CCAR
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ASEN 5070
Statistical Orbit determination I
Fall 2012

Colorado Center for
Astrodynamics Research

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Professor Jeffrey S. Parker

Lecture 11: Batch and Sequential

University of Colorado
Boulder

Announcements

- Homework 4 Graded
- Homework 5 due Thursday
- LaTex example on the ccar.colorado.edu/ASEN5070 website
- Bobby Braun will be a guest speaker Thursday.
  - NASA Chief Technologist
  - Now at Georgia Tech.
- Exam on 10/11.
- I'll be out of the state 10/9 – 10/16; we'll have special lecturers then.

Homework 5

- Problem (1c) clarification

  (c) For this problem, $\Phi(t_i, t_0)$ is symplectic. Demonstrate this for $\Phi(t_{100}, t_0)$ by multiplying it by $\Phi^{-1}(t_{100}, t_0)$, given by Equation 4.2.22 in the text. Show that the result is the identity matrix.

- Don't just multiply Phi by its inverse. Rather, use Equation 4.2.22 to define the inverse:

- $$\Phi^{-1}(t, t_k) = -(J\Phi(t, t_k)J)^T$$

- Then multiply Phi by this matrix and show that it's equal to the identity matrix.

Quick Phi discussion

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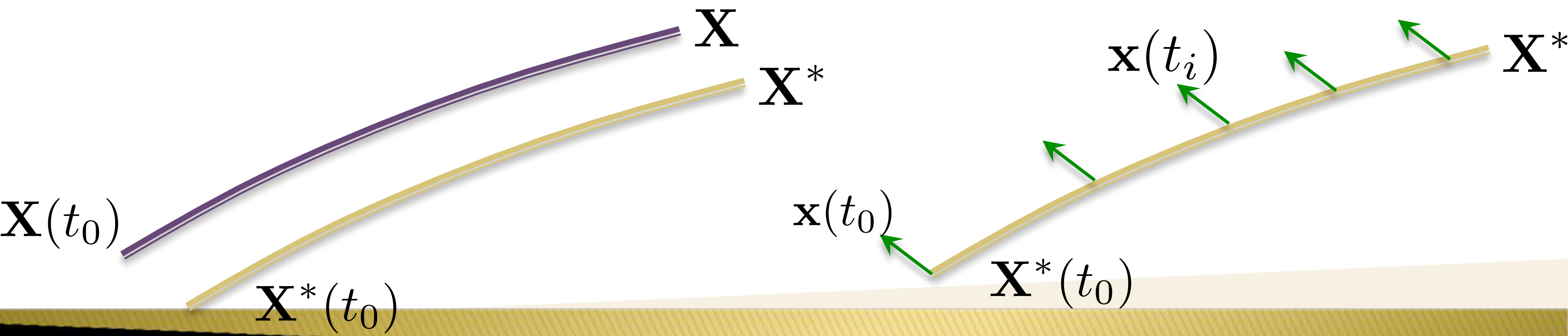
- This is a very common mistake, so we'll review it.

- Integrators propagate state vectors.
- State transition matrices map state deviation vectors.

$$\mathbf{x}(t_2) = \Phi(t_2, t_1)\mathbf{x}(t_1)$$

$$\mathbf{x}(t_i) = \mathbf{X}(t_i) - \mathbf{X}^*(t_i)$$



Quiz Results

Information

Spacecraft states often include position, velocity, mass, coefficient of reflectivity, etc.

For this quiz, we'll set the spacecraft state to 7 parameters:

X = [x, y, z, vx, vy, vz, mass]^T

The following quiz questions consider different scenarios. Which of these scenarios permit a navigator to estimate the spacecraft's mass?

Assume that the spacecraft is being tracked from normal ground stations using normal range and Doppler tracking. There are no accelerometers or other special instruments collecting data for Stat OD's purposes.

Quiz Results

Question 1 (1 point)

Scenario: Spacecraft in ballistic orbit about point-mass (or "spherical Earth") with no other forces present.

