ASEN 5070
Statistical Orbit determination I
Fall 2012

Colorado Center for
Astrodynamics Research

Professor George H. Born
Professor Jeffrey S. Parker

Lecture 8: Stat OD Processes

Announcements

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- Homework 1 Graded
  - Comments included on D2L
  - Any questions, talk with us this week
- Homework 2 CAETE due Today
  - Graded soon after
- Homework 3 due Today
- Homework 4 due next week

Homework 3

- A few questions on Problem 4's Laplace Transform.
- There's one Transform missing from the table:

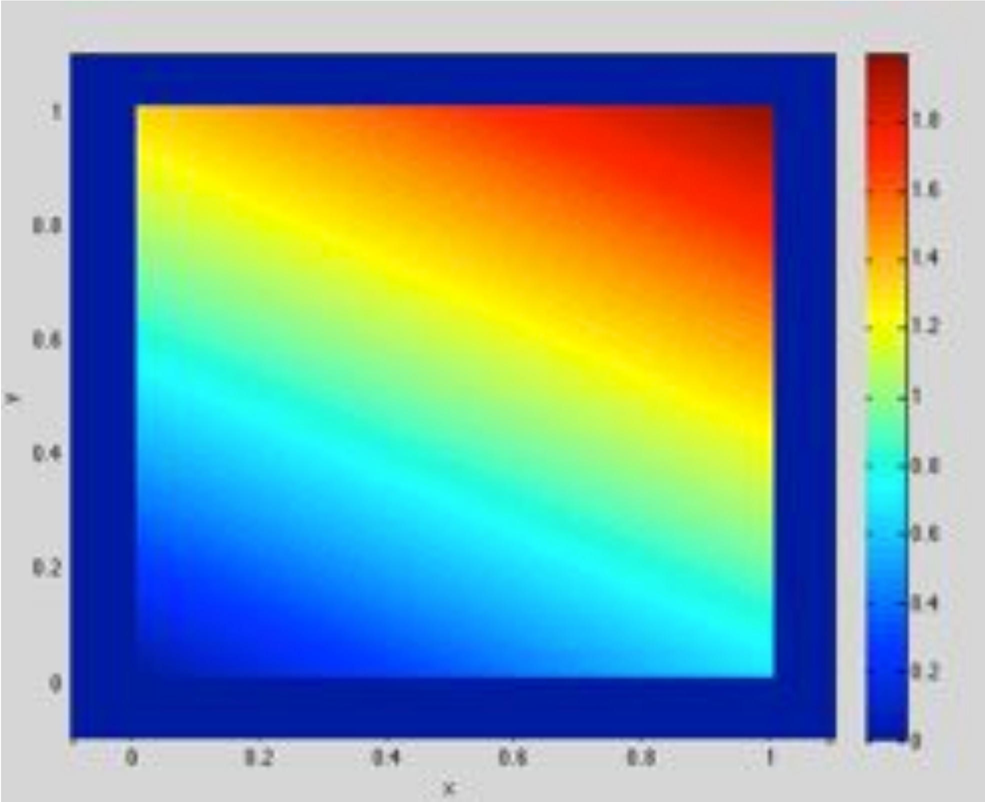
$$\mathcal{L}\left(e^{at} - e^{bt}\right) = \frac{a-b}{(s-a)(s-b)}$$

- Note: This has been posted in the HW3 discussions on D2L for a few days now. Check there if you have a commonly-asked question!

What does the joint density function MEAN?

Given

$$f(x,y)=\begin{cases} k(x+2y) & 0\le x\le 1, 0\le y\le 1 \\ 0 & elsewhere \end{cases}$$



Quiz Results

Question 1 (1 point)

If a spacecraft's state is very well known, then the trace of its variance-covariance matrix is:

- ◯ Large
- ◯ Small

Quiz Results

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Quiz Results

Question 2 (1 point)

Given the 1-dimensional probability density function:

f(x) = {

2(1-x), for 0 < x < 1

0, elsewhere

}

Don't do the math, but imagine you randomly sample this probability density function. Imagine where the probability is equal to 0 and where it is maximized.

What is the probability that your random sample fall in the range 0 - 0.5? i.e., what is P[0 < X < 1/2]?

Again, don't bother doing the math.

- Less than 1/2
- Equal to 1/2
- Greater than 1/2

Quiz Results

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Quiz Results

Question 3 (1 point)

If you are interested in fitting a lot of data with a curve, you typically compute the error for each data point by:

Error = Observed Data - Fitted Curve

aka

Error = "Observed" - "Computed"

aka

e = O-C.

When you fit the curve, you want to minimize the errors.

Question: Which of the following methods is the best way to minimize the errors?

- Minimize the sum of the errors:

  Sum(e)

- Minimize the sum of the square of the errors:

  Sum(e.^2)

- Minimize the sum of the cube of the errors:

  Sum(e.^3)

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  Sum(e)^2

Quiz Results

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Quiz Results

Question 4 (1 point)

What does it mean for x-hat to be an "unbiased estimator" of x?

(If you haven't heard this phrase before, don't worry. But give it a go.)

- The expected value of x-hat is equal to 0, i.e., E[x-hat]=0
- The expected value of x-hat is equal to x, i.e., E[x-hat]=x
- The standard deviation of x-hat is equal to 1, i.e., sigma(x-hat)=1
- The expected value of x-hat prefers and estimates prime numbers better than any other number.

