ASEN 5070
Statistical Orbit Determination I
Fall 2012

Professor Jeffrey S. Parker
Professor George H. Born

Lecture 16: Numerical Compensations

Announcements

- Homework 5 (?) and the test are graded.
  - Contact me within the week to challenge any grading.
- Homework 6 will be graded shortly
- Homework 7 due this week.
  - Points for quality

- Guest lecturer this Thursday
  - Jason Leonard will speak about LiAISON Navigation

- I'm unavailable Wednesday – Monday
  - Will be around sometimes, so you may catch me.
  - But I will be missing office hours – email me or call me if you have questions.

Exam 1 Debrief Again

- I have a few corrections regarding last week's debrief. Sorry for any confusion – I made a mistake when I announced the answers!

1. 20% Answer the following and be sure to provide sufficient explanation. If you need more space, use an additional sheet of paper.

   a. If one column of the $\tilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

      If it is always singular, provide an explanation for why.
      If it is not always singular, provide an example when it is not.

Exam 1 ReDebrief

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1. 20% Answer the following and be sure to provide sufficient explanation. If you need more space, use an additional sheet of paper.

   a. If one column of the $\tilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

      If it is always singular, provide an explanation for why.
      If it is not always singular, provide an example when it is not.

True. The transpose of a column of zeros becomes a row of zeros. That row propagates through the whole system and the H^TH matrix becomes singular.

Exam 1 ReDebrief

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b. If one row of the $\tilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

If it is always singular, provide an explanation for why.
If it is not always singular, provide an example when it is not.

b. If one row of the $\tilde{H}_i$ matrix is zero, the information matrix $H_i^T H_i$ will always be singular. T or F

If it is always singular, provide an explanation for why.
If it is not always singular, provide an example when it is not.

False. Consider H-tilde = [a, b, 0]^T and Phi = 1.

Exam 1 Debrief

2. 15% A random variable has the probability density function given by:

$$f(x) = \begin{cases} 1/3, & 0 \leq x < 1, \\ k\left(x^3 + 1/4\right), & 1 \leq x \leq 2, \\ 0, & \text{elsewhere} \end{cases}$$

Find k

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$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 \frac{1}{3}dx + \int_1^2 k\left(x^3 + \frac{1}{4}\right)dx = 1$$

$$\frac{1}{3}[x]_0^1 + k\left[\frac{x^4}{4} + \frac{1}{4}x\right]_1^2 = 1$$

$$\frac{1}{3} + k(4 + \frac{1}{2} - \frac{1}{4} - \frac{1}{4}) = 1$$

$$\frac{1}{3} + 4k = 1$$

$$k = \frac{2/3}{4} = \frac{1}{6}$$

Exam 1 Debrief

3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

where a, b, c, d, x_0, and x_1 are constants and t_i are given. Which of the following state vectors are observable (assuming that the constants that aren't in the state are known perfectly):

1. $\mathbf{X} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ 2. $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ 3. $\begin{bmatrix} b \\ x_0 \\ d \end{bmatrix}$ 4. $\begin{bmatrix} a \\ c \\ x_1 \end{bmatrix}$ 5. $\begin{bmatrix} b \\ c \\ d \end{bmatrix}$

Exam 1 Debrief

3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

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1. $\mathbf{X} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ 2. $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ 3. $\begin{bmatrix} b \\ x_0 \\ d \end{bmatrix}$ 4. $\begin{bmatrix} a \\ c \\ x_1 \end{bmatrix}$ 5. $\begin{bmatrix} b \\ c \\ d \end{bmatrix}$

answer: 1 and 5 are observable

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3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

Consider estimating c and x_1

$$\tilde{H}(t_i) = \begin{bmatrix} \frac{\partial y(t_i)}{\partial c} & \frac{\partial y(t_i)}{\partial x_1} \end{bmatrix}$$
$$= \begin{bmatrix} t_i & t_i \end{bmatrix}$$

$$H^T H = \begin{bmatrix} t_i & t_2 & \cdots & t_l \\ t_i & t_2 & \cdots & t_l \end{bmatrix} \begin{bmatrix} t_i & t_i \\ t_2 & t_2 \\ \vdots & \vdots \\ t_l & t_l \end{bmatrix}$$
$$= \begin{bmatrix} \sum_1^l t_i^2 & \sum_1^l t_i^2 \\ \sum_1^l t_i^2 & \sum_1^l t_i^2 \end{bmatrix}$$

Exam 1 Debrief

3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

Consider estimating a and b

$$\tilde{H}(t_i) = \begin{bmatrix} \frac{\partial y(t_i)}{\partial a} & \frac{\partial y(t_i)}{\partial b} \end{bmatrix} = \begin{bmatrix} t_i^2 & x_0 t_i^2 \end{bmatrix}$$

$$H^T H = \begin{bmatrix} t_1^2 & t_2^2 & \cdots & t_l^2 \\ x_0 t_1^2 & x_0 t_2^2 & \cdots & x_0 t_l^2 \end{bmatrix} \begin{bmatrix} t_1^2 & x_0 t_1^2 \\ t_2^2 & x_0 t_2^2 \\ \vdots & \vdots \\ t_l^2 & x_0 t_l^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_1^l t_i^4 & \sum_1^l x_0 t_i^4 \\ \sum_1^l x_0 t_i^4 & \sum_1^l x_0^2 t_i^4 \end{bmatrix} = \begin{bmatrix} \sum_1^l t_i^4 & x_0 \sum_1^l t_i^4 \\ x_0 \sum_1^l t_i^4 & x_0^2 \sum_1^l t_i^4 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_1^l t_i^4 & x_0 \sum_1^l t_i^4 \\ \sum_1^l t_i^4 & x_0 \sum_1^l t_i^4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x_0 \end{bmatrix}$$

Exam 1 Debrief

3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

Consider estimating b and x_0

$$\tilde{H}(t_i) = \begin{bmatrix} \frac{\partial y(t_i)}{\partial b} & \frac{\partial y(t_i)}{\partial x_0} \end{bmatrix} = \begin{bmatrix} x_0 t_i^2 & b t_i^2 \end{bmatrix}$$

$$H^T H = \begin{bmatrix} x_0 t_1^2 & x_0 t_2^2 & \cdots & x_0 t_l^2 \\ b t_1^2 & b t_2^2 & \cdots & b t_l^2 \end{bmatrix} \begin{bmatrix} x_0 t_1^2 & b t_1^2 \\ x_0 t_2^2 & b t_2^2 \\ \vdots & \vdots \\ x_0 t_l^2 & b t_l^2 \end{bmatrix}$$

$$= \begin{bmatrix} x_0^2 \sum_1^l t_i^4 & b x_0 \sum_1^l t_i^4 \\ b x_0 \sum_1^l t_i^4 & b^2 \sum_1^l t_i^4 \end{bmatrix}$$

$$= \begin{bmatrix} x_0 \sum_1^l t_i^4 & b \sum_1^l t_i^4 \\ x_0 \sum_1^l t_i^4 & b \sum_1^l t_i^4 \end{bmatrix} \begin{bmatrix} x_0 & 0 \\ 0 & b \end{bmatrix}$$

Exam 1 Debrief

3. 15% Given the observation-state equation

$$y(t_i) = (a + bx_0)t_i^2 + (c + x_1)t_i + d \qquad i = 1, 2, \ldots, l$$

Consider estimating a and c

$$\tilde{H}(t_i) = \begin{bmatrix} \frac{\partial y(t_i)}{\partial a} & \frac{\partial y(t_i)}{\partial c} \end{bmatrix} = \begin{bmatrix} t_i^2 & t_i \end{bmatrix}$$

$$H^T H = \begin{bmatrix} t_1^2 & t_2^2 & \cdots & t_l^2 \\ t_1 & t_2 & \cdots & t_l \end{bmatrix} \begin{bmatrix} t_1^2 & t_1 \\ t_2^2 & t_2 \\ \vdots & \vdots \\ t_l^2 & t_l \end{bmatrix} = \begin{bmatrix} \sum_1^l t_i^4 & \sum_1^l t_i^3 \\ \sum_1^l t_i^3 & \sum_1^l t_i^2 \end{bmatrix}$$

Exam 1 Debrief

4. 10% Which weighting matrix is the most appropriate to use given two observations and the second is twice as accurate as the first:

1. $W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2. $W = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ 3. $W = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ 4. $W = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ 5. $W = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

Exam 1 Debrief

4. 10% Which weighting matrix is the most appropriate to use given two observations and the second is twice as accurate as the first:

1. $W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2. $W = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ 3. $W = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ 4. $W = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ 5. $W = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

answer: 3

Exam 1 Debrief

5. 40% The equation of motion for a particular system is given by

$$\ddot{z} - 3z^2\dot{z} = c$$

and we wish to estimate the state

$$\mathbf{X} = [\ z(t_0),\ \dot{z}(t_0),\ c\ ]^T$$

using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

a. 5% What is the degree and order of this equation of motion?

Exam 1 Debrief

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

a. 5% What is the degree and order of this equation of motion?

   a. 1st degree, 2nd order

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

b. 5% Is this equation of motion linear or nonlinear?

5. 40% The equation of motion for a particular system is given by

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

b. 5% Is this equation of motion linear or nonlinear?

b. Nonlinear

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c. 10% What is the $A(t)$ matrix?

Exam 1 Debrief

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$$\ddot{z} - 3z^2\dot{z} = c$$
and we wish to estimate the state
$$\mathbf{X} = [\ z(t_0),\ \dot{z}(t_0),\ c\ ]^T$$
using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

c. 10% What is the $A(t)$ matrix?

$$\dot{\mathbf{X}}(t) = \begin{bmatrix} \dot{z} \\ \ddot{z} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} \dot{z} \\ 3z^2\dot{z} + c \\ 0 \end{bmatrix}$$

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial \dot{z}} & \frac{\partial \dot{z}}{\partial c} \\ \frac{\partial \ddot{z}}{\partial z} & \frac{\partial \ddot{z}}{\partial \dot{z}} & \frac{\partial \ddot{z}}{\partial c} \\ \frac{\partial \dot{c}}{\partial z} & \frac{\partial \dot{c}}{\partial \dot{z}} & \frac{\partial \dot{c}}{\partial c} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 6z\dot{z} & 3z^2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Exam 1 Debrief

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and we wish to estimate the state

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

d. 5% What is the $\widetilde{H}$ matrix?

5. 40% The equation of motion for a particular system is given by

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

d. 5% What is the $\tilde{H}$ matrix?

$$G(t) = z(t)$$

$$\tilde{H}(t) = \frac{\partial z}{\partial \mathbf{X}} = \left[\ \frac{\partial z}{\partial z} \quad \frac{\partial z}{\partial \dot{z}} \quad \frac{\partial z}{\partial c}\ \right]$$

$$= \left[\ 1 \quad 0 \quad 0\ \right]$$

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5. 40% The equation of motion for a particular system is given by

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and we wish to estimate the state

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

d. 5% What is the $\tilde{H}$ matrix?

$$G(t) = z(t) = \left(\frac{\ddot{z} - c}{3\dot{z}}\right)^{1/2}$$

$$\tilde{H}(t) = \frac{\partial z}{\partial \mathbf{X}} = \left[\ \frac{\partial z}{\partial z} \quad \frac{\partial z}{\partial \dot{z}} \quad \frac{\partial z}{\partial c}\ \right]$$

$$= \left[\ 1 \quad -\frac{1}{2}\left(\frac{\ddot{z}-c}{3\dot{z}^3}\right)^{1/2} \quad -\frac{1}{2}\left(\frac{\ddot{z}-c}{3\dot{z}}\right)^{1/2}\ \right]$$

Exam 1 Debrief

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$$\mathbf{X} = [\ z(t_0),\ \ \dot{z}(t_0),\ \ c\ ]^T$$

using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

e. 10% How would you determine the state transition matrix, $\Phi(t_i, t_0)$? Note: don't attempt to compute it. Can you use a Laplace Transform? Why/why not?

Exam 1 Debrief

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

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e.

$$\begin{aligned}\dot{\Phi}(t, t_0) &= A(t)\Phi(t, t_0) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 6z\dot{z} & 3z^2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Phi(t, t_0)\end{aligned}$$

Can't use a Laplace Transform because A(t) is time-dependent!

Exam 1 Debrief

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

f. 5% What is the minimum number of independent observations of z that is required to generate an estimate of the state?

Exam 1 Debrief

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using observations of $z(t)$. That is, $z(t)$ is the observation. Answer the following.

f. 5% What is the minimum number of independent observations of z that is required to generate an estimate of the state?

f. 3

Quiz Review

Question 1 (1 point)

If you have a perfectly observable system and infinite precision, the Batch process and the Sequential process (i.e., the conventional Kalman filter) will generate identical estimates of your state, given the same observations and a priori information.

- True
- False

Quiz Review

Question 1 (1 point)

If you have a perfectly observable system and infinite precision, the Batch process and the Sequential process (i.e., the conventional Kalman filter) will generate identical estimates of your state, given the same observations and a priori information.

- True
- False

Quiz Review

Question 2 (1 point)

If you have a perfectly observable system and infinite precision, the Batch process and the Extended Kalman Filter will generate identical estimates of your state, given the same observations and a priori information.

- True
- False

Quiz Review

Question 2 (1 point)

If you have a perfectly observable system and infinite precision, the Batch process and the Extended Kalman Filter will generate identical estimates of your state, given the same observations and a priori information.

- True
- False

Quiz Review

Question 3 (1 point)

The Extended Kalman Filter differs from the Conventional Kalman Filter (Sequential) in such a way that it permits the estimated state's variance-covariance matrix (P) to be asymmetric.

- True
- False

Quiz Review

Question 3 (1 point)

The Extended Kalman Filter differs from the Conventional Kalman Filter (Sequential) in such a way that it permits the estimated state's variance-covariance matrix (P) to be asymmetric.

- True
- False

Quiz Review

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Question 4 (1 point)

Say we have a machine that has 16 digits of precision. That is, it can recognize the difference between 1.000000000000000 (15 zeros) and 1+e where e=1e-15. (please remember we're conceptualizing).

In this machine, if d=1e-16, then 1+d=1 and the machine loses the information in d.

Question: What do I get on this machine if I take the number 1.034e-15 and add it to 1, and then subtract 1 again?

- 1.034e-15
- 0
- 1
- 1e-15

Quiz Review

Question 4 (1 point)

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In this machine, if d=1e-16, then 1+d=1 and the machine loses the information in d.

Question: What do I get on this machine if I take the number 1.034e-15 and add it to 1, and then subtract 1 again?

- 1.034e-15
- 0
- 1
- 1e-15

NOTE: This is a demo for why machines have limits.
Computer round-off = bad.

Topics

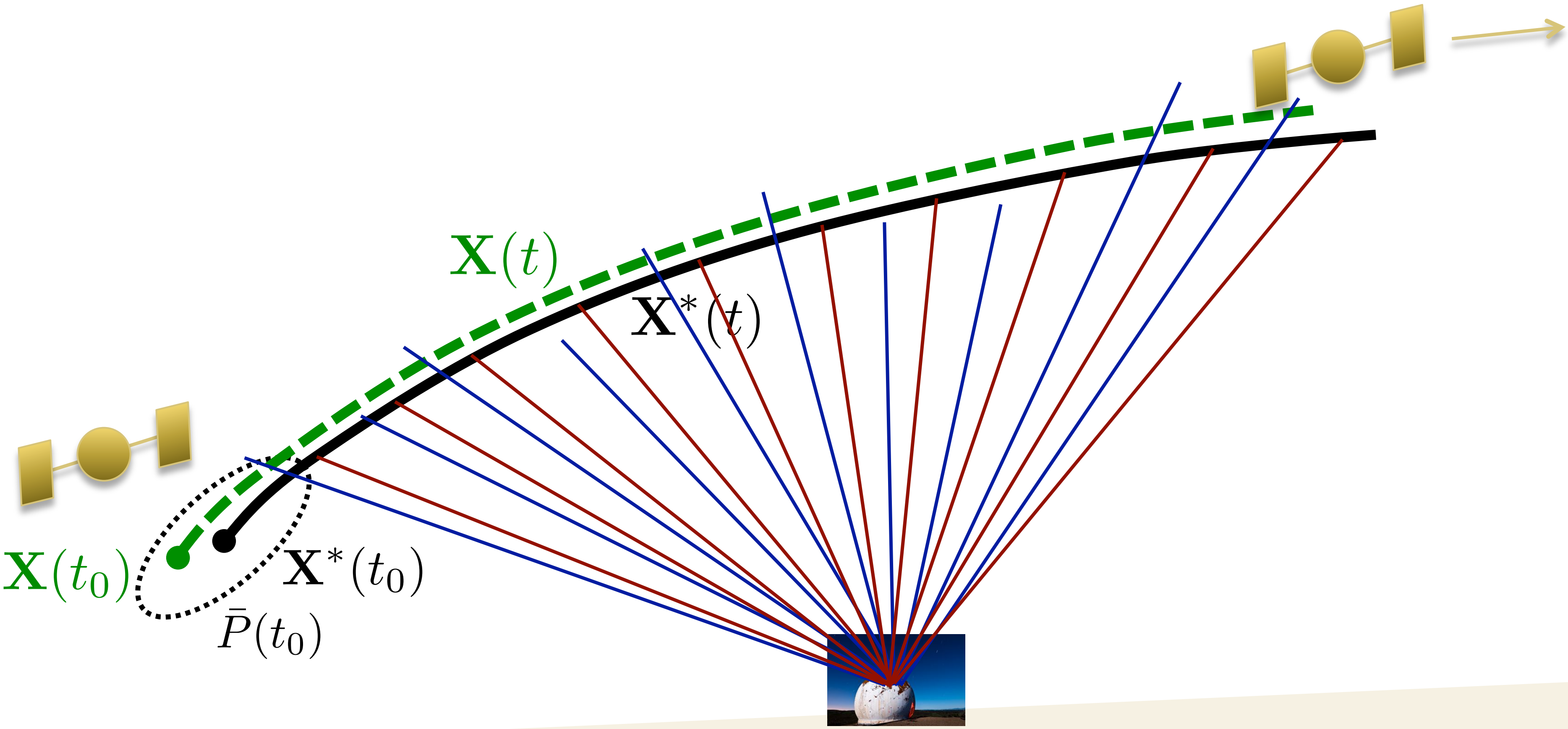
- Conventional Kalman Filter (CKF)
- Extended Kalman Filter (EKF)

- Numerical Issues
  - Machine precision
  - Covariance collapse

- Numerical Compensation
  - Joseph, Potter, Cholesky, Square-root free, unscented, Givens, orthogonal transformation, SVD
  - State Noise Compensation, Dynamical Model Compensation

Stat OD Conceptualization

- Full, nonlinear system:



Stat OD Conceptualization

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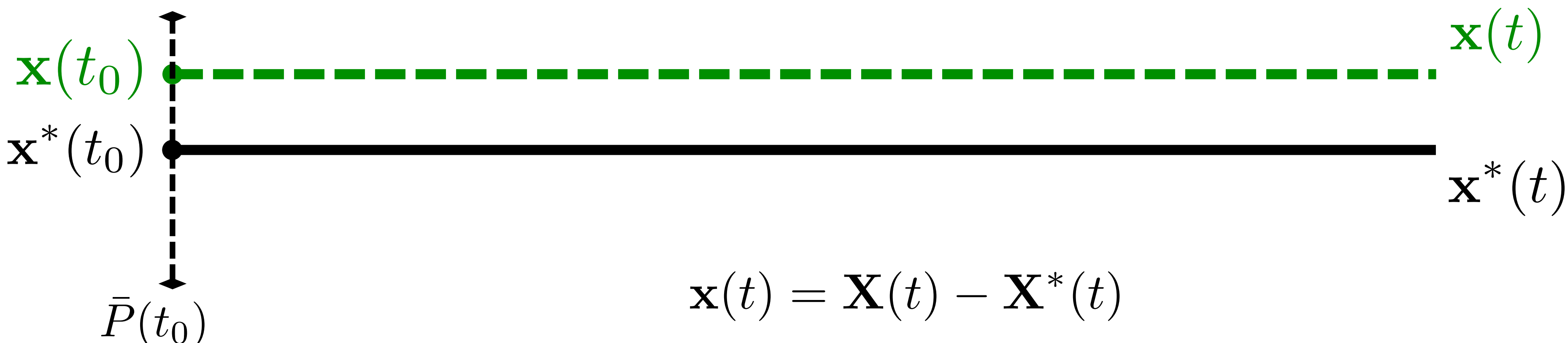
- Linearization