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 1 (1 point)

Scenario: Spacecraft in ballistic orbit about point-mass (or "spherical Earth") with no other forces present.

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 2 (1 point)

Scenario: Spacecraft in ballistic orbit about aspherical body (e.g., asteroid), with no other forces present.

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 2 (1 point)

Scenario: Spacecraft in ballistic orbit about aspherical body (e.g., asteroid), with no other forces present.

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 3 (1 point)

Scenario: Spacecraft performing frequent maneuvers in orbit about a point-mass, with a well-characterized and accurate engine model (no other forces present).

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 3 (1 point)

Scenario: Spacecraft performing frequent maneuvers in orbit about a point-mass, with a well-characterized and accurate engine model (no other forces present).

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 4 (1 point)

Scenario: Spacecraft aerobraking about Mars w/ very poor data about the atmospheric conditions from one orbit to the next.

T/F: We can estimate this spacecraft's mass (accurately).

- True
- False

Quiz Results

Question 4 (1 point)

Scenario: Spacecraft aerobraking about Mars w/ very poor data about the atmospheric conditions from one orbit to the next.

T/F: We can estimate this spacecraft's mass (accurately).

- True
- False

Quiz Results

Question 5 (1 point)

Scenario: Spacecraft aerobraking about Mars w/ very GOOD data about the atmospheric conditions from one orbit to the next. That is, we have great models and can easily predict the density of the atmosphere as the spacecraft passes through it.

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 5 (1 point)

Scenario: Spacecraft aerobraking about Mars w/ very GOOD data about the atmospheric conditions from one orbit to the next. That is, we have great models and can easily predict the density of the atmosphere as the spacecraft passes through it.

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 6 (1 point)

Scenario: Spacecraft flying under the propulsion of a solar electric engine, i.e., near-continuous low-thrust engine, and the engine is indeed burning during the measurement arc.

T/F: We can estimate this spacecraft's mass.

- True
- False

Quiz Results

Question 6 (1 point)

Scenario: Spacecraft flying under the propulsion of a solar electric engine, i.e., near-continuous low-thrust engine, and the engine is indeed burning during the measurement arc.

T/F: We can estimate this spacecraft's mass.

- True
- False

Least Squares Options

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- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

- Least Squares with *a priori*

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

- Min Variance

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

- Min Variance with *a priori*

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

Propagating the estimate and covariance

- Given a state deviation and/or a covariance, how do we map them from one time to another?

$$\bar{\mathbf{x}}_k = \Phi(t_k, t_j)\hat{\mathbf{x}}_j$$

$$\bar{P}_k \equiv E\left[(\bar{\mathbf{x}}_k - \mathbf{x}_k)(\bar{\mathbf{x}}_k - \mathbf{x}_k)^T\right]$$

$$\bar{P}_k = \Phi(t_k, t_j)P_j\Phi^T(t_k, t_j)$$

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- Minimum Variance Estimate with *a priori*

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Minimum Variance w/a priori

- Given: System of state-propagation equations and observation state equations:

$$\mathbf{x}_i = \Phi(t_i, t_k)\mathbf{x}_k$$

$$\mathbf{y}_i = \tilde{H}_i \mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

- Also given *a priori* information about the state and covariance at some time

$$\bar{\mathbf{x}}_k \qquad \bar{P}_k$$

- Such that $E[\epsilon_k] = 0 \qquad E[\epsilon_k \epsilon_j^T] = R_k \delta_{kj} \qquad E[(\bar{\mathbf{x}}_j - \mathbf{x}_j)\epsilon_k^T] = 0$

- Find: The linear, unbiased, minimum variance estimate, $\hat{\mathbf{x}}_k$, of the state $\mathbf{x}_k$.

