ASEN 5070
Statistical Orbit Determination I
Fall 2012

Professor George H. Born
Professor Jeffrey S. Parker

Lecture 12: The Kalman Filter

Announcements

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- Homework 5 due Today
- Exam on 10/11. (Anyone going to miss it?)
  - Eduardo and/or Paul will be reviewing subjects on Tuesday – send them emails with questions/subjects that you'd like them to cover.
  - 1 hour, open book, open notes.
  - Topics: Definitions of variables, Probability/Statistics Observability, Linearization Least squares, Batch processor

$$
\begin{array}{lll}
\mathbf{X}(t_i), & \mathbf{X}^*(t_i), & \hat{\mathbf{X}}(t_i) \\
\mathbf{x}(t_i), & \mathbf{x}^*(t_i), & \hat{\mathbf{x}}(t_i) \\
\mathbf{Y}(t_i), & \epsilon(t_i), & \hat{\epsilon}(t_i) \\
\Phi(t_i, t_j), & H(t_i), & \tilde{H}(t_i) \\
A(t_i), & P(t_i), & W(t_i) \\
\bar{\mathbf{x}}(t_i), & \bar{P}(t_i), & \bar{W}(t_i)
\end{array}
$$

Quiz Results

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If you took the quiz, you scored 100%

Quiz Results

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- Some feedback:
- Biggest issues:
  - Moving pretty fast
  - Lectures and HW don't correlate well
- Review next week:
  - Review of variables
  - Stat OD example from start to finish (something plain and easy to follow)

Review of the Stat OD Process

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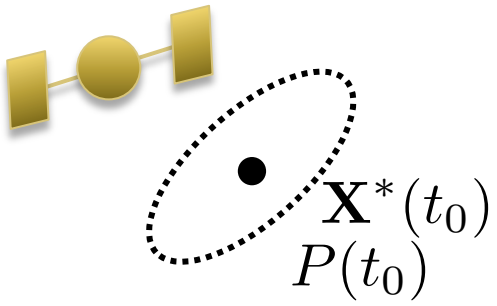
- Setup.
  - Given: an initial state $\mathbf{X}^*(t_0)$
  - Optional: an initial covariance $P(t_0)$

$$\mathbf{X}^*(t_0) = [\ x_0,\ y_0,\ z_0,\ \dot{x}_0,\ \dot{y}_0,\ \dot{z}_0,\ c_1,\ c_2\ ]^T$$



Review of the Stat OD Process

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- Setup.
  - Given: an initial state $\mathbf{X}^*(t_0)$
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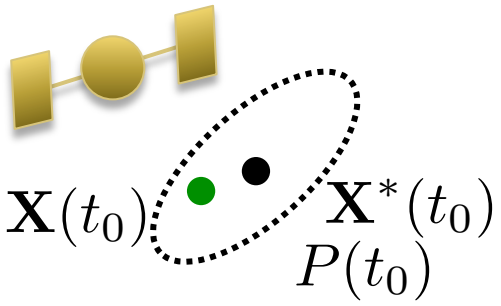
$$\mathbf{X}^*(t_0) = [\ x_0,\ y_0,\ z_0,\ \dot{x}_0,\ \dot{y}_0,\ \dot{z}_0,\ c_1,\ c_2\ ]^T$$

  - The satellite will *not* be there, but will (hopefully) be nearby
    - True state = $\mathbf{X}(t_0)$



Review of the Stat OD Process

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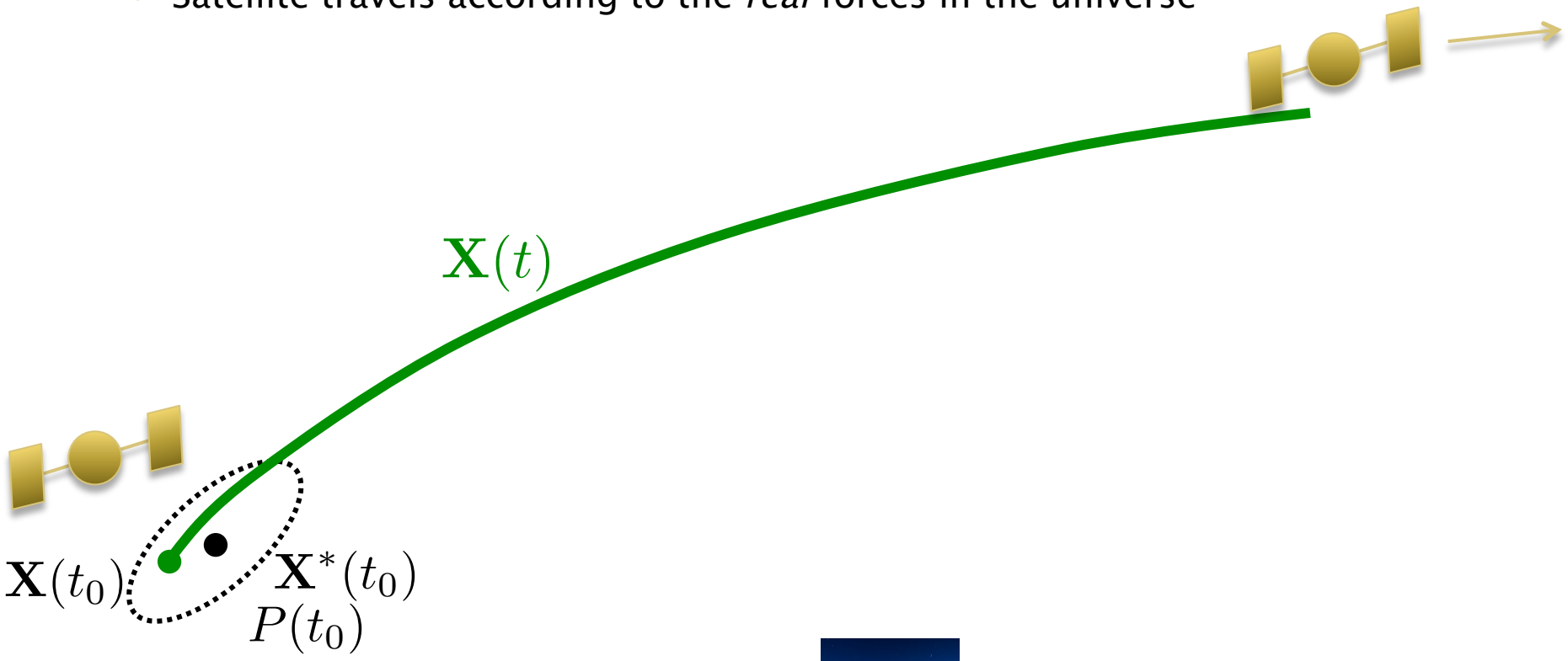
- What really happens
  - Satellite travels according to the *real* forces in the universe



Review of the Stat OD Process

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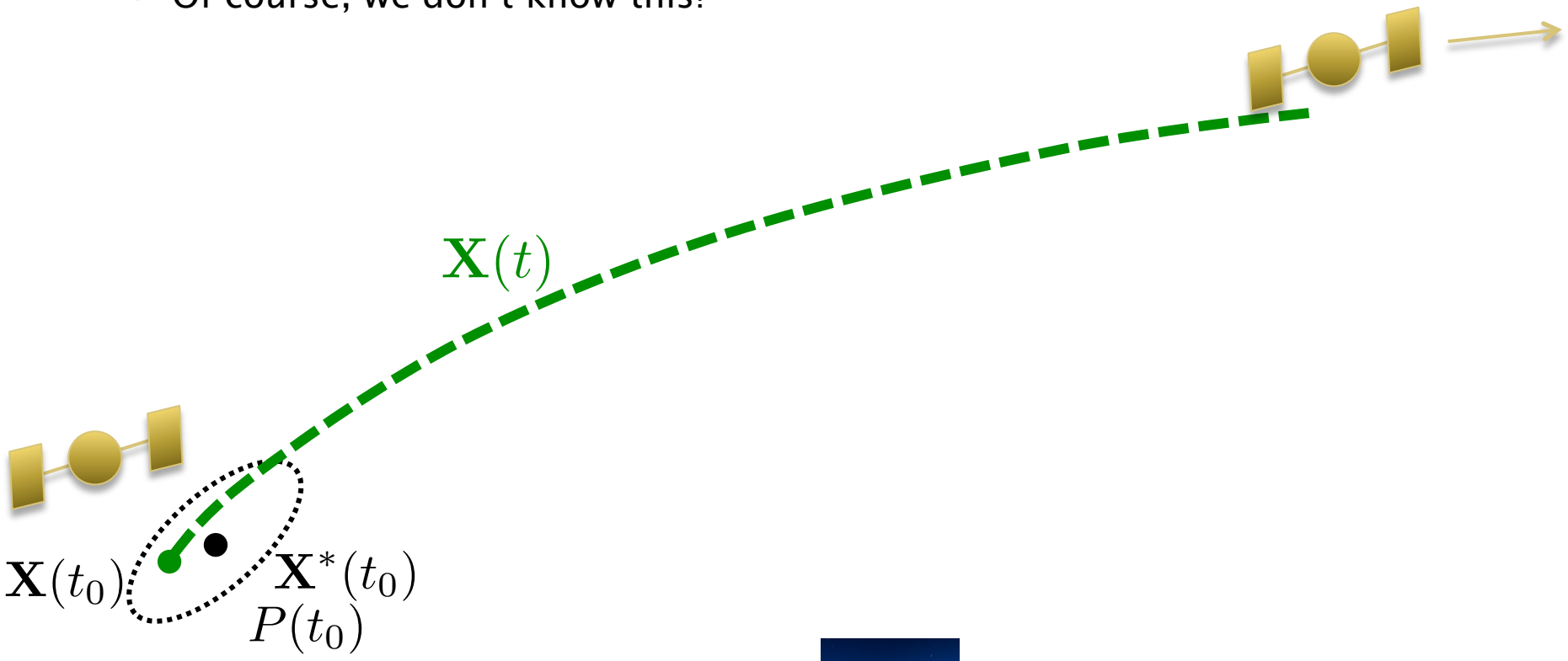
- What really happens
  - Of course, we don't know this!



Review of the Stat OD Process

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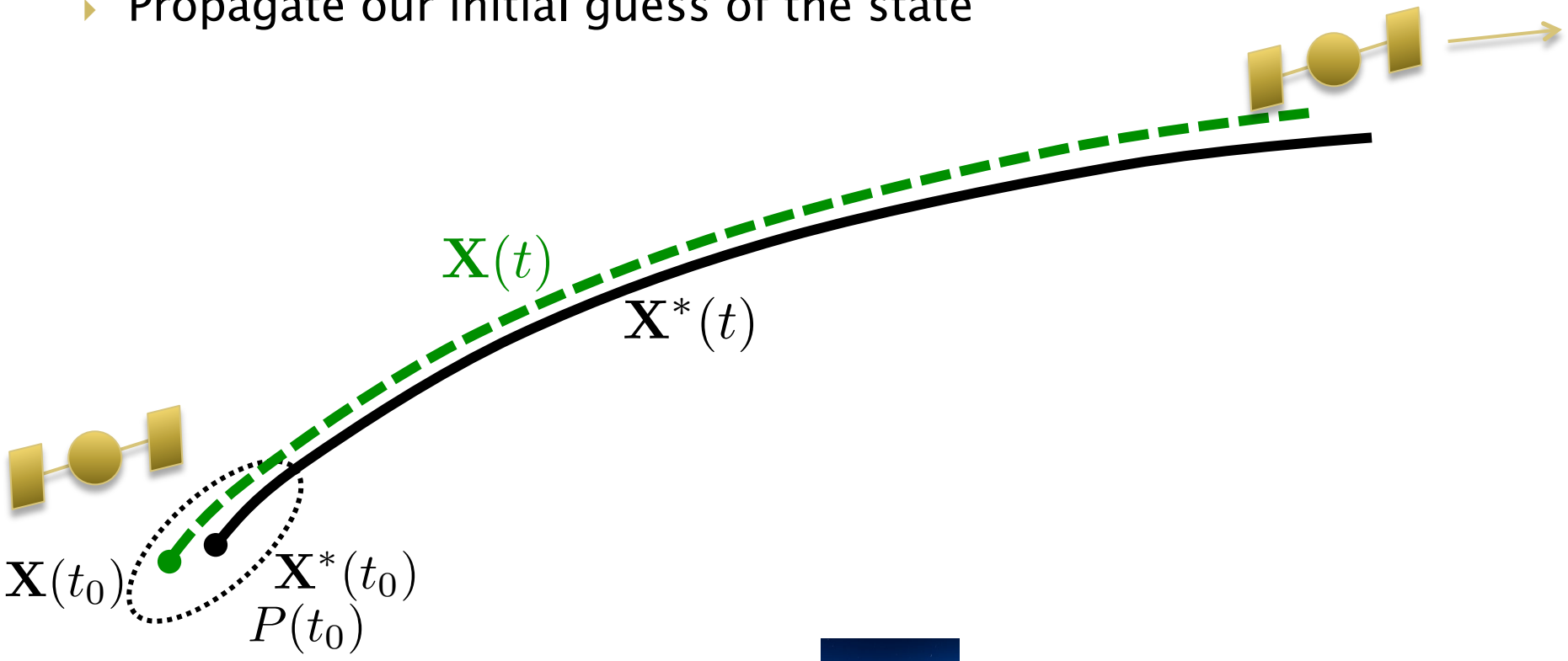
- Model reality as best as possible
- Propagate our initial guess of the state



Review of the Stat OD Process

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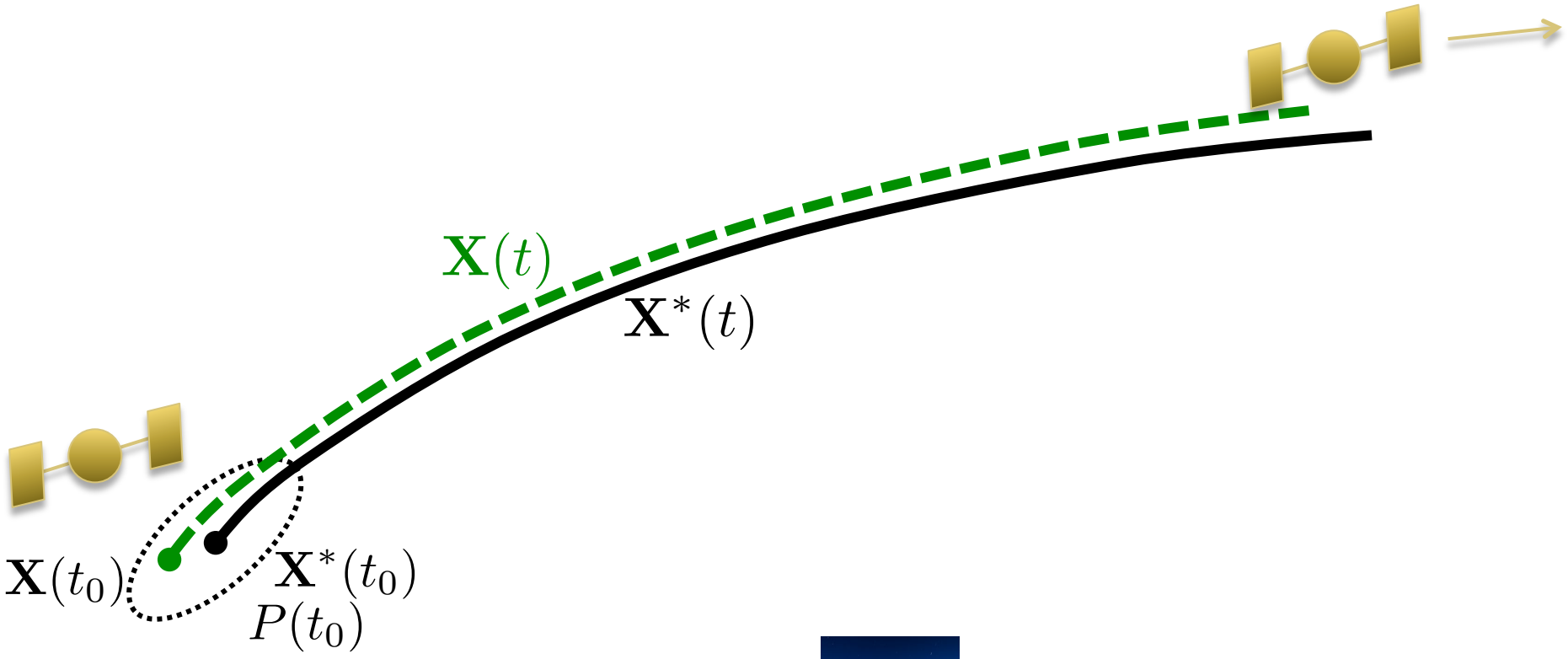
- Goal: Determine how to modify $\mathbf{X}^*(t)$ to match $\mathbf{X}(t)$



