Colorado Center for Astrodynamics Research

ASEN 5070
Statistical Orbit Determination I
Fall 2012

Professor Jeffrey S. Parker

Professor George H. Born

Lecture 20: Exam 2 Review

Announcements

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- Homework 8 due this week.
  - Make sure you spend time studying for the exam
- Homework 9 out today. You're not busy, are you? This one is easy and will push you toward the completion of the final project.
- Exam 2 on Thursday.
  - A-H in this classroom
  - I-Z in ECEE 265
- Exam 2 will cover:
  - Batch vs. CKF vs. EKF
  - Probability and statistics (good to keep this up!)
    - Haven't settled on a question yet, but it will probably be a conditional probability question. I.e., what's the probability of X given that Y occurs?
  - Observability
  - Numerical compensation techniques, such as the Joseph and Potter formulation.
  - No calculators should be necessary
  - Open Book, Open Notes

Quiz 16 Review

Question 1 (1 point)

Given a scenario where we have 6 state parameters (n=6) and 2 data types (p=2): The Batch processor can be set up to process observations without any a priori information about the state or covariance (or equivalently P-bar = infinity).

- True
- False

Quiz 16 Review

Question 1 (1 point)

Given a scenario where we have 6 state parameters (n=6) and 2 data types (p=2): The Batch processor can be set up to process observations without any a priori information about the state or covariance (or equivalently P-bar = infinity).

- True
- False

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

Quiz 16 Review

Question 2 (1 point)

Same scenario (n=6, p=2, m=lots). The Sequential processor (Conventional Kalman Filter) can be set up to process observations without any a priori information about the state or covariance (or equivalently P-bar = infinity).

- True
- False

Quiz 16 Review

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Question 2 (1 point)

Same scenario (n=6, p=2, m=lots). The Sequential processor (Conventional Kalman Filter) can be set up to process observations without any a priori information about the state or covariance (or equivalently P-bar = infinity).

- True
- False

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1}$$

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$

$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]$$

$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k .$$

HW#9

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- Due a week from Thursday

This assignment builds on Homework 7 and the final project. Use the code and experiences from Homework 7 as a starting point for this assignment.

Use the nominal trajectory and the state transition matrix generated in Homework 7, along with the observation data, to compute the following using the Batch Processor. Note that the weighting matrix W_i does not change over time, but remains a constant matrix.

1. Compute $\sum_{i=1}^{n} H_i^T W_i H_i$
2. Compute $\sum_{i=1}^{n} H_i^T W_i y_i$
3. Compute $\hat{x}_0$

For each of the above, turn in the value after one pass through the data. That is, don't iterate the process, though you will be doing so for the final project so prepare your code to support that feature. There are hints and example solutions on the course website to check your code.

Turn in the code and the values you obtain for each of these answers. You don't have to include any more write-up than the values, as long as your code is clear and commented well enough for us to follow.

- Review
- Lots of questions of CKF vs. EKF
- Lots of questions on observability
- Some questions on clarifications of parameters (bar, hat, P vs R, etc.), n / m / p

Parameters

- First off, conceptual parameters
- If you have *n* parameters to estimate, you require at least *n* pieces of information to uniquely estimate those parameters.
  - If you don't have that you can use the min-norm estimate

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  - Range = 1 piece
  - Doppler = 1 piece
  - An optical observation may involve 2 pieces (RA and Dec)

Parameters

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- If you have n parameters to estimate, you require at least n pieces of information to uniquely estimate those parameters.
  - If you don't have that you can use the min-norm estimate
- The sum of all observations = m pieces of information
  - Range = 1 piece
  - Doppler = 1 piece
  - An optical observation may involve 2 pieces (RA and Dec)
- Number of observation data types = p
- Number of observations = l
- $l \times p = m$

Parameters

- So, say you have *n* parameters and *m* total observations.
  - If $m < n$, min-norm
  - If $m = n$, deterministic
  - If $m > n$, least squares
- Each observation has an error associated with it, which introduces more unknowns. You end up with $n+m$ unknowns and *m* pieces of information → least squares to minimize the errors.

Stat OD Conceptualization

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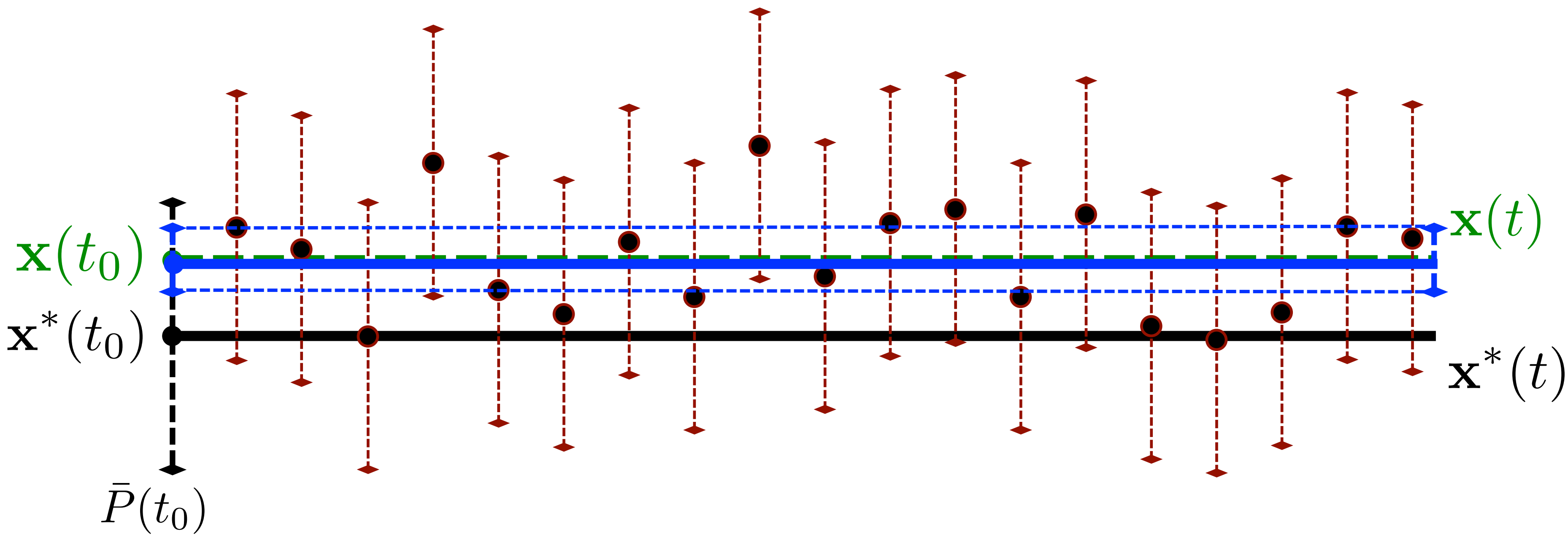
- Least Squares (Batch)



$$\hat{\mathbf{x}}(t_0) = \left(H^T R^{-1} H + \bar{P}^{-1}\right)^{-1} \left(H^T R^{-1} \mathbf{y} + \bar{P}^{-1} \bar{\mathbf{x}}\right)$$

$P(t)$

Least Squares Options

- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

- Least Squares with *a priori*

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

- Min Variance

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

- Min Variance with *a priori*

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

Batch

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- The Batch processor is just a wrapper around Least Squares.
- Accumulate information from all observations and simultaneously process them all (in a *batch*).

$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} H_i \right) + \bar{P}_0^{-1} \right)^{-1} \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} \mathbf{y}_i \right) + \bar{P}_0^{-1} \bar{\mathbf{x}}_0 \right)$$

Note the sizes of each matrix

Batch

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- Any numerical issues with the Batch?

$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} H_i \right) + \bar{P}_0^{-1} \right)^{-1} \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} \mathbf{y}_i \right) + \bar{P}_0^{-1} \bar{\mathbf{x}}_0 \right)$$

Batch

- What if the *a priori* covariance is huge? Tiny?

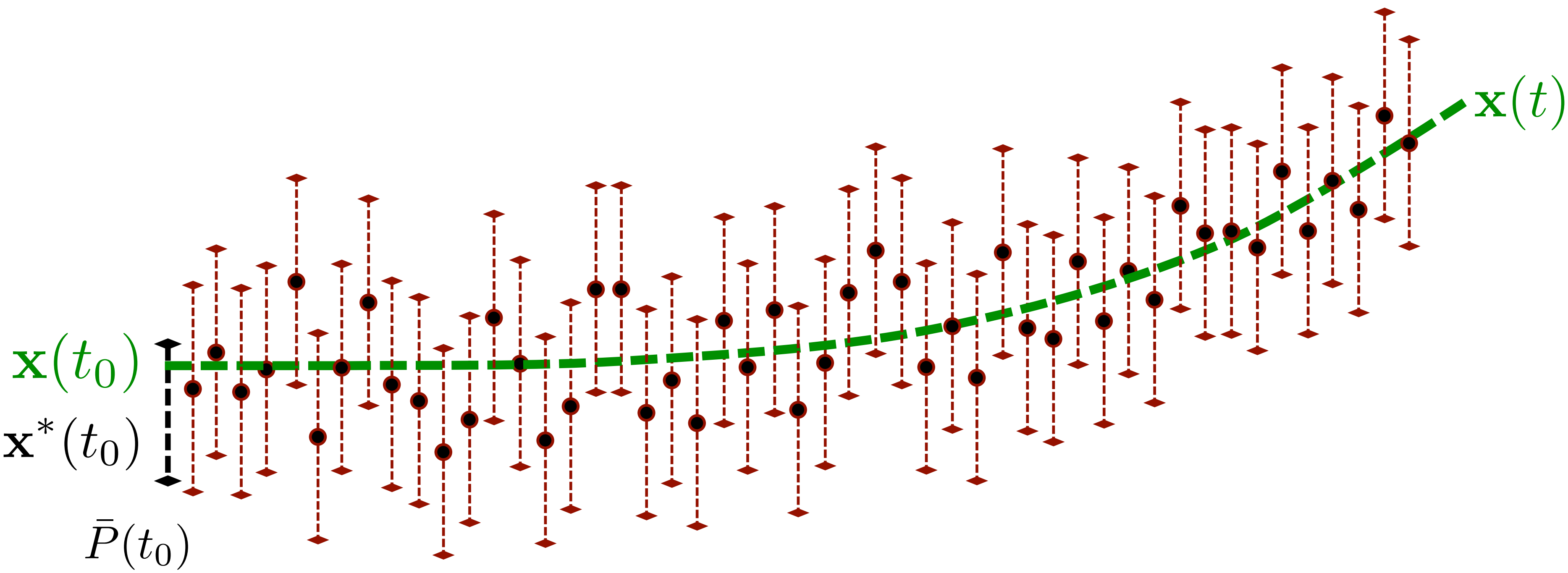
$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} H_i\right) + \bar{P}_0^{-1}\right)^{-1} \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} \mathbf{y}_i\right) + \bar{P}_0^{-1} \bar{\mathbf{x}}_0\right)$$

Batch

- What if we have poorly-modeled dynamics?

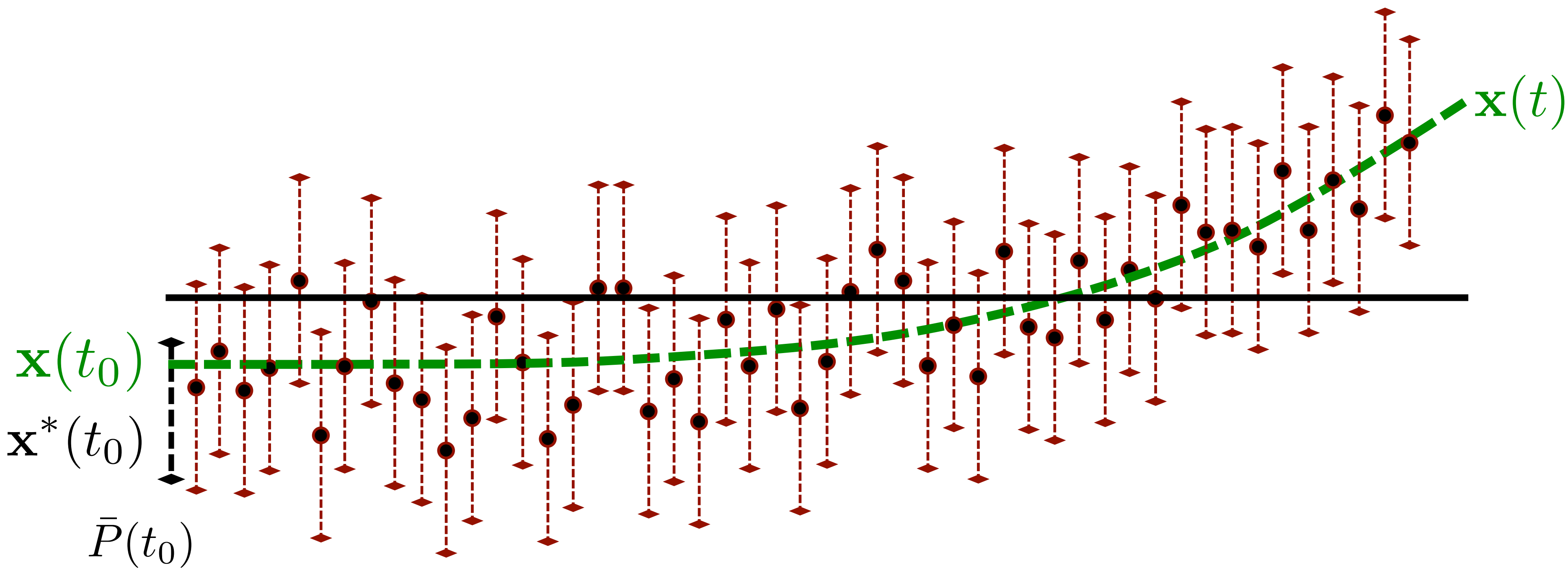
$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} H_i \right) + \bar{P}_0^{-1} \right)^{-1} \left(\sum_{i=1}^{l} \left(H_i^T R_i^{-1} \mathbf{y}_i \right) + \bar{P}_0^{-1} \bar{\mathbf{x}}_0 \right)$$

Stat OD Conceptualization



Batch fits a line to this data. (CONCEPTUAL)

Stat OD Conceptualization



Algorithm Options

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- Batch
  - Process all observations at once

$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{p} \left(H_i^T R^{-1} H_i \right) + \bar{P}_0^{-1} \right)^{-1} \left(\sum_{i=1}^{p} \left(H_i^T R^{-1} \mathbf{y}_i \right) + \bar{P}_0^{-1} \bar{\mathbf{x}}_0 \right)$$

- Sequential
  - Process one observation at a time

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1} \right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k \right)$$

Algorithm Options

- Sequential
  - Process one observation at a time

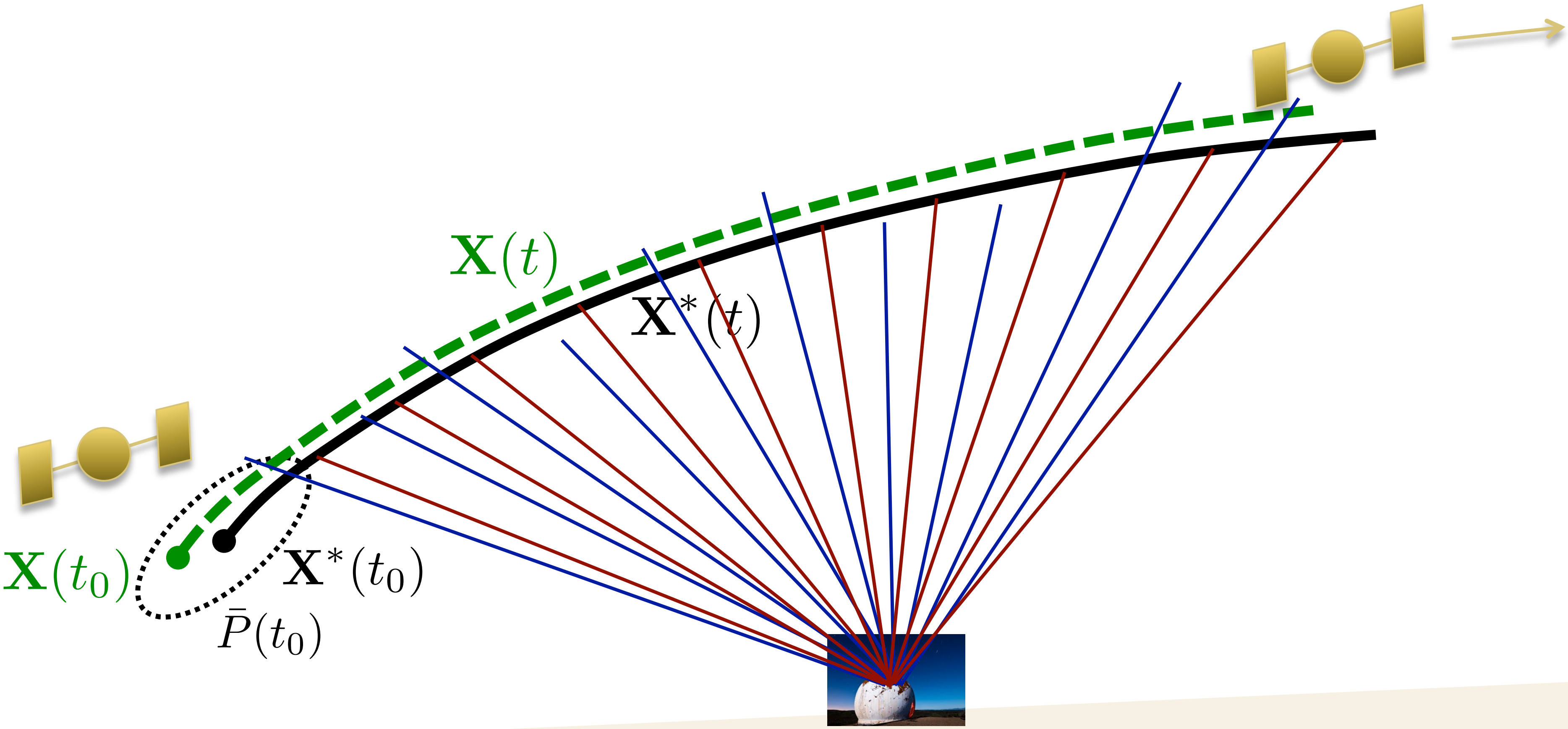
$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

  - Reformulation

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$
$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]$$
$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k.$$

Stat OD Conceptualization

- Full, nonlinear system:



Stat OD Conceptualization

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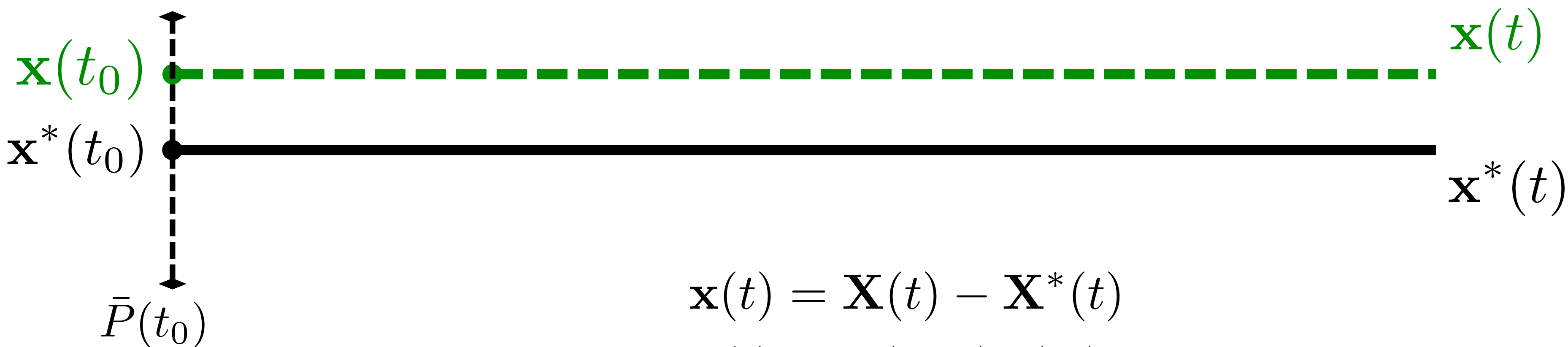
- Linearization



$$\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t)$$

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$$

Stat OD Conceptualization

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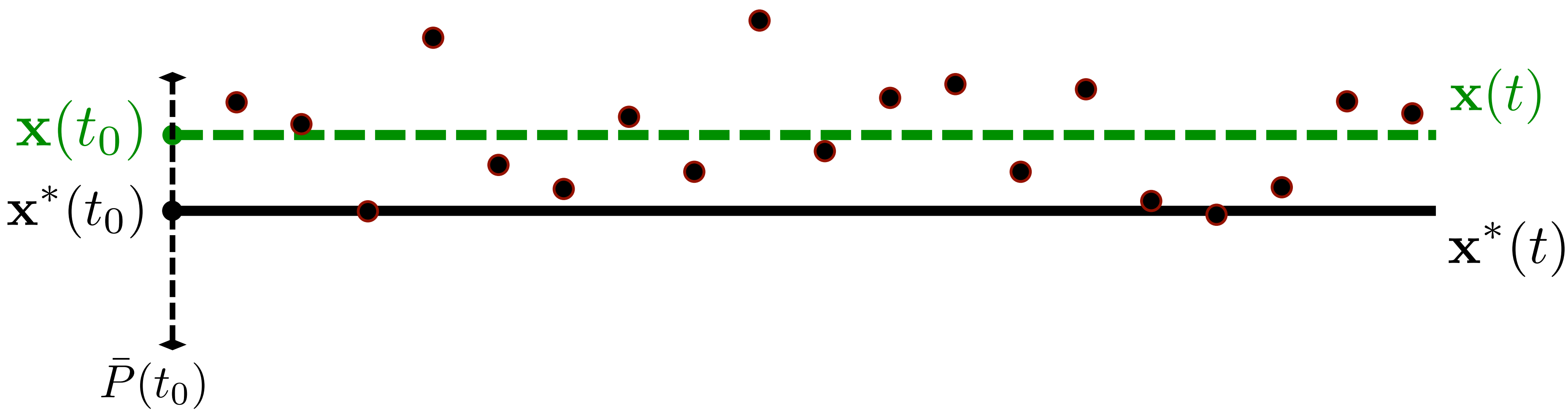
- Observations



