Colorado Center for Astrodynamics Research

# ASEN 5070
# Statistical Orbit Determination I
# Fall 2012

Professor Jeffrey S. Parker
Professor George H. Born

## Lecture 15: Numerical Compensations

# Announcements

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- Exam 1
- Final Project

- Homework 5 is not graded…neither is the test.
- Homework 6 due Today
- Homework 7 due next week (Tuesday!)
  - It's okay to use Matlab to compute partials and to output them. But verify them.

- Concept Quizzes to resume Monday!

- Guest lecturer next week 10/25

# Exam 1 Debrief



- Too easy?

- Too short?

# Exam 1 Debrief

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1. 20% Answer the following and be sure to provide sufficient explanation. If you need more space, use an additional sheet of paper.

   a. If one column of the $\widetilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

      If it is always singular, provide an explanation for why.
      If it is not always singular, provide an example when it is not.

# Exam 1 Debrief

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1. 20% Answer the following and be sure to provide sufficient explanation. If you need more space, use an additional sheet of paper.

   a. If one column of the $\tilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

      If it is always singular, provide an explanation for why.
      If it is not always singular, provide an example when it is not.

True. The transpose of a column of zeros becomes a row of zeros. That row propagates through the whole system and the HTH matrix becomes singular.

b. If one row of the $\tilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

If it is always singular, provide an explanation for why.
If it is not always singular, provide an example when it is not.

# Exam 1 Debrief



b. If one row of the $\tilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

If it is always singular, provide an explanation for why.
If it is not always singular, provide an example when it is not.

False. Consider H-tilde = [a, b, 0]^T and Phi = 1.

# Exam 1 Debrief

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2. 15% A random variable has the probability density function given by:

$$f(x) = \begin{cases} 1/3, & 0 \leq x < 1, \\ k\left(x^3 + 1/4\right), & 1 \leq x \leq 2, \\ 0, & \text{elsewhere} \end{cases}$$

Find $k$

# Exam 1 Debrief

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$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 \frac{1}{3}dx + \int_1^2 k\left(x^3 + \frac{1}{4}\right)dx = 1$$

$$\frac{1}{3}[x]_0^1 + k\left[\frac{x^4}{4} + \frac{1}{4}x\right]_1^2 = 1$$

$$\frac{1}{3} + k(4 + \frac{1}{2} - \frac{1}{4} - \frac{1}{4}) = 1$$

$$\frac{1}{3} + 4k = 1$$

$$k = \frac{2/3}{4} = \frac{1}{6}$$

3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

where $a$, $b$, $c$, $d$, $x_0$, and $x_1$ are constants and $t_i$ are given. Which of the following state vectors are observable (assuming that the constants that aren't in the state are known perfectly):

1. $\mathbf{X} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ 2. $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ 3. $\begin{bmatrix} b \\ x_0 \\ d \end{bmatrix}$ 4. $\begin{bmatrix} a \\ c \\ x_1 \end{bmatrix}$ 5. $\begin{bmatrix} b \\ c \\ d \end{bmatrix}$

# Exam 1 Debrief



3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

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1. $\mathbf{X} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ 2. $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ 3. $\begin{bmatrix} b \\ x_0 \\ d \end{bmatrix}$ 4. $\begin{bmatrix} a \\ c \\ x_1 \end{bmatrix}$ 5. $\begin{bmatrix} b \\ c \\ d \end{bmatrix}$

answer: 1 and 5 are observable

# Exam 1 Debrief



4. 10% Which weighting matrix is the most appropriate to use given two observations and the second is twice as accurate as the first:

1. $W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2. $W = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ 3. $W = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ 4. $W = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ 5. $W = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

# Exam 1 Debrief



4. 10% Which weighting matrix is the most appropriate to use given two observations and the second is twice as accurate as the first:

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answer: 3

5. 40% The equation of motion for a particular system is given by

$$\ddot{z} - 3z^2\dot{z} = c$$

and we wish to estimate the state

$$\mathbf{X} = [\ z(t_0),\ \dot{z}(t_0),\ c\ ]^T$$

using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

a. 5% What is the degree and order of this equation of motion?

# Exam 1 Debrief



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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

a. 5% What is the degree and order of this equation of motion?

   a. 1st degree, 2nd order

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b. 5% Is this equation of motion linear or nonlinear?

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b. 5% Is this equation of motion linear or nonlinear?

b. Nonlinear

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

c. 10% What is the $A(t)$ matrix?

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

c. 10% What is the $A(t)$ matrix?

$$\dot{\mathbf{X}}(t) = \begin{bmatrix} \dot{z} \\ \ddot{z} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} \dot{z} \\ 3z^2\dot{z} + c \\ 0 \end{bmatrix}$$

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial \dot{z}} & \frac{\partial \dot{z}}{\partial c} \\ \frac{\partial \ddot{z}}{\partial z} & \frac{\partial \ddot{z}}{\partial \dot{z}} & \frac{\partial \ddot{z}}{\partial c} \\ \frac{\partial \dot{c}}{\partial z} & \frac{\partial \dot{c}}{\partial \dot{z}} & \frac{\partial \dot{c}}{\partial c} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 6z\dot{z} & 3z^2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

# Exam 1 Debrief

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5. 40% The equation of motion for a particular system is given by

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and we wish to estimate the state

$$\mathbf{X} = [\ z(t_0),\ \dot{z}(t_0),\ c\ ]^T$$

using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

d. 5% What is the $\widetilde{H}$ matrix?

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and we wish to estimate the state

$$\mathbf{X} = [\ z(t_0),\ \dot{z}(t_0),\ c\ ]^T$$

using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

d. 5% What is the $\tilde{H}$ matrix?

$$\tilde{H}(t) = \frac{\partial z}{\partial \mathbf{X}} = \left[\ \frac{\partial z}{\partial z} \quad \frac{\partial z}{\partial \dot{z}} \quad \frac{\partial z}{\partial c}\ \right]$$

$$= \left[\ 1 \quad 0 \quad 0\ \right]$$

G(t) = z(t) is the simplest representation of the observation. If you solve the EOM for z, you can also get partials wrt z-dot and c, which is more information (optional).

5. 40% The equation of motion for a particular system is given by

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and we wish to estimate the state

$$\mathbf{X} = [\ z(t_0),\ \dot{z}(t_0),\ c\ ]^T$$

using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

e. 10% How would you determine the state transition matrix, $\Phi(t_i, t_0)$? Note: don't attempt to compute it. Can you use a Laplace Transform? Why/why not?

# Exam 1 Debrief

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e. 10% How would you determine the state transition matrix, $\Phi(t_i, t_0)$? Note: don't attempt to compute it. Can you use a Laplace Transform? Why/why not?

e.

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 6z\dot{z} & 3z^2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Phi(t, t_0)$$

Can't use a Laplace Transform because A(t) is time-dependent!

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and we wish to estimate the state

$$\mathbf{X} = [\ z(t_0),\ \dot{z}(t_0),\ c\ ]^T$$

using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

f. 5% What is the minimum number of independent observations of $z$ that is required to generate an estimate of the state?

# Exam 1 Debrief



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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

f. 5% What is the minimum number of independent observations of $z$ that is required to generate an estimate of the state?

f. 3

# Topics coming up

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- Conventional Kalman Filter (CKF)
- Extended Kalman Filter (EKF)

- Numerical Issues
  - Machine precision
  - Covariance collapse

- Numerical Compensation
  - Joseph, Potter, Cholesky, Square-root free, unscented, Givens, orthogonal transformation, SVD
  - State Noise Compensation, Dynamical Model Compensation

# Stat OD Conceptualization

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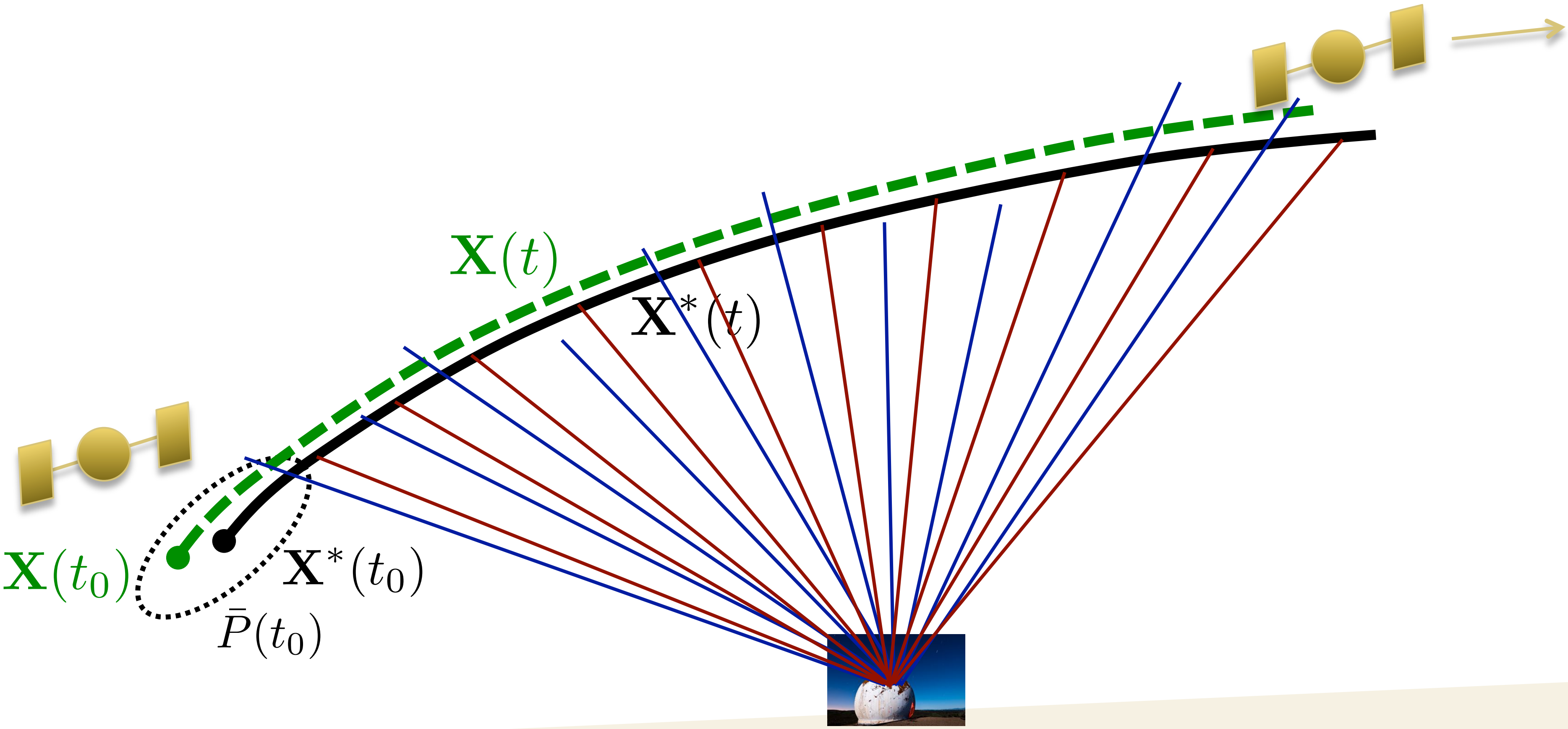
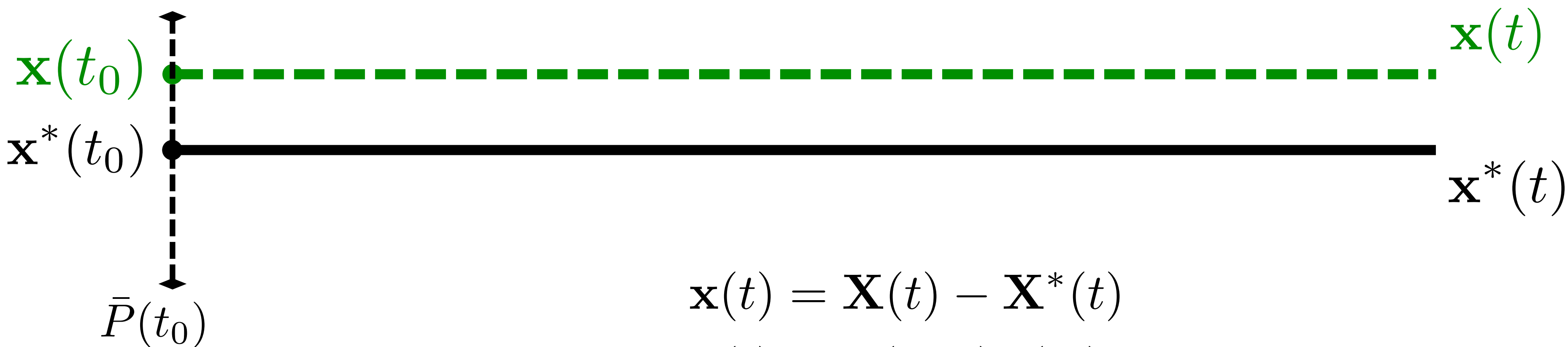
- Full, nonlinear system:



# Stat OD Conceptualization

- Linearization




$$\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t)$$

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$$

# Stat OD Conceptualization

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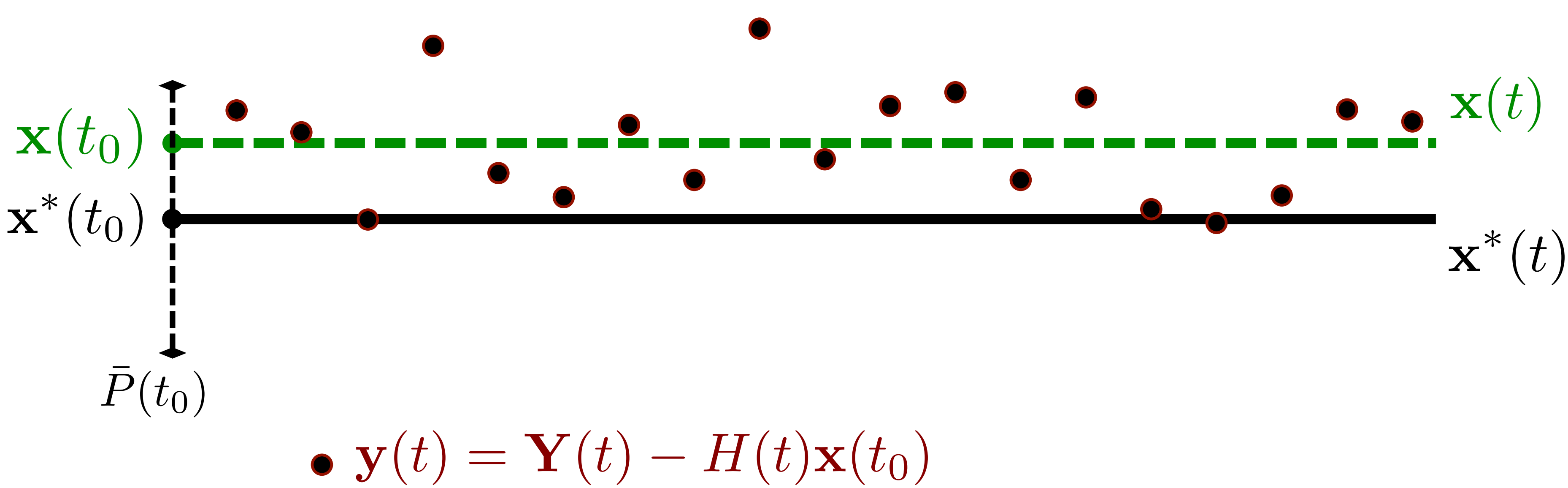
- Observations



# Stat OD Conceptualization

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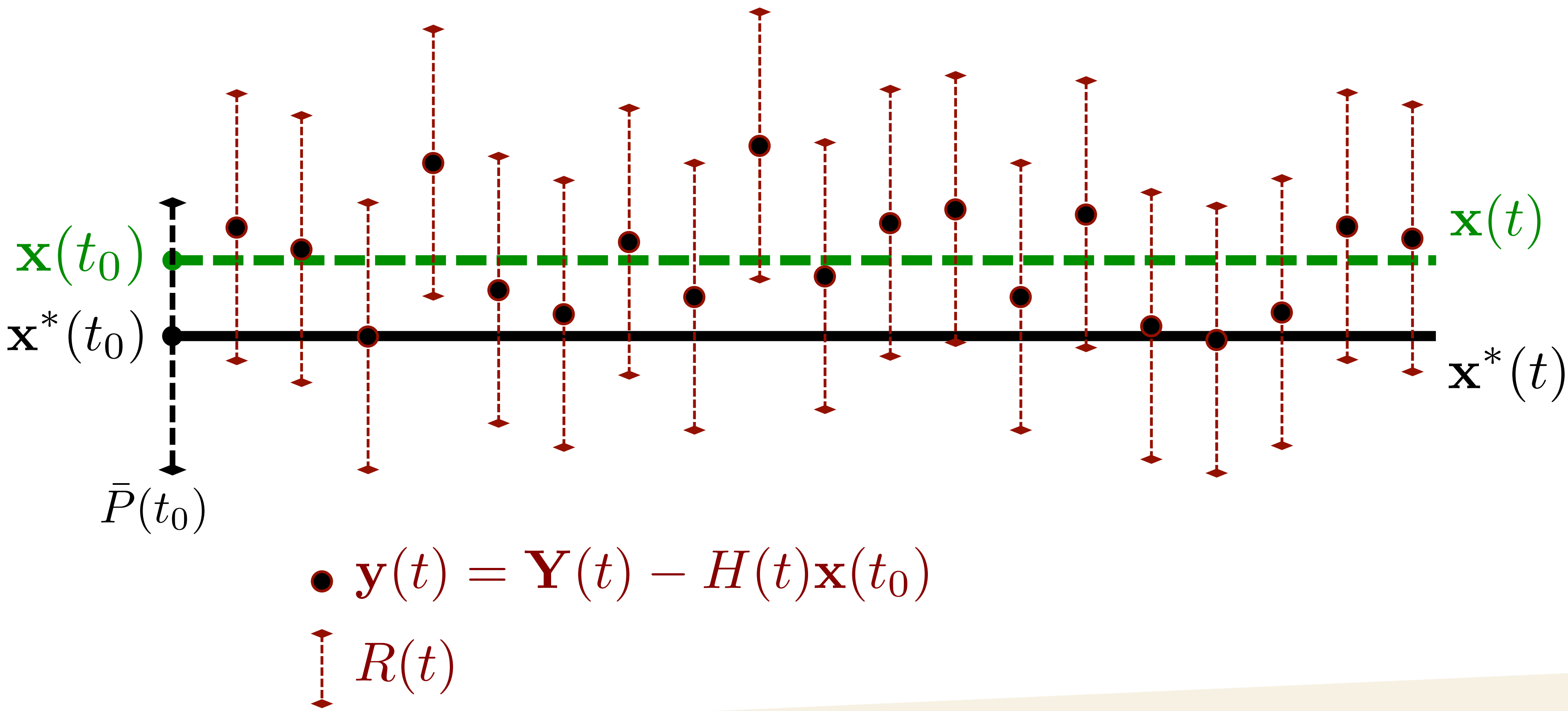
- Observation Uncertainties



# Stat OD Conceptualization

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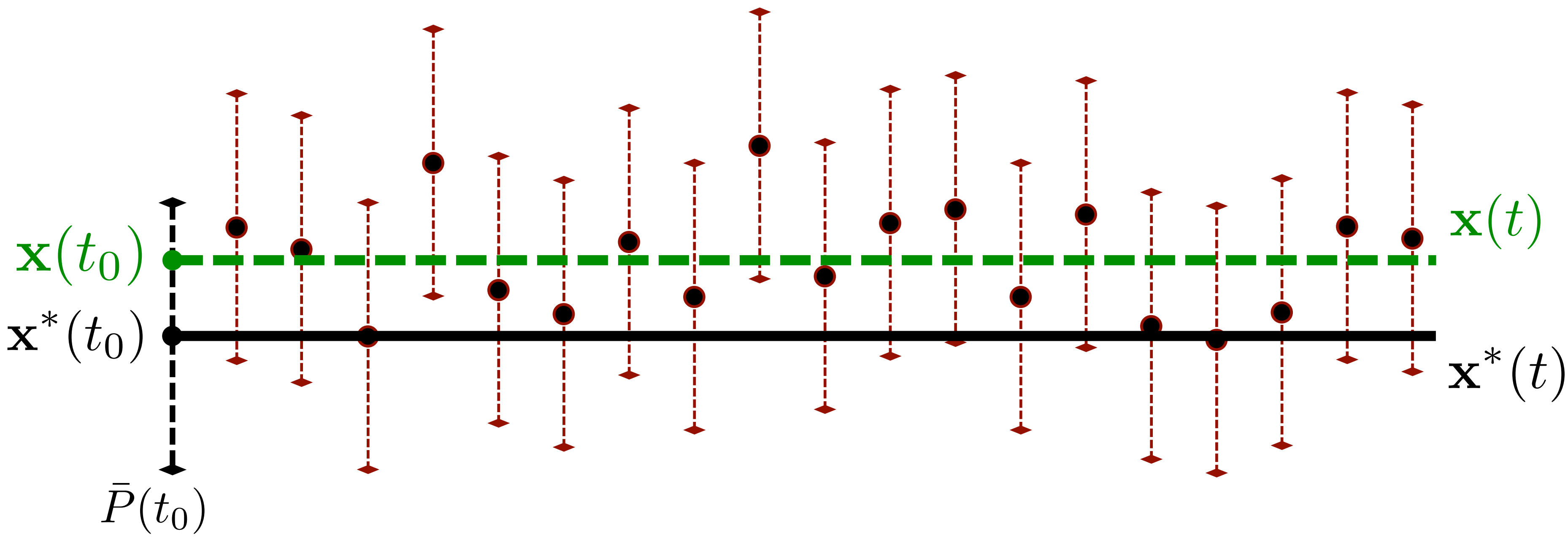
- Least Squares (Batch)




$$\hat{\mathbf{x}}(t_0) = \left(H^T R^{-1} H + \bar{P}^{-1}\right)^{-1} \left(H^T R^{-1} \mathbf{y} + \bar{P}^{-1} \bar{\mathbf{x}}\right)$$