Review of the Stat OD Process

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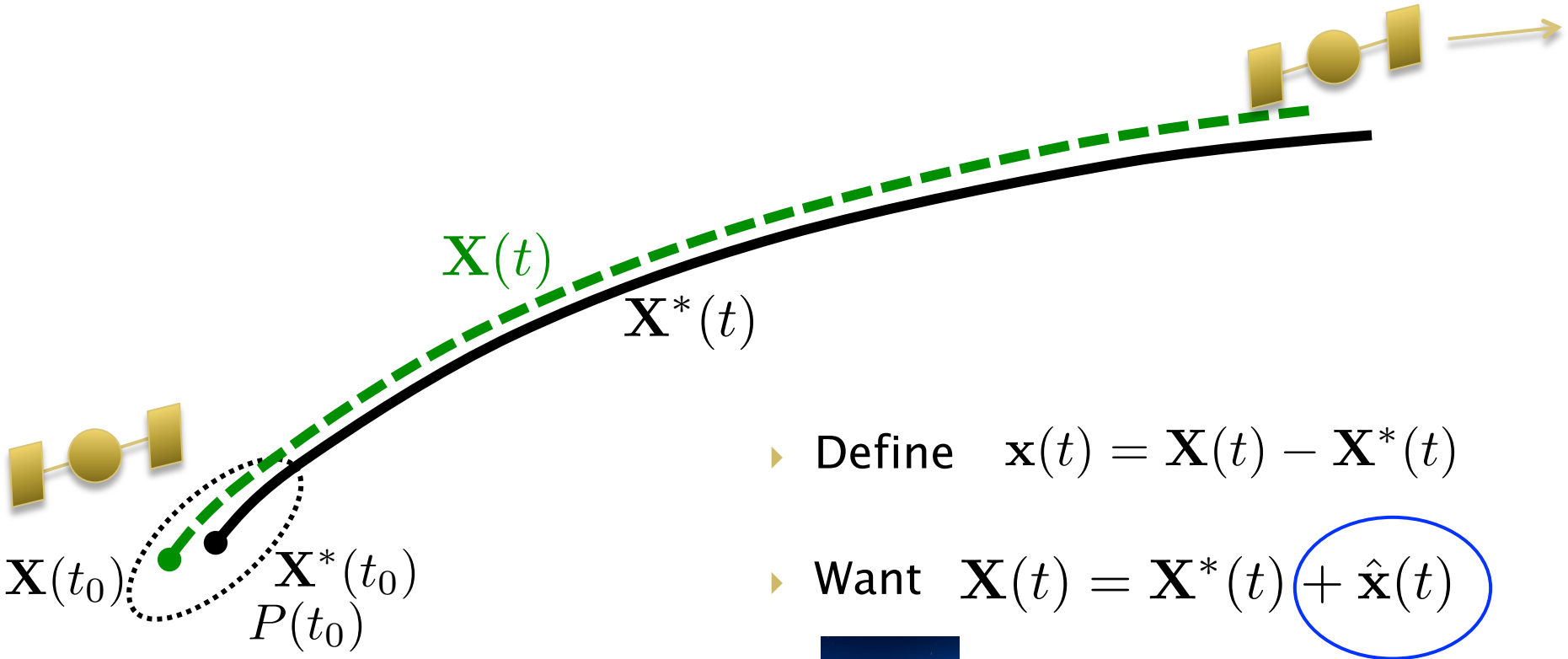

- Define $\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t)$
- Want $\mathbf{X}(t) = \mathbf{X}^*(t) + \hat{\mathbf{x}}(t)$

Review of the Stat OD Process

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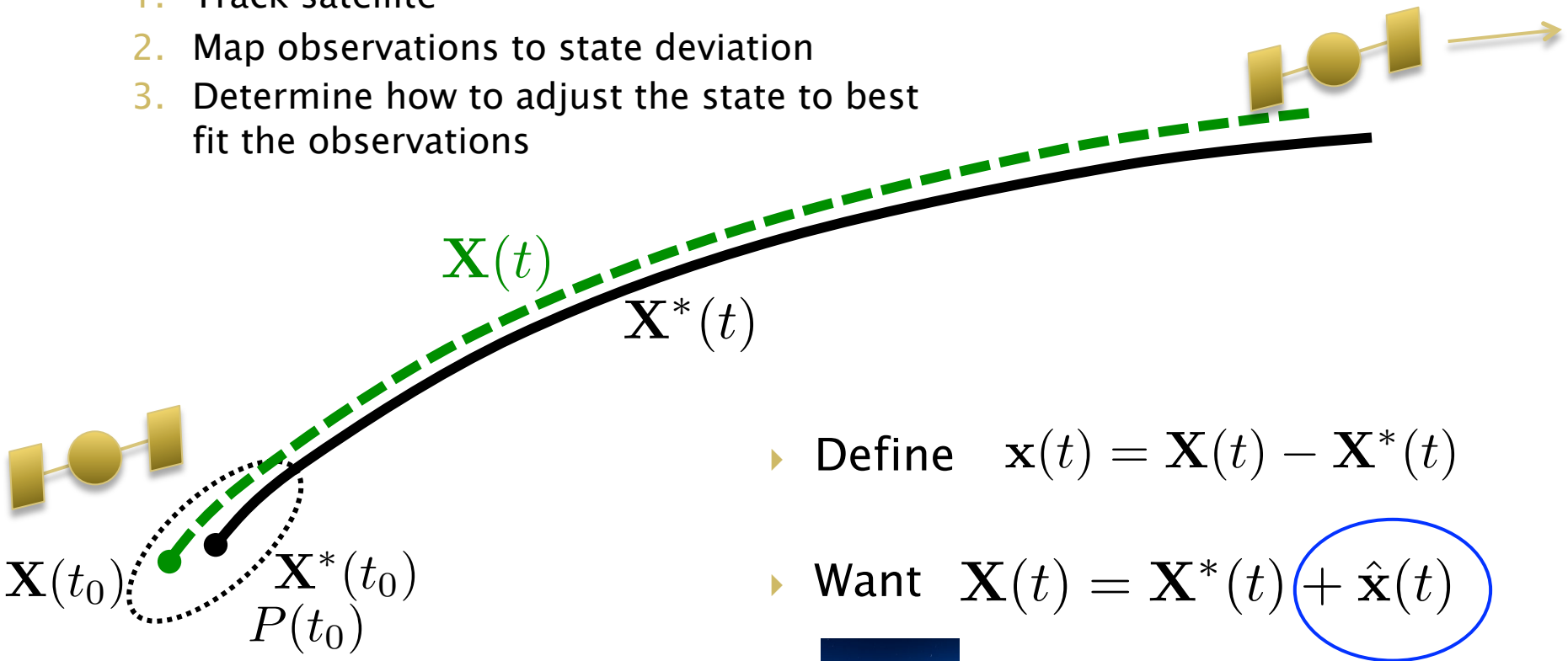
- Process:
  1. Track satellite
  2. Map observations to state deviation
  3. Determine how to adjust the state to best fit the observations



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Review of the Stat OD Process

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- Process:
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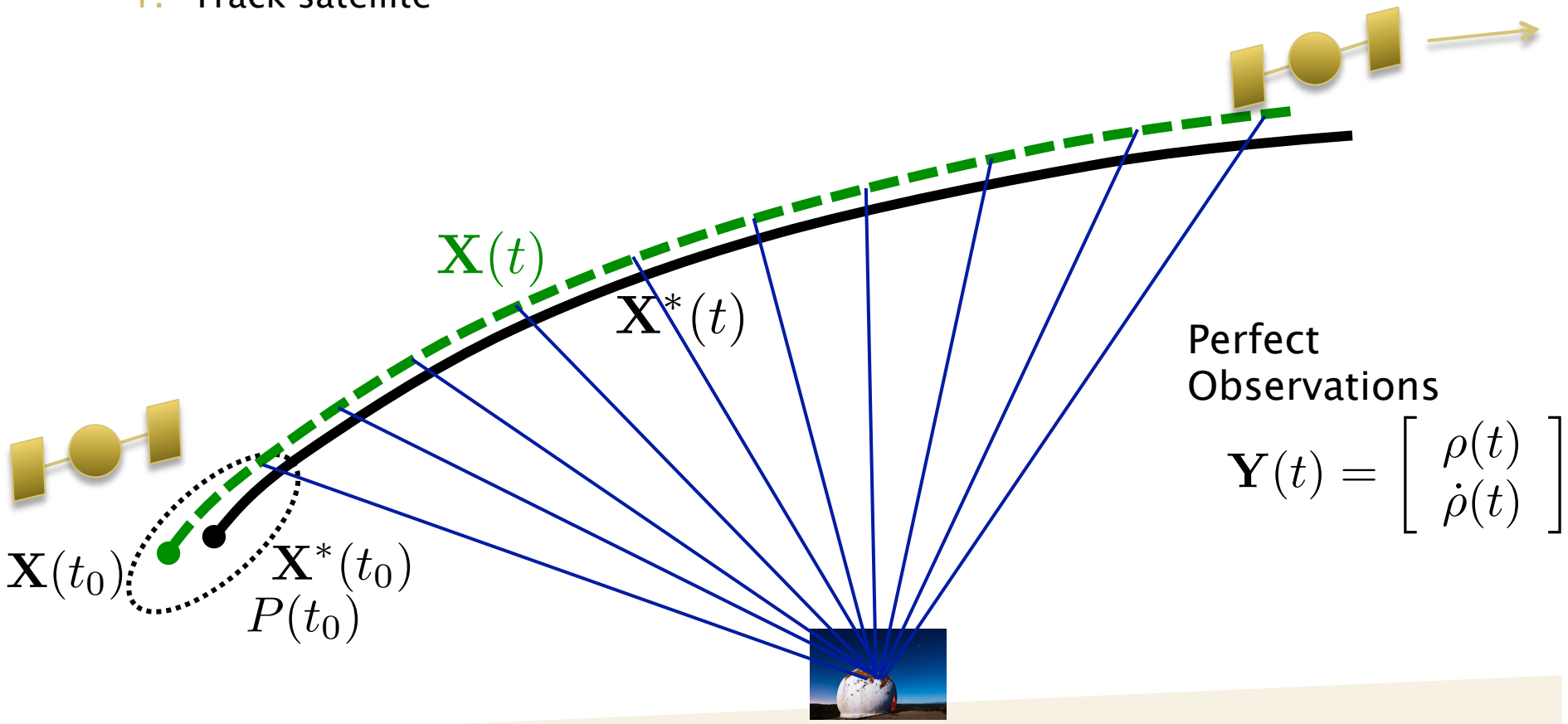

Review of the Stat OD Process

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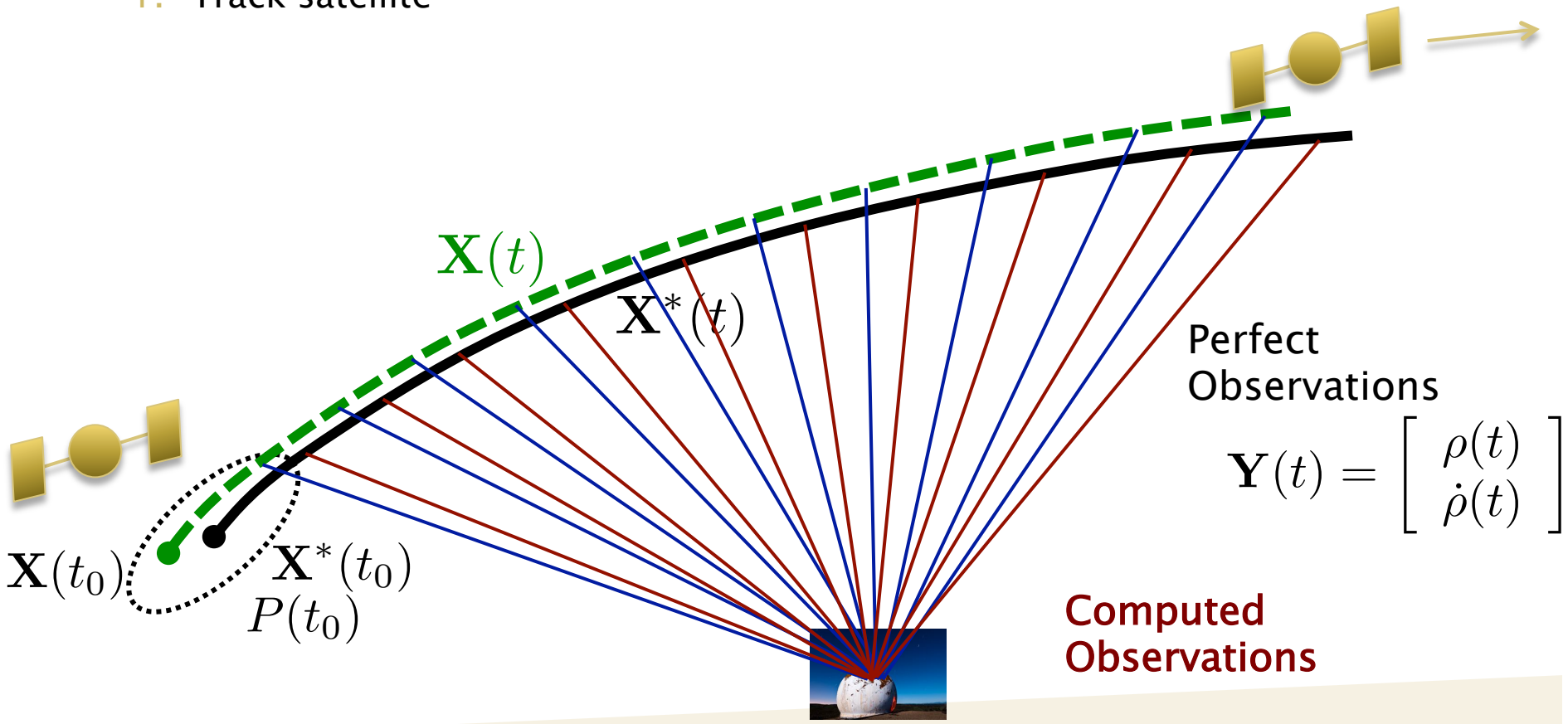

Review of the Stat OD Process

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- Process:
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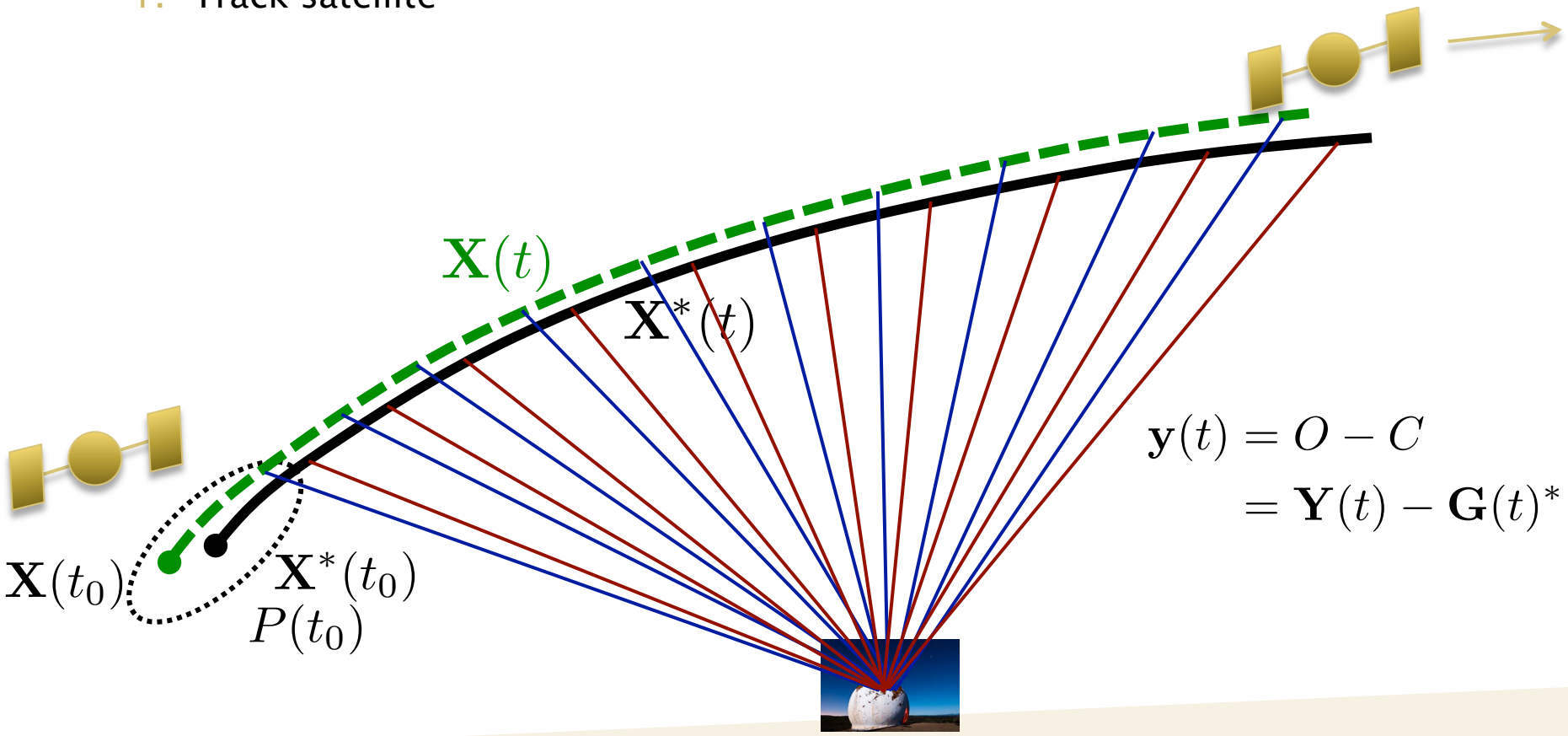

