ASEN 5070

Statistical Orbit Determination I

Fall 2012

Professor Jeffrey S. Parker

Professor George H. Born

Lecture 21: Exam 2 Debrief and More Fun

Announcements

CCAR
Colorado Center for Astrodynamics Research

- Homework 9 due this week.
  - Make sure you spend time studying for the exam
- Homework 10 out Thursday.
  - Give you a small reprieve to focus on HW9.

Quiz 17 Review

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Information

Congrats on surviving Exam #2!

While writing the exam, I tried out a lot of different concept questions. Here's one that didn't make it onto the exam, but I thought it was still worthy of a concept quiz study.

Let's say that we have a ground station that can track a satellite using both radiometric range observations and laser range observations. Further, let's assume that the observation-state relationships for both observation data types are identical (they originate at the exact same location at the ground station and are processed on the satellite at the exact same location!). The laser range observations are twice as accurate as the radiometric range observations.

Our Stat OD scenario is set up to estimate the position and velocity of the satellite (n=6) and the ground station's state is known perfectly.

Question 1 (1 point)

If we process both observation data types simultaneously, what is the size of the H-tilde matrix?

- 6x1
- 1x6
- 6x2
- 2x6

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Question 2 (1 point)

If we process both observation data types simultaneously, what is the rank of the H-tilde matrix?

- 0
- 1
- 2
- 6

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- 0
- 1
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- 6

The matrix of partials of one observation relative to the state parameters is identical to the other matrix.

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Question 3 (1 point)

If we process both observation data types simultaneously such that the laser ranging is the first type and the radiometric is the second, what is the most appropriate R matrix?

I'm using Matlab syntax: read these from top-left, to top-right, then bottom-left to bottom-right. Units for every value is distance^2.

- [1, 0; 0, 2]
- [2, 0; 0, 1]
- [1, 0; 0, 4]
- [4, 0; 0, 1]

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Question 4 (1 point)

Assume we have an a priori estimate of the state with a large a priori covariance. Which scenario should statistically provide the most accurate estimate of the state?

- Processing 100 observations of the laser ranging.
- Processing 100 observations of the radiometric ranging.
- Processing 100 observations of both.
- Guessing

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- ◯ Processing 100 observations of the laser ranging.
- ◯ Processing 100 observations of the radiometric ranging.
- ◯ Processing 100 observations of both.
- ◯ Guessing

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1}$$

HW#9

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- Due this Thursday

This assignment builds on Homework 7 and the final project. Use the code and experiences from Homework 7 as a starting point for this assignment.

Use the nominal trajectory and the state transition matrix generated in Homework 7, along with the observation data, to compute the following using the Batch Processor. Note that the weighting matrix W_i does not change over time, but remains a constant matrix.

1. Compute $\sum_{i=1}^{n} H_i^T W_i H_i$
2. Compute $\sum_{i=1}^{n} H_i^T W_i y_i$
3. Compute $\hat{x}_0$

For each of the above, turn in the value after one pass through the data. That is, don't iterate the process, though you will be doing so for the final project so prepare your code to support that feature. There are hints and example solutions on the course website to check your code.

Turn in the code and the values you obtain for each of these answers. You don't have to include any more write-up than the values, as long as your code is clear and commented well enough for us to follow.

University of Colorado Boulder

HW#9 Tip

- Try building x-hat from the data given online. If you can get that to work then you'll have a better chance of getting your own x-hat to match the solutions!

- Grab the accumulated matrices HTWH and HTWY.
- Try computing inv(HTWH+P0bar)*HTWY

```
sum HTWH 18340

               15642892.7919084               96445345.965
39035427396.1596            85380624521.8574
286051171293847     -7713945.19997599                    9
1837092.69316719               678092.191685084
733971.67113219          635723.942233195
               96445345.9658138               647567114.09
261992461796.065            568653325080.957
1.99572902376194e+15        -52282989.4335981
10595645.9232998               7392018.98738737
6309531.29539101                  1145232.754678
               87553864.9910207               591096282.62
239589755363.463            517506251255.126
1.83458038894187e+15        -48506921.1140942
8376682.98363485               7194081.84735885
6065287.38949461                1084370.86168306
               39035427396.1596               261992461796
106661556893419     230019296694618                    -24
```

```
sum HTWY 18340
          5487739212.46244
          36595706760.8576
          33419205123.3187
          14839386643350.3
           32168502424883
        -34097157842912.1
        -605.174326023799
     1.12352606137068e+17
         -2949593511.2199
        -27779918.1841308
          67061310.3507744
          603641482.691014
          400303872.615135
           91725182.49837
        -484248662.453383
          369187354.386611
          75484294.4256515
        -13500051.7585946
```

```
xhat (first pass)

     -0.0355074965116273
       -0.27341590598553
      -0.179970761622627
      0.0409377124301569
      0.0327563065901537
     -0.0147606324122311
       -9310103.07490921
   -6.57354368573488e-07
       0.108504936878731
    1.86320404339314e-06
    1.37869926339216e-06
   -2.54491428042453e-07
       -10.5636844179305
        9.98328899684054
        5.79442720656618
       -5.78200041893004
         2.3443163443211
         1.5100298917132
```

HW#9 Tip

- Your x-hat should match to at least 1 digit of precision in each parameter (hopefully 3). It will not be identical!
  - Different integrator
  - Different tolerance
  - Different computer
  - Different inverter

- inv(HTWH+P0bar) is very poorly conditioned (e-34 I believe)
- Matlab's "inv" function will not produce the right answer.