Minimum Variance w/a priori

- Observations:

$$\mathbf{y}_i = \tilde{H}_i \mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

- Can $\bar{\mathbf{x}}_k$ be treated as an observation? We certainly have some information about it: $\bar{P}_k \quad E[(\bar{\mathbf{x}}_j - \mathbf{x}_j)\epsilon_k^T] = 0$

Minimum Variance w/a priori

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- Observations:

$$\mathbf{y}_i = \tilde{H}_i \mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

- Can $\bar{\mathbf{X}}_k$ be treated as an observation? We certainly have some information about it: $\bar{P}_k \quad E[(\bar{\mathbf{x}}_j - \mathbf{x}_j)\epsilon_k^T] = 0$

- Yes, it can. Since $\hat{\mathbf{x}}_j$ is unbiased, then $\bar{\mathbf{X}}_k$ will be unbiased.

$$E[\overline{\mathbf{x}}_k] = \Phi(t_k, t_j) E[\hat{\mathbf{x}}_j] = \mathbf{x}_k$$

- Therefore, we may treat $\bar{\mathbf{X}}_k$ as an additional data equation.

Minimum Variance w/a priori

- Observations:

$$\mathbf{y}_i = \tilde{H}_i \mathbf{x}_i + \epsilon_i \qquad i = 1, \ldots, l.$$

$$\overline{\mathbf{x}}_k = \mathbf{x}_k + \boldsymbol{\eta}_k$$

- where

$$E[\boldsymbol{\epsilon}_k] = 0, \quad E[\boldsymbol{\epsilon}_k \boldsymbol{\epsilon}_k^T] = R_k, \quad E[\boldsymbol{\eta}_k] = 0,$$

$$E[\boldsymbol{\eta}_k \boldsymbol{\epsilon}_k^T] = 0, \text{ and } E[\boldsymbol{\eta}_k \boldsymbol{\eta}_k^T] = \overline{P}_k.$$

Note, for this formulation, we're assuming that each observation is independent.

$$E[\epsilon_j \epsilon_k^T] = 0$$

Minimum Variance w/a priori

- We can thus build an appended set of relationships:

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_k \\ \dots \\ \overline{\mathbf{x}}_k \end{bmatrix}; H = \begin{bmatrix} \widetilde{H}_k \\ \dots \\ I \end{bmatrix};$$

$$\epsilon = \begin{bmatrix} \epsilon_k \\ \dots \\ \eta_k \end{bmatrix}; R = \begin{bmatrix} R_k & 0 \\ & \dots \\ 0 & \overline{P}_k \end{bmatrix};$$

- Given these relationships, we already know what the solution is!

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

Though of course “H”, “R”, and “y” are a bit different than previous definitions. Let’s clean this up.

Minimum Variance w/a priori

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$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_k \\ \cdots \\ \overline{\mathbf{x}}_k \end{bmatrix}; H = \begin{bmatrix} \widetilde{H}_k \\ \cdots \\ I \end{bmatrix};$$

$$\epsilon = \begin{bmatrix} \epsilon_k \\ \cdots \\ \eta_k \end{bmatrix}; R = \begin{bmatrix} R_k & 0 \\ & \cdots \\ 0 & \overline{P}_k \end{bmatrix};$$

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

$$\hat{\mathbf{x}}_k = \left\{ [\widetilde{H}_k^T : I] \begin{bmatrix} R_k^{-1} & 0 \\ \cdots & \cdots \\ 0 & \overline{P}_k^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{H}_k \\ \cdots \\ I \end{bmatrix} \right\}^{-1} \left\{ [\widetilde{H}_k^T : I] \begin{bmatrix} R_k^{-1} & 0 \\ \cdots & \cdots \\ 0 & \overline{P}_k^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_k \\ \cdots \\ \overline{\mathbf{x}}_k \end{bmatrix} \right\}$$

or

$$\hat{\mathbf{x}}_k = (\widetilde{H}_k^T R_k^{-1} \widetilde{H}_k + \overline{P}_k^{-1})^{-1} (\widetilde{H}_k^T R_k^{-1} \mathbf{y}_k + \overline{P}_k^{-1} \overline{\mathbf{x}}_k)$$