# Stat OD Conceptualization

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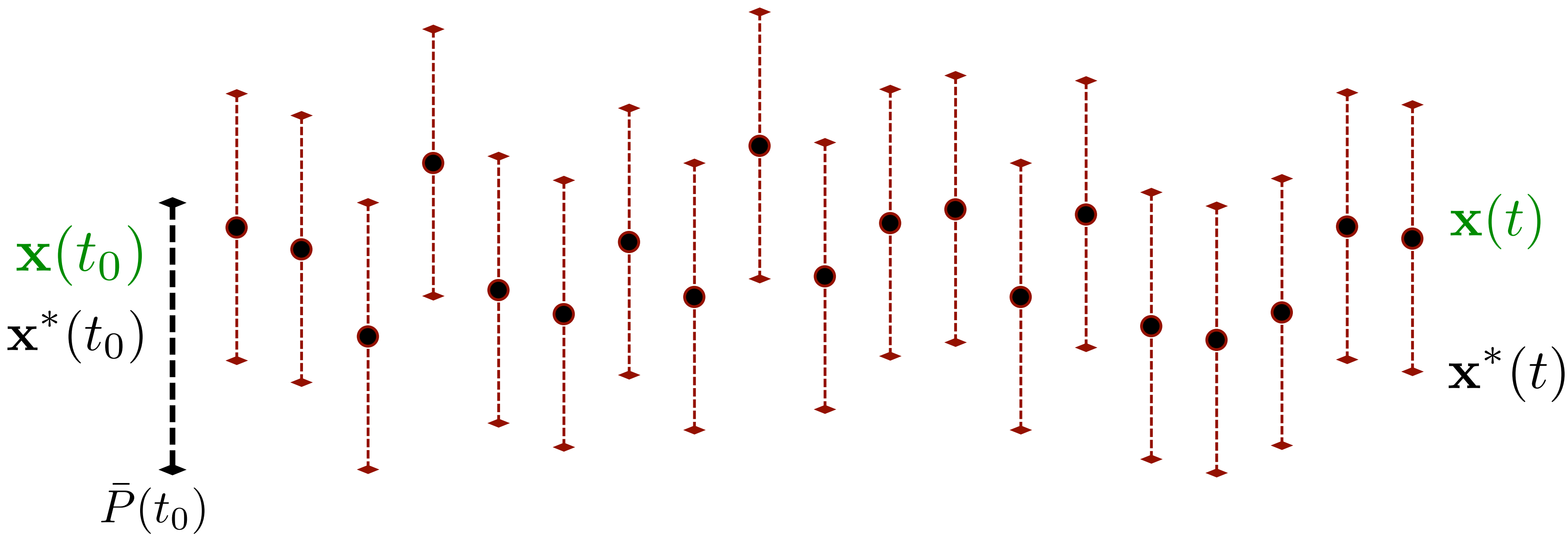
- Least Squares (Batch)

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# Stat OD Conceptualization



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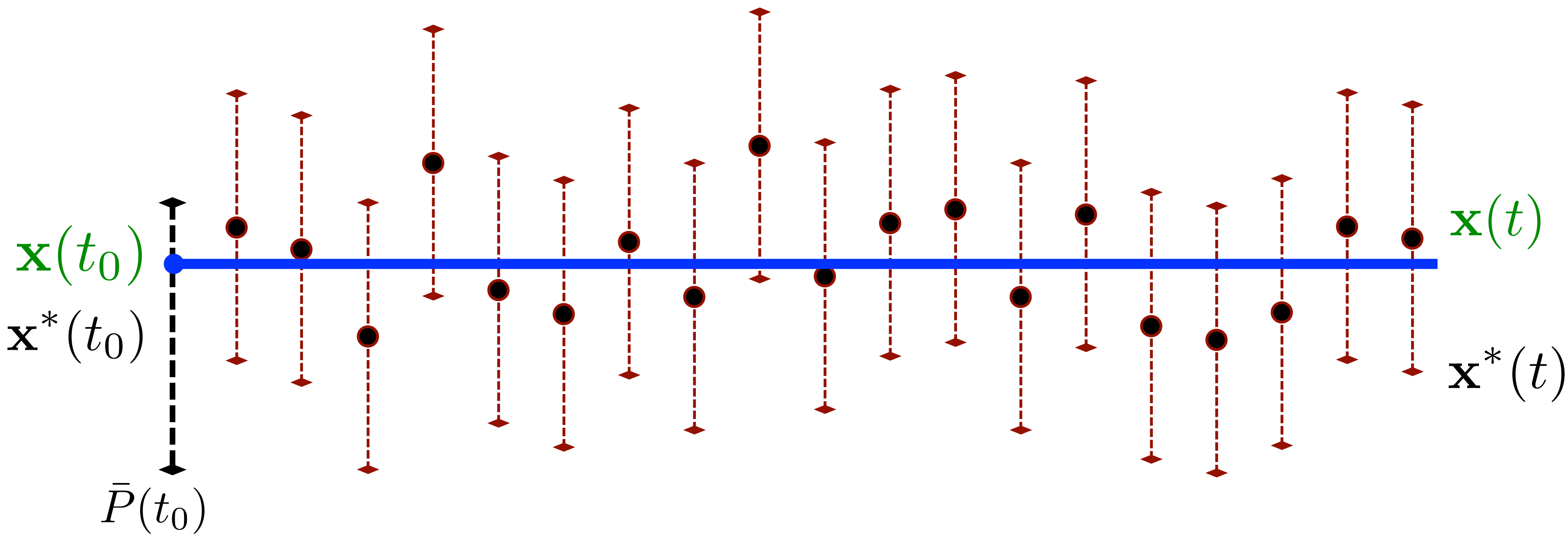


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# Stat OD Conceptualization

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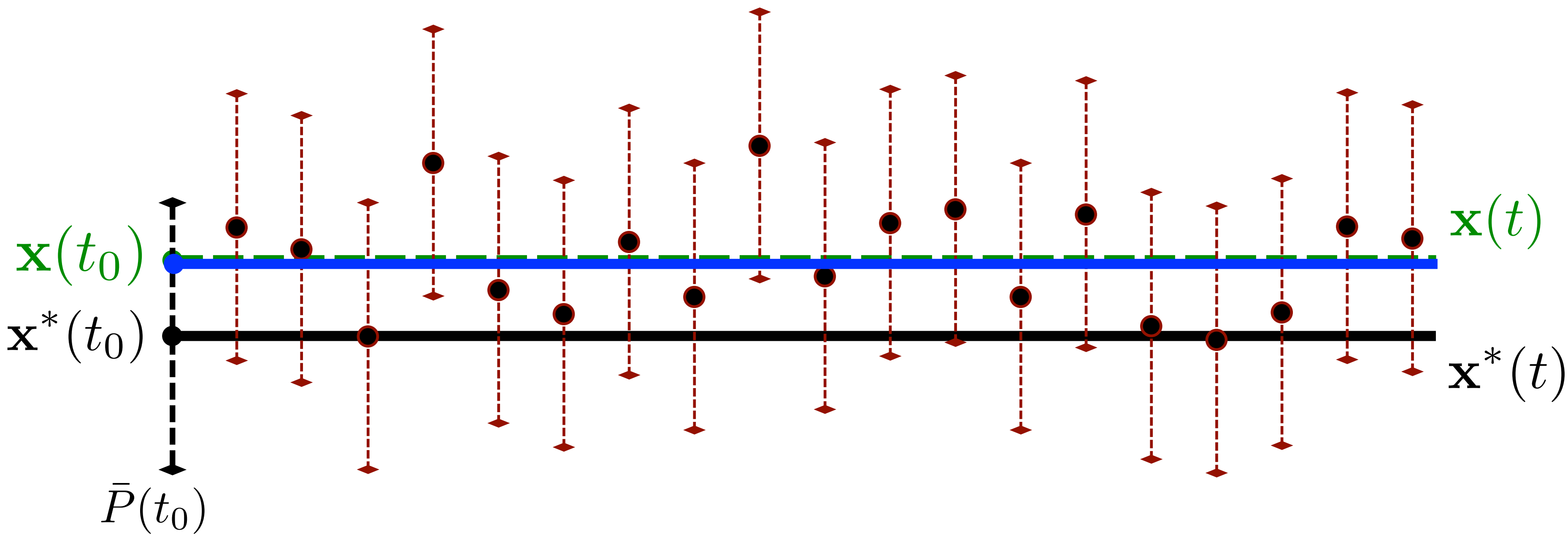


$$\hat{\mathbf{x}}(t_0) = \left(H^T R^{-1} H + \bar{P}^{-1}\right)^{-1} \left(H^T R^{-1} \mathbf{y} + \bar{P}^{-1} \bar{\mathbf{x}}\right)$$

# Stat OD Conceptualization



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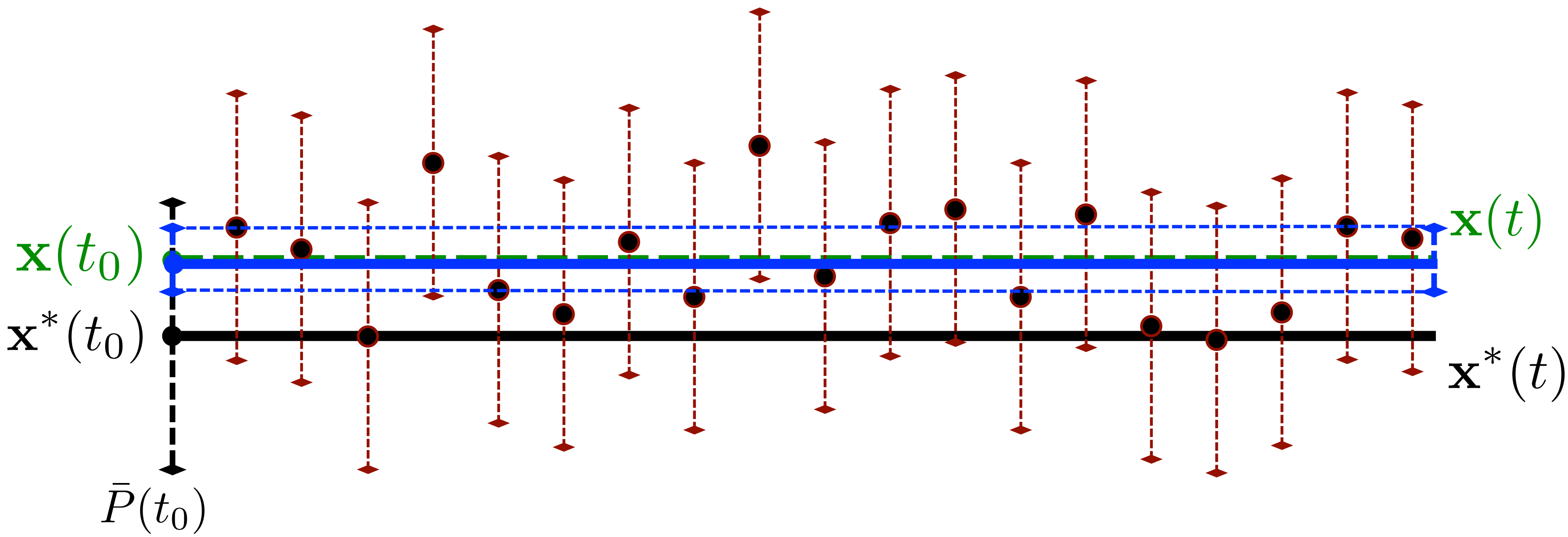


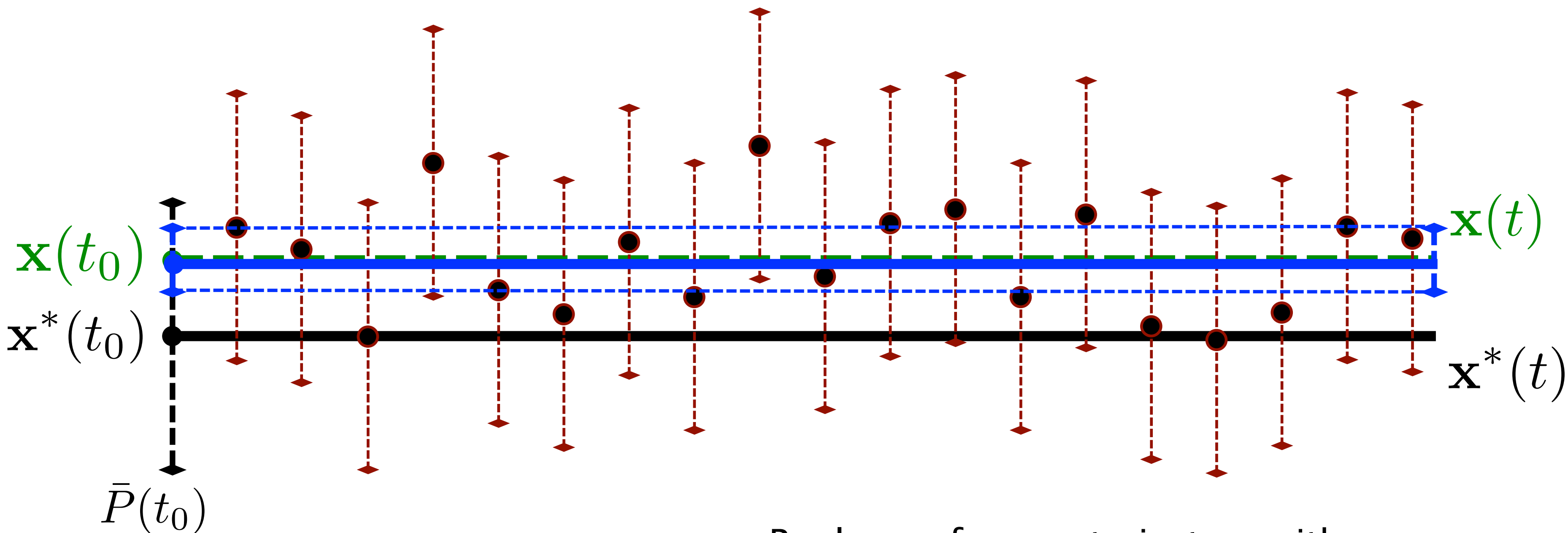

$$\hat{\mathbf{x}}(t_0) = \left(H^T R^{-1} H + \bar{P}^{-1}\right)^{-1} \left(H^T R^{-1} \mathbf{y} + \bar{P}^{-1} \bar{\mathbf{x}}\right)$$

$P(t)$

# Stat OD Conceptualization



- Least Squares (Batch)



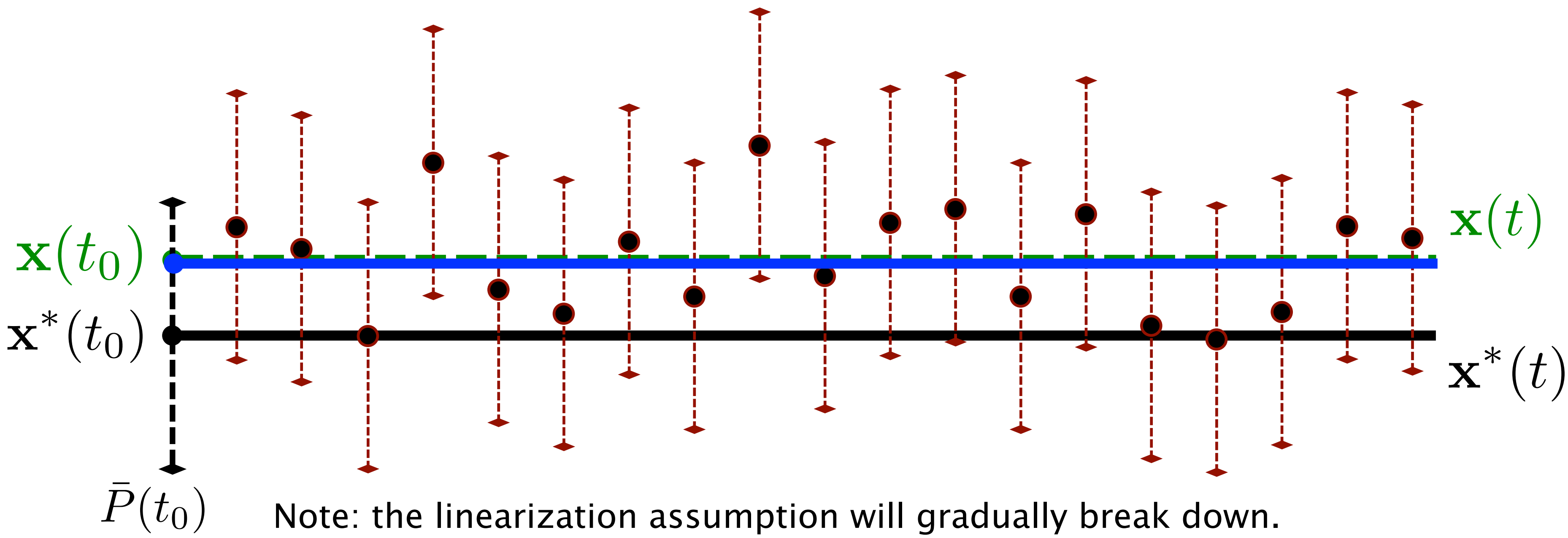

Iterate a few times.

- Replace reference trajectory with best-estimate
- Update *a priori* state
- Generate new computed observations

# Stat OD Conceptualization



- Least Squares (Batch)




Note: the linearization assumption will gradually break down.

Toward the end, the truth will begin to deviate further from the nominal.

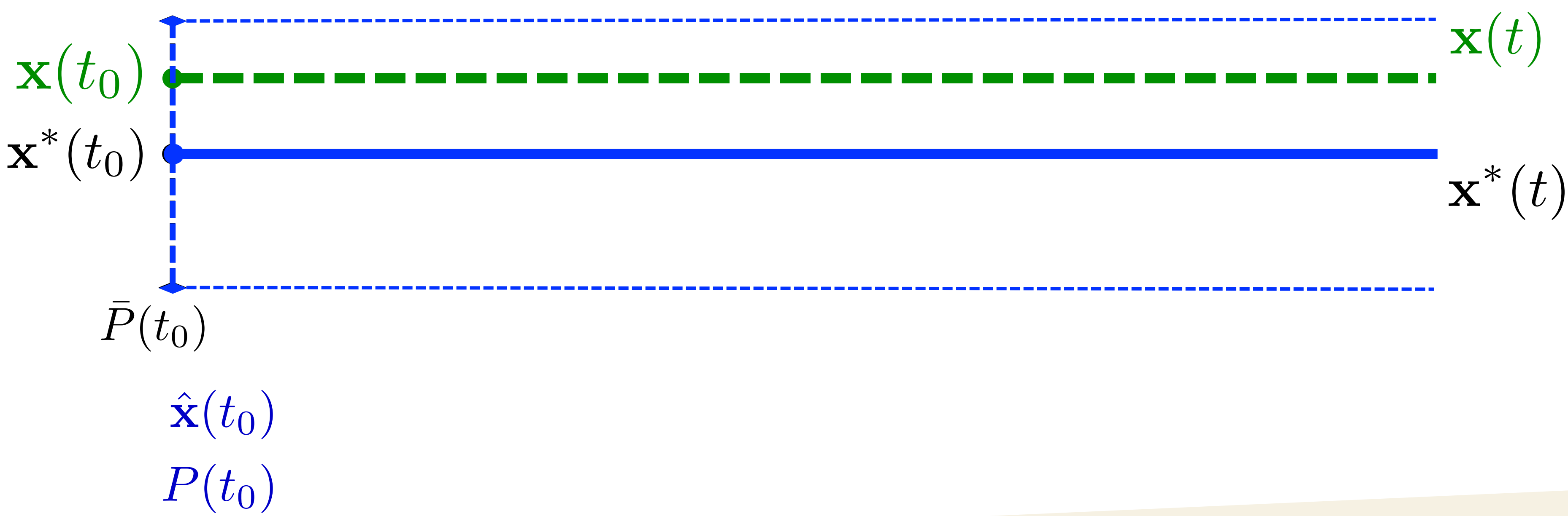
Select a measurement interval that balances the number of observations with the length of time being used.