Review of the Stat OD Process

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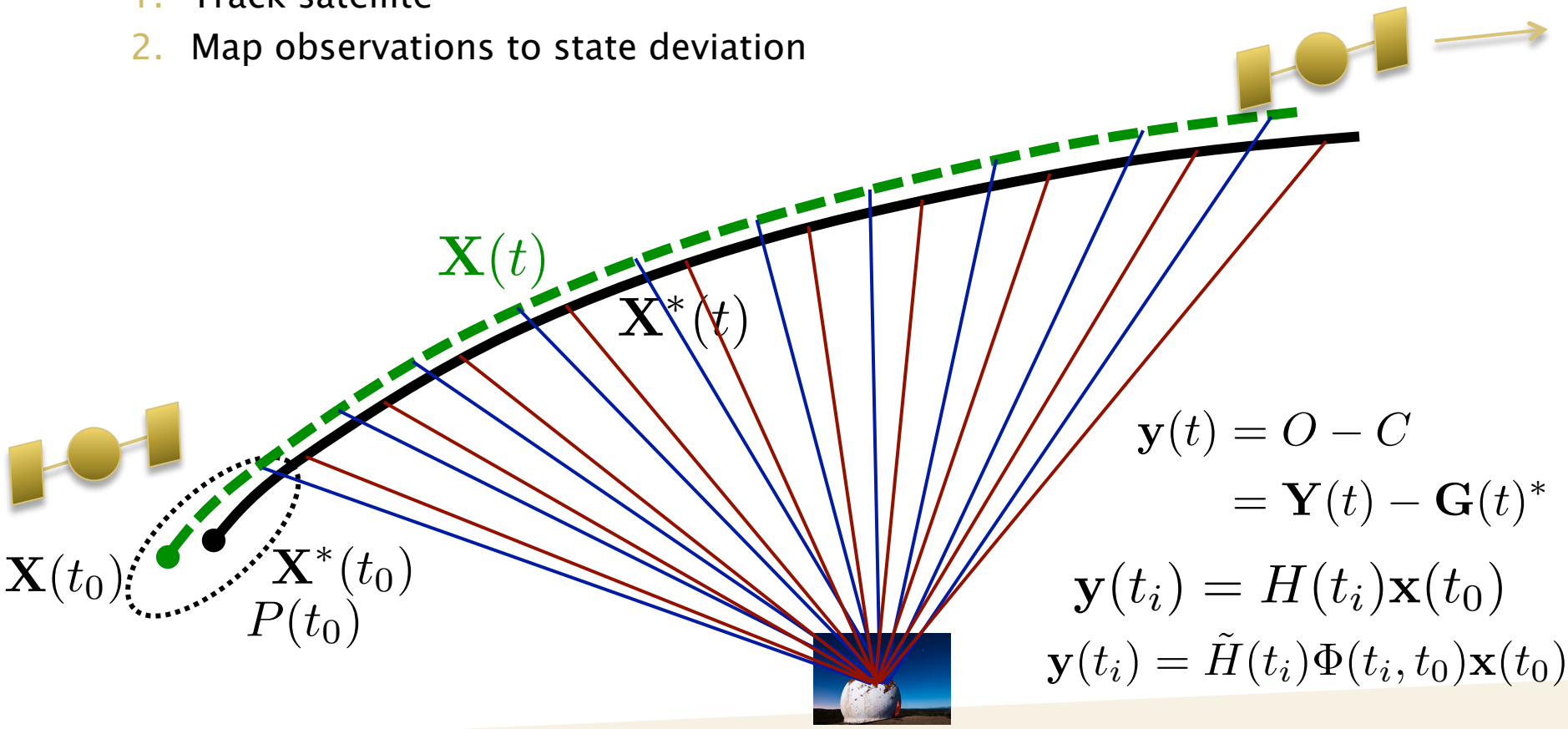

$$\mathbf{y}(t) = O - C$$
$$= \mathbf{Y}(t) - \mathbf{G}(t)^*$$
$$\mathbf{y}(t_i) = H(t_i)\mathbf{x}(t_0)$$
$$\mathbf{y}(t_i) = \tilde{H}(t_i)\Phi(t_i, t_0)\mathbf{x}(t_0)$$

Review of the Stat OD Process

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- Process:
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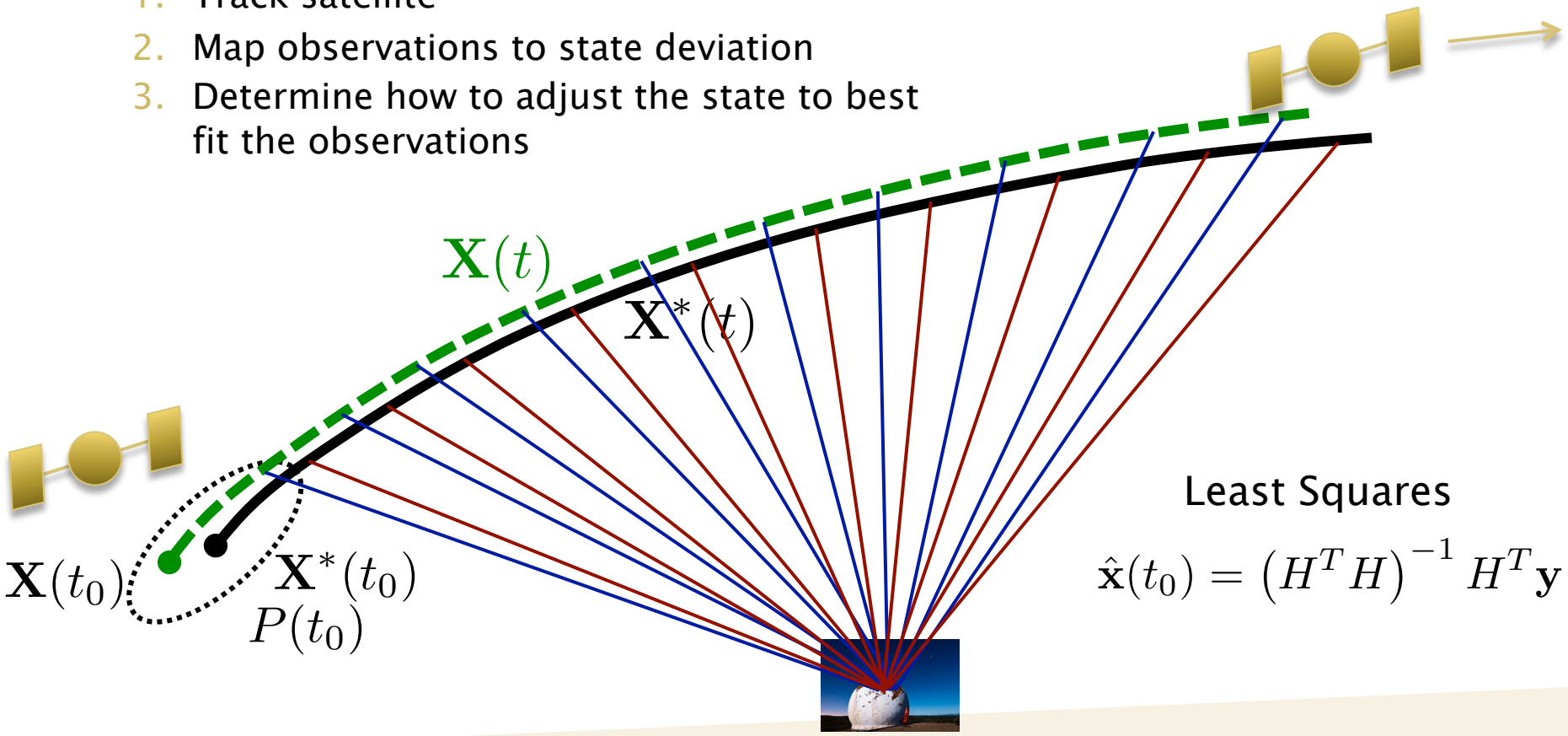
  2. Map observations to state deviation
  3. Determine how to adjust the state to best fit the observations



Least Squares

$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Review of the Stat OD Process

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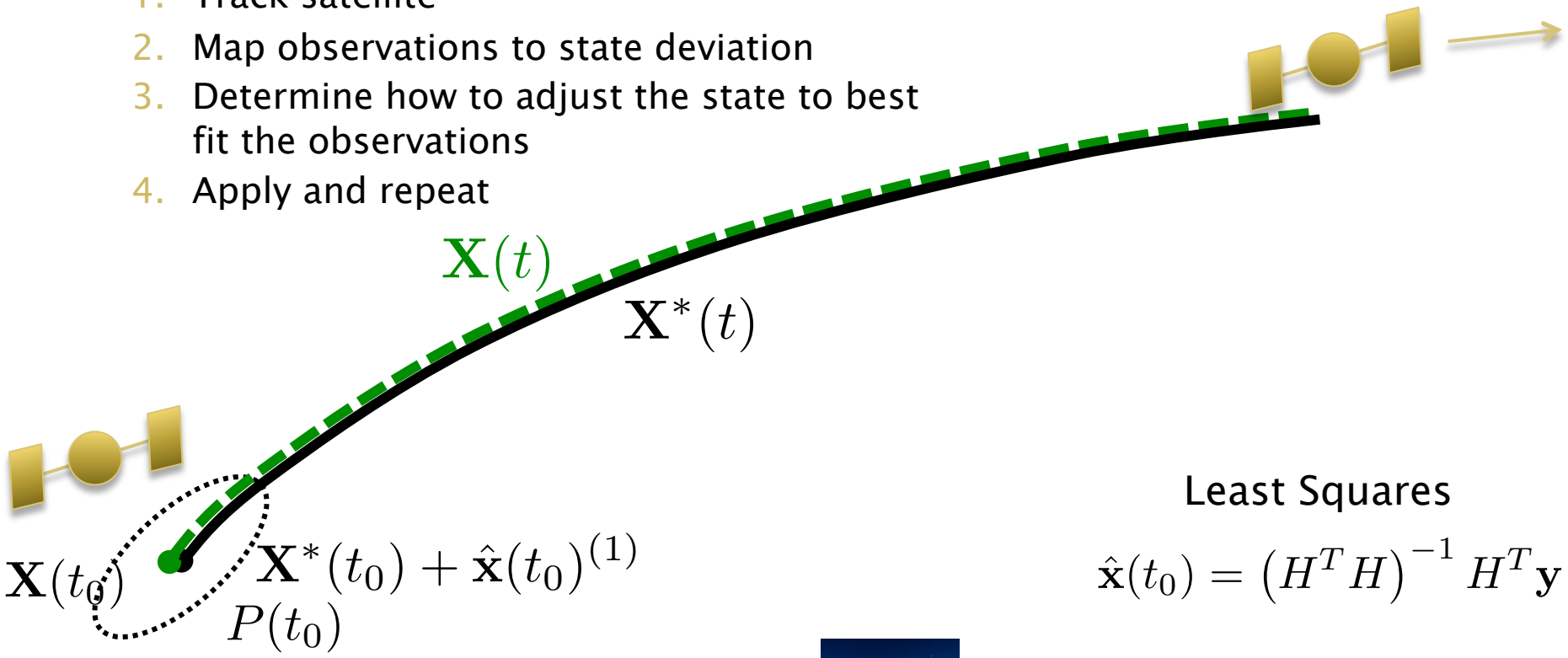
- Process:
  1. Track satellite
  2. Map observations to state deviation
  3. Determine how to adjust the state to best fit the observations
  4. Apply and repeat



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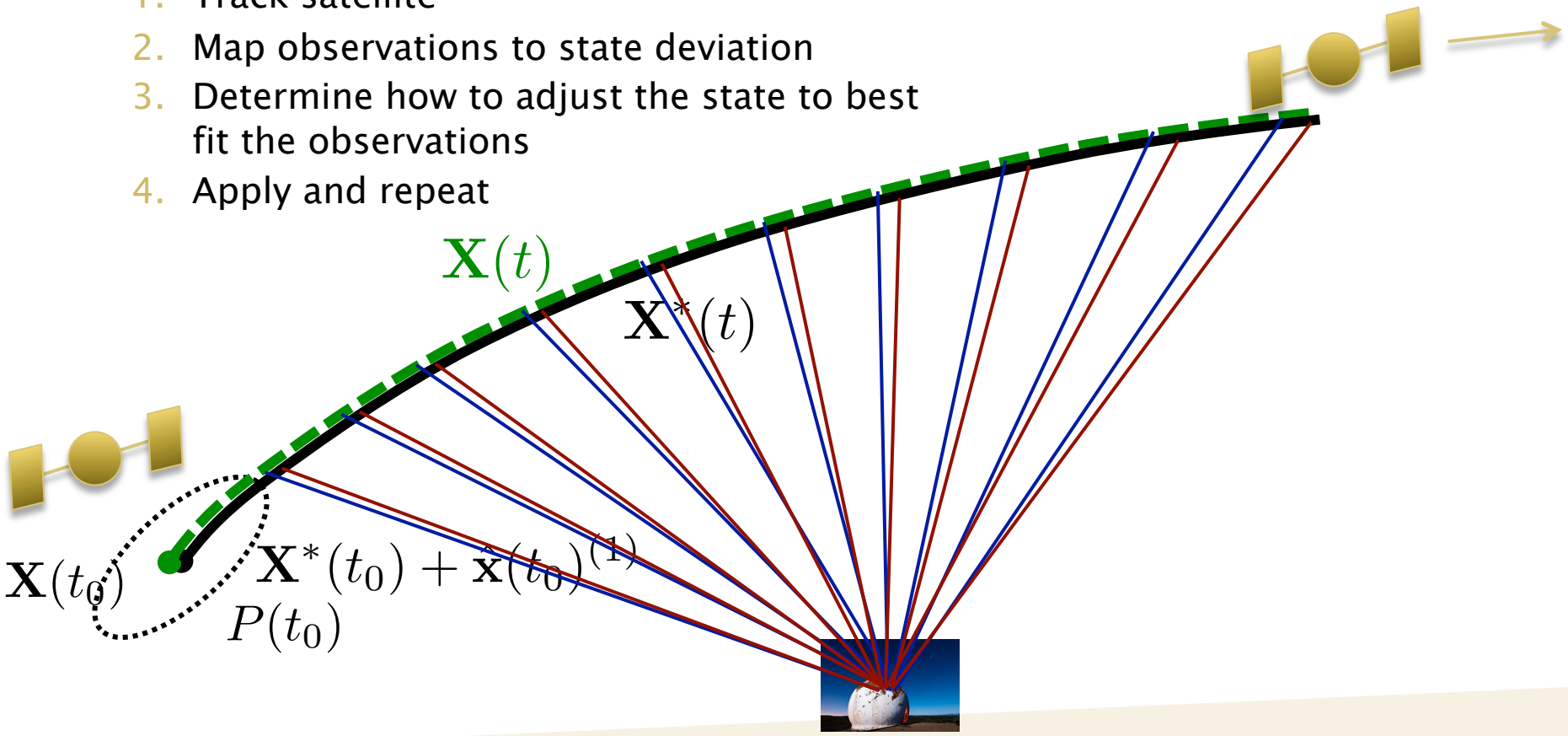

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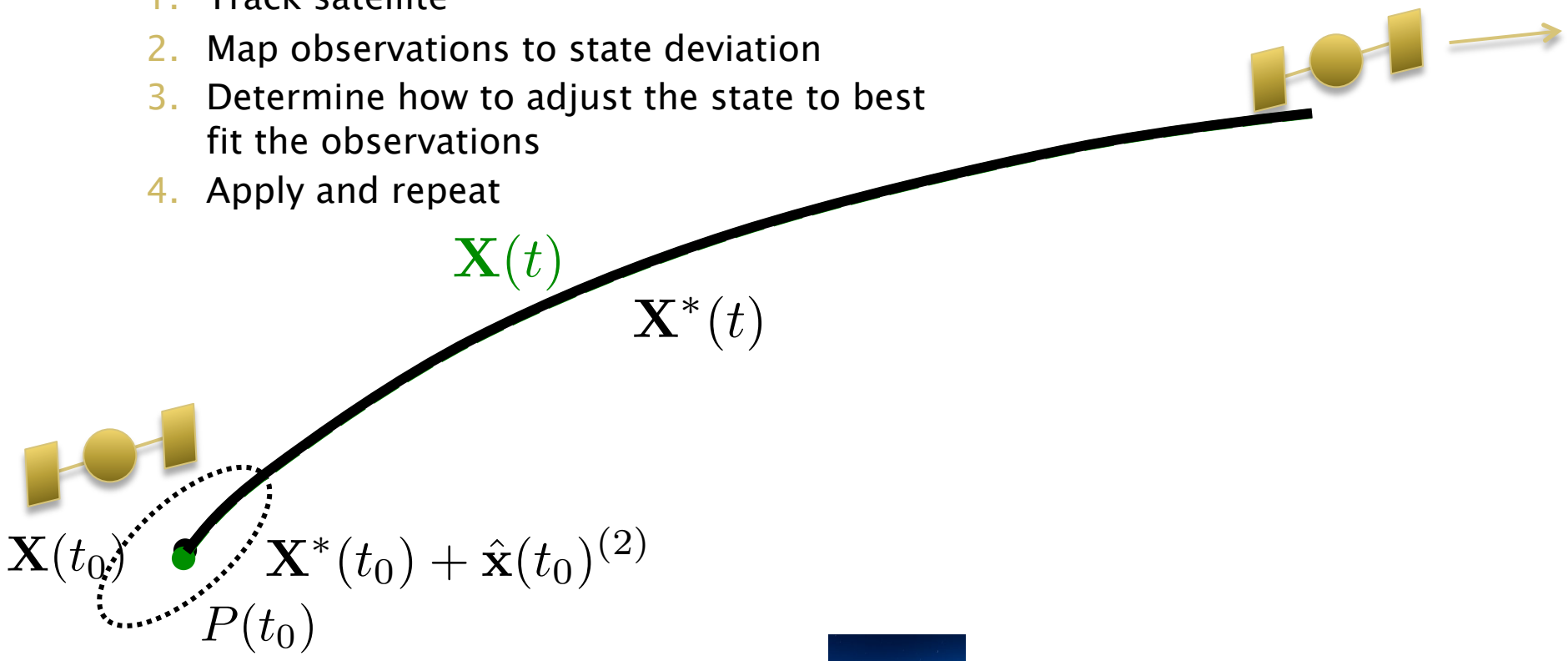