(Eq 4.4.29)

Minimum Variance w/a priori

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- The covariance matrix associated with the estimation error in this estimate is:

$$P_k = E[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^T]$$
$$= (\widetilde{H}_k^T R_k^{-1} \widetilde{H}_k + \overline{P}_k^{-1})^{-1}.$$

- Information Matrix:

$$\Lambda_k = P_k^{-1}$$

$$\Lambda_k \hat{\mathbf{x}}_k = \tilde{H}_k^T R_k^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k$$
$$\Lambda_k \hat{\mathbf{x}}_k = N_k$$

Minimum Variance w/a priori

- Review of variables

$$\Lambda_k \hat{\mathbf{x}}_k = \tilde{H}_k^T R_k^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k$$

Tools at Hand

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- We now have tools to do the following:

- Propagate a state
- Propagate a state transition matrix
- Map a state deviation vector
- Map an observation deviation vector
- Map a variance-covariance matrix

- Generate an estimate of a state given any sort of information (*a priori*, weighted, observation variance, etc)

Least Squares Options

- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

- Least Squares with *a priori*

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

- Min Variance

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

- Min Variance with *a priori*

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

- The Batch Processor

The Batch Processor

- Process:
  - Collect all observations and residuals over an arc
  - Process them using Least Squares
  - Generate a best estimate
  - Iterate.

The Batch Processor

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- Process:
  - Collect all observations and residuals over an arc
  - Process them using Least Squares
  - Generate a best estimate
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$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

Computational Algorithm for the Batch Processor

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}\right)^{-1}_{n\times n}\left(H^T W \mathbf{y} + \bar{W}\bar{\mathbf{x}}_k\right)_{n\times 1}$$

Look at $H^T W H$,

$$\left[H_1^T \; H_2^T \ldots H_l^T\right]\begin{bmatrix} W_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W_l \end{bmatrix}\begin{bmatrix} H_1 \\ \vdots \\ H_l \end{bmatrix} = H_1^T W_1 H_1 + H_2^T W_2 H_2 + \cdots H_l^T W_l H_l$$

$$= \sum_{i=1}^{l} H_i^T W_i H_i$$

Computational Algorithm for the Batch Processor

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}\right)^{-1}_{n\times n} \left(H^T W \mathbf{y} + \bar{W}\bar{\mathbf{x}}_k\right)_{n\times 1}$$

Look at $H^T W H$,

$$\left[H_1^T \; H_2^T \ldots H_l^T\right] \begin{bmatrix} W_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W_l \end{bmatrix} \begin{bmatrix} H_1 \\ \vdots \\ H_l \end{bmatrix} = H_1^T W_1 H_1 + H_2^T W_2 H_2 + \cdots H_l^T W_l H_l$$