- Conceptualization of the Conventional Kalman Filter (Sequential Filter)

# Stat OD Conceptualization



- Conventional Kalman



# Stat OD Conceptualization

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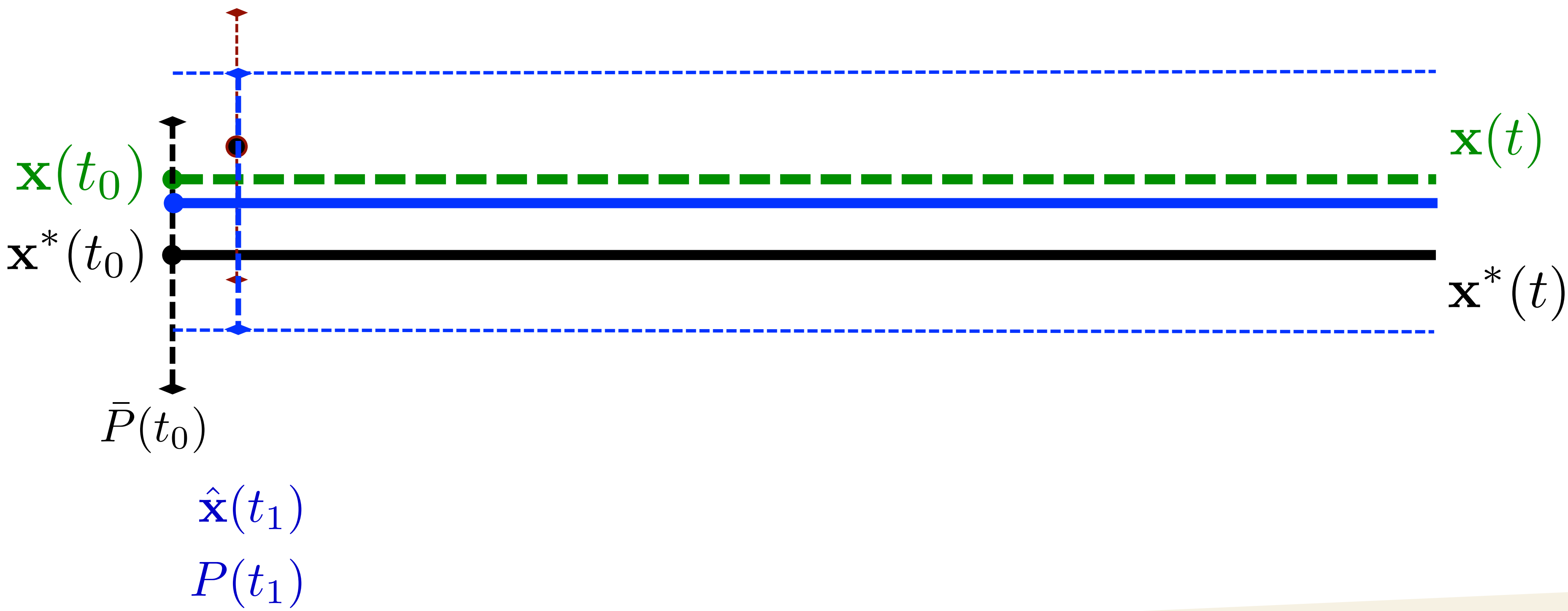

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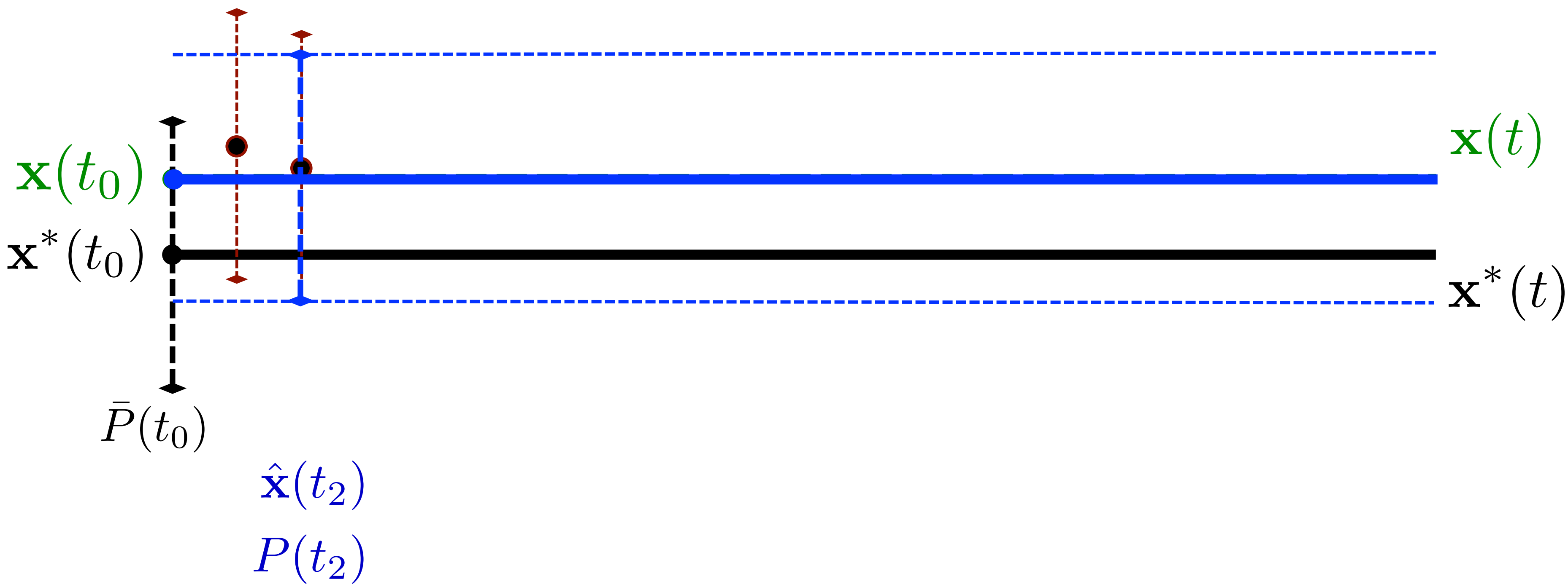

# Stat OD Conceptualization



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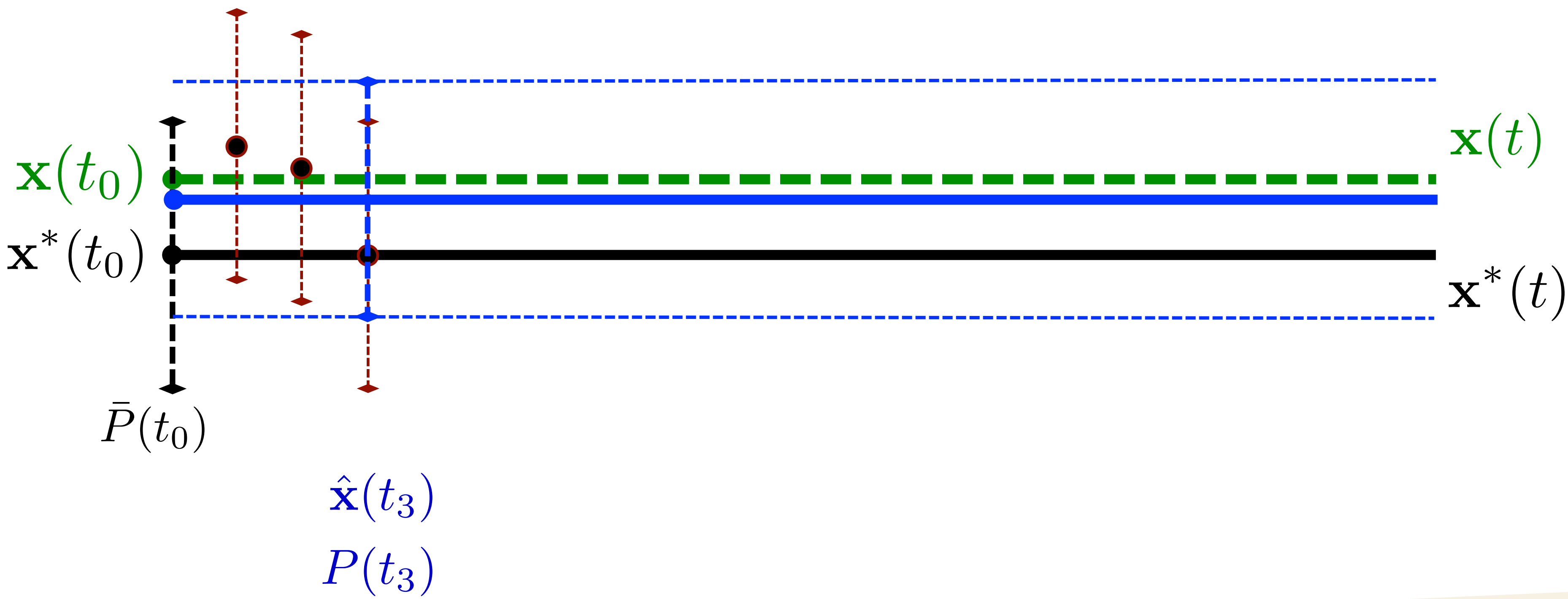

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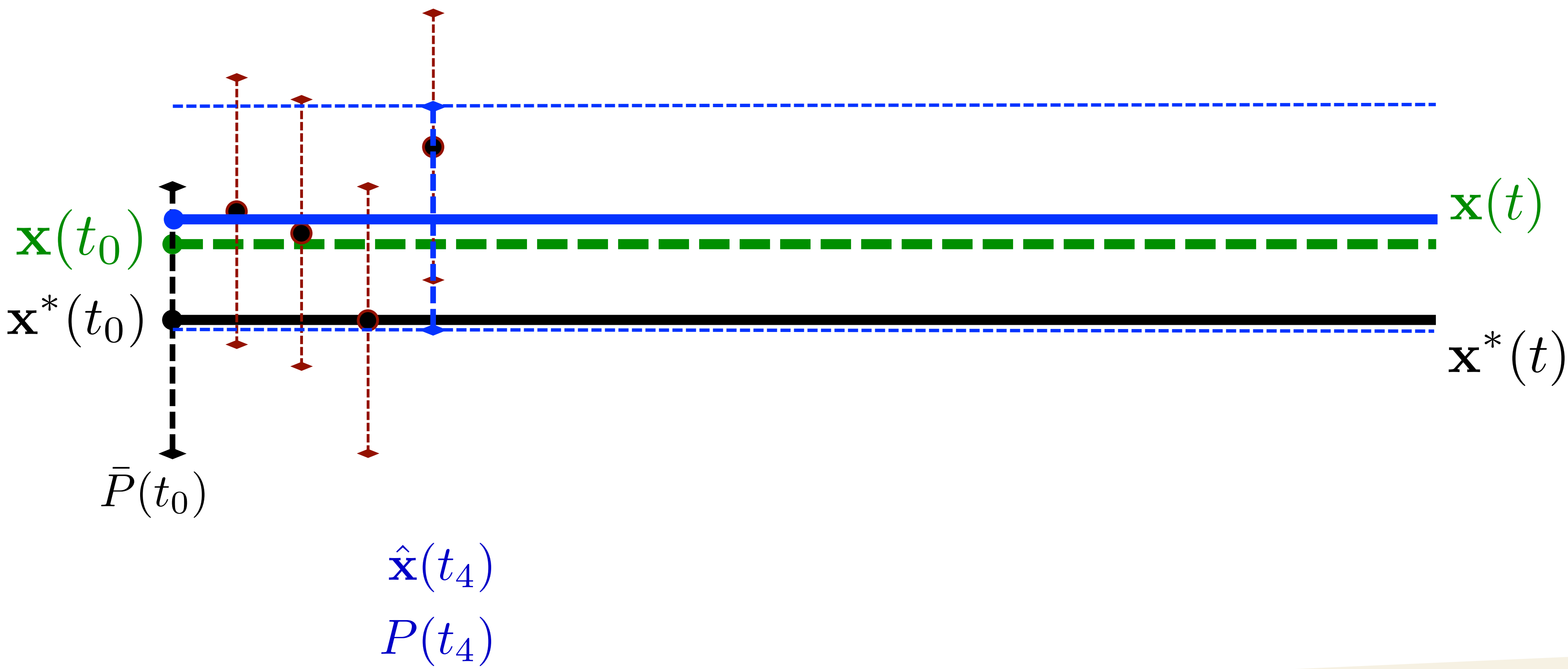
- Conventional Kalman



# Stat OD Conceptualization



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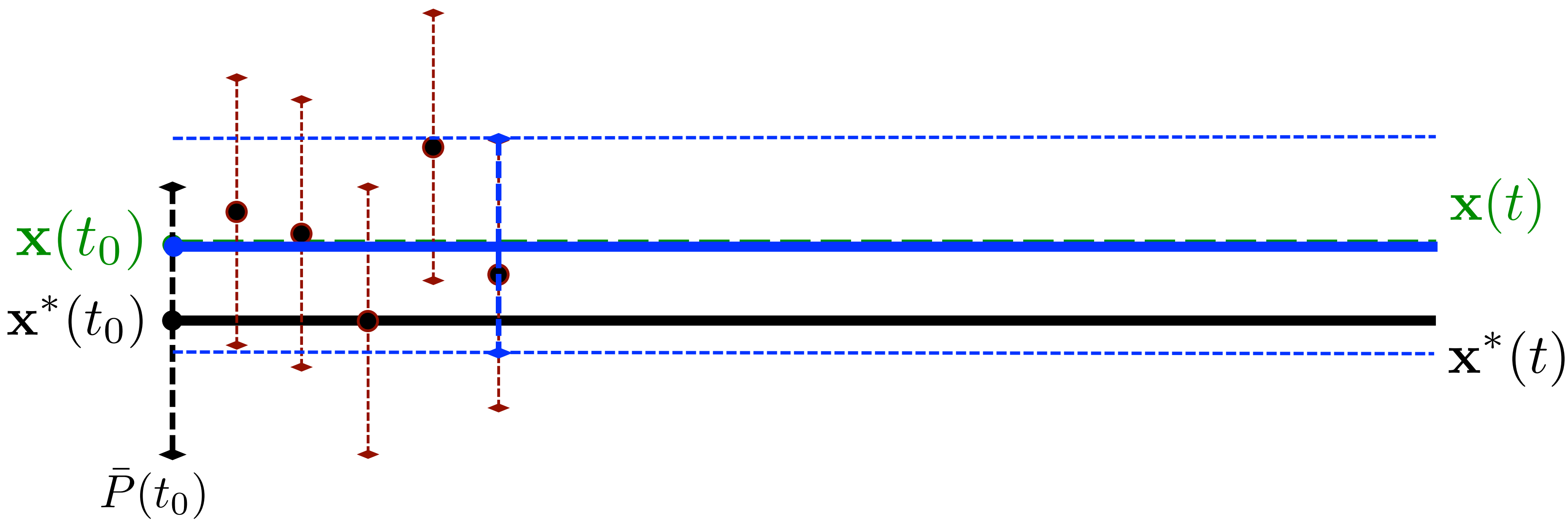

# Stat OD Conceptualization



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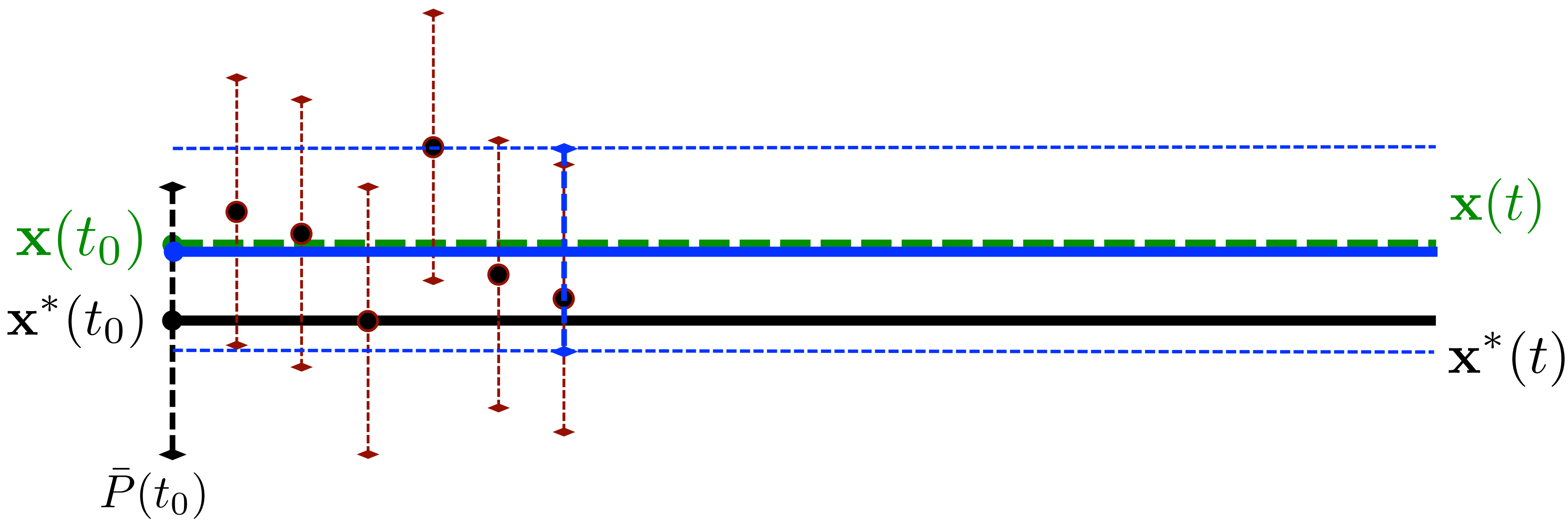

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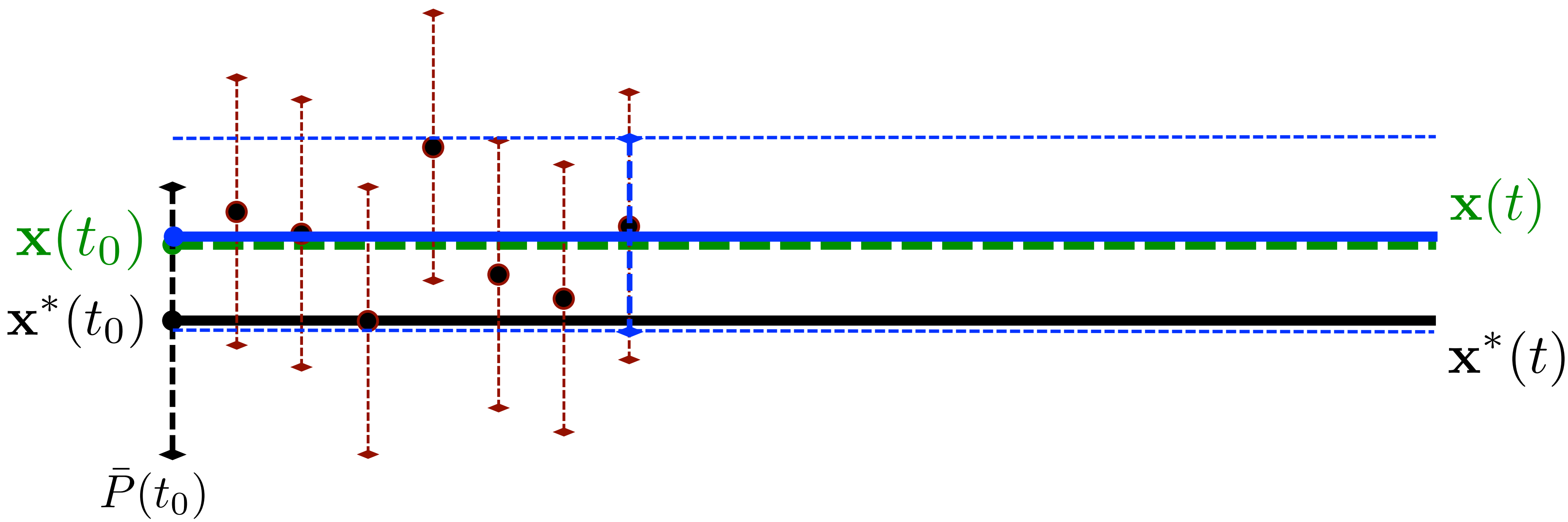

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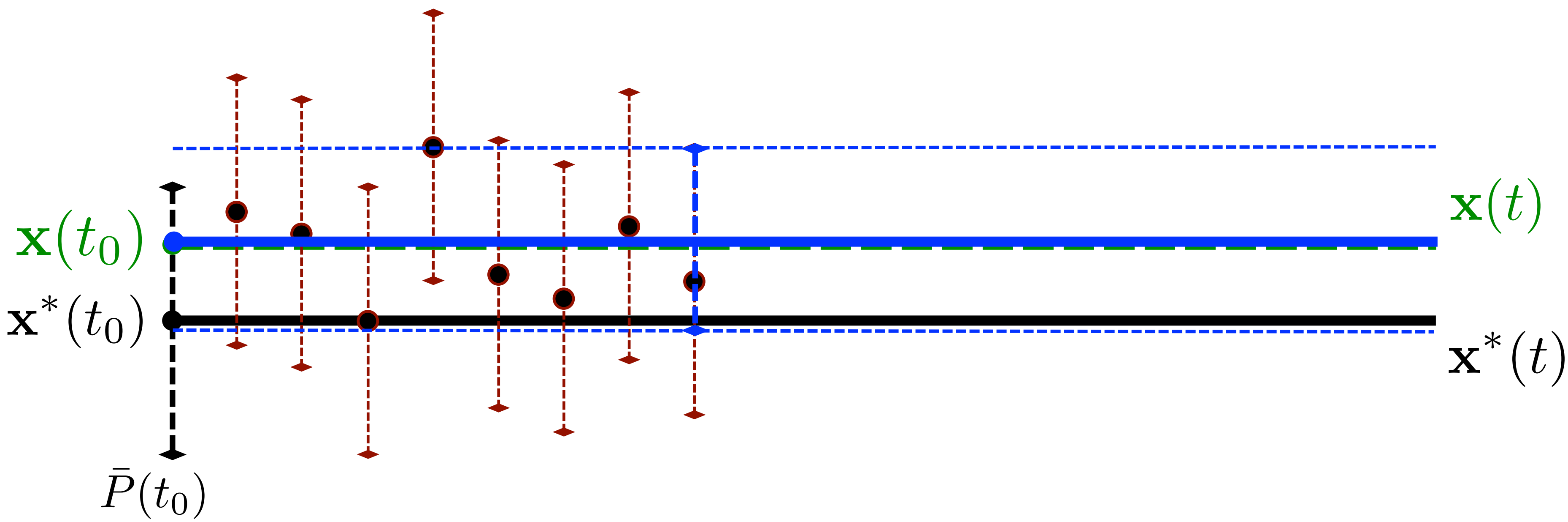