Review of the Stat OD Process

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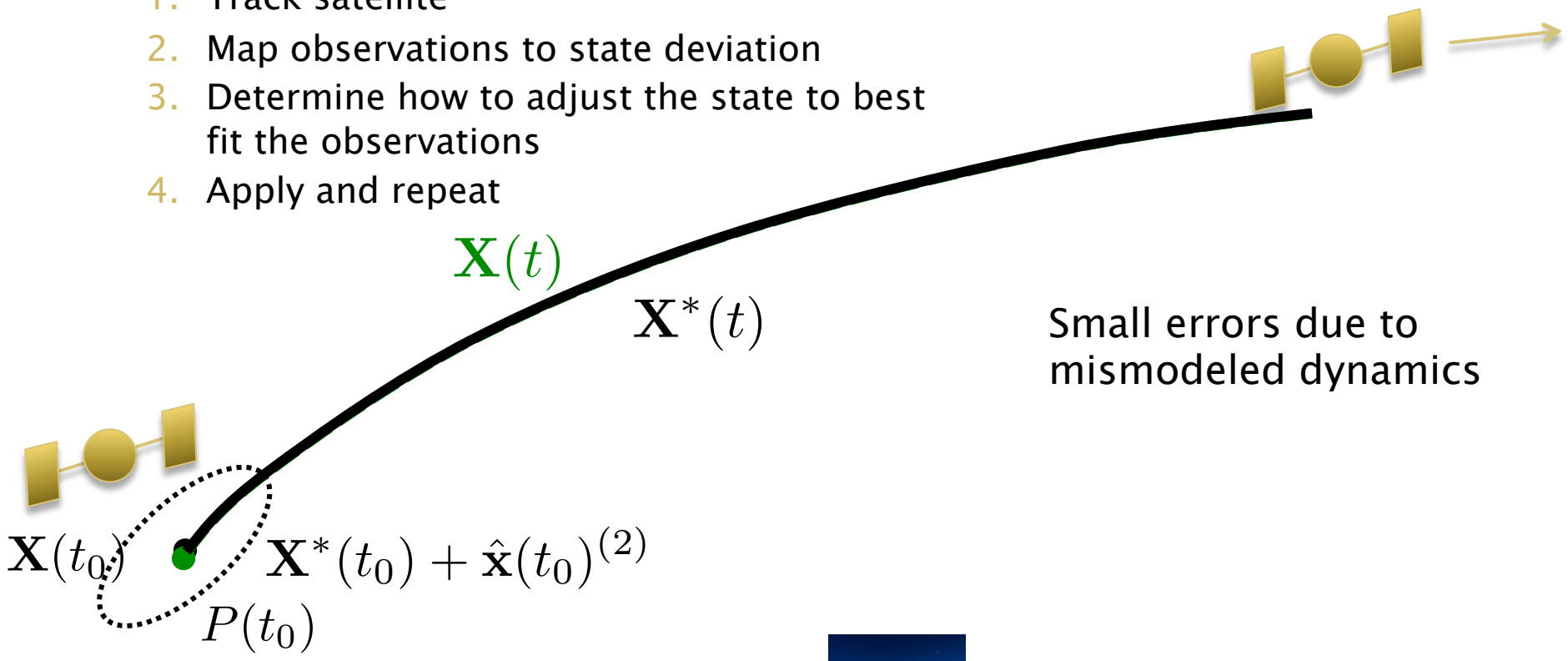

Small errors due to mismodeled dynamics

Review of the Stat OD Process

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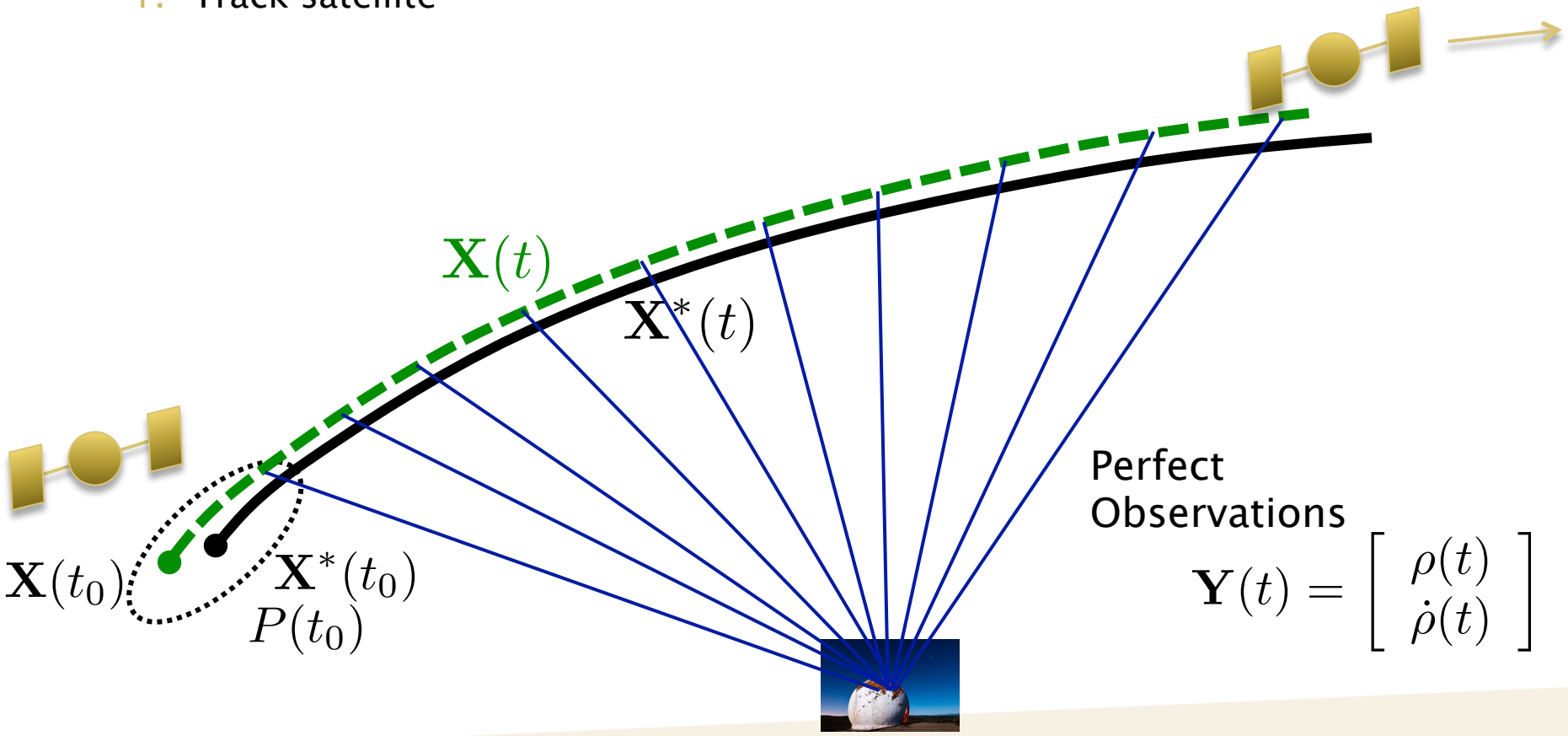
- Process:
  1. Track satellite



Review of the Stat OD Process

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- Process:
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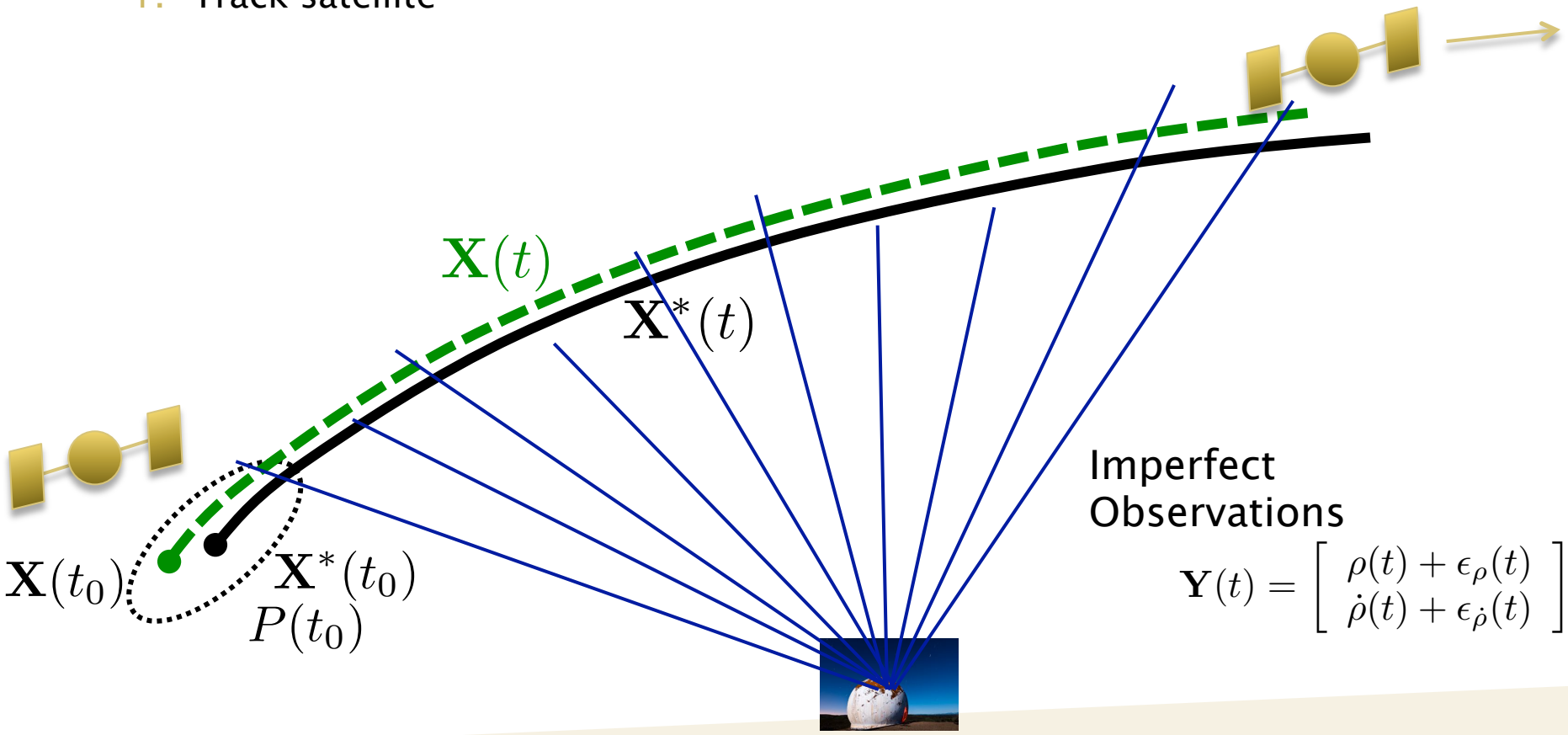

Review of the Stat OD Process

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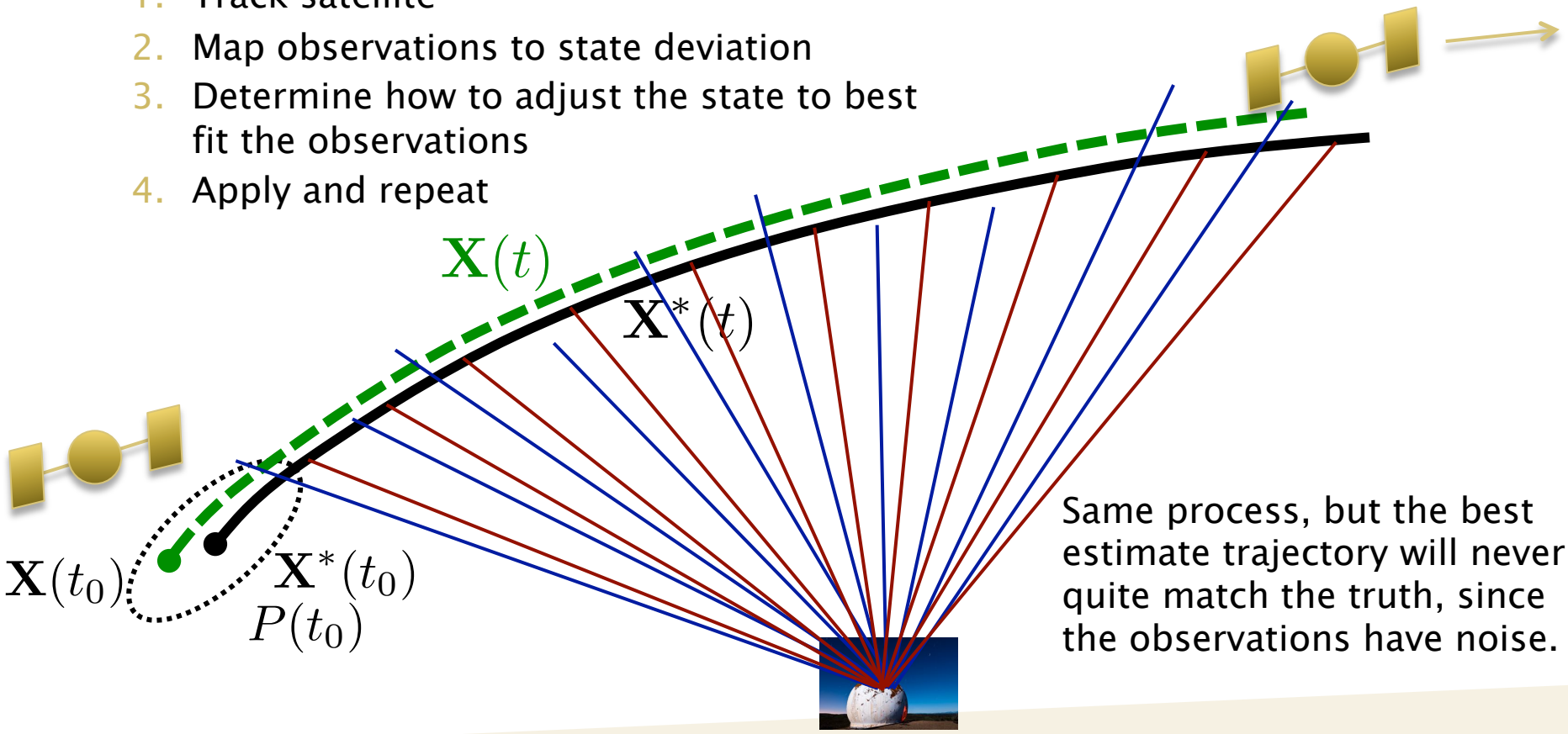

Same process, but the best estimate trajectory will never quite match the truth, since the observations have noise.

Review of the Stat OD Process

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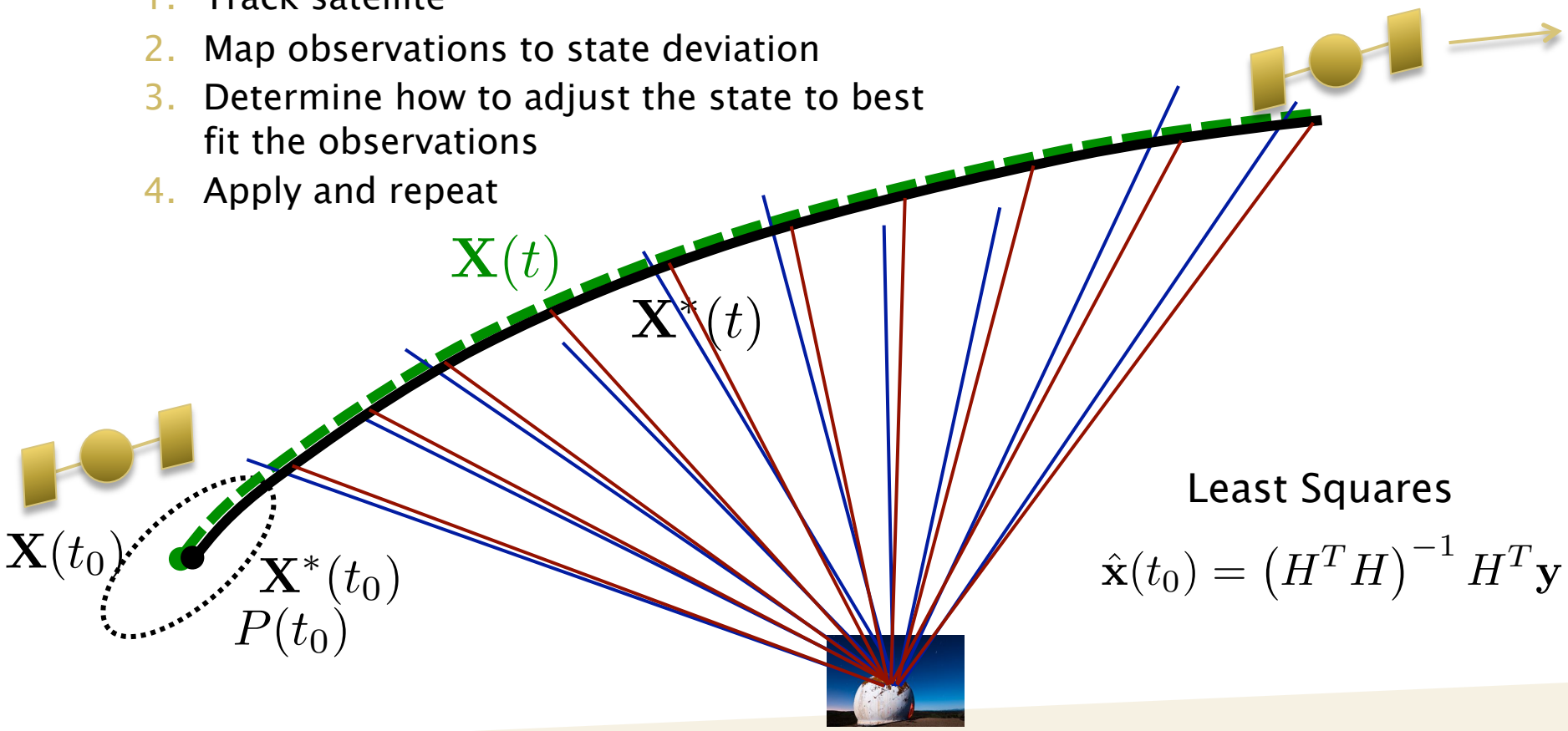