$$= \sum_{i=1}^{l} H_i^T W_i H_i$$

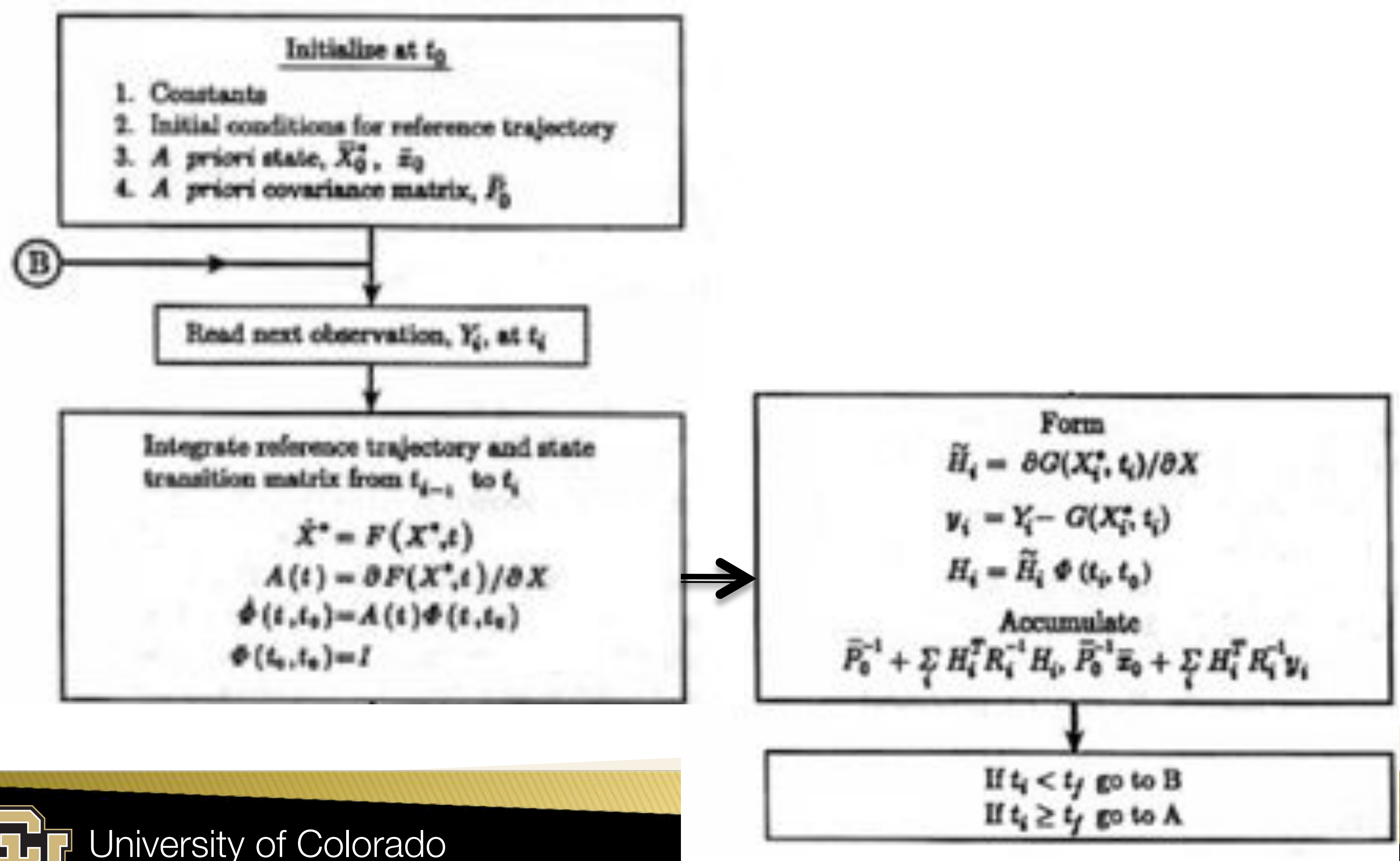
Likewise,

$$H^T W y = \sum_{i=1}^{l} H_i^T W_i y_i$$

and

$$\hat{\mathbf{x}}_k = \left(\sum_{i=1}^{l} H_i^T W_i H_i + \bar{W}\right)^{-1} \left(\sum_{i=1}^{l} H_i^T W_i \mathbf{y}_i + \bar{W}\bar{\mathbf{x}}_k\right)$$

Computational Algorithm for the Batch Processor



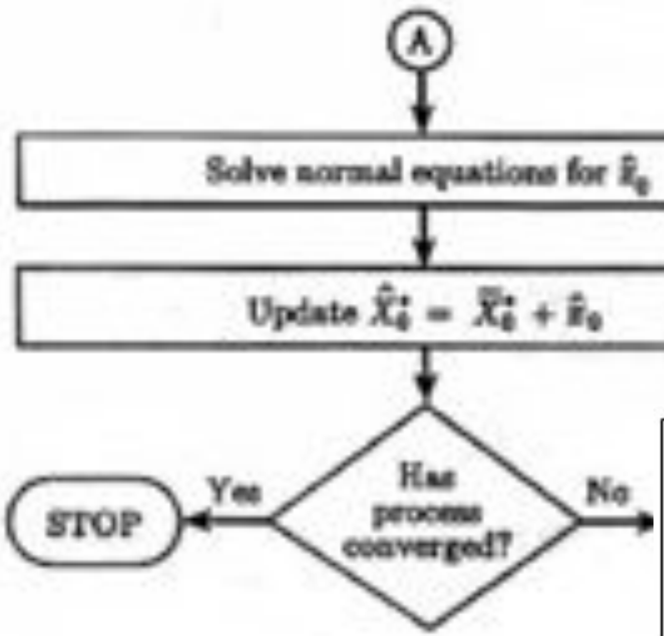
Computational Algorithm for the Batch Processor