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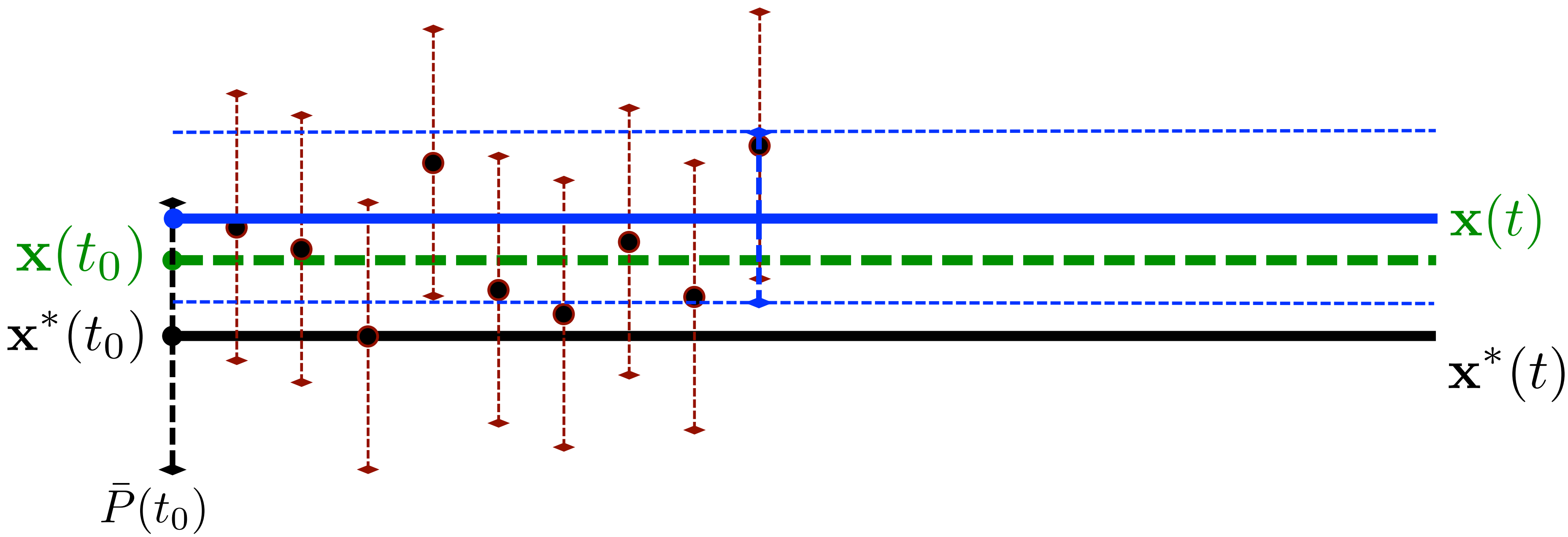

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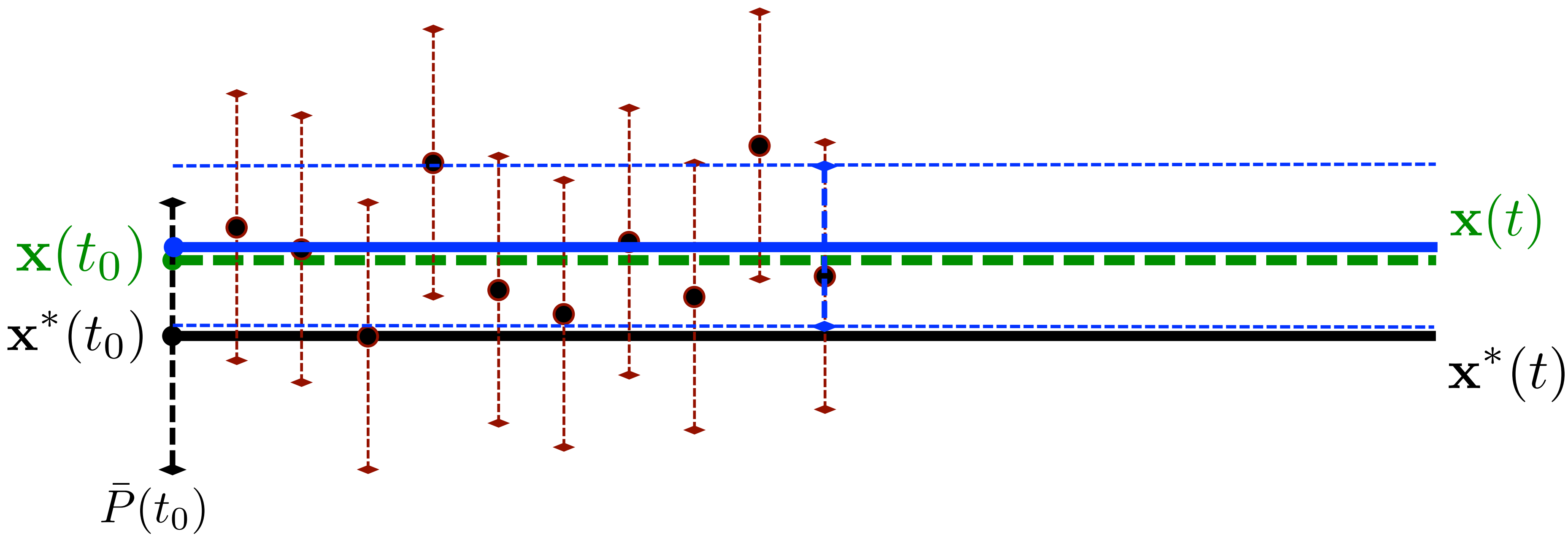

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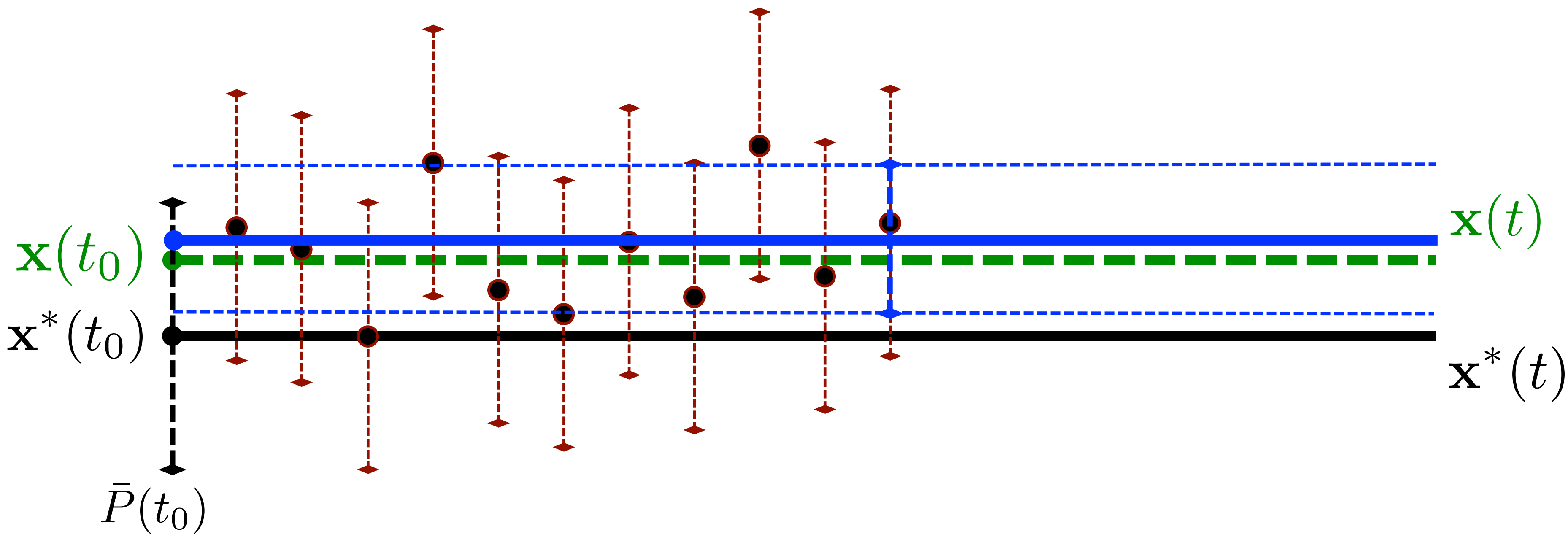

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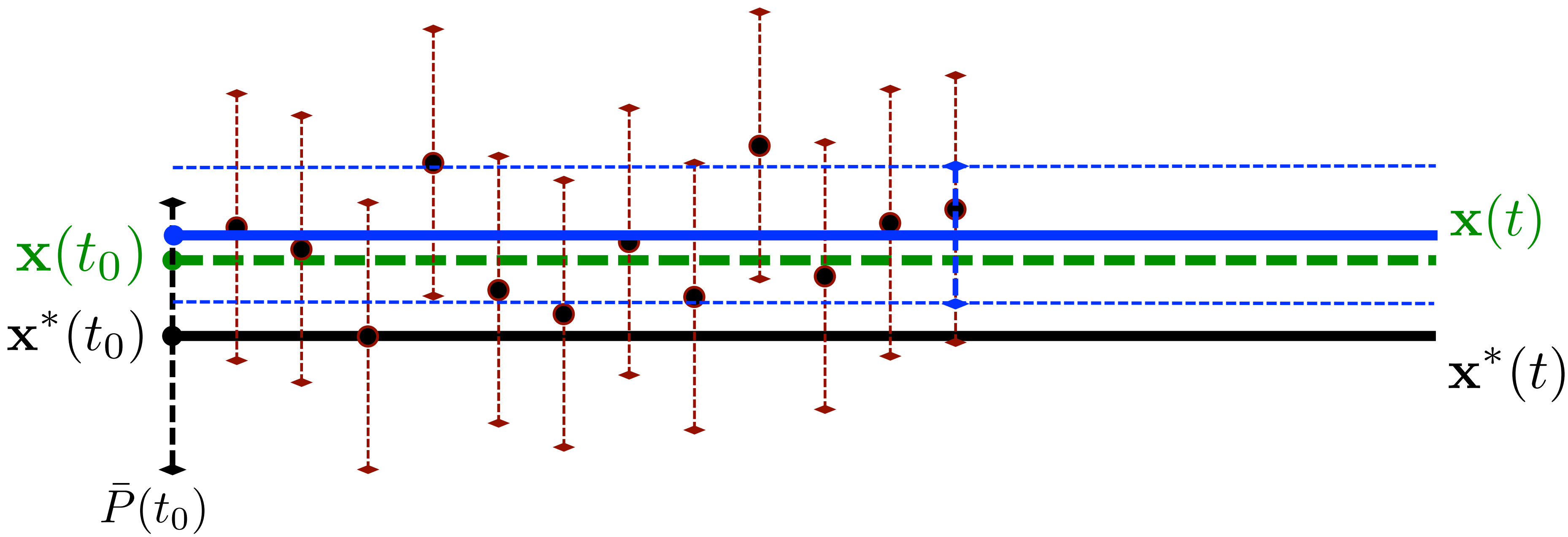

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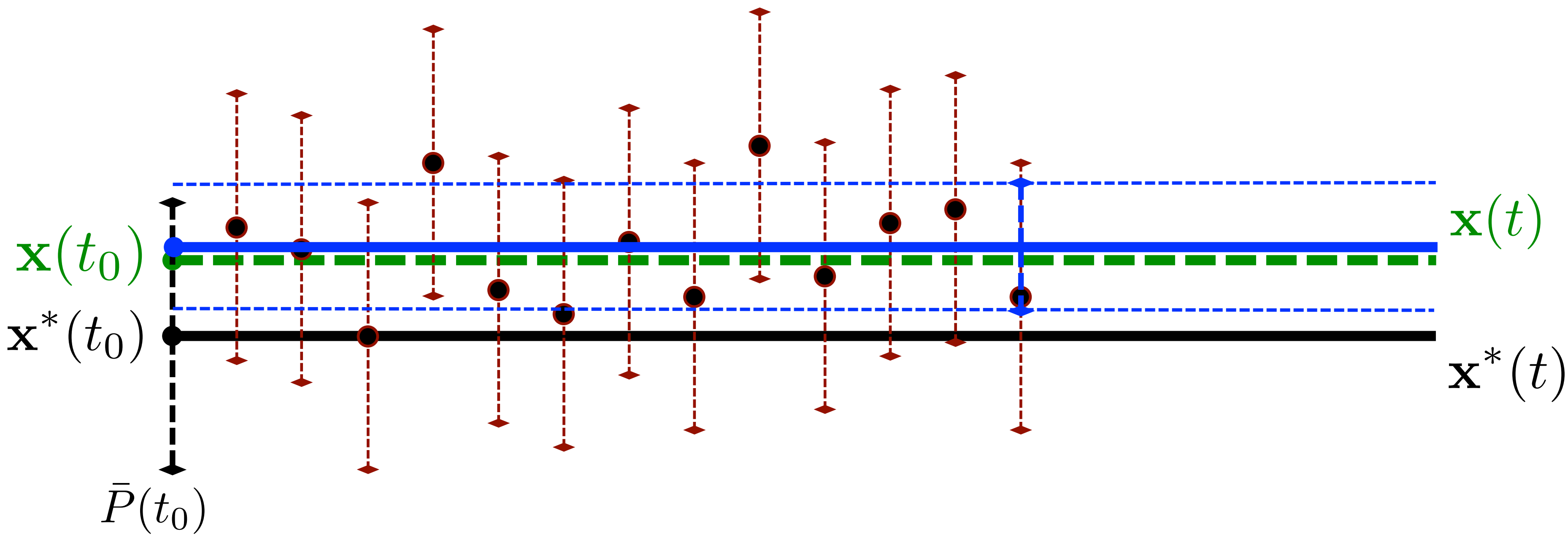

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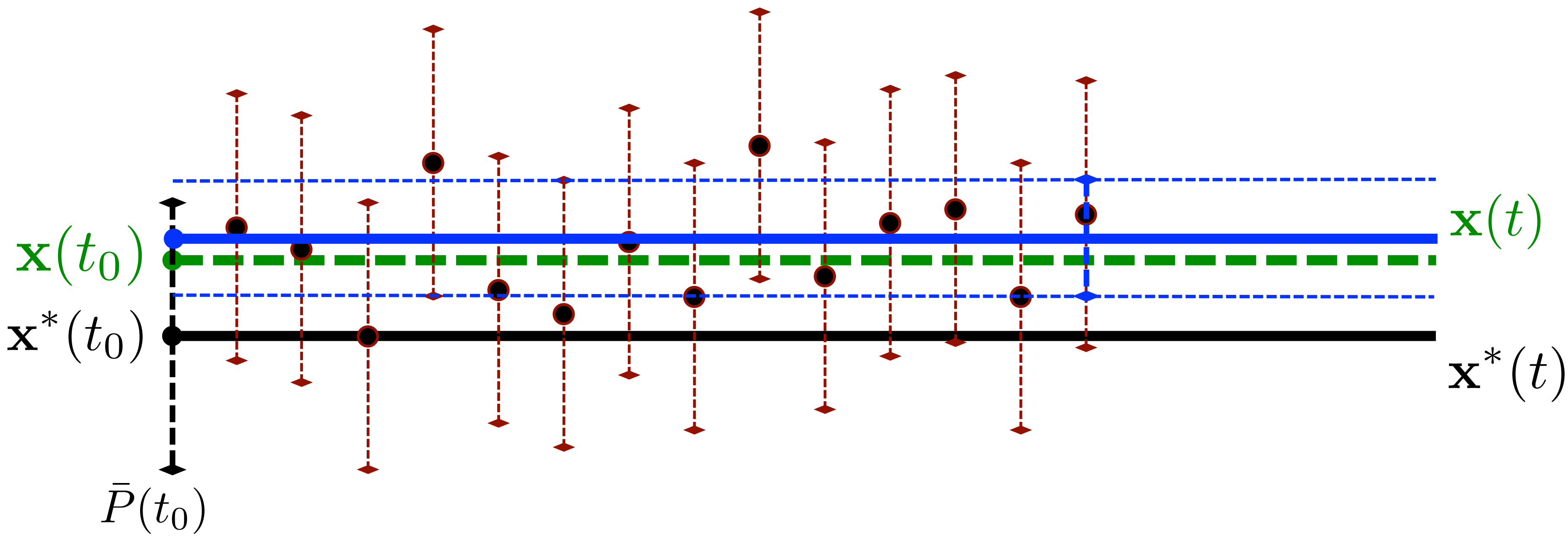

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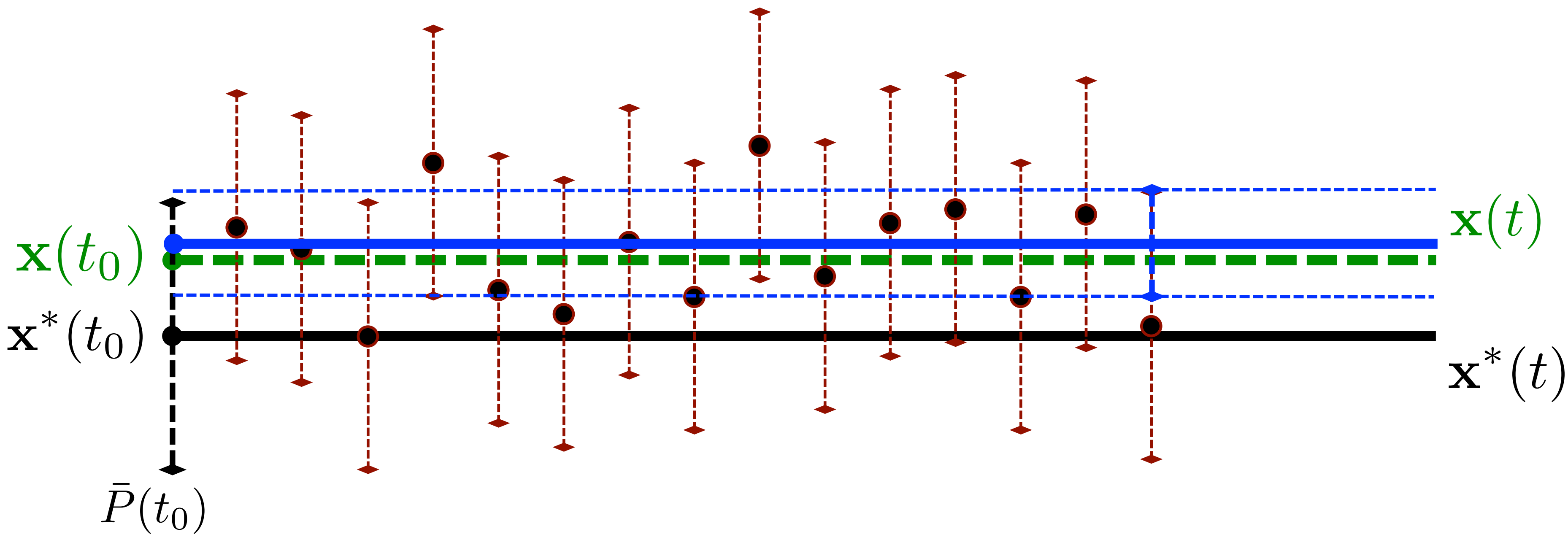

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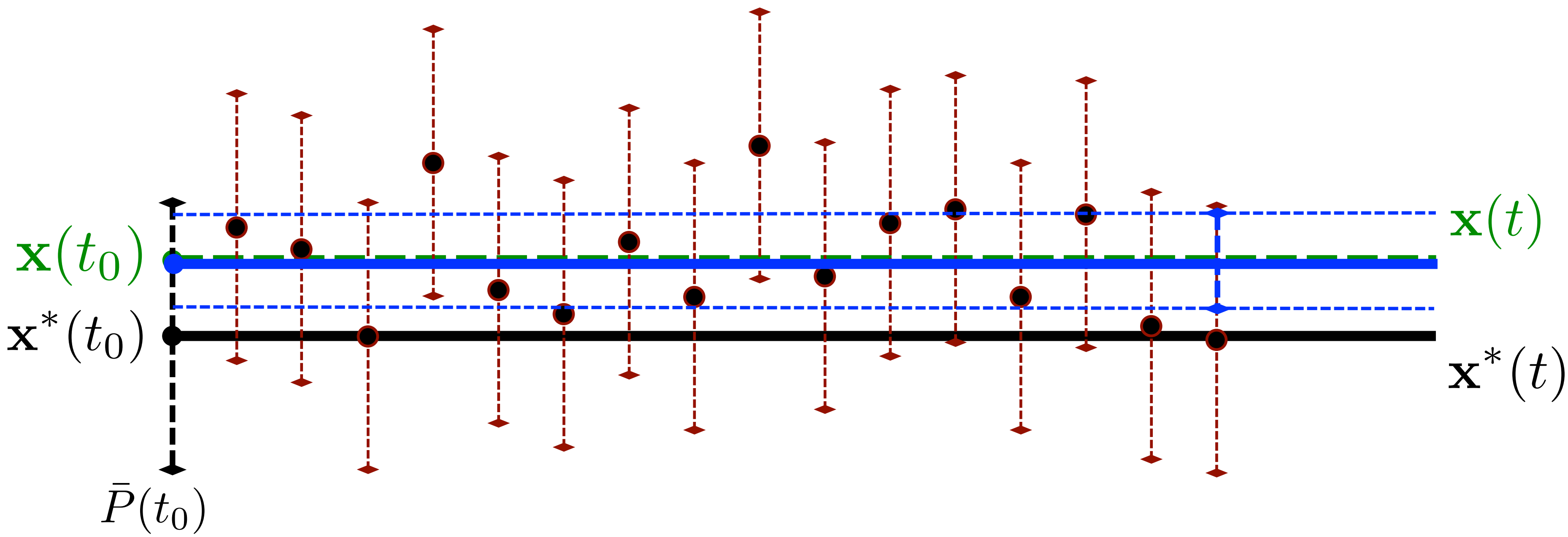

