Zach Dischner | HW 3 | Born, Ch 4 # 1-6

1) Problem Statement/Givens

$$\dot{\vec{X}}(t) = A\vec{X}(t)$$

$$\Phi = \begin{bmatrix} e^{-2at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

Solve for $\underline{A}$

Solution

$$\dot{\vec{X}} = \dot{\Phi}\vec{x} = A\Phi\vec{x}$$

$$\dot{\Phi}\cancel{\vec{x}}(\vec{x})^{-1} = A\Phi\vec{x}(\vec{x})^{-1}$$

$$\dot{\Phi} = A\Phi$$

$$\begin{bmatrix} -2ae^{-2at} & 0 \\ 0 & be^{bt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} e^{-2at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

from the above

$$-2ae^{-2at} = A_{11}e^{-2at} + A_{12}\cdot 0$$

$$0 = A_{11}\cdot 0 + A_{12}\cdot e^{bt}$$

$$0 = A_{21}e^{-2at} + A_{22}\cdot 0$$

$$be^{bt} = A_{21}\cdot 0 + A_{22}\cdot e^{bt}$$

$A_{12} = 0$

$A_{21} = 0$

$A_{11} = -2a$

$A_{22} = b$

Then $\boxed{A = \begin{bmatrix} -2a & 0 \\ 0 & b \end{bmatrix}}$

2) Problem Statement / Givens

$$\dot{\vec{x}}(t_i) = A(t_i)\vec{x}(t_i)$$

$$\downarrow \text{solution}$$

$$\vec{x}(t_i) = \Phi(t_i, t_k)\, \vec{x}(t_k)$$

where $\Phi(t_k, t_k) = \bar{I}$

Show:

(a): $\dot{\Phi}(t_i, t_k) = A(t_i)\,\Phi(t_i, t_k)$

rewrite: $\dot{\vec{x}}(t_i) = \dot{\Phi}(t_i, t_k)\, \vec{x}(t_k)$

substitute: $A(t_i)\, x(t_i) = \dot{\Phi}(t_i, t_k)\, \Phi(t_i, t_k)\, \vec{x}(t_k)$

substitute: $A(t_i)\, \Phi(t_i, t_k)\, x(t_k) = \dot{\Phi}(t_i, t_k)\, x(t_k)$

mult by x_k^{-1}

$$\boxed{A(t_i)\, \Phi(t_i, t_k) = \dot{\Phi}(t_i, t_k)}$$

? (b): $\Phi(t_i, t_j) = \Phi(t_i, t_k)\,\Phi(t_k, t_j)$

from eq 4.2.7 $\dot{x}(t_j) = \Phi(t_j, t_k)\, x(t_k)$

$\dot{x}(t_i) = \Phi(t_i, t_k)\, x(t_k)$

use substitution

$x(t_j) = \Phi(t_j, t_k)\, \Phi(t_k, t_i)\, x(t_i)$

$= \Phi(t_j, t_i)\, x(t_i)$

so $\boxed{\Phi(t_i, t_j) = \Phi(t_i, t_k)\, \Phi(t_k, t_j)}$

1) (c) $\Phi^{-1}(t_i, t_k) = \Phi(t_k, t_i)$

From given: $x(t_i) = \Phi(t_i, t_k) \cdot x(t_k)$

becomes

$$x(t_i)\,\Phi(t_i, t_k)^{-1} = \cancel{\Phi(t_i, t_k)}\;\cancel{\Phi(t_i, t_k)^{-1}}\; x(t_k)$$

$$x(t_i)\,\Phi(t_i, t_k)^{-1} = x(t_k)$$

looks like orignal equation statement where we would see:

$$x_k = \Phi(t_k, t_i)\,x(t_i)$$

so $\boxed{\Phi^{-1}(t_i, t_k) = \Phi(t_k, t_i)}$

(d) $\dot{\Phi}(t_i, t_k) = -\Phi^{-1}(t_i, t_k)\,A(t_i)$

From above

$$\dot{x}(t_k) = \Phi^{-1}(t_i, t_k)\,A(t_i)$$

differentiate $\dot{x}(t_k) = \dot{\Phi}^{-1}(t_i, t_k)\;A(t_i)$

4.2.6 $A(t_i)\,x(t_k) = \dot{\Phi}^{-1}(t_i, t_k)\,A(t_i)$

given $A(t_i)\,\Phi(t_i, t_k) = \Phi^{-1}(t_i, t_k)\,A(t_i)$

from eq 4.2.10: $\boxed{\dot{\Phi}(t_i, t_k) = \Phi^{-1}(t_i, t_k)\,A(t_i)}$

3) problem Statement / givens

given vector of observations $y = H x + \varepsilon$

weighting Matrix W

priori information $(\bar{x}, \bar{W})$

→ Determine least squares estimate for $\hat{x}$

$W \Rightarrow R^{-1}$
$\bar{W} \Rightarrow P_0^{-1}$

performance index

$$J(x) = \frac{1}{2} \varepsilon^T W \varepsilon + \frac{1}{2} \eta^T \bar{W} \eta$$