Least Squares

$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Least Squares Options

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- Least Squares

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Weighted Least Squares

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}$$

- Least Squares with *a priori*

$$\hat{\mathbf{x}}_k = \left(H^T W H + \bar{W}_k\right)^{-1} \left(H^T W \mathbf{y} + \bar{W}_k \bar{\mathbf{x}}_k\right)$$

- Min Variance

$$\hat{\mathbf{x}}_k = \left(H^T R^{-1} H\right)^{-1} H^T R^{-1} \mathbf{y}$$

- Min Variance with *a priori*

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

Algorithm Options

- Batch
  - Process all observations at once

$$\hat{\mathbf{x}}_0 = \left(\sum_{i=1}^{p} \left(H_i^T R^{-1} H_i \right) + \bar{P}_0^{-1} \right)^{-1} \left(\sum_{i=1}^{p} \left(H_i^T R^{-1} \mathbf{y}_i \right) + \bar{P}_0^{-1} \bar{\mathbf{x}}_0 \right)$$

- Sequential
  - Process one observation at a time

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1} \right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k \right)$$

Batch Processor

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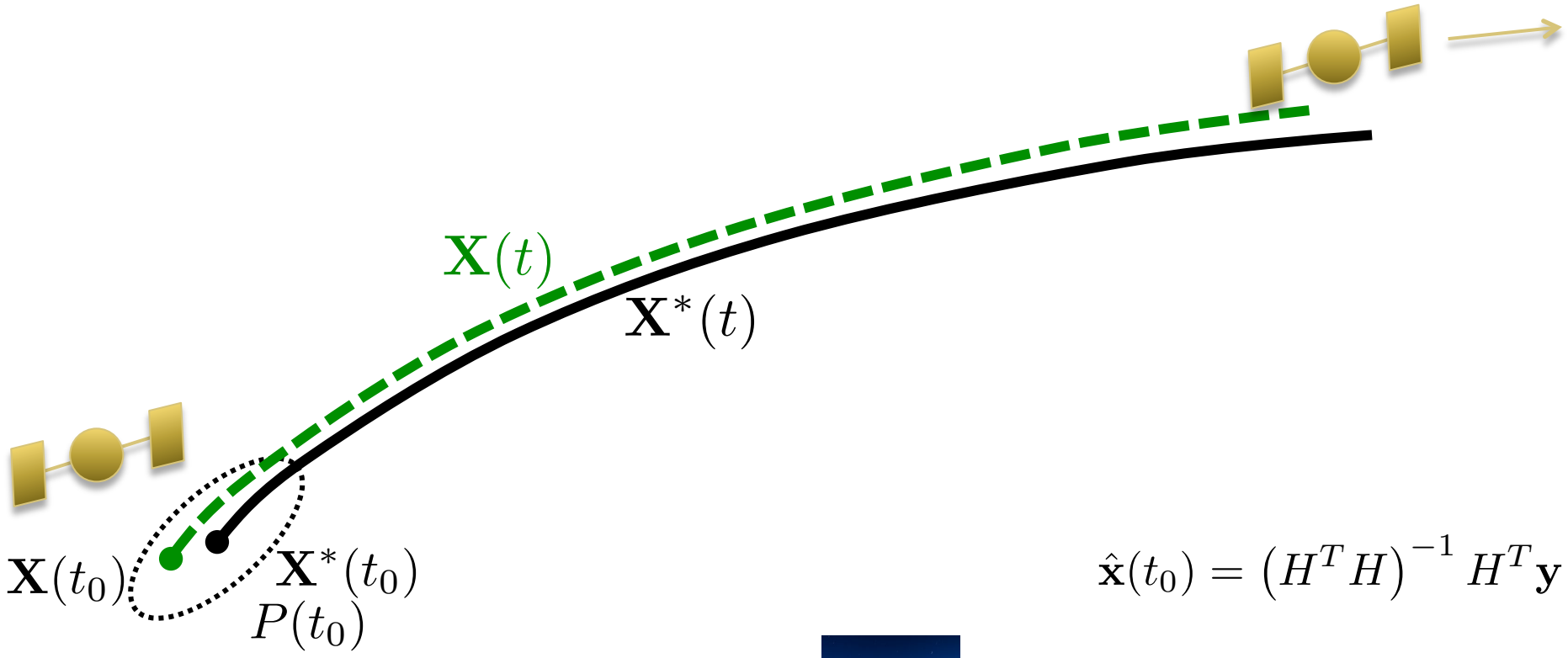
- Collect mapped information



$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Batch Processor

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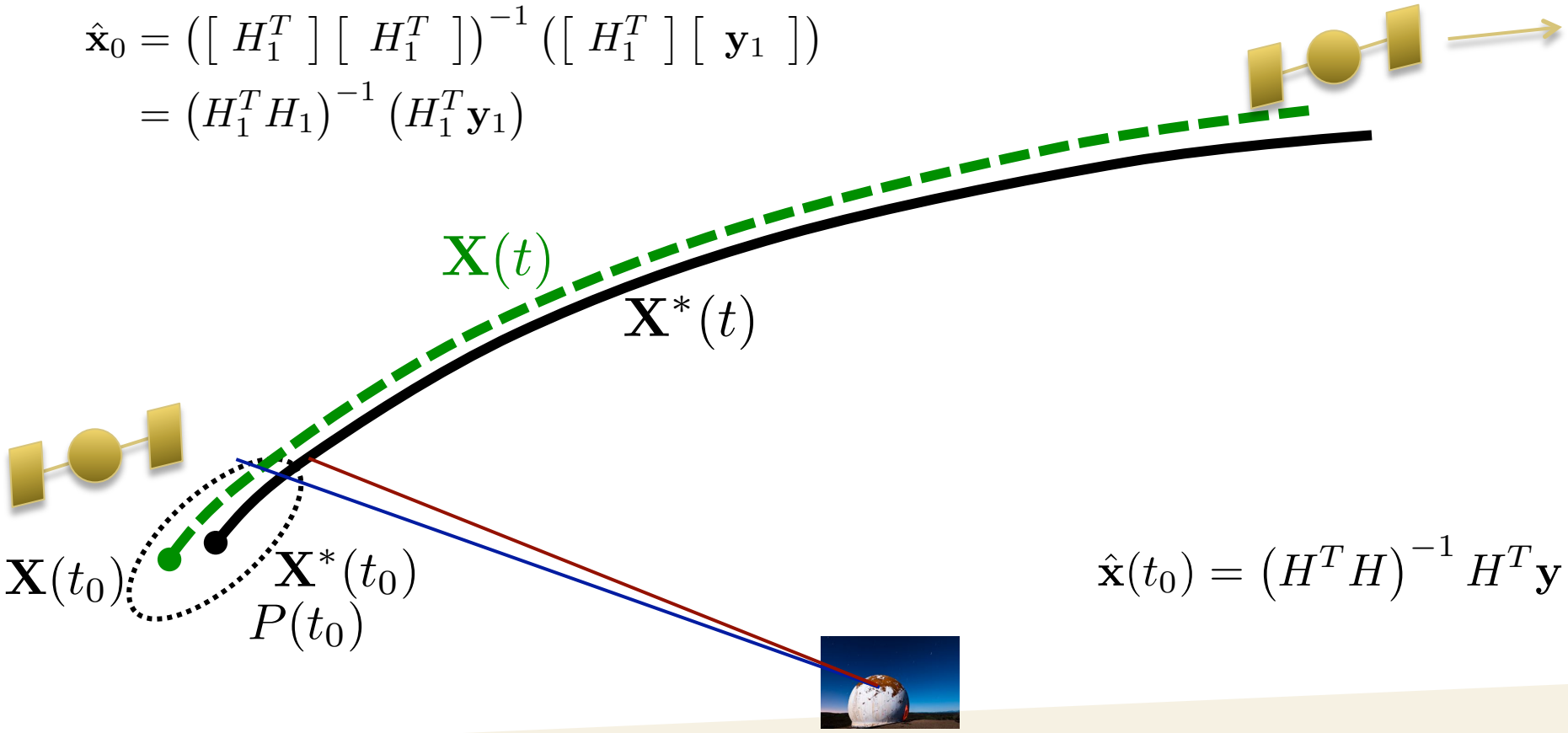
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\left[\; H_1^T \; \right] \left[\; H_1^T \; \right] \right)^{-1} \left(\left[\; H_1^T \; \right] \left[\; \mathbf{y}_1 \; \right] \right)$$
$$= \left(H_1^T H_1 \right)^{-1} \left(H_1^T \mathbf{y}_1 \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H \right)^{-1} H^T \mathbf{y}$$

Batch Processor

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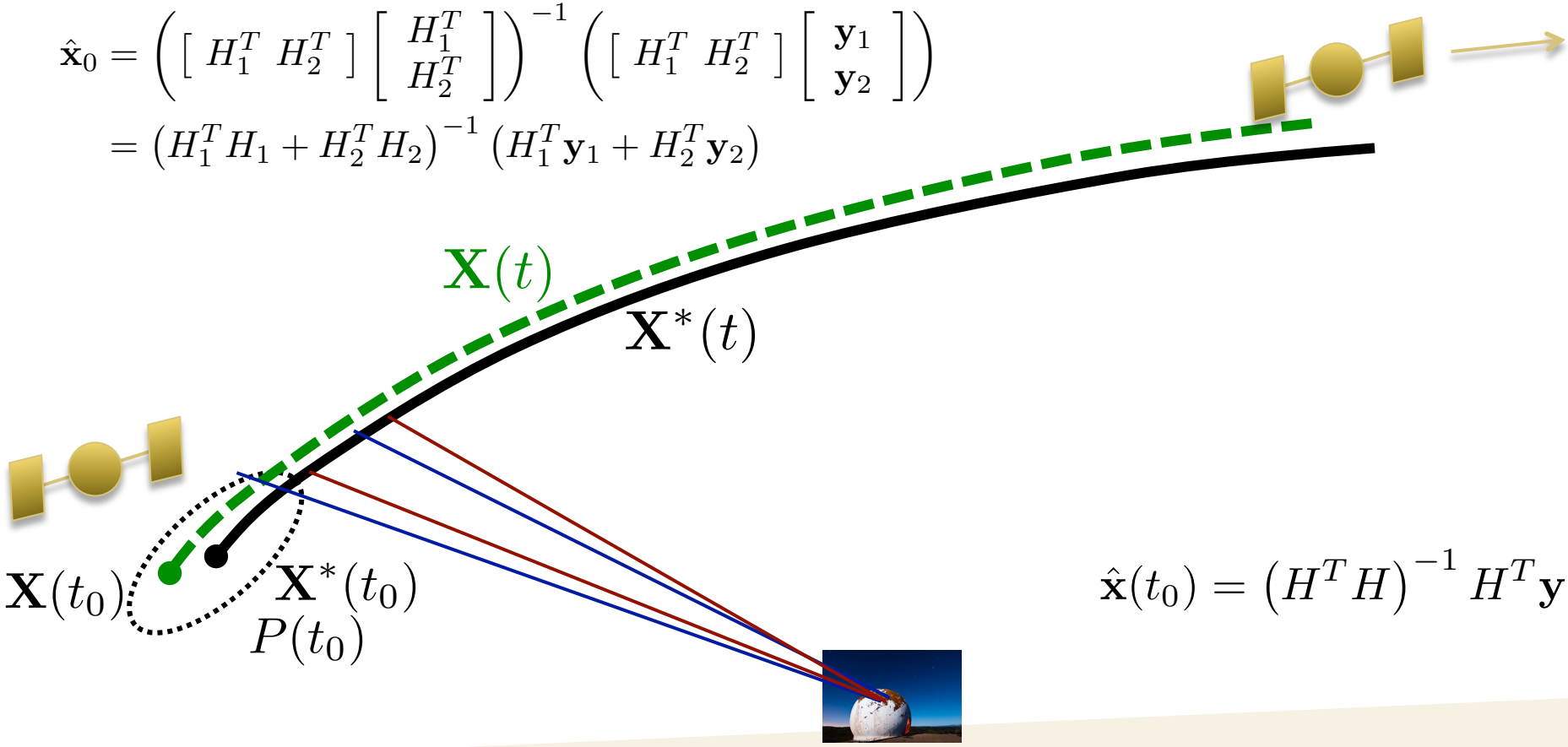
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\begin{bmatrix} H_1^T & H_2^T \end{bmatrix} \begin{bmatrix} H_1^T \\ H_2^T \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} H_1^T & H_2^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \right)$$
$$= \left(H_1^T H_1 + H_2^T H_2 \right)^{-1} \left(H_1^T \mathbf{y}_1 + H_2^T \mathbf{y}_2 \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H \right)^{-1} H^T \mathbf{y}$$

Batch Processor

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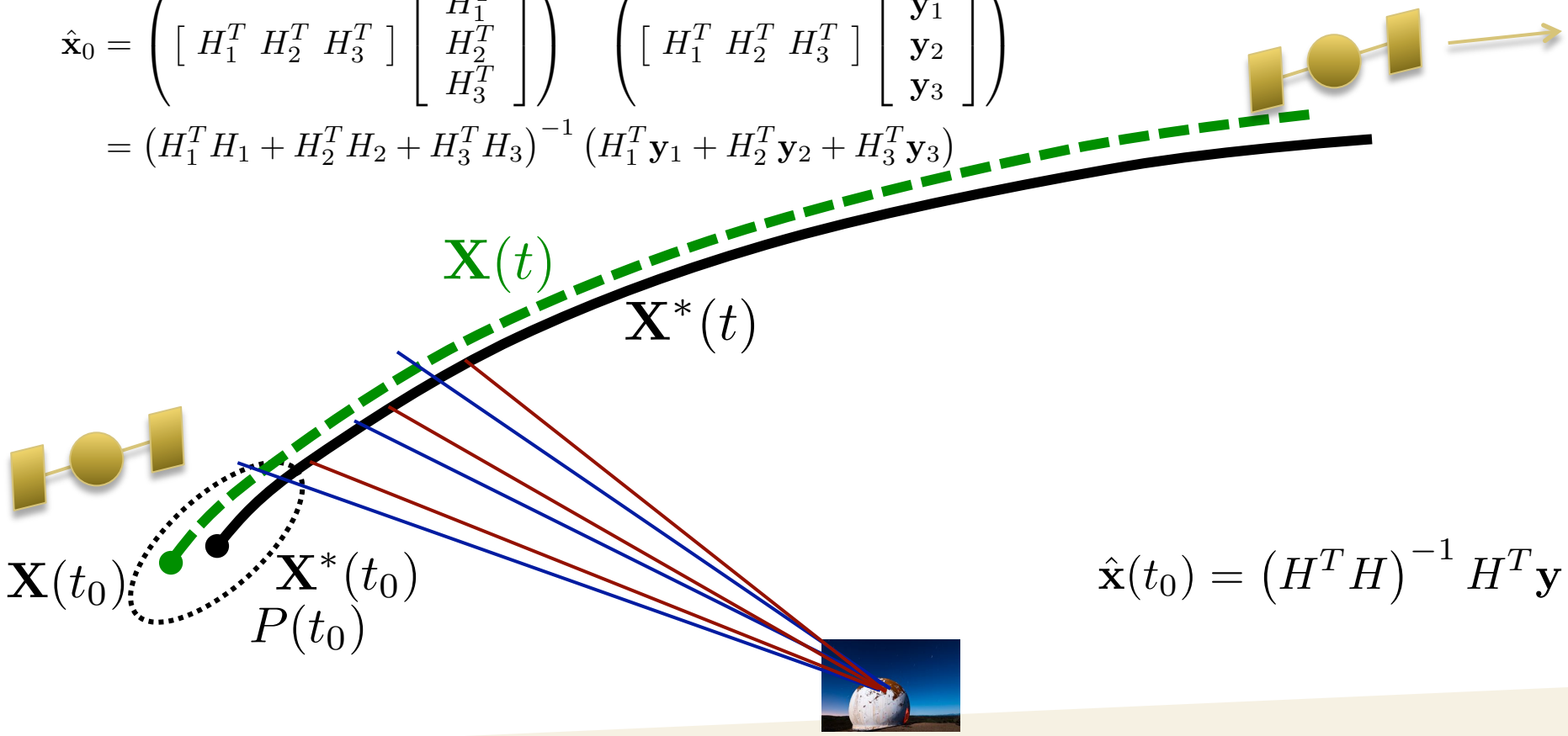
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\begin{bmatrix} H_1^T & H_2^T & H_3^T \end{bmatrix} \begin{bmatrix} H_1^T \\ H_2^T \\ H_3^T \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} H_1^T & H_2^T & H_3^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix} \right)$$

$$= \left(H_1^T H_1 + H_2^T H_2 + H_3^T H_3 \right)^{-1} \left(H_1^T \mathbf{y}_1 + H_2^T \mathbf{y}_2 + H_3^T \mathbf{y}_3 \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H \right)^{-1} H^T \mathbf{y}$$