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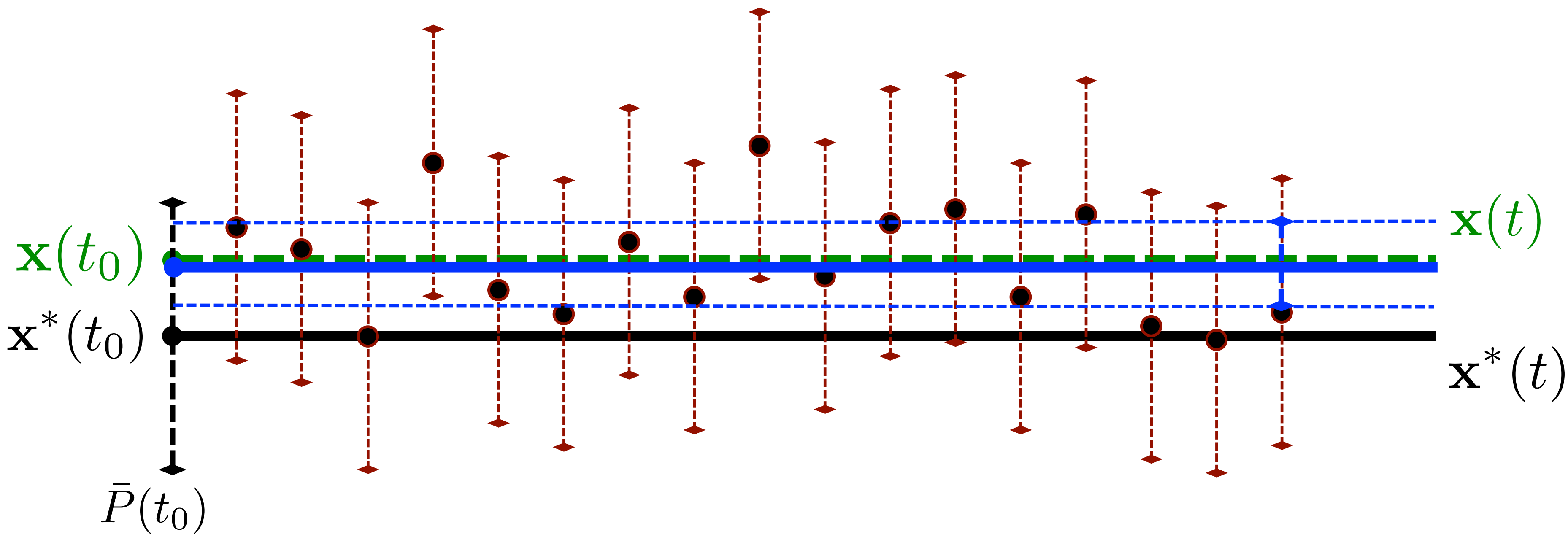

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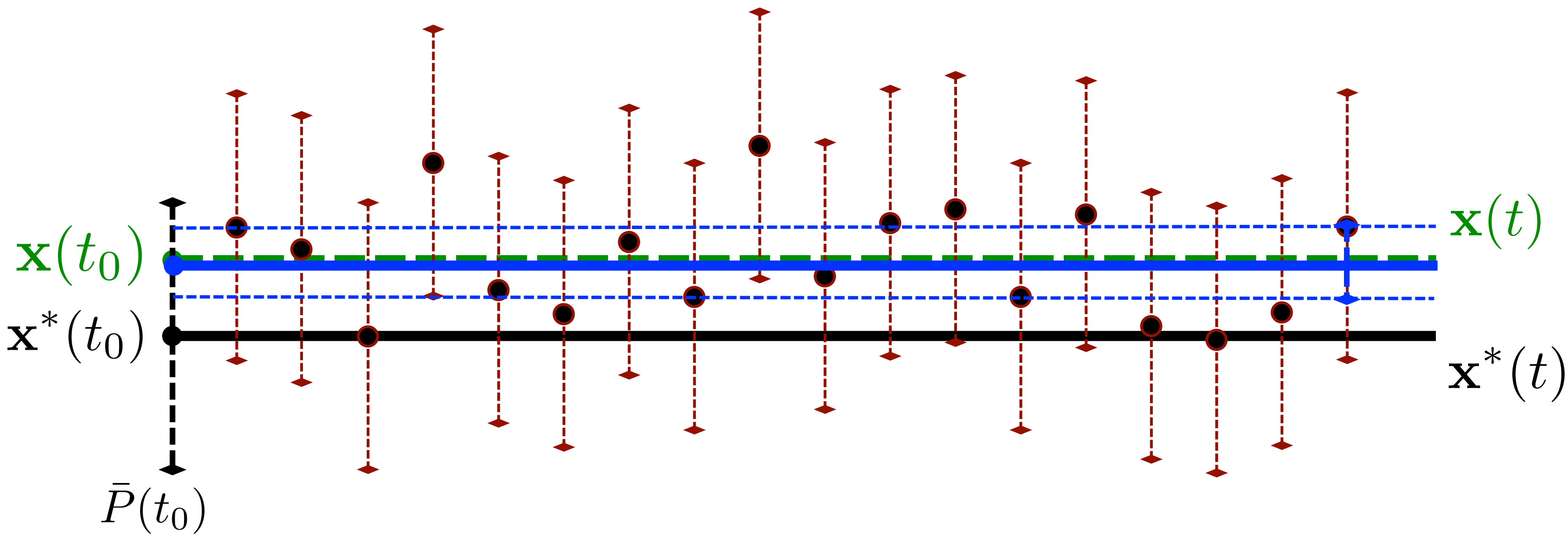
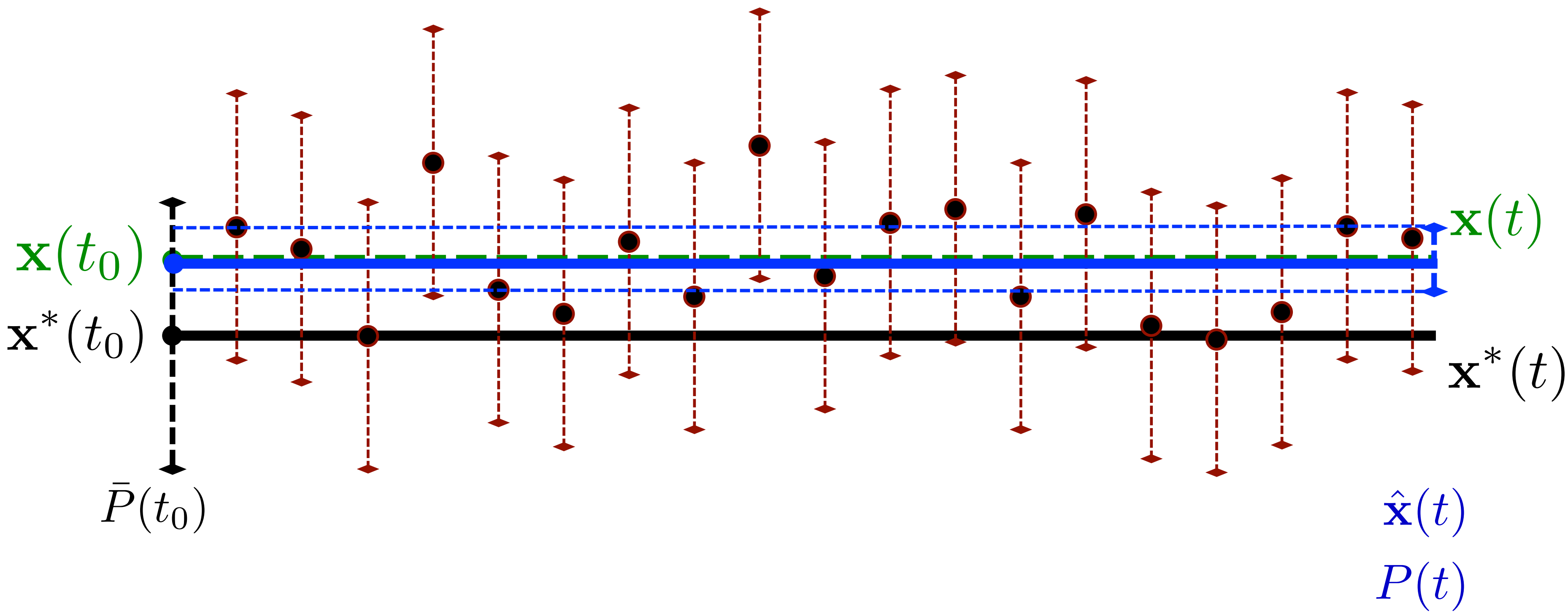
- Conventional Kalman



- Conventional Kalman



# Stat OD Conceptualization

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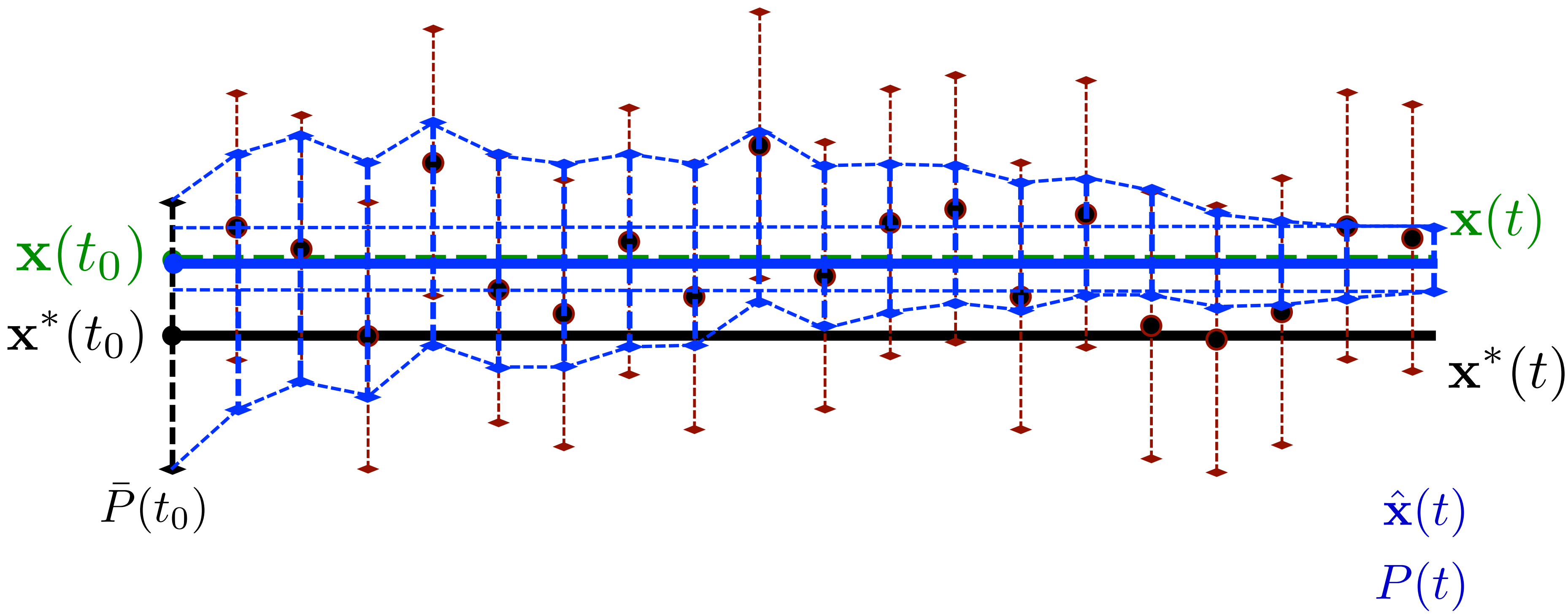
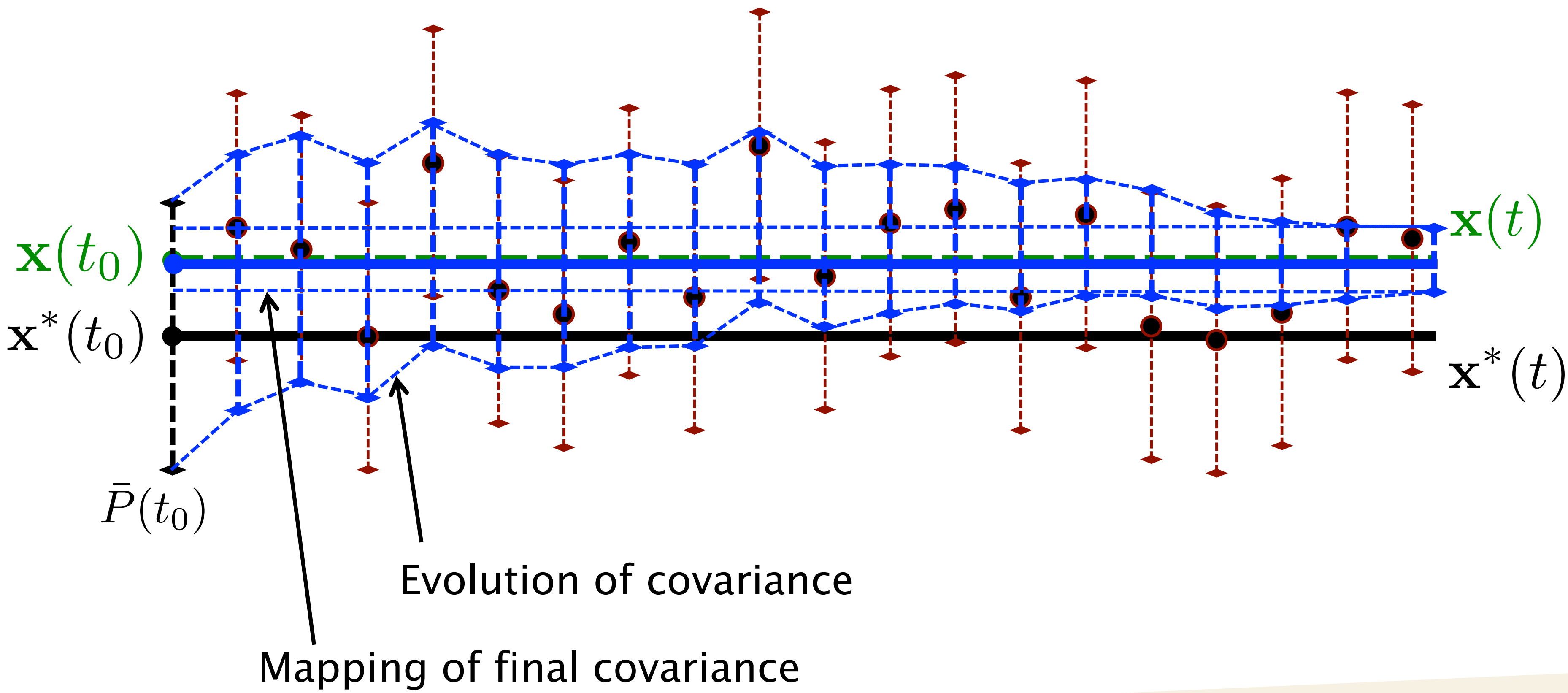
- Conventional Kalman



# Stat OD Conceptualization



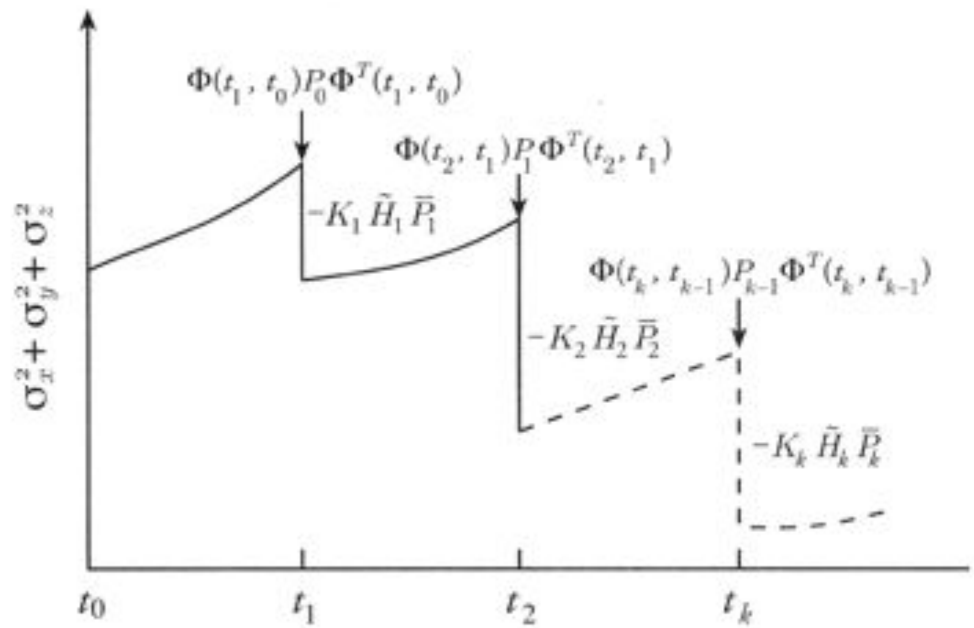
- Conventional Kalman



# Evolution of Covariance Matrix



- Evolution of the covariance matrix as observations are processed.

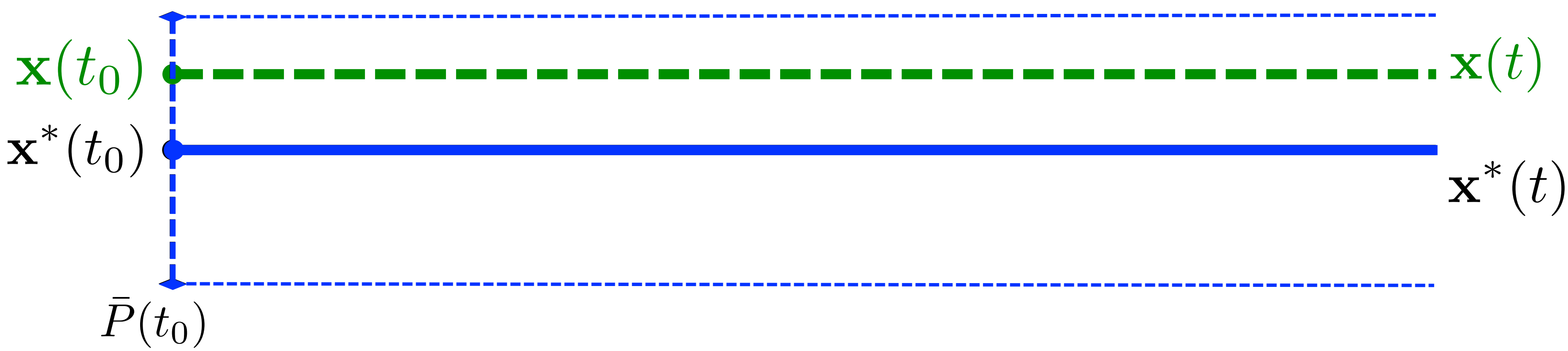


- Conceptualization of the Extended Kalman Filter (EKF)
- Major change: the reference trajectory is updated by the best estimate after every measurement.