HW#9 Tip

- inv(HTWH+P0bar) is very poorly conditioned (e-34 I believe)

- R = chol(HTWH+P0bar)
- Inv(R) is also poorly conditioned, but only e-1.
  This is far better.

- If $R^T R$ = (H^TWH+P0bar), what is inv($R^T R$)?

Exam 2 Debrief

- Overall, the class did well. Most everyone grasped the concepts.
- Nobody got 100% – so don't worry if your grade was lower than 90. (curve TBD)

Exam 2 Debrief

1. 20% Two random variables have the joint density function given by:

$$f(x,y) = \begin{cases} k\left(x^2 y + z\right), & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

where z is some constant. Answer the following:

A. 10% Find k as a function of z. Use a separate sheet of paper as needed.

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where z is some constant. Answer the following:

A. 10% Find k as a function of z. Use a separate sheet of paper as needed.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k\left(x^2 y + z\right) dx dy = 1$$

$$k\int_0^1\int_0^1 \left(x^2 y + z\right) dx dy = 1$$

$$k\int_0^1 \left[\left(\frac{1}{3}x^3 y + zx\right)\right]_0^1 dy = 1$$

$$k\int_0^1 \left(\frac{1}{3}y + z\right) dy = 1$$

$$k\left[\left(\frac{1}{6}y^2 + zy\right)\right]_0^1 = 1$$

$$k\left(\frac{1}{6} + z\right) = 1$$

$$k = \frac{1}{\frac{1}{6} + z}$$

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1. 20% Two random variables have the joint density function given by:

$$f(x,y) = \begin{cases} k\left(x^2 y + z\right), & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

B. 10% Let's say that $z = 0$. Consider the conditional probability that $x < ½$ given that $y < ½$.

$$p\left(x < \frac{1}{2} \,/\, y < \frac{1}{2}\right)$$

a. 5% How would you set up this problem to solve it?
b. 3% Looking at the joint density function, do you expect that the answer to this conditional probability will be < ½ or > ½?
c. 2% What is the answer? I'd suggest waiting until the end of the test to do this because it is a bit ugly to do this by hand. (1 pt partial credit if you show the steps without completing it or getting the right answer; 1 pt for getting the right answer)

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$$\frac{\int\limits_{y=0}^{1/2}\int\limits_{x=0}^{1/2} f(x,y)\,dx\,dy}{\int\limits_{y=0}^{1/2}\int\limits_{x=0}^{1} f(x,y)\,dx\,dy} = \frac{\int\limits_{y=0}^{1/2}\int\limits_{x=0}^{1/2} 6\left(x^2 y\right)dx\,dy}{\int\limits_{y=0}^{1/2}\int\limits_{x=0}^{1} 6\left(x^2 y\right)dx\,dy} = \frac{6\int\limits_{y=0}^{1/2}\left[\left(\frac{1}{3}x^3 y\right)\right]_{x=0}^{1/2} dy}{6\int\limits_{y=0}^{1/2}\left[\left(\frac{1}{3}x^3 y\right)\right]_{x=0}^{1} dy}$$

$$= \frac{\int\limits_{y=0}^{1/2}\left(\frac{1}{24}y\right)dy}{\int\limits_{y=0}^{1/2}\left(\frac{1}{3}y\right)dy} = \frac{\left[\frac{1}{48}y^2\right]_{y=0}^{1/2}}{\left[\frac{1}{6}y^2\right]_{y=0}^{1/2}} = \frac{\frac{1}{192}}{\frac{1}{24}} = \frac{24}{192} = \frac{1}{8} = 0.125$$

Exam 2 Debrief

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2. 20% Consider a satellite that is being tracked by a ground station under the influence of a simple 2-body gravity field where μ is perfectly known. The state **X** includes 9 parameters, including the 3 position coordinates of the satellite, the 3 velocity coordinates of the satellite, and the 3 position coordinates of the ground station:

$$\mathbf{X} = [x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, xs, ys, zs]^T$$

The satellite's *a priori* position at some reference epoch t_0 is known to within 10 meters in each axis (1σ); its *a priori* velocity is known to within 0.01 m/s in each axis (1σ); the ground station's *a priori* position is known to within 5 meters in each axis (1σ); all *a priori* correlations between these parameters are equal to 0.

A. 5% Write out the matrix $\bar{P}_0$. Be sure to include proper units (though you can ignore units on any zero elements).