We want to maintain apriori Information; $\bar{x}_0$, *and* $\bar{P}_0$ hence,

$$X^*_{n+1} + \bar{x}_{n+1} = X^*_n + \bar{x}_n \quad (1)$$

$$BUT,\ X^*_{n+1} = X^*_n + \hat{x}_n \quad (2)$$

$$Subst\ (2) \rightarrow (1),\quad \bar{x}_{n+1} = \bar{x}_n - \hat{x}_n$$



Repeat initialization using $\hat{X}^*_0$ as the initial conditions for the reference trajectory.
Set $\bar{x}_0 = \bar{x}_0 - \hat{x}_0$ and use the original value of $\bar{P}_0$.
Reprocess all observations.

Computational Algorithm for the Batch Processor

- The Batch Processor depends on assumptions of linearity.
- Provided that the reference trajectory is near the truth, this holds just fine.

- However, the Batch must be iterated 2-3 times to get the *best* best estimate.

- Check by computing the RMS of the observation residuals each run. Keep iterating until the RMS stops decreasing.

LEO Orbit Determination Example

- Instantaneous observation data is taken from three Earth fixed tracking stations over an approximate 5 hour time span (light time is ignored).

$$\rho = \left(x^2 + y^2 + z^2 + x_{s_i}^2 + y_{s_i}^2 + z_{s_i}^2 - 2(xx_{s_i} + yy_{s_i})\cos\theta + 2(xy_{s_i} - yx_{s_i})\sin\theta - 2zz_{s_i}\right)^{\frac{1}{2}}$$

$$\dot{\rho} = \frac{x\dot{x} + y\dot{y} + z\dot{z} - (\dot{x}x_{s_i} + \dot{y}y_{s_i})\cos\theta + \dot{\theta}(xx_{s_i} + yy_{s_i})\sin\theta + (\dot{x}y_{s_i} - \dot{y}x_{s_i})\sin\theta + \dot{\theta}(xy_{s_i} - yx_{s_i})\cos\theta - \dot{z}z_{s_i}}{\rho}$$

- where x, y, and z represent the spacecraft Earth Centered Inertial (ECI) coordinates and x_s, y_s, and z_s are the tracking station Earth Centered, Earth Fixed (ECEF) coordinates.