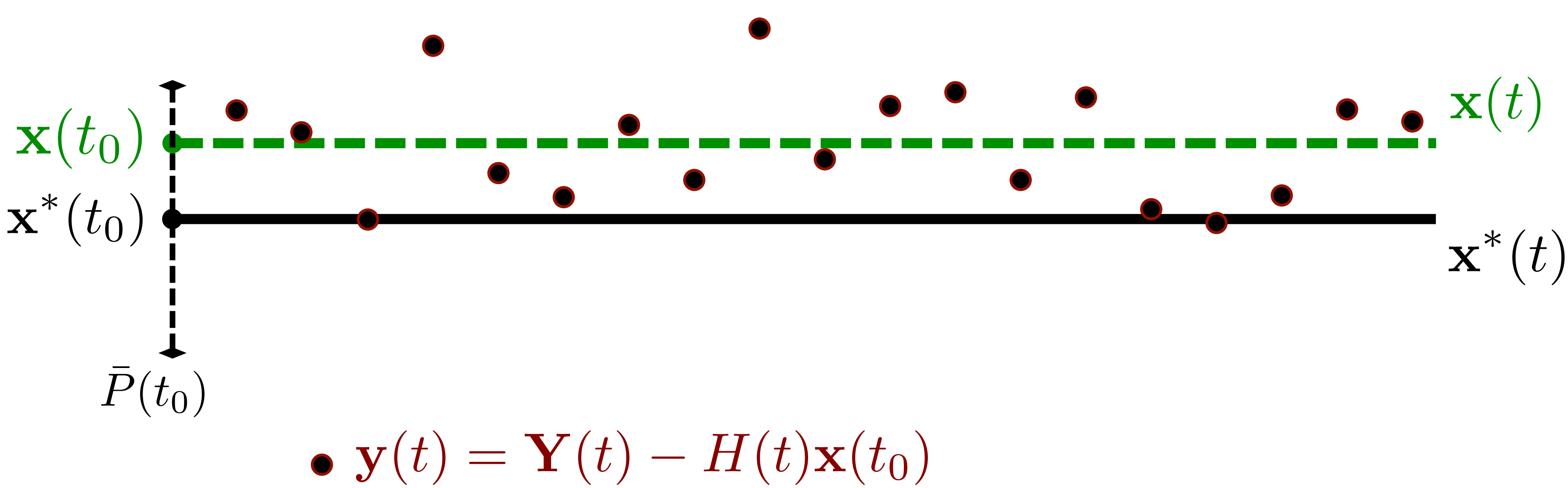


$$\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t)$$

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$$

Stat OD Conceptualization

- Observations



Stat OD Conceptualization

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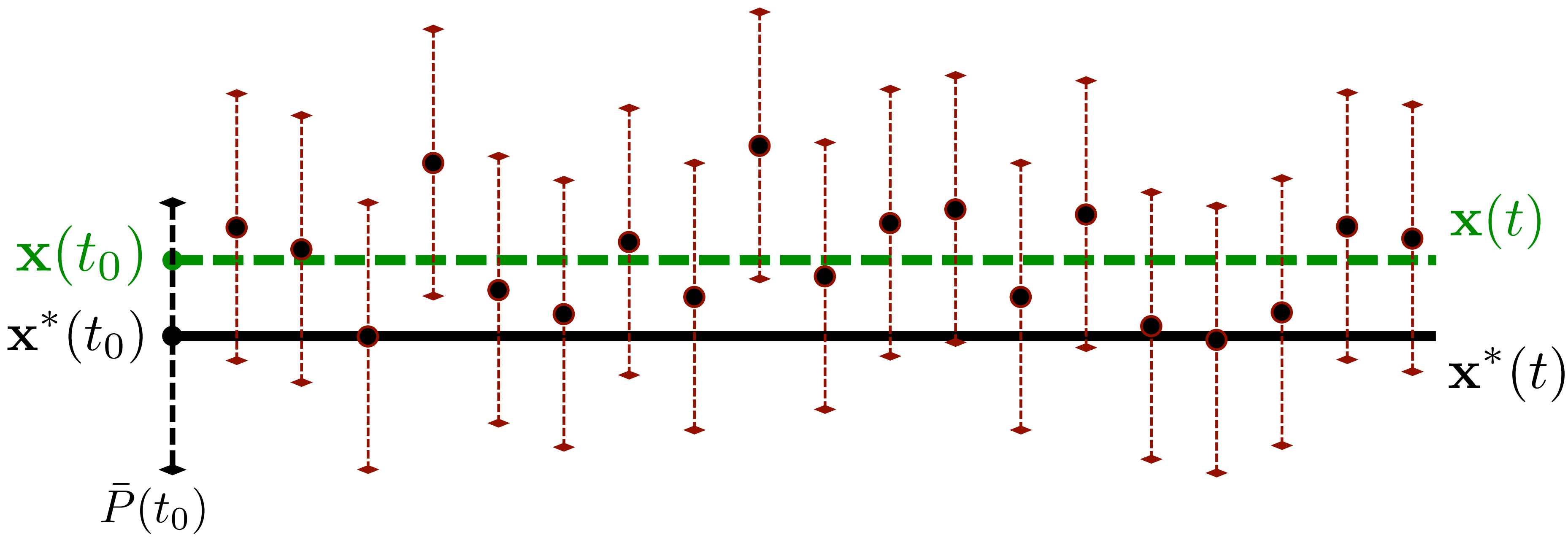
- Observation Uncertainties

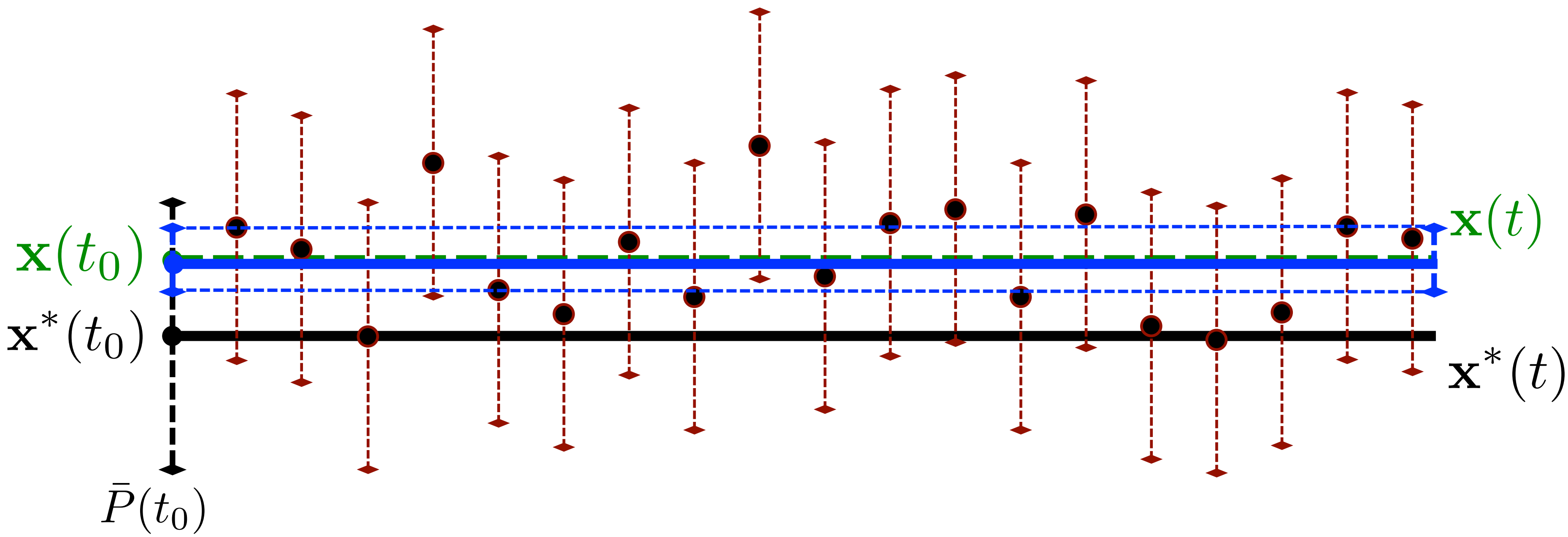


- $\mathbf{y}(t) = \mathbf{Y}(t) - H(t)\mathbf{x}(t_0)$

$R(t)$

Stat OD Conceptualization

- Least Squares (Batch)

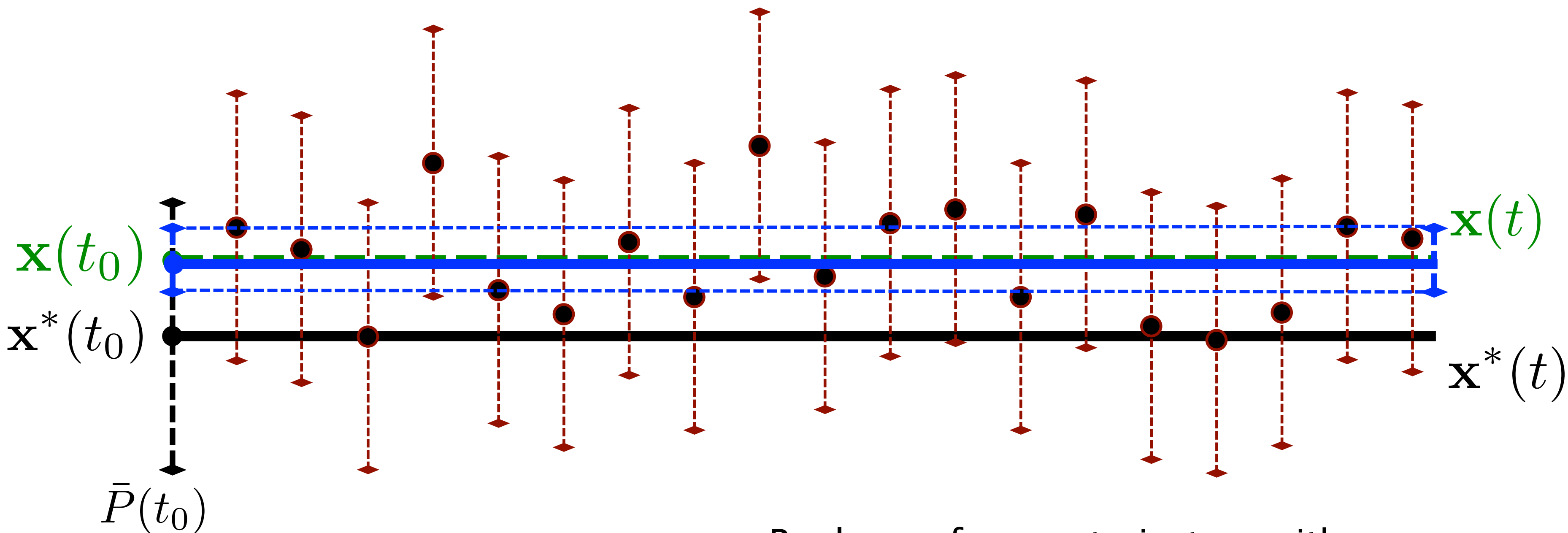


$$\hat{\mathbf{x}}(t_0) = \left(H^T R^{-1} H + \bar{P}^{-1}\right)^{-1} \left(H^T R^{-1} \mathbf{y} + \bar{P}^{-1} \bar{\mathbf{x}}\right)$$

$P(t)$

Stat OD Conceptualization

- Least Squares (Batch)



Iterate a few times.

- Replace reference trajectory with best-estimate
- Update *a priori* state
- Generate new computed observations

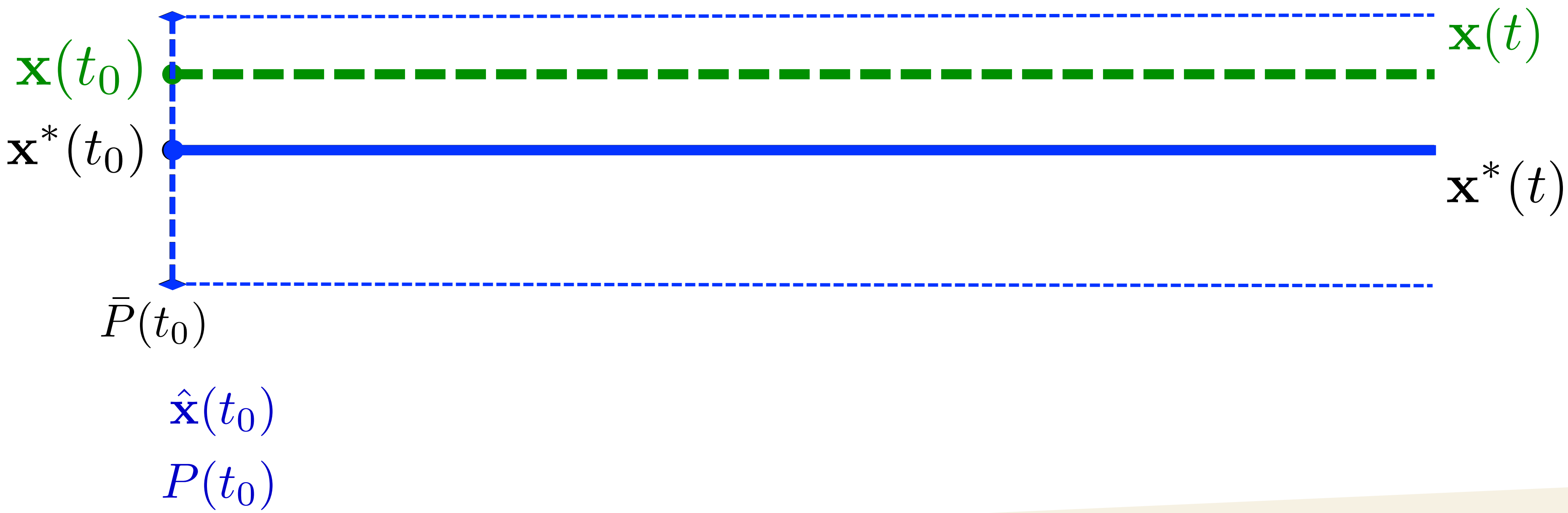
Side Note

- Trajectory remains within the linear region longer if you model the dynamics very well.
- Always a limit
- ARTEMIS addressed this by deweighting older measurements
    - Makes sense because ARTEMIS Nav only cared about where the spacecraft is and not where it has been.

- Conceptualization of the Conventional Kalman Filter (Sequential Filter)

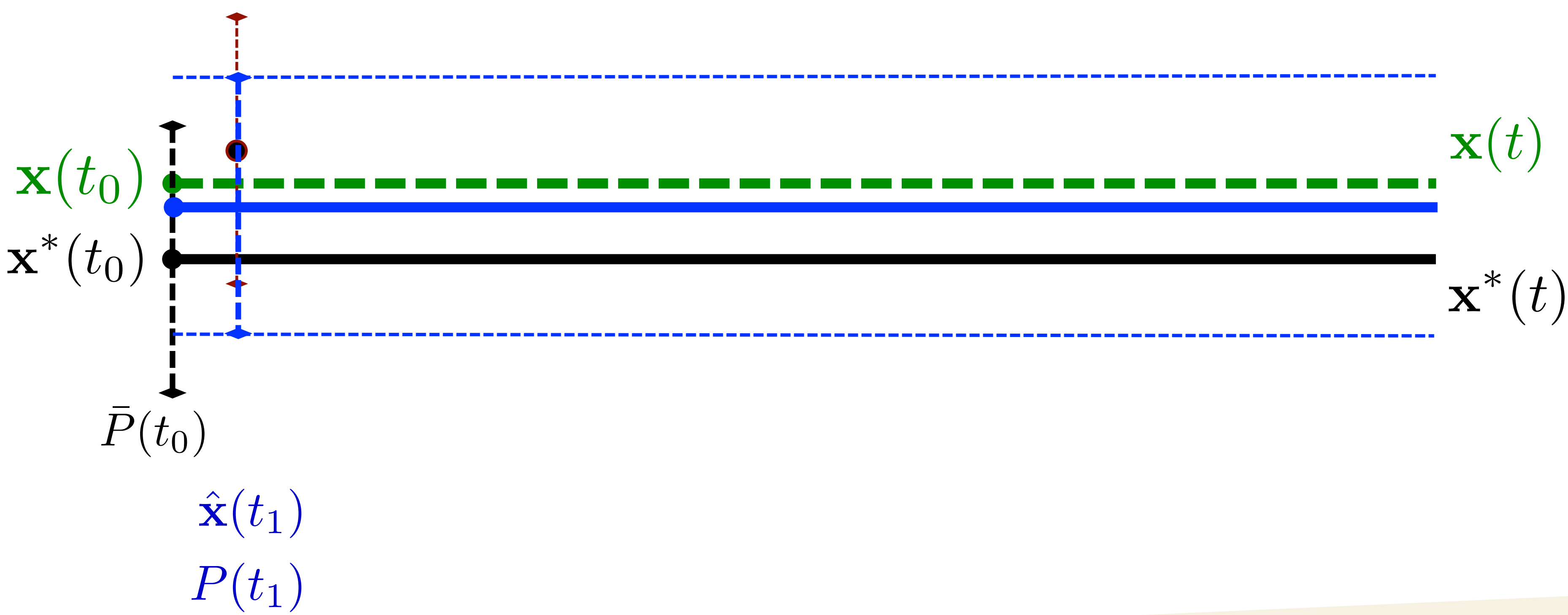
Stat OD Conceptualization

- Conventional Kalman



Stat OD Conceptualization

- Conventional Kalman



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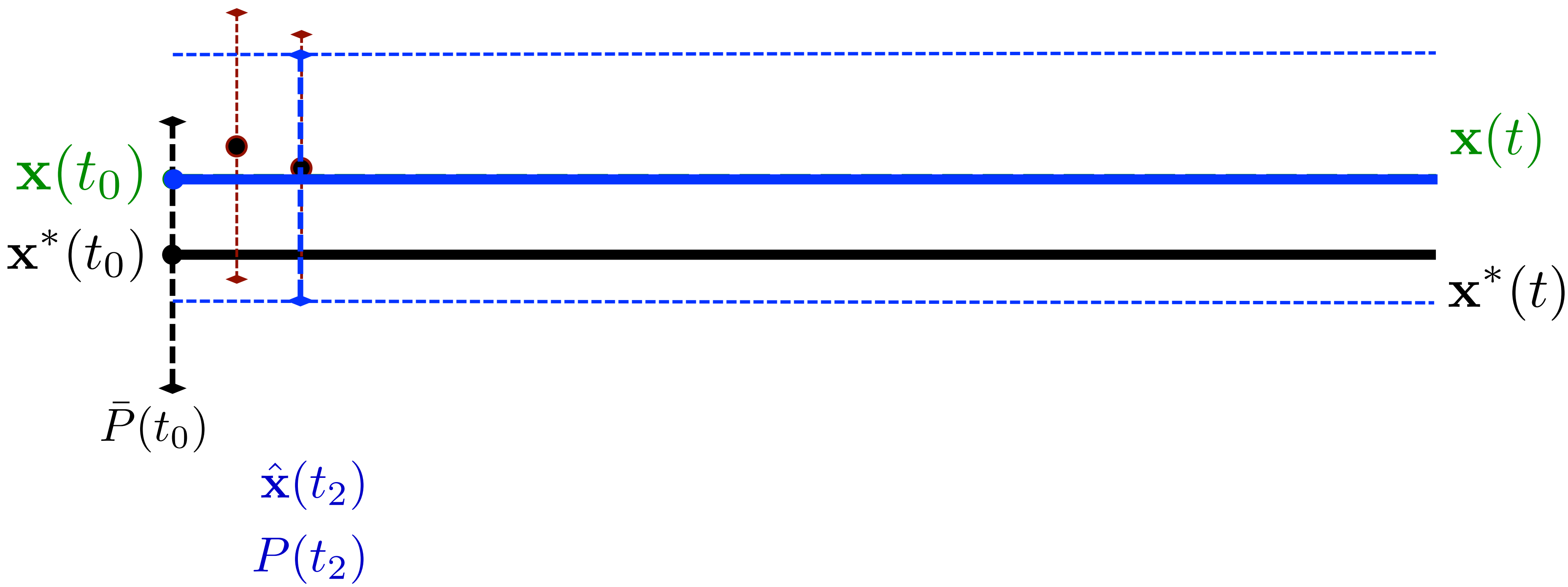
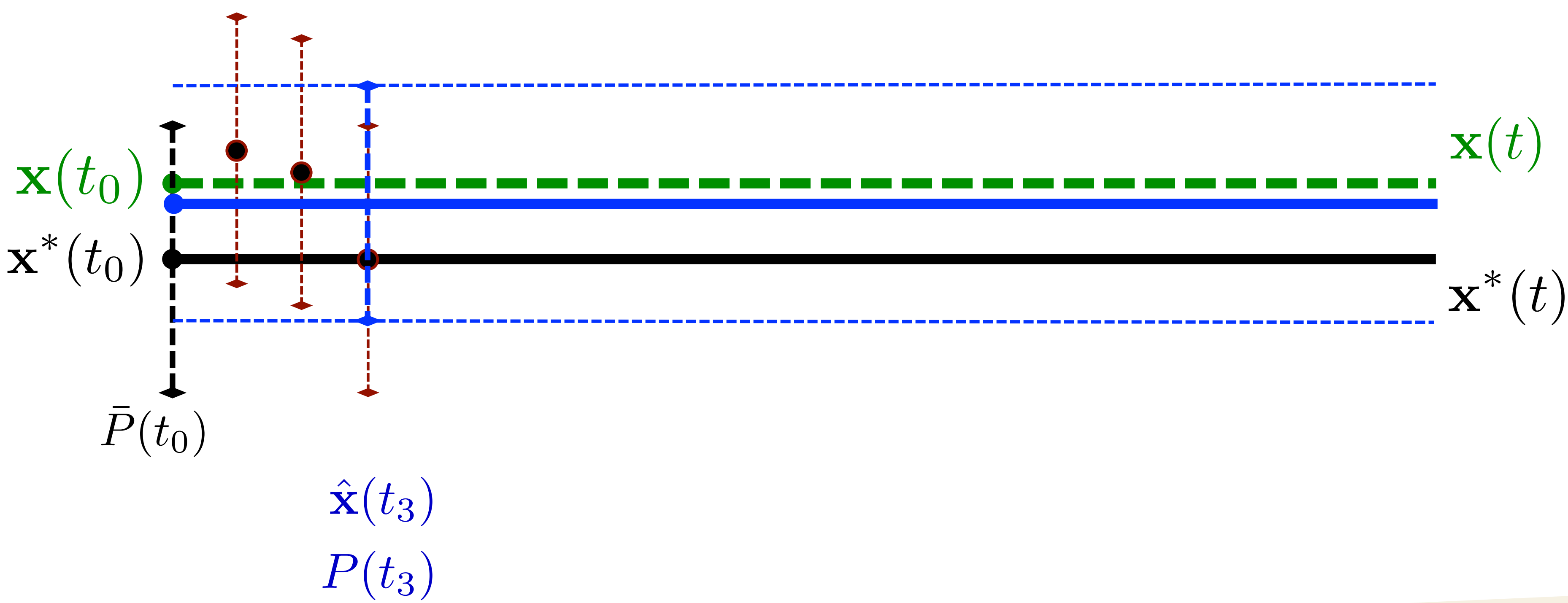
- Conventional Kalman