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A. 5% Write out the matrix P_0. Be sure to include proper units (though you can ignore units on any zero elements).

$$\mathrm{Diag}\Big(a, a, a, b, b, b, c, c, c\Big)$$

$$a = 100m^2, \quad b = 1 \times 10^{-4} m^2/s^2, \quad c = 25m^2$$

2. 20% Consider a satellite that is being tracked by a ground station under the influence of a simple 2-body gravity field where μ is perfectly known. The state $\mathbf{X}$ includes 9 parameters, including the 3 position coordinates of the satellite, the 3 velocity coordinates of the satellite, and the 3 position coordinates of the ground station:

$$\mathbf{X} = [x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, xs, ys, zs]^T$$

B. 5% The ground station tracks the satellite using laser ranging and radiometric Doppler observations; the two data types are independent; there are lots of observations. Each laser ranging observation is accurate to within 0.1 m (1σ); each Doppler observation is accurate to within 0.001 m/s (1σ). Write out the matrix R. Be sure to include proper units (once again you can ignore units on any zero elements).

2. 20% Consider a satellite that is being tracked by a ground station under the influence of a simple 2-body gravity field where μ is perfectly known. The state $\mathbf{X}$ includes 9 parameters, including the 3 position coordinates of the satellite, the 3 velocity coordinates of the satellite, and the 3 position coordinates of the ground station:

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$$R = \begin{bmatrix} 0.01m^2 & 0 \\ 0 & 1 \times 10^{-6} m^2/s^2 \end{bmatrix}$$

Exam 2 Debrief

C. 5% A customer wishes to track the satellite to within 0.2 m in position (1σ) relative to the ground station using this set-up. Is this possible? Why / why not? If you need more space, use another sheet of paper.

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C. 5% A customer wishes to track the satellite to within 0.2 m in position (1σ) relative to the ground station using this set-up. Is this possible? Why / why not? If you need more space, use another sheet of paper.

$$\bar{P}_0 = \text{Diag}\Big(a, a, a, b, b, b, c, c, c\Big)$$

$$a = 100m^2, \quad b = 1 \times 10^{-4} m^2/s^2, \quad c = 25m^2$$

$$R = \begin{bmatrix} 0.01m^2 & 0 \\ 0 & 1 \times 10^{-6} m^2/s^2 \end{bmatrix}$$

4 pts: This is certainly possible since the observations are plenty accurate and there are plenty of them.
5 pts: (same as above but also:) It will take some time since the a priori covariance begins far above 0.2 meters, but the system will converge. Also good: the model dynamics are perfectly known – if they weren't this may NOT be possible.

D. 5% A customer wishes to track the satellite to within 0.2 m in position (1σ) relative to the Earth (specifically an inertial frame of reference such as J2000) using this set-up. Is this possible? Why / why not? If you need more space, use another sheet of paper.

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D. 5% A customer wishes to track the satellite to within 0.2 m in position (1σ) relative to the Earth (specifically an inertial frame of reference such as J2000) using this set-up. Is this possible? Why / why not? If you need more space, use another sheet of paper.

$$\bar{P}_0 = \text{Diag}\Big(a, a, a, b, b, b, c, c, c\Big)$$

$$a = 100m^2, \quad b = 1 \times 10^{-4} m^2/s^2, \quad c = 25m^2$$

$$R = \begin{bmatrix} 0.01m^2 & 0 \\ 0 & 1 \times 10^{-6} m^2/s^2 \end{bmatrix}$$

This is not possible. The variance of the station's position is far higher than 0.2 m and nothing is tying the state of the satellite to the Earth any tighter than that. The position uncertainty may fall below 0.2 m, but it's wrong, since there is nothing tying the system to the Earth. In fact, the H^TH matrix is only rank 6.

i.e., High Precision but low accuracy!

Exam 2 Debrief

3. 12% You are a navigator on a mission, running a sequential filter to process your observations, and you notice that your covariance matrix routinely loses its symmetric and positive definite properties. You took this class so you know that that's a bad thing; it is indicative that your system is sensitive to numerical round-off errors. You recall that there are several ways you can modify your code to avoid these problems.

A. The Joseph formulation is a method that you can use to update your covariance matrix.
   a. 2% True/False: The Joseph formulation always results in a symmetric covariance matrix.
   b. 2% True/False: The Joseph formulation always results in a positive definite covariance matrix.

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B. Square root formulations such as Potter and Givens are methods that you can also use to update your covariance matrix.
   a. 2% True/False: Square root formulations always results in a symmetric covariance matrix.
   b. 2% True/False: Square root formulations always results in a positive definite covariance matrix.

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B. Square root formulations such as Potter and Givens are methods that you can also use to update your covariance matrix.
   a. 2% True/False: Square root formulations always results in a symmetric covariance matrix.
   b. 2% True/False: Square root formulations always results in a positive definite covariance matrix.

Only guarantees a nonnegative definite!

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C. Another way to update your covariance matrix to avoid numerical issues is to reset it to some specified diagonal matrix when you detect that it has lost its symmetric positive definite properties.
   a. 2% True/False: Resetting the covariance matrix always results in a symmetric covariance matrix.
   b. 2% True/False: Resetting the covariance matrix always results in a positive definite covariance matrix.