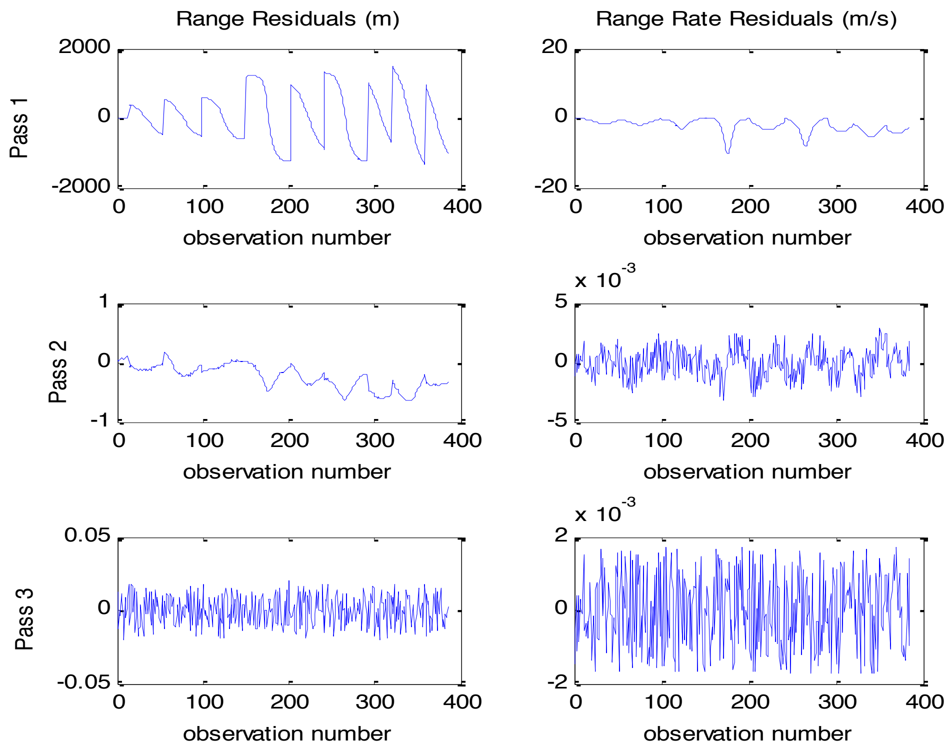
LEO Orbit Determination Example

State Deviation ($\hat{x}_0$)		
Pass 1	**Pass 2**	**Pass 3**
-0.036383127158274	0.326725527230386	2.16404427676782e-007
-0.274523873228247	-0.1480641161433	-2.62437517889209e-007
-0.18013801260102	-0.0809938800450721	1.64403340270235e-007
0.0409346491303605	-0.000317153824609643	-4.38244130278385e-010
0.0327489396676593	-3.92221769835543e-005	1.94288829502823e-010
-0.0147524440475379	0.000338033953408322	2.49099138592487e-010
-9463031.96726608	-33306147.3255543	-16.31113339128591
-6.57367472046001e-007	2.98810508464489e-008	2.27735423982078e-015
0.147516774454061	0.0411458796251399	-4.59840606133234e-007
1.86265905206096e-006	-1.86970469476067e-006	8.73427771458123e-013
1.37829598816653e-006	-1.38350949129108e-006	6.46310696714768e-013
-2.54260084011498e-007	2.54404878085509e-007	2.24461403220845e-013
-10.5635758182854	0.555182863038883	9.07302237892118e-008
9.98322113724009	0.0201549881875307	-2.43030096543763e-007
5.79485500114413	0.18208202802521	2.37370032711892e-007
-5.78184708032509	0.773198712686432	2.35005539856875e-008
2.34411863803746	-0.323056978825581	-8.52191767278272e-008
1.51222736642103	1.46363482699289	5.24955707897414e-007

Batch State Deviation for Passes 1, 2, and 3

LEO Orbit Determination Example

Batch (Least Squares) Residuals



RMS Values			
	Pass 1	**Pass 2**	**Pass 3**
Range (m)	732.748350225264	0.319570766726265	0.00974562719122707
Range Rate (m/s)	2.90016531897711	0.00119972978584721	0.000997930398398708

Batch Issues

- Inverting a potentially poorly scaled matrix

$$\hat{\mathbf{x}}_k = \left(\sum_{i=1}^{l} H_i^T W_i H_i + \bar{W} \right)^{-1} \left(\sum_{i=1}^{l} H_i^T W_i \mathbf{y}_i + \bar{W} \bar{\mathbf{x}}_k \right)$$

- Solutions: Decompose the matrix, Cholesky, Givens, Householder, etc.
  - Square-root free transformations

- Another issue comes up with numerical computations of the covariance matrix. It can sometimes drift away from being symmetric or nonnegative.
  - Solutions: Joseph, Potter

- (break)

- Sequential Processors

Sequential Processor

- Consider

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

- Rather than mapping all observations to one epoch and processing them simultaneously, what if we processed each separately and mapped the best estimate through each?