Batch Processor

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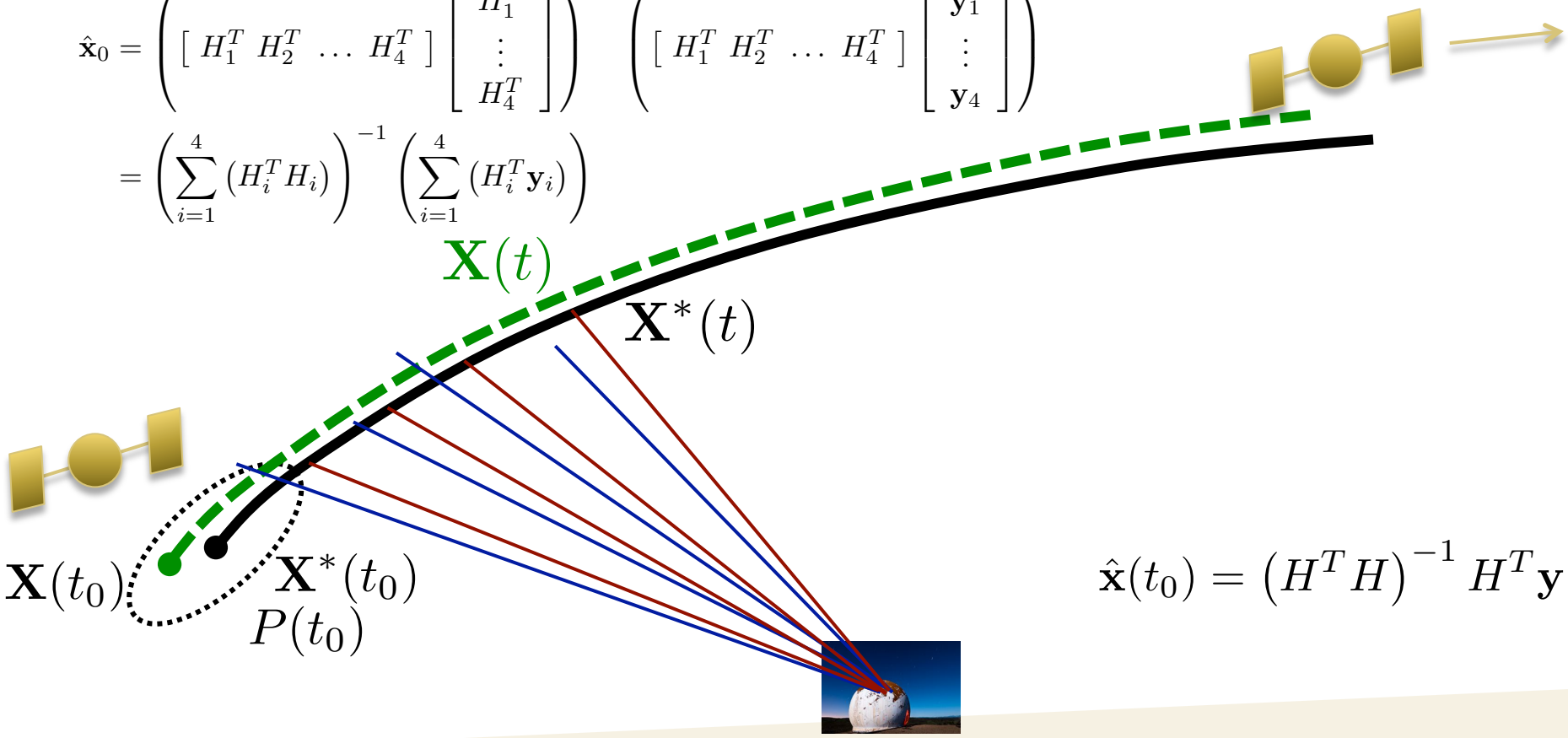
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\begin{bmatrix} H_1^T & H_2^T & \dots & H_4^T \end{bmatrix} \begin{bmatrix} H_1^T \\ \vdots \\ H_4^T \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} H_1^T & H_2^T & \dots & H_4^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_4 \end{bmatrix} \right)$$

$$= \left(\sum_{i=1}^{4} (H_i^T H_i) \right)^{-1} \left(\sum_{i=1}^{4} (H_i^T \mathbf{y}_i) \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Batch Processor

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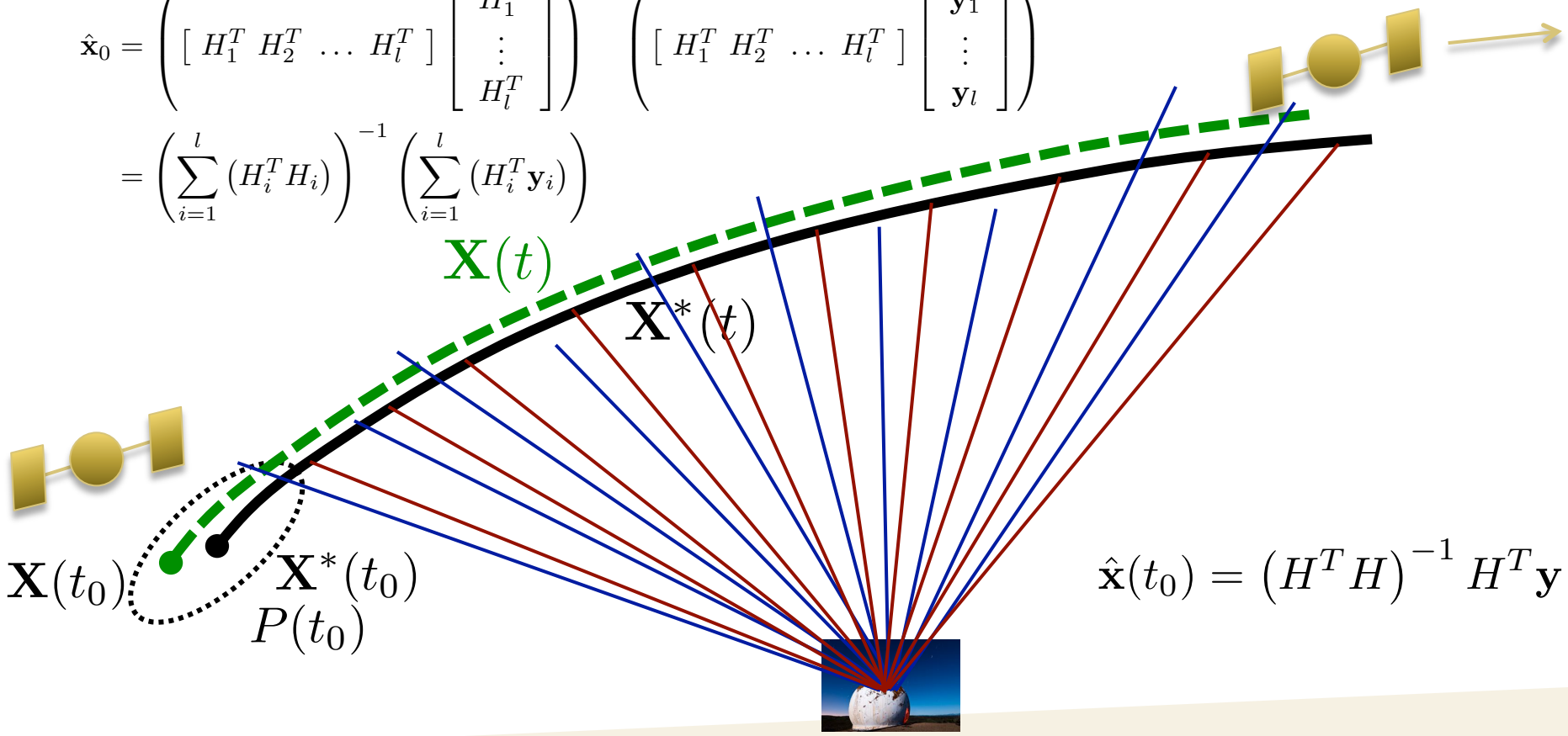
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\begin{bmatrix} H_1^T & H_2^T & \dots & H_l^T \end{bmatrix} \begin{bmatrix} H_1^T \\ \vdots \\ H_l^T \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} H_1^T & H_2^T & \dots & H_l^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_l \end{bmatrix} \right)$$

$$= \left(\sum_{i=1}^{l} \left(H_i^T H_i \right) \right)^{-1} \left(\sum_{i=1}^{l} \left(H_i^T \mathbf{y}_i \right) \right)$$



- (Break)

- TAs will go through an end-to-end Batch run to demo equations, etc.

- Next up: Kalman Filter

Sequential Processor

- Consider

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

- Rather than mapping all observations to one epoch and processing them simultaneously, what if we processed each separately and mapped the best estimate through each?

Sequential Processor

- Consider

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

- Rather than mapping all observations to one epoch and processing them simultaneously, what if we processed each separately and mapped the best estimate through each?

$$\bar{\mathbf{x}}_k = \Phi(t_k, t_j)\hat{\mathbf{x}}_j$$
$$\bar{P}_k = \Phi(t_k, t_j) P_j \Phi^T(t_k, t_j)$$

Sequential Processor

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- Given an *a priori* state and covariance, we know how to generate a new estimate of the state:

$$\hat{\mathbf{x}}_k = \left(\tilde{H}_k^T R^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1} \left(\tilde{H}_k^T R^{-1} \mathbf{y}_k + \bar{P}_k^{-1} \bar{\mathbf{x}}_k\right)$$

- Need a way to generate the *a posteriori* covariance matrix as well.

- Recall

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1}\right)^{-1}$$

- The trouble is inverting the *n x n* matrix.

Sequential Processor

- We can use the Schur Identity and a bunch of math (see Section 4.7) and obtain:

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1} \right)^{-1}$$

$$P_k = \bar{P}_k - \bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1} \tilde{H}_k \bar{P}_k$$

Sequential Processor

- We can use the Schur Identity and a bunch of math (see Section 4.7) and obtain:

$$P_k = \Lambda_k^{-1} = \left(\tilde{H}_k^T R_k^{-1} \tilde{H}_k + \bar{P}_k^{-1} \right)^{-1}$$

$$P_k = \bar{P}_k - \underbrace{\bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1}}_{K_k} \tilde{H}_k \bar{P}_k$$

Kalman Gain

Sequential Processor

- After some more math, we can simplify to obtain:

$$P_k = \left[I - K_k \tilde{H}_k^T \right] \bar{P}_k$$

$$\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + K_k \left[\mathbf{y}_k - \tilde{H}_k \bar{\mathbf{x}}_k \right]$$

Sequential Algorithm

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1. Initialize the first run
2. Start at the reference epoch
3. Time Update
   - Integrate from the current time to the next time of interest
   - Map the state estimate and the covariance to the new time
4. Measurement Update
   - If there is a new measurement, process it.
   - Update the state estimate and covariance with this new information

- Repeat 3-4 until all measurements have been processed and all times of interest have been recorded.
- Optional: Map the estimate and covariance back to the reference epoch and iterate the whole process.

Sequential Algorithm

- Initialization

Given: $\hat{\mathbf{x}}_{k-1}$, P_{k-1}, $\mathbf{X}^*_{k-1}$, and R_k, and the observation $\mathbf{Y}_k$, at t_k (at the initial time t_0, these would be $\mathbf{X}^*_0$, $\hat{\mathbf{x}}_0$, and P_0).

Sequential Algorithm

- Time Update
  - Integration

$$\dot{\mathbf{X}}^* = F(\mathbf{X}^*, t), \qquad \mathbf{X}^*(t_{k-1}) = \mathbf{X}^*_{k-1}$$
$$\dot{\Phi}(t, t_{k-1}) = A(t)\Phi(t, t_{k-1}), \qquad \Phi(t_{k-1}, t_{k-1}) = I.$$

  - Mapping

$$\overline{\mathbf{x}}_k = \Phi(t_k, t_{k-1})\hat{\mathbf{x}}_{k-1} \qquad \overline{P}_k = \Phi(t_k, t_{k-1})P_{k-1}\Phi^T(t_k, t_{k-1})$$

Sequential Algorithm

- Measurement Update
  - Collect measurement

$$\mathbf{y}_k = \mathbf{Y}_k - G(\mathbf{X}_k^*, t_k) \qquad \widetilde{H}_k = \frac{\partial G(\mathbf{X}_k^*, t_k)}{\partial \mathbf{X}}$$

  - Compute update

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$
$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]$$
$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k.$$

Sequential Processor

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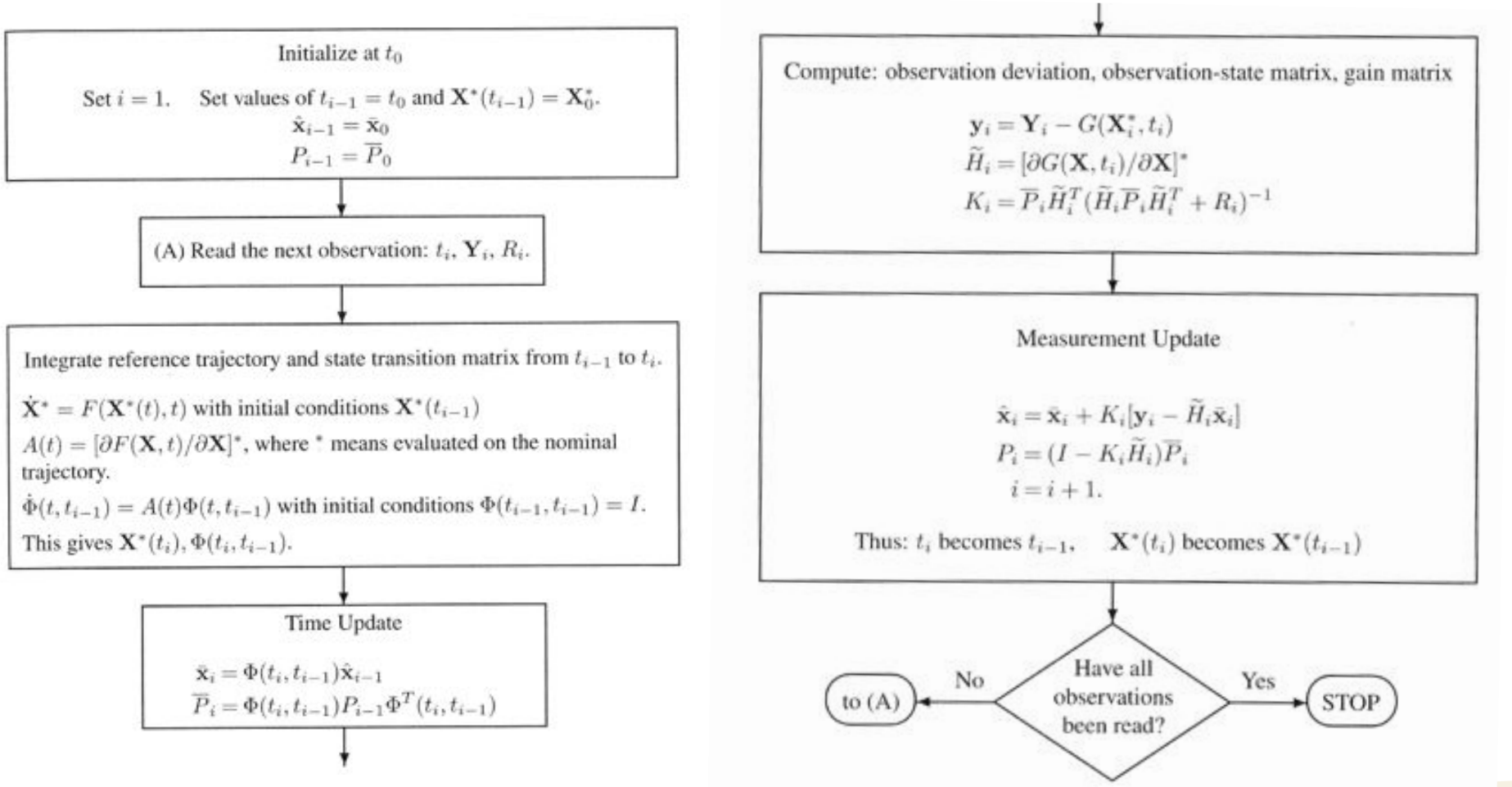
- Repeat just like the Batch
  - Replace the reference trajectory with the new best estimate.
  - Make sure to update the *a priori* state deviation vector to retain any information.
  - Recompute all observation residuals