where

η = error in priori estimate $\bar{x}$

$$\eta = \bar{x} - x$$

Solution

want $\frac{\partial J}{\partial x} = 0$

$$= \frac{\partial J}{\partial x} \left[\frac{1}{2} (y - Hx)^T W (y - Hx) + \frac{1}{2} (\bar{x} - x)^T \bar{W} (\bar{x} - x) \right]$$

from Appendix B

$$\left((A^T)(B) \Rightarrow \frac{\partial (A^T B)}{\partial x} = B^T \frac{\partial A}{\partial x} + A^T \frac{\partial B}{\partial x} \right)$$

$$= -H^T W (y - H\hat{x}) + \bar{W} (\bar{x} - \hat{x})(-1)$$

$$= (-H^T W H \hat{x}) - H^T W y + -\bar{W} \bar{x} + \bar{W} \hat{x}$$

$$= (H^T W H + \bar{W}) \hat{x} - H^T W y - \bar{W} \bar{x}$$

$$\boxed{\hat{x}_{min} = \left[H^T W H + \bar{W} \right]^{-1} \left[H^T W y + \bar{W} \bar{x} \right]}$$

4) Problem Statement/Givens

$$A = \begin{bmatrix} a & 0 \\ b & g \end{bmatrix} \quad \text{where} \quad a \neq g$$

$$\dot{\Phi} = A\Phi, \quad \Phi(t_0, t_0) = I$$

Find State Transition Matrix

$$\dot{\Phi} = A\Phi$$

$$\begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & g \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Use Laplace

$$sI\Phi_{\mathcal{L}} - A\Phi_{\mathcal{L}} = 0$$

$$\Phi_{\mathcal{L}} = (sI_{\mathcal{L}} - A)^{-1}$$

$$\Phi = \mathcal{L}^{-1} \frac{1}{sI - A}$$

$$= \mathcal{L}^{-1} \left[\frac{1}{\begin{bmatrix} s-a & 0 \\ -b & s-g \end{bmatrix}} \right]$$

$X_{\mathcal{L}}$ marks laplace domain

Inverse laplace performed in MATLAB

$$\boxed{\Phi = \begin{bmatrix} \left(e^{at}\right) & (0) \\ \left(\frac{be^{at}}{a-g} - \frac{be^{gt}}{a-g}\right) & \left(e^{gt}\right) \end{bmatrix}}$$

5) problem statement/givens

express: $\ddot{x} = -abx$

in form $\dot{x} = A\vec{x}$

where $\vec{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

find State Transition Matrix

say $\dot{\vec{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$

$= \begin{bmatrix} 0 \cdot x \\ -ab x \end{bmatrix}$

$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -ab & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ looks like $\dot{x} = Ax$

$\dot{\Phi} = A\Phi$

laplace transform

$\Phi = \mathcal{L}^{-1}[sI - A]$

$= \mathcal{L}^{-1} \begin{bmatrix} s & 1 \\ -ab & s \end{bmatrix}$

INverse LAPLACE
IN MATLAB

$$\Phi = \begin{bmatrix} [\cos(t\sqrt{ab})] & [\sin(t\sqrt{ab})/\sqrt{ab}] \\ [\sqrt{ab}\sin(\sqrt{ab}\,t)] & [\cos(\sqrt{ab}\,t)] \end{bmatrix}$$

6) Problem Statement / Givens

show that:

$$\Phi(t, t_0) = \begin{bmatrix} 3e^{at} & 0 \\ 0 & 2e^{-bt} \end{bmatrix}$$

satisfies:

$$\dot{\Phi} = A\Phi$$

but that $\Phi(t, t_0)$ is not a transition matrix

soln

$$\dot{\Phi} = A\Phi$$

$$\begin{bmatrix} 3ae^{at} & 0 \\ 0 & -2be^{-bt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 3e^{at} & 0 \\ 0 & 2e^{-bt} \end{bmatrix}$$

solveable

Matrix Math, $\dot{\Phi}\Phi^{-1} = A \underbrace{\cancel{\Phi\Phi^{-1}}}_{\text{Identity}}$

done in MATLAB

$$\begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix} = A$$

$$\boxed{\text{so yes There is a solution} \quad \dot{\Phi} = A\Phi} \checkmark$$

$\Phi(t_0, t_0)$

$$= \begin{bmatrix} 3e^0 & 0 \\ 0 & 2e^0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \neq I$$

$$\boxed{\Phi(t_0, t_0) \neq \text{identity}} \checkmark$$

$\dot{\Phi} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow$ so $\dot{\Phi} = A\Phi$ is unique for $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$