Sequential Processor

- Consider

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

- Rather than mapping all observations to one epoch and processing them simultaneously, what if we processed each separately and mapped the best estimate through each?

$$\bar{\mathbf{x}}_k = \Phi(t_k, t_j)\hat{\mathbf{x}}_j$$

$$\bar{P}_k = \Phi(t_k, t_j) P_j \Phi^T(t_k, t_j)$$

Sequential Processor

- Given an *a priori* state and covariance, we know how to generate a new estimate of the state:

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

- Need a way to generate the *a posteriori* covariance matrix as well.

- Recall

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1}$$

- The trouble is inverting the *n x n* matrix.

Sequential Processor

- We can use the Schur Identity and a bunch of math (see Section 4.7) and obtain:

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1} \right)^{-1}$$

$$P_k = \bar{P}_k - \bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1} \tilde{H}_k \bar{P}_k$$

Sequential Processor

- We can use the Schur Identity and a bunch of math (see Section 4.7) and obtain:

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1} \right)^{-1}$$

$$P_k = \bar{P}_k - \underbrace{\bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1}}_{K_k} \tilde{H}_k \bar{P}_k$$

Kalman Gain

Sequential Processor

- After some more math, we can simplify to obtain:

$$P_k = \left[I - K_k \tilde{H}_k^T \right] \bar{P}_k$$

$$\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + K_k \left[\mathbf{y}_k - \tilde{H}_k \bar{\mathbf{x}}_k \right]$$

Sequential Algorithm

- 1. Initialize the first run
- 2. Start at the reference epoch
- 3. Time Update
  - Integrate from the current time to the next time of interest
  - Map the state estimate and the covariance to the new time
- 4. Measurement Update
  - If there is a new measurement, process it.
  - Update the state estimate and covariance with this new information
- Repeat 3-4 until all measurements have been processed and all times of interest have been recorded.
- Optional: Map the estimate and covariance back to the reference epoch and iterate the whole process.

Sequential Algorithm

- Initialization

Given: $\hat{\mathbf{x}}_{k-1}$, P_{k-1}, $\mathbf{X}^*_{k-1}$, and R_k, and the observation $\mathbf{Y}_k$, at t_k (at the initial time t_0, these would be $\mathbf{X}^*_0$, $\hat{\mathbf{x}}_0$, and P_0).

Sequential Algorithm

- Time Update
  - Integration

$$\dot{\mathbf{X}}^* = F(\mathbf{X}^*, t), \qquad \mathbf{X}^*(t_{k-1}) = \mathbf{X}^*_{k-1}$$
$$\dot{\Phi}(t, t_{k-1}) = A(t)\Phi(t, t_{k-1}), \qquad \Phi(t_{k-1}, t_{k-1}) = I.$$

  - Mapping

$$\overline{\mathbf{x}}_k = \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1} \qquad \overline{P}_k = \Phi(t_k, t_{k-1})P_{k-1}\Phi^T(t_k, t_{k-1})$$

Sequential Algorithm

- Measurement Update
  - Collect measurement

$$\mathbf{y}_k = \mathbf{Y}_k - G(\mathbf{X}_k^*, t_k) \qquad \widetilde{H}_k = \frac{\partial G(\mathbf{X}_k^*, t_k)}{\partial \mathbf{X}}$$

  - Compute update

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$
$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]$$
$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k.$$

Sequential Processor

- Repeat just like the Batch
  - Replace the reference trajectory with the new best estimate.
  - Make sure to update the *a priori* state deviation vector to retain any information.
  - Recompute all observation residuals

Final Statements

- Homework 4 Graded
- Homework 5 due Thursday
- LaTex example on the ccar.colorado.edu/ASEN5070 website
- Bobby Braun will be a guest speaker Thursday.
  - NASA Chief Technologist
  - Now at Georgia Tech.
- Exam on 10/11.
- I'll be out of the state 10/9 – 10/16; we'll have special lecturers then.