- $\mathbf{y}(t) = \mathbf{Y}(t) - G(\mathbf{X}^*(t), t)$

Stat OD Conceptualization

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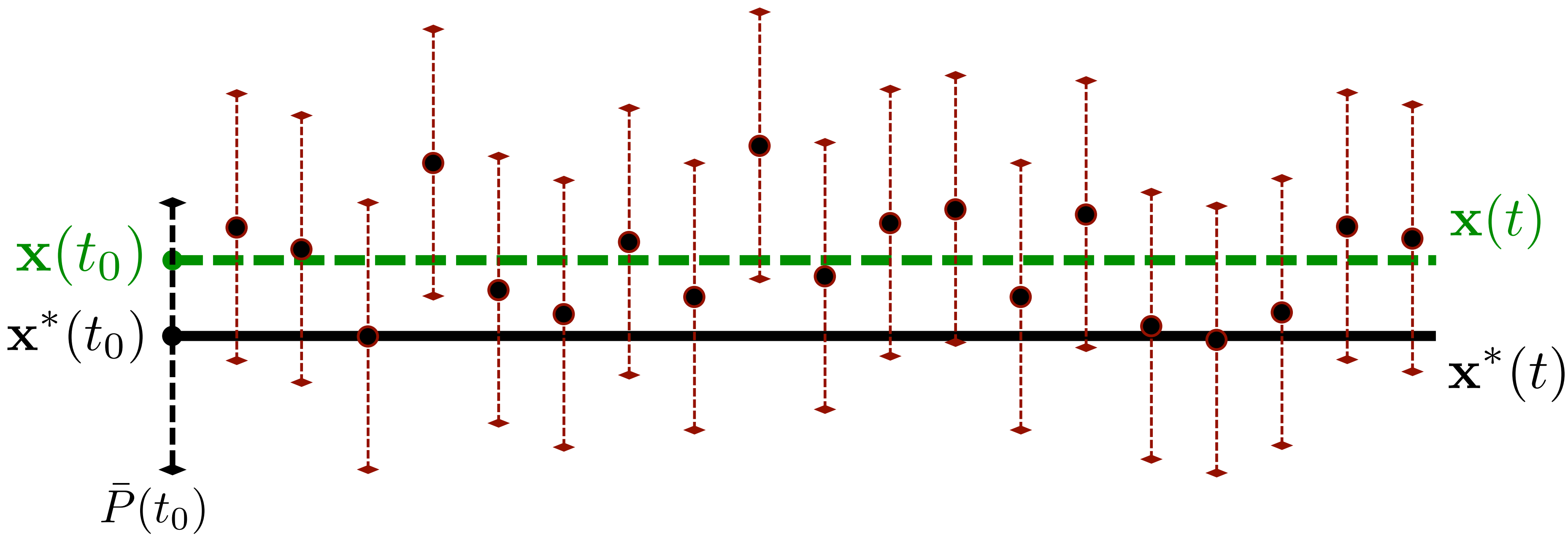
- Observation Uncertainties

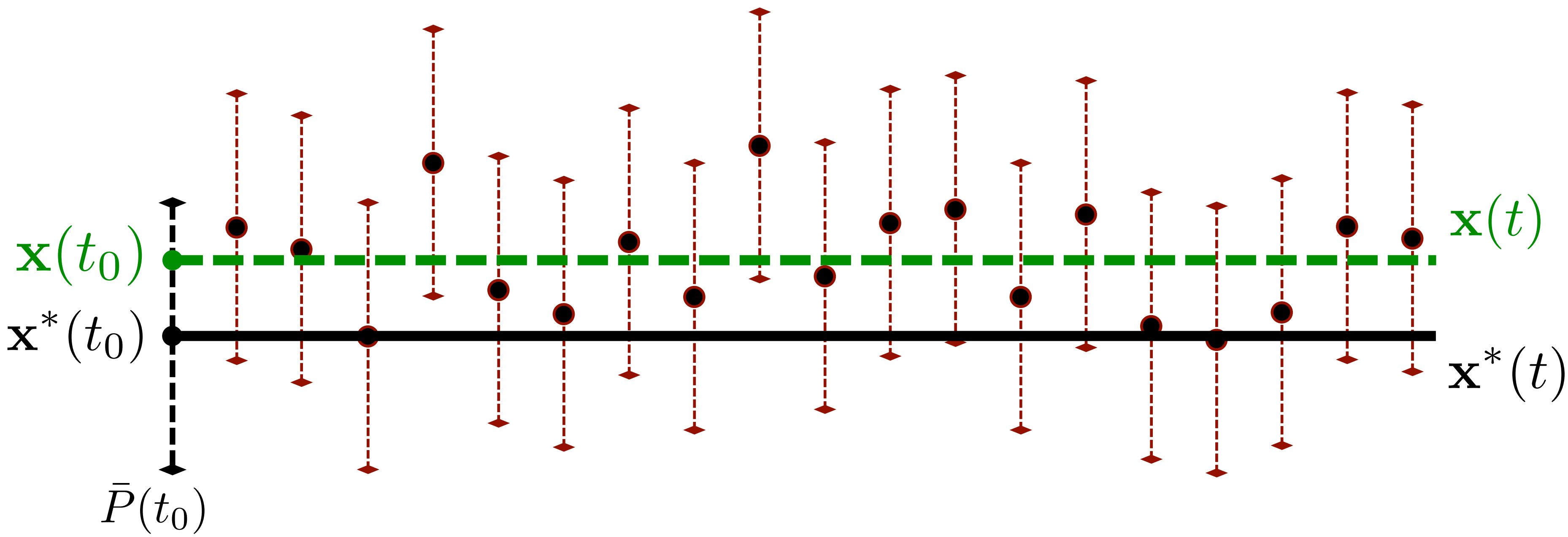


$\bullet\ \mathbf{y}(t) = \mathbf{Y}(t) - G(\mathbf{X}^*(t), t)$

$R(t)$

Stat OD Conceptualization

- Least Squares (Batch)



$$\hat{\mathbf{x}}(t_0) = \left(H^T R^{-1} H + \bar{P}^{-1}\right)^{-1} \left(H^T R^{-1} \mathbf{y} + \bar{P}^{-1} \bar{\mathbf{x}}\right)$$

Stat OD Conceptualization

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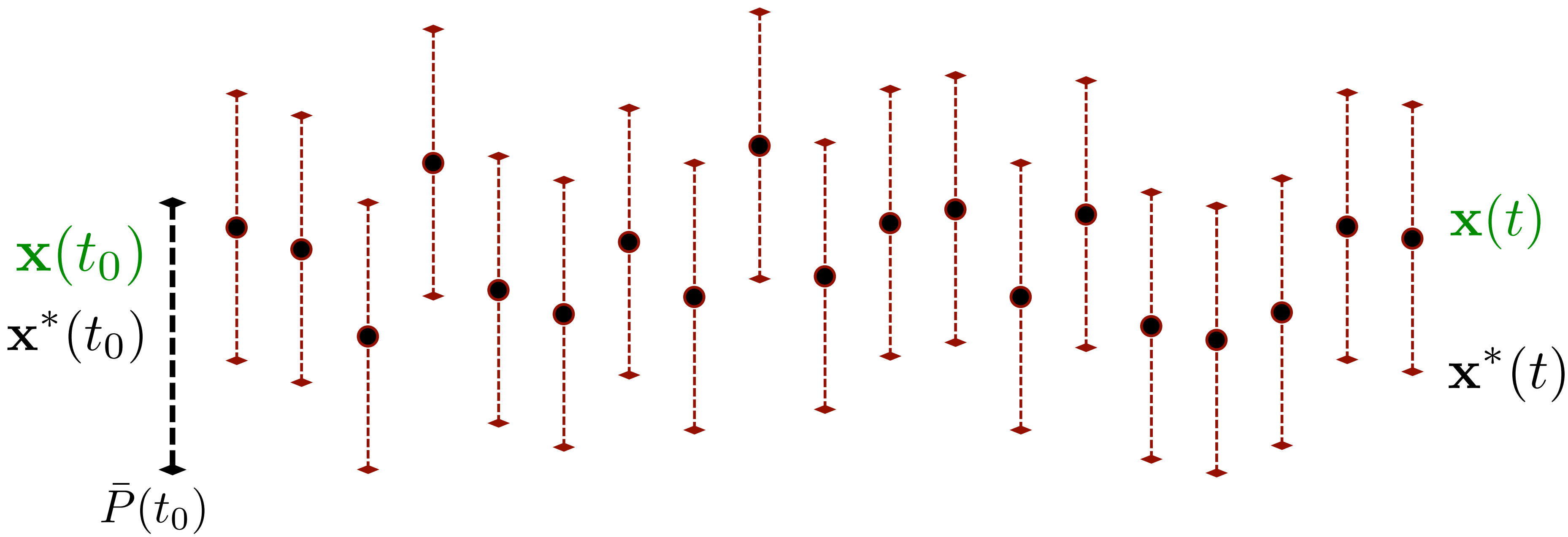

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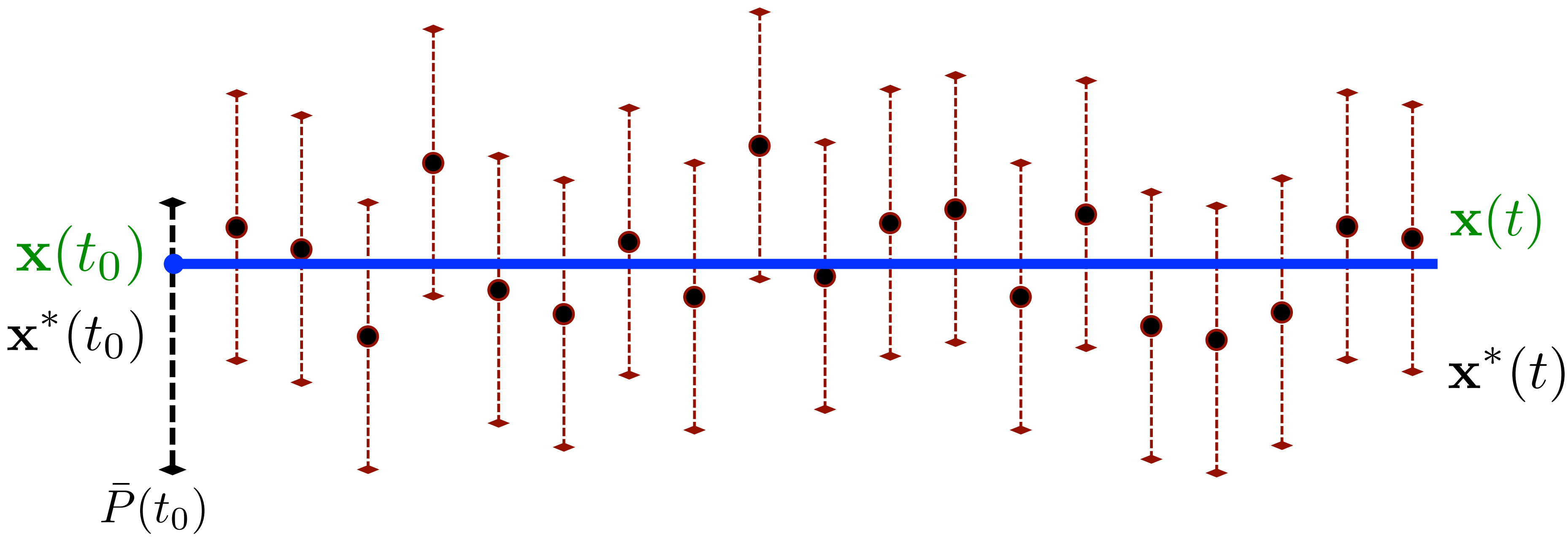

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Stat OD Conceptualization

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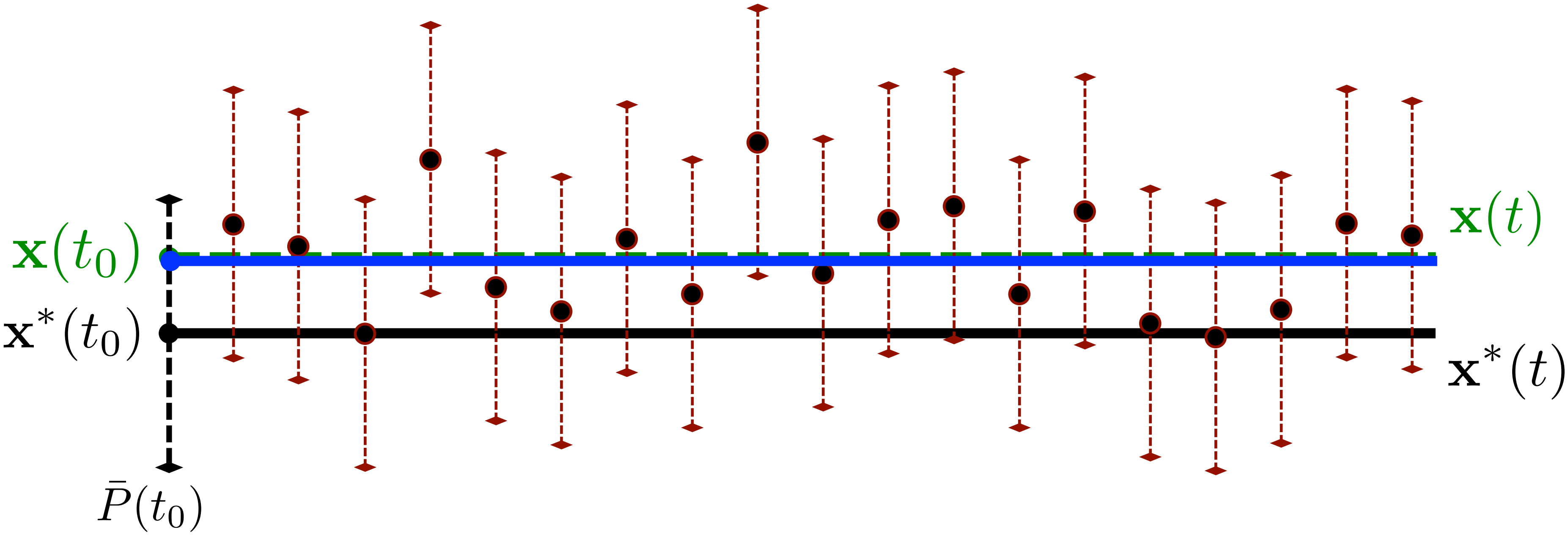

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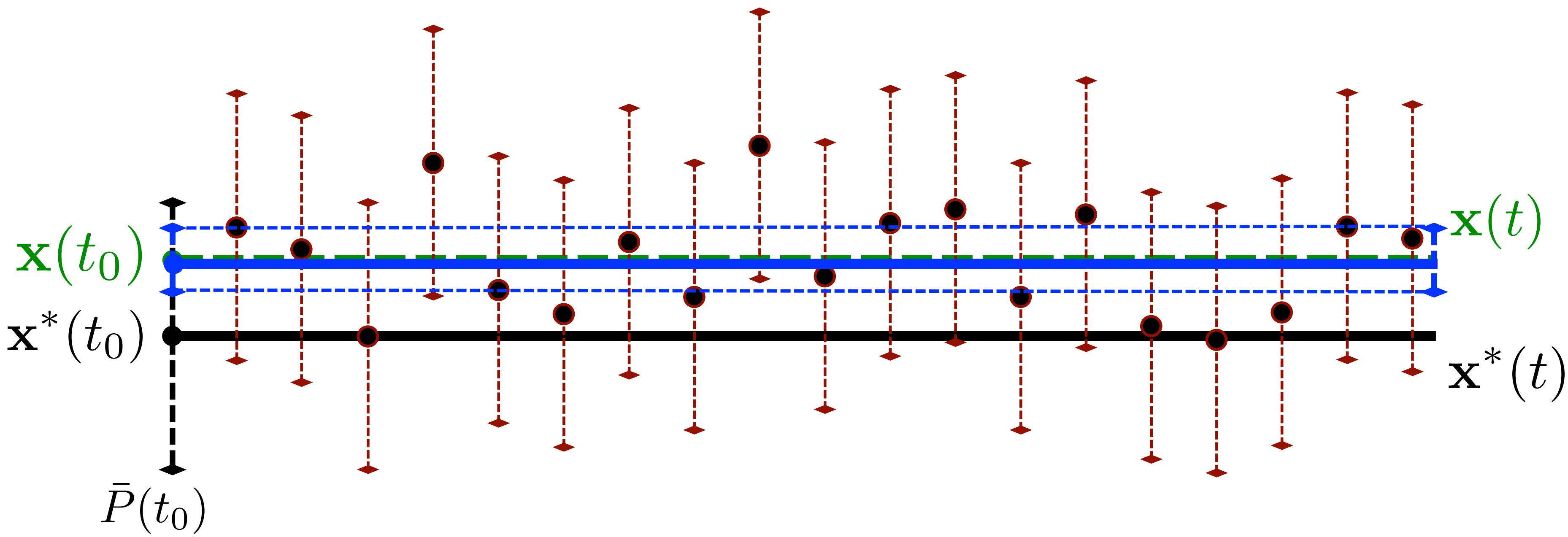

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$P(t)$

Stat OD Conceptualization

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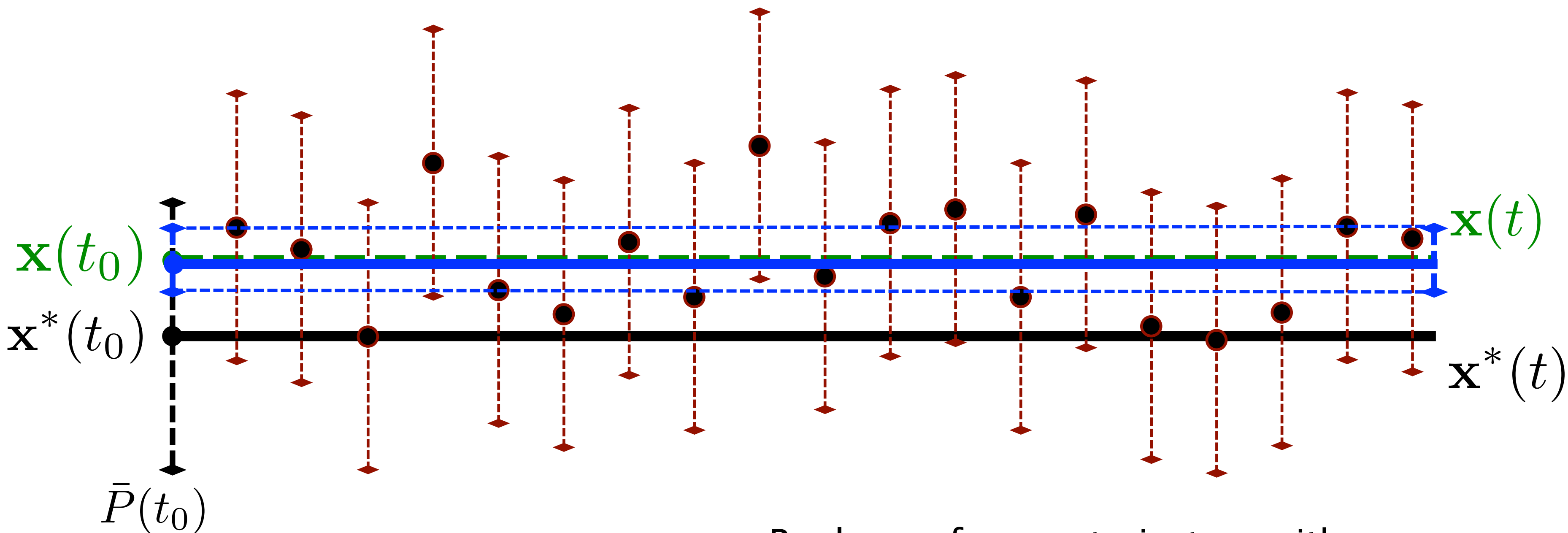

Iterate a few times.