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   b. 2% True/False: Resetting the covariance matrix always results in a positive definite covariance matrix.

Exam 2 Debrief

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4. 15% You are a navigator on a mission (which means you don't have the true trajectory and you don't want to mess up!). You have a large amount of tracking data at your disposal including observations at times $t_1, t_2, \ldots, t_l$ (that's an "el"). Each observation time includes 4 independent data types ($p = 4$) from one tracking station: range, Doppler, and angular information (right ascension and declination). You are estimating 3 state parameters at the moment (say, the position of the ground station). *A priori* information is available as well as the observation covariance matrix at each time (R_i).

A. 5% What is the size of the $\widetilde{H}$-matrix?

Exam 2 Debrief

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4. 15% You are a navigator on a mission (which means you don't have the true trajectory and you don't want to mess up!). You have a large amount of tracking data at your disposal including observations at times $t_1, t_2, \ldots, t_l$ (that's an "el"). Each observation time includes 4 independent data types ($p = 4$) from one tracking station: range, Doppler, and angular information (right ascension and declination). You are estimating 3 state parameters at the moment (say, the position of the ground station). *A priori* information is available as well as the observation covariance matrix at each time (R_i).

A. 5% What is the size of the $\widetilde{H}$-matrix?

4x3

Exam 2 Debrief

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B. 5% What is the size of the matrix $H^T H$?

Exam 2 Debrief

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B. 5% What is the size of the matrix $H^T H$?

[3x4]*[4x3] = [3x3] (hint: it's always nxn)

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C. 5% How many observation times must be used to acquire a unique estimate of the state at a given time, assuming that the $\widetilde{H}$-matrix is full rank? For this problem, ignore all *a priori* information.

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C. 5% How many observation times must be used to acquire a unique estimate of the state at a given time, assuming that the $\widetilde{H}$-matrix is full rank? For this problem, ignore all *a priori* information.

1. one observation vector includes 4 independent pieces of information. We only need 3 pieces of information.

Exam 2 Debrief

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5. 33% We're given the following dynamic system

$$\ddot{x} = -ax - bx^3 + c\cos\omega t$$

where x is the position of an object, $\dot{x}$ is the velocity of the object, $\ddot{x}$ is the acceleration of the object, and a, b, and c are unknown constants. We have one observation data type at each time (where the subscript i is shorthand for the value of the parameter at time t_i):

$$\rho_i = \sqrt{d^2 + (e + x_i)^2} + \epsilon_i; \quad i = 1,2,\ldots,l$$

where d and e are two more unknown constants and ϵ_i is the observation error at time t_i, which has the statistical properties:

$$E[\epsilon_i] = 0, \qquad E\left[\epsilon_i\epsilon_j\right] = \sigma_i^2\delta_{ij}$$

We're interested in estimating the position and velocity of the system as well as each of the unknown constants. Hence, we have the state X_i at any given time:

$$X_i = [\, x_i, \dot{x}_i, a, b, c, d, e \,]^T$$

Exam 2 Debrief

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We're interested in estimating the position and velocity of the system as well as each of the unknown constants. Hence, we have the state X_i at any given time:

$$X_i = [\, x_i, \dot{x}_i, a, b, c, d, e \,]^T$$

A. 8% Set up all necessary equations for sequentially estimating the state X using the conventional Kalman filter. Don't solve them. For the A matrix, only show the upper-left 3x3 matrix, but indicate how the rest of the A matrix should be filled out.

They should show pretty much everything from page 203-204. At a minimum they should show:

$$\dot{\mathbf{X}}_i = \begin{bmatrix} \dot{x}_i \\ \ddot{x}_i \\ \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \dot{x}_i \\ -ax - bx^3 + c\cos\omega t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then Phi, A, y, H-tilde
x-hat, P, x-bar, P-bar

Exam 2 Debrief

B. 8% Solve the upper-left 3x3 portion of the A matrix.

$$\ddot{x} = -ax - bx^3 + c\cos\omega t$$

$$A = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial a} & \cdots & \frac{\partial \dot{x}}{\partial e} \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial a} & \cdots & \frac{\partial \ddot{x}}{\partial e} \\ \frac{\partial \dot{a}}{\partial x} & \frac{\partial \dot{a}}{\partial \dot{x}} & \frac{\partial \dot{a}}{\partial a} & \cdots & \frac{\partial \dot{a}}{\partial e} \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{\partial \dot{e}}{\partial x} & \frac{\partial \dot{e}}{\partial \dot{x}} & \frac{\partial \dot{e}}{\partial a} & \cdots & \frac{\partial \dot{e}}{\partial e} \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -a - 3bx^2 & 0 & -x & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Exam 2 Debrief

C. 8% State clearly all information that must be given as input information beyond what is given in this problem statement.