Quiz Results

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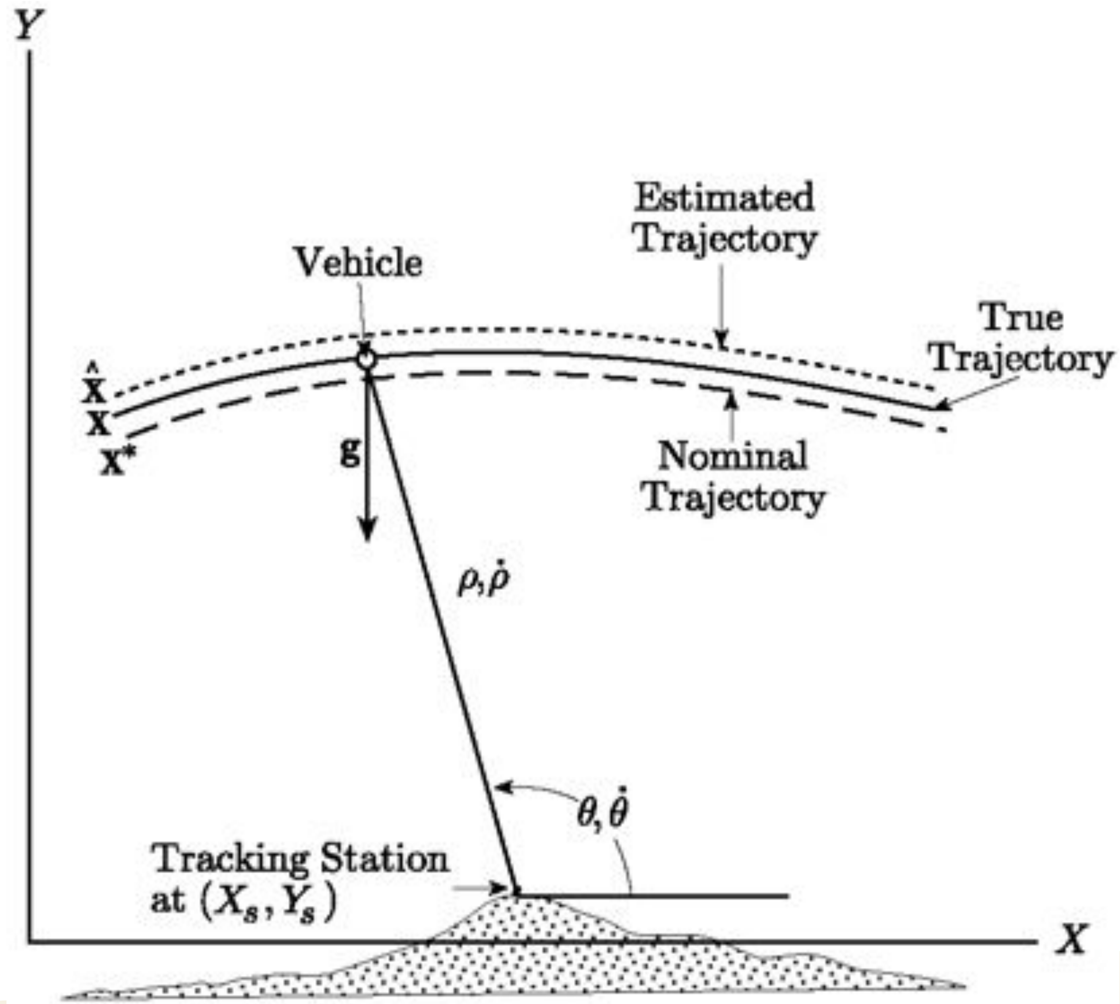
Stat OD in a Nutshell

- Truth
- Reference
- Best Estimate

- Observations are functions of state parameters, but usually NOT state parameters.

- Mismodeled dynamics

- Underdetermined system.



Stat OD in a Nutshell

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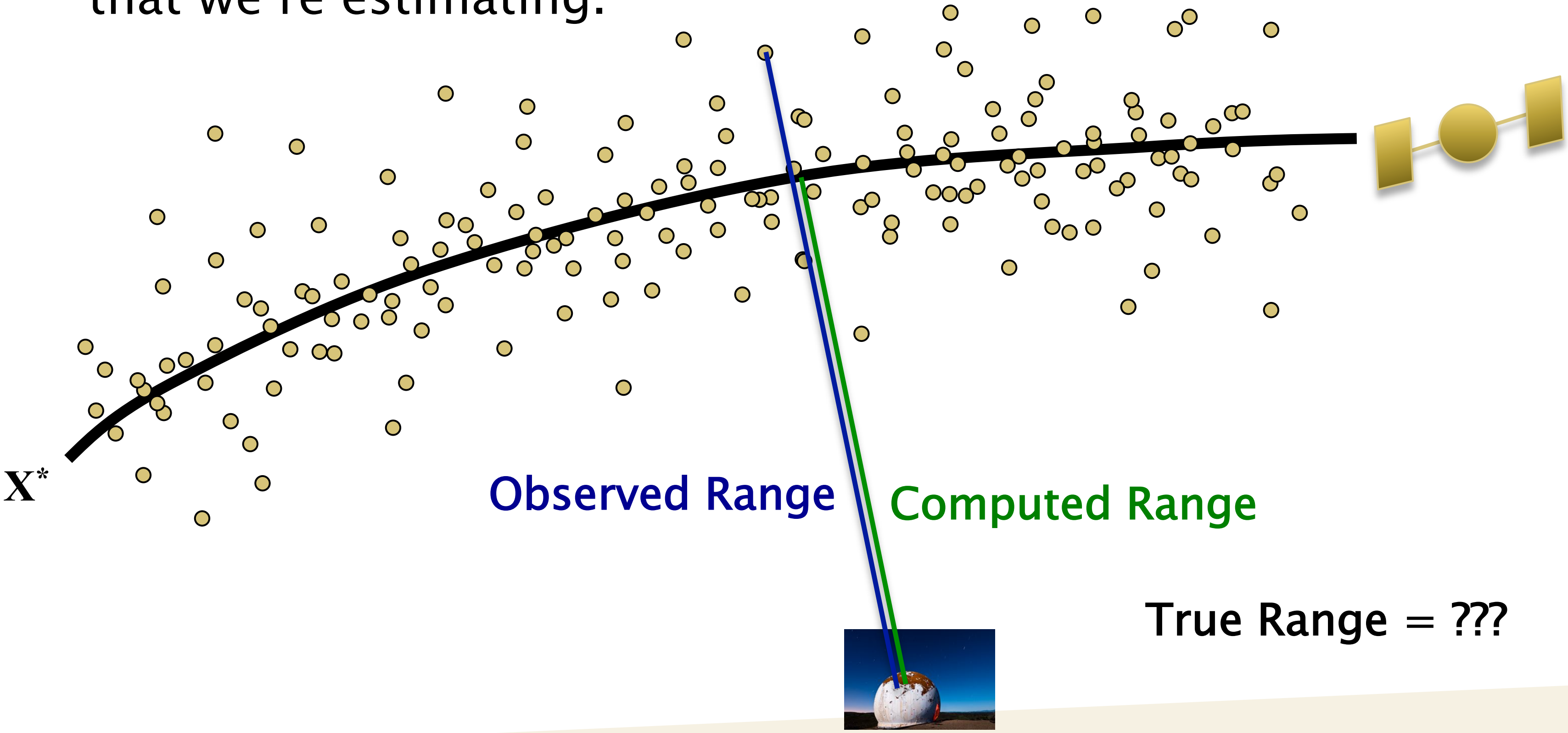
- We have noisy observations of certain aspects of the system.
- We need some way to relate each observation to the trajectory that we're estimating.