# Stat OD Conceptualization



- EKF



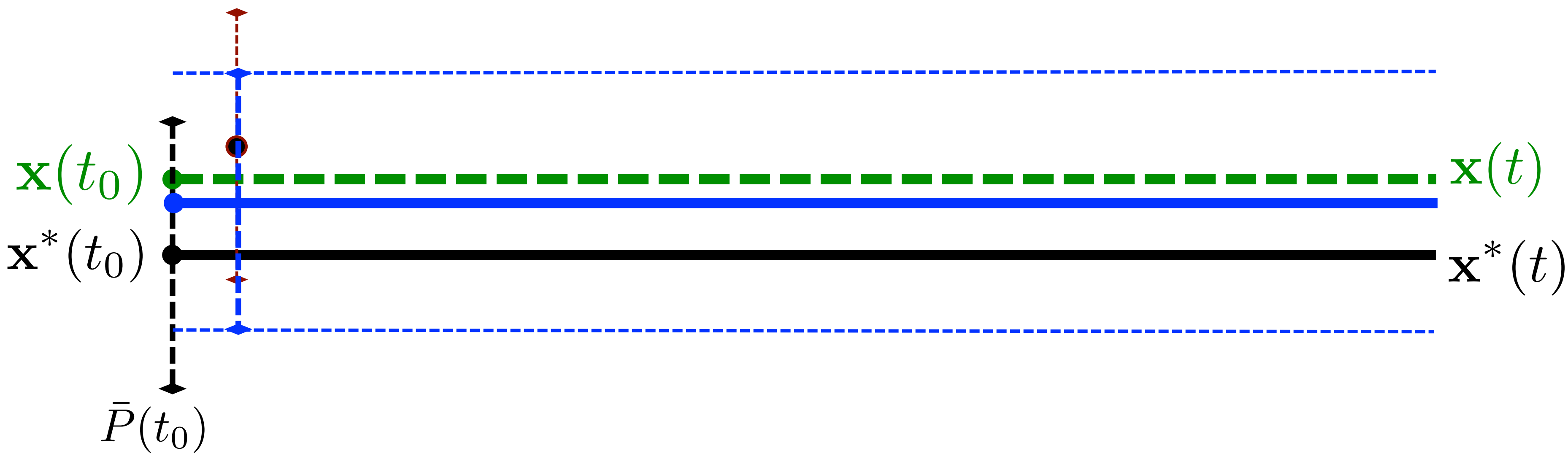

$\hat{\mathbf{x}}(t_0)$

$P(t_0)$

# Stat OD Conceptualization



- EKF



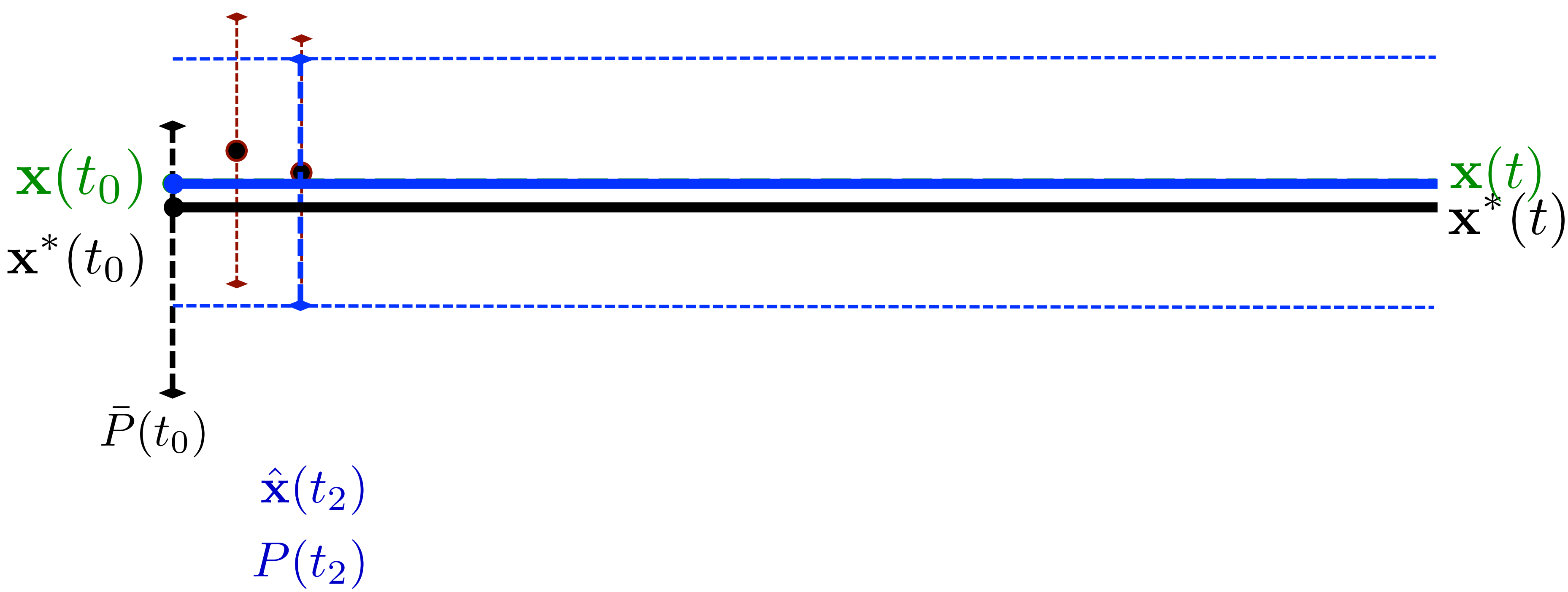

$\hat{\mathbf{x}}(t_1)$

$P(t_1)$

# Stat OD Conceptualization



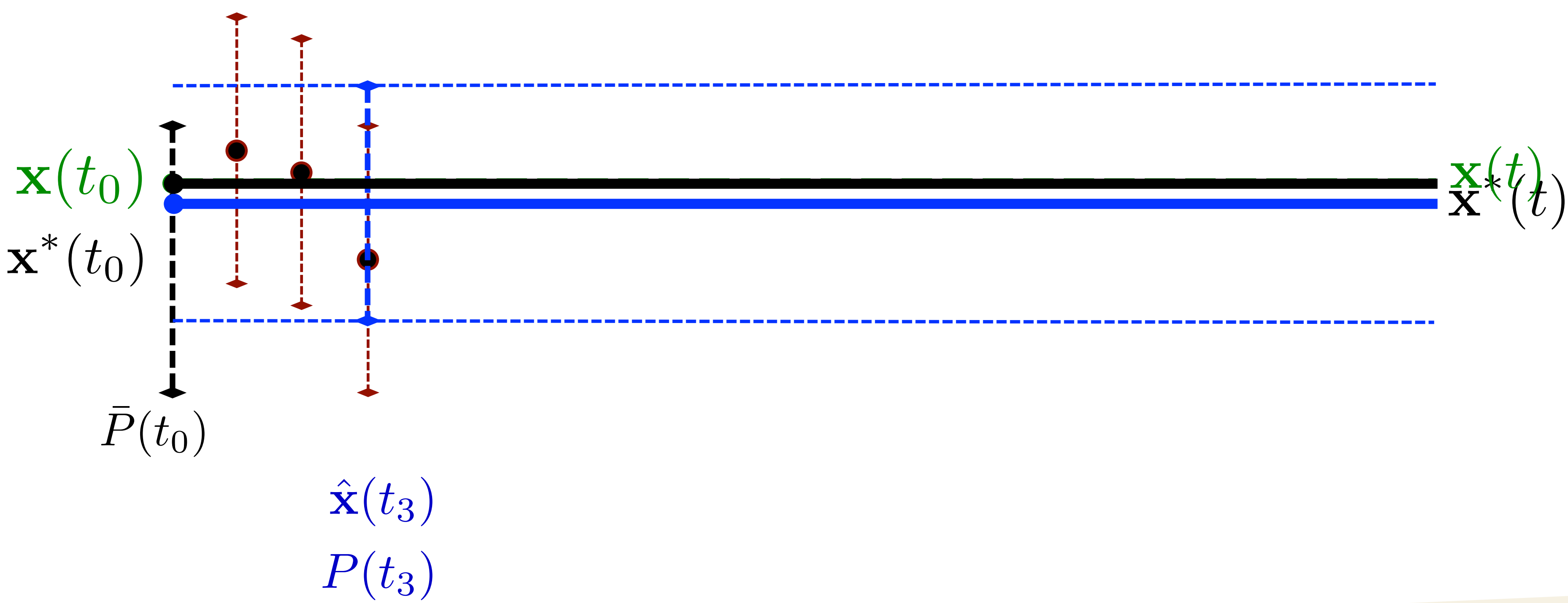
- EKF



# Stat OD Conceptualization



- EKF



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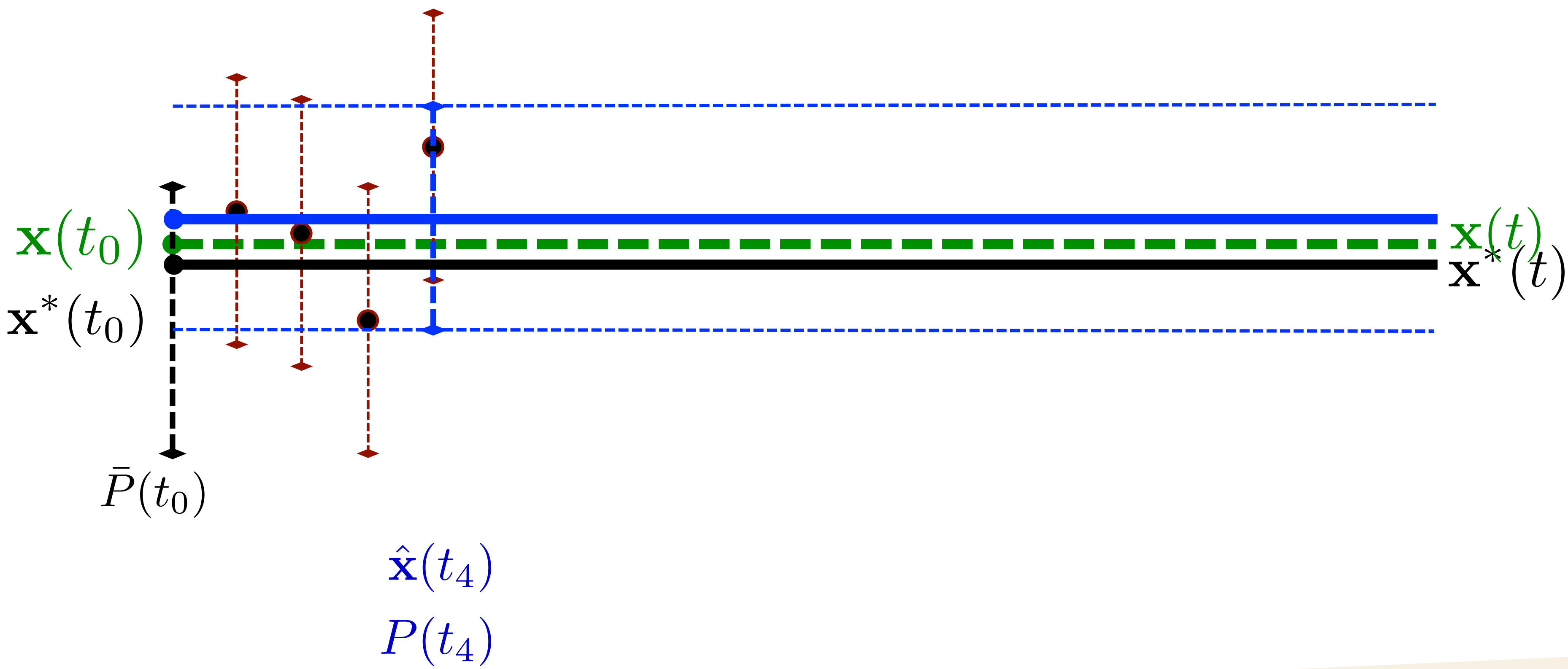
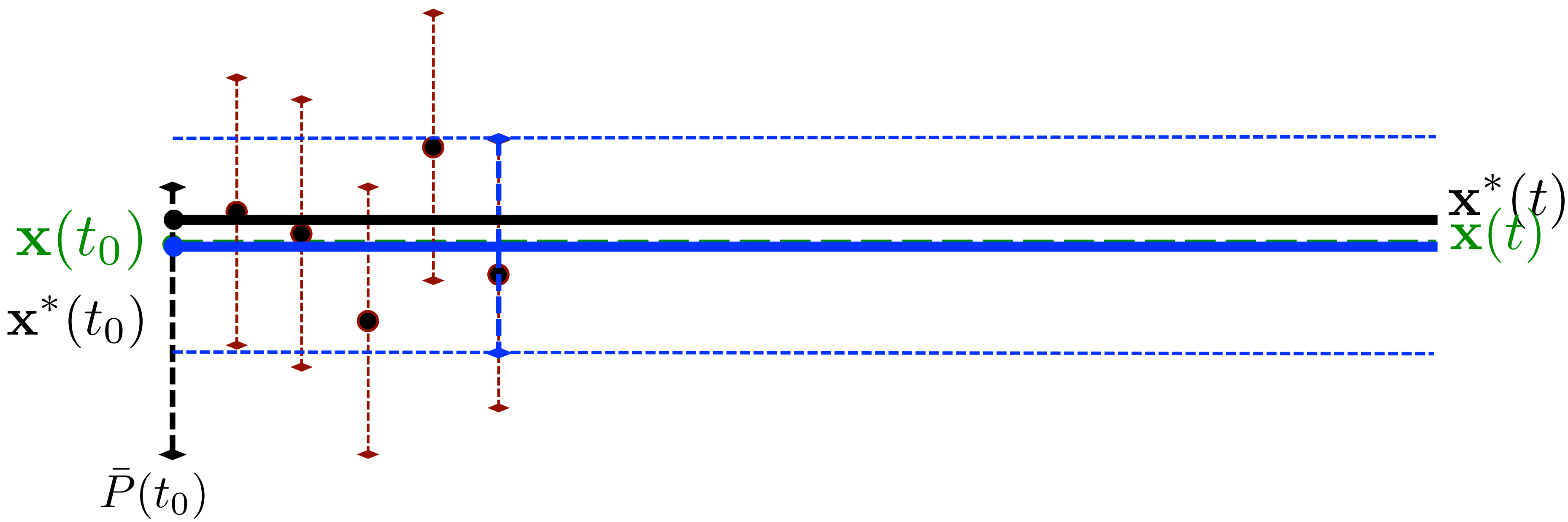
- EKF



# Stat OD Conceptualization



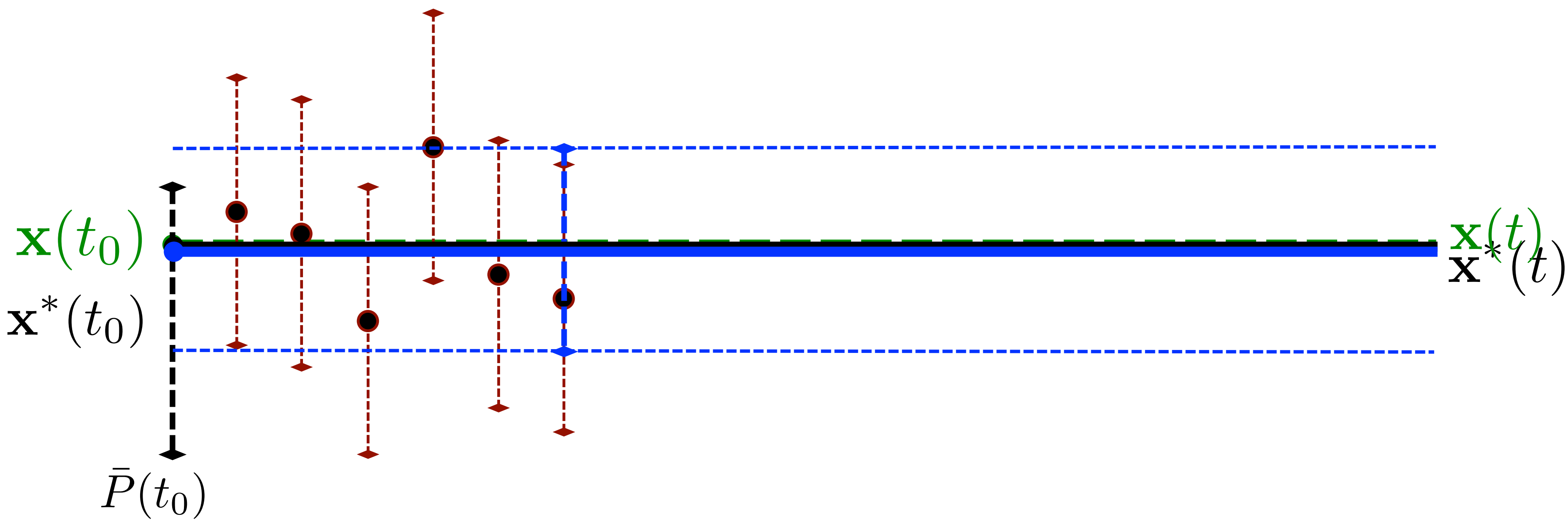
- EKF



# Stat OD Conceptualization



- EKF



# Stat OD Conceptualization

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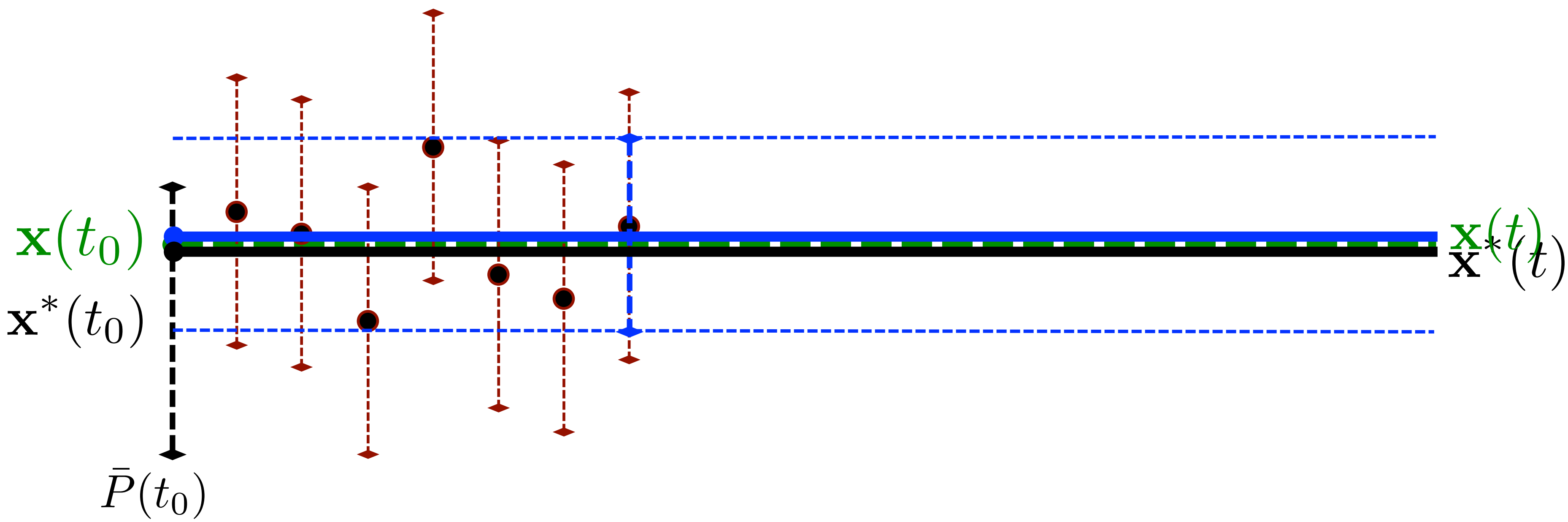
- EKF



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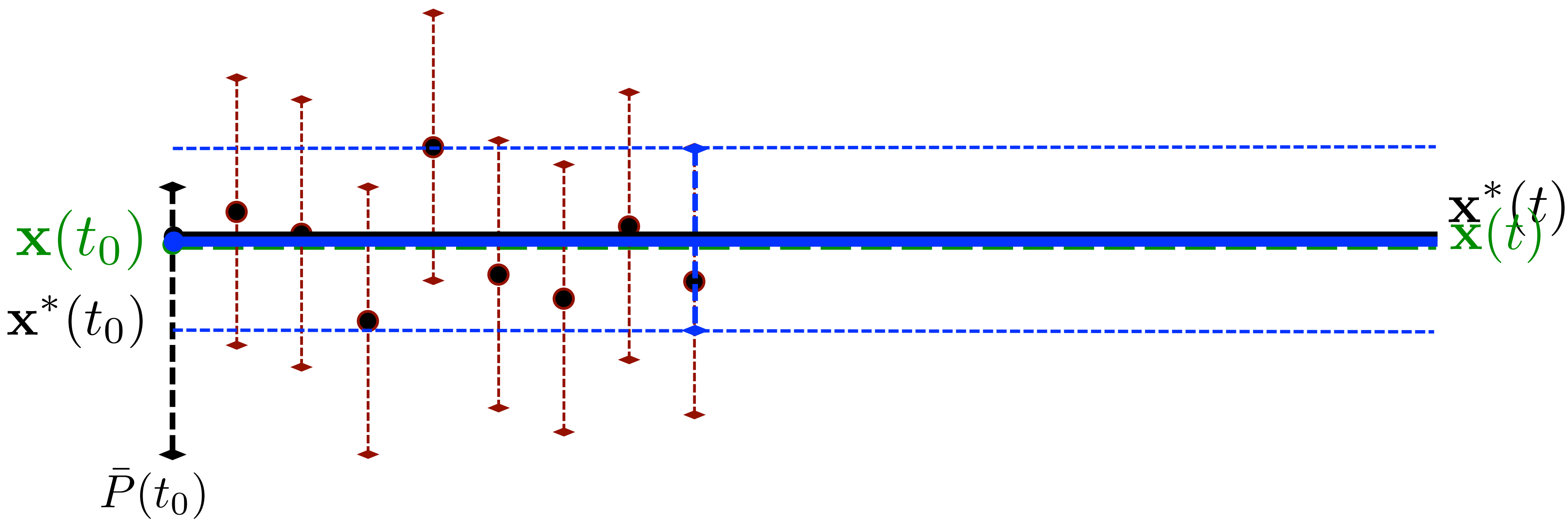
- EKF