- Replace reference trajectory with best-estimate
- Update *a priori* state
- Generate new computed observations

- Conceptualization of the Conventional Kalman Filter (Sequential Filter)

Stat OD Conceptualization

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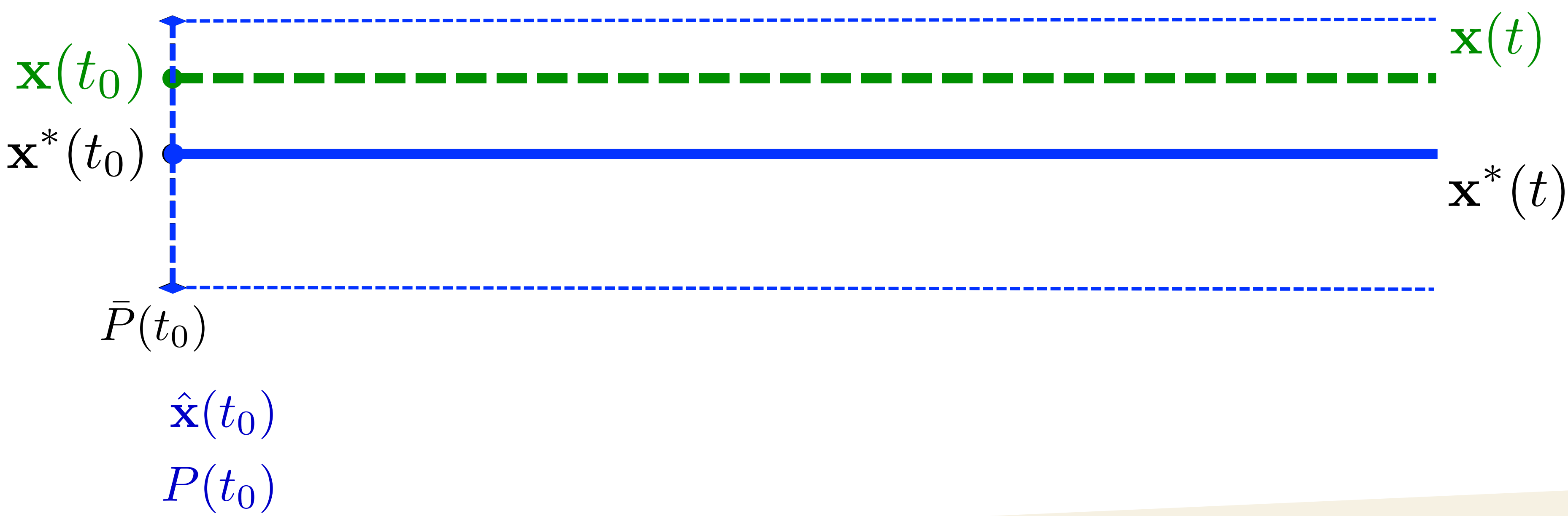
- Conventional Kalman



Stat OD Conceptualization

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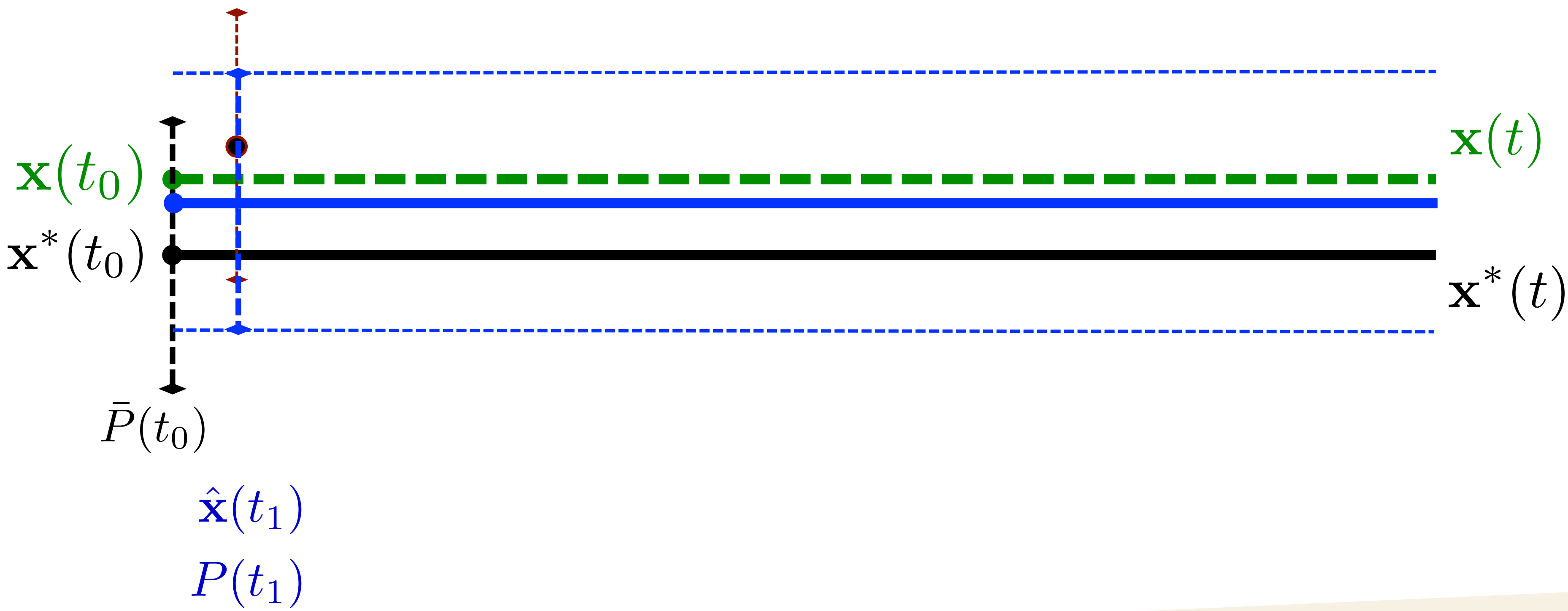

Stat OD Conceptualization

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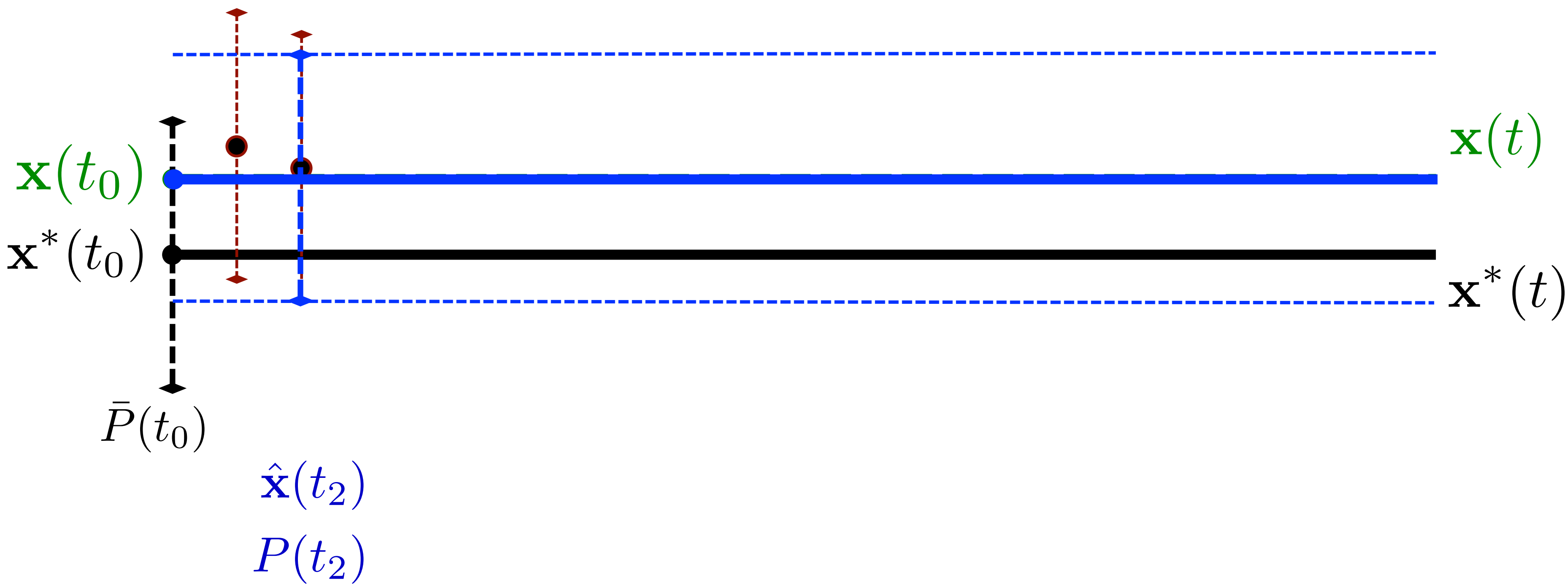

Stat OD Conceptualization

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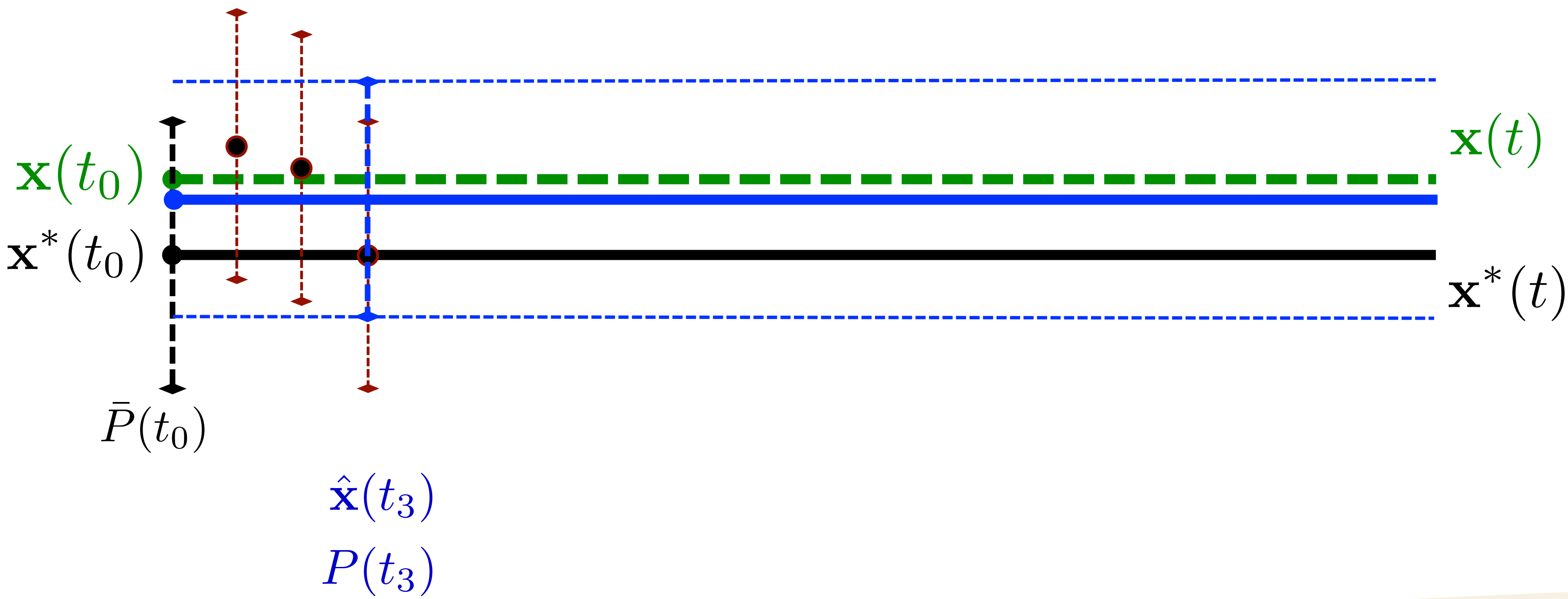

Stat OD Conceptualization

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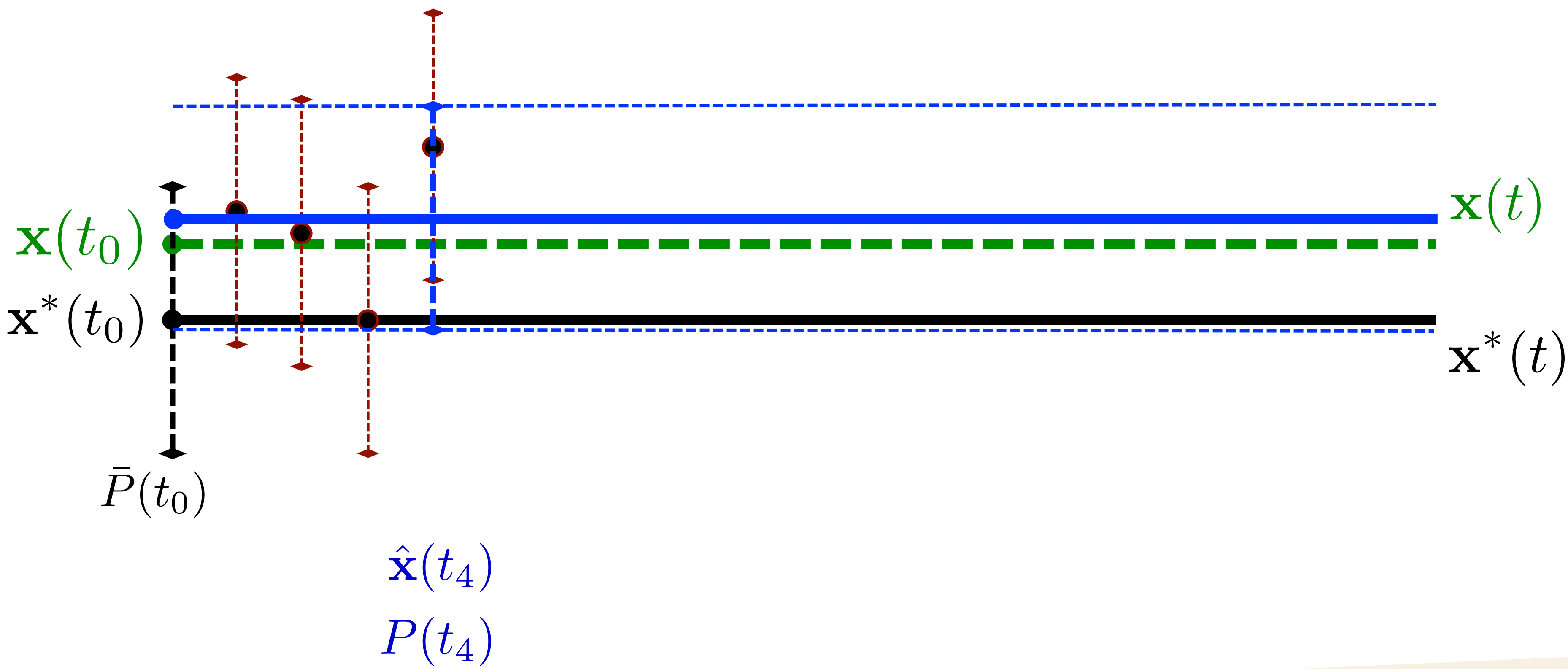

Stat OD Conceptualization

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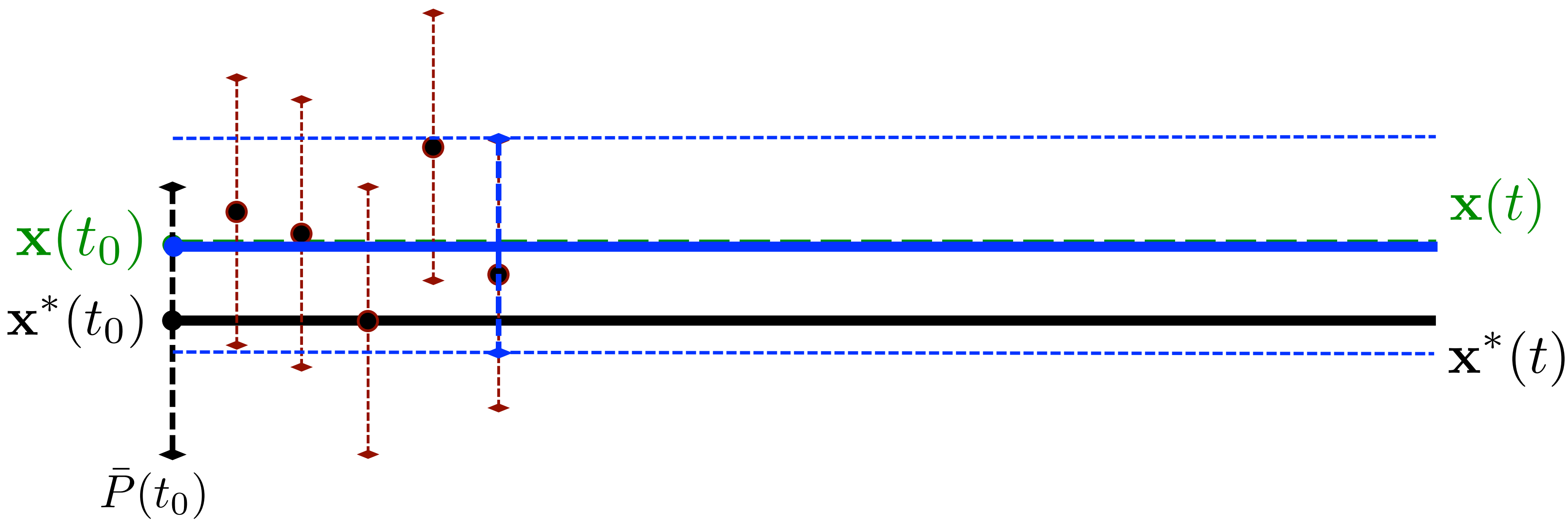

Stat OD Conceptualization

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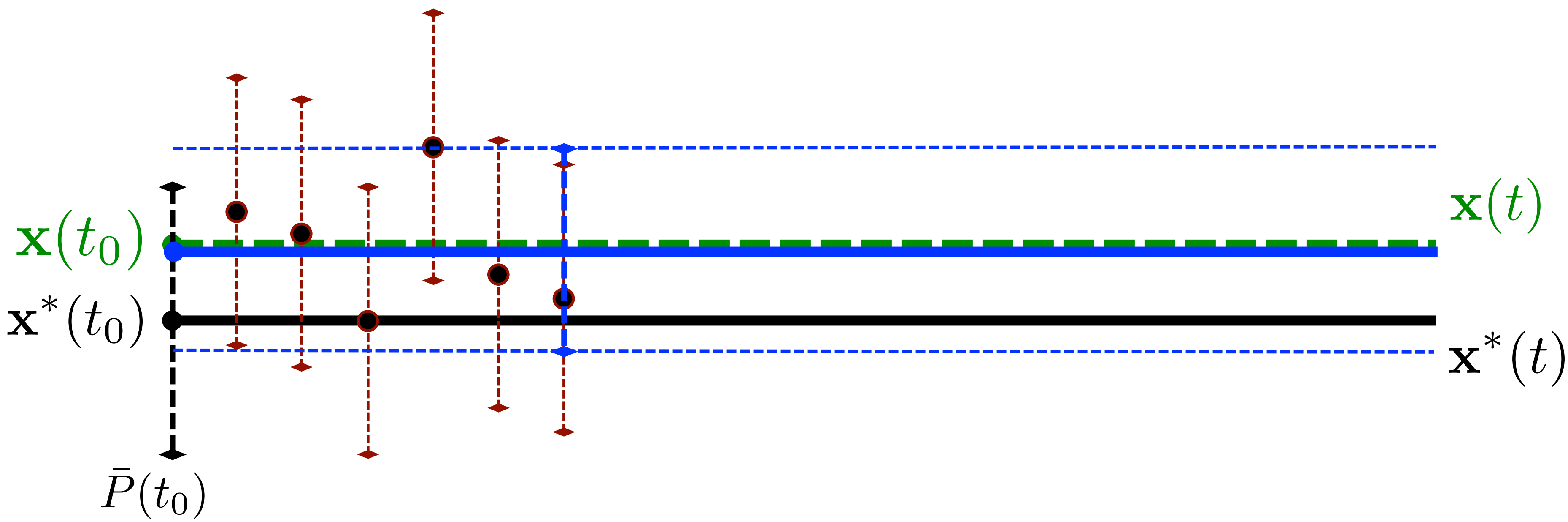

Stat OD Conceptualization

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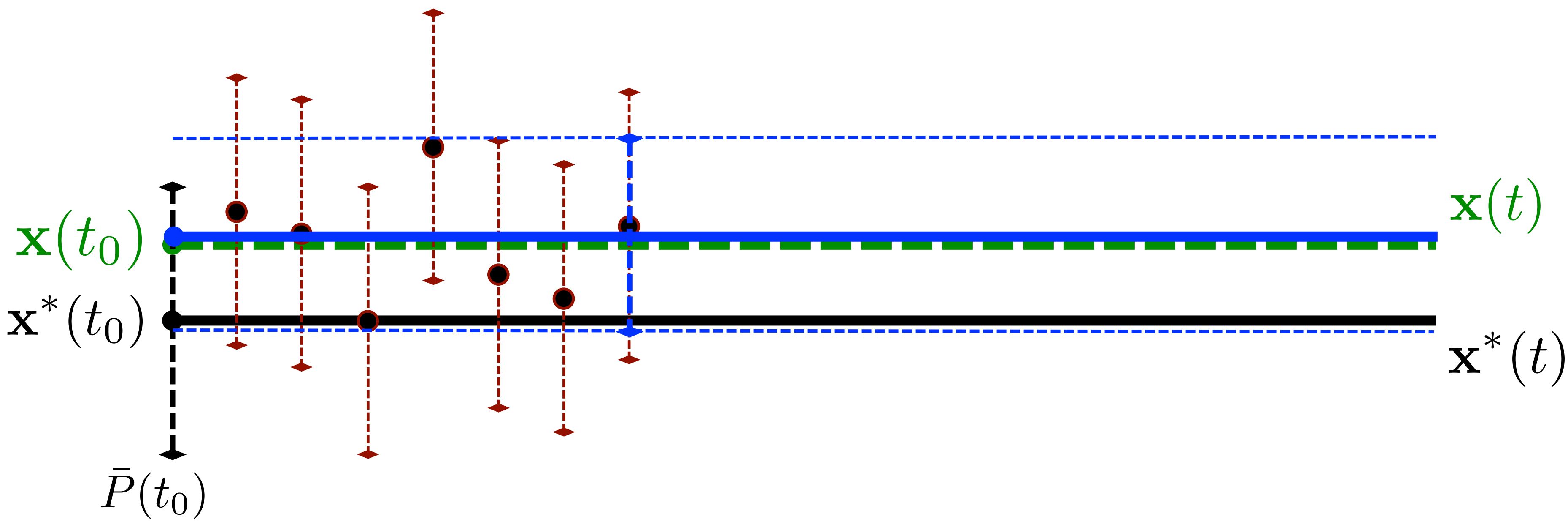

Stat OD Conceptualization

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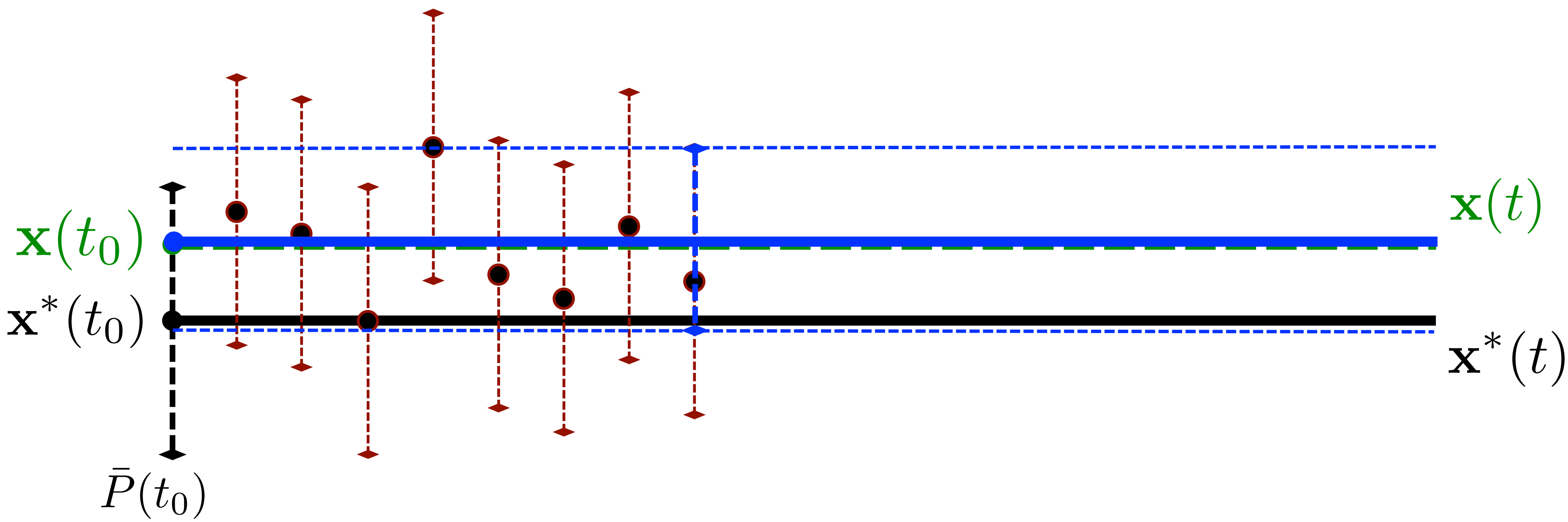