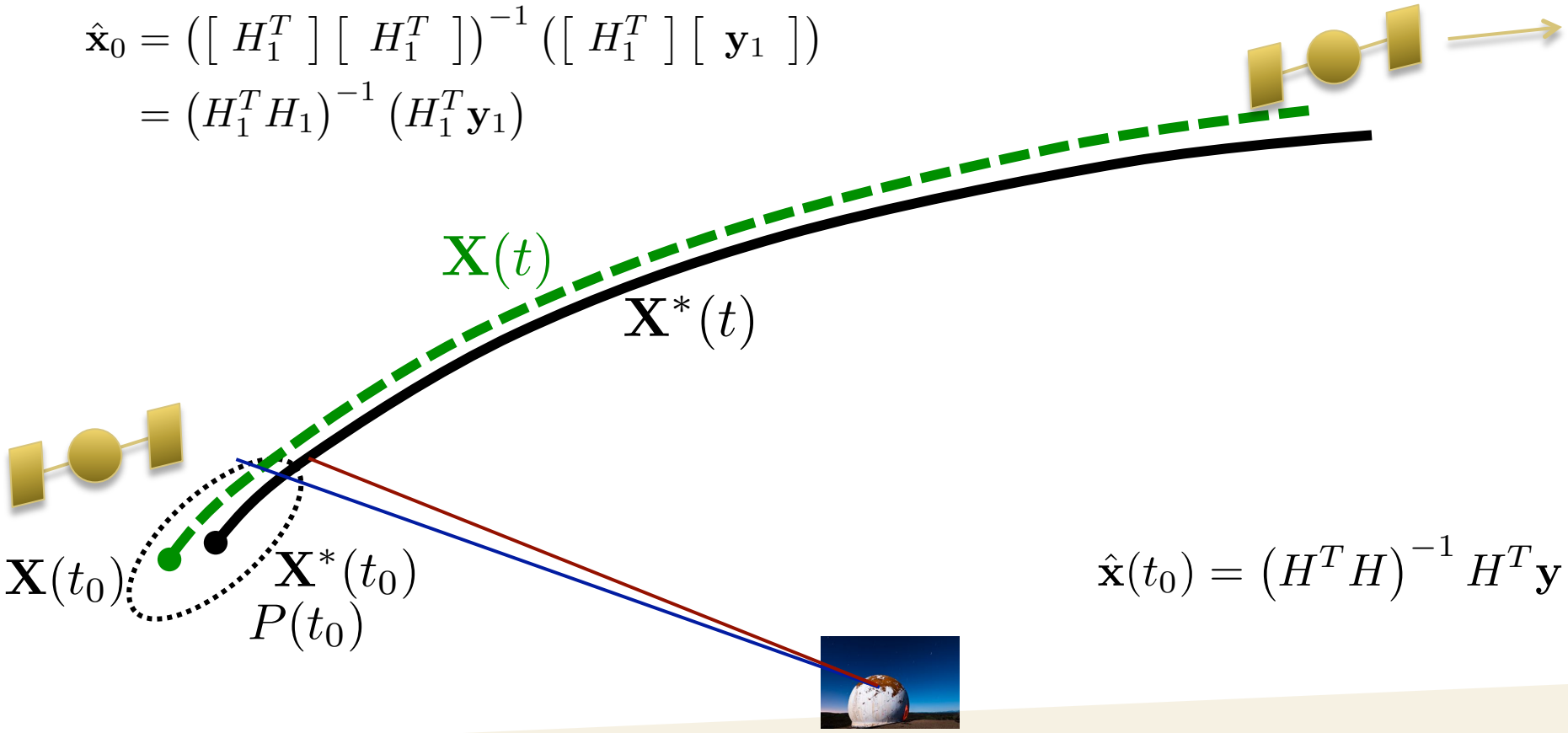
Sequential Flow-Chart



Batch Processor

- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\left[\; H_1^T \; \right] \left[\; H_1^T \; \right] \right)^{-1} \left(\left[\; H_1^T \; \right] \left[\; \mathbf{y}_1 \; \right] \right)$$
$$= \left(H_1^T H_1 \right)^{-1} \left(H_1^T \mathbf{y}_1 \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H \right)^{-1} H^T \mathbf{y}$$

Batch Processor

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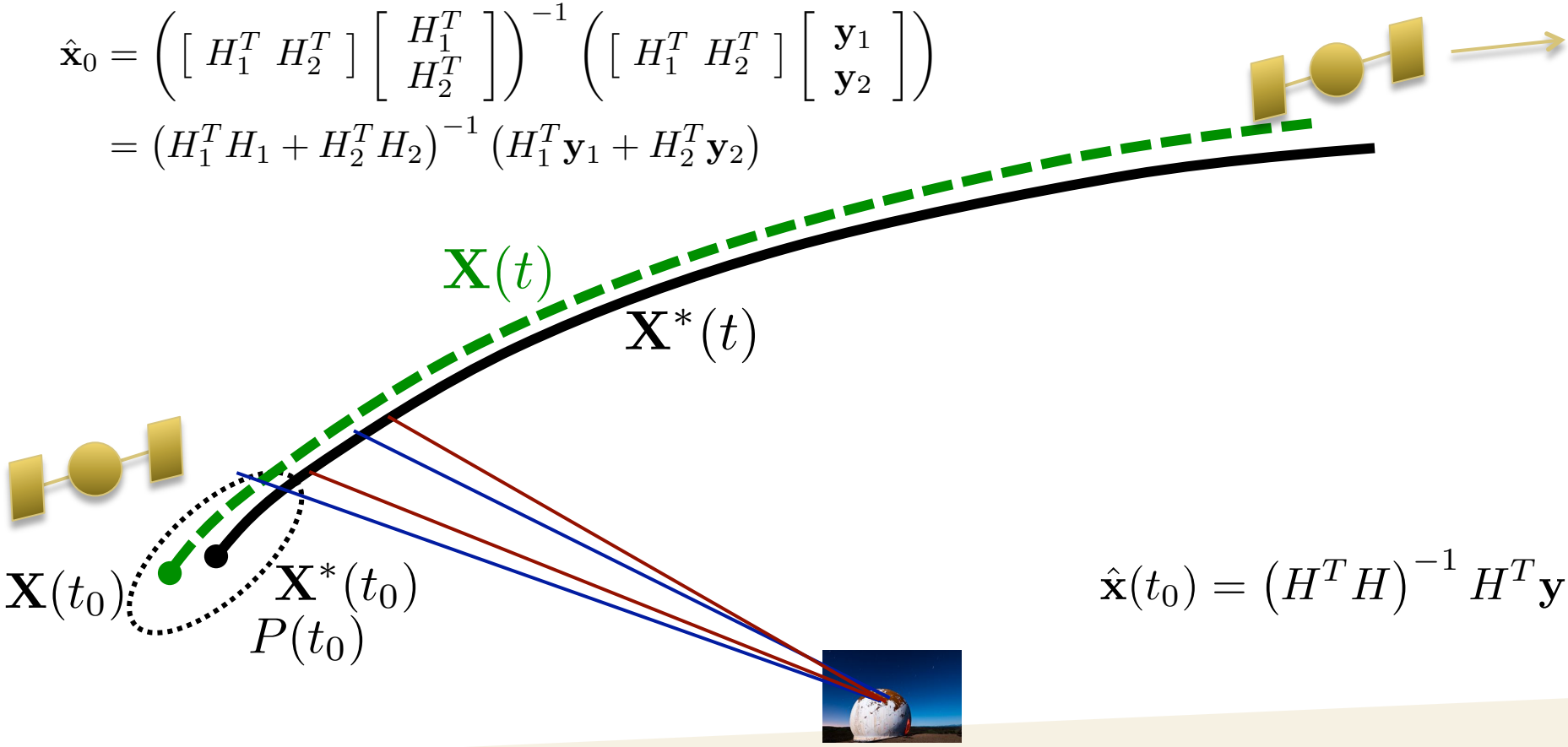
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\begin{bmatrix} H_1^T & H_2^T \end{bmatrix} \begin{bmatrix} H_1^T \\ H_2^T \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} H_1^T & H_2^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \right)$$
$$= \left(H_1^T H_1 + H_2^T H_2 \right)^{-1} \left(H_1^T \mathbf{y}_1 + H_2^T \mathbf{y}_2 \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H \right)^{-1} H^T \mathbf{y}$$

Batch Processor

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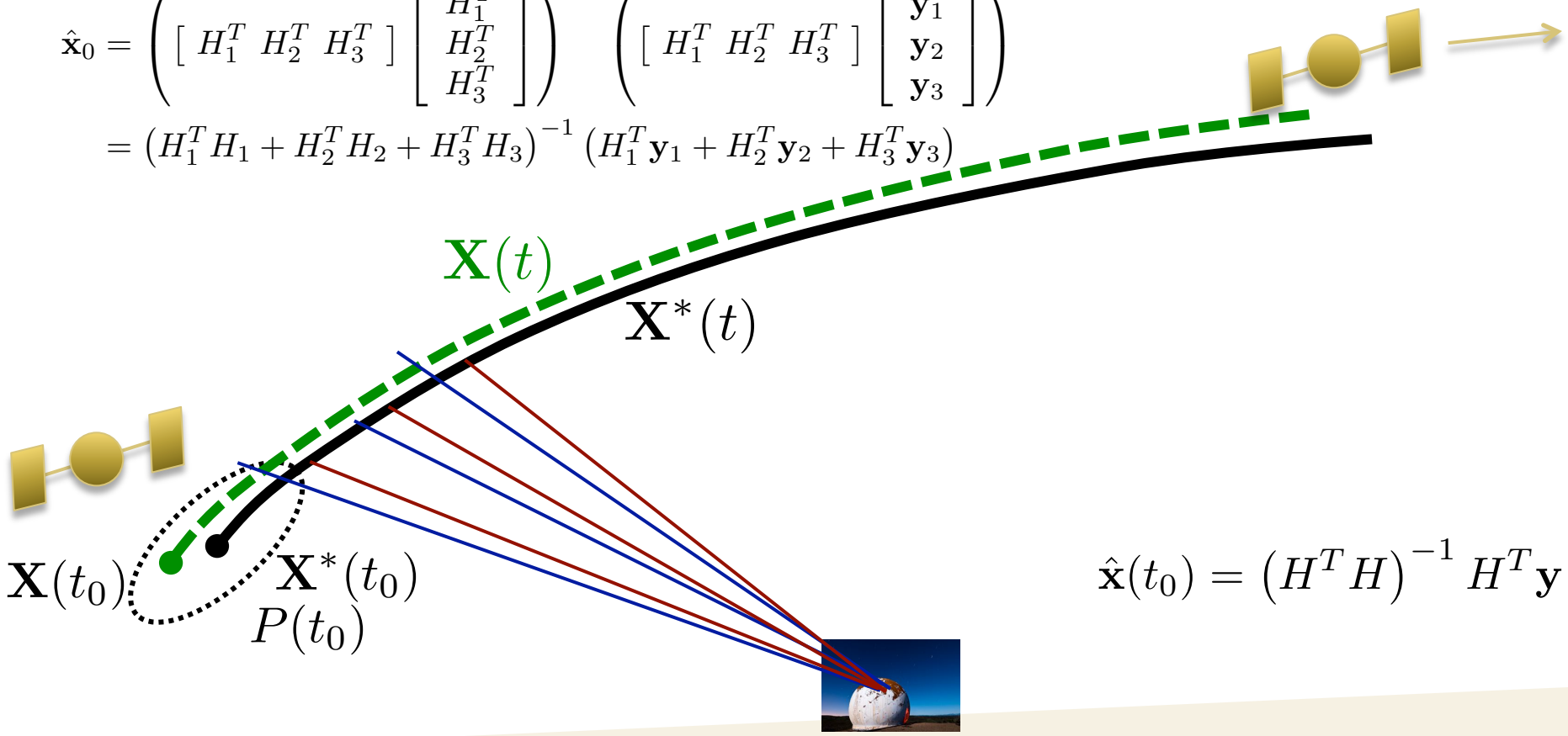
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\begin{bmatrix} H_1^T & H_2^T & H_3^T \end{bmatrix} \begin{bmatrix} H_1^T \\ H_2^T \\ H_3^T \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} H_1^T & H_2^T & H_3^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix} \right)$$

$$= \left(H_1^T H_1 + H_2^T H_2 + H_3^T H_3 \right)^{-1} \left(H_1^T \mathbf{y}_1 + H_2^T \mathbf{y}_2 + H_3^T \mathbf{y}_3 \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H \right)^{-1} H^T \mathbf{y}$$

Batch Processor

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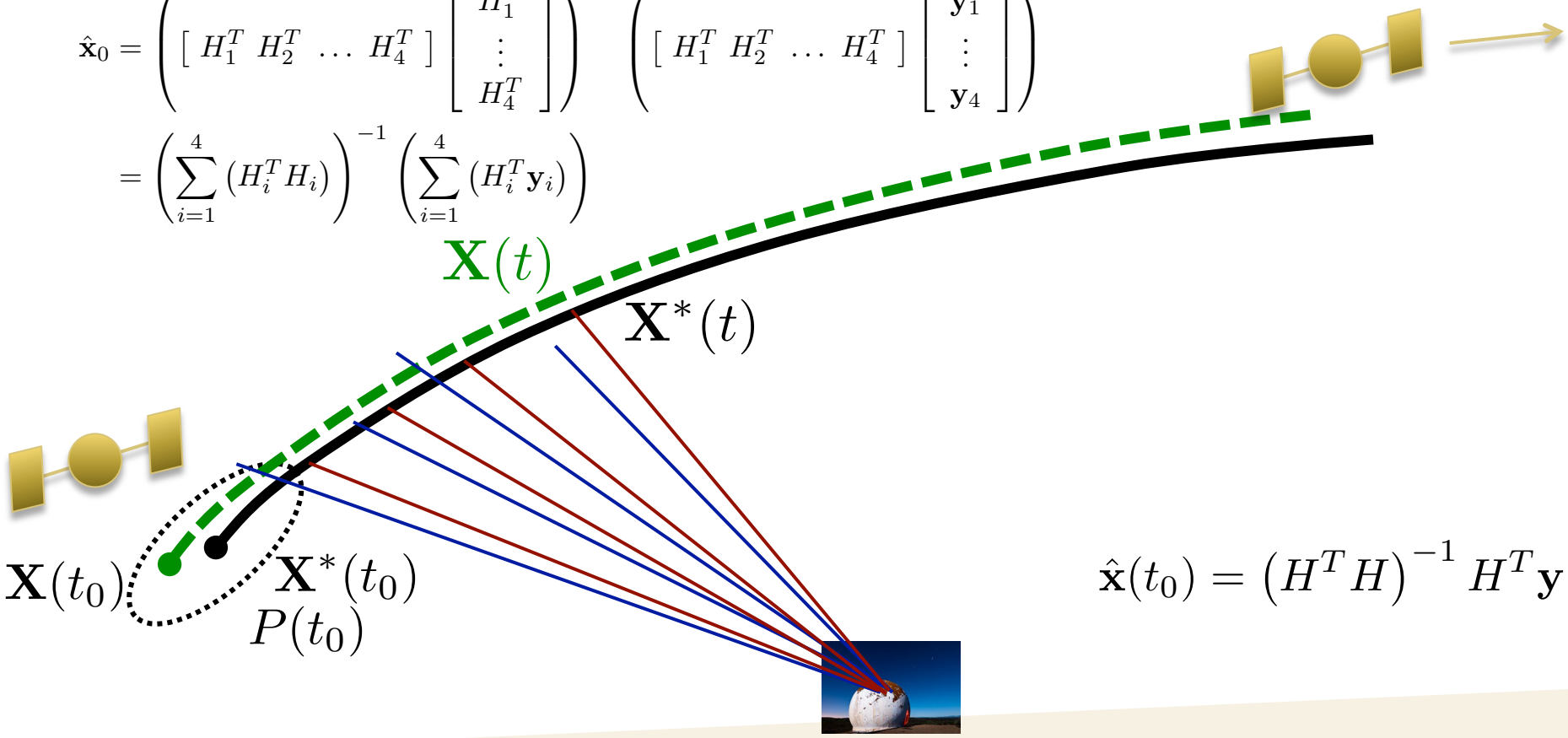
- Collect mapped information

$$\hat{\mathbf{x}}_0 = \left(\begin{bmatrix} H_1^T & H_2^T & \dots & H_4^T \end{bmatrix} \begin{bmatrix} H_1^T \\ \vdots \\ H_4^T \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} H_1^T & H_2^T & \dots & H_4^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_4 \end{bmatrix} \right)$$

$$= \left(\sum_{i=1}^{4} (H_i^T H_i) \right)^{-1} \left(\sum_{i=1}^{4} (H_i^T \mathbf{y}_i) \right)$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Kalman Filter

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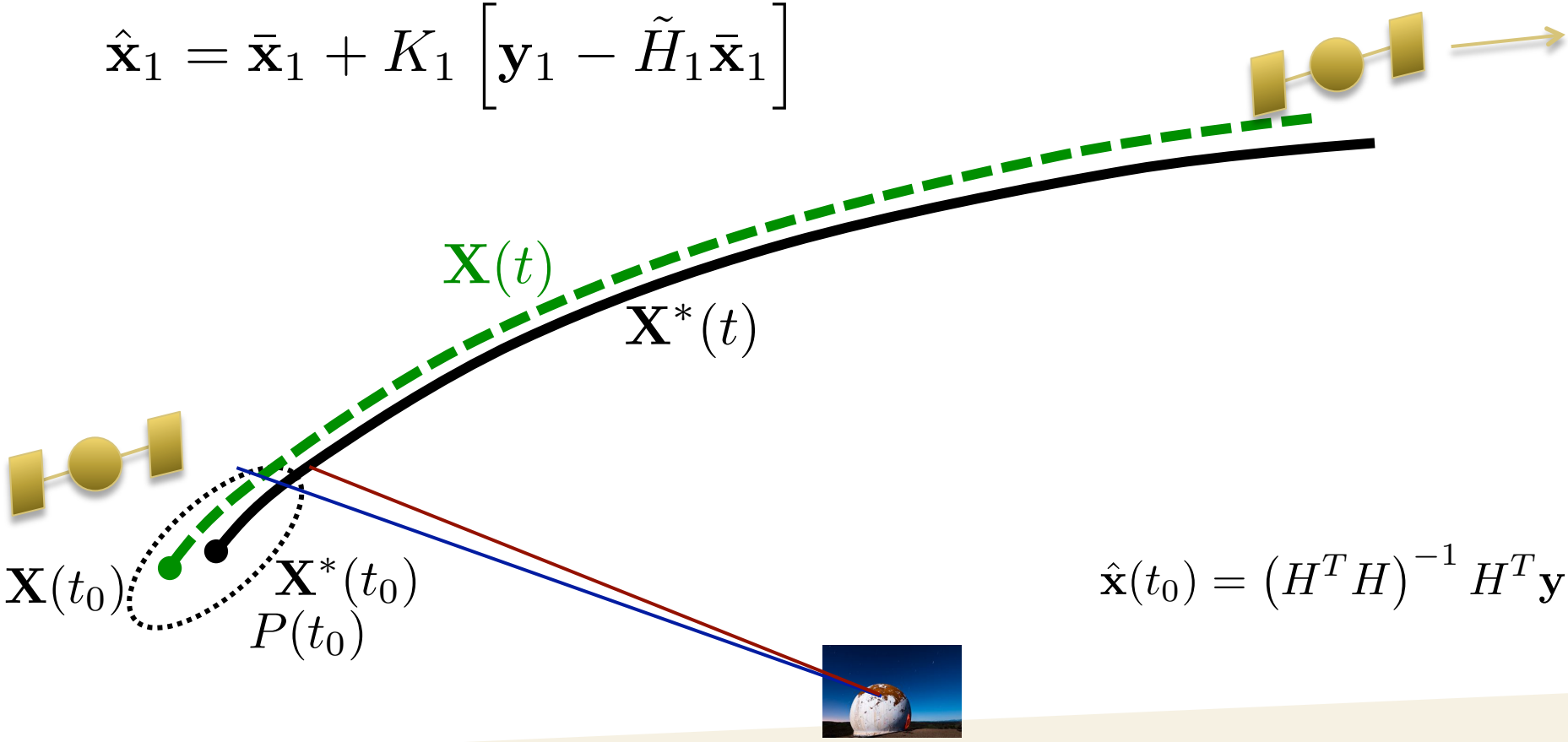
- Collect mapped information

$$\hat{\mathbf{x}}_1 = \bar{\mathbf{x}}_1 + K_1 \left[\mathbf{y}_1 - \tilde{H}_1 \bar{\mathbf{x}}_1 \right]$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Kalman Filter