Stat OD Conceptualization

- Conventional Kalman



Stat OD Conceptualization

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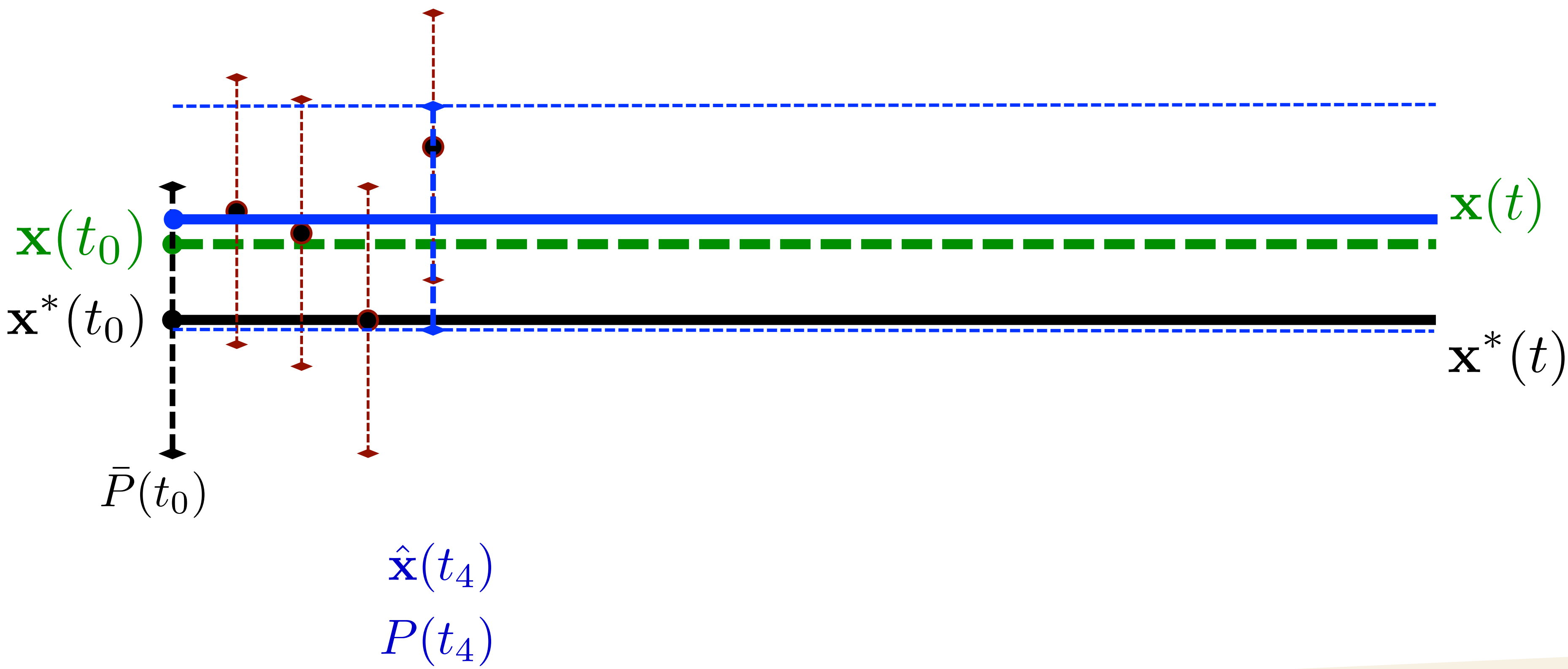
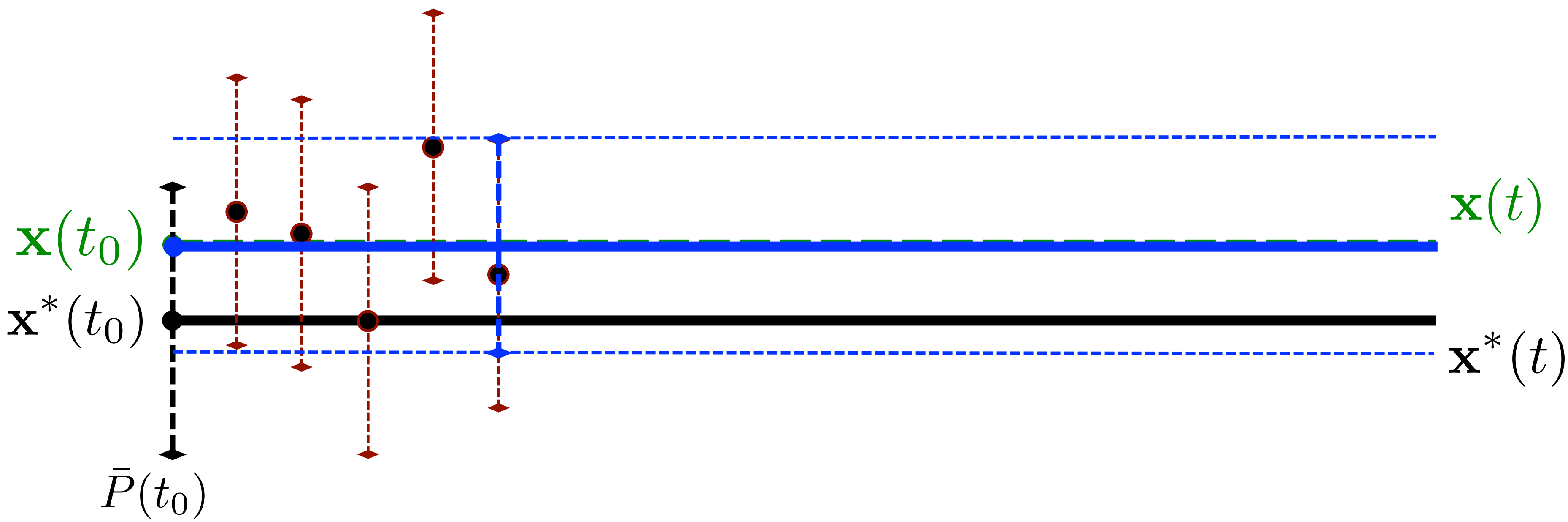
- Conventional Kalman



Stat OD Conceptualization

- Conventional Kalman



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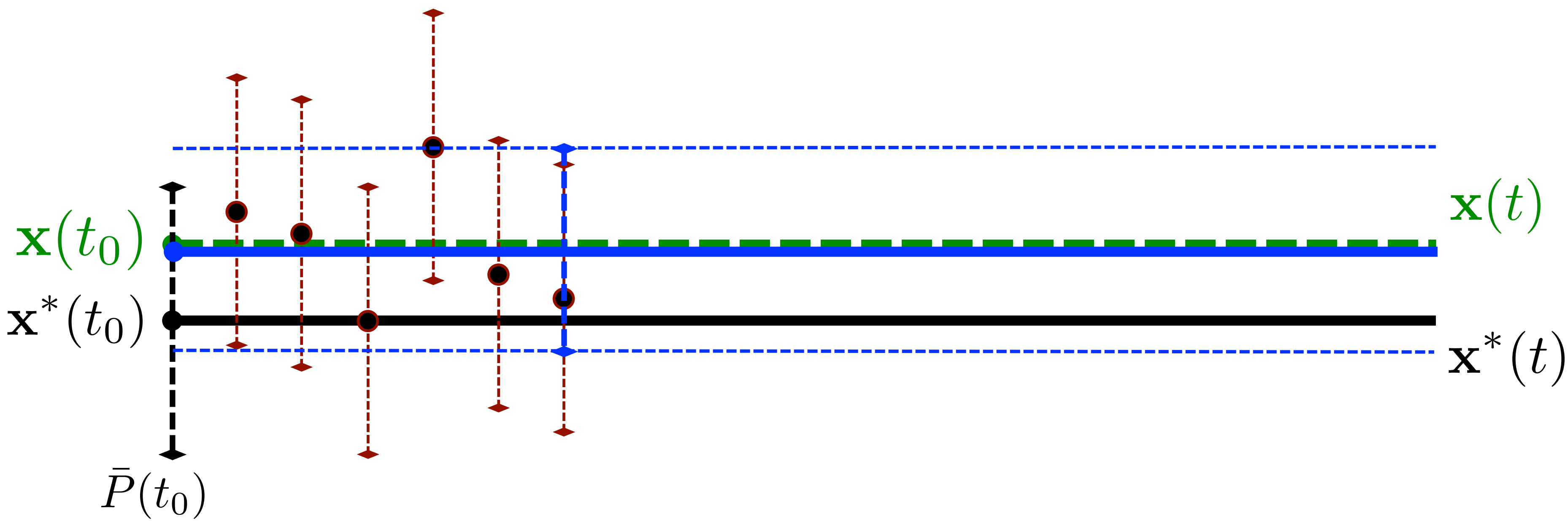

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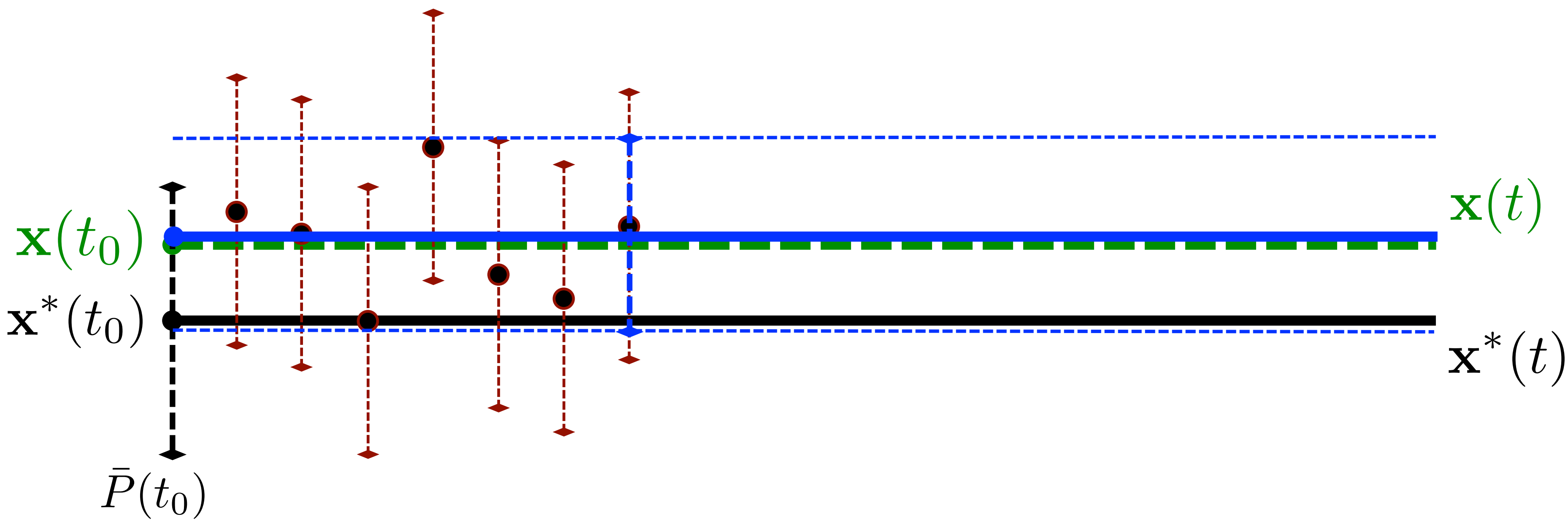

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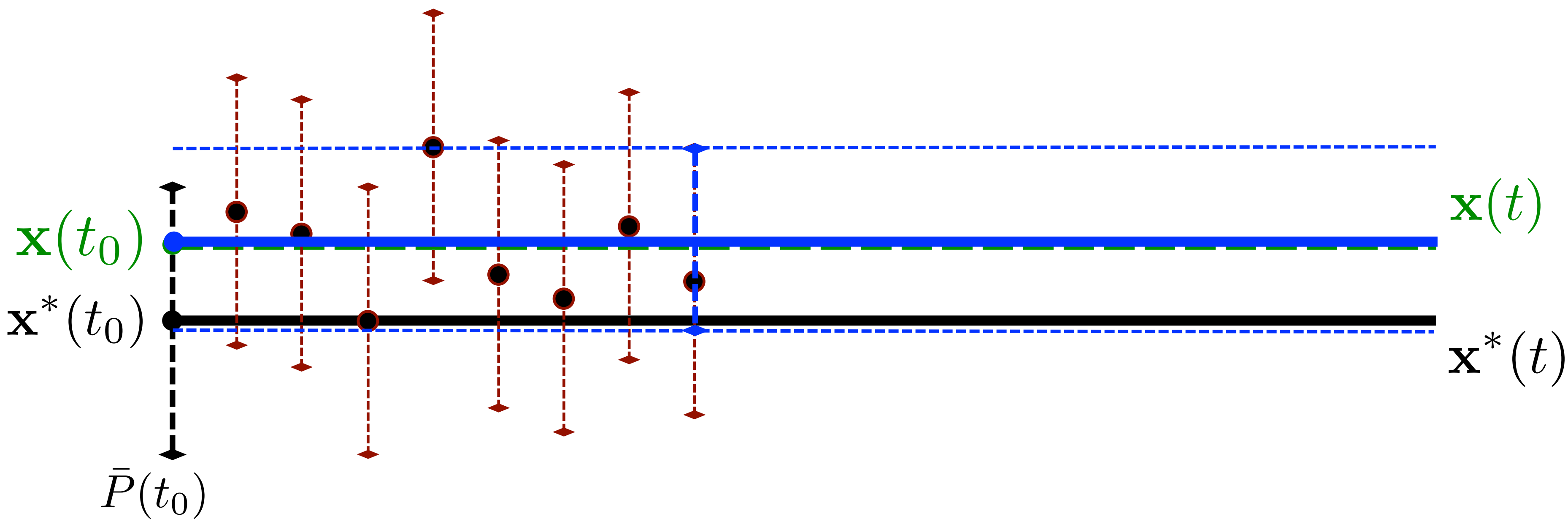

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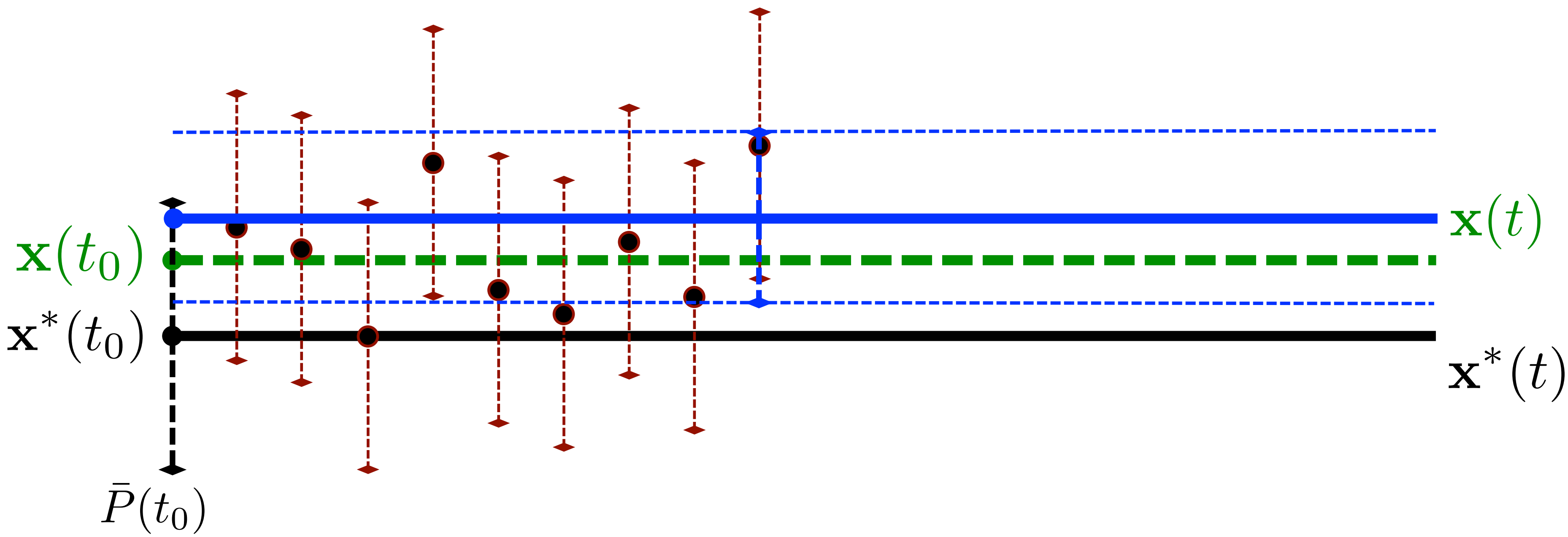

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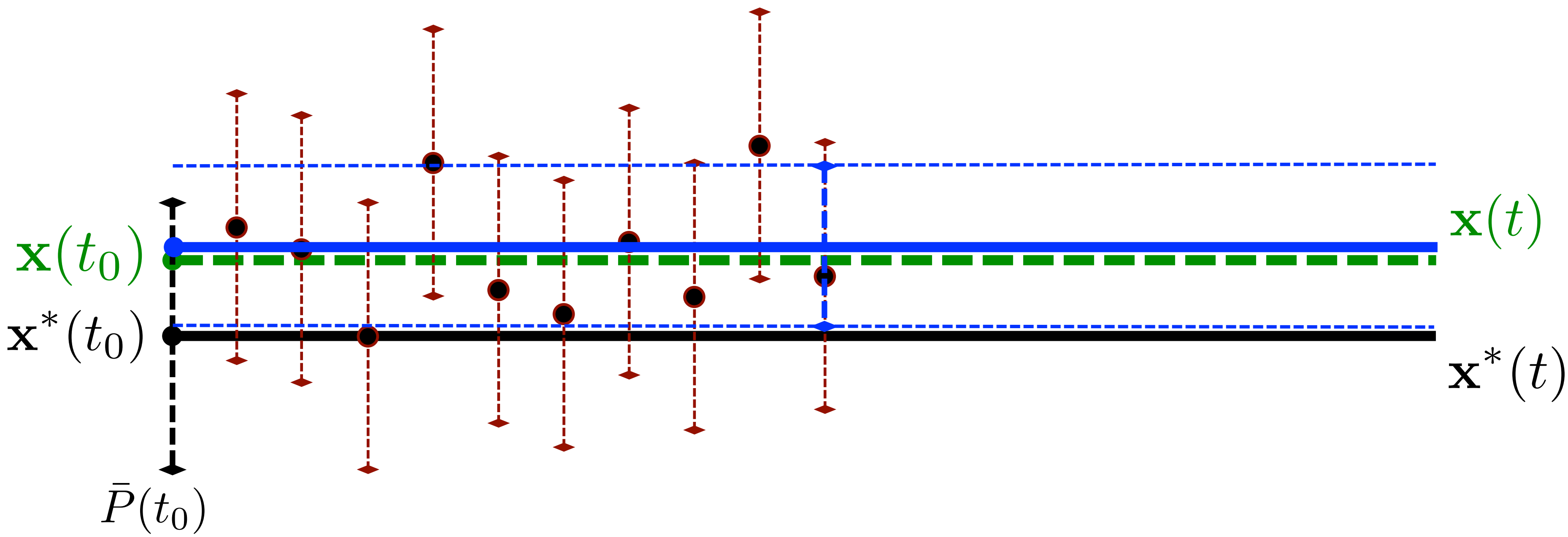

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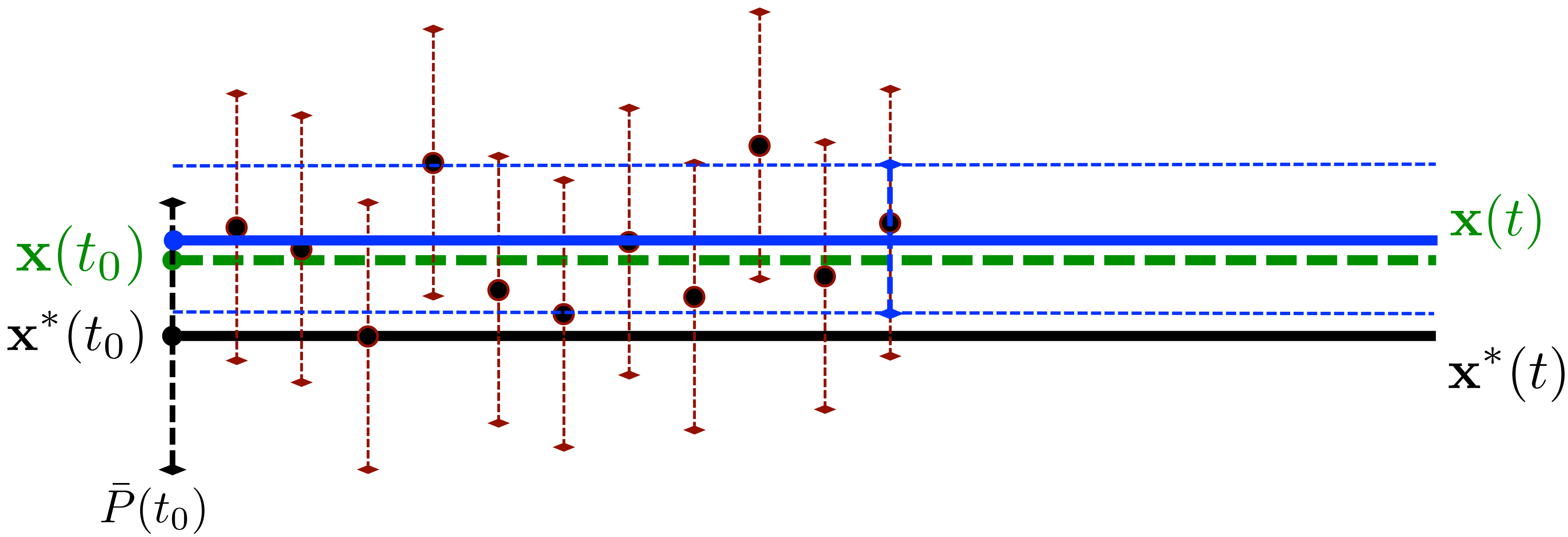

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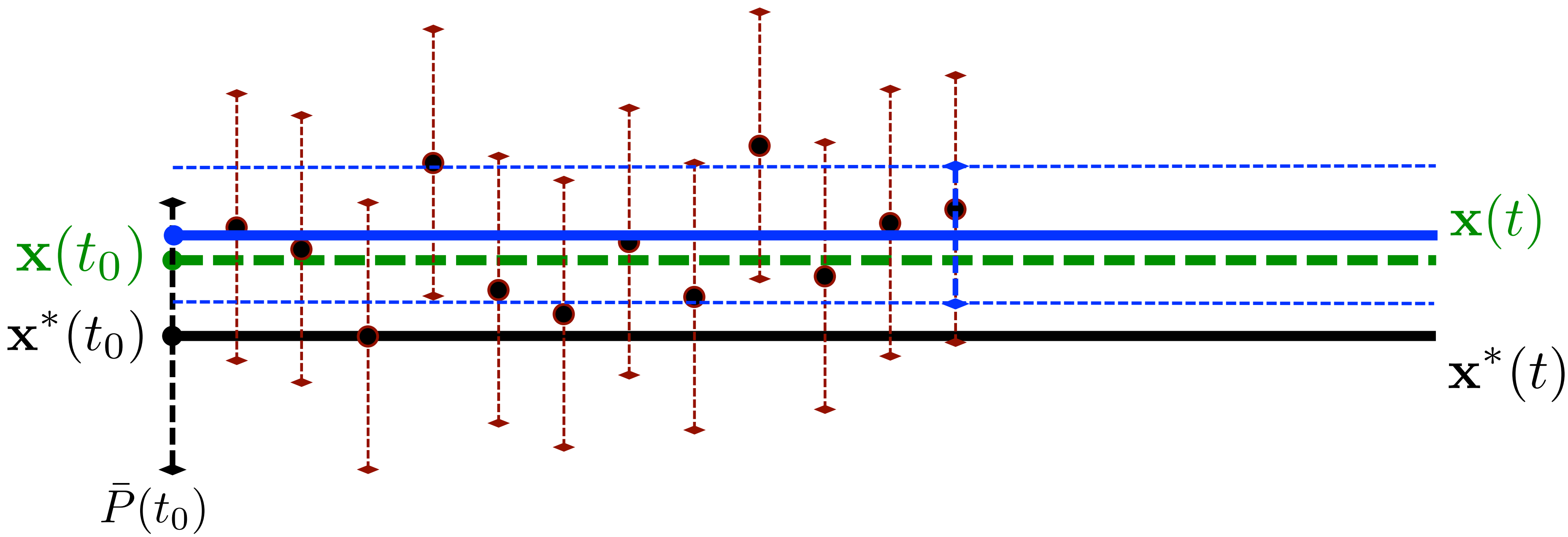