X* (the reference trajectory)
x-bar (the *a priori* deviation, nominally zero)
P-bar (the *a priori* covariance)
Y_i (the observations)
omega and sigma (though that's specific to this problem and it's okay if you didn't include that!

D. 9% What would change if you estimated the state using an Extended Kalman filter?

The things that would change:

1. The best estimate of the state deviation vector would be used to update the best estimate of the state after each observation. That is,
   $\mathbf{X}_i^* = \mathbf{X}_i^* + \hat{x}_i$
2. The a priori state deviation vector would go to zero, x-bar = 0.

- Quick Break

- Next up: Stuff.

  - Prediction Residual
  - Givens
  - Householder
  - Future: Process Noise, Smoothing

The Prediction Residual

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4.7.4 The Prediction Residual

It is of interest to examine the variance of the predicted residuals, which are sometimes referred to as the *innovation*, or new information, which comes from each measurement. The *predicted residual*, or innovation, is the observation residual based on the *a priori* or predicted state , $\bar{\mathbf{x}}_k$, at the observation time, t_k, and is defined as

$$\beta_k = \mathbf{y}_k - \tilde{H}_k \bar{\mathbf{x}}_k. \tag{4.7.33}$$

As noted previously,

$$\begin{aligned}\bar{\mathbf{x}}_k &= \mathbf{x}_k + \eta_k \\ \mathbf{y}_k &= \tilde{H}_k \mathbf{x}_k + \epsilon_k\end{aligned}$$

The Prediction Residual

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As noted previously,

$$\begin{aligned}\bar{\mathbf{x}}_k &= \mathbf{x}_k + \eta_k\\ \mathbf{y}_k &= \tilde{H}_k \mathbf{x}_k + \epsilon_k\end{aligned}$$

$$\begin{aligned}\beta_k &= \tilde{H}_k \mathbf{x}_k + \epsilon_k - \tilde{H}_k \bar{\mathbf{x}}_k\\ &= \tilde{H}(\mathbf{x}_k - \bar{\mathbf{x}}_k) + \epsilon_k\end{aligned}$$

The Prediction Residual

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where $\mathbf{x}_k$ is the true value of the state deviation vector and η_k is the error in $\overline{\mathbf{x}}_k$. Also,

$$E[\eta_k] = 0,\ E[\eta_k \eta_k^T] = \overline{P}_k$$

and

$$E[\epsilon_k] = 0,\ E[\epsilon_k \epsilon_k^T] = R_k$$
$$E[\epsilon_k \eta_k^T] = 0.$$

From these conditions it follows that β_k has mean

$$\begin{aligned} E[\beta_k] \equiv \overline{\beta}_k &= E(\widetilde{H}_k \mathbf{x}_k + \epsilon_k - \widetilde{H}_k \overline{\mathbf{x}}_k) \\ &= E(\epsilon_k - \widetilde{H} \eta_k) = 0 \end{aligned}$$

The Prediction Residual

and variance-covariance

$$
\begin{aligned}
P_{\beta_k} &= E[(\beta_k - \overline{\beta}_k)(\beta_k - \overline{\beta}_k)^T] = E[\beta_k \beta_k^T] \\
&= E[(\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k)(\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k)^T] \\
&= E[(\epsilon_k - \widetilde{H}_k \eta_k)(\epsilon_k - \widetilde{H}_k \eta_k)^T] \\
P_{\beta_k} &= R_k + \widetilde{H}_k \overline{P}_k \widetilde{H}_k^T . \qquad (4.7.34)
\end{aligned}
$$

Hence, for a large prediction residual variance-covariance, the Kalman gain

$$
K_k = \overline{P}_k \widetilde{H}_k^T P_{\beta_k}^{-1} \qquad (4.7.35)
$$

will be small, and the observation will have little influence on the estimate of the state. Also, large values of the prediction residual relative to the prediction residual standard deviation may be an indication of bad tracking data and hence may be used to edit data from the solution.

This would be especially important in the case of the EKF

- Next: Orthogonal transformations: Givens, Householder

5.3 LEAST SQUARES SOLUTION VIA ORTHOGONAL TRANSFORMATION

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An alternate approach that avoids some of the numerical problems encountered in the normal equation approach is described in the following discussion. The method obtains the solution by applying successive orthogonal transformations to the information array, $(H, \mathbf{y})$. Enhanced numerical accuracy is obtained by this approach. Consider the quadratic performance index, $J(\mathbf{x})$, which minimizes the weighted sum of squares of the observation errors, $\epsilon = \mathbf{y} - H\mathbf{x}$ (for the moment we will assume no *a priori* information; i.e., $\overline{P}^{-1} = 0, \overline{\mathbf{x}} = 0$):

$$J(\mathbf{x}) = \epsilon^T W \epsilon = \left\| W^{\frac{1}{2}}(H\mathbf{x} - \mathbf{y}) \right\|^2 = (H\mathbf{x} - \mathbf{y})^T W (H\mathbf{x} - \mathbf{y}). \quad (5.3.1)$$

If W is not diagonal, $W^{1/2}$ can be computed by the Cholesky decomposition. Or the prewhitening transformation described at the end of Section 5.7.1 can be applied so that $W = I$. For notational convenience we are using $-\epsilon$ in Eq. (5.3.1).