Stat OD Conceptualization

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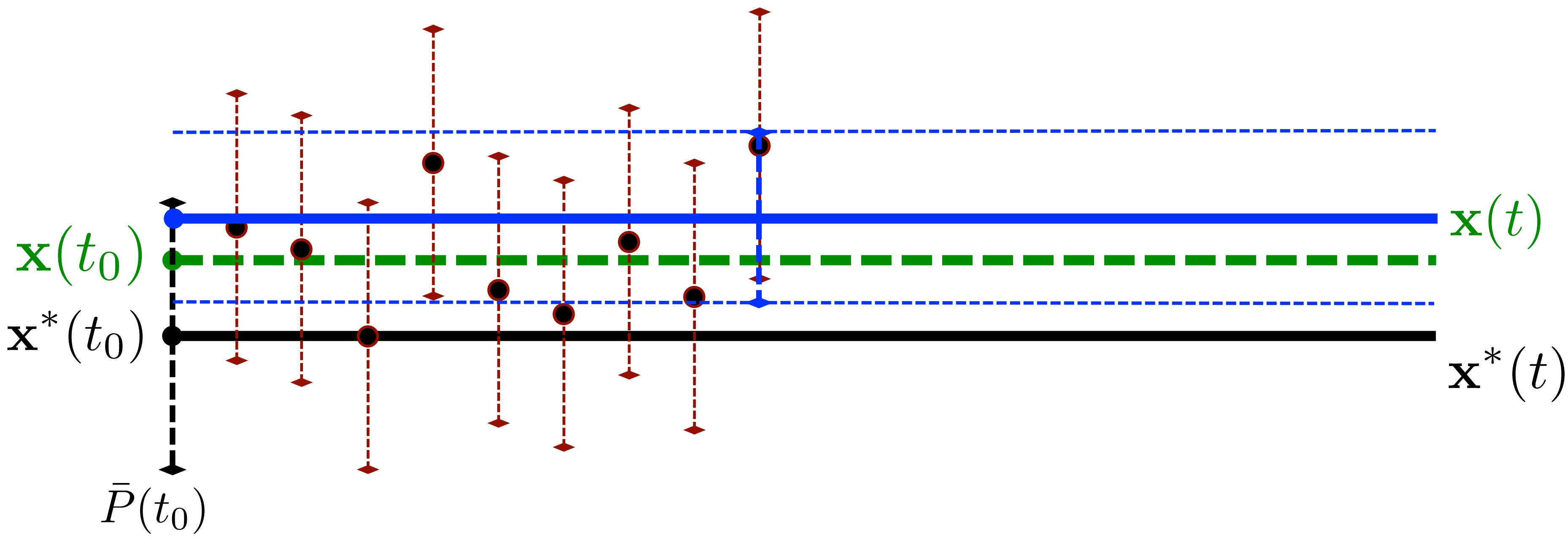

Stat OD Conceptualization

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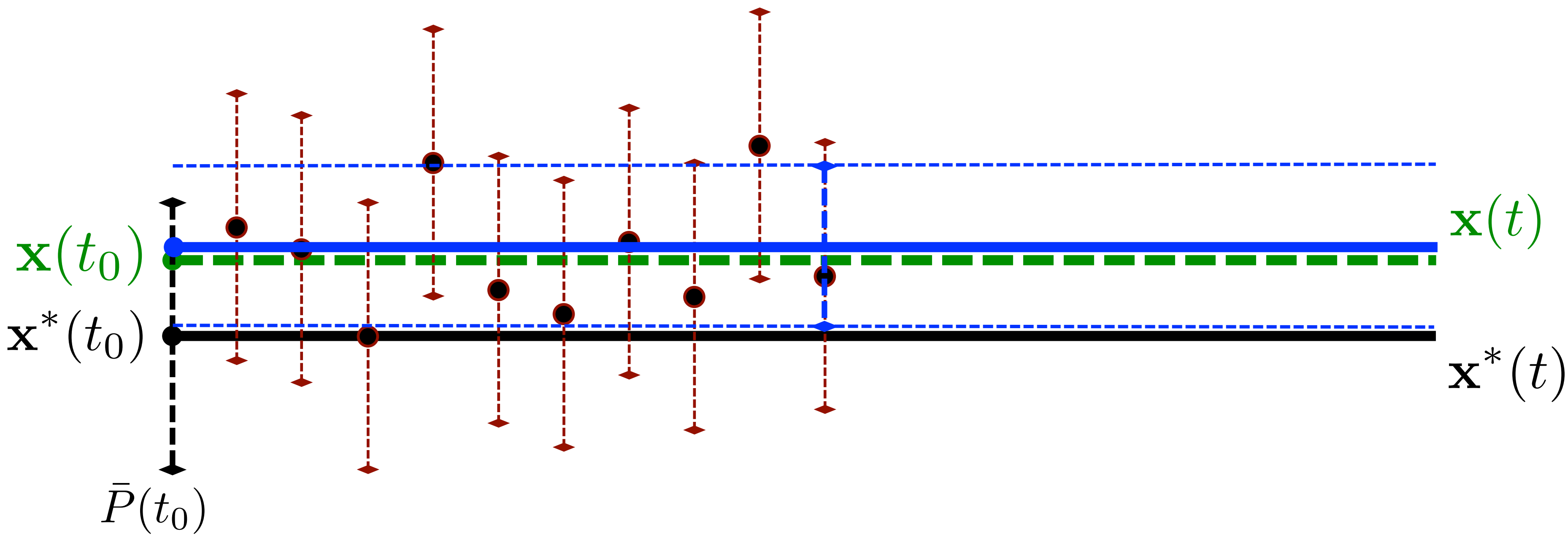

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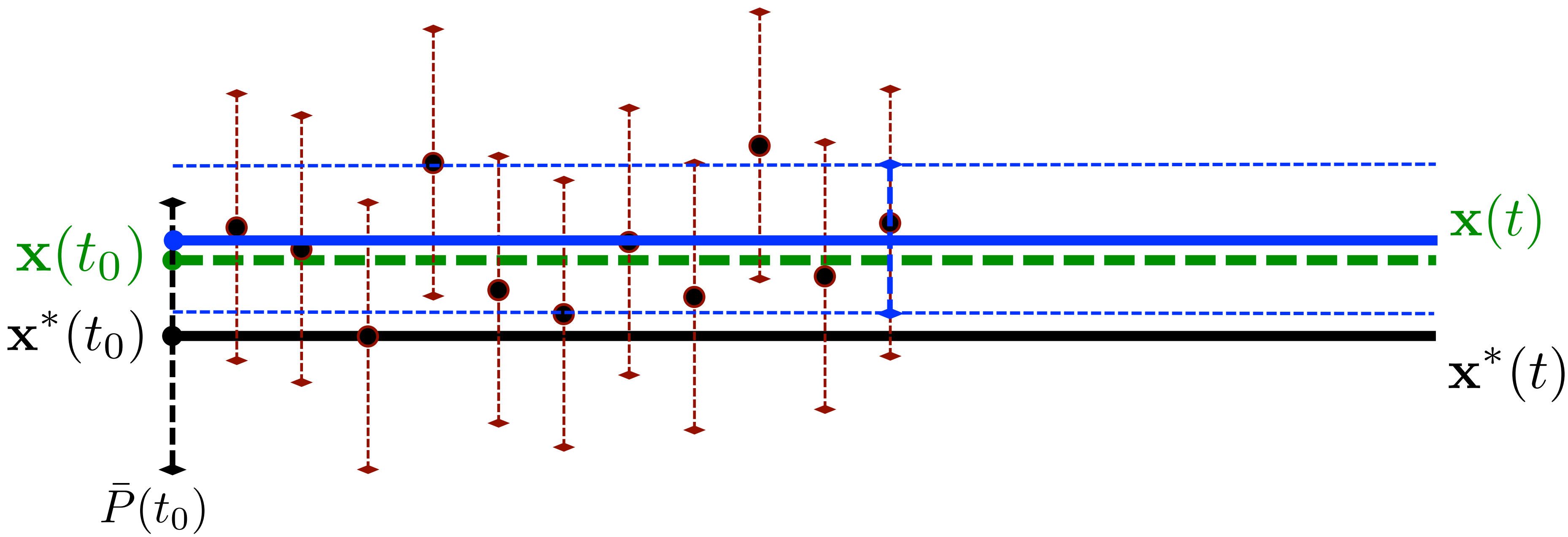

Stat OD Conceptualization

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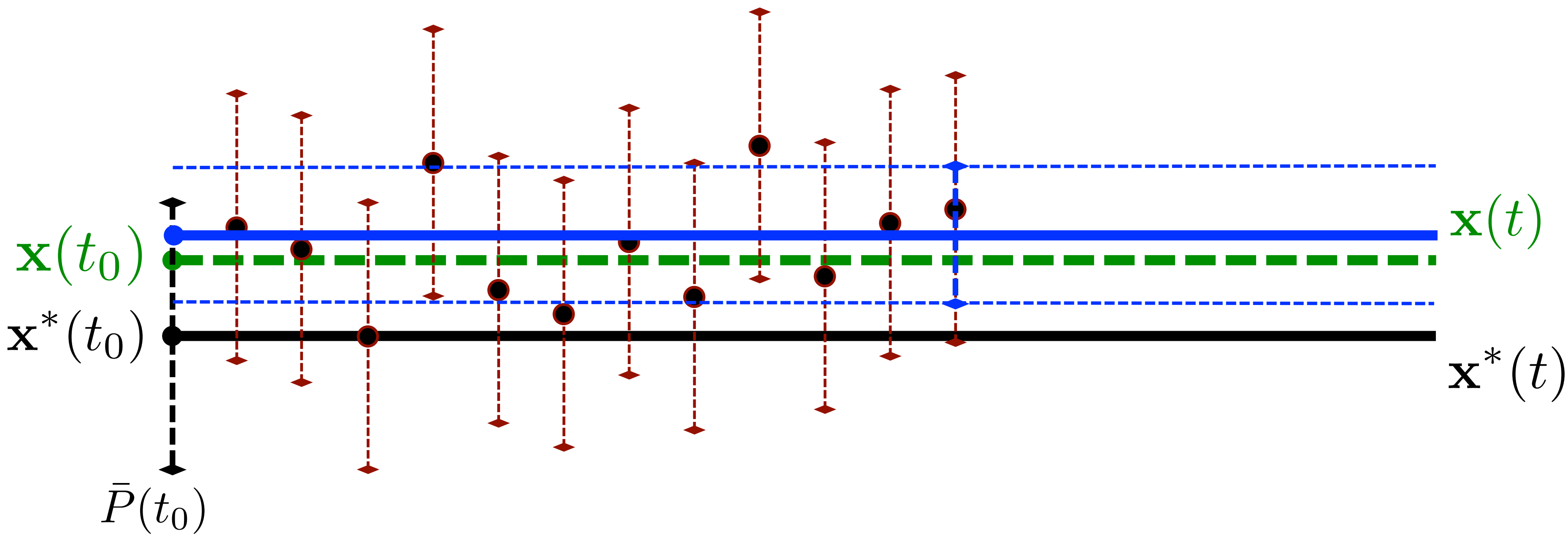

Stat OD Conceptualization

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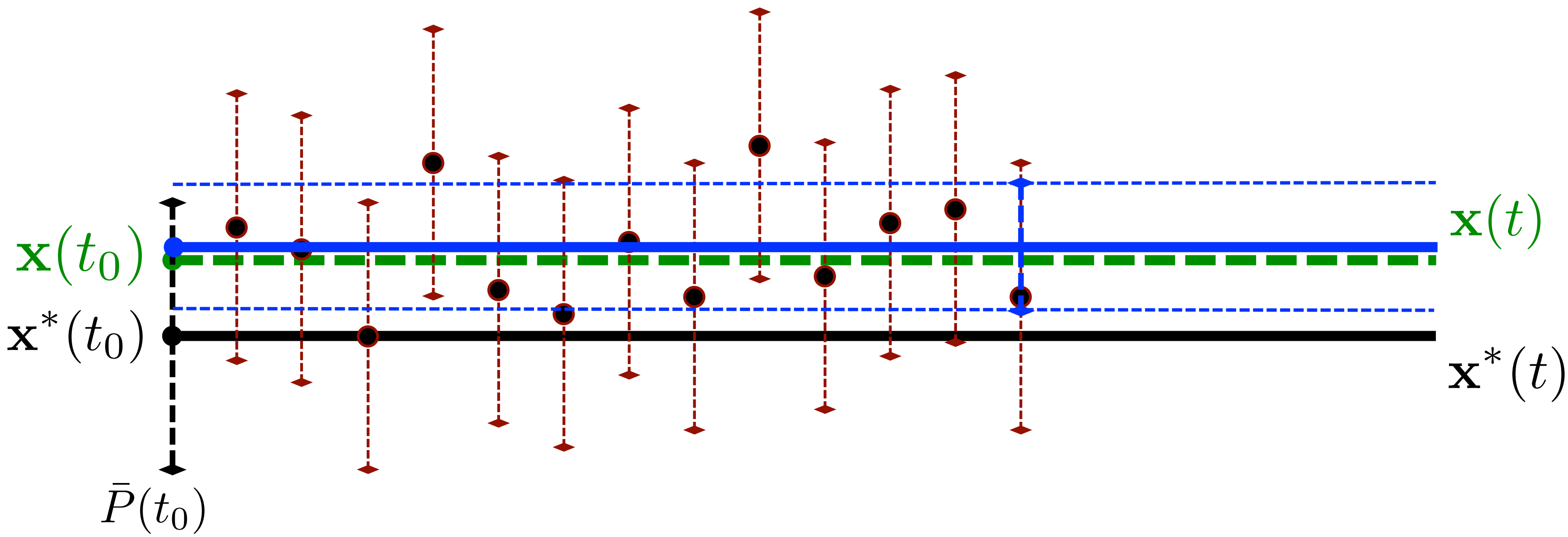

Stat OD Conceptualization

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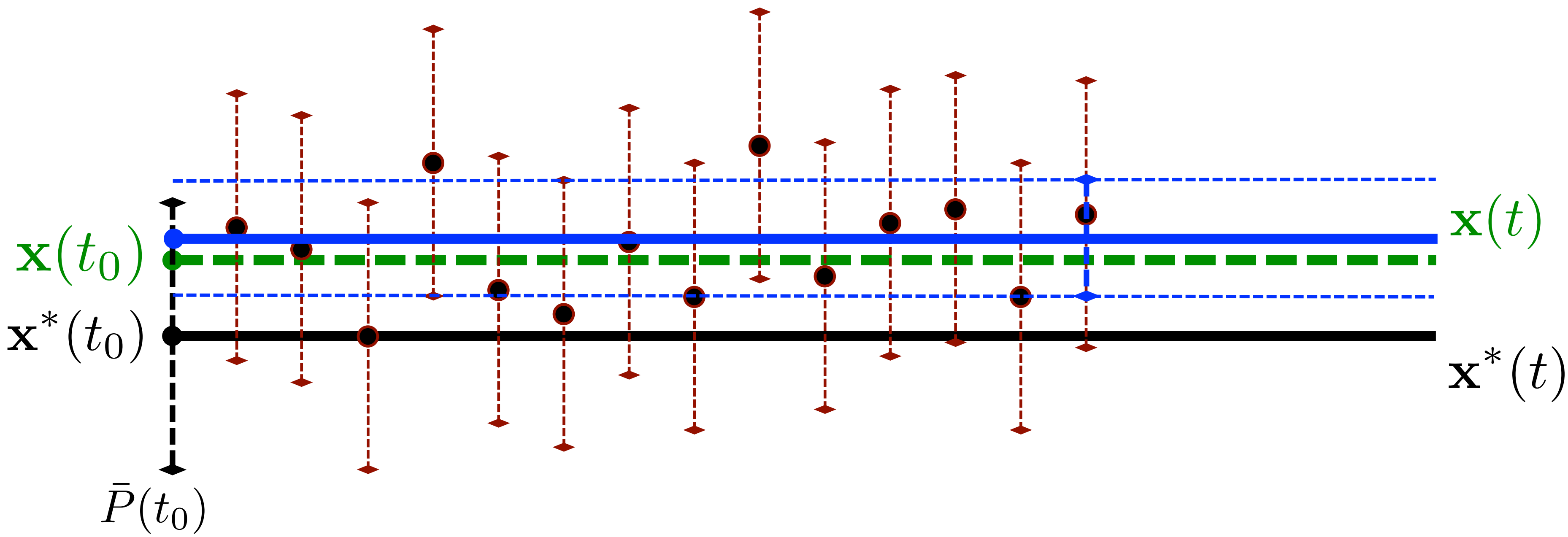

Stat OD Conceptualization

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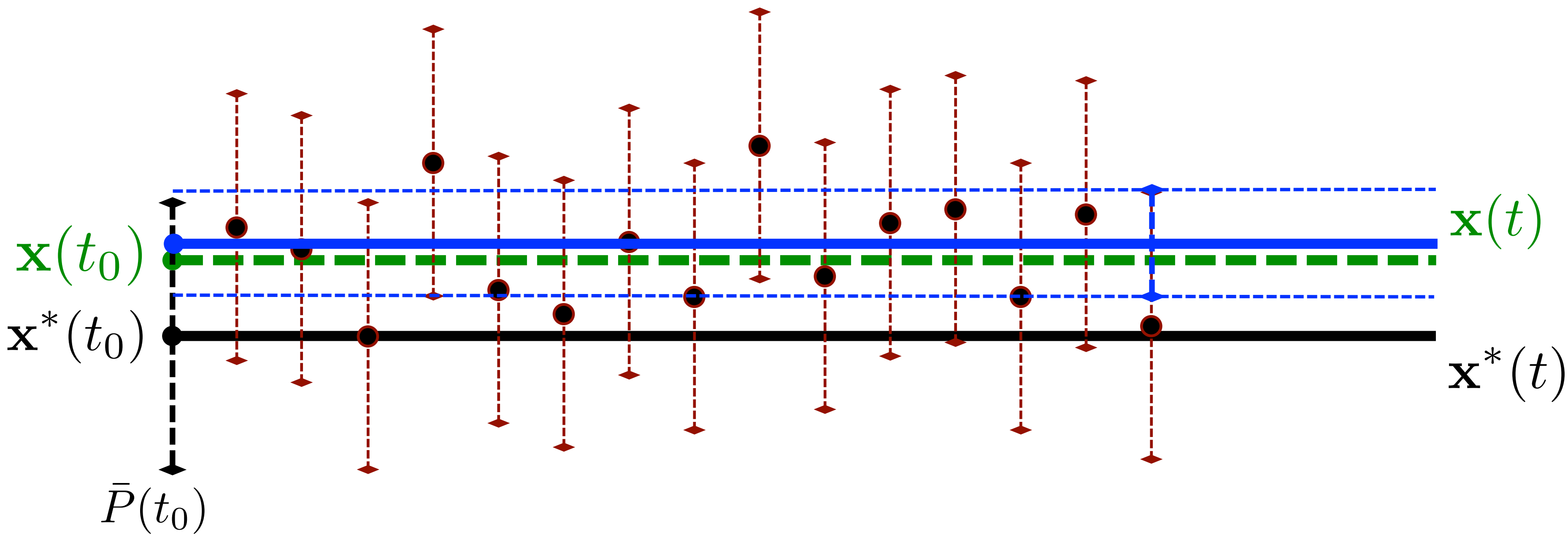

Stat OD Conceptualization

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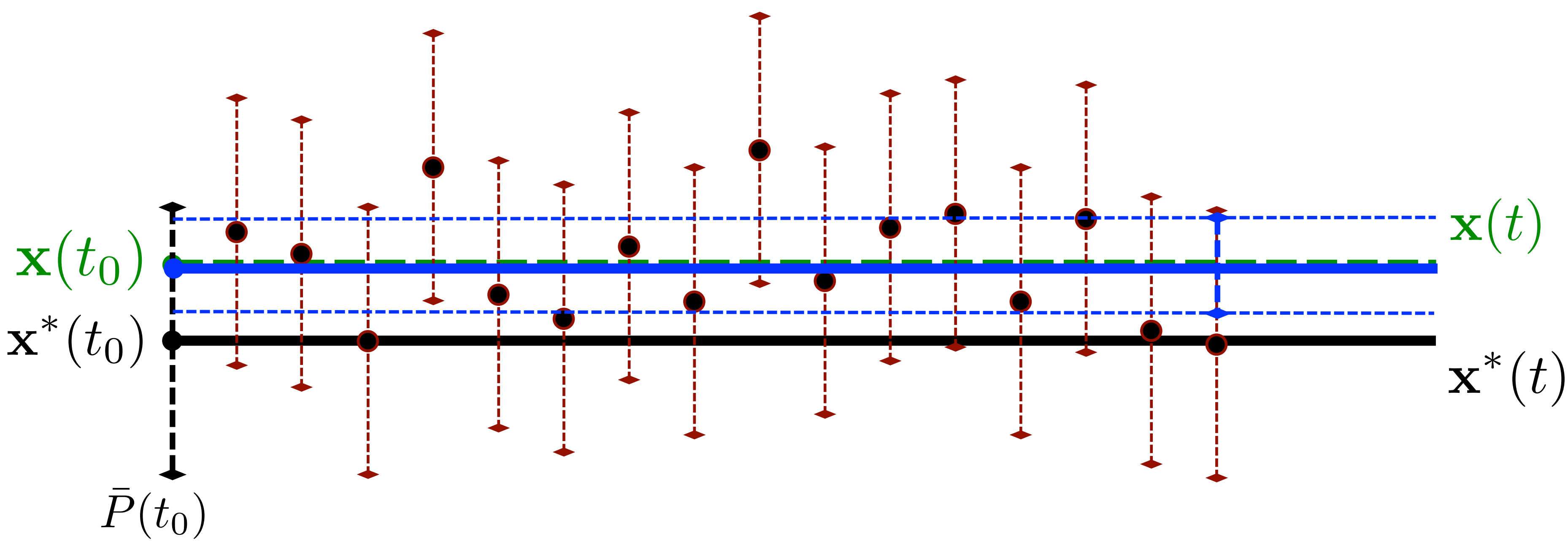