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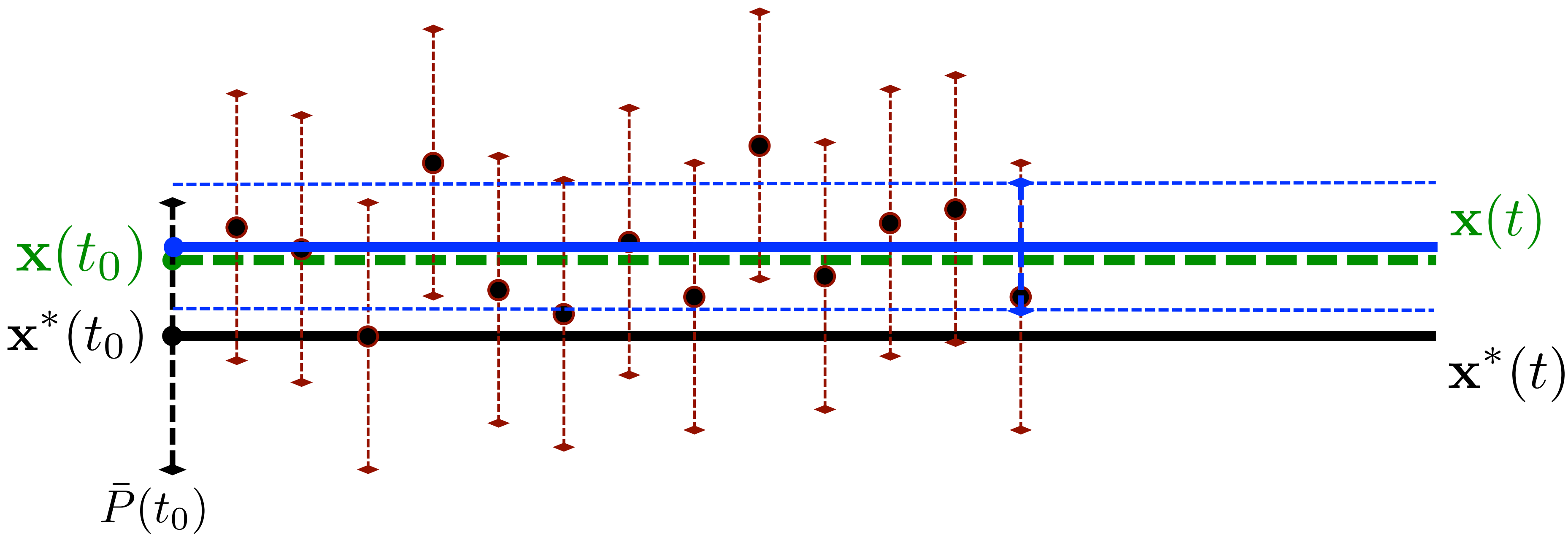

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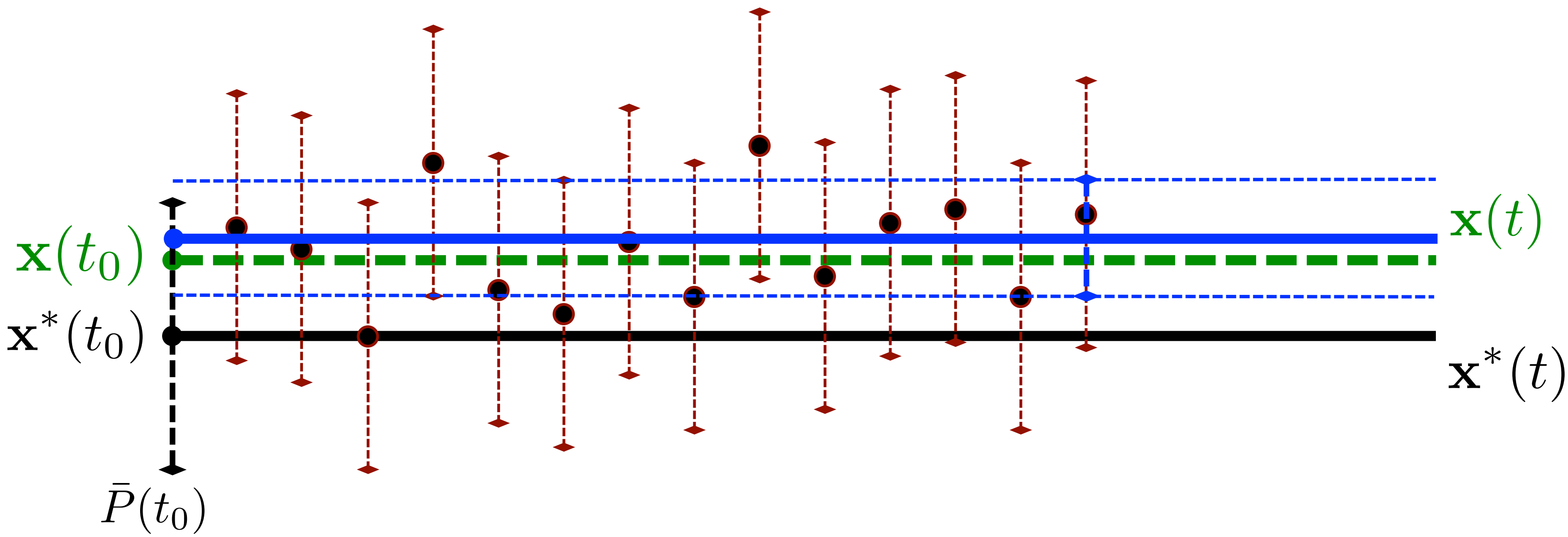

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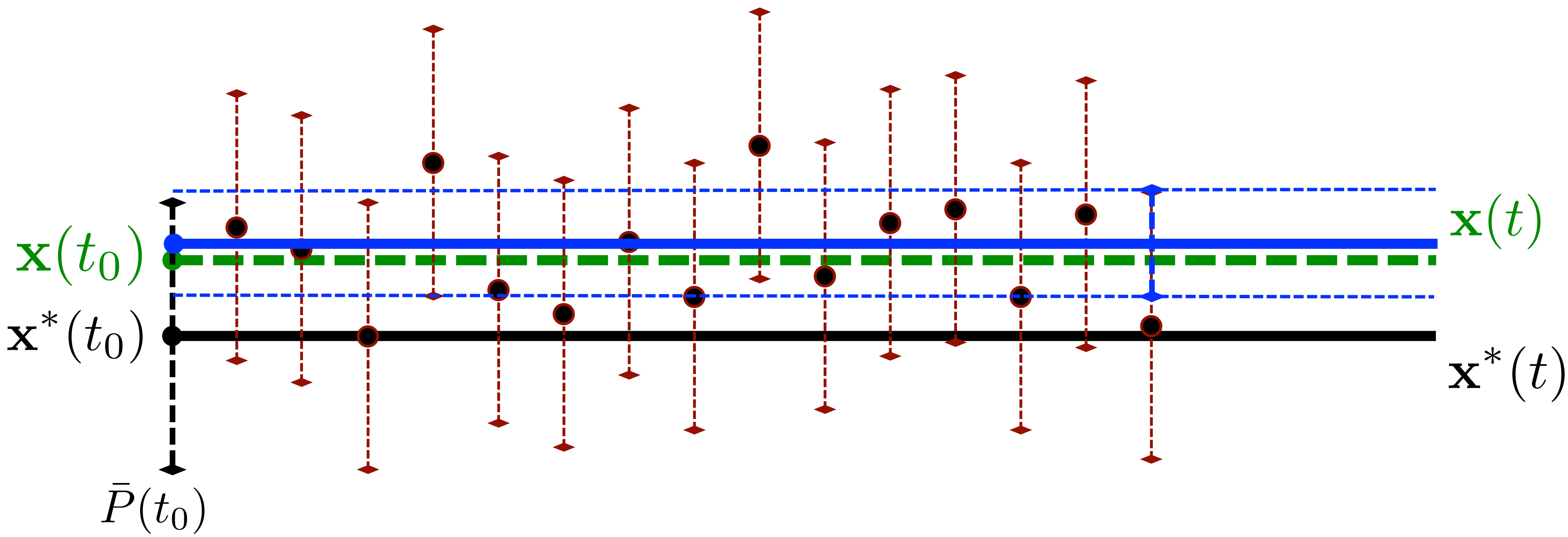

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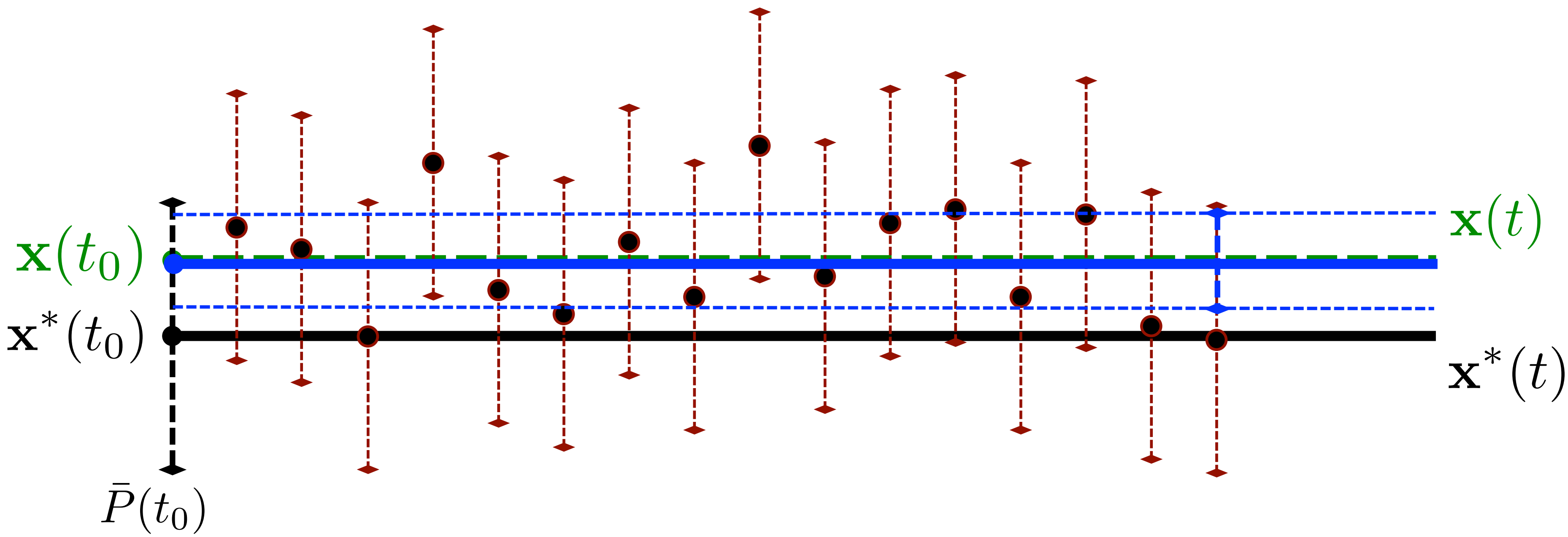

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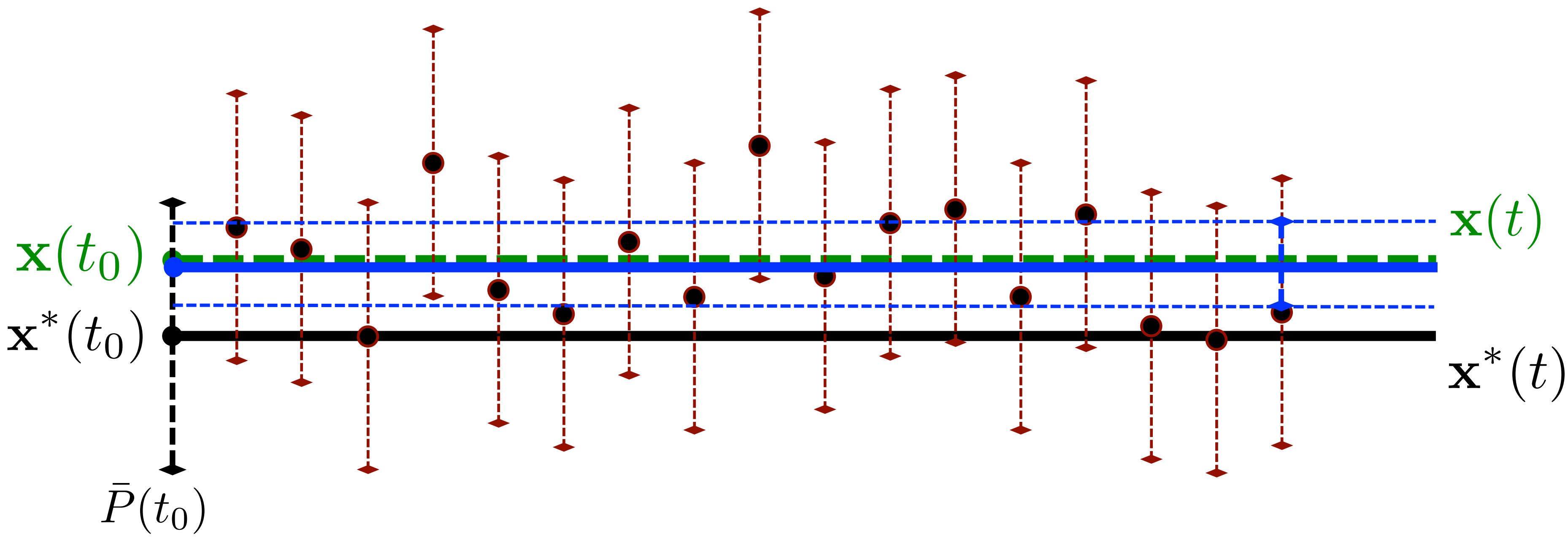
- Conventional Kalman



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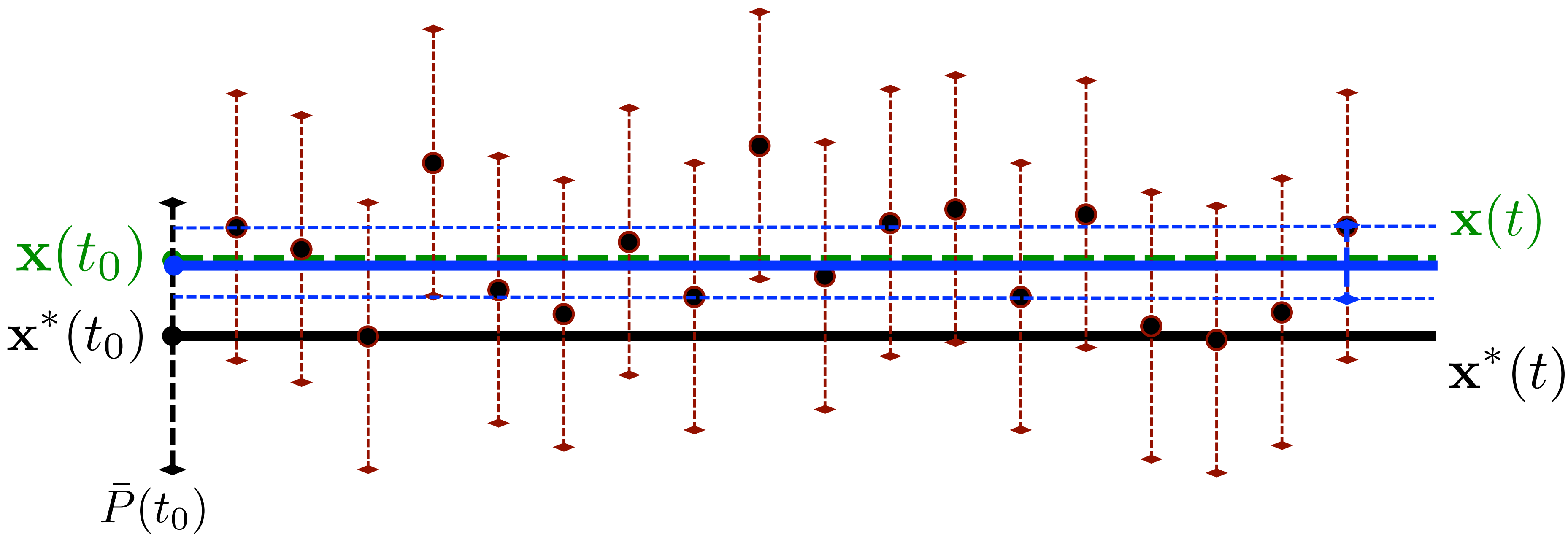

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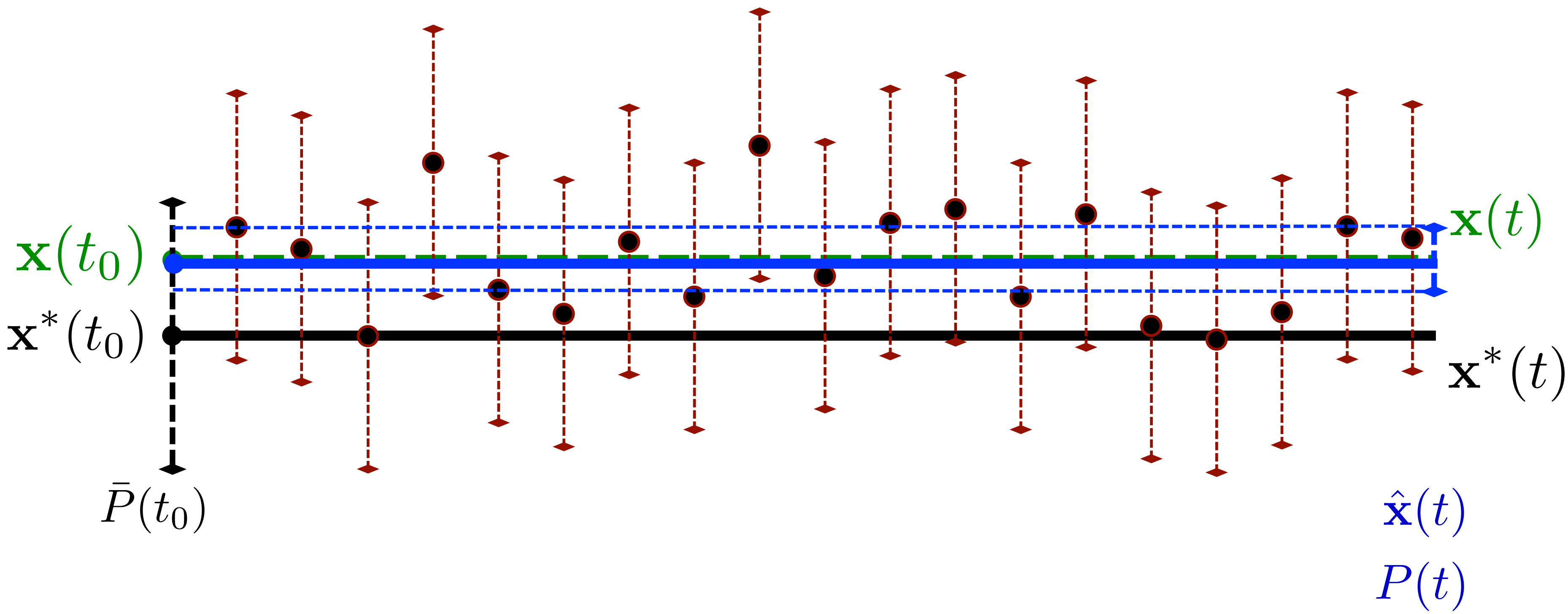

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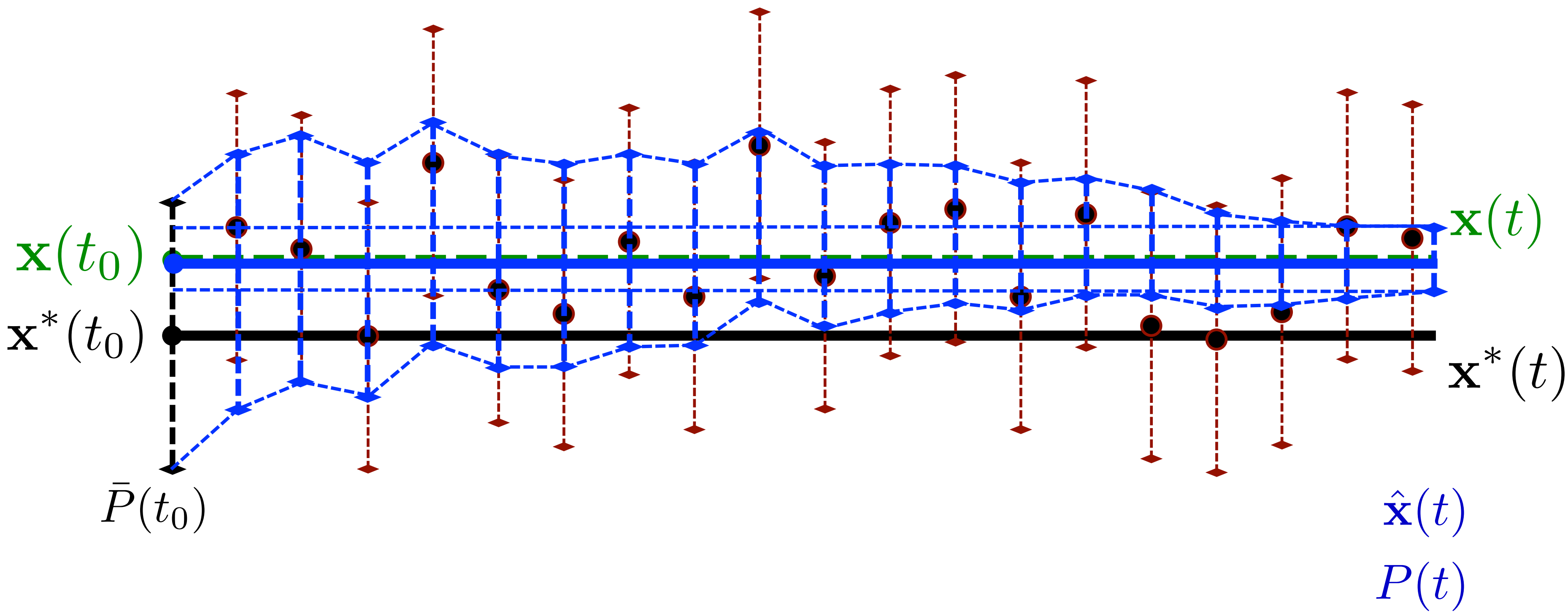