5.3 LEAST SQUARES SOLUTION VIA ORTHOGONAL TRANSFORMATION

The solution to the least squares estimation problem (as well as the minimum variance and the maximum likelihood estimation problem, under certain restrictions) is obtained by finding the value $\hat{\mathbf{x}}$ that minimizes the performance index $J(\mathbf{x})$. To achieve the minimum value of $J(\mathbf{x})$, we introduce the $m \times m$ orthogonal matrix, Q. An orthogonal matrix has the following properties:

1. $QQ^T = I.$ (5.3.2)

2. $Q^{-1} = Q^T$ hence $Q^T Q = I.$

3. If Q_1 and Q_2 are orthogonal matrices, then so is $Q_1 Q_2$.

5.3 LEAST SQUARES SOLUTION VIA ORTHOGONAL TRANSFORMATION

4. For any vector x,

$$\|Qx\| = \|x\| = (x^T x)^{\frac{1}{2}}. \tag{5.3.3}$$

Multiplying by Q does not change the Euclidean norm of a vector.

5. If ϵ is an m vector of random variables with $\epsilon \sim (O, I)$ (i.e., $E(\epsilon) = 0$ and $E(\epsilon\epsilon^T) = I$), then $\overline{\epsilon} = Q\epsilon$ has the same properties,

$$E(\overline{\epsilon}) = QE(\epsilon) = 0,\ E(\overline{\epsilon}\,\overline{\epsilon}^T) = QE(\epsilon\epsilon^T)Q^T = I. \tag{5.3.4}$$

It follows then that (5.3.1) can be expressed as

$$\begin{aligned} J(x) &= \left\|QW^{\frac{1}{2}}(Hx - y)\right\|^2 \\ &= \left(Hx - y\right)^T W^{\frac{1}{2}} Q^T Q W^{\frac{1}{2}} \left(Hx - y\right). \end{aligned} \tag{5.3.5}$$

5.3 LEAST SQUARES SOLUTION VIA ORTHOGONAL TRANSFORMATION

Select Q such that

$$QW^{\frac{1}{2}}H = \begin{bmatrix} R \\ O \end{bmatrix} \text{ and define } QW^{\frac{1}{2}}\mathbf{y} \equiv \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix} \tag{5.3.6}$$

where

R is a $n \times n$ upper-triangular matrix of rank n

O is a $(m-n) \times n$ null matrix

$\mathbf{b}$ is a $n \times 1$ column vector

$\mathbf{e}$ is a $(m-n) \times 1$ column vector.

The results given by Eq. (5.3.6) assume that $m > n$ and H is of rank n. Using Eq. (5.3.6), Eq. (5.3.5) can be written as

$$J(\mathbf{x}) = \left\| \begin{bmatrix} R \\ O \end{bmatrix} \mathbf{x} - \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix} \right\|^2 . \tag{5.3.7}$$

Expanding leads to

$$J(\mathbf{x}) = \left\| R\mathbf{x} - \mathbf{b} \right\|^2 + \left\| \mathbf{e} \right\|^2 . \tag{5.3.8}$$

Only the first term in Eq. (5.3.8) is a function of x, so the value of x that minimizes $J(\mathbf{x})$ is obtained by requiring that

$$R\hat{\mathbf{x}} = \mathbf{b} \tag{5.3.9}$$

5.3 LEAST SQUARES SOLUTION VIA ORTHOGONAL TRANSFORMATION

and the minimum value of the performance index becomes (equating $J(\hat{x})$ in Eq. (5.3.1) and Eq. (5.3.8))

$$J(\hat{\mathrm{x}}) = \left\|\mathrm{e}\right\|^2 = \left\|W^{\frac{1}{2}}(H\hat{\mathrm{x}} - \mathrm{y})\right\|^2. \qquad (5.3.10)$$

That is, $\|\mathrm{e}\|$ is the norm of the observation residual vector, which for a linear system will be equal to the weighted sum of the squares of observation residuals determined by using $\hat{\mathrm{x}}$ in Eq. (5.3.10).

Choices from here

- So how do we select Q?

$$J(\mathrm{x}) = \left\| QW^{\frac{1}{2}}(H\mathrm{x} - \mathrm{y}) \right\|^2$$
$$= \left(H\mathrm{x} - \mathrm{y}\right)^T W^{\frac{1}{2}} Q^T Q W^{\frac{1}{2}} \left(H\mathrm{x} - \mathrm{y}\right)$$

Select Q such that

$$QW^{\frac{1}{2}}H = \begin{bmatrix} R \\ O \end{bmatrix} \text{ and define } QW^{\frac{1}{2}}\mathrm{y} \equiv \begin{bmatrix} \mathrm{b} \\ \mathrm{e} \end{bmatrix}$$