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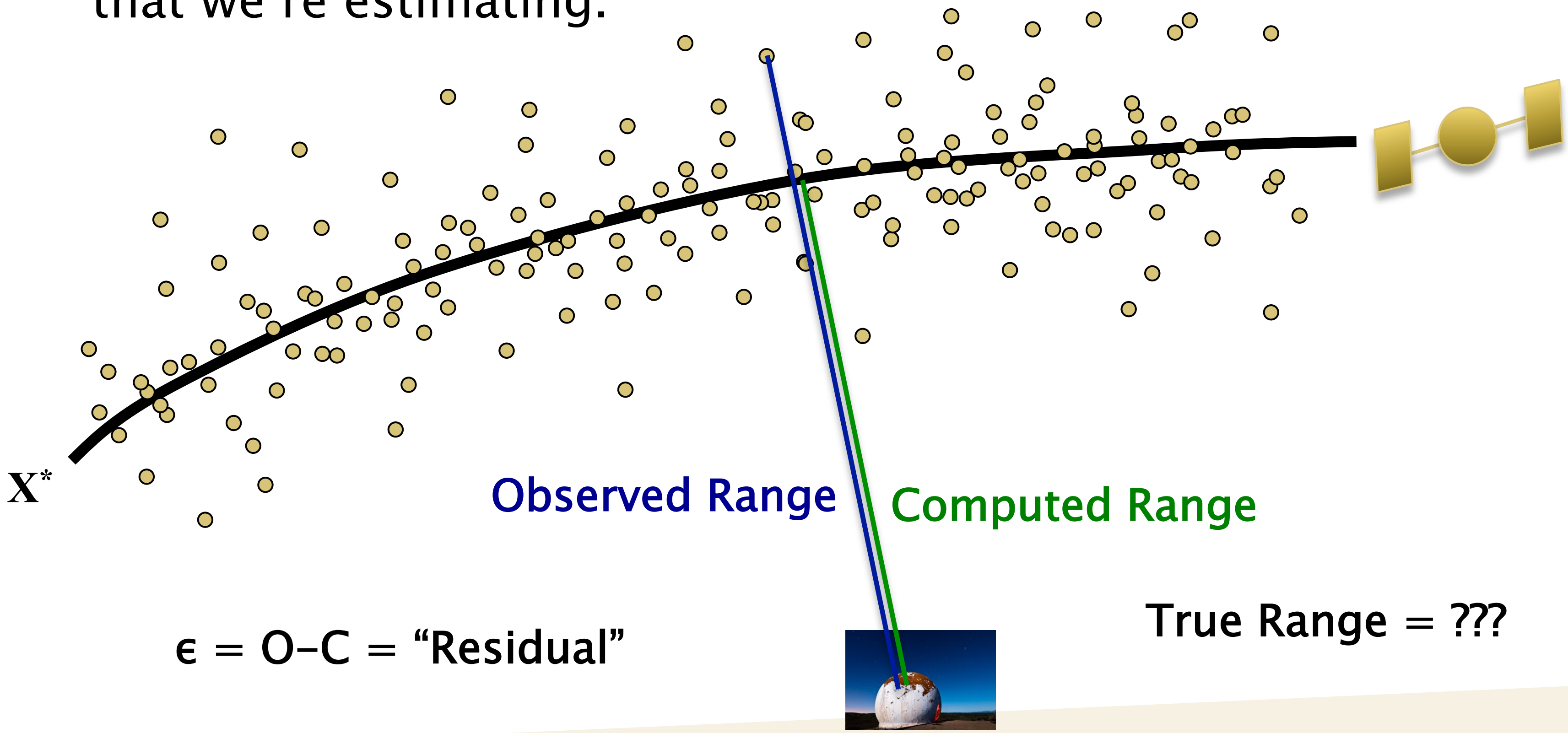


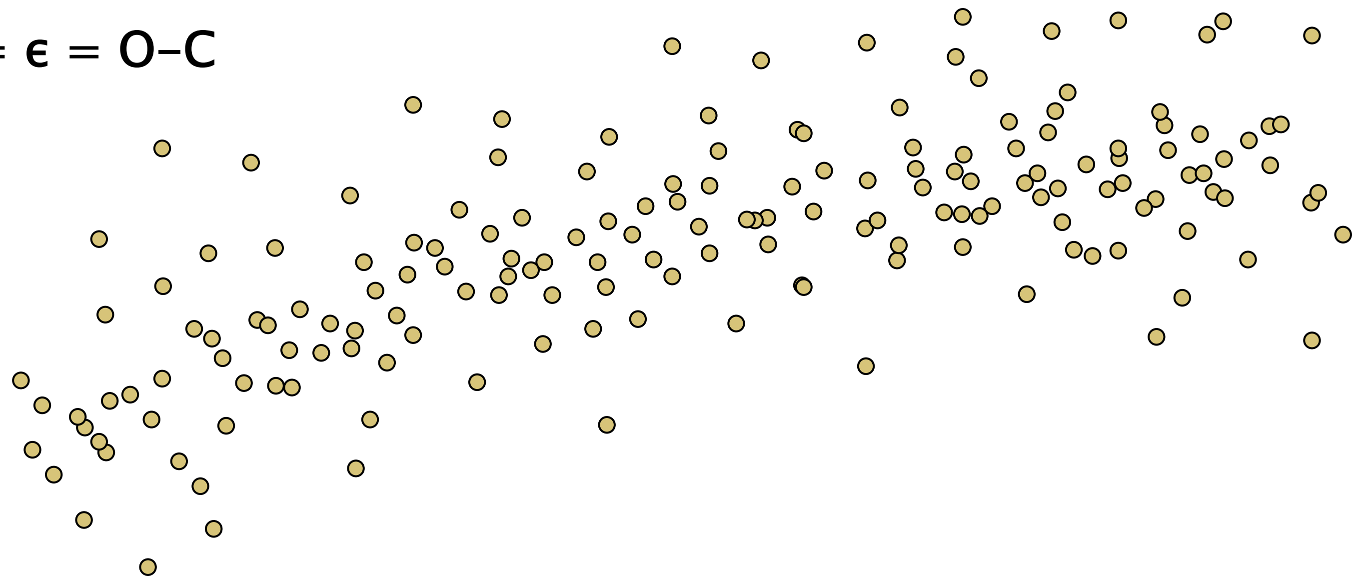
Stat OD in a Nutshell

- Assumptions:
  - The reference/nominal trajectory is near the truth trajectory.
    - Linear approximations are decent
  - Force models are good approximations for the duration of the measurement arc.
  - The filter that we're using is unbiased:
    - The filter's best estimate is consistent with the true trajectory.