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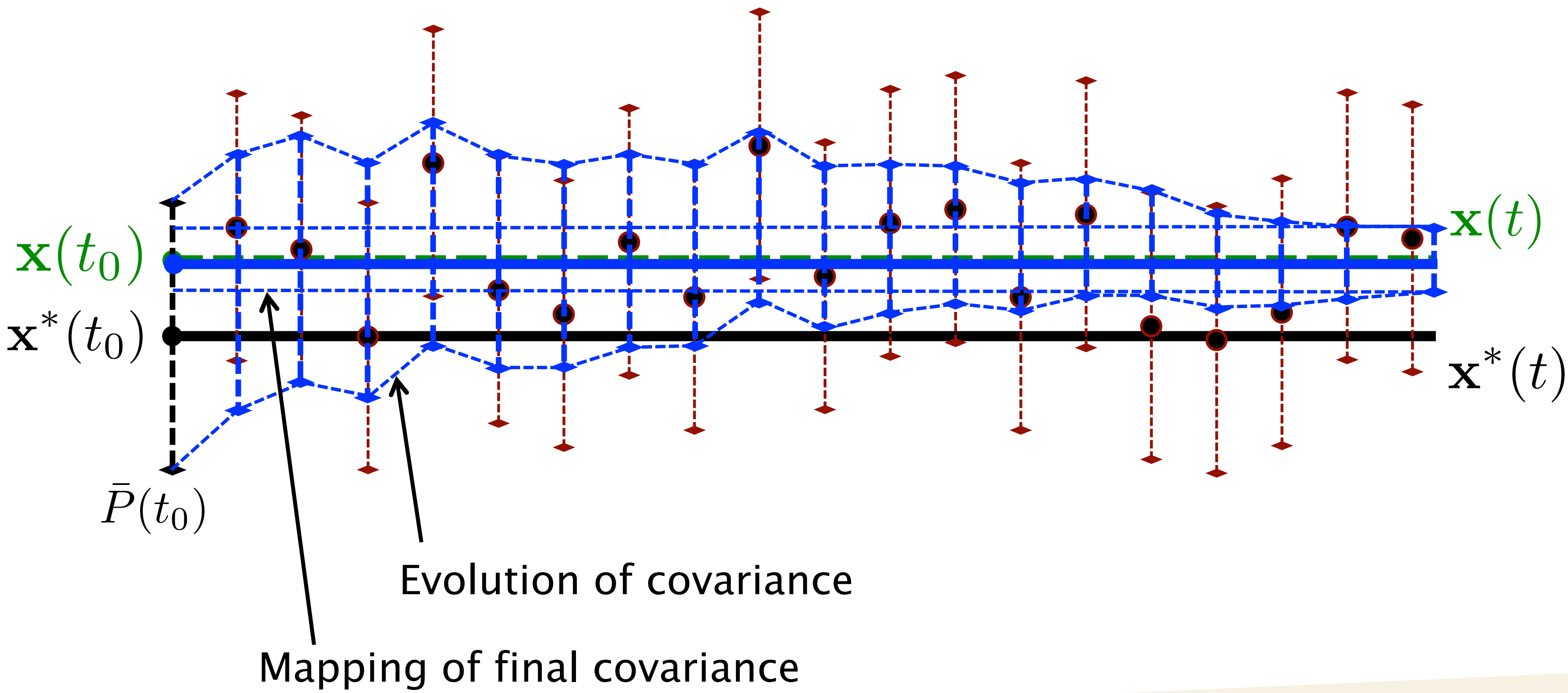
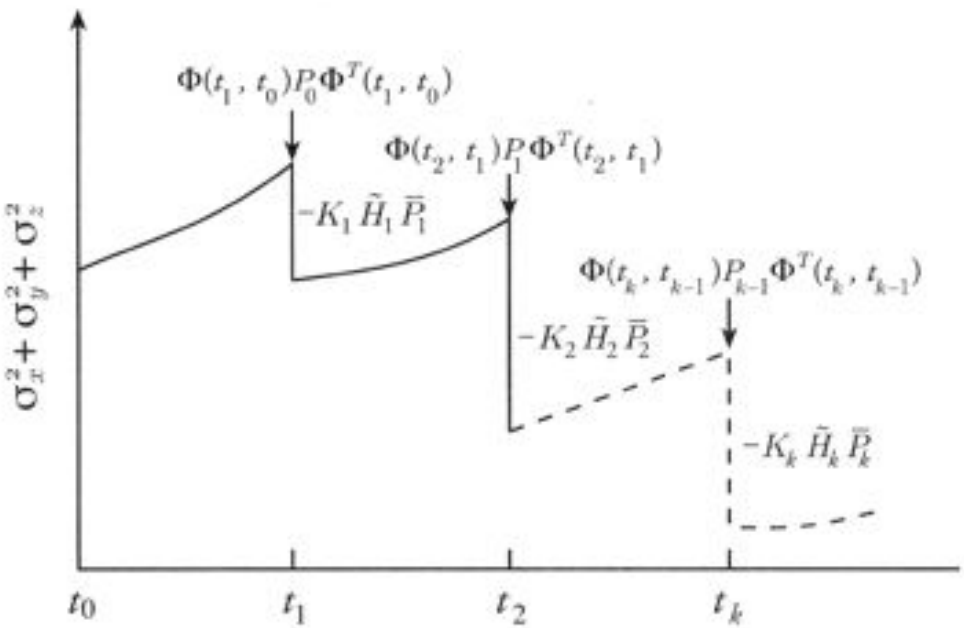


Evolution of Covariance Matrix

- Evolution of the covariance matrix as observations are processed.



- Conceptualization of the Extended Kalman Filter (EKF)
- Major change: the reference trajectory is updated by the best estimate after every measurement.

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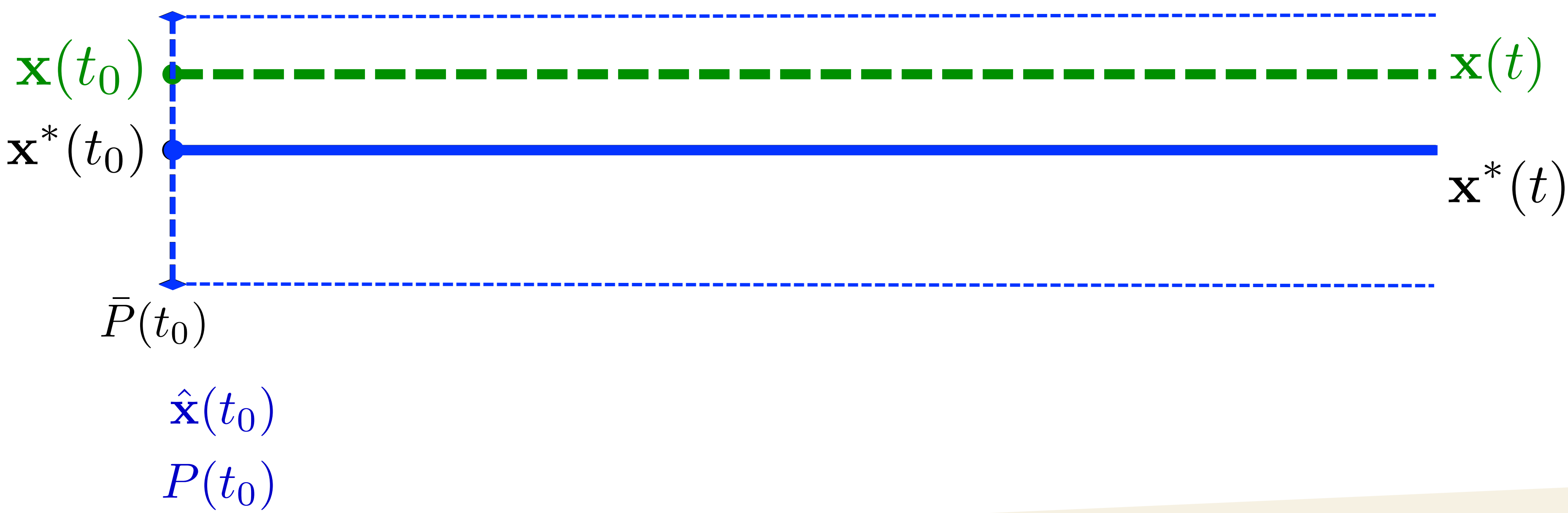
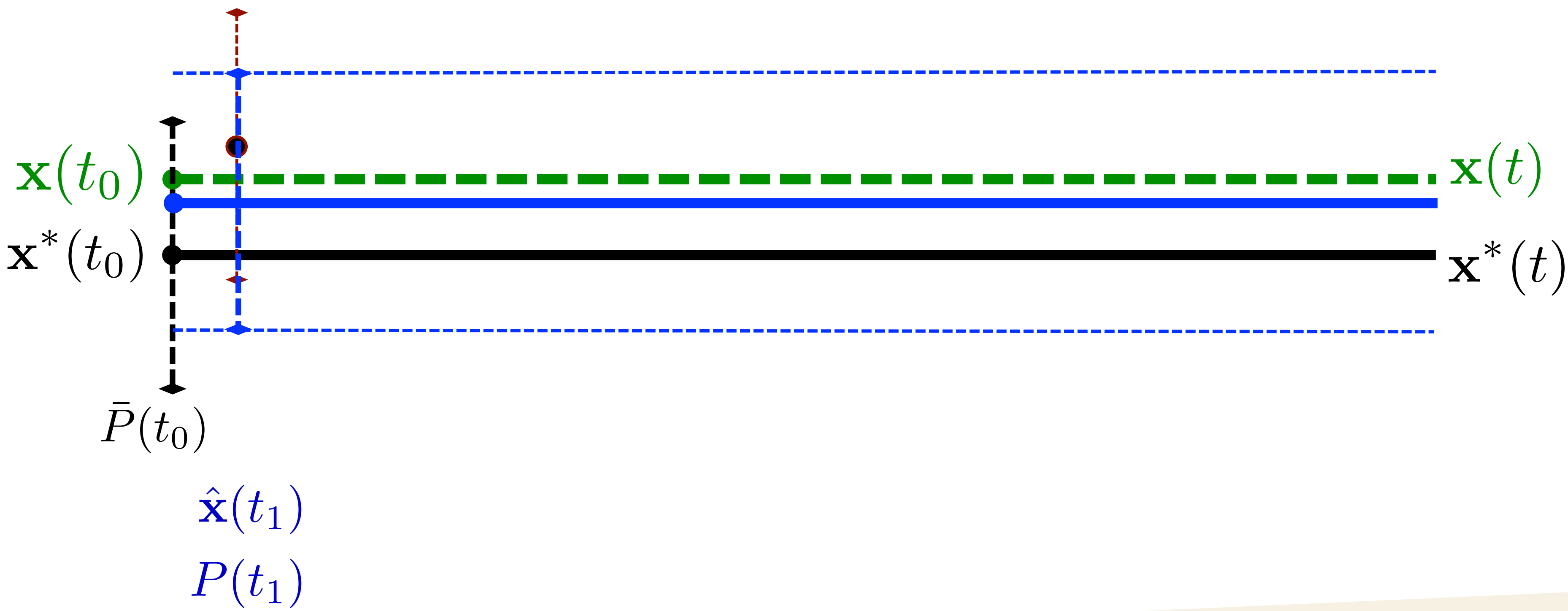
- EKF



Stat OD Conceptualization

- EKF



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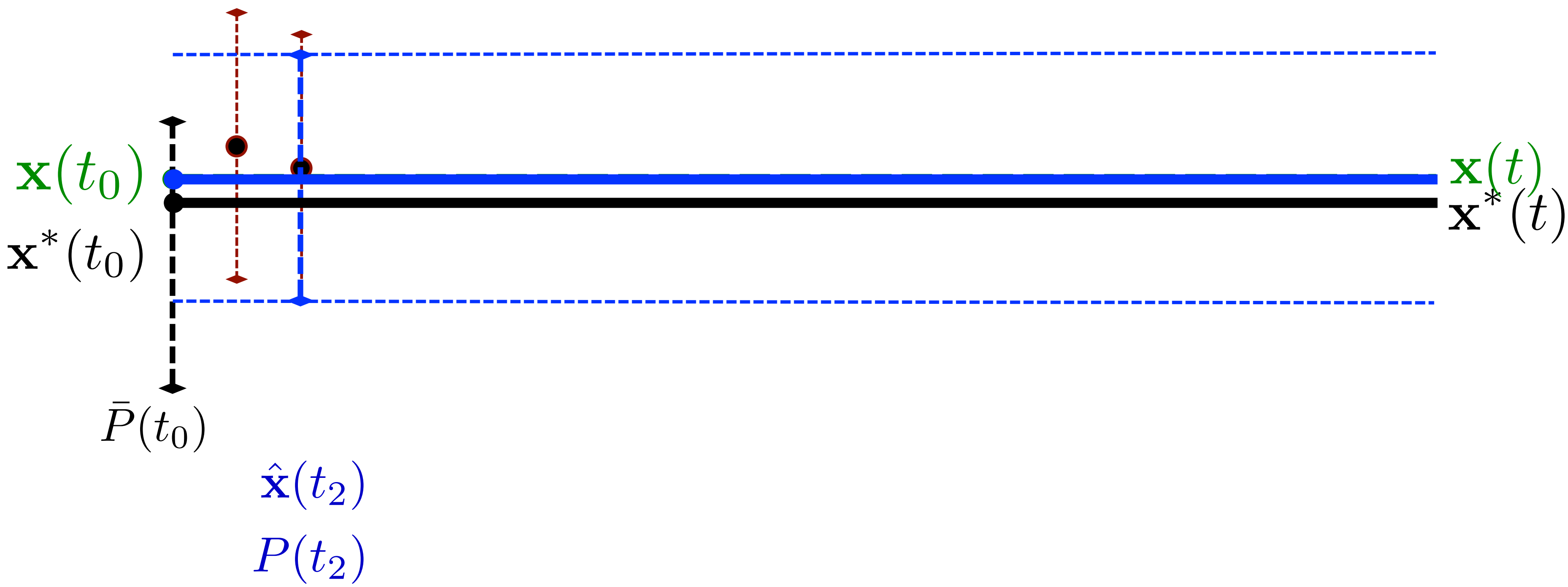

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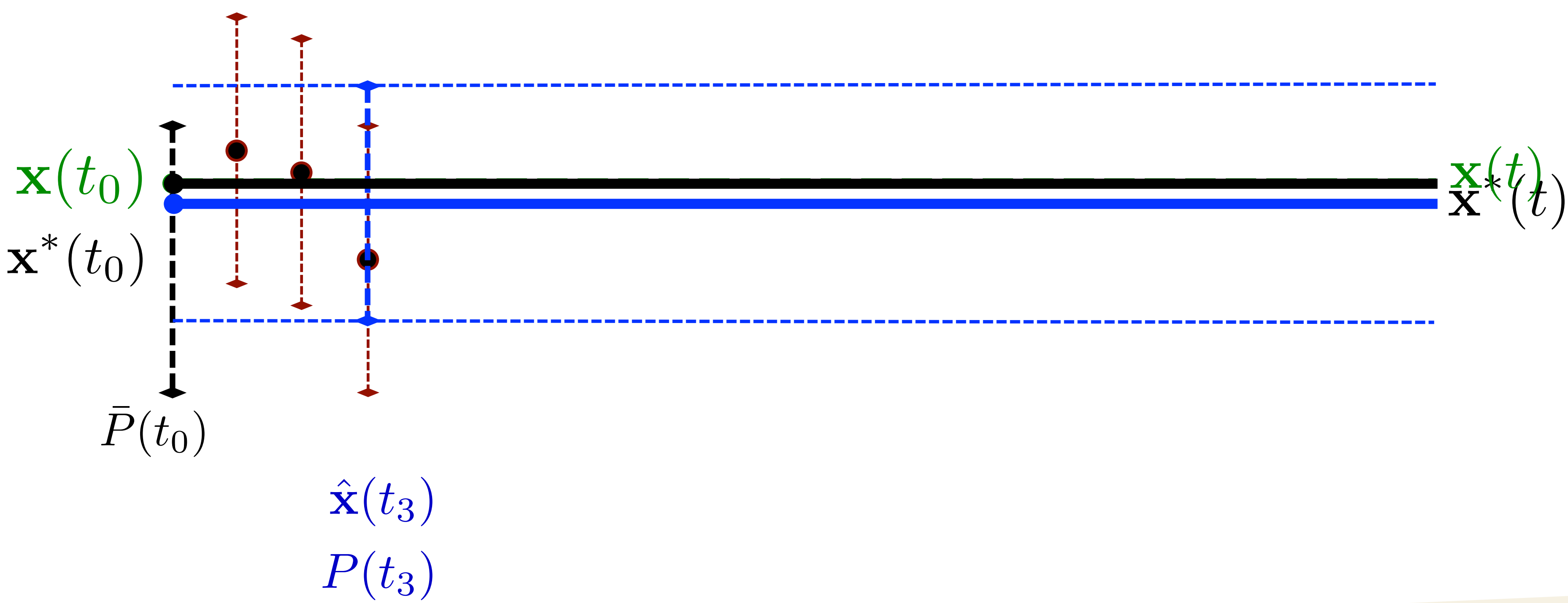

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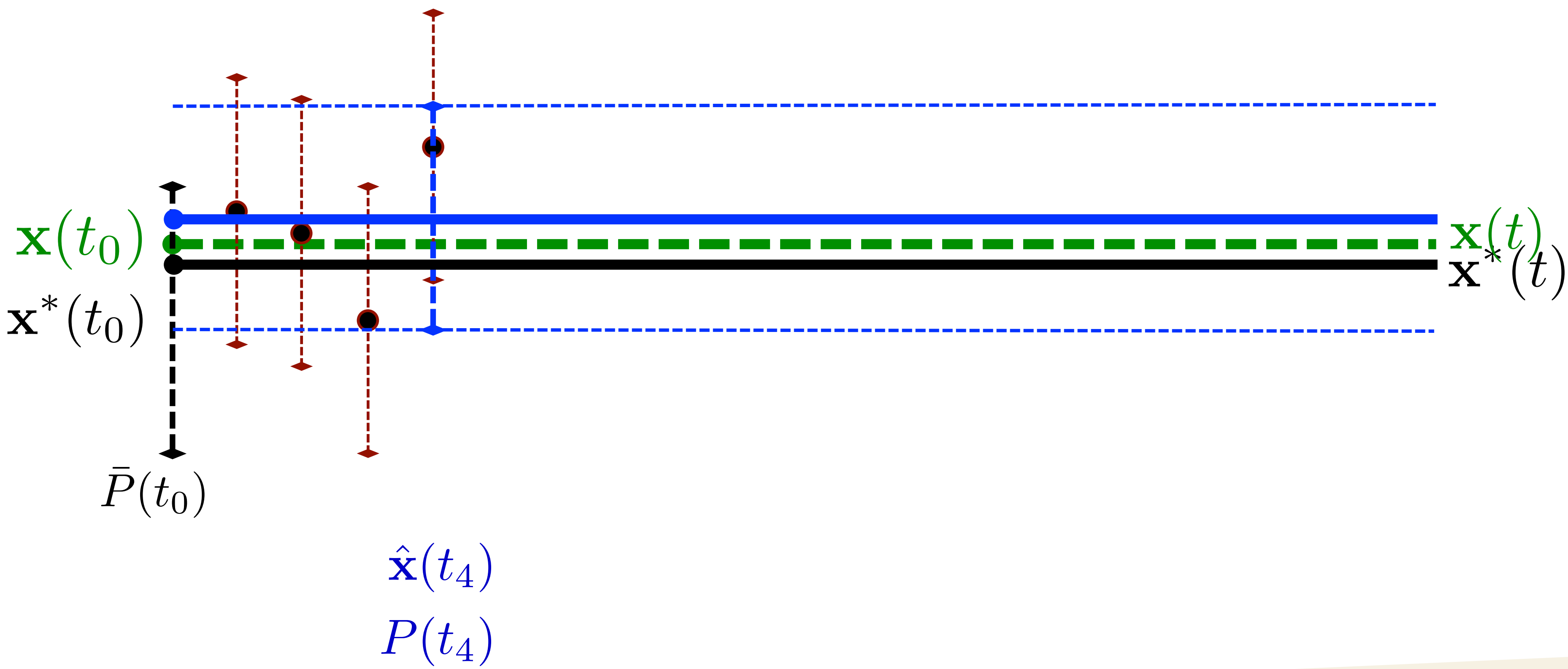