Stat OD Conceptualization

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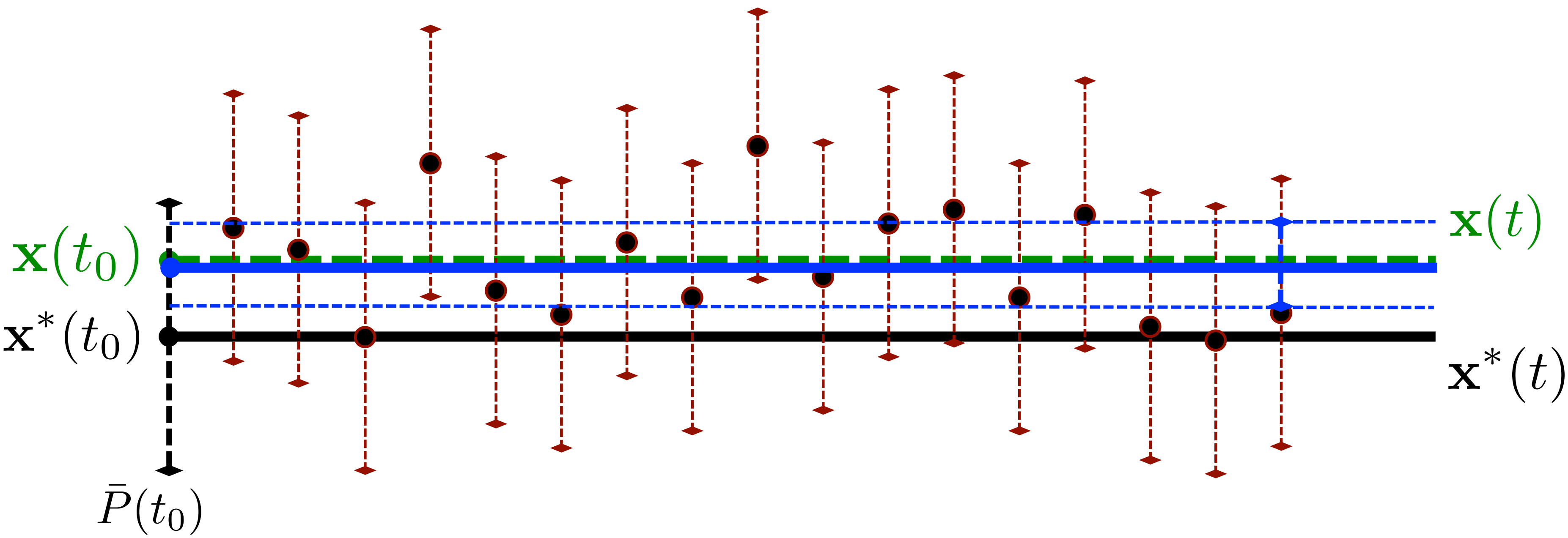

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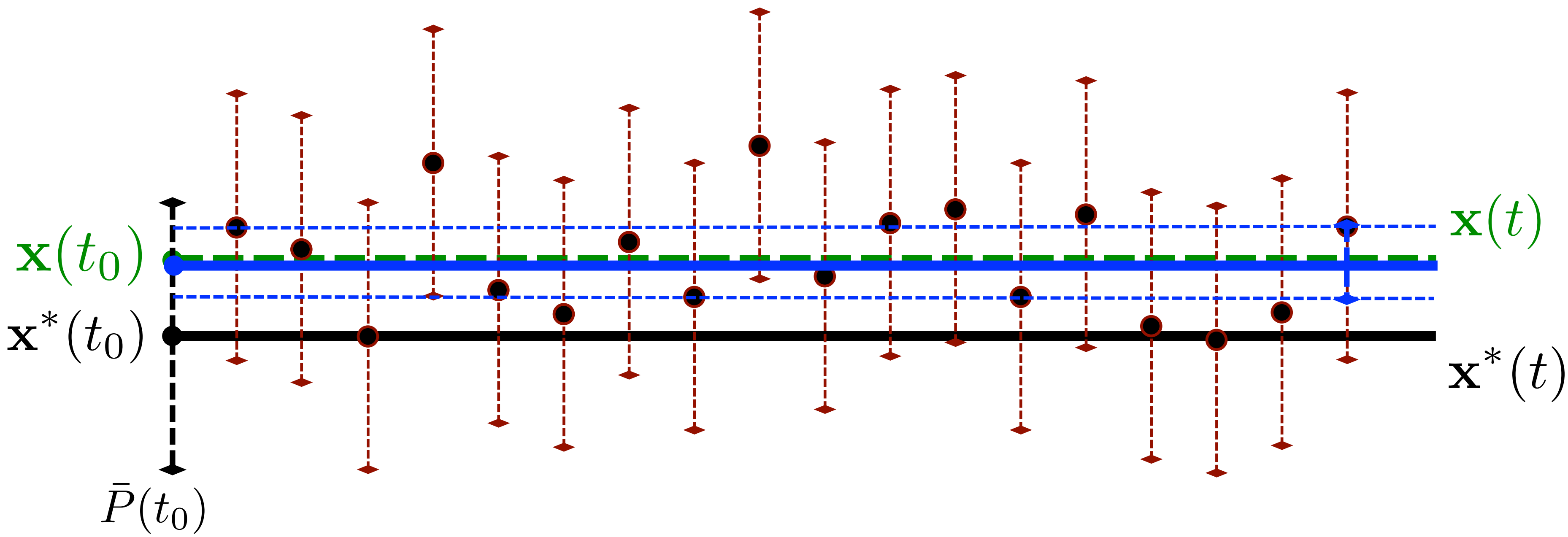

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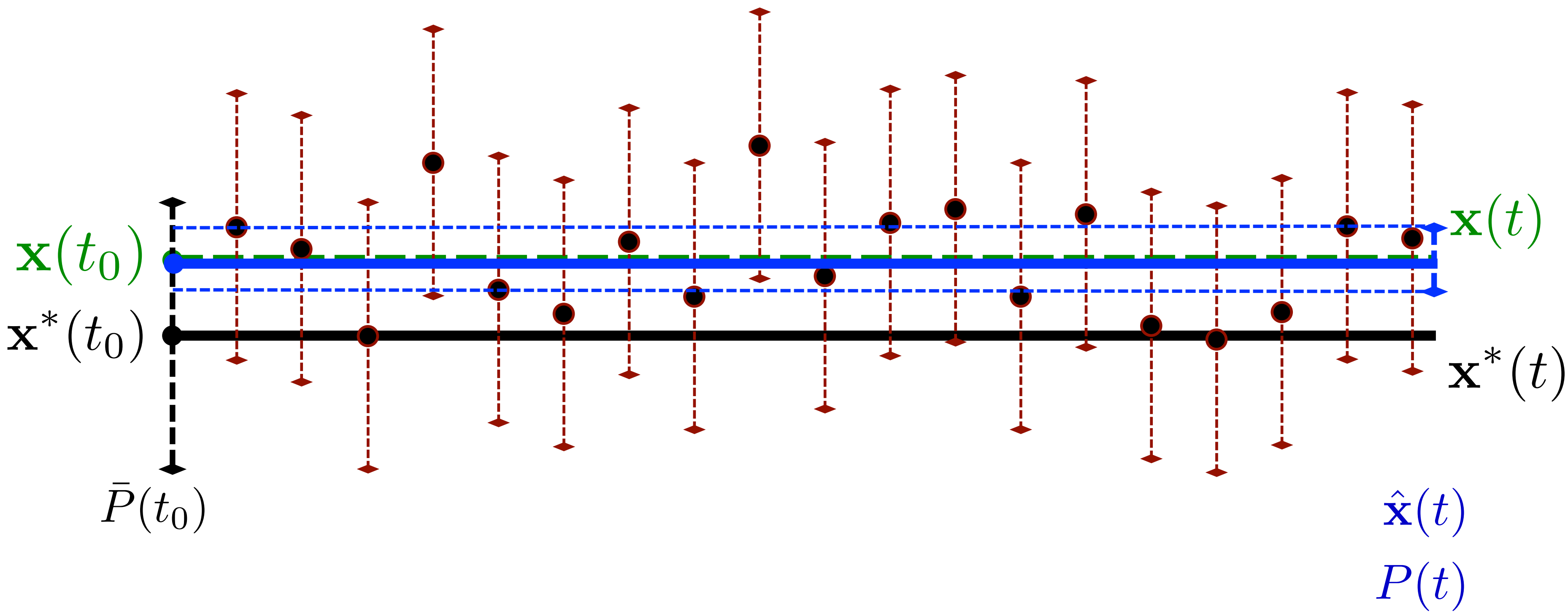

Stat OD Conceptualization

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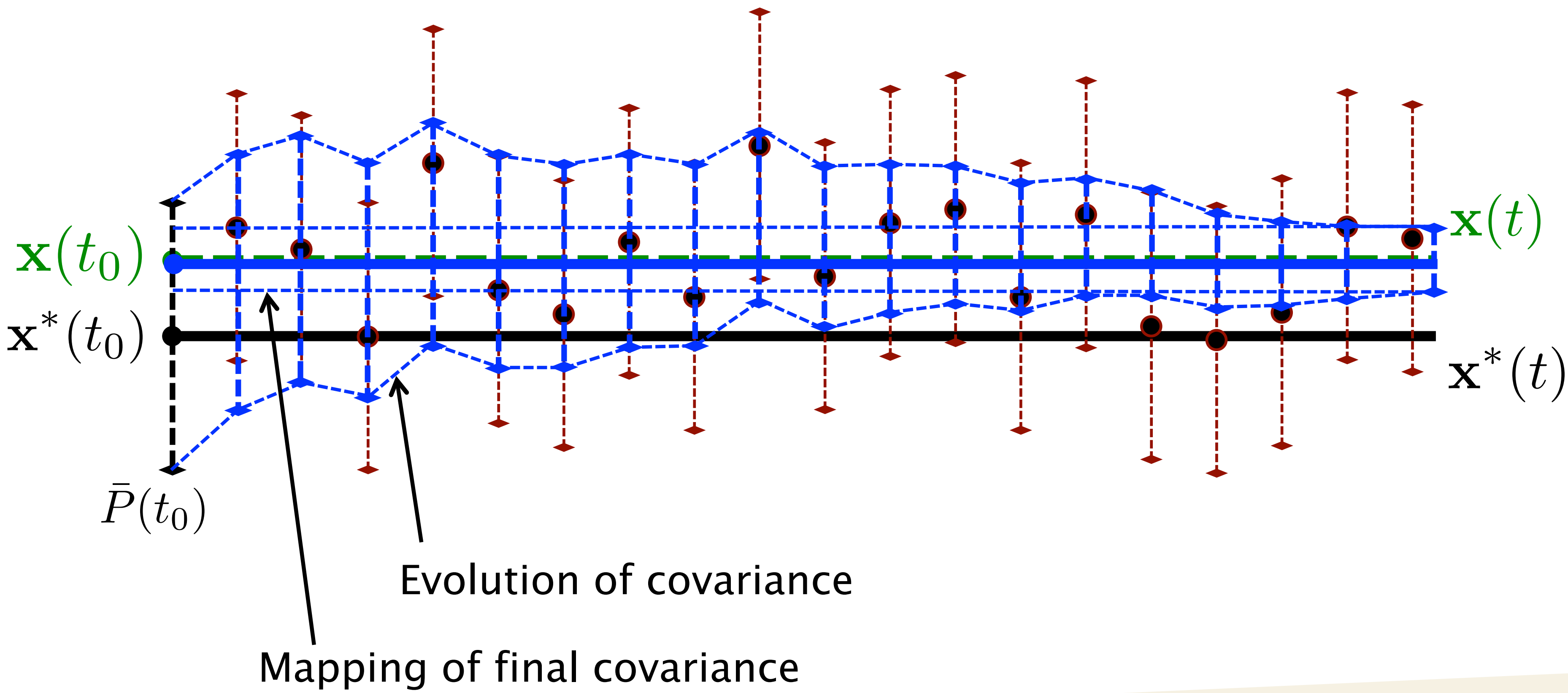
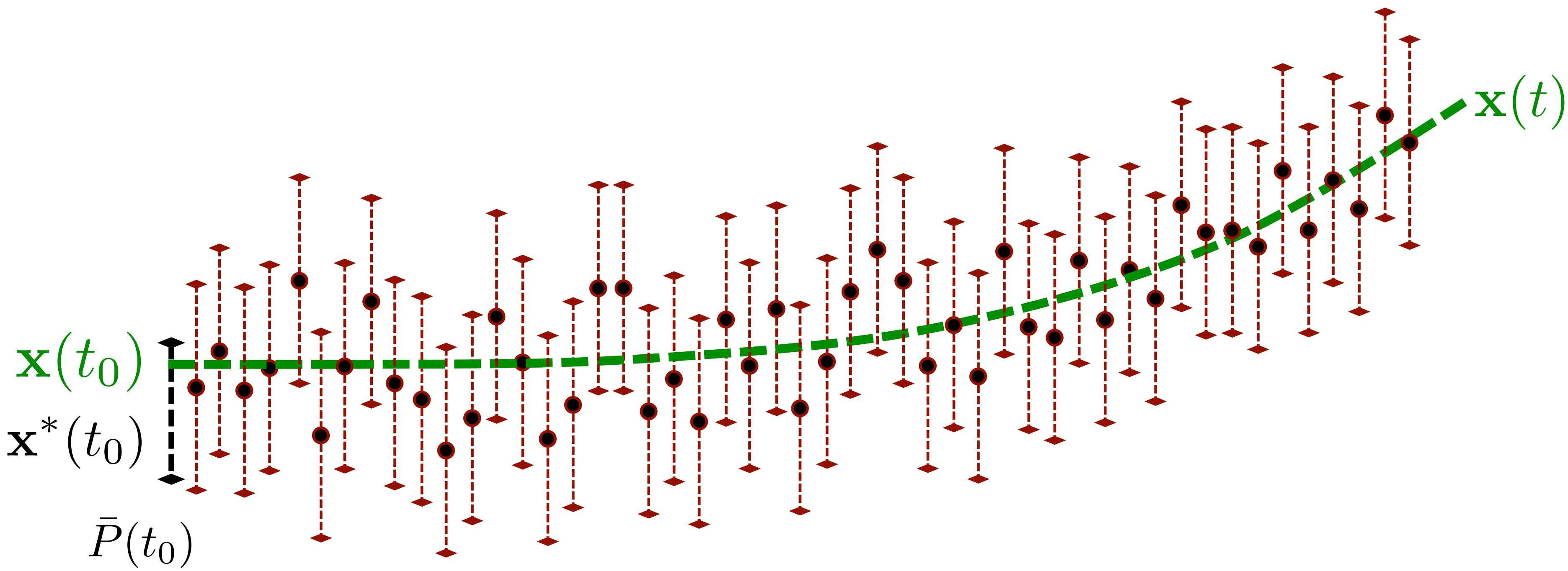
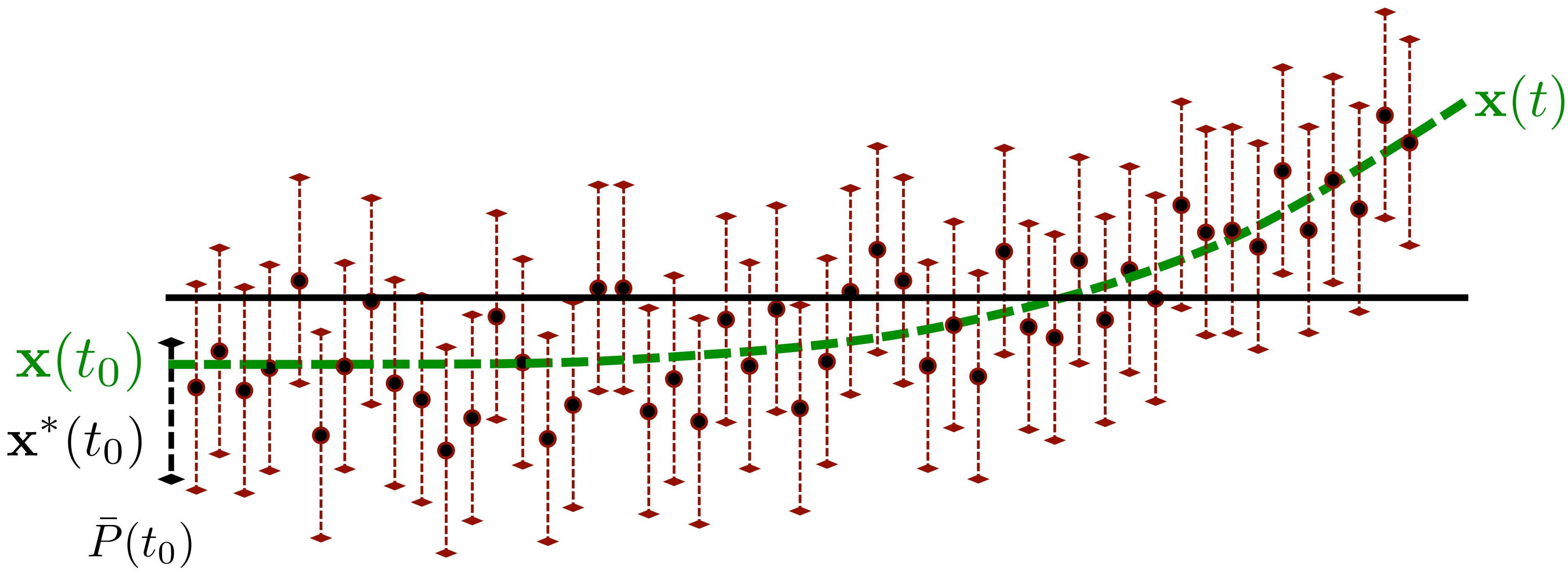


Stat OD Conceptualization



CKF fits a line to this data. (CONCEPTUAL)

Stat OD Conceptualization



AFTER all observations have been processed.

Imagine what it would look like DURING the process.

Sequential

- Filter Saturation
  - Causes?
  - Fixes?

- Any numerical issues with the Kalman filter?

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$

$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]$$

$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k .$$

Sequential

- Any numerical issues with the Kalman filter?

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$
$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]$$
$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k.$$

- Joseph:

$$P_k = \left(I - K_k \tilde{H}_k^T\right) \bar{P}_k \left(I - K_k \tilde{H}_k^T\right)^T + K_k R_k K_k^T$$

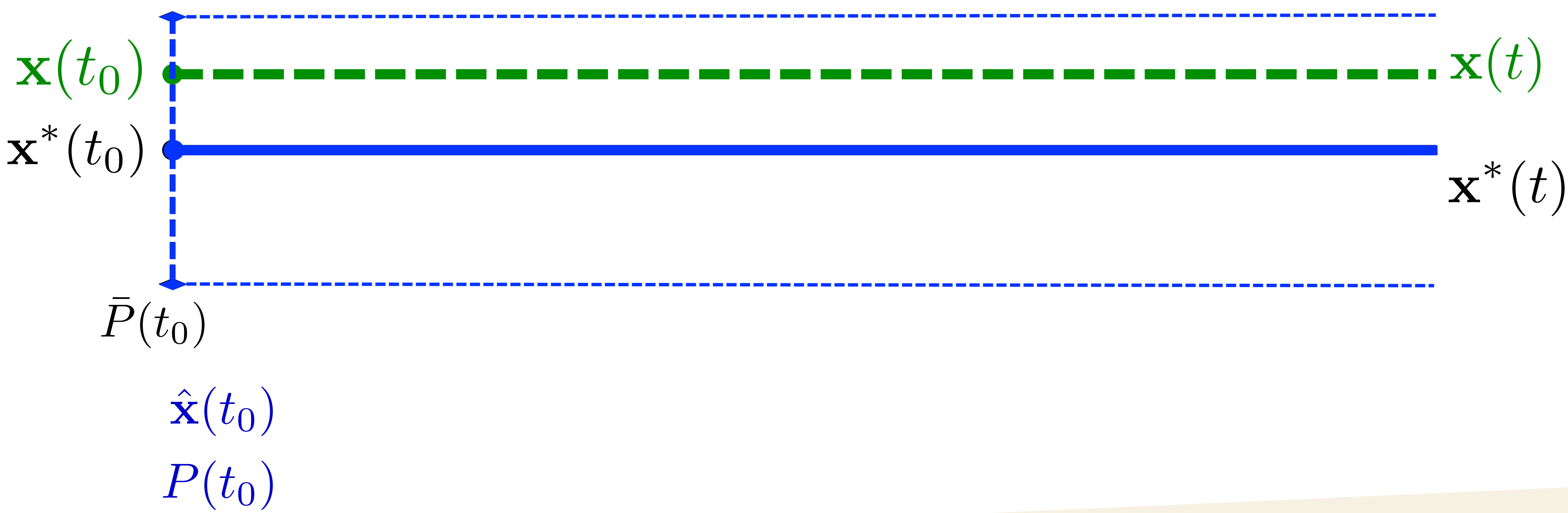
- Square Root $\quad P_k = W_k W_k^T$
  - Potter

- Conceptualization of the Extended Kalman Filter (EKF)

- Major change: the reference trajectory is updated by the best estimate after every measurement.