Fitting the data

- How do we best fit the data?

Residuals = ϵ = O–C



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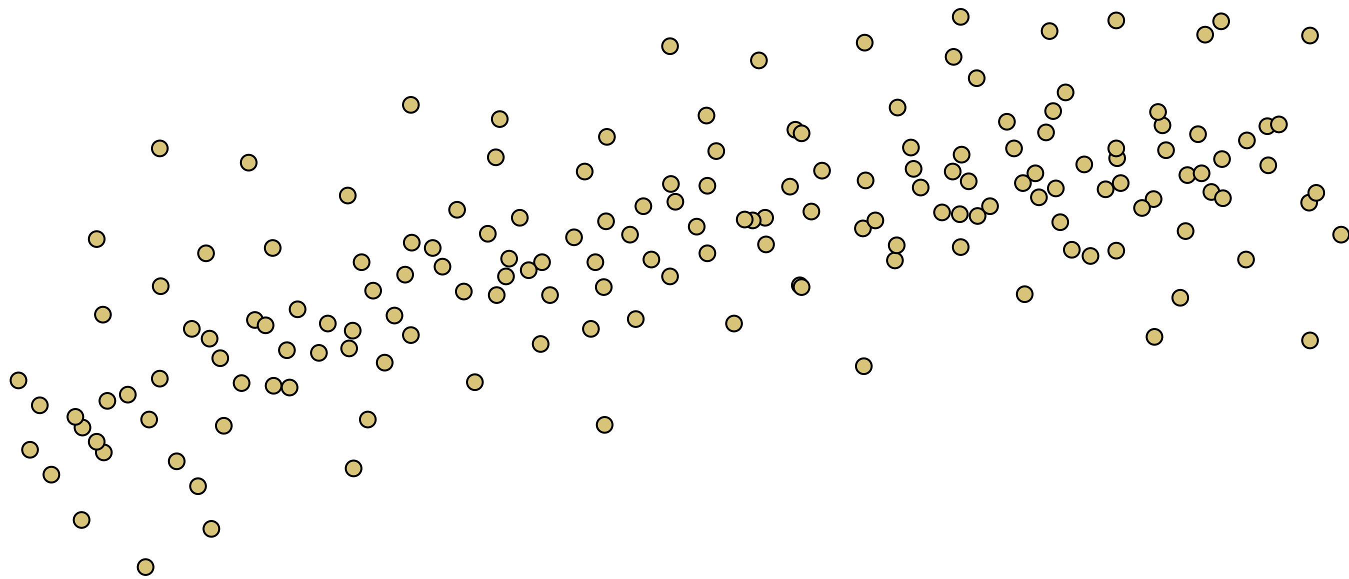
$$\min\left(\sum_{i=1}^{N} \epsilon_i\right)?$$



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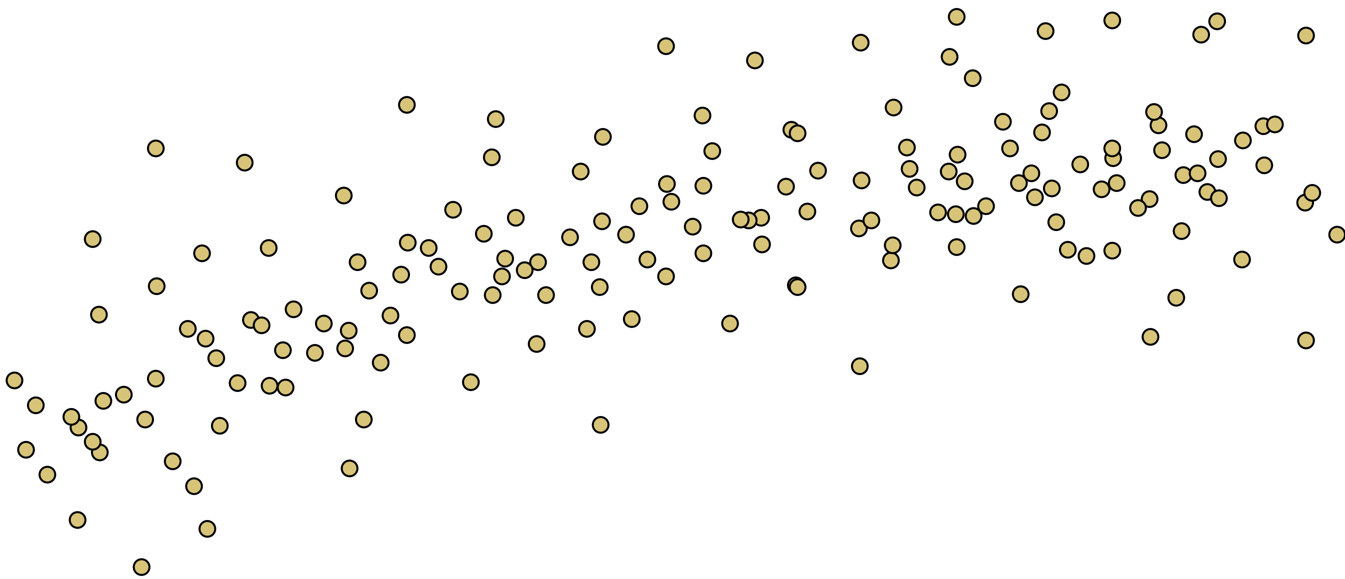
$$\min\left(\sum_{i=1}^{N}\epsilon_i\right)^2 ?$$