# Stat OD Conceptualization

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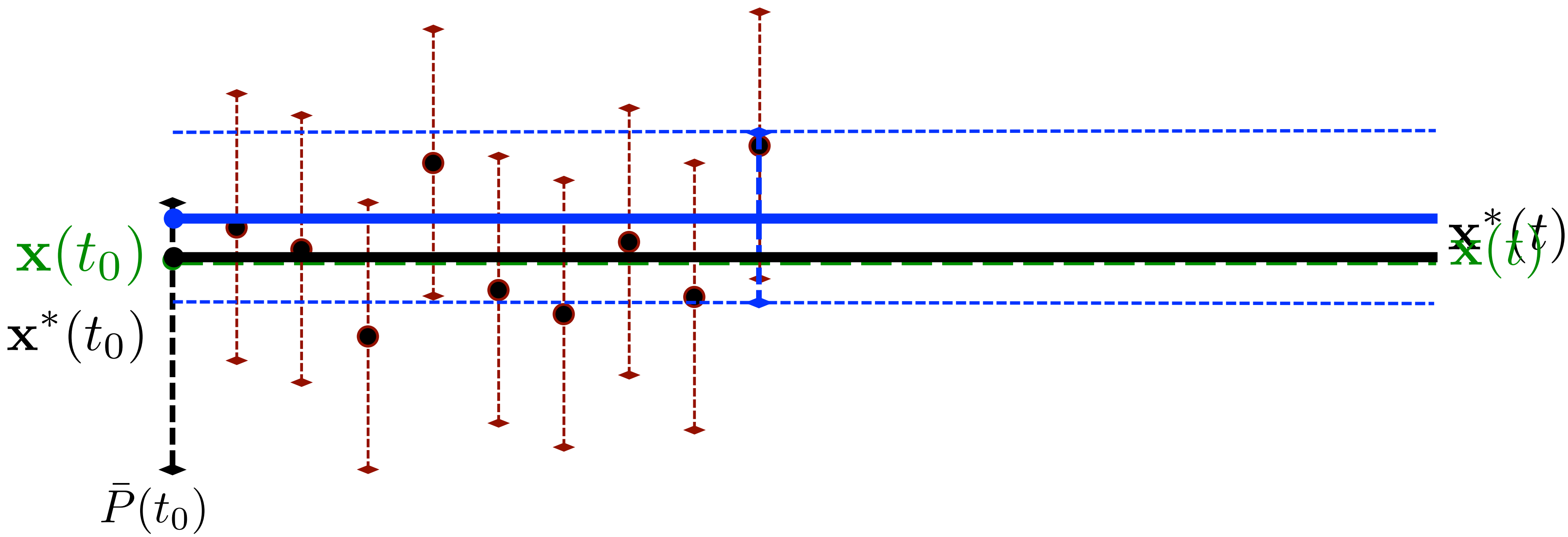

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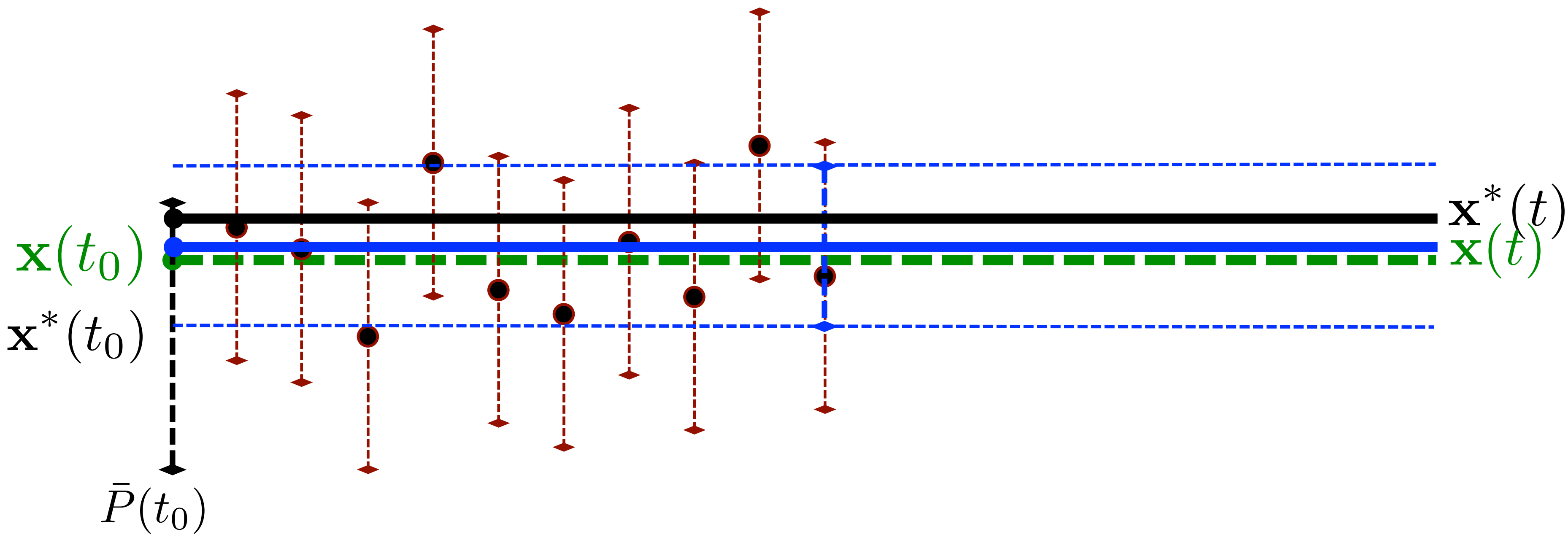

# Stat OD Conceptualization

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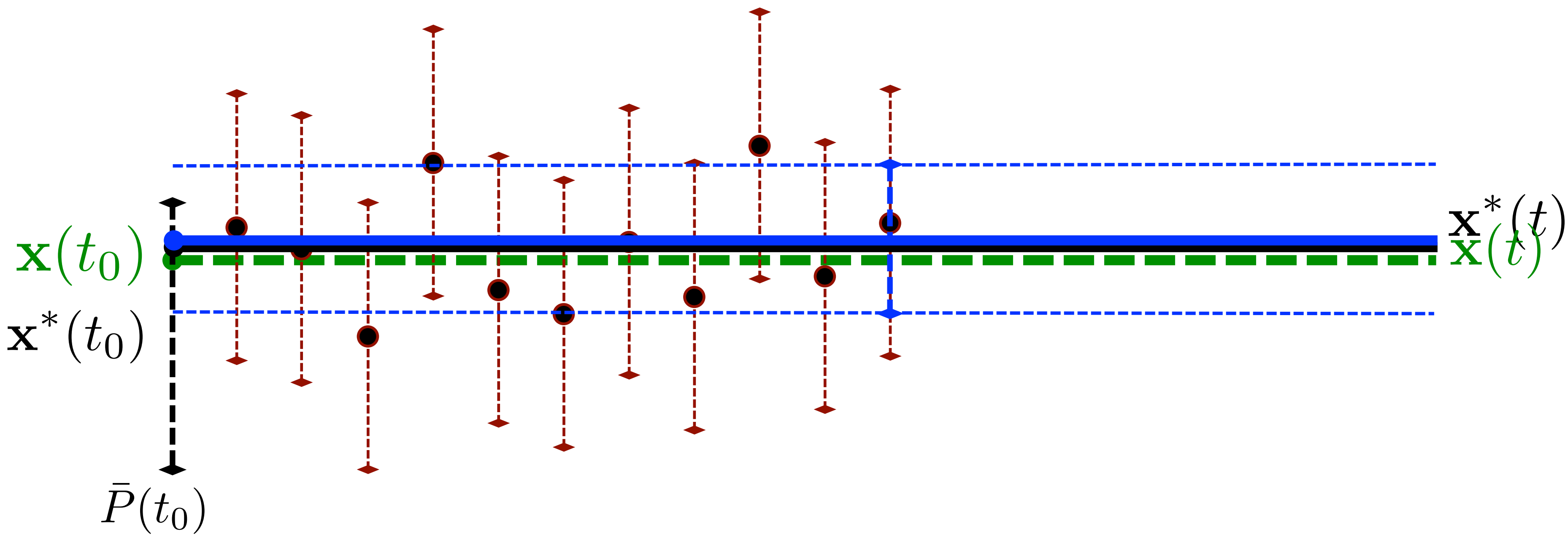
- EKF



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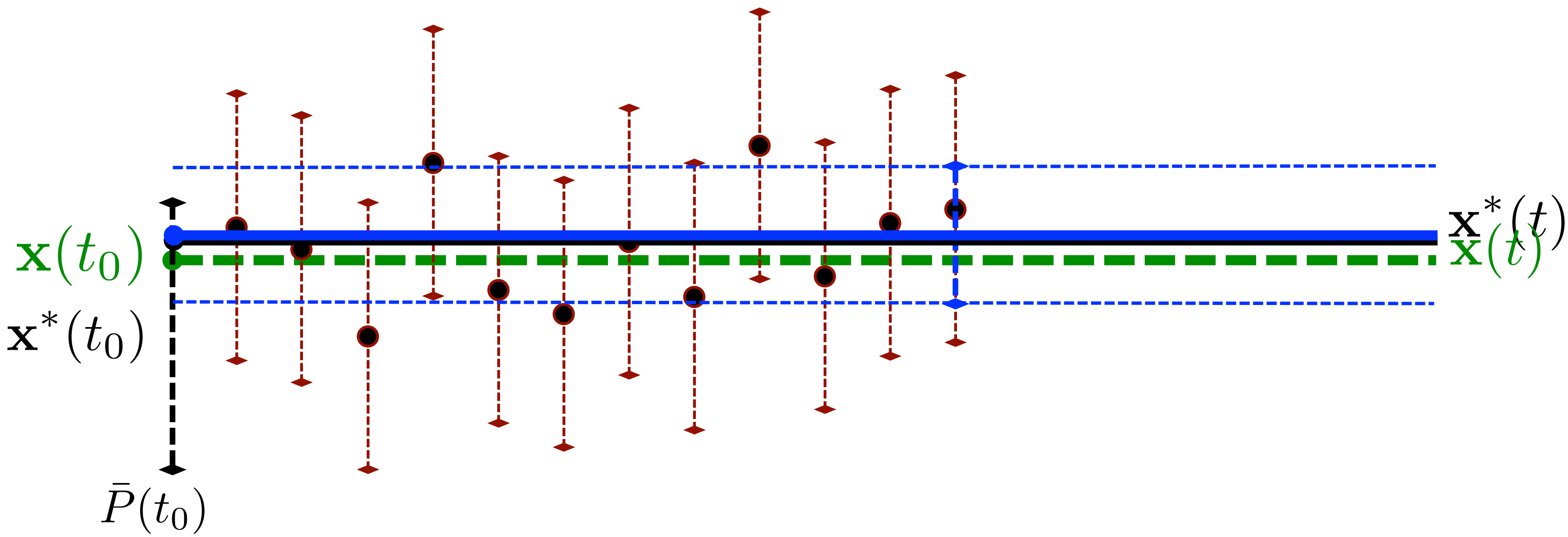
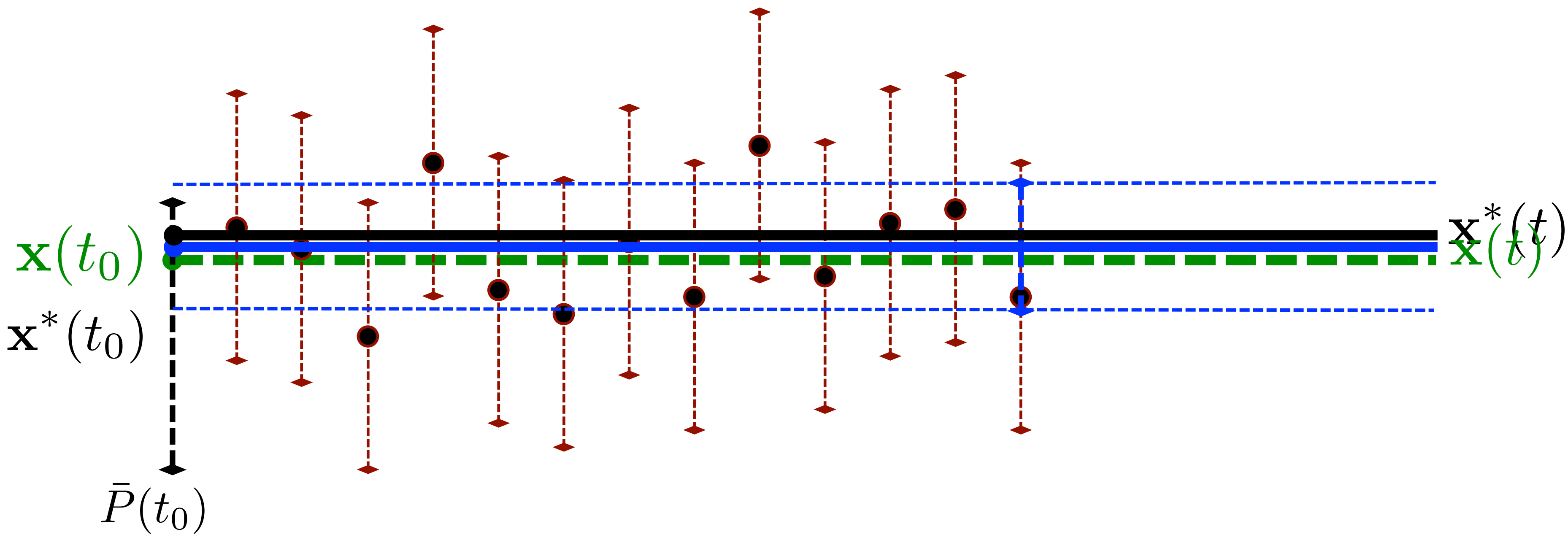


# Stat OD Conceptualization



- EKF



# Stat OD Conceptualization

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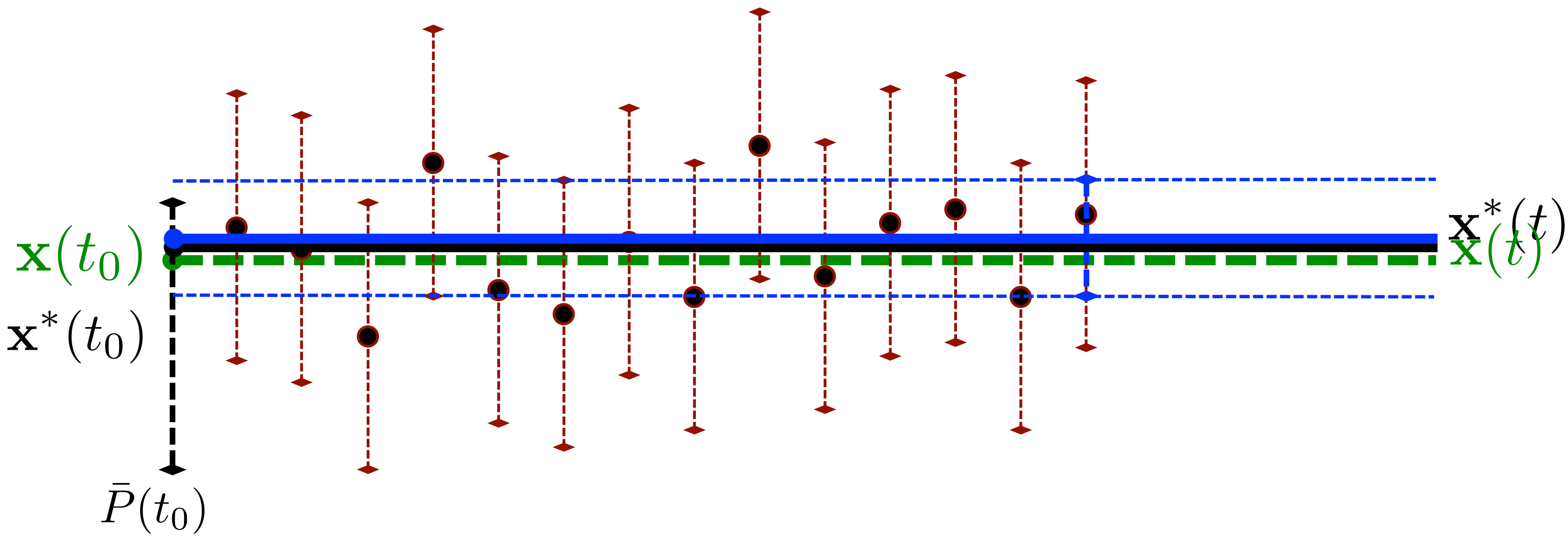
Astrodynamics Research

- EKF



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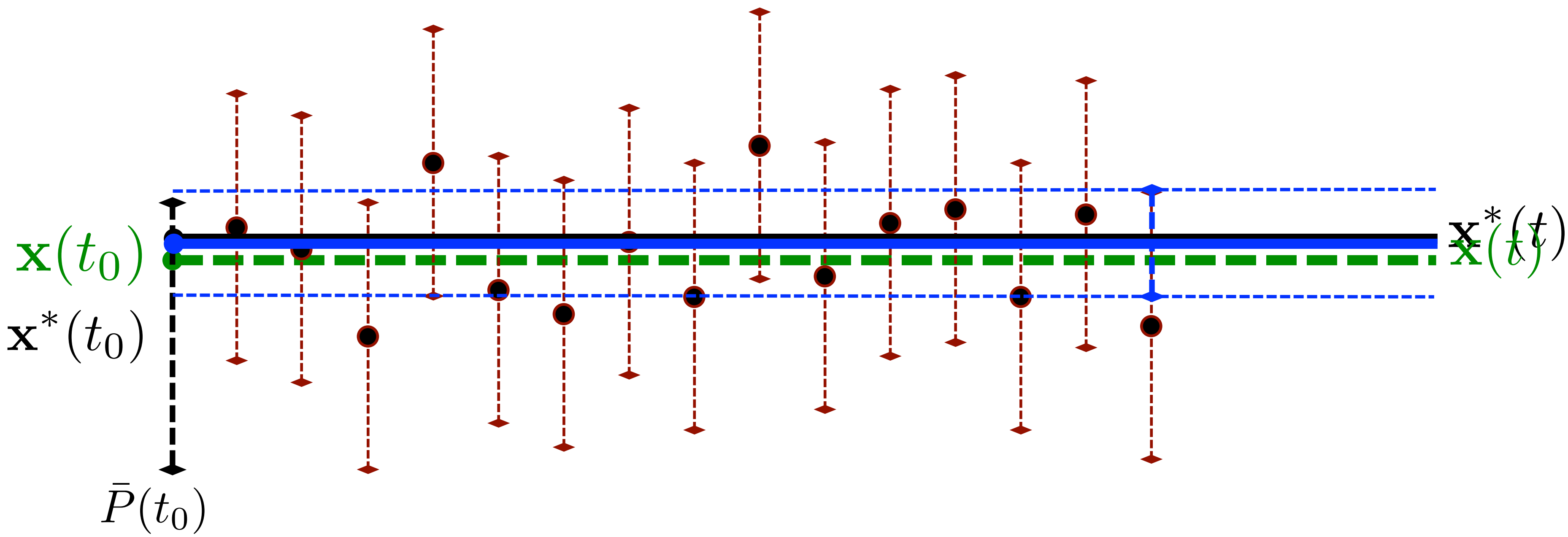
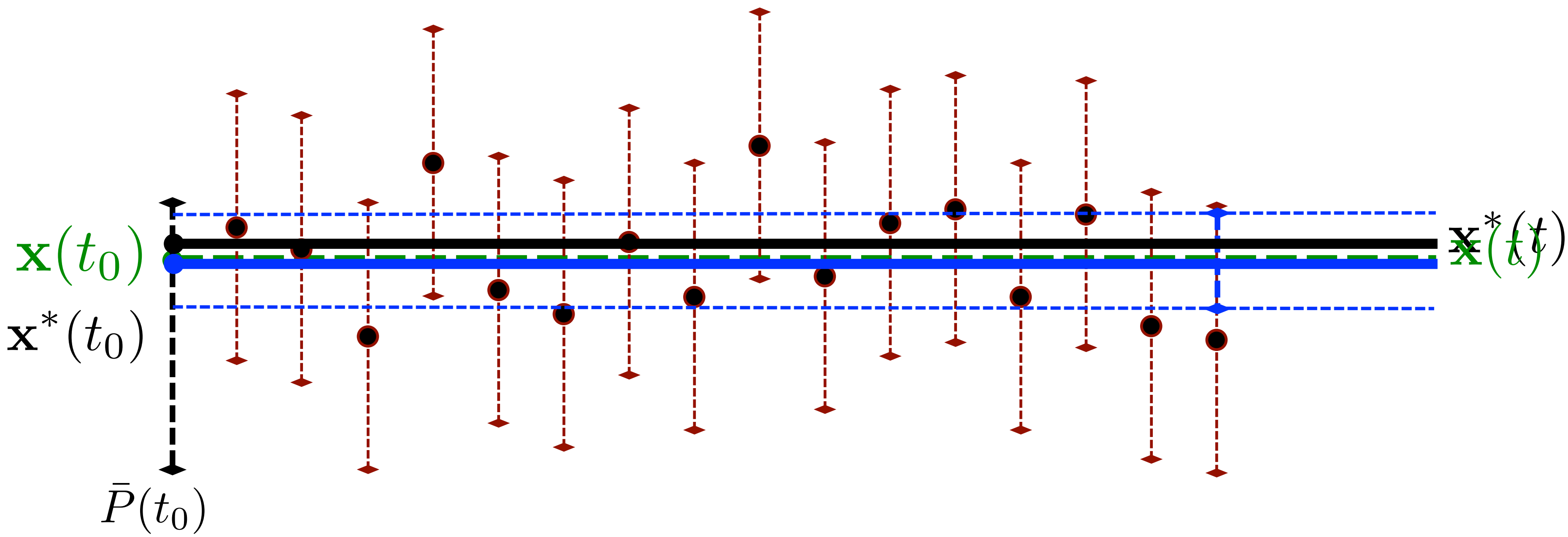