- Givens, Householder, many methods

Givens

The procedure described in the previous section is direct and for implementation requires only that a convenient procedure for computing Q be obtained. One such procedure can be developed based on the Givens plane rotation (Givens, 1958). Let $\mathbf{x}$ be a 2×1 vector having components $\mathbf{x}^T = [x_1 \;\; x_2]$ and let G be a 2×2 orthogonal matrix associated with the plane rotation through the angle θ. Then select G such that

$$G\mathbf{x} = \mathbf{x}' = \begin{pmatrix} x_1' \\ 0 \end{pmatrix}. \tag{5.4.1}$$

Givens

To this end, consider the transformation

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.4.2}$$

or

$$\begin{aligned} x_1' &= \quad \cos\theta x_1 + \sin\theta x_2 \\ x_2' &= -\sin\theta x_1 + \cos\theta x_2. \end{aligned} \tag{5.4.3}$$

Equations (5.4.3) represent a system of two equations in three unknowns; that is, x_1', x_2', and θ. The Givens rotation is defined by selecting the rotation θ such that $x_2' = 0$. That is, let

$$x_1' = \quad \cos\theta x_1 + \sin\theta x_2 \tag{5.4.4}$$

$$0 = -\sin\theta x_1 + \cos\theta x_2. \tag{5.4.5}$$

Givens

From Eq. (5.4.5), it follows that

$$\tan\theta = \frac{x_2}{x_1}, \sin\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \cos\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}. \tag{5.4.6}$$

The positive value associated with the square root operation is selected for the following discussion. Substituting the expression for $\sin\theta$ and $\cos\theta$ into Eq. (5.4.4) leads to

$$x_1' = \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}} + \frac{x_2^2}{\sqrt{x_1^2 + x_2^2}} = \sqrt{x_1^2 + x_2^2}. \tag{5.4.7}$$

Givens

Consider the application of the transformation

$$G(\theta) = \begin{bmatrix} \cos\theta \ \sin\theta \\ -\sin\theta \ \cos\theta \end{bmatrix} \tag{5.4.8}$$

to two general row vectors $\mathbf{h}_i$ and $\mathbf{h}_k$; for example,

$$G \begin{bmatrix} h_{ii} \ h_{ii+1} \ \ldots h_{in} \\ h_{ki} \ h_{ki+1} \ldots h_{kn} \end{bmatrix} = \begin{bmatrix} h'_{ii} \ h'_{ii+1} \ \ldots h'_{in} \\ 0 \quad h'_{ki+1} \ldots h'_{kn} \end{bmatrix}. \tag{5.4.9}$$

Givens

That is, for any two general row vectors, $\mathbf{h}_i$ and $\mathbf{h}_k$, the transformation is applied to the first column so as to null h_{ki}. The transformation that accomplishes this is applied to each remaining column to obtain the transformed matrix. Hence,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} h_{ii} \\ h_{ki} \end{bmatrix} = \begin{bmatrix} h'_{ii} \\ 0 \end{bmatrix} \tag{5.4.10}$$

or

$$\begin{aligned} \sin\theta &= h_{ki}/\sqrt{h_{ii}^2 + h_{ki}^2} = h_{ki}/h'_{ii} \\ \cos\theta &= h_{ii}/\sqrt{h_{ii}^2 + h_{ki}^2} = h_{ii}/h'_{ii} \\ h'_{ii} &= \sqrt{h_{ii}^2 + h_{ki}^2}. \end{aligned} \tag{5.4.11}$$

Then for all other columns,

$$\begin{aligned} h'_{ij} &= h_{ij}\cos\theta + h_{kj}\sin\theta \\ & \qquad\qquad\qquad\qquad j = i+1, \ldots, n \\ h'_{kj} &= -h_{ij}\sin\theta + h_{kj}\cos\theta\,. \end{aligned} \tag{5.4.12}$$

Givens

- Apply the rotation across the matrix, converting it into a triangular matrix

$$
\begin{bmatrix}
1 & & & & & & & & \\
 & 1 & & & & & & & \\
 & & C^{4,3} & S^{4,3} & & & & & \\
 & & -S^{4,3} & C^{4,3} & & & & & \\
 & & & & 1 & & & & \\
 & & & & & 1 & & & \\
 & & & & & & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\
0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\
0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\
0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\
0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\
0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & h_{m3} & \cdots & h_{mn} & y_m
\end{bmatrix}
$$

Givens

$$
\begin{bmatrix}
1 & & & & & & & & \\
& 1 & & & & & & & \\
& & C^{4,3} & S^{4,3} & & & & & \\
& & -S^{4,3} & C^{4,3} & & & & & \\
& & & & 1 & & & & \\
& & & & & 1 & & & \\
& & & & & & 1 & & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\
0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\
0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\
0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\
0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\
0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & h_{m3} & \cdots & h_{mn} & y_m
\end{bmatrix}
=
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\
0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\
0 & 0 & h'_{33} & \cdots & h'_{3n} & y'_3 \\
0 & 0 & 0 & \cdots & h'_{4n} & y'_4 \\
0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\
0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & h_{m3} & \cdots & h_{mn} & y_m
\end{bmatrix}
$$