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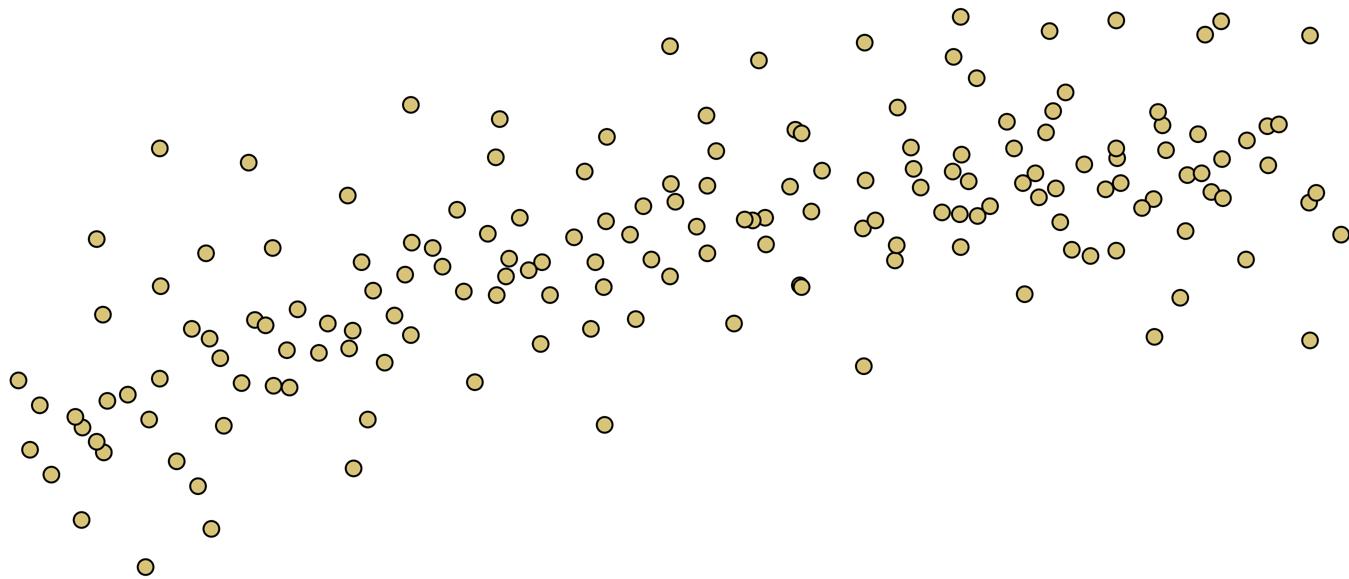
$$\min \left(\sum_{i=1}^{N} |\epsilon_i| \right) ?$$



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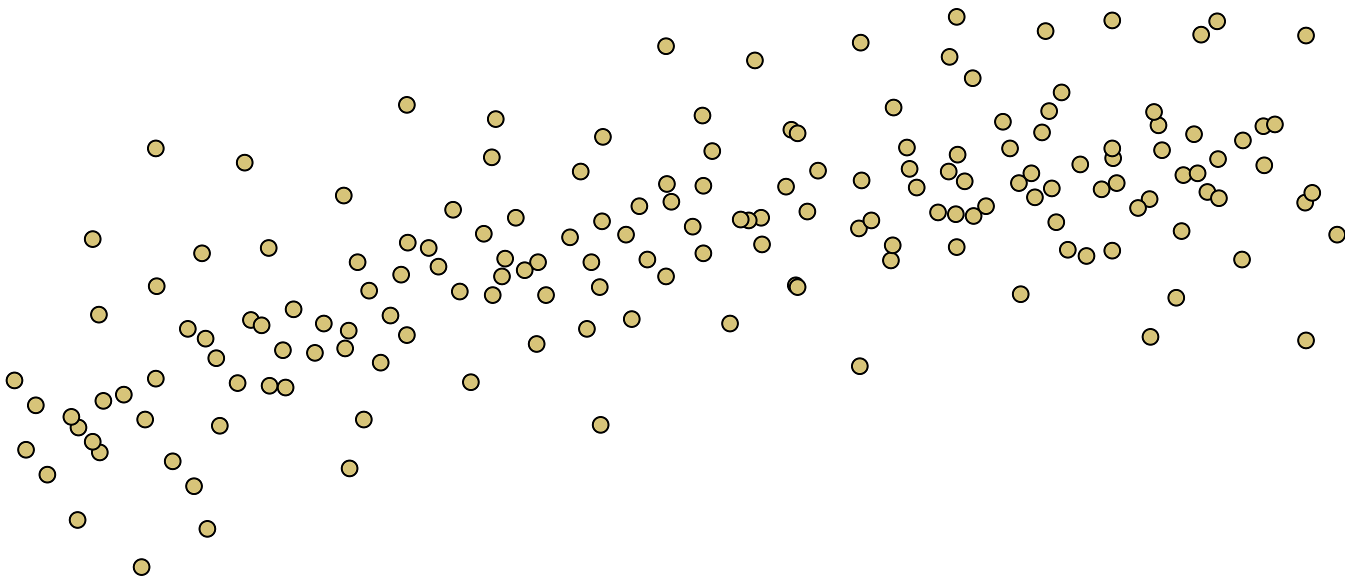
$$\min\left(\sum_{i=1}^{N}|\epsilon_i|\right)?$$ Not bad

$$\min\left(\sum_{i=1}^{N}\epsilon_i^2\right)?$$



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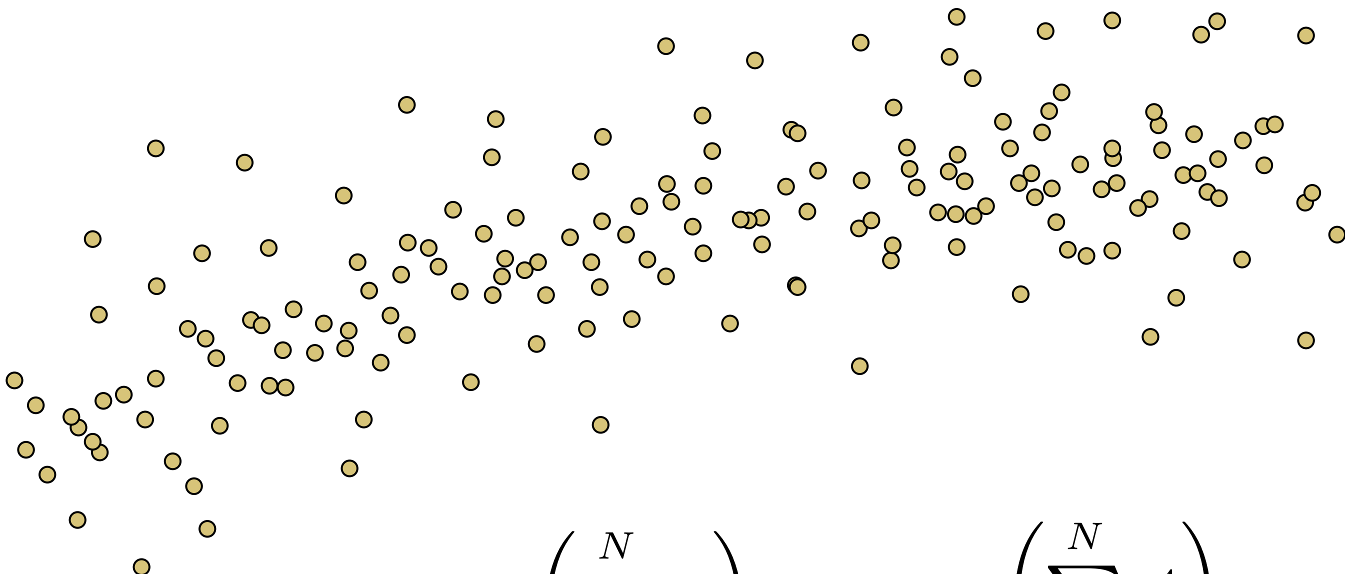

$$\min\left(\sum_{i=1}^{N}\epsilon_i^3\right)?$$

$$\min\left(\sum_{i=1}^{N}\epsilon_i^4\right)?$$

Fitting the data

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- How do we best fit the data?