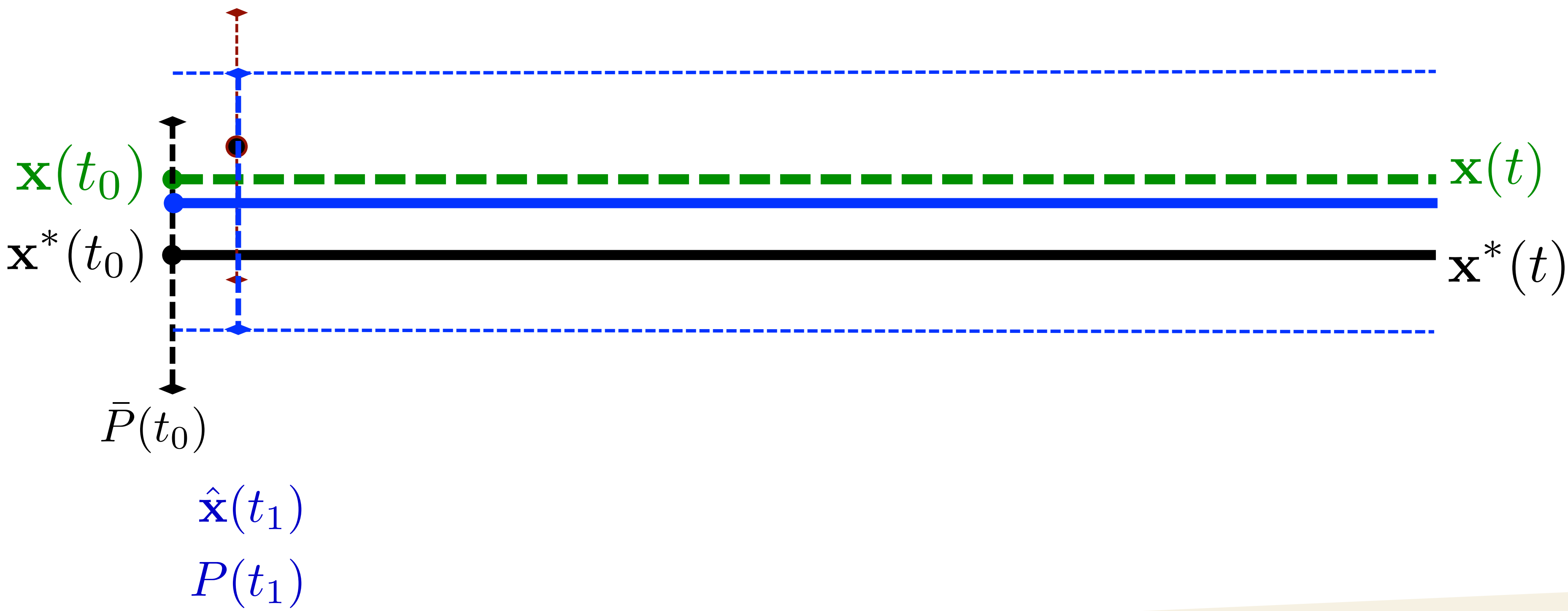
Stat OD Conceptualization

- EKF



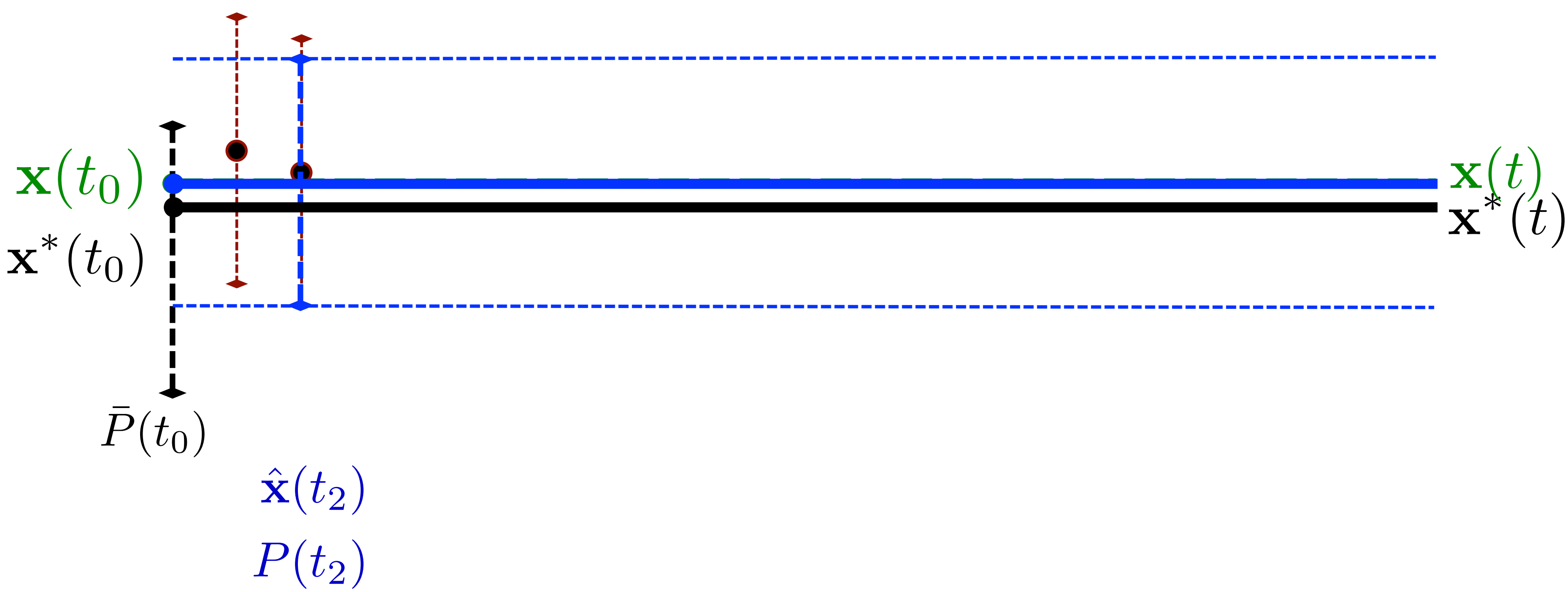
Stat OD Conceptualization

- EKF



Stat OD Conceptualization

- EKF



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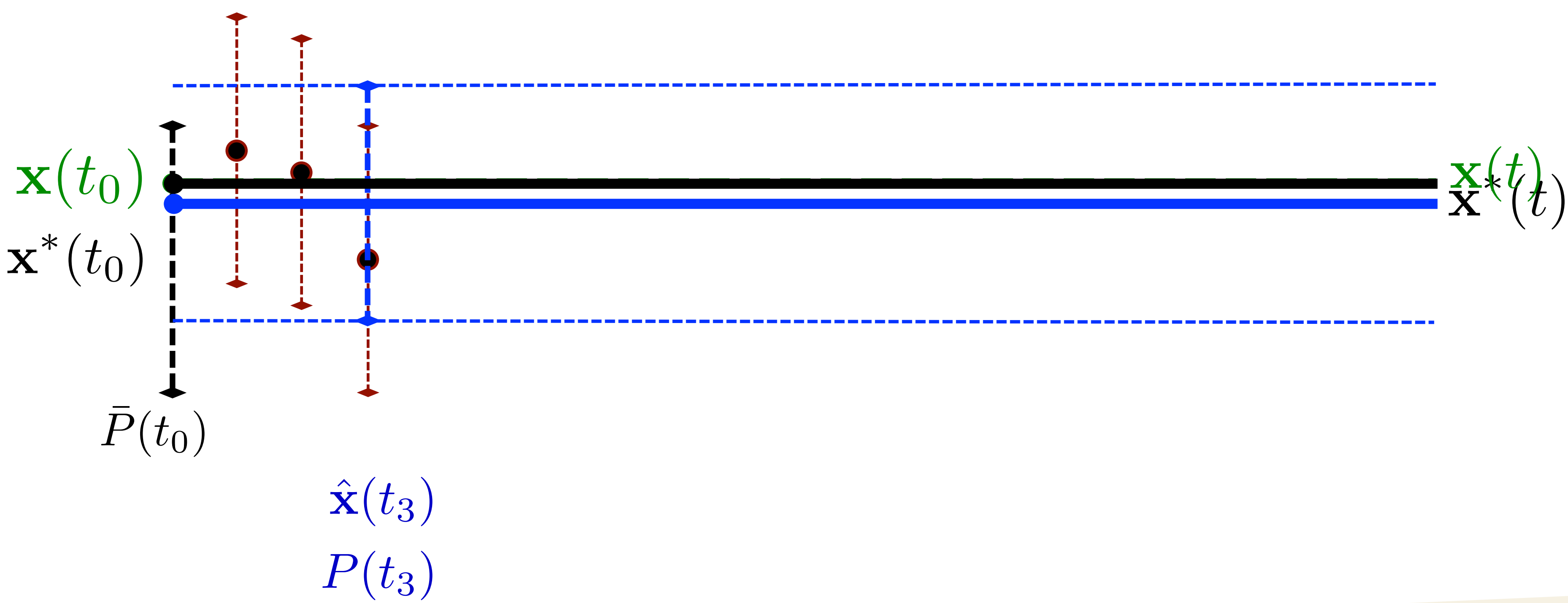
- EKF



Stat OD Conceptualization

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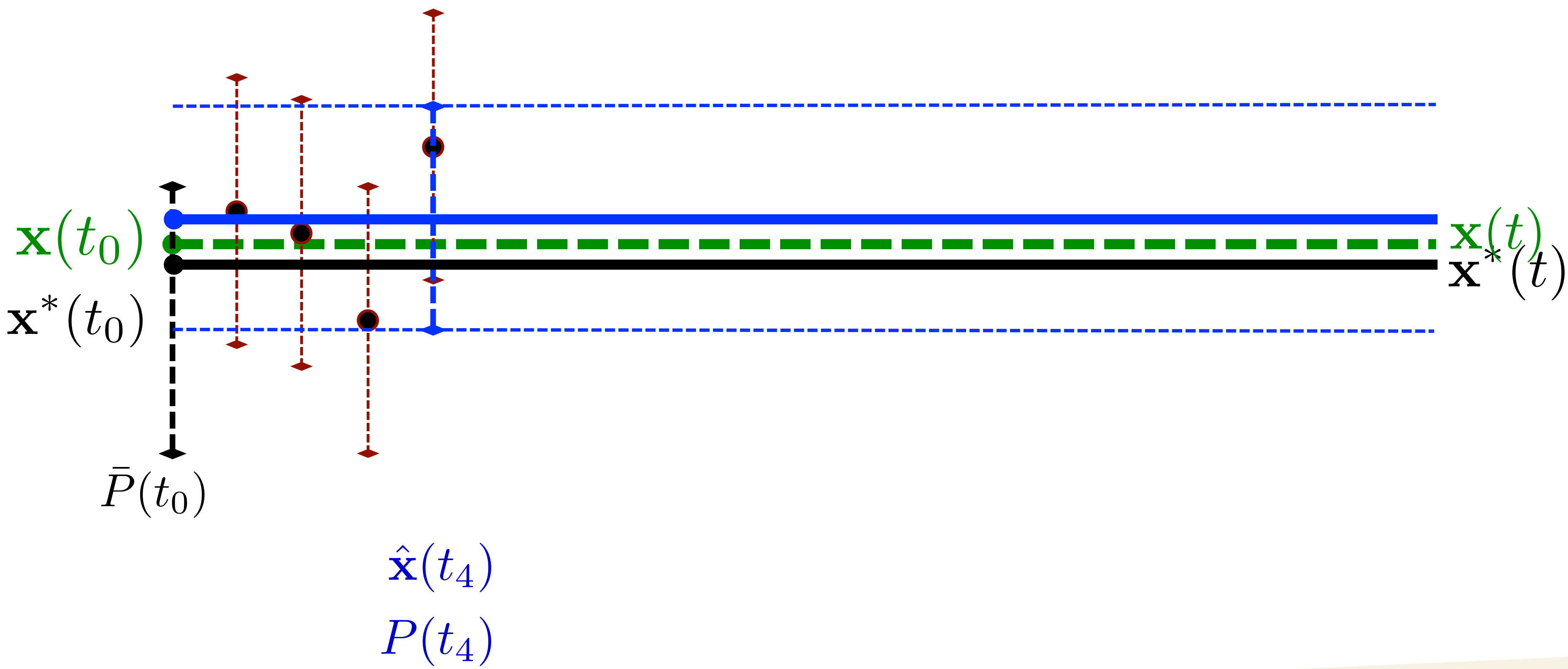
- EKF



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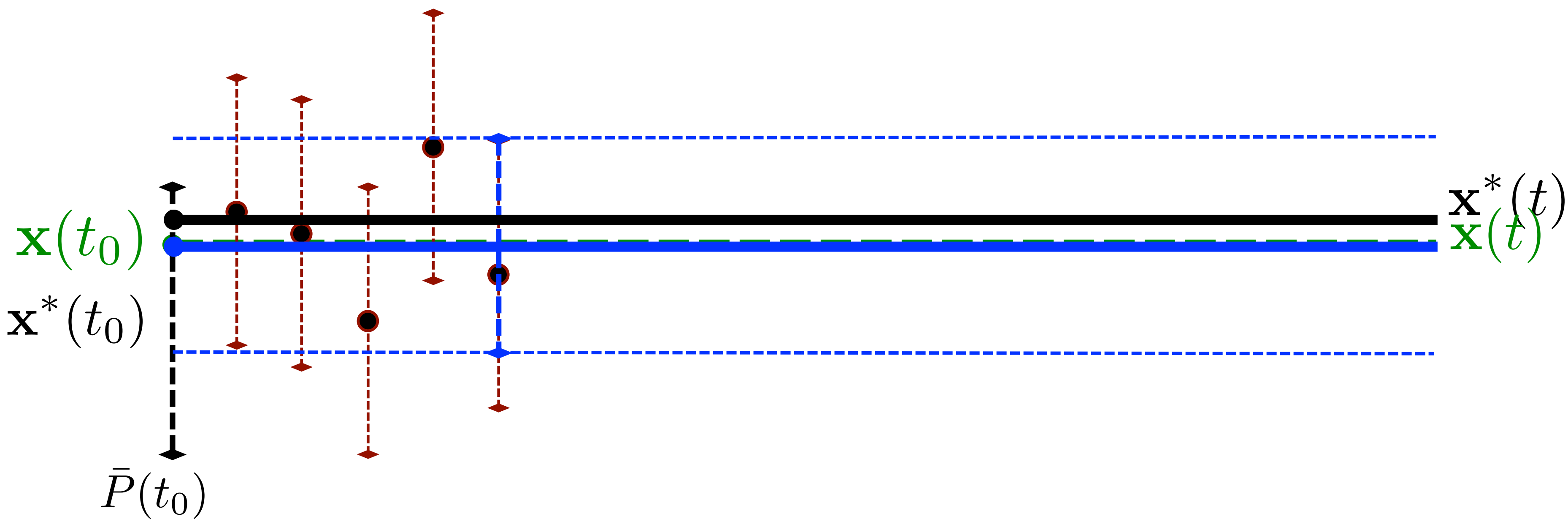
- EKF



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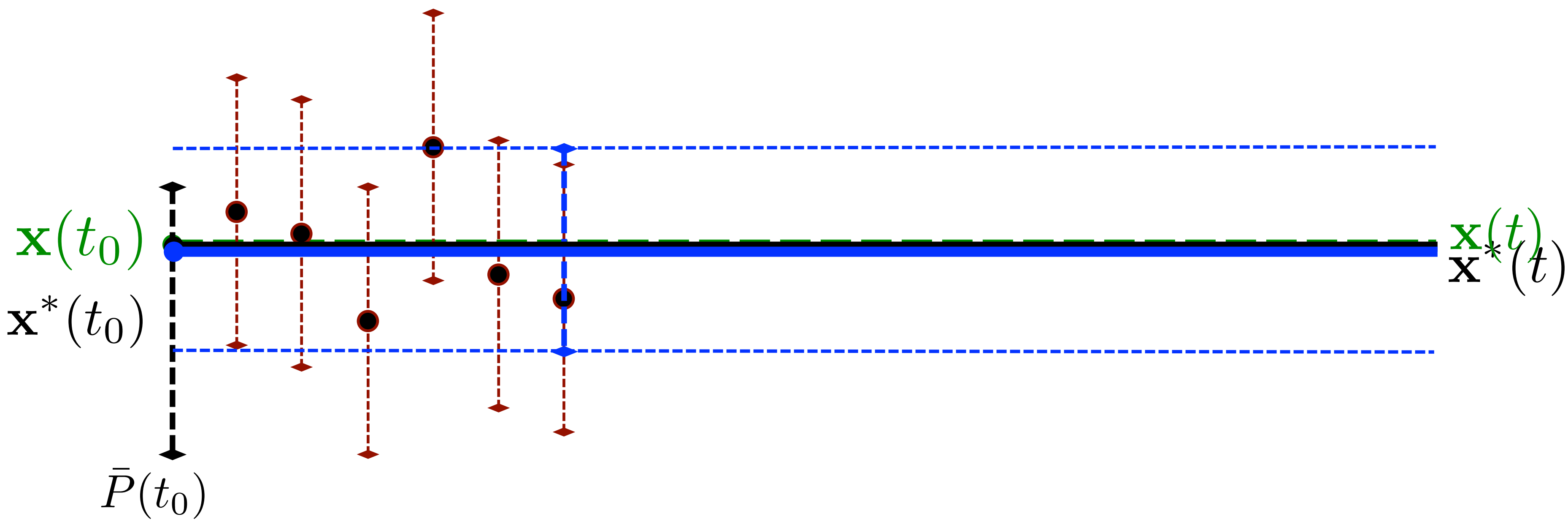
- EKF



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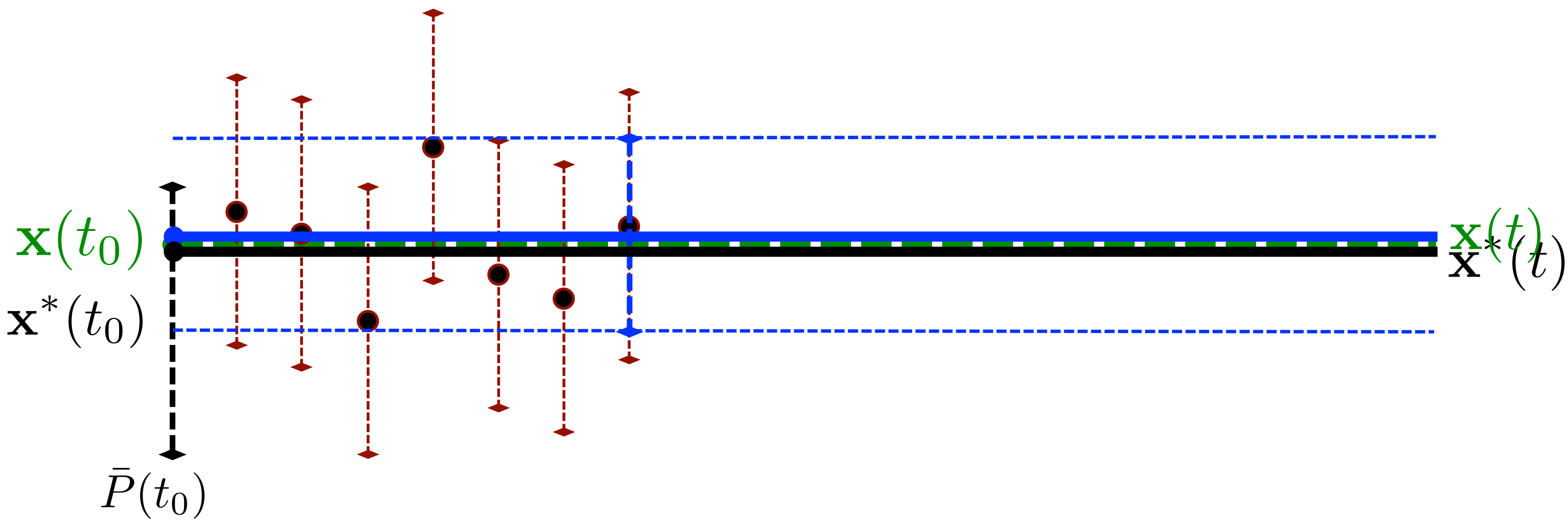

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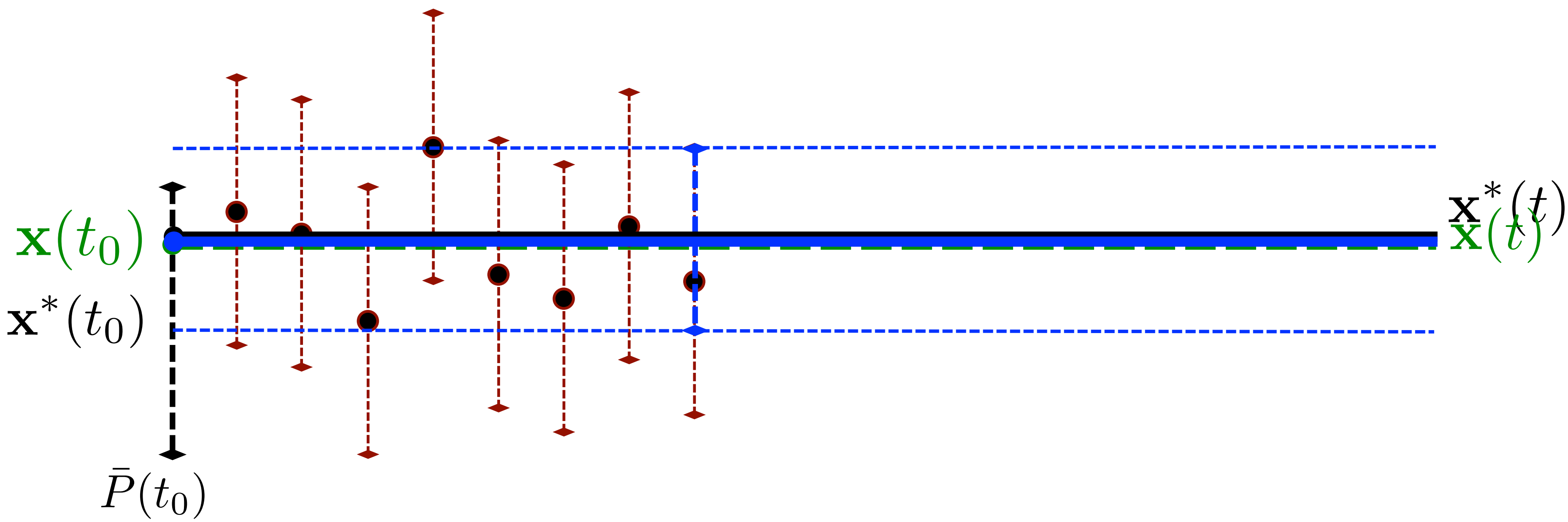