- EKF



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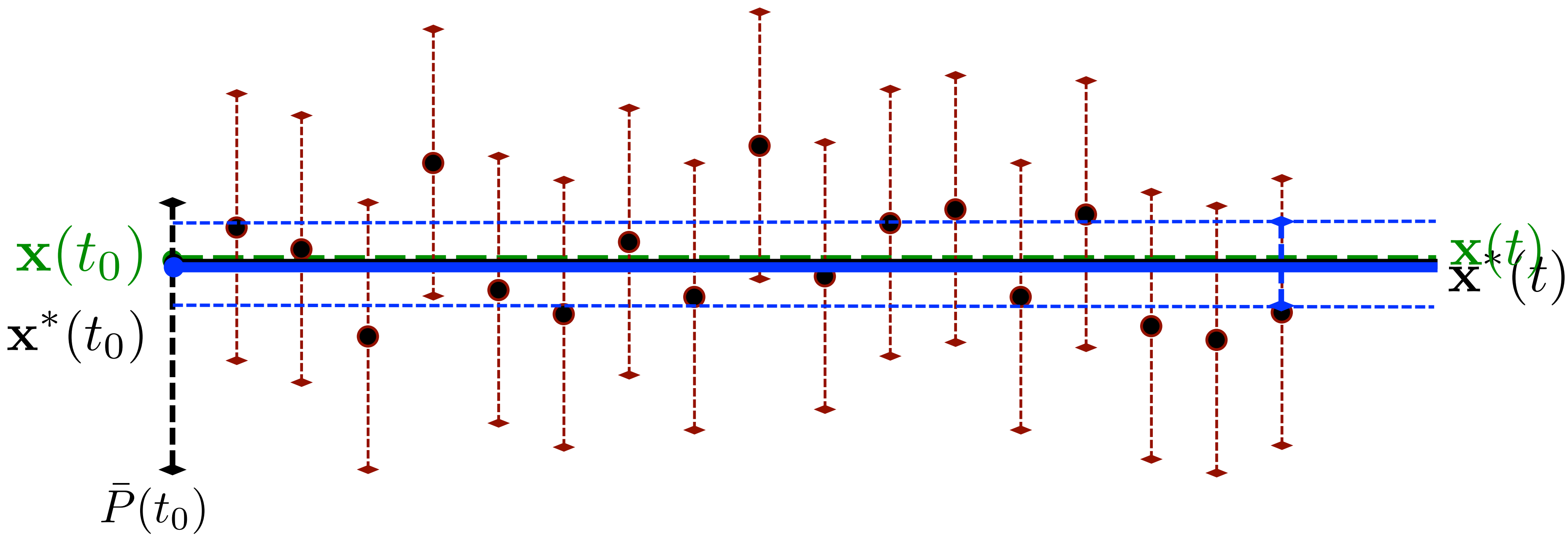

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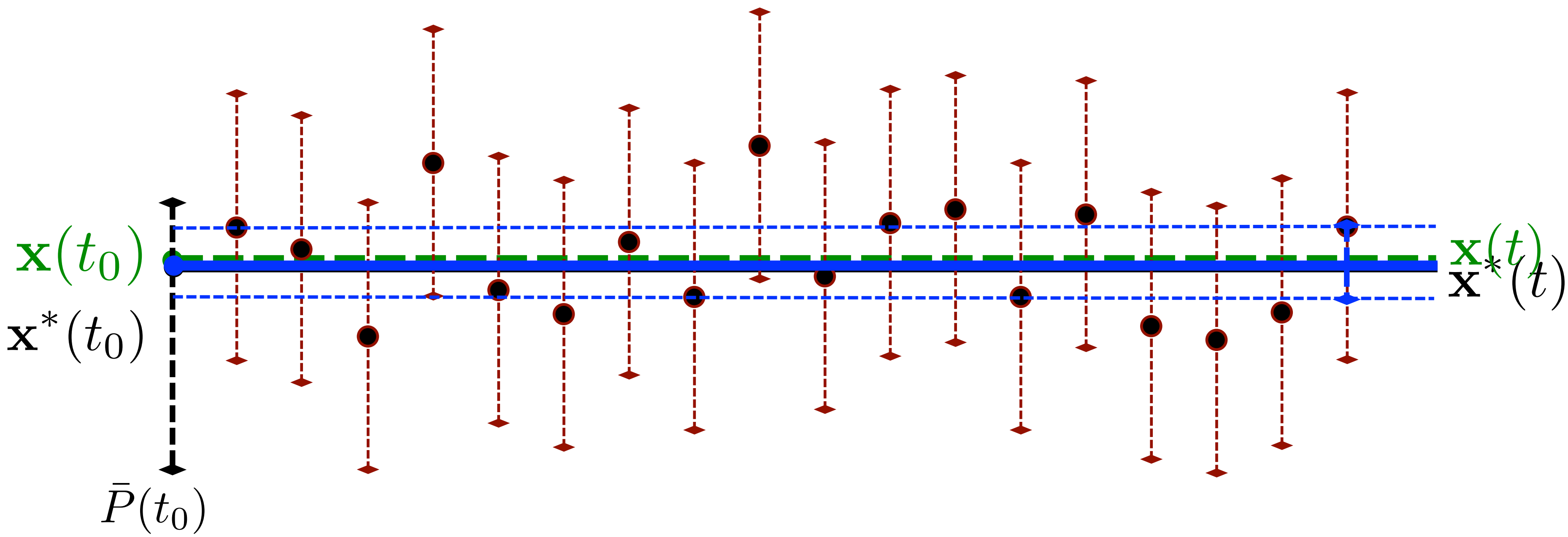

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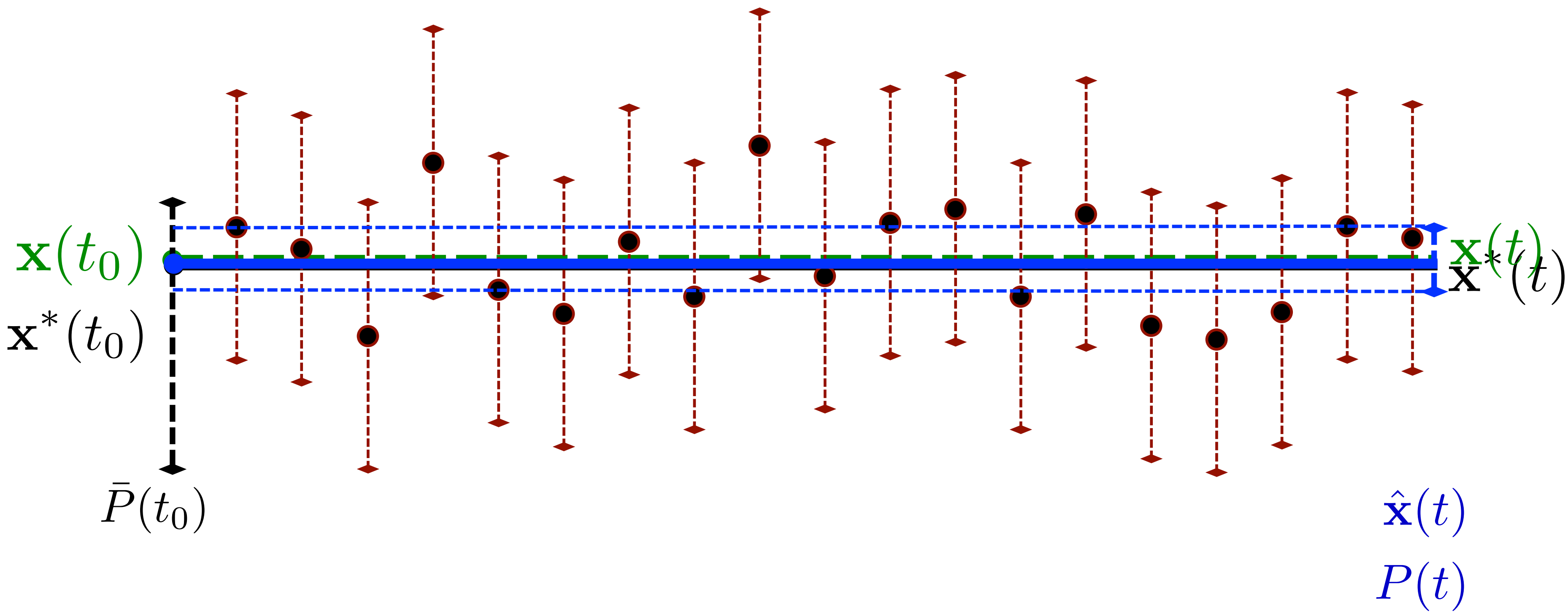
- EKF



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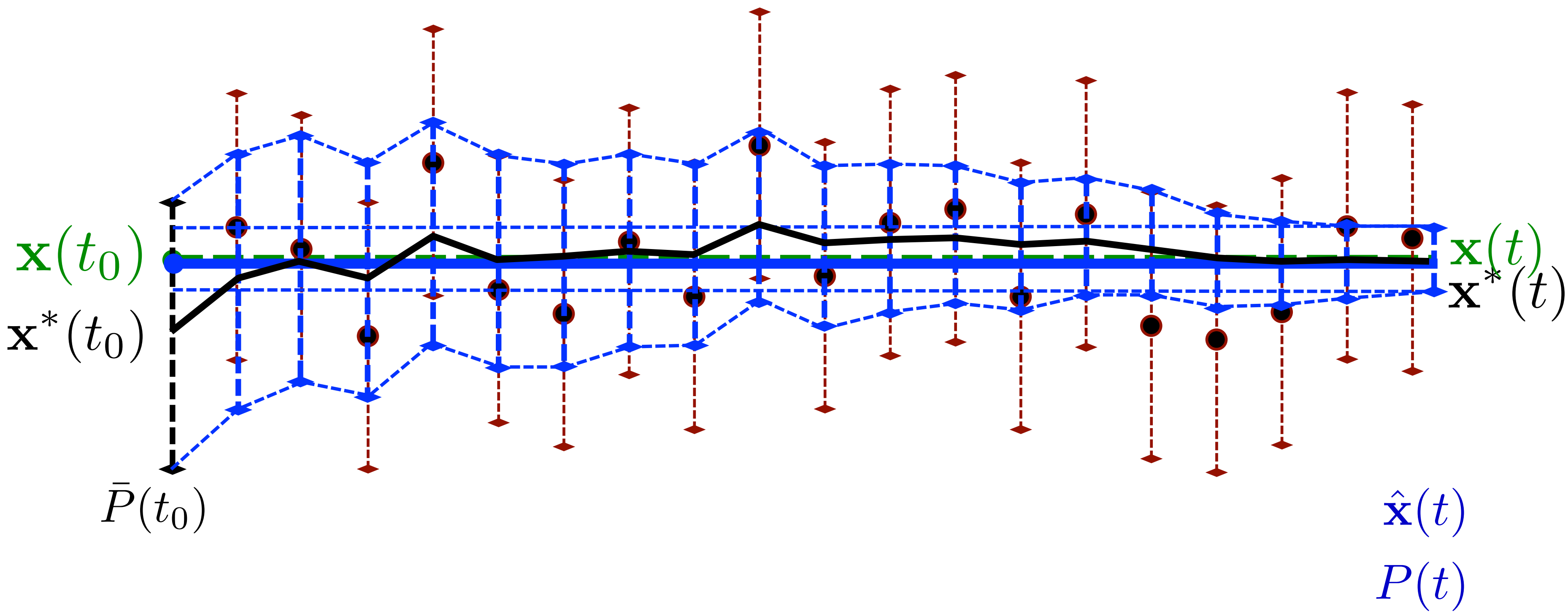

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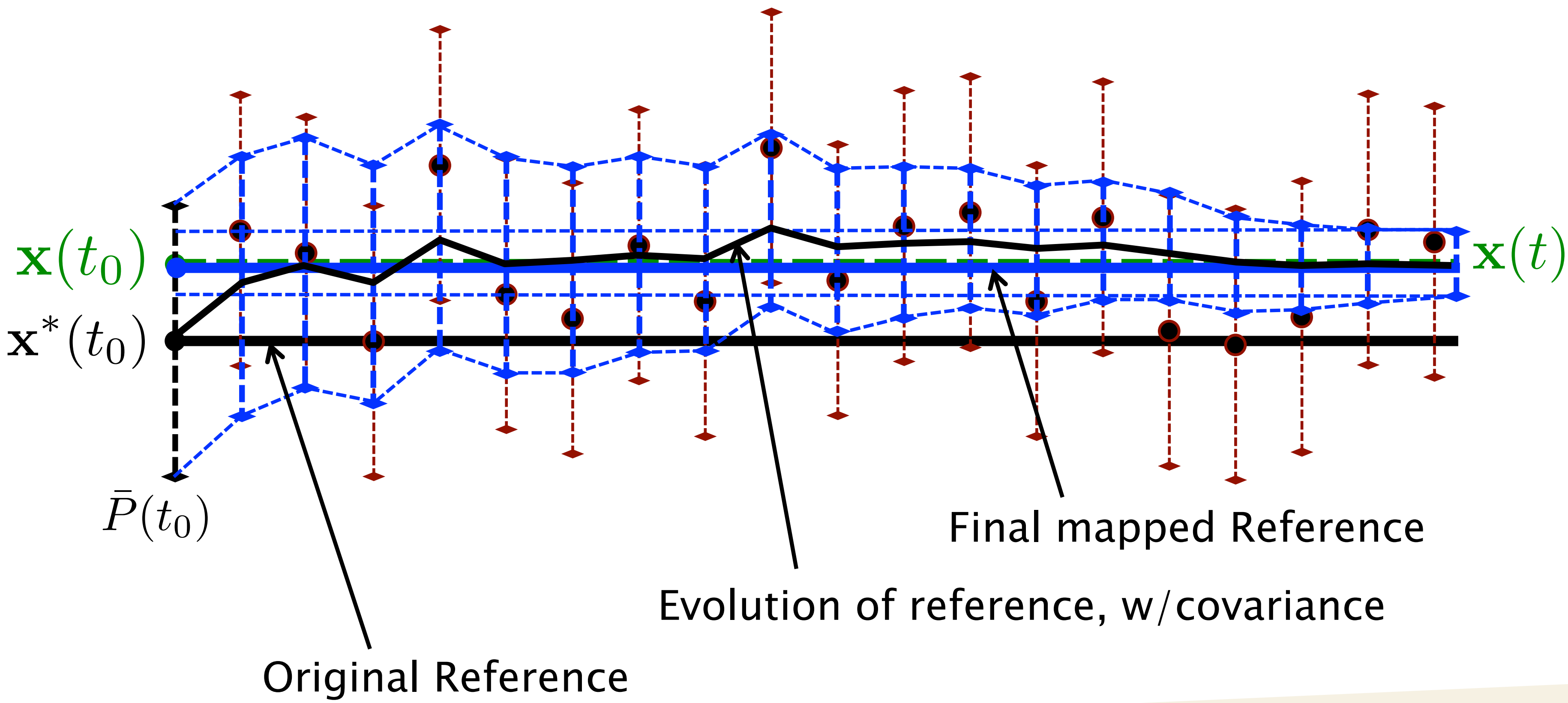
- EKF



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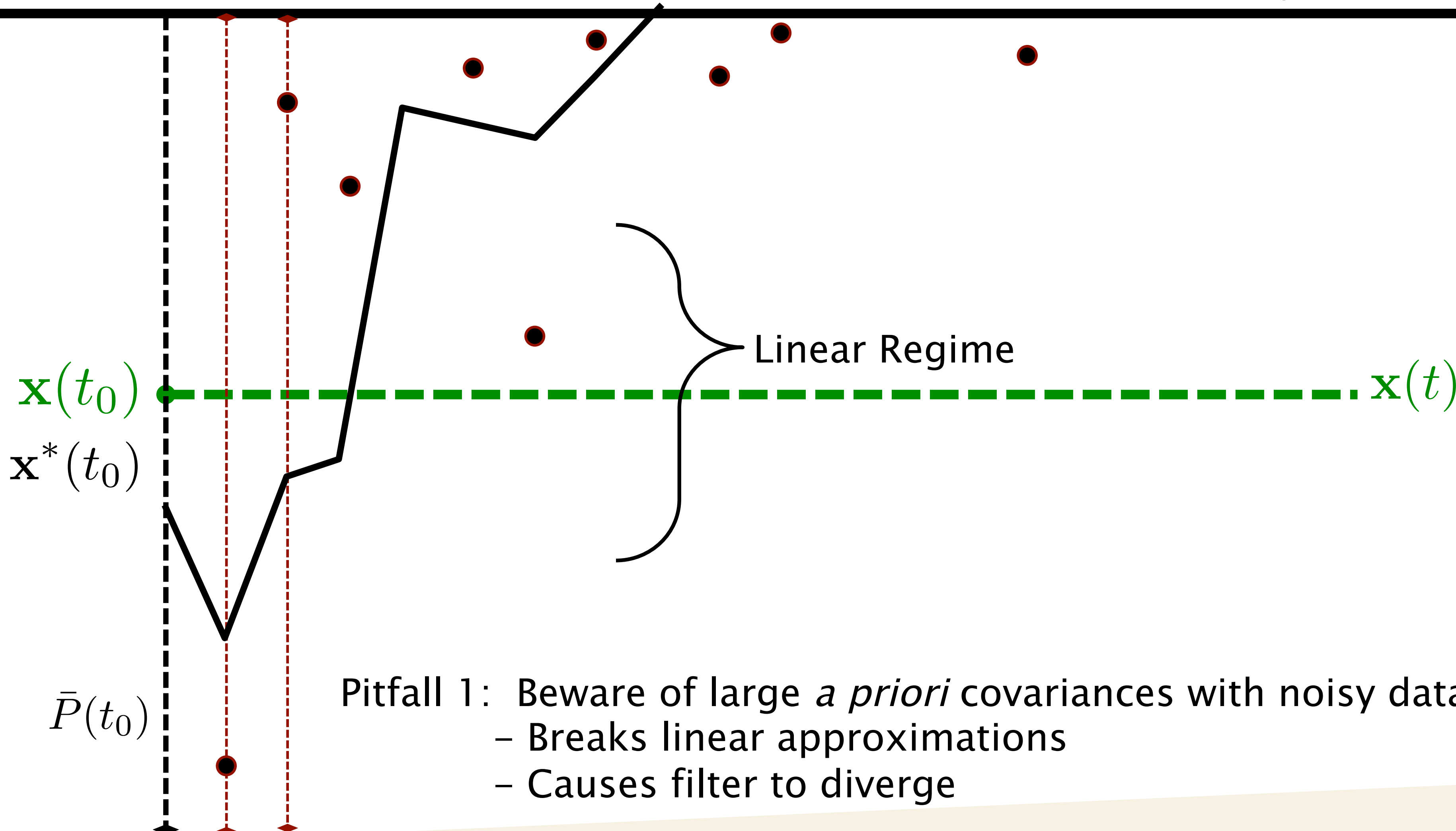
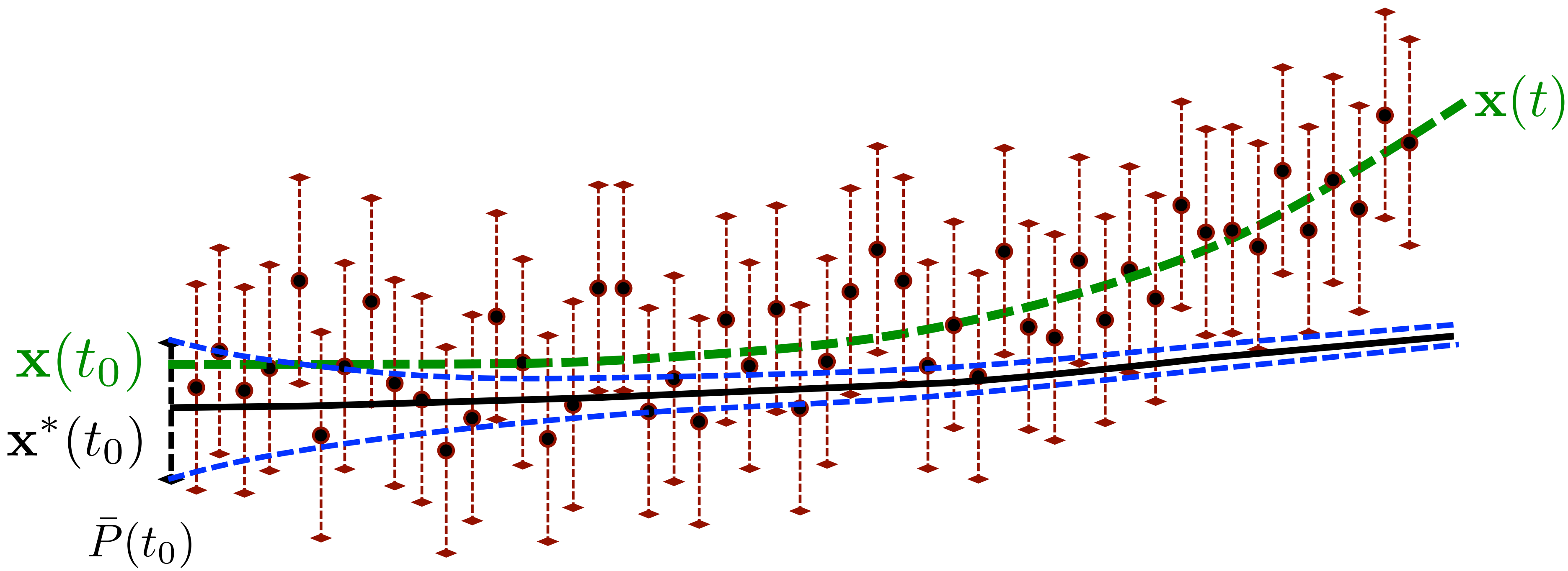



Pitfall 1: Beware of large *a priori* covariances with noisy data

- Breaks linear approximations
- Causes filter to diverge

# Stat OD Conceptualization






Pitfall 2: Beware of collapsing covariance
- Prevents new data from influencing solution
- More prevalent for longer time-spans

# The End



- Final Project

- Homework 5 is not graded…neither is the test.
- Homework 6 due Today
- Homework 7 due next week (Tuesday!)
  - It's okay to use Matlab to compute partials and to output them. But verify them.

- Concept Quizzes to resume Monday!

- Guest lecturer next week 10/25