- A good solution, and one easy to code up, is the least-squares solution

$$\text{Minimize } J = \frac{1}{2}\epsilon^T\epsilon$$

$$\frac{\partial J}{\partial X} = 0$$

$$\frac{\partial^2 J}{\partial^2 X} \text{ is positive definite}$$

Mapping an observation

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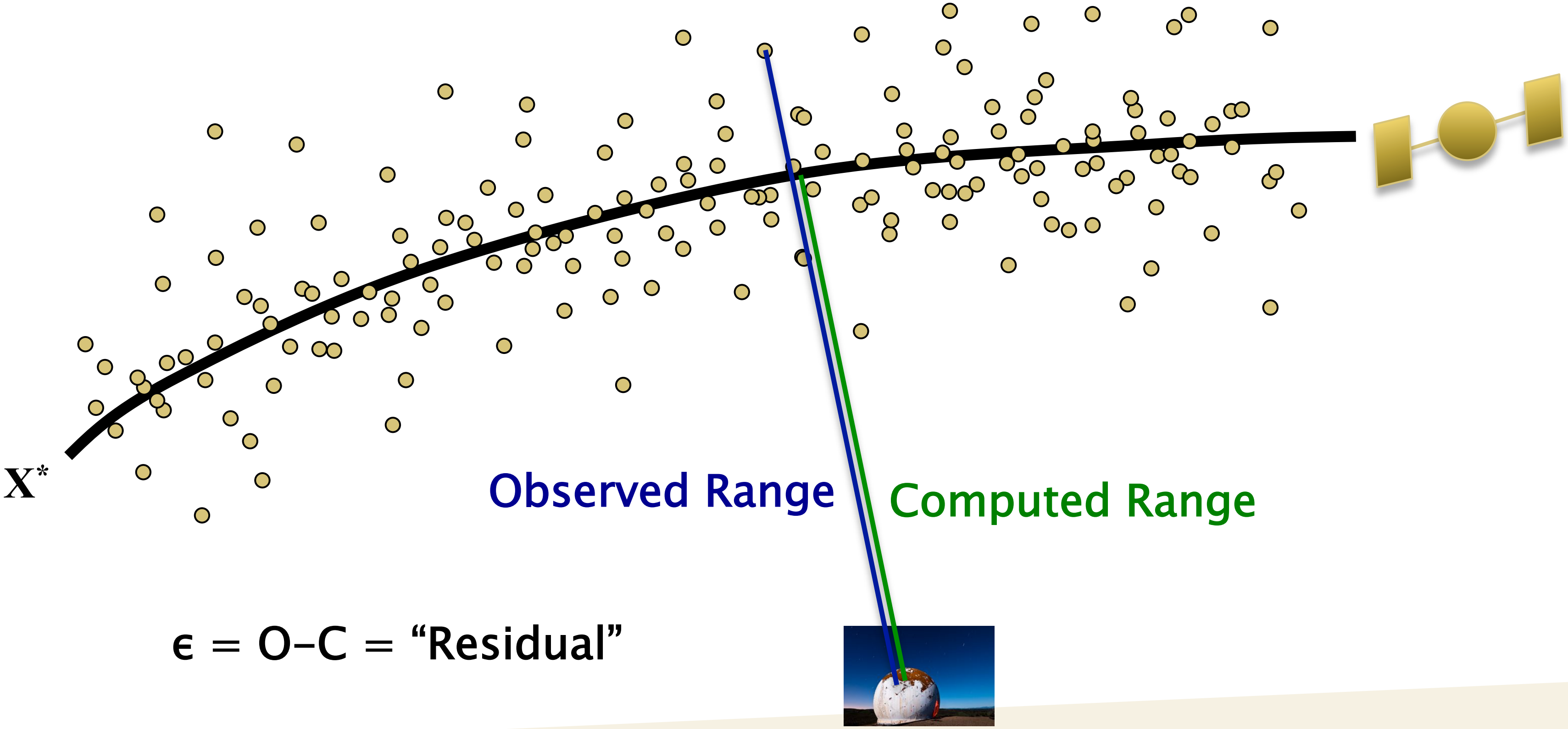
- How do we map an observation to the trajectory?



ϵ = O–C = "Residual"

Mapping an observation

- How do we map an observation to the trajectory?

$$\mathbf{Y}_O(t) = \boldsymbol{h}(t, \mathbf{X}(t)) + \boldsymbol{\nu} = \begin{bmatrix} \rho(t) + \rho_{\text{bias}} + \rho_{\text{noise}} \\ \dot{\rho}(t) + \dot{\rho}_{\text{bias}} + \dot{\rho}_{\text{noise}} \end{bmatrix}$$

$$\begin{aligned} \rho &= \sqrt{(x_h - x_g)^2 + (y_h - y_g)^2 + (z_h - z_g)^2} \\ \dot{\rho} &= \frac{(x_h - x_g)(\dot{x}_h - \dot{x}_g) + (y_h - y_g)(\dot{y}_h - \dot{y}_g) + (z_h - z_g)(\dot{z}_h - \dot{z}_g)}{\sqrt{(x_h - x_g)^2 + (y_h - y_g)^2 + (z_h - z_g)^2}} \end{aligned}$$

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$$\rho = \sqrt{(x_h - x_g)^2 + (y_h - y_g)^2 + (z_h - z_g)^2}$$

$$\dot{\rho} = \frac{(x_h - x_g)(\dot{x}_h - \dot{x}_g) + (y_h - y_g)(\dot{y}_h - \dot{y}_g) + (z_h - z_g)(\dot{z}_h - \dot{z}_g)}{\sqrt{(x_h - x_g)^2 + (y_h - y_g)^2 + (z_h - z_g)^2}}$$

- Very non-linear relationships!
- Need to linearize to make a practical algorithm.

State Deviation and Linearization

- Linearization
- Introduce the state deviation vector

$$\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t)$$

- If the reference/nominal trajectory is close to the truth trajectory, then a linear approximation is reasonable.