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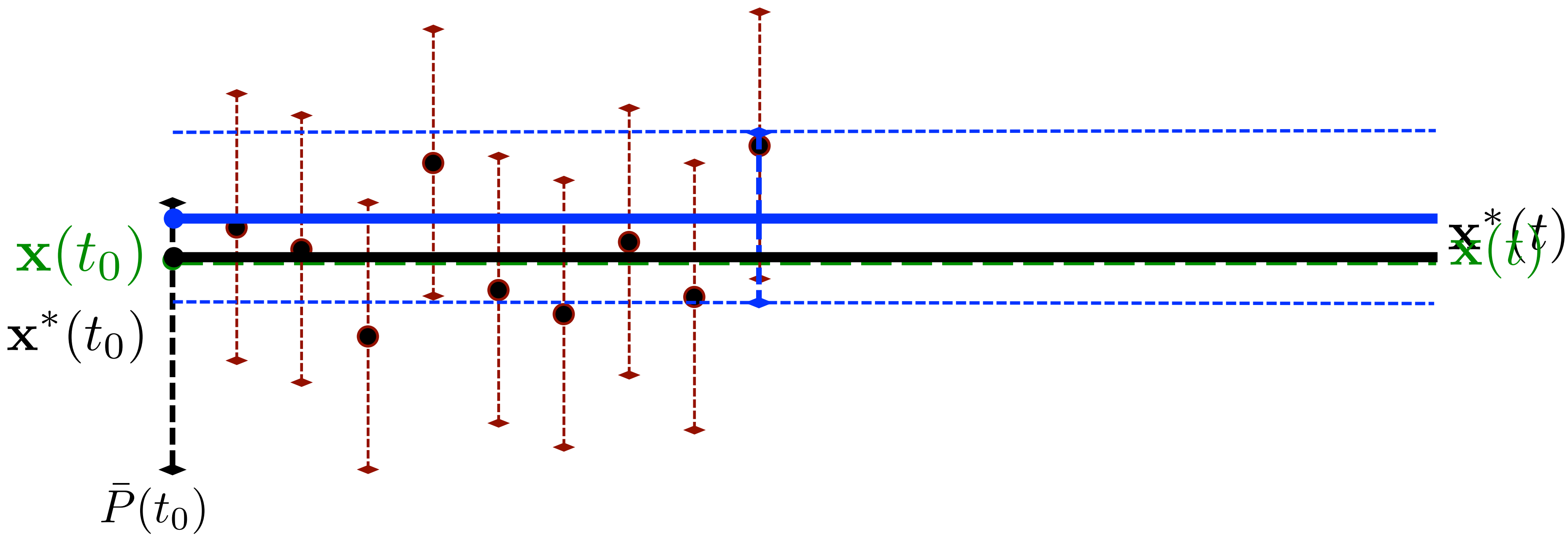

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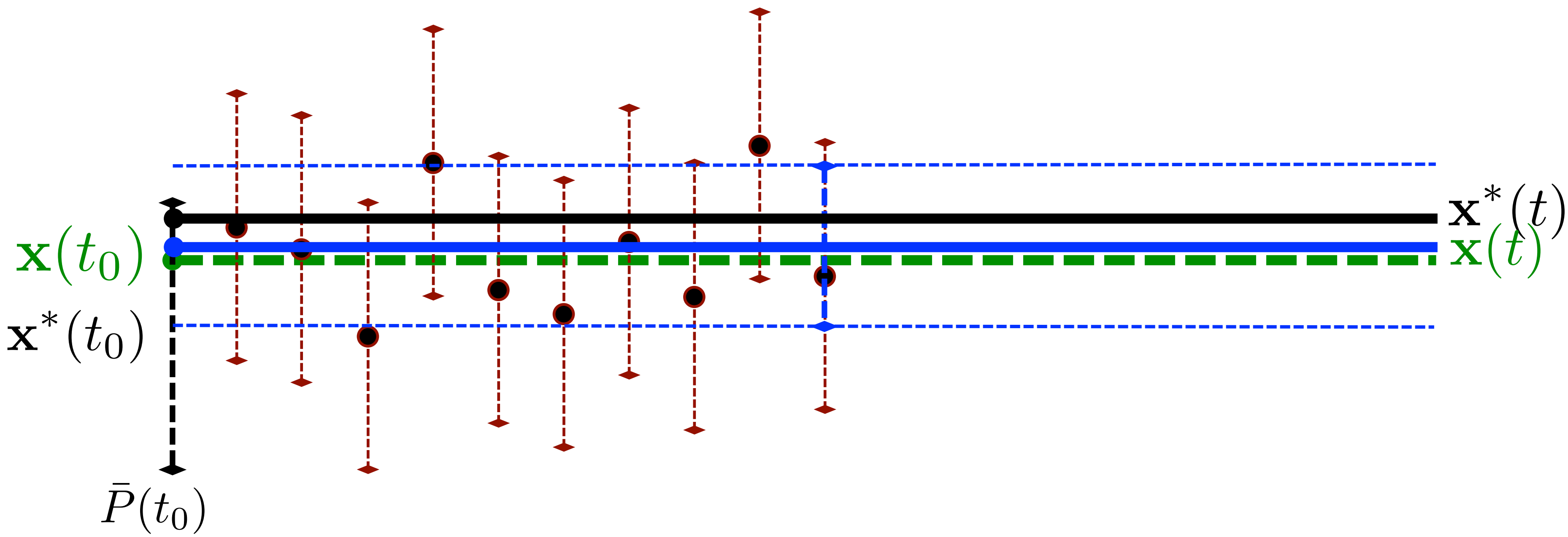

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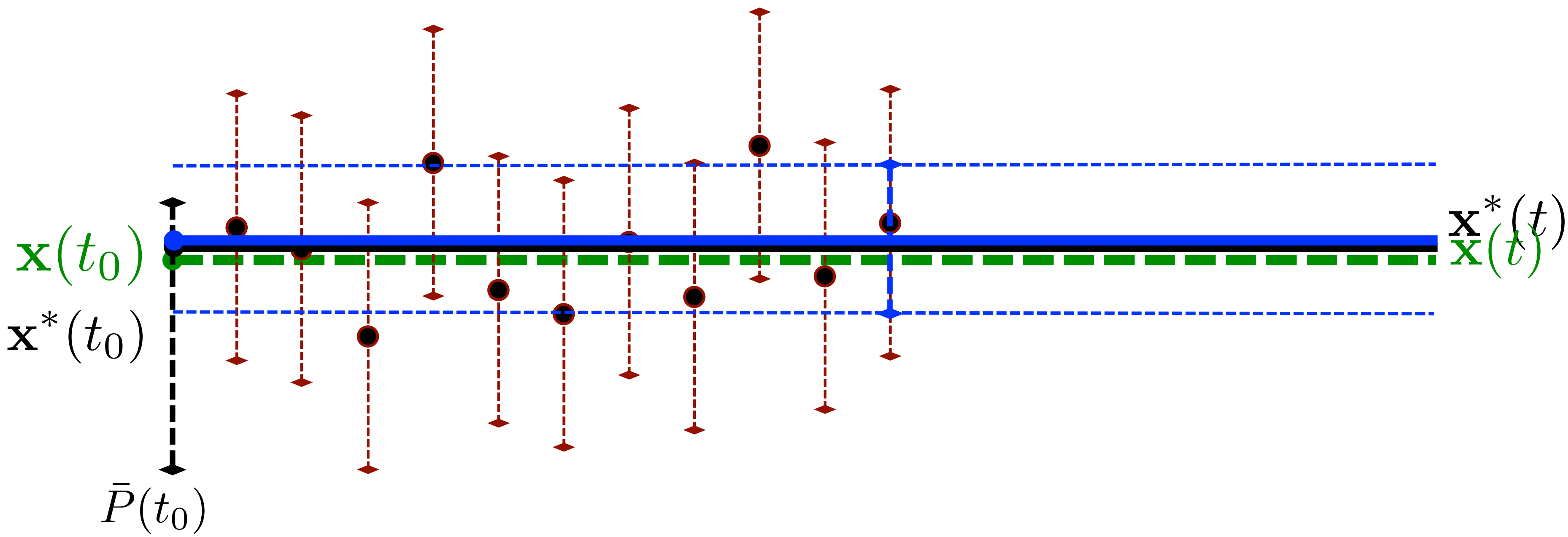

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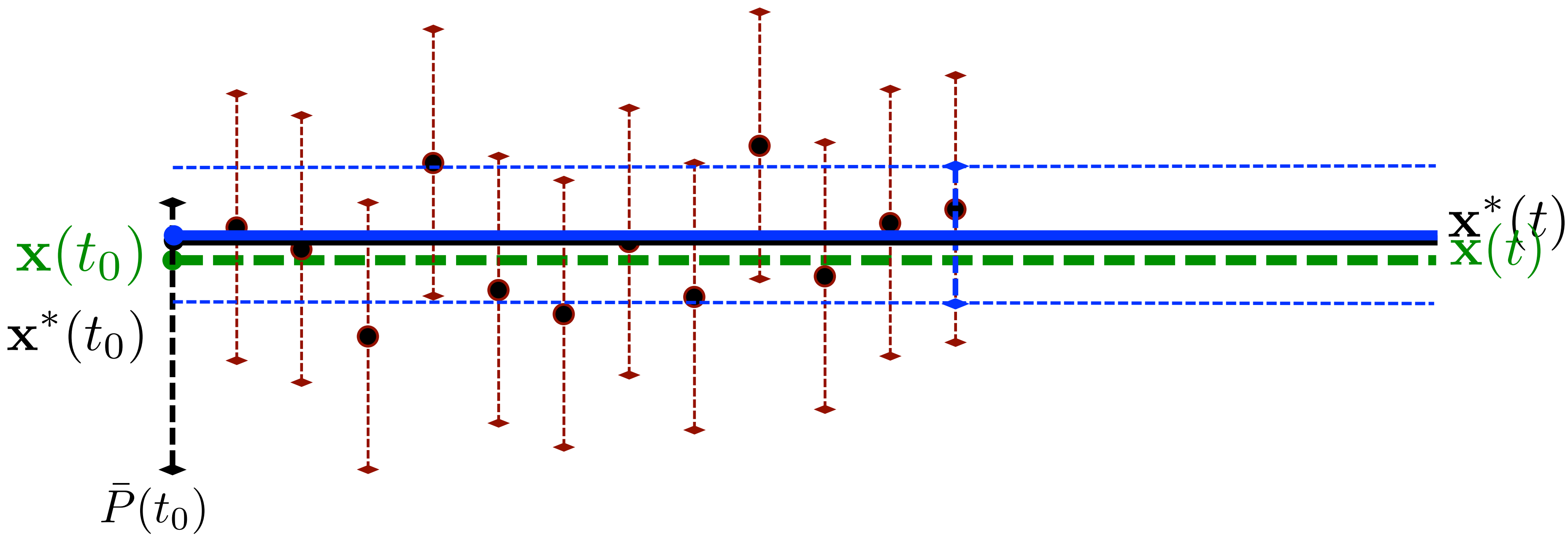

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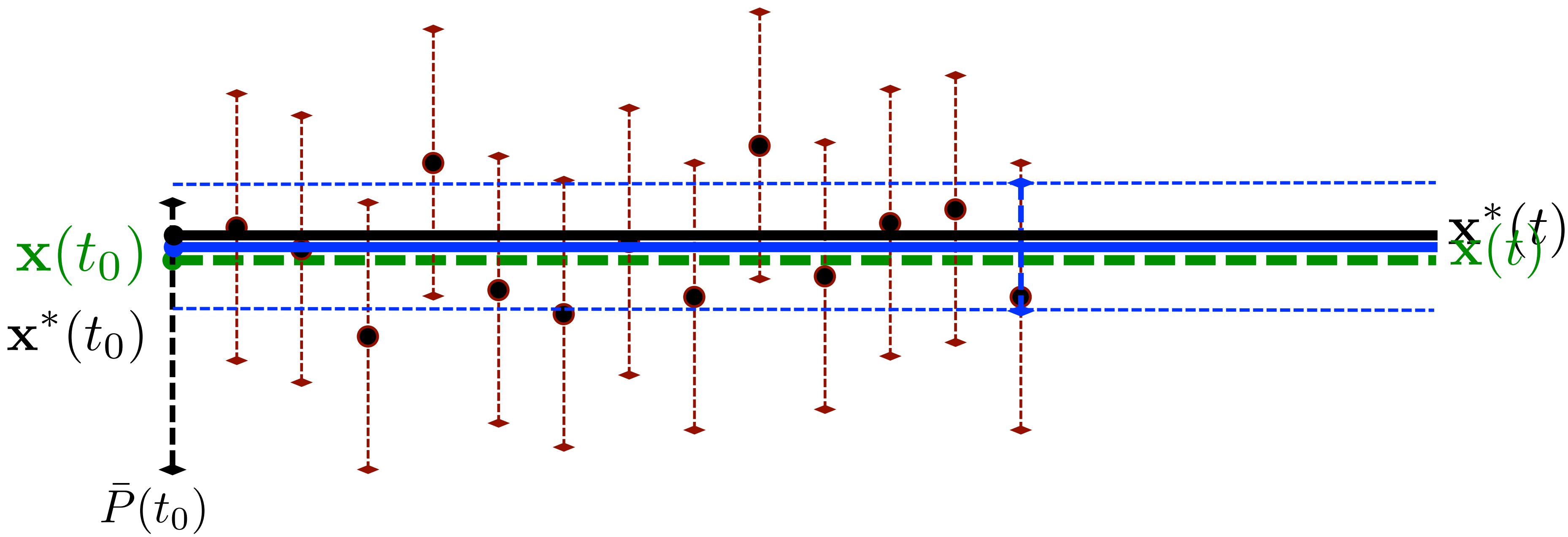

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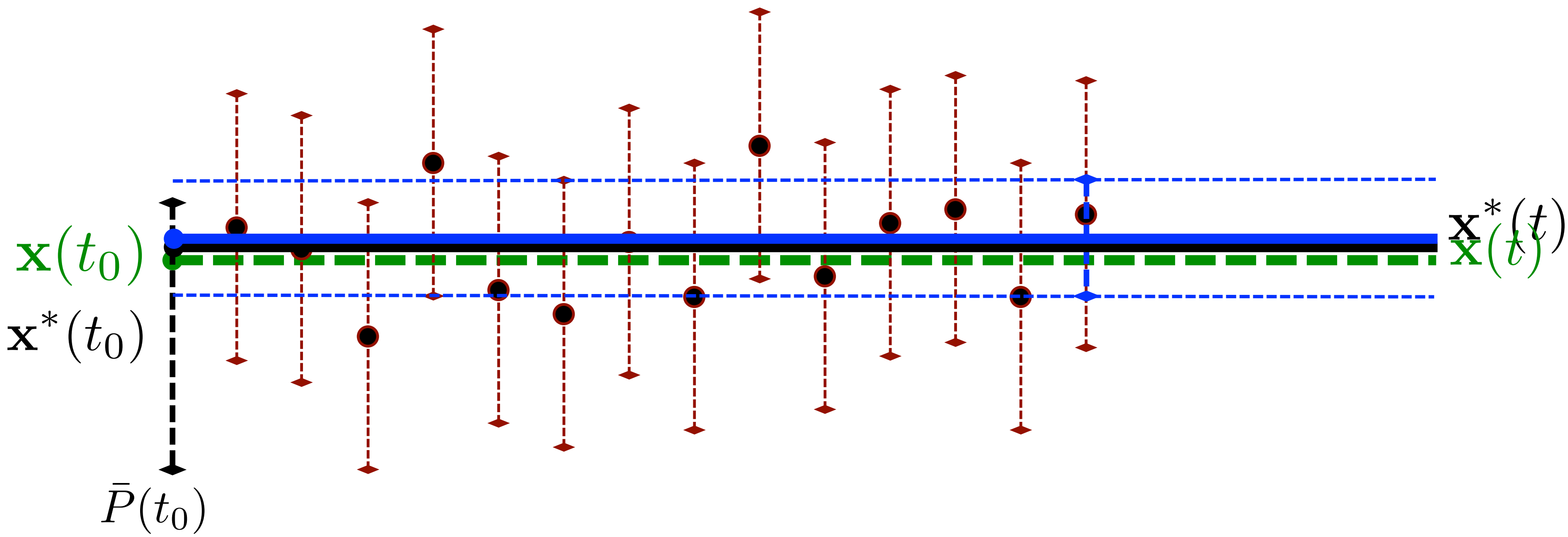

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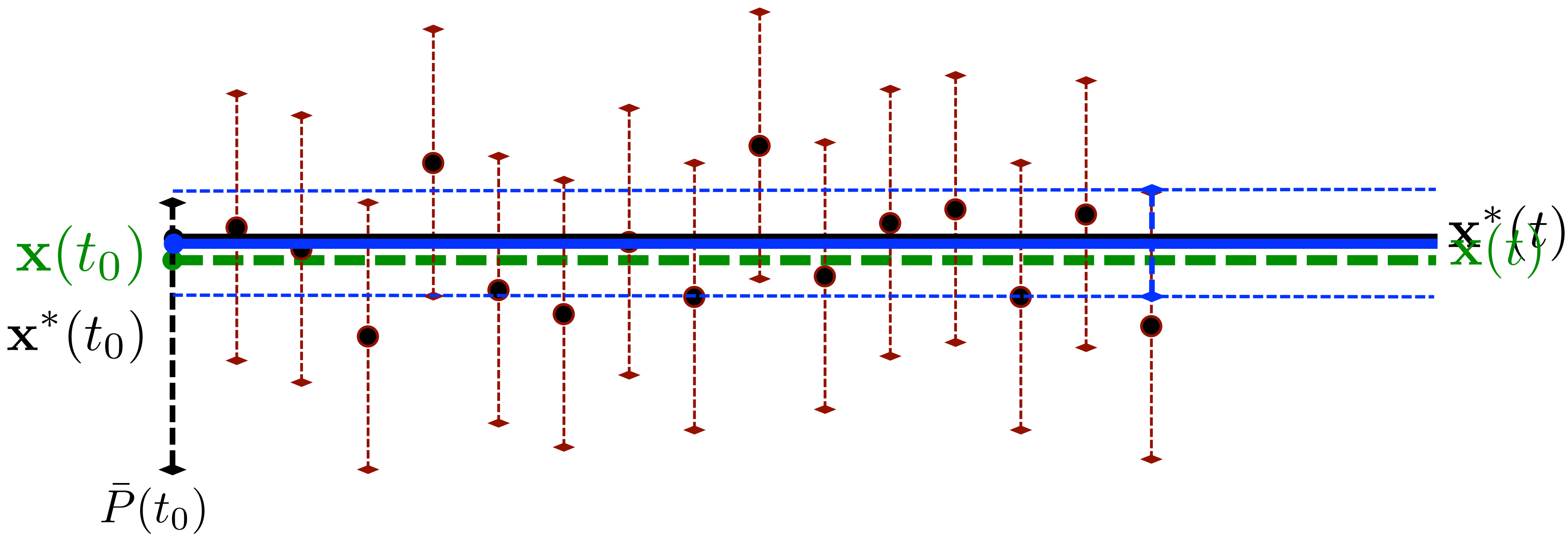

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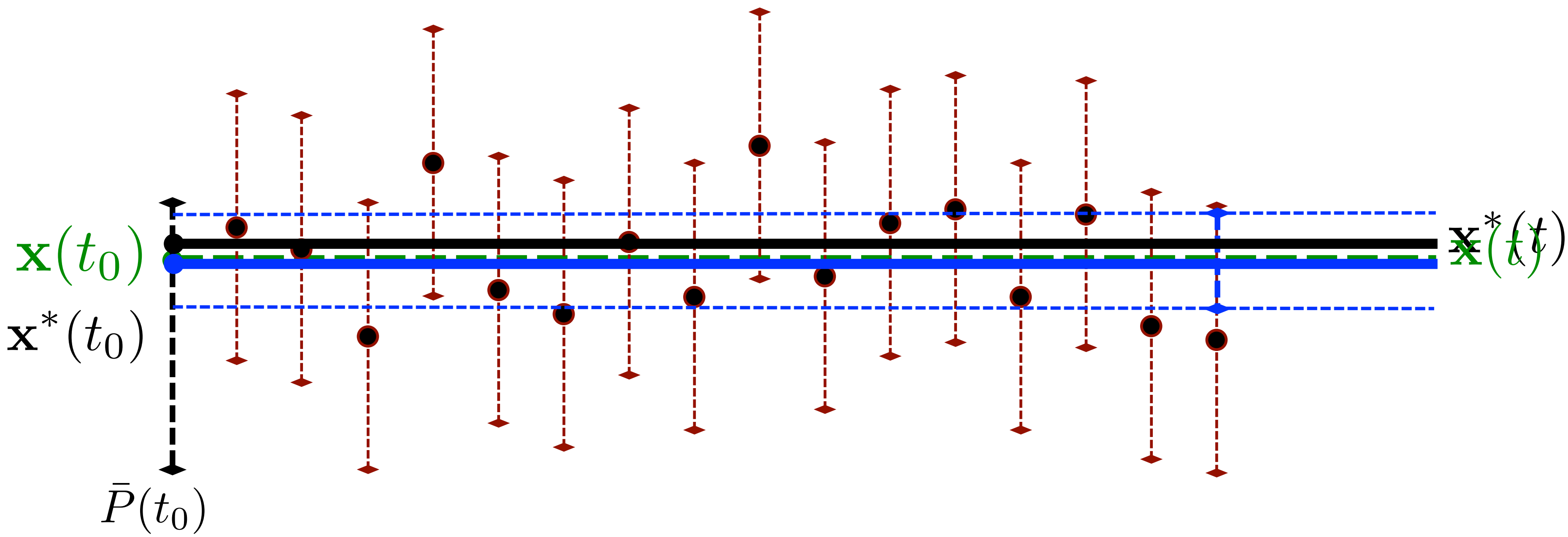