Givens

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$$
\begin{bmatrix}
1 & & & & & & & & \\
& 1 & & & & & & & \\
& & C^{4,3} & S^{4,3} & & & & & \\
& & -S^{4,3} & C^{4,3} & & & & & \\
& & & & 1 & & & & \\
& & & & & 1 & & & \\
& & & & & & 1 & & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\
0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\
0 & 0 & h_{33} & \cdots & h_{3n} & y_3 \\
0 & 0 & h_{43} & \cdots & h_{4n} & y_4 \\
0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\
0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & h_{m3} & \cdots & h_{mn} & y_m
\end{bmatrix}
=
\begin{bmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1n} & y_1 \\
0 & h_{22} & h_{23} & \cdots & h_{2n} & y_2 \\
0 & 0 & h'_{33} & \cdots & h'_{3n} & y'_3 \\
0 & 0 & 0 & \cdots & h'_{4n} & y'_4 \\
0 & 0 & h_{53} & \cdots & h_{5n} & y_5 \\
0 & 0 & h_{63} & \cdots & h_{6n} & y_6 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & h_{m3} & \cdots & h_{mn} & y_m
\end{bmatrix}
$$

$$QW^{\frac{1}{2}}H = \begin{bmatrix} R \\ O \end{bmatrix} \text{ and define } QW^{\frac{1}{2}}\mathbf{y} \equiv \begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix}$$

Accuracy Comparison for Batch and Givens – Finite Precision Computer

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Consider

$$H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}$$

Notice that a vector of Observations is not needed. Why?

Machine precision is such that

$$1 \pm \epsilon^2 = 1$$

The normal matrix is given by

$$H^T H = \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon+\varepsilon^2 \end{bmatrix}; \quad \text{exact solution}$$

our computer will drop the ε^2 and

$$H^T H = \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon \end{bmatrix}$$

To order ε $|H^T H| = 0$, hence it is singular

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Consequently, the Batch Processor will fail to yield a solution. Note that this Illustrates the problem with forming H^TH, i.e. numerical problems are amplified.

The Cholesky decomposition yields:

$$R = \begin{bmatrix} \sqrt{3} & \dfrac{3-\varepsilon}{\sqrt{3}} \\ 0 & 0 \end{bmatrix}$$

R is singular and will not yield a solution for $\hat{x}$.

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Use the Givens transformation to determine *R*

1st zero element (2,1) of *H*

$$\begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}$$

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Use the Givens transformation to determine *R*

1st zero element (2,1) of *H*

$$\begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}$$

$$x_1 = 1 \quad x_2 = 1$$

$$S_\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{\sqrt{2}} \qquad C_\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{\sqrt{2}}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}_{3\times2} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix}$$

Note that the magnitude of the columns of [H y] are unchanged.

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$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}_{3\times 2} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix}$$

Next zero element (3,1) $\qquad x_1 = \dfrac{2}{\sqrt{2}}, \quad x_2 = 1$

$$S_\theta = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}, \quad C_\theta = \frac{2/\sqrt{2}}{\sqrt{3}} = \frac{2}{\sqrt{6}}$$

$$\begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix} = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & 0 \\ 0 & -\sqrt{2}\varepsilon/\sqrt{3} \end{bmatrix}$$

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$$\begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 \\ 1 & 1-\varepsilon \end{bmatrix} = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & 0 \\ 0 & -\sqrt{2}\varepsilon/\sqrt{3} \end{bmatrix}$$

Next zero element (3,2) $x_1 = 0\,,\ x_2 = -\dfrac{\sqrt{2}}{\sqrt{3}}\varepsilon \rightarrow \quad S_\theta = -1,\, C_\theta = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & 0 \\ 0 & -\sqrt{2}\varepsilon/\sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & \sqrt{2}\varepsilon/\sqrt{3} \\ 0 & 0 \end{bmatrix}$$

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The Givens transformations yield

$$R = \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix}$$

Which will yield a valid solution for $\hat{x}$

In fact

$$R^T R = \begin{bmatrix} \sqrt{3} & 0 \\ (3-\varepsilon)/\sqrt{3} & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix} \begin{bmatrix} \sqrt{3} & (3-\varepsilon)/\sqrt{3} \\ 0 & \sqrt{\frac{2}{3}}\varepsilon \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3-\varepsilon \\ 3-\varepsilon & 3-2\varepsilon+\varepsilon^2 \end{bmatrix}$$

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Which is the exact solution result for $H^T H$. Hence, for this example the orthogonal transformations would yield the correct solution. However, the estimation error covariance matrix would be incorrect because our computer would drop the ε^2 term.

Other Orthogonal Transformations

- Givens used rotations to null values until R became upper-triangular

- Householder uses reflections to accomplish the same goal

The End

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- Homework 9 due this week.
  - Make sure you spend time studying for the exam

- Homework 10 out Thursday.
  - Give you a small reprieve to focus on HW9.