State Deviation and Linearization

- Goal of the Stat OD process:

- Find a new state/trajectory that best fits the observations:

$$\hat{\mathbf{X}}(t) = \mathbf{X}^*(t) + \hat{\mathbf{x}}(t)$$

- If the reference is near the truth, then we can assume:

$$\hat{\mathbf{X}}(t) = \mathbf{X}(t) + \hat{\mathbf{x}}(t)$$

State Deviation and Linearization

- Goal of the Stat OD process:
- The best fit trajectory

$$\hat{\mathbf{X}}(t) = \mathbf{X}^*(t) + \hat{\mathbf{x}}(t)$$

is represented by

$$\hat{\mathbf{X}}(t_0) = \mathbf{X}^*(t_0) + \hat{\mathbf{x}}(t_0)$$

This is what we want

State Deviation Mapping

- How do we map the state deviation vector from one time to another?

State Deviation Mapping

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- How do we map the state deviation vector from one time to another?

$$\Phi(t_2, t_1) = \frac{\partial \mathbf{X}(t_2)}{\partial \mathbf{X}(t_1)}$$

- The state transition matrix.

State Deviation Mapping

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$$\Phi(t_2, t_1) = \frac{\partial \mathbf{X}(t_2)}{\partial \mathbf{X}(t_1)}$$

- The state transition matrix.

- It permits: $\mathbf{x}(t_2) = \Phi(t_2, t_1)\mathbf{x}(t_1)$

$$\Phi(t_2, t_1) = \Phi(t_1, t_2)^{-1}$$

State Transition Matrix

- The state transition matrix maps a deviation in the state from one epoch to another.

$$\mathbf{x}(t_2) = \Phi(t_2, t_1)\mathbf{x}(t_1)$$

$$\Phi(t_2, t_1) = \Phi(t_1, t_2)^{-1}$$

- It is constructed via numerical integration, in parallel with the trajectory itself.

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I$$

The A Matrix

- The "A" Matrix:

$$A(t) = \frac{\partial \boldsymbol{f}(t, \mathbf{X}(t))}{\partial \mathbf{X}(t)} = \frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}(t)}$$

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \qquad A(t) = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \cdots & \frac{\partial \dot{x}}{\partial \dot{z}} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \cdots & \frac{\partial \dot{y}}{\partial \dot{z}} \\ \vdots & & \ddots & \vdots \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial y} & \cdots & \frac{\partial \ddot{x}}{\partial \dot{z}} \end{bmatrix}$$

Measurement Mapping

- Still need to know how to map measurements from one time to a state at another time!

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- Would like this:

$$\mathbf{y}(t) = H(t)\mathbf{x}(t_0) + \epsilon(t)$$

Measurement Mapping

- Still need to know how to map measurements from one time to a state at another time!

$$\mathbf{y}(t) = H(t)\mathbf{x}(t_0) + \epsilon(t)$$

$$\mathbf{y}(t) = H(t)\Phi(t_0, t)\mathbf{x}(t) + \epsilon(t)$$

Measurement Mapping

- Still need to know how to map measurements from one time to a state at another time!

$$\mathbf{y}(t) = H(t)\mathbf{x}(t_0) + \epsilon(t)$$

$$\mathbf{y}(t) = H(t)\Phi(t_0, t)\mathbf{x}(t) + \epsilon(t)$$

- Define: $H(t) = \tilde{H}(t)\Phi(t, t_0)$

$$\mathbf{y}(t) = \tilde{H}(t)\mathbf{x}(t) + \epsilon(t)$$

Measurement Mapping Matrix

$$\mathbf{y}(t) = \tilde{H}(t)\mathbf{x}(t) + \epsilon(t)$$

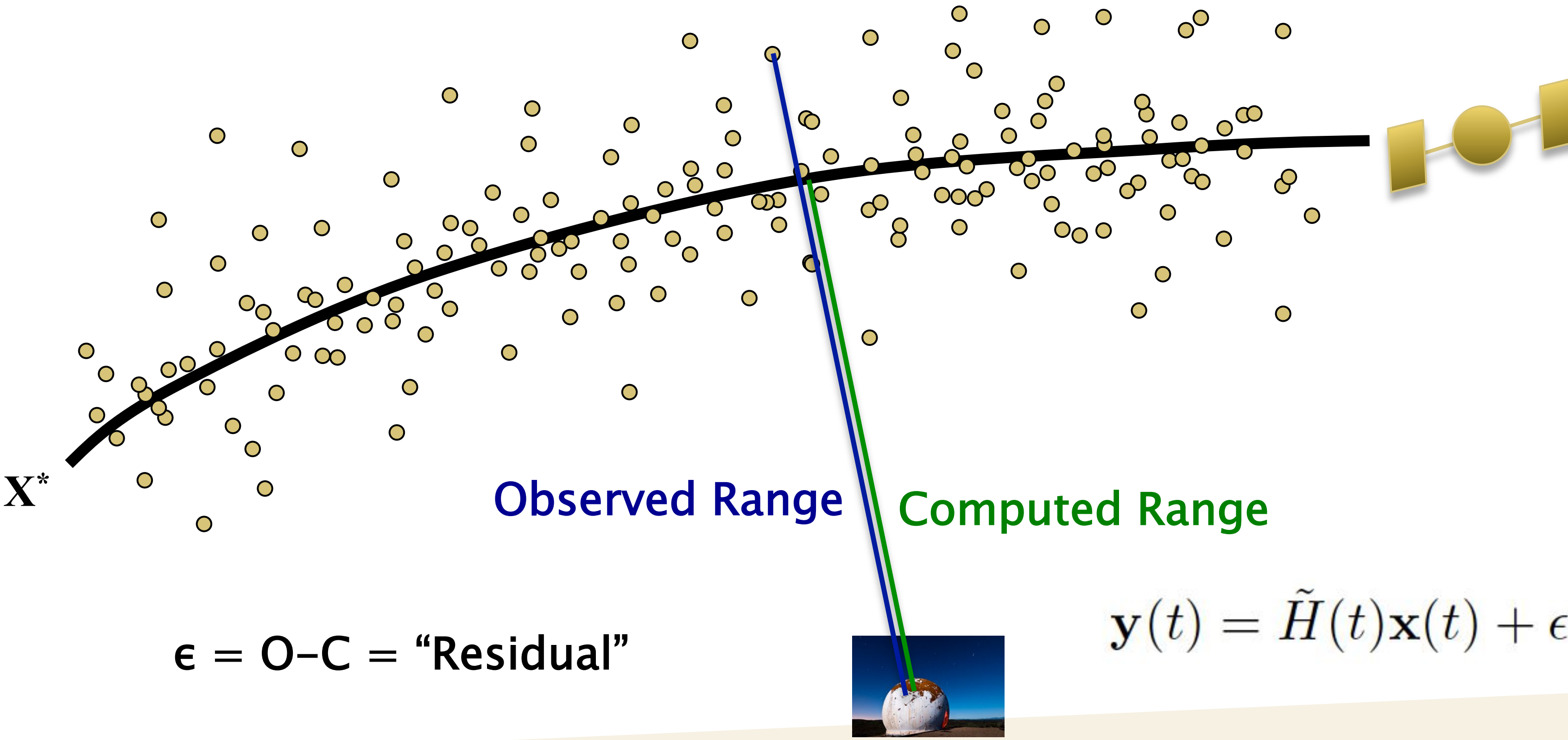
- The Mapping Matrix

$$\tilde{H}(t) = \frac{\partial \boldsymbol{h}(t, \mathbf{X}(t))}{\partial \mathbf{X}(t)}$$