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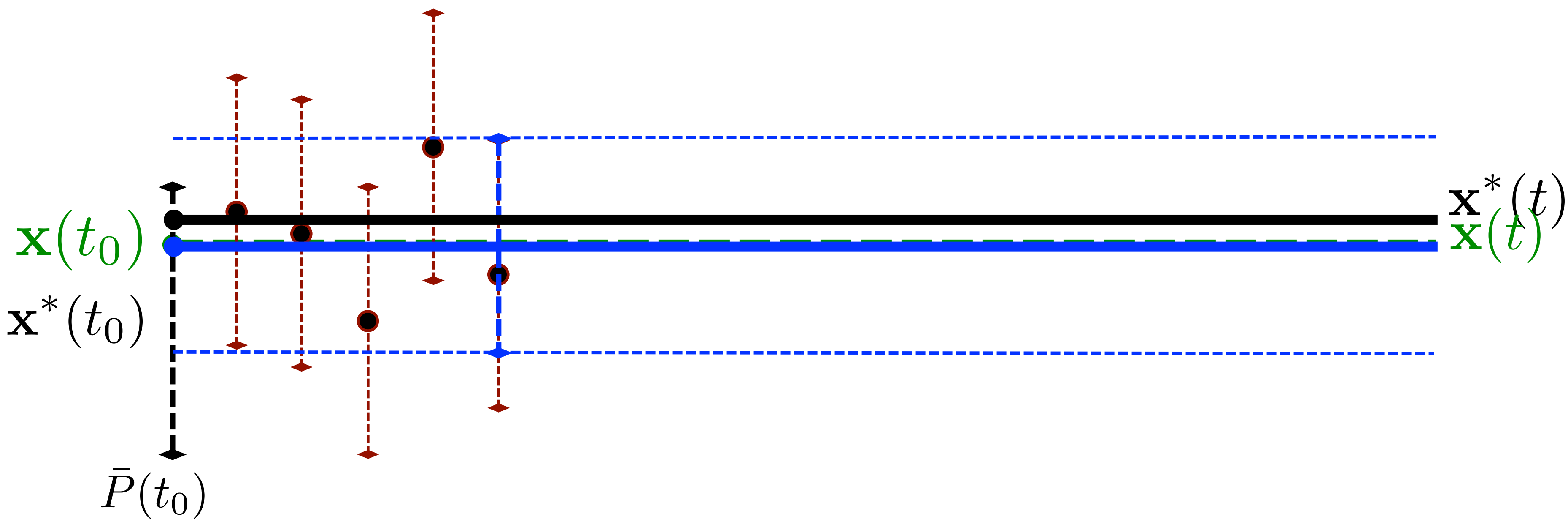
- EKF



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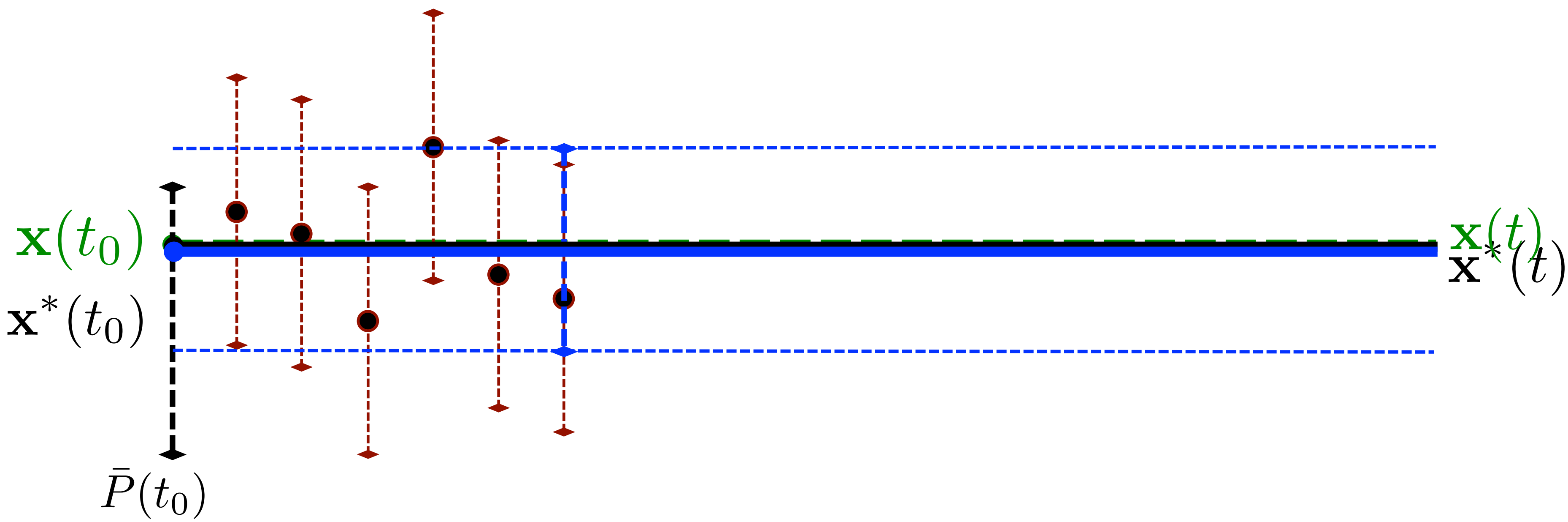

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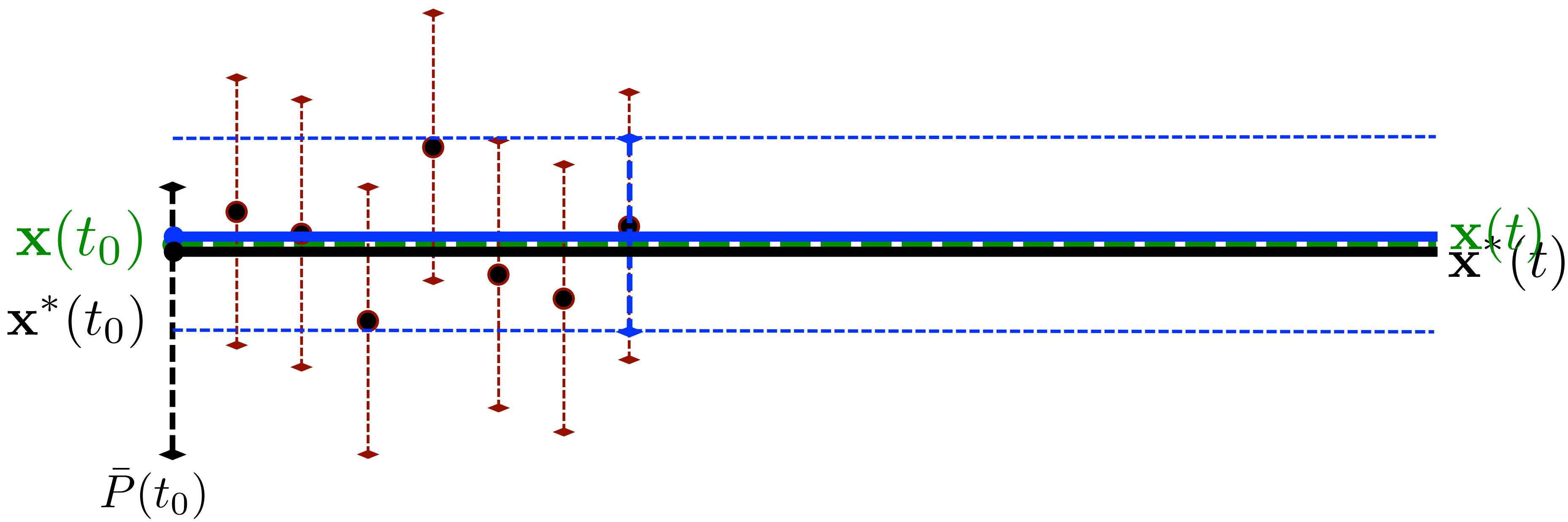

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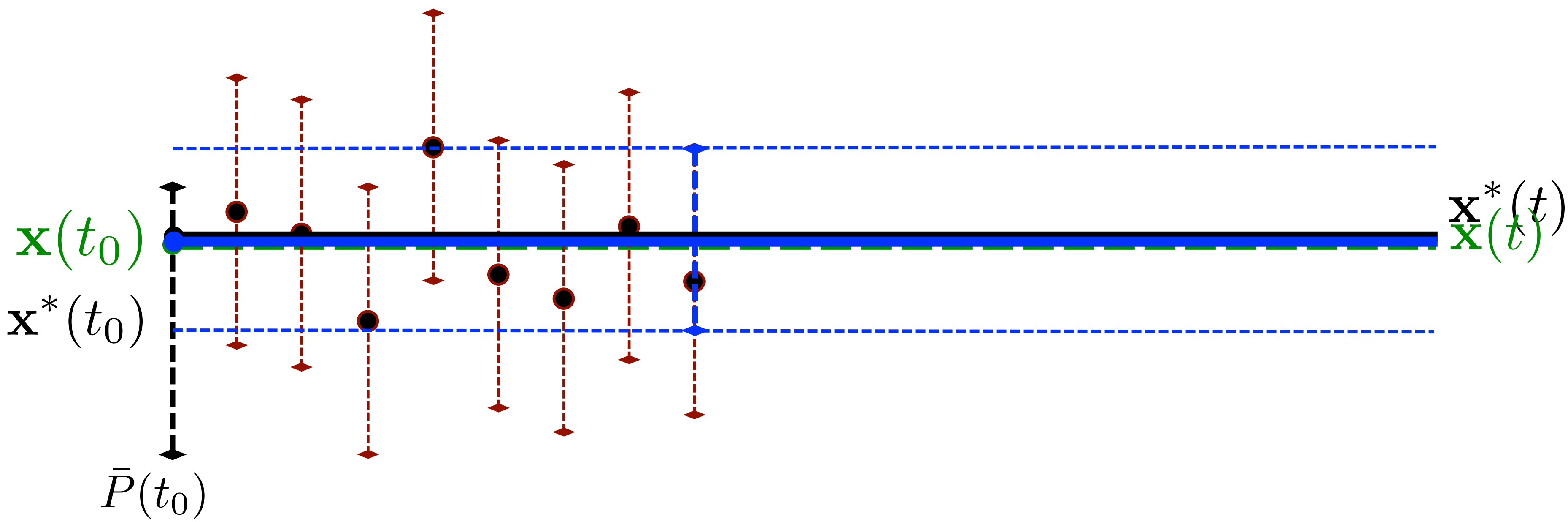
- EKF



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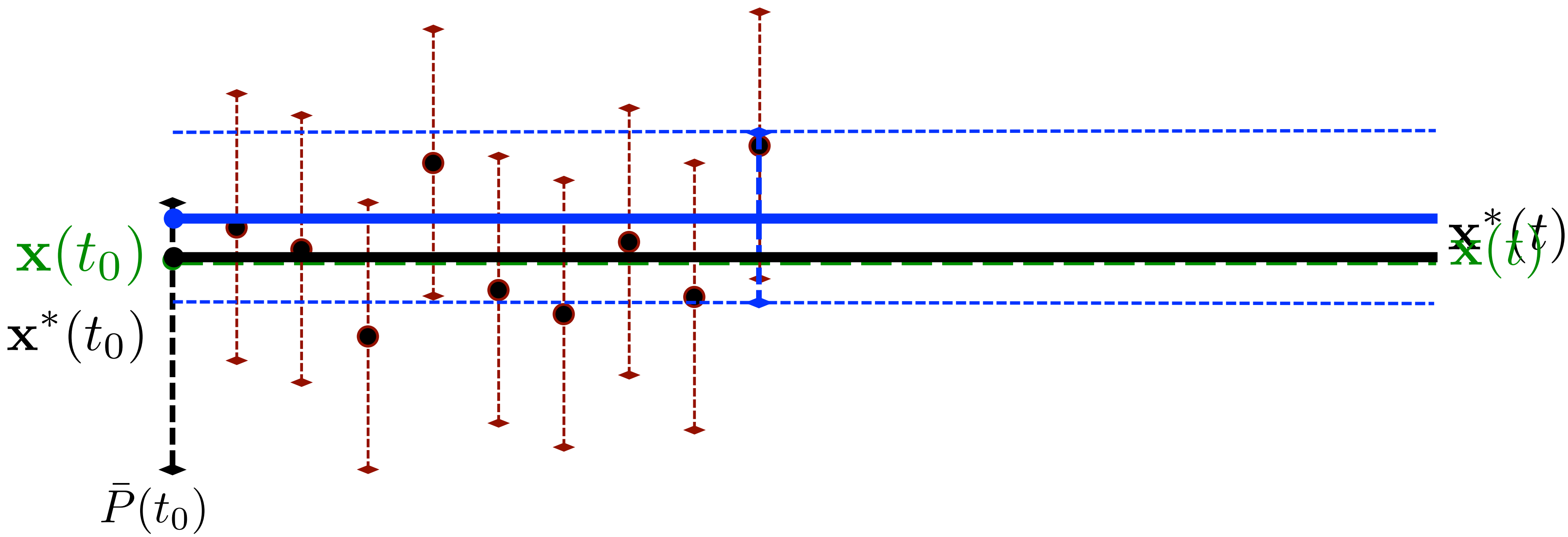

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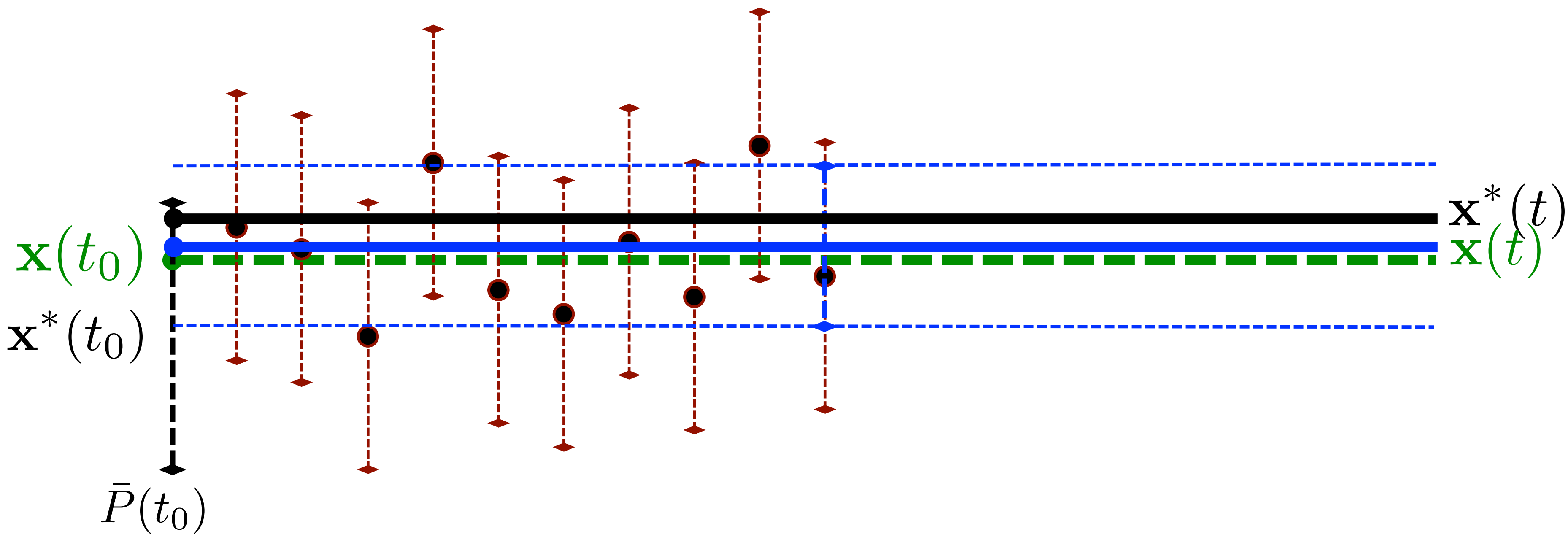

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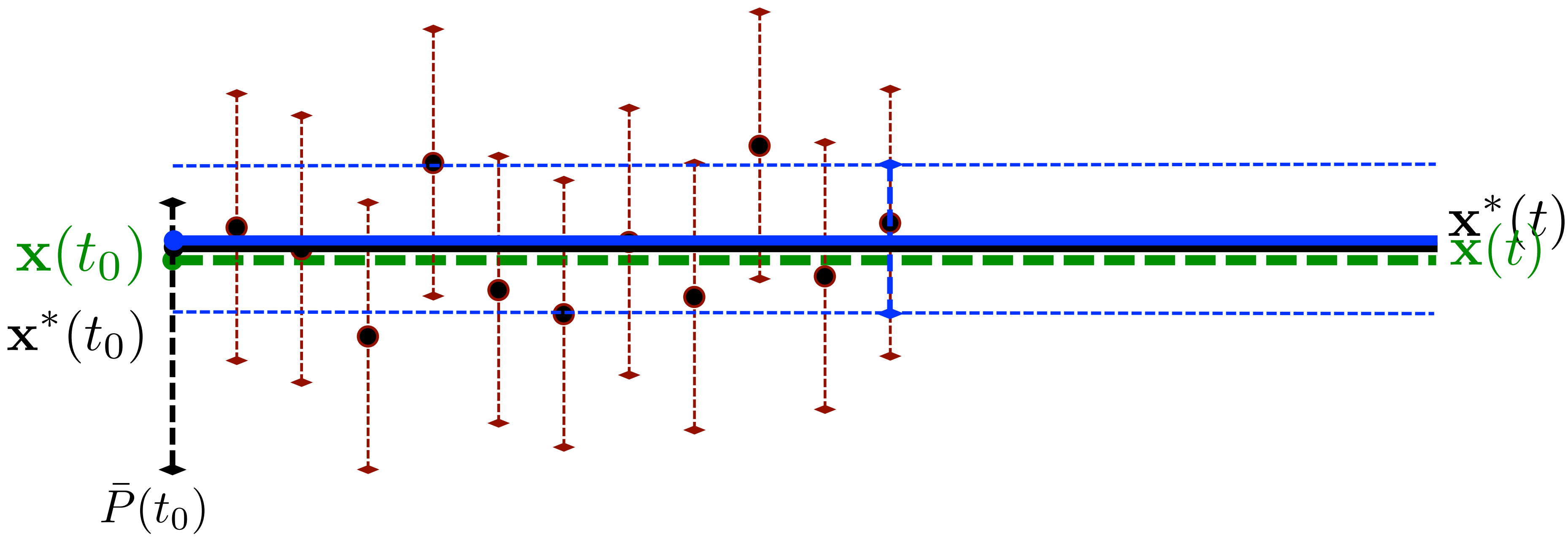

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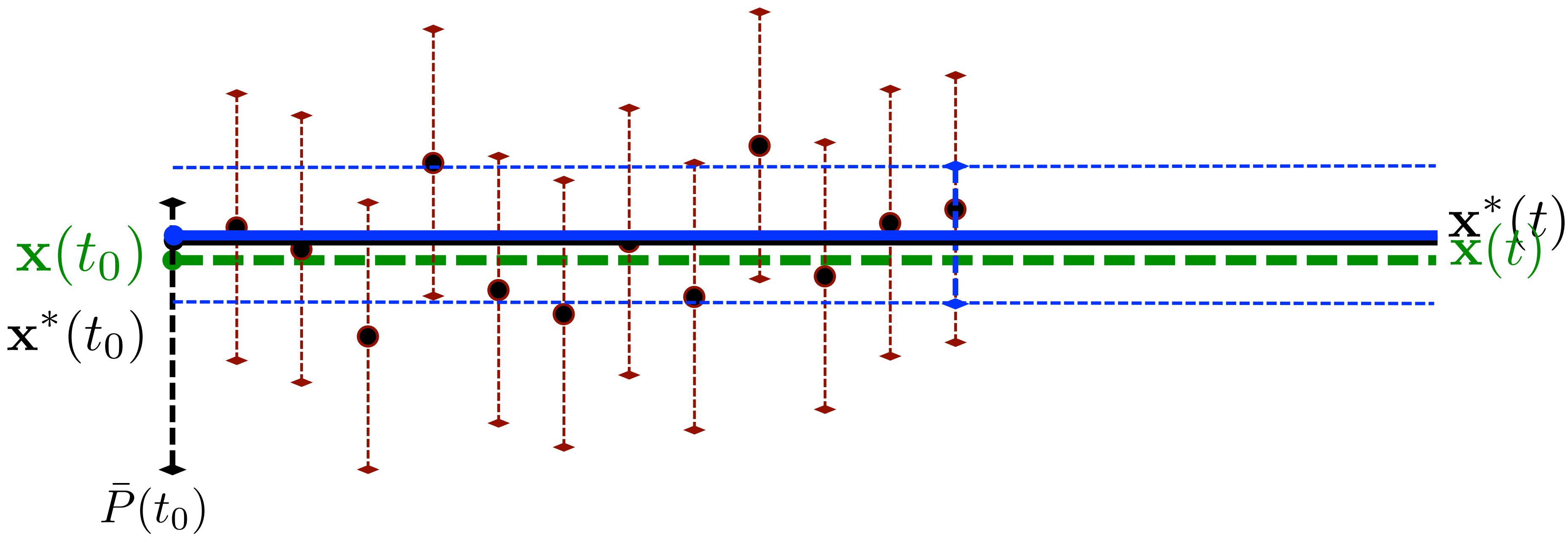

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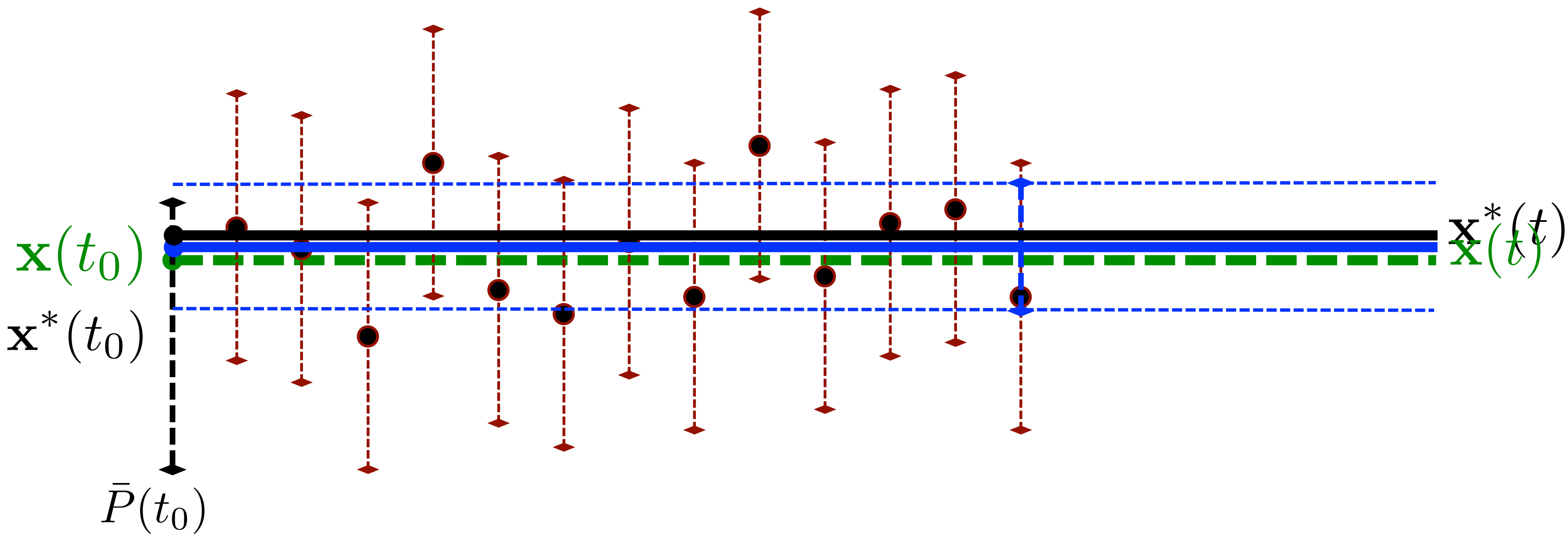