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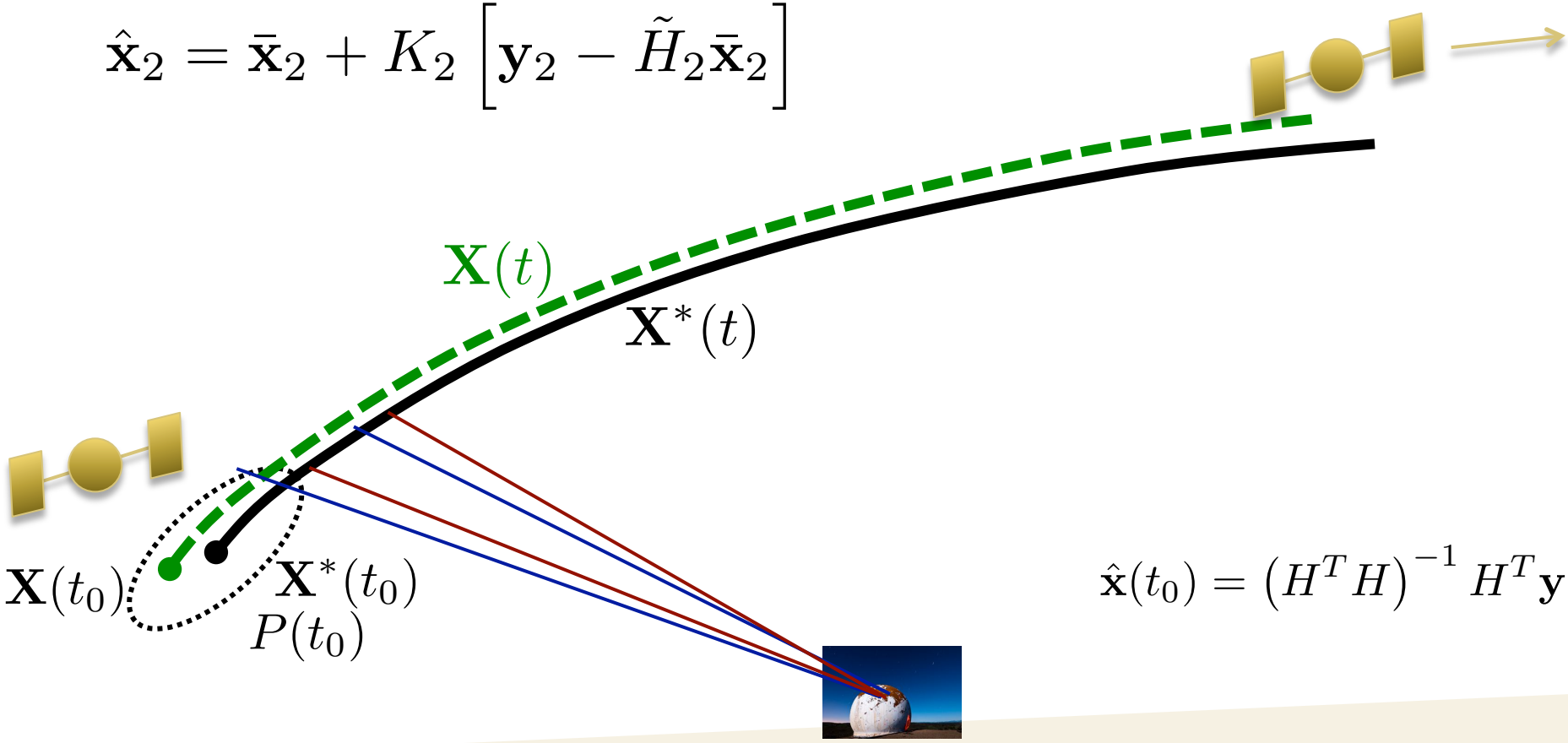
- Collect mapped information

$$\hat{\mathbf{x}}_2 = \bar{\mathbf{x}}_2 + K_2 \left[\mathbf{y}_2 - \tilde{H}_2 \bar{\mathbf{x}}_2 \right]$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Kalman Filter

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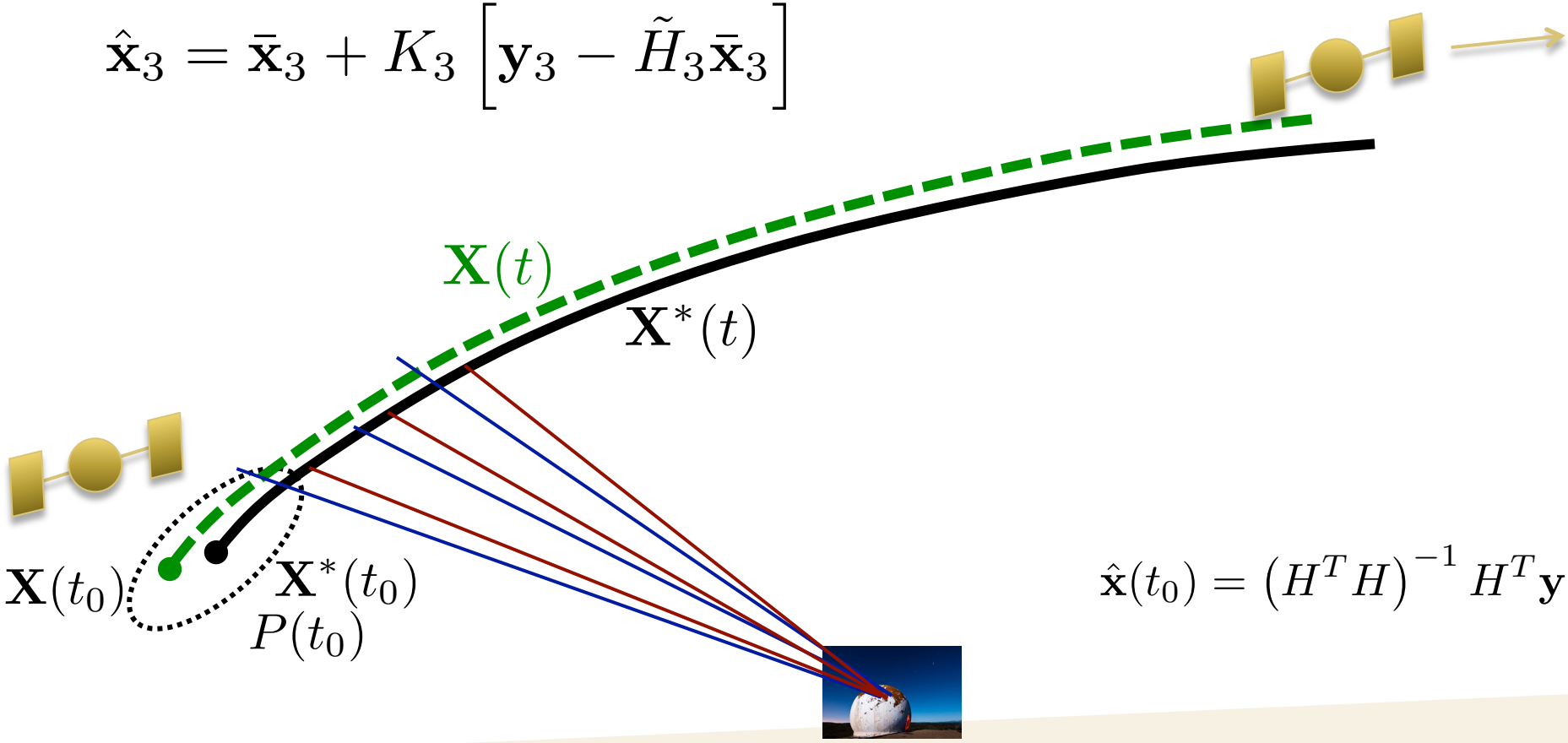
- Collect mapped information

$$\hat{\mathbf{x}}_3 = \bar{\mathbf{x}}_3 + K_3 \left[\mathbf{y}_3 - \tilde{H}_3 \bar{\mathbf{x}}_3 \right]$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Kalman Filter

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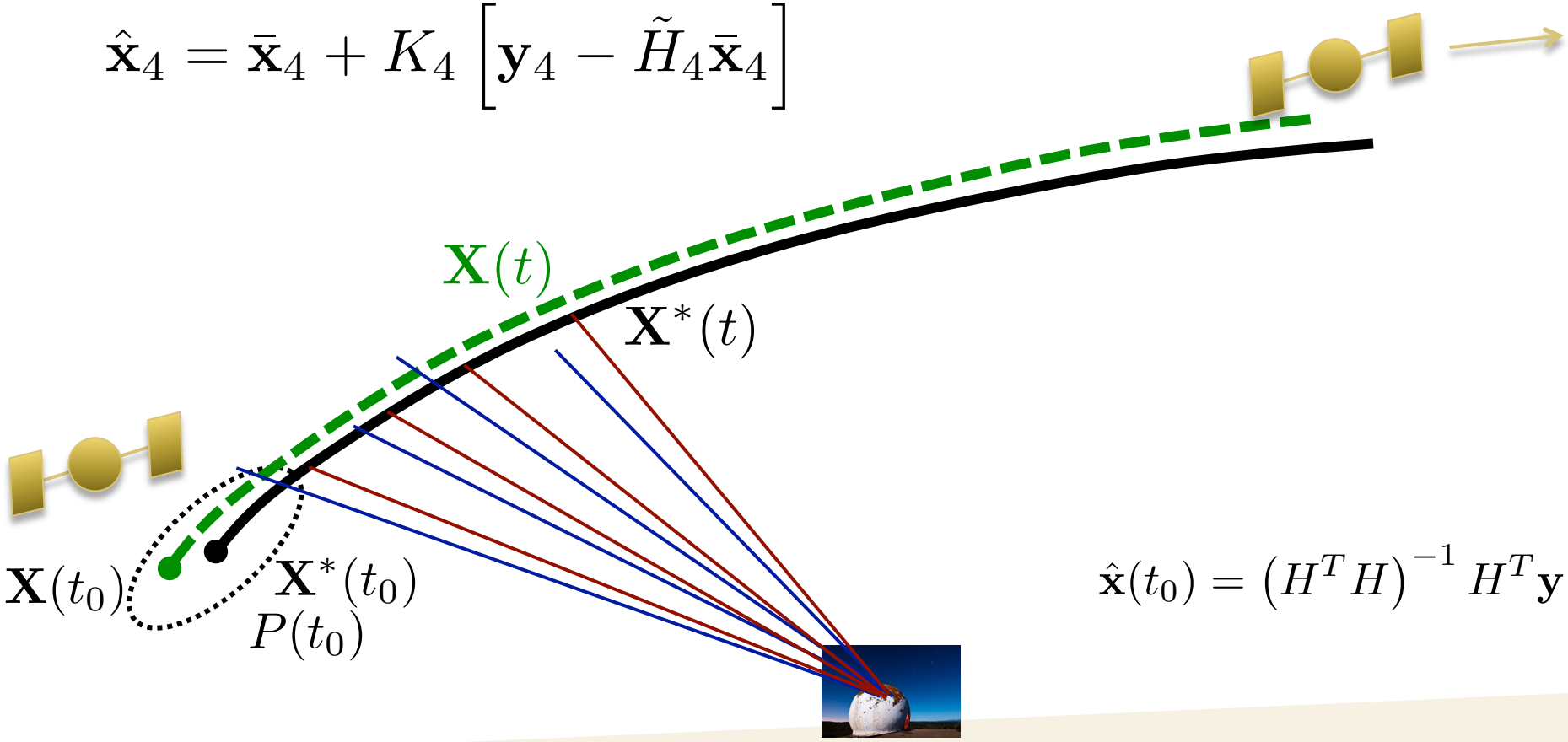
- Collect mapped information

$$\hat{\mathbf{x}}_4 = \bar{\mathbf{x}}_4 + K_4 \left[\mathbf{y}_4 - \tilde{H}_4 \bar{\mathbf{x}}_4 \right]$$



$$\hat{\mathbf{x}}(t_0) = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

Kalman Filter

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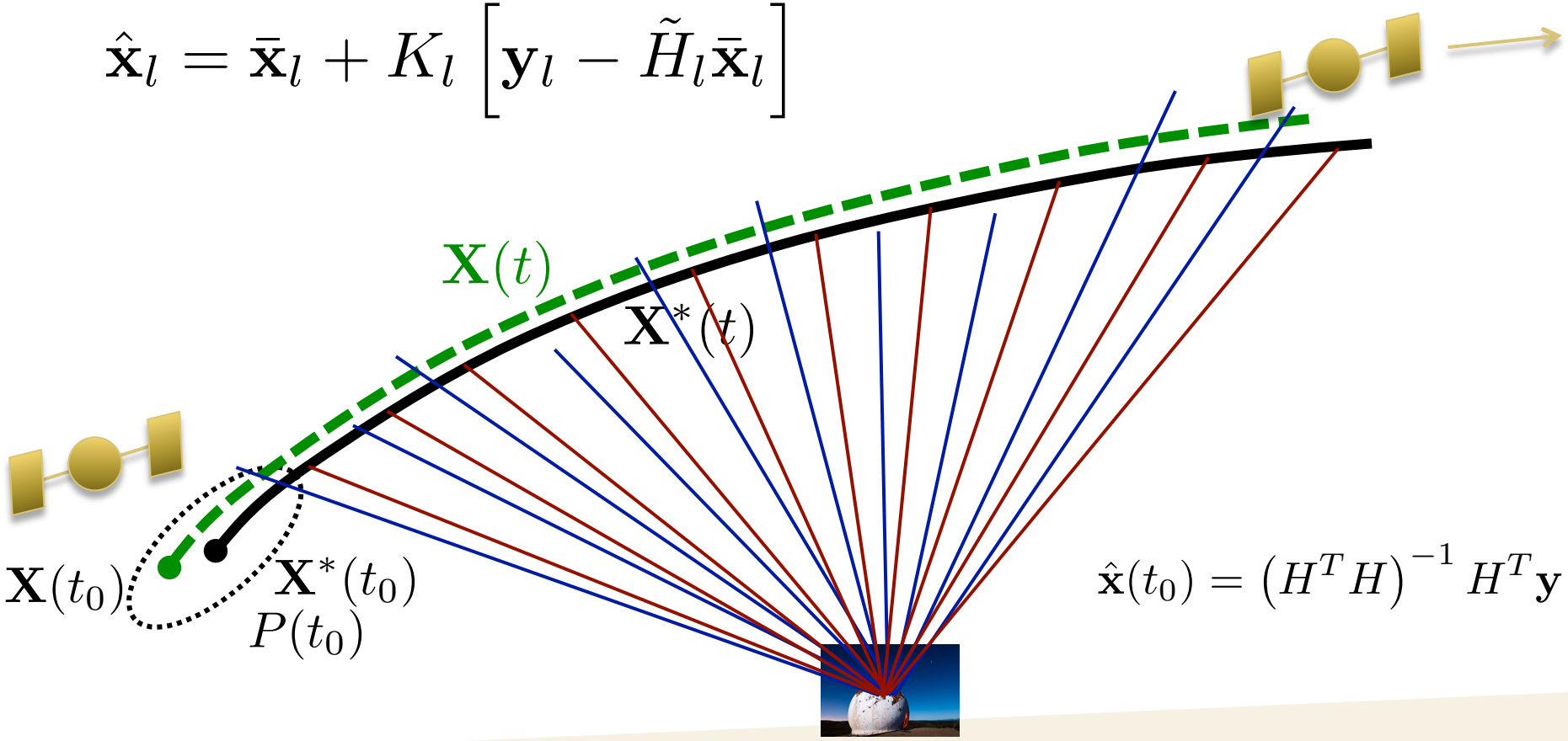
- Collect mapped information

$$\hat{\mathbf{x}}_l = \bar{\mathbf{x}}_l + K_l \left[\mathbf{y}_l - \tilde{H}_l \bar{\mathbf{x}}_l \right]$$



Kalman Filter

- Evolution of the covariance matrix as observations are processed.
  - Q: How do you imagine it would change?