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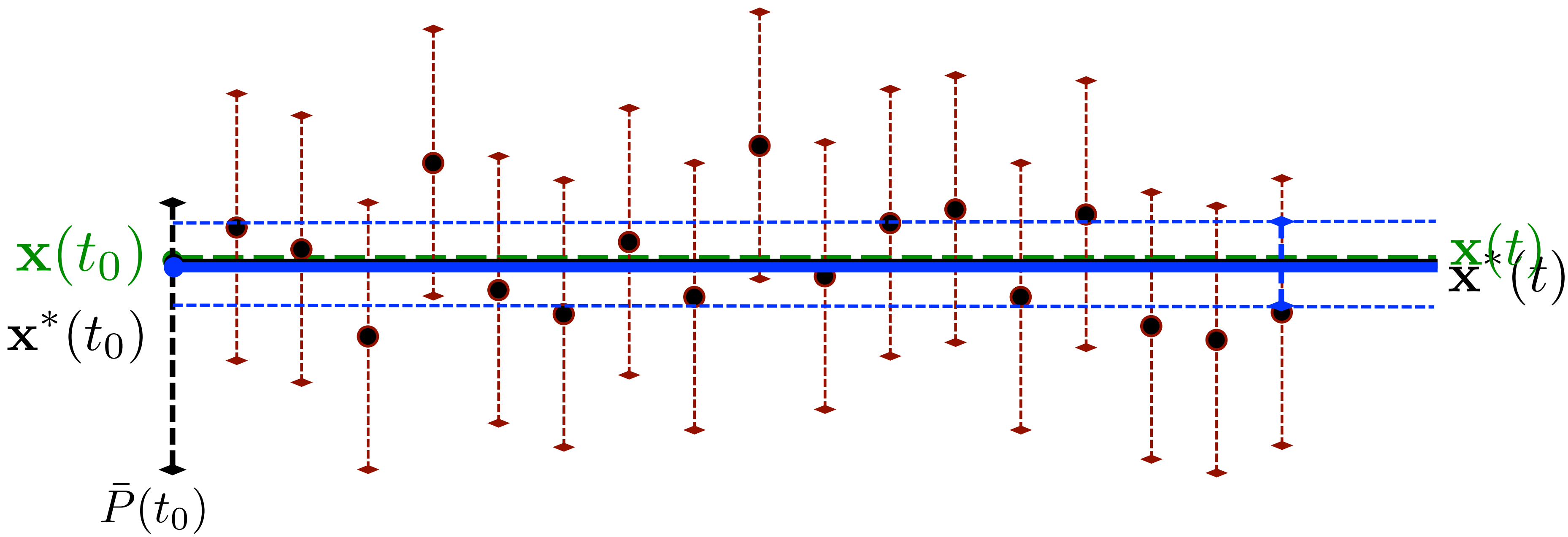

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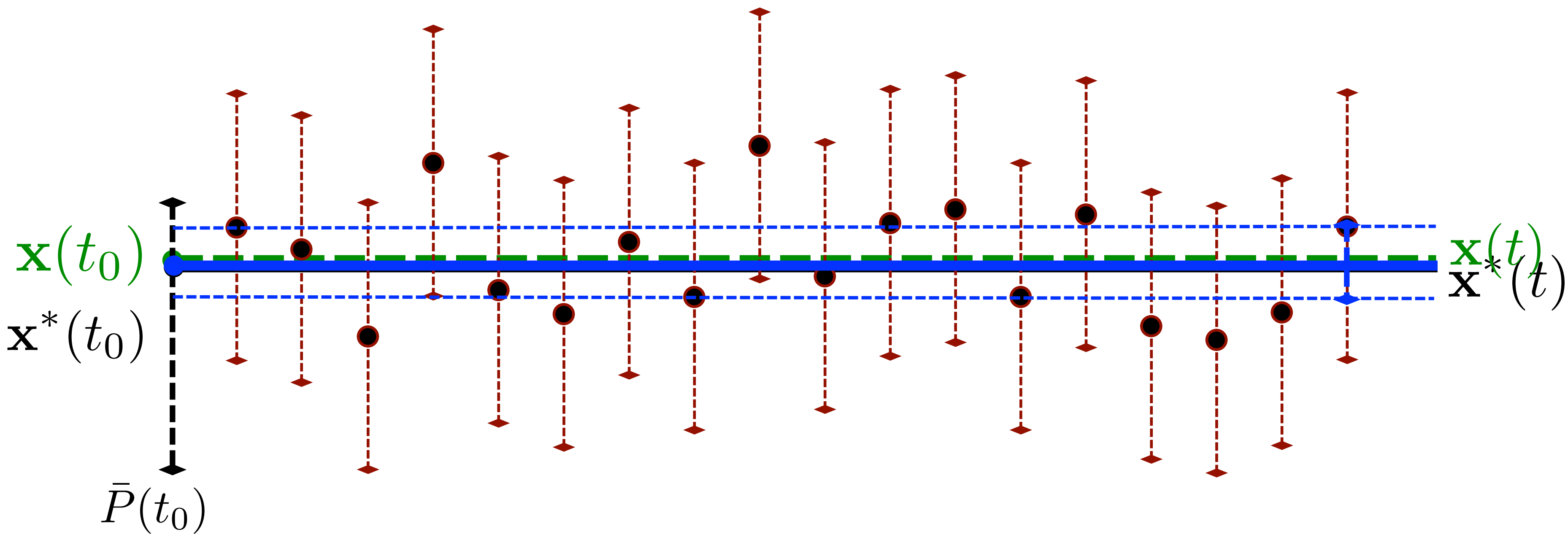

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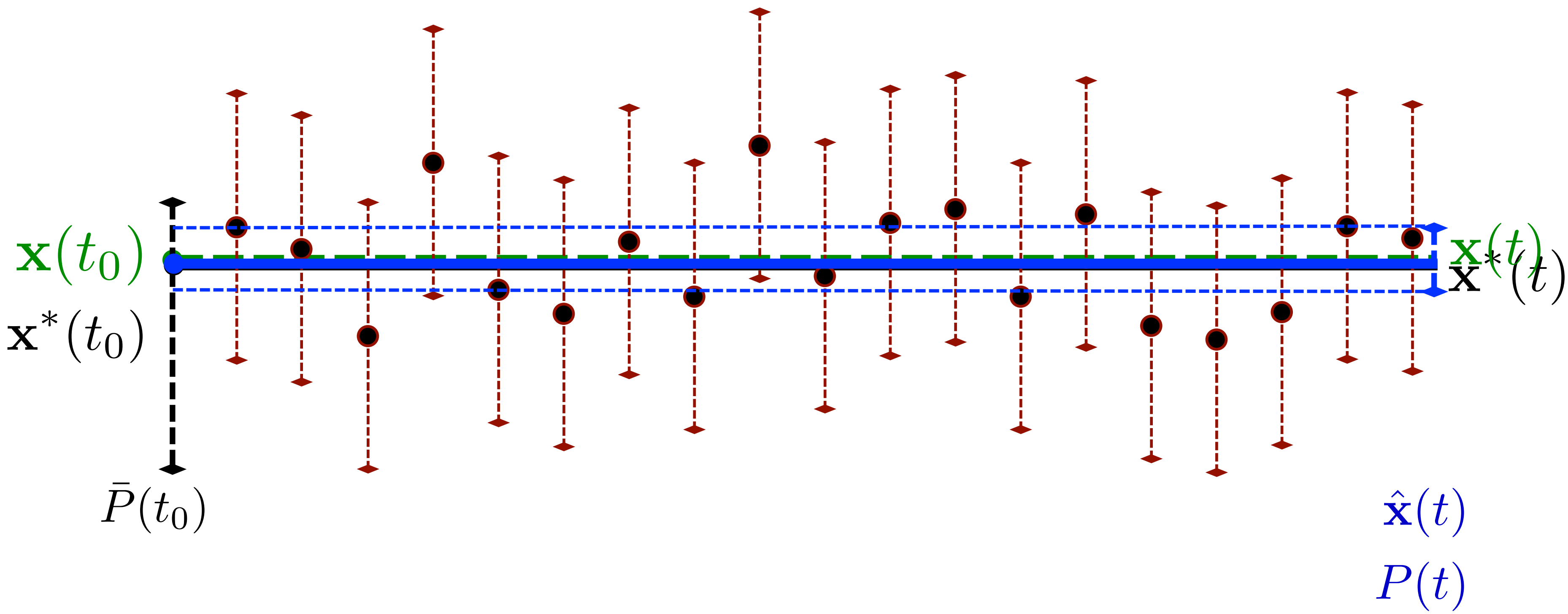

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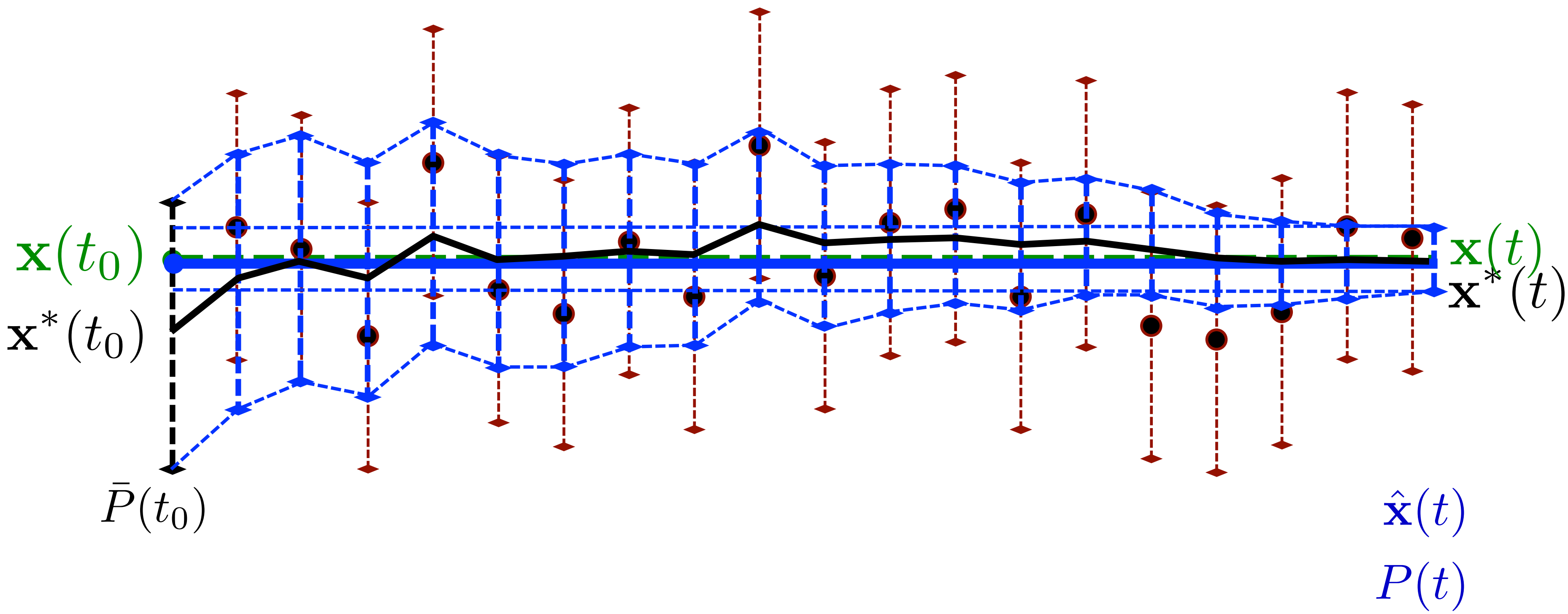

Stat OD Conceptualization

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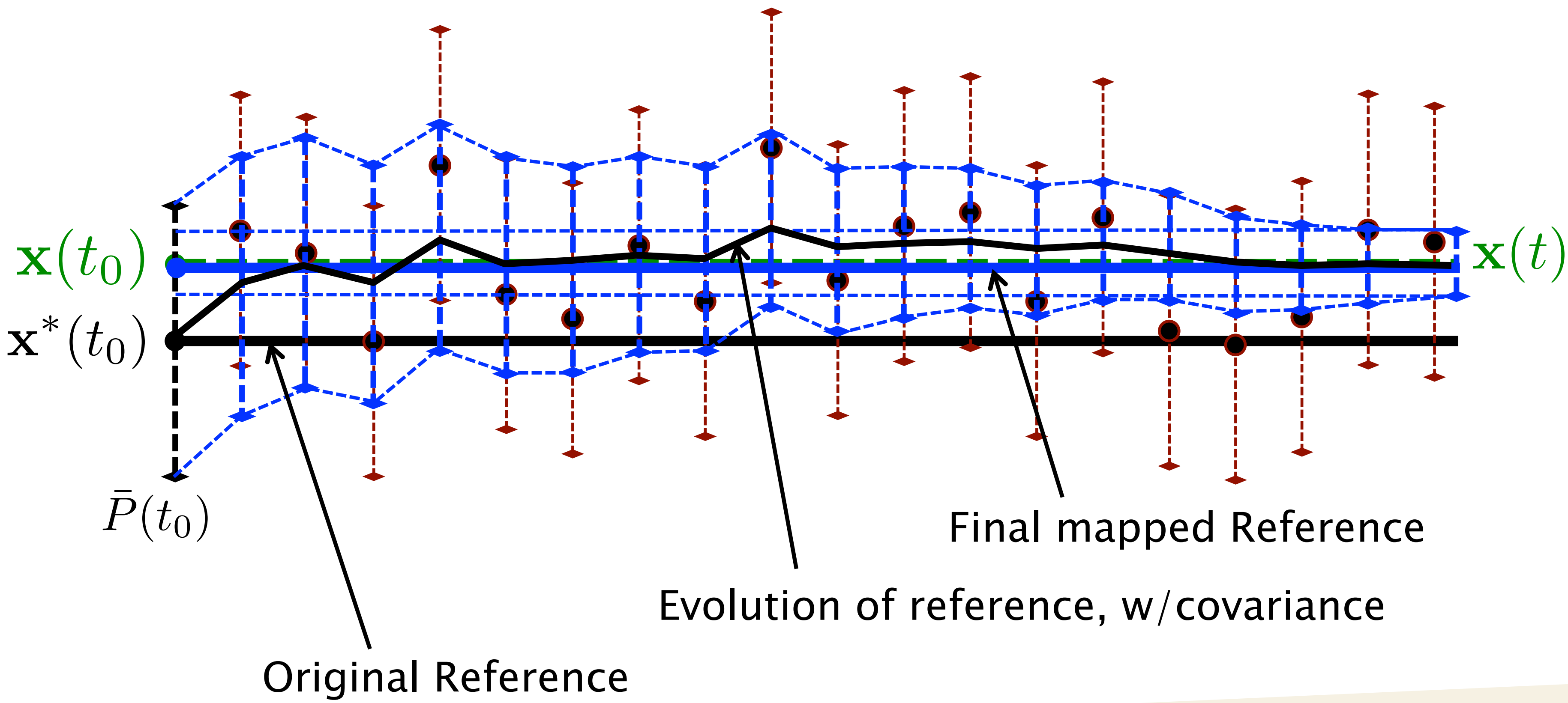

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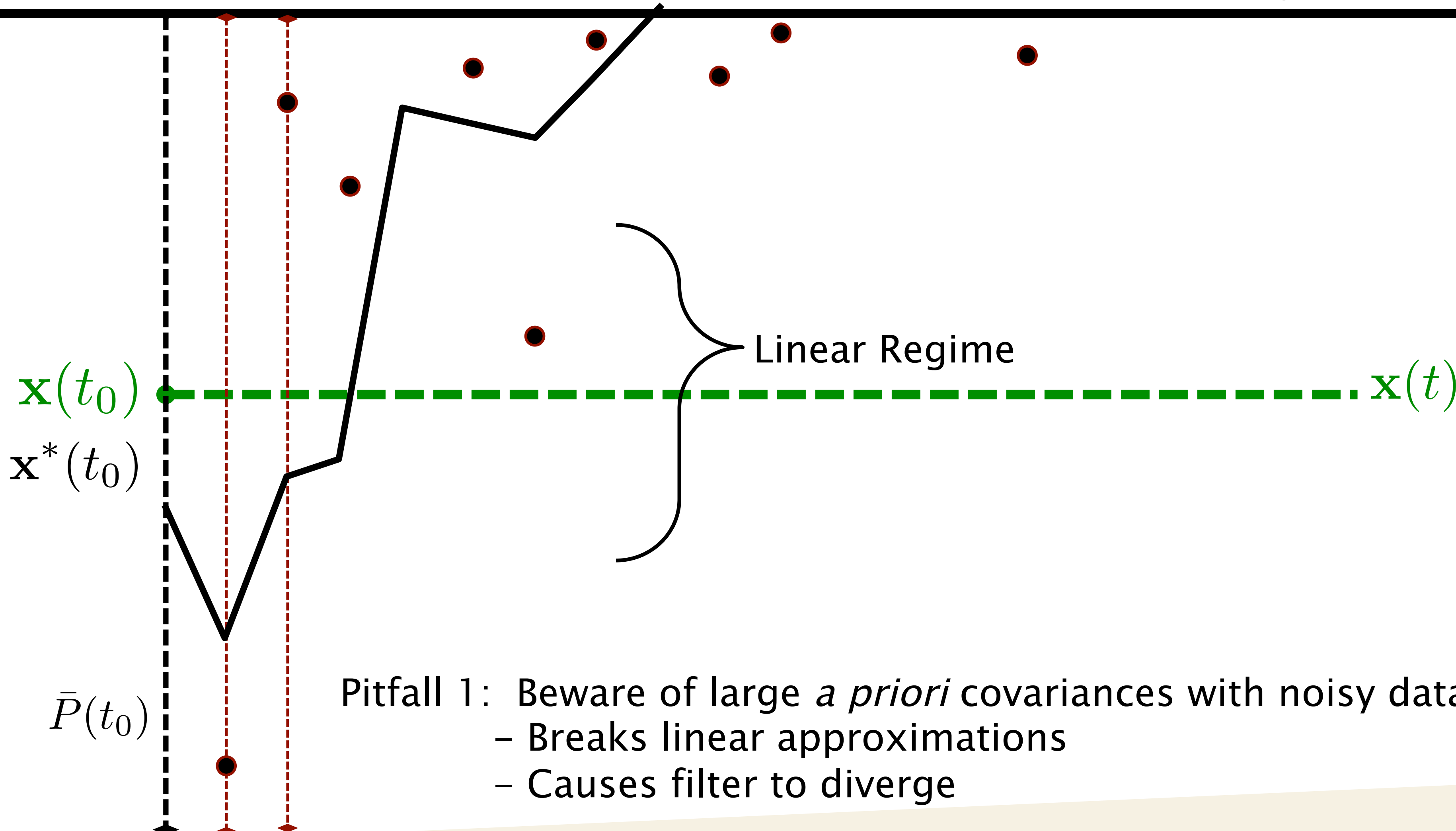
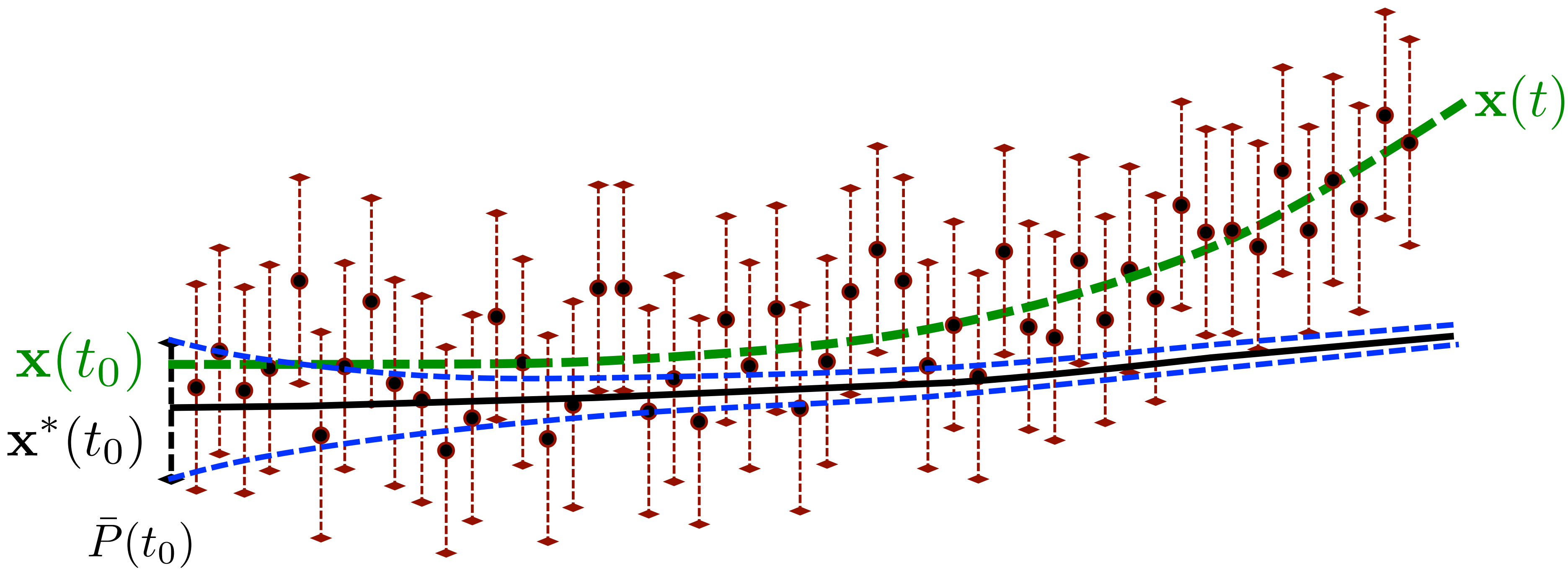


Pitfall 1: Beware of large *a priori* covariances with noisy data

- Breaks linear approximations
- Causes filter to diverge

Stat OD Conceptualization



Pitfall 2: Beware of collapsing covariance

- Prevents new data from influencing solution
- More prevalent for longer time-spans

Observability

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- Every state parameter must be observed somehow
  - Either the observations must be a function of that parameter, or the observation-state relationship changes over time according to the effects of that parameter.
  - I.e., it has to be in the A or H matrix!

- There have to be enough observations

- The state parameters must be distinguishable. That is, they can't be linearly dependent.

Observability

- Basic $$Y = a + bt + ct^2 + dt^3$$

- Linearly Dependent

$$Y = a + (b_1 + b_2)t + (c_1 c_2)t^2 + (d_1/d_2)t^3$$

Questions?

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- Homework 8 due this week.
  - Make sure you spend time studying for the exam
- Homework 9 out today. You're not busy, are you? This one is easy and will push you toward the completion of the final project.
- Exam 2 on Thursday.
  - A–H in this classroom
  - I–Z in ECEE 265
- Exam 2 will cover:
  - Batch vs. CKF vs. EKF
  - Probability and statistics (good to keep this up!)
    - Haven't settled on a question yet, but it will probably be a conditional probability question. I.e., what's the probability of X given that Y occurs?
  - Observability
  - Numerical compensation techniques, such as the Joseph and Potter formulation.
  - No calculators should be necessary
  - Open Book, Open Notes