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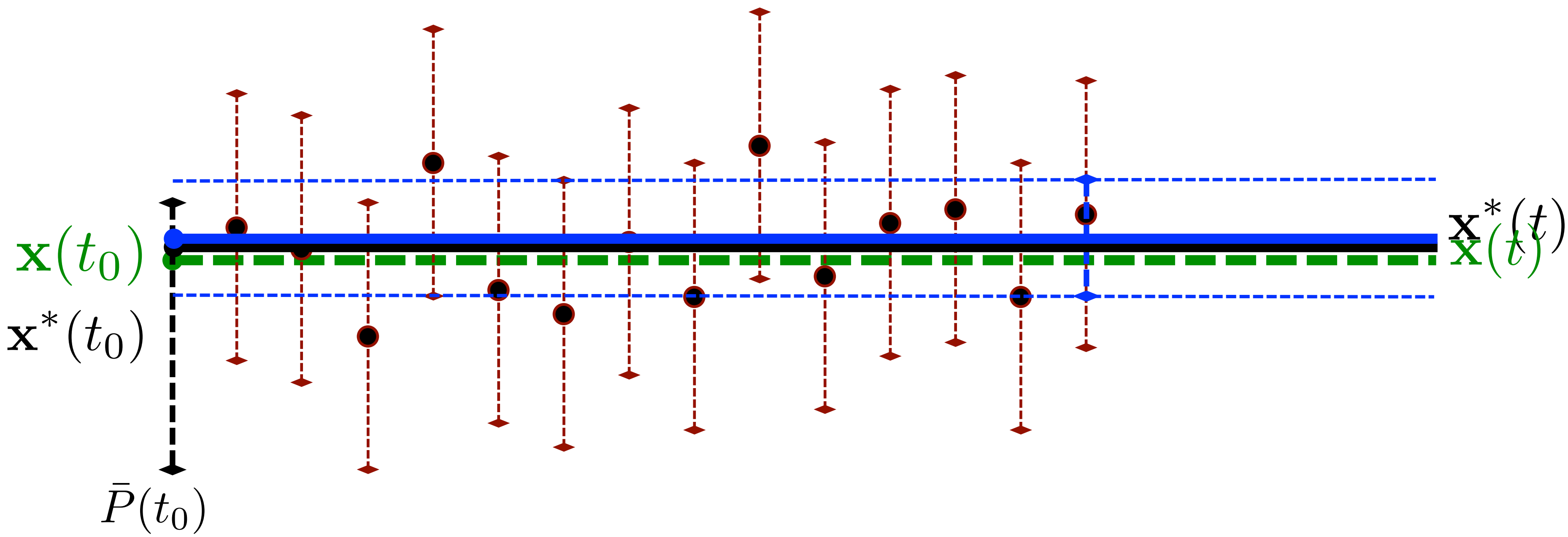

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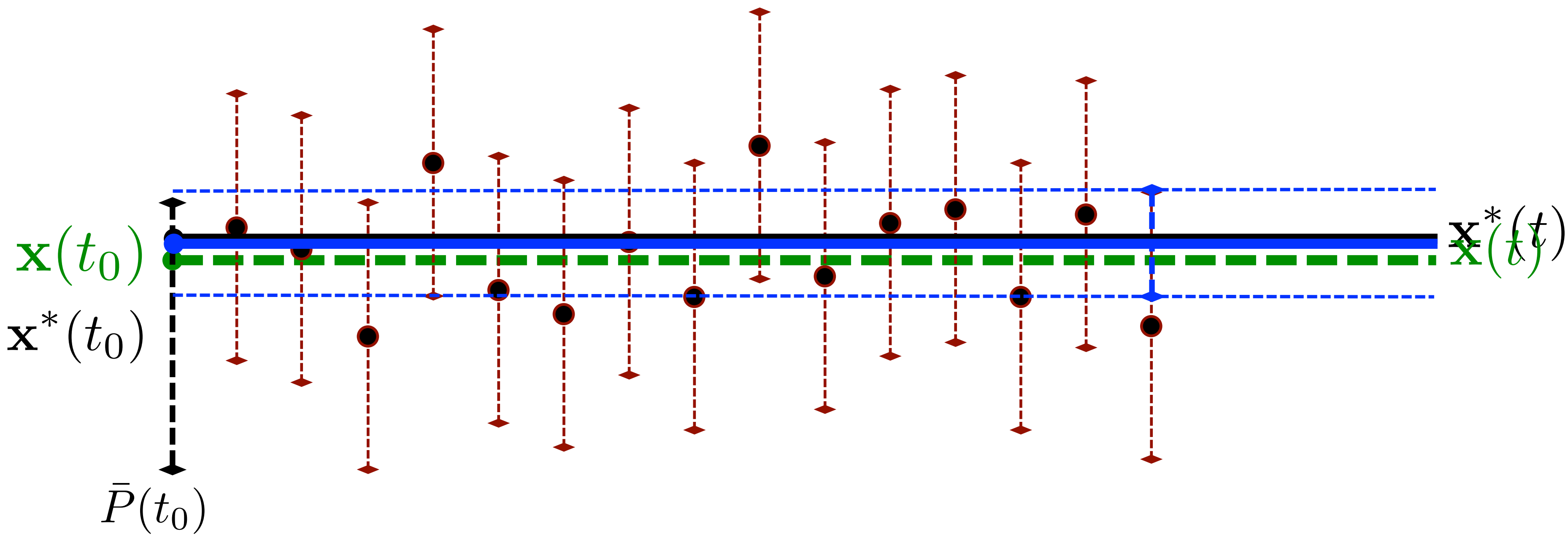

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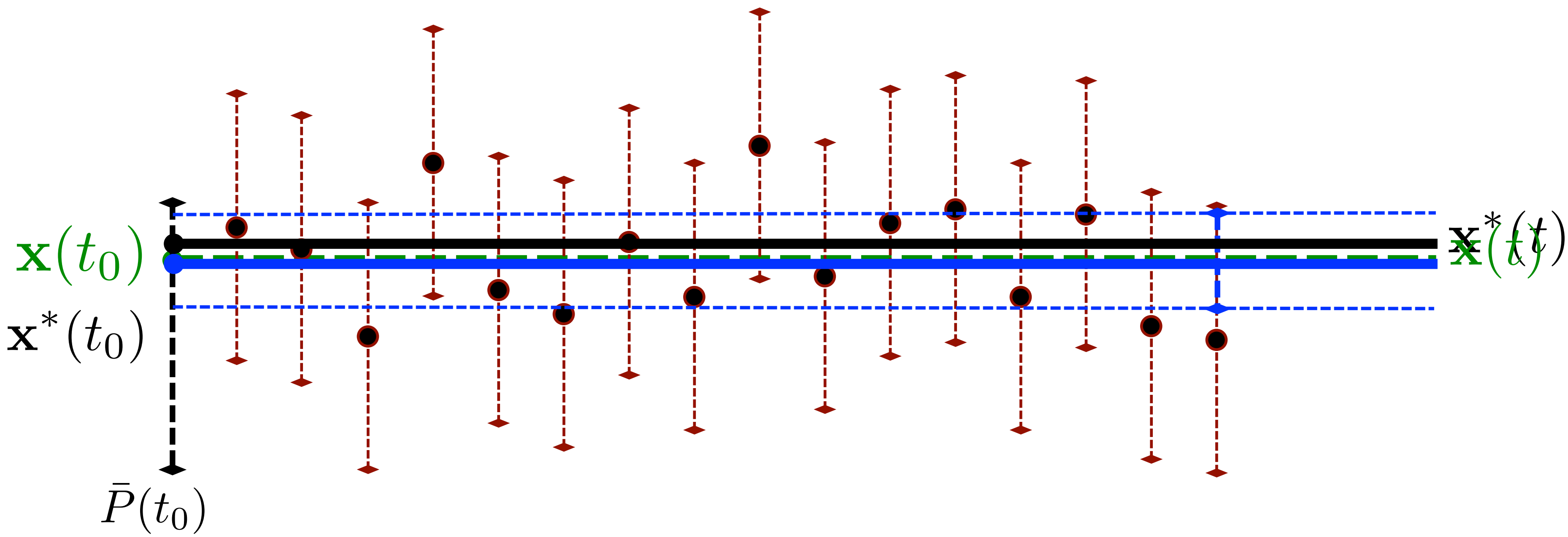

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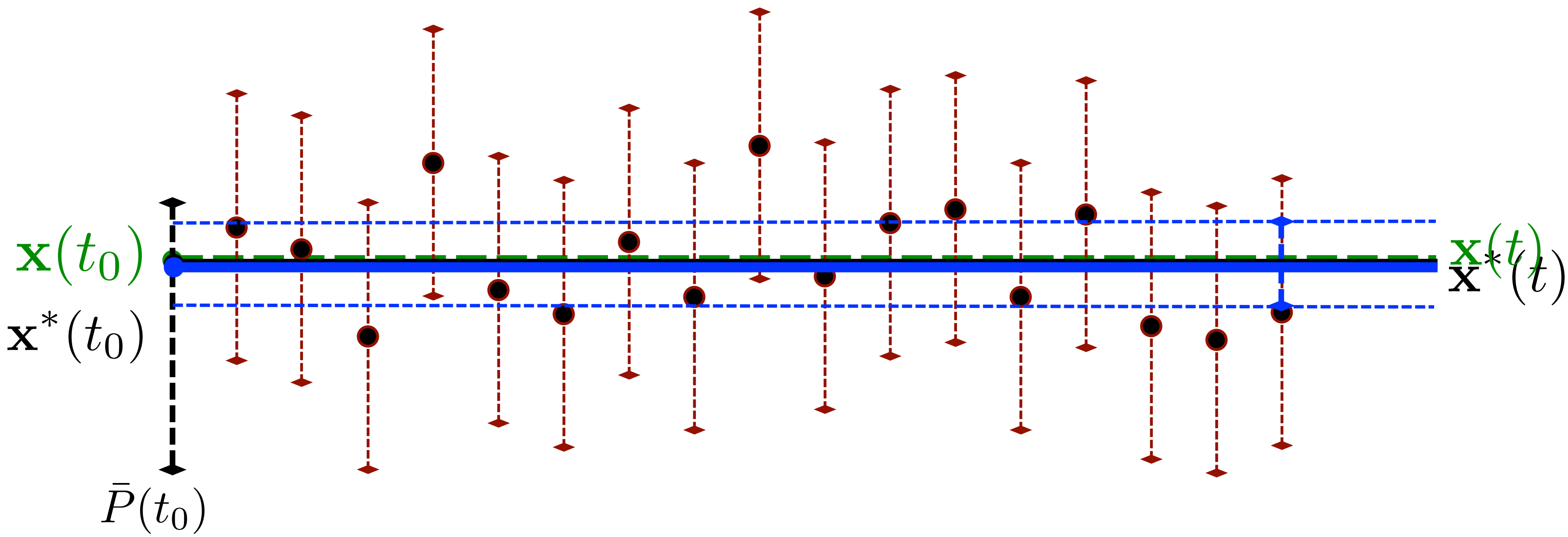

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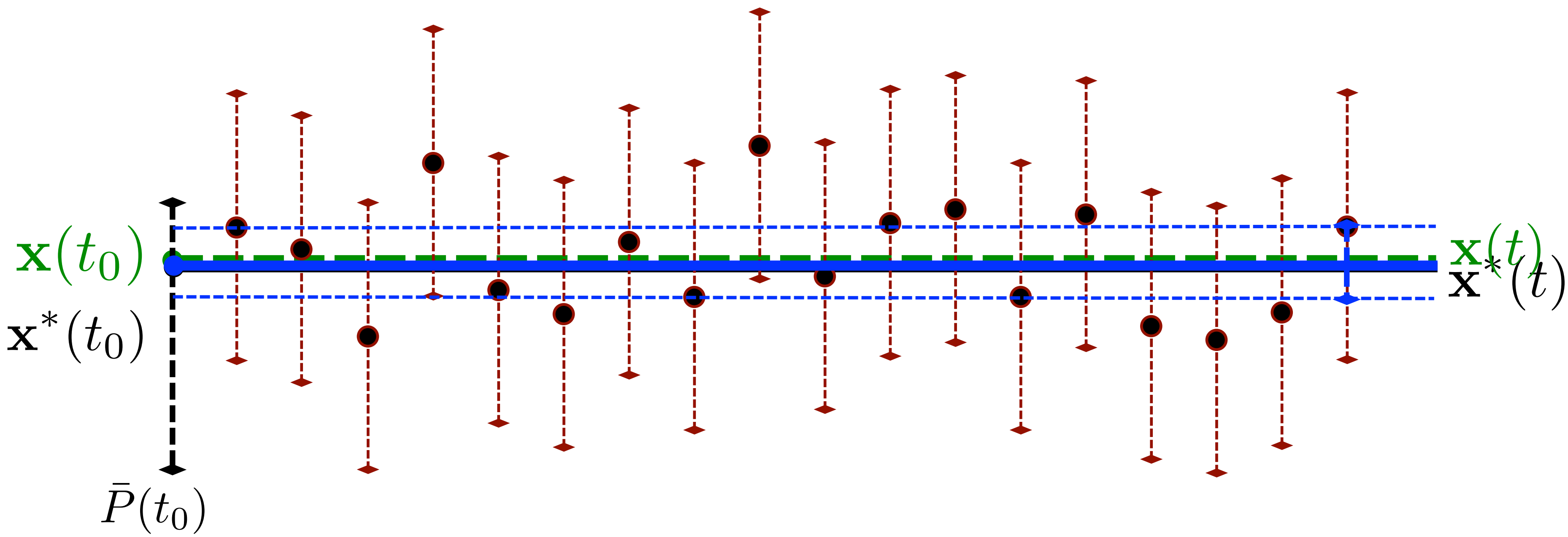

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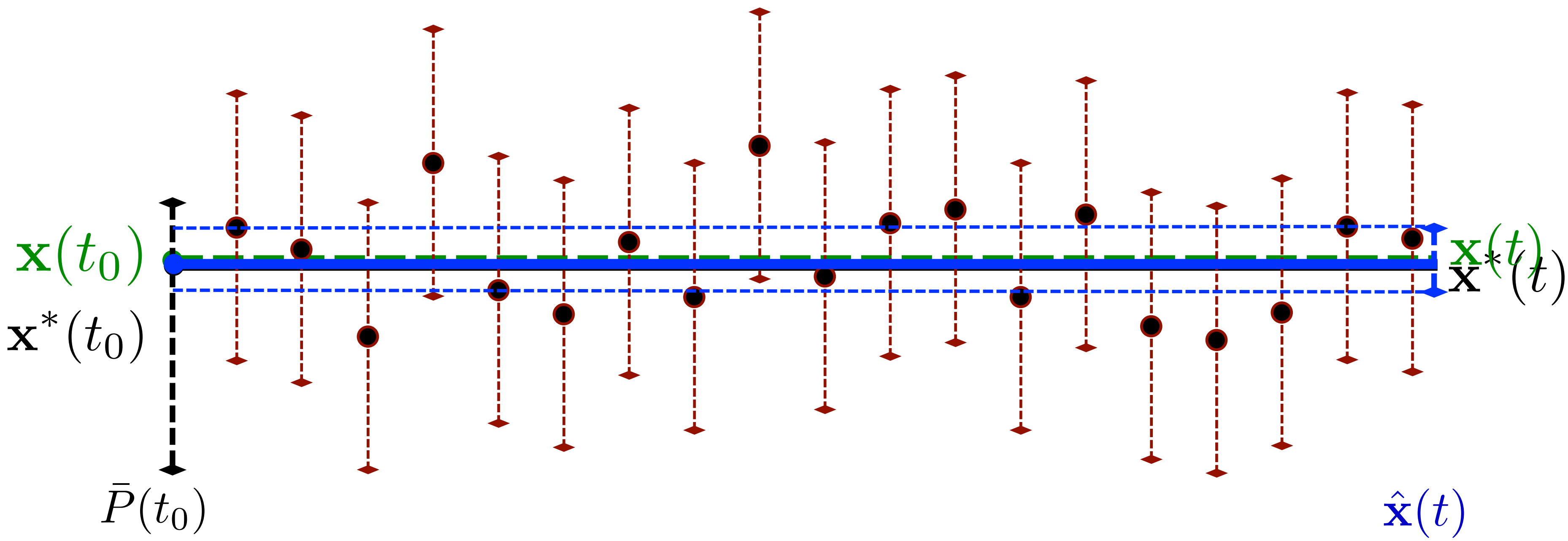

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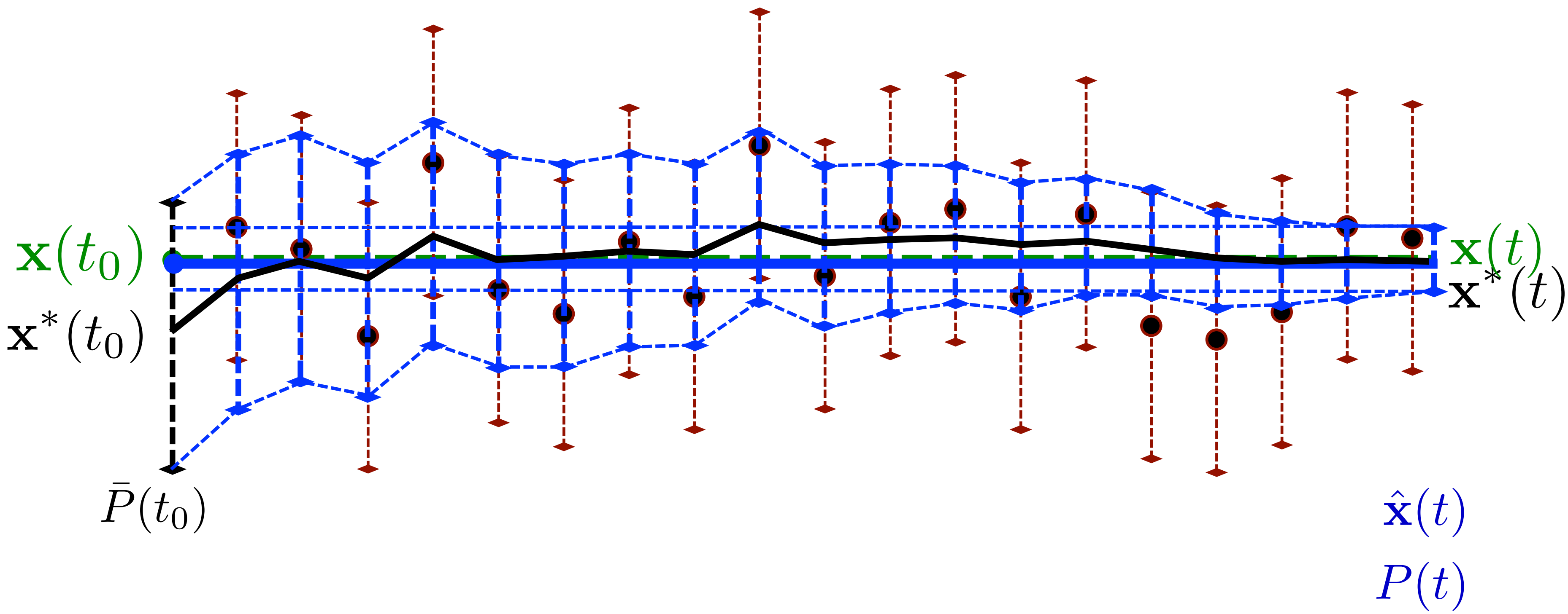

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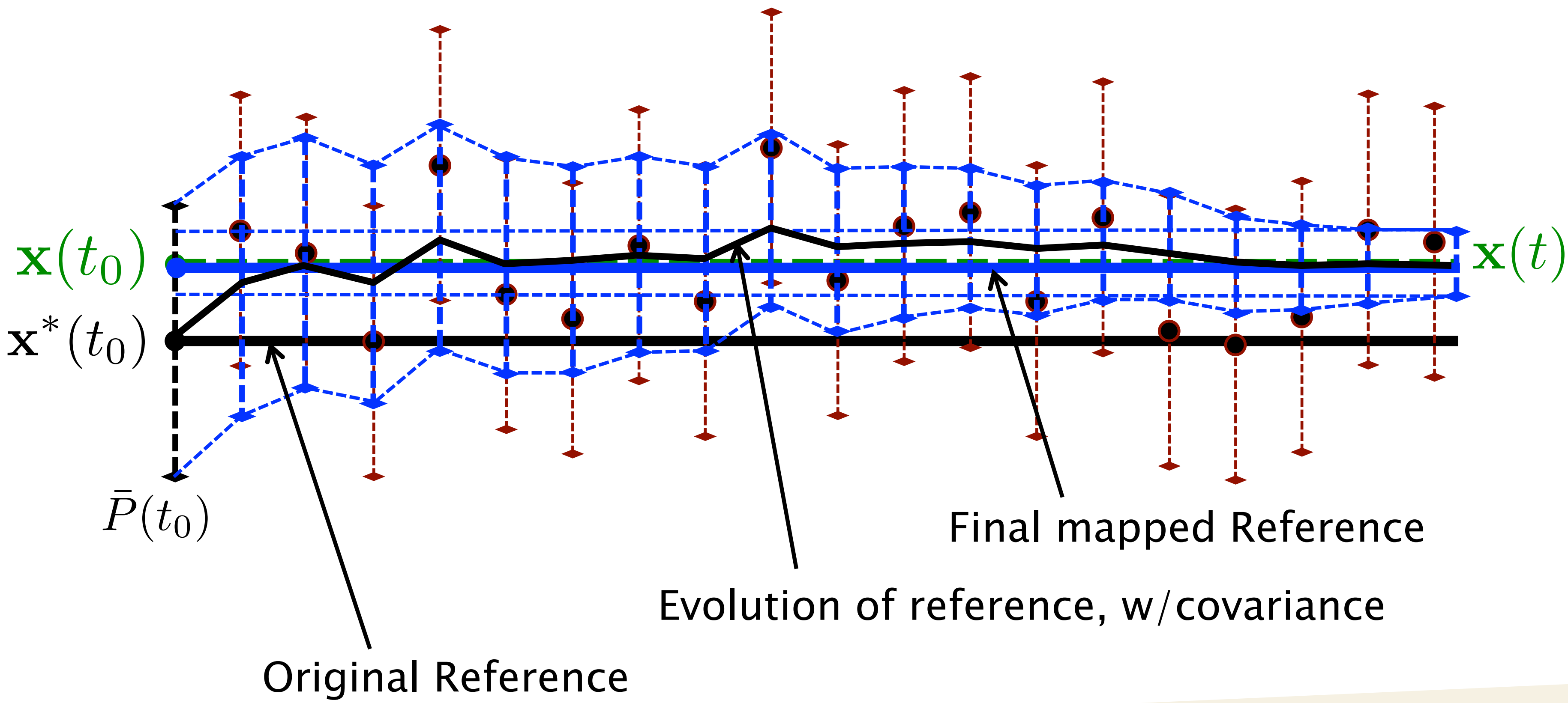

