithm represent $\mathbf{b}$ and $\mathbf{e}$, respectively, in Eq. (5.4.41). Also, $\text{Sum} = \sum_{k=1}^{m} e_k^2$.

Once the array has been reduced to the form given by Eq. (5.4.41), subsequent observations can be included by considering the following array:

$$\begin{bmatrix} R & \mathbf{b} \\ H_{m+1} & y_{m+1} \\ 0 & e^2 \end{bmatrix} = \tag{5.4.42}$$