$$\tilde{H}(t) = \left[\begin{array}{cccc} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \cdots & \frac{\partial \rho}{\partial \dot{z}} \\ \frac{\partial \dot{\rho}}{\partial x} & \frac{\partial \dot{\rho}}{\partial y} & \cdots & \frac{\partial \dot{\rho}}{\partial \dot{z}} \end{array} \right]$$

Mapping an observation

- Now we can map an observation to the state at an epoch.



ϵ = O–C = "Residual"

$$\mathbf{y}(t) = \tilde{H}(t)\mathbf{x}(t) + \epsilon(t)$$

How do we solve the problem?

- We have the method of least squares

$$\text{Minimize } J = \frac{1}{2}\epsilon^T\epsilon$$

$$\frac{\partial J}{\partial X} = 0$$

$$\frac{\partial^2 J}{\partial^2 X} \text{ is positive definite}$$

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$$\text{Minimize } J = \frac{1}{2}\epsilon^T\epsilon$$

$$\epsilon = O - C = (\mathbf{y} - H\mathbf{x})$$

$$J = \frac{1}{2}(\mathbf{y} - H\mathbf{x})^T(\mathbf{y} - H\mathbf{x})$$

How do we solve the problem?

- We have the method of least squares

$$\text{Minimize } J = \frac{1}{2}\epsilon^T\epsilon$$

$$J = \frac{1}{2}\left(\mathbf{y} - H\mathbf{x}\right)^T\left(\mathbf{y} - H\mathbf{x}\right)$$

$$\frac{\partial J}{\partial X} = 0$$

$$\left(\mathbf{y} - H\hat{\mathbf{x}}_k\right)^T H = 0$$

$$H^T H\hat{\mathbf{x}}_k = H^T\mathbf{y}.$$

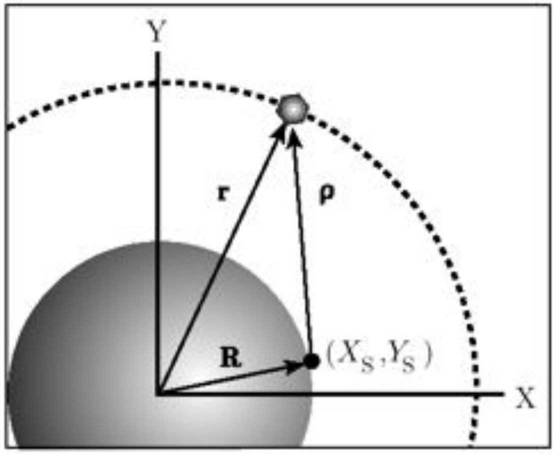
$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1}H^T\mathbf{y}.$$

Example 4.2.1

Example 4.2.1

Compute the A matrix and the $\widetilde{H}$ matrix for a satellite in a plane under the influence of only a *central force*. Assume that the satellite is being tracked with range observations, ρ, from a single ground station. Assume that the station coordinates, (X_S, Y_S), and the gravitational parameter are unknown. Then, the state vector, $\mathbf{X}$, is given by

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ U \\ V \\ \mu \\ X_S \\ Y_S \end{bmatrix}$$



where U and V are velocity components and X_S and Y_S are coordinates of the tracking station. From Newton's Second Law and the law of gravitation,

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}$$

Example 4.2.1

Or in component form:

$$\ddot{X} = -\frac{\mu X}{r^3} \qquad \ddot{Y} = -\frac{\mu Y}{r^3}$$

Expressed in first order form:

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{U} \\ \dot{V} \\ \dot{\mu} \\ \dot{X}_S \\ \dot{Y}_S \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} U \\ V \\ -\frac{\mu X}{r^3} \\ -\frac{\mu Y}{r^3} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example 4.2.1

$$A(t) = \frac{\partial F(\mathbf{X}^*, t)}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial U} & \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial \mu} & \frac{\partial F_1}{\partial X_S} & \frac{\partial F_1}{\partial Y_S} \\ \frac{\partial F_2}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_2}{\partial Y_S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_7}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_7}{\partial Y_S} \end{bmatrix}^*$$

Example 4.2.1

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$$
= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{\mu}{r^3} + \frac{3\mu X^2}{r^5} & \frac{3\mu XY}{r^5} & 0 & 0 & -\frac{X}{r^3} & 0 & 0 \\
\frac{3\mu XY}{r^5} & -\frac{\mu}{r^3} + \frac{3\mu Y^2}{r^5} & 0 & 0 & -\frac{Y}{r^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^*
$$

Example 4.2.1

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The $\widetilde{H}$ matrix is given by

$$\widetilde{H} = \frac{\partial \rho}{\partial \mathbf{X}} = \left[\begin{array}{ccccccc} \frac{\partial \rho}{\partial X} & \frac{\partial \rho}{\partial Y} & \frac{\partial \rho}{\partial U} & \frac{\partial \rho}{\partial V} & \frac{\partial \rho}{\partial \mu} & \frac{\partial \rho}{\partial X_{\mathrm{S}}} & \frac{\partial \rho}{\partial Y_{\mathrm{S}}} \end{array} \right]^*$$

where

$$\rho = \left[(X - X_{\mathrm{S}})^2 + (Y - Y_{\mathrm{S}})^2 \right]^{1/2}.$$

It follows then that

$$\widetilde{H} = \left[\begin{array}{ccccccc} \frac{X - X_{\mathrm{S}}}{\rho} & \frac{Y - Y_{\mathrm{S}}}{\rho} & 0 & 0 & 0 & -\frac{(X - X_{\mathrm{S}})}{\rho} & -\frac{(Y - Y_{\mathrm{S}})}{\rho} \end{array} \right]^*$$

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