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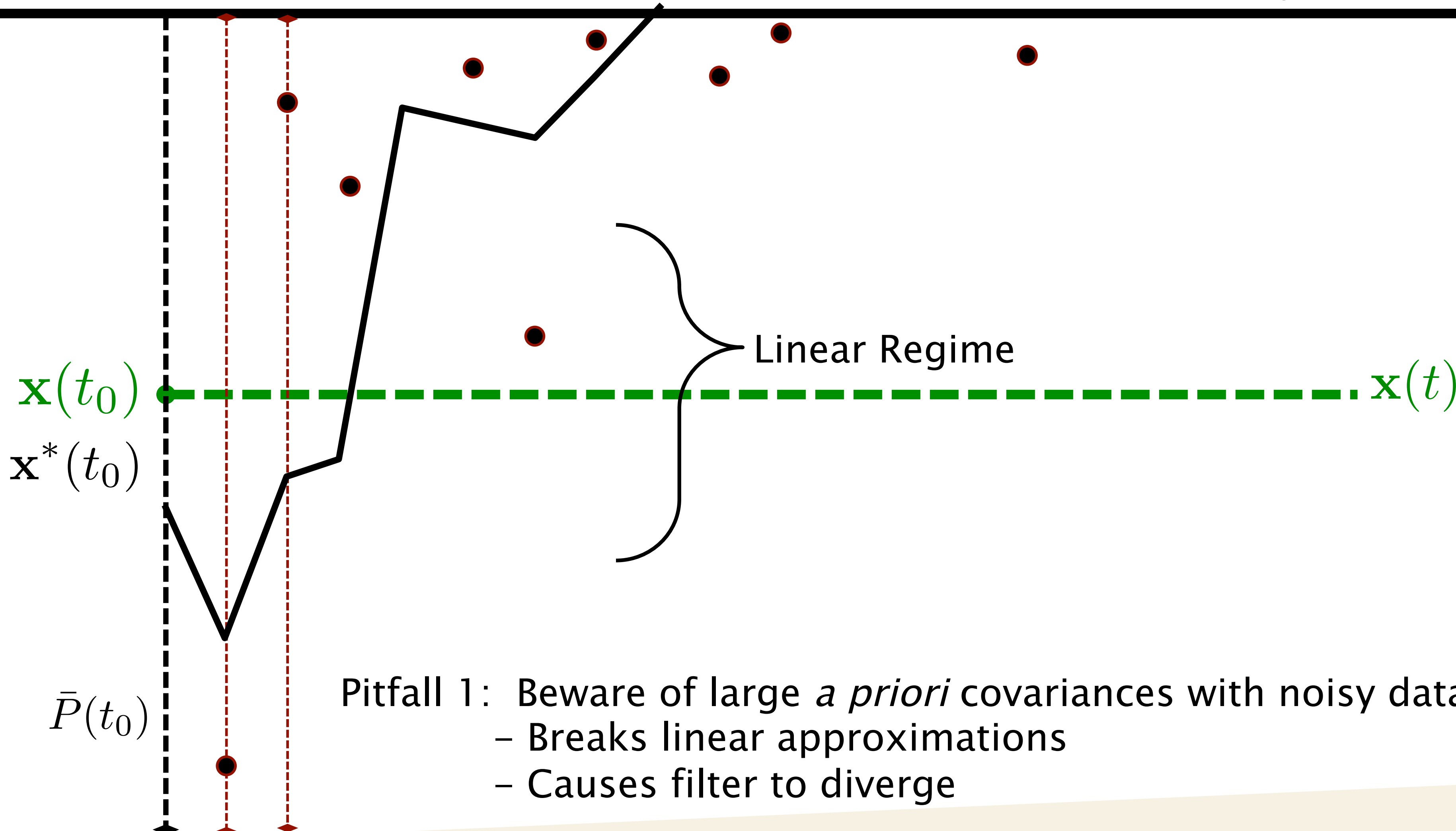

Pitfall 1: Beware of large *a priori* covariances with noisy data

- Breaks linear approximations
- Causes filter to diverge

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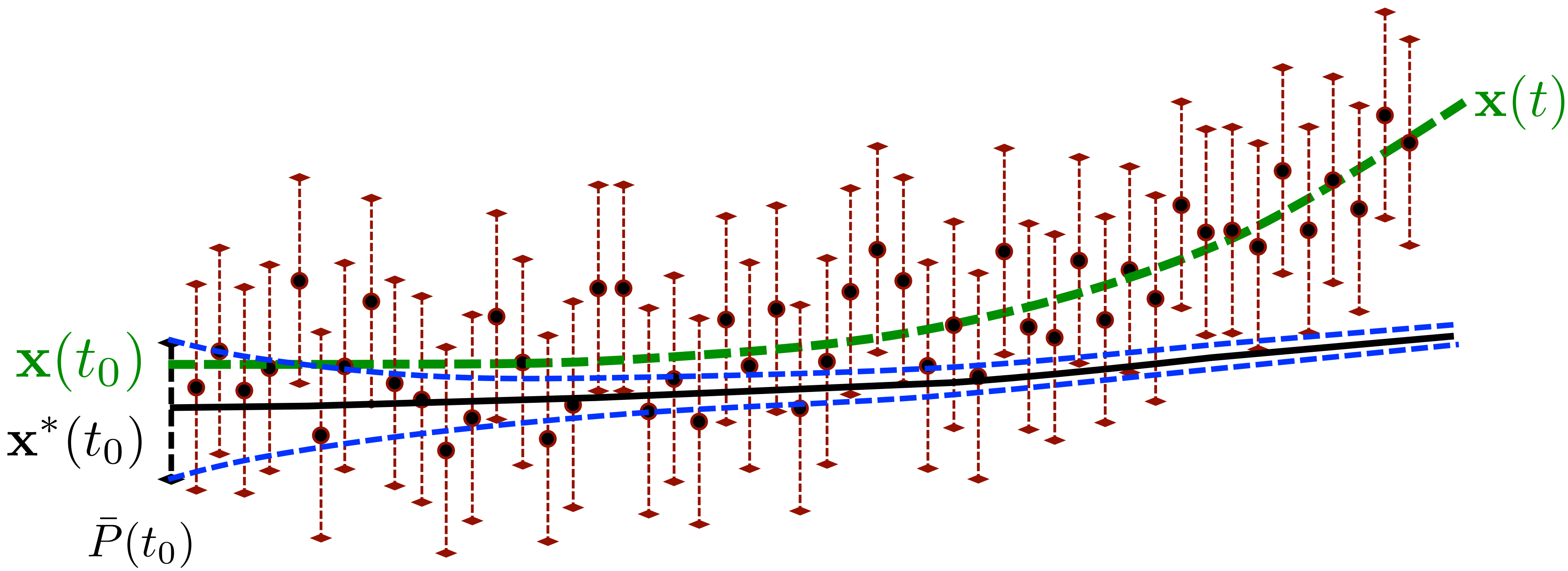

Pitfall 2: Beware of collapsing covariance
- Prevents new data from influencing solution
- More prevalent for longer time-spans

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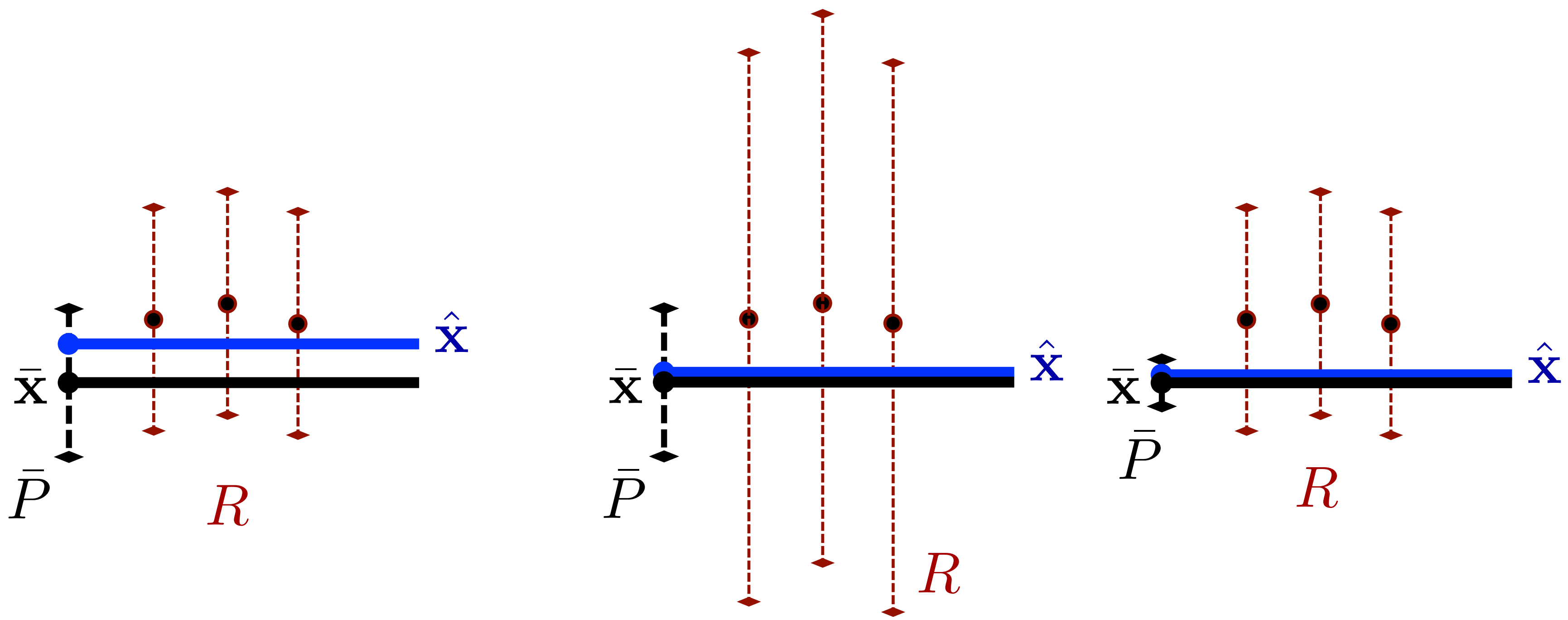

Pitfall 2: Beware of collapsing covariance

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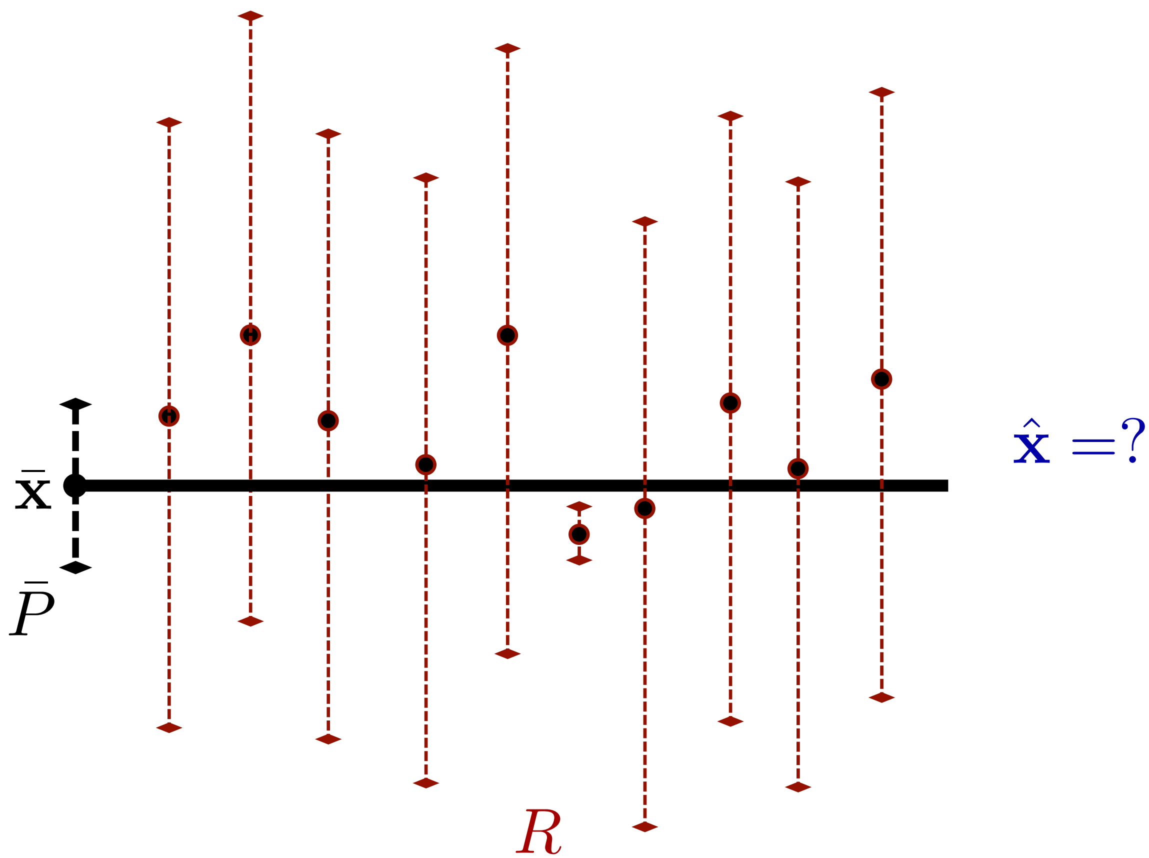

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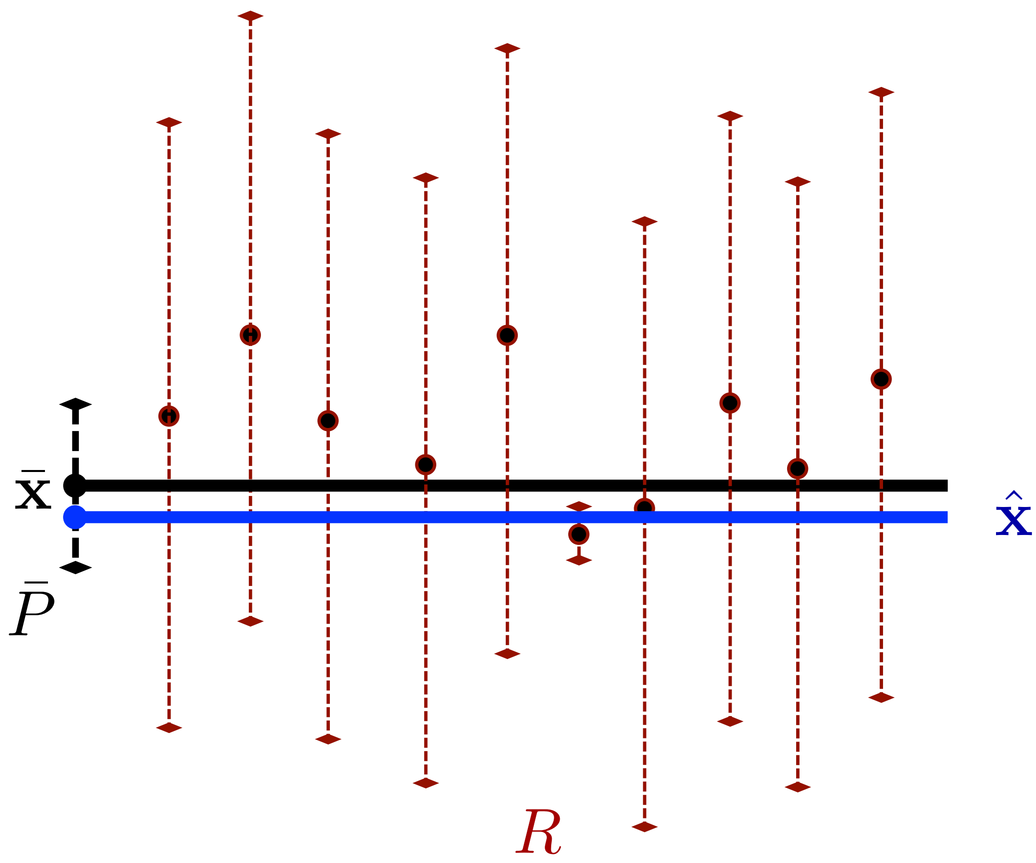

Kalman Gain's Influence

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- Recall the Kalman Gain, which was formed in the derivation of the Sequential Algorithm.

$$P_k = \bar{P}_k - \underbrace{\bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1}}_{K_k} \tilde{H}_k \bar{P}_k$$

$$K_k = \bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1}$$

Kalman Gain's Influence

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- Recall the Kalman Gain, which was formed in the derivation of the Sequential Algorithm.
- What happens if $R_k \gg \bar{P}_k$ or $R_k \ll \bar{P}_k$

$$K_k = \bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1}$$

$$\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + K_k \left[\mathbf{y}_k - \tilde{H}_k \bar{\mathbf{x}}_k \right]$$

$$P_k = \left[I - K_k \tilde{H}_k^T \right] \bar{P}_k$$

What is the Role of the Kalman Gain?

Let's examine a simple case where $\mathbf{x}(t)$ & $\mathbf{y}(t)$ are both scalars and we observe $\mathbf{x}(t)$ directly, i.e.

$$y_k = x_k + \varepsilon_k, \qquad E\left[\epsilon\epsilon^T\right] = E[\epsilon^2] = \sigma_\epsilon^2$$

so

$$\tilde{H}_k = 1$$

Furthermore assume that $\mathbf{x}(t)$ is a constant so that $\Phi\left(t_i, t_j\right) = 1$.

Then at time *k*,

$$\bar{x}_k = \hat{x}_j$$

$$\bar{P}_k = P_j \equiv \sigma_x^2$$

What is the Role of the Kalman Gain?

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The Kalman gain @ t_k is

$$K_k = \bar{P}_k \tilde{H}_k^T \left(\tilde{H}_k \bar{P}_k \tilde{H}^T + \sigma_\varepsilon^2 \right)^{-1}$$

$$= \sigma_x^2 \left(\sigma_x^2 + \sigma_\varepsilon^2 \right)^{-1}$$

The measurement update becomes

$$\hat{x}_k = \bar{x}_k + K_k \left(y_k - \tilde{H}_k \bar{x}_k \right)$$

$$= \bar{x}_k + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\varepsilon^2} \left(y_k - \bar{x}_k \right)$$

$$= \frac{\sigma_\varepsilon^2}{\sigma_x^2 + \sigma_\varepsilon^2} \bar{x}_k + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\varepsilon^2} y_k$$

What is the Role of the Kalman Gain?

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Hence, the best estimate of x_k is a weighted average of the predicted estimate and the measurement.

$$\hat{x}_k = \frac{\sigma_\varepsilon^2}{\sigma_x^2 + \sigma_\varepsilon^2}\bar{x}_k + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\varepsilon^2} y_k$$

If the measurement is very noisy (inaccurate) relative to the predicted state ($\sigma_\varepsilon^2 >> \sigma_x^2$)

Then $$\frac{\sigma_\varepsilon^2}{\sigma_x^2 + \sigma_\varepsilon^2} \approx 1, \qquad \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\varepsilon^2} \approx 0$$

and $$\hat{x}_k \approx \bar{x}_k$$

Hence, the measurement has little effect.

What is the Role of the Kalman Gain?

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Hence, the best estimate of x_k is a weighted average of the predicted estimate and the measurement.

$$\hat{x}_k = \frac{\sigma_\varepsilon^2}{\sigma_x^2 + \sigma_\varepsilon^2} \bar{x}_k + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_\varepsilon^2} y_k$$

Conversely if $\sigma_x^2 >> \sigma_\varepsilon^2$

$$\hat{x}_k \approx y_k$$

and the predicted estimate has little effect.

In general the Kalman gain provides a relative weighting between the *a priori* and the tracking data.

Summary

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- Batch
  - Process all observations at once

$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{p} (H_i^T R^{-1} H_i) + \bar{P}_0^{-1} \right)^{-1} \left(\sum_{i=1}^{p} (H_i^T R^{-1} \mathbf{y}_i) + \bar{P}_0^{-1} \bar{\mathbf{x}}_0 \right)$$

- Sequential / CKF
  - Process one observation at a time

$$\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + K_k \left[\mathbf{y}_k - \tilde{H}_k \bar{\mathbf{x}}_k \right]$$

$$P_k = \left[I - K_k \tilde{H}_k^T \right] \bar{P}_k$$

- Extended Kalman Filter (EKF)
  - Process one observation at a time and update ref.
  - Can be good for long arcs, if properly implemented

The End

- Homework 5 (?) and the test are graded.
  - Contact me within the week to challenge any grading.
- Homework 6 will be graded shortly
- Homework 7 due this week.
  - Points for quality

- Guest lecturer this Thursday
  - Jason Leonard will speak about LiAISON Navigation

- I'm unavailable Wednesday – Monday
  - Will be around sometimes, so you may catch me.
  - But I will be missing office hours – email me or call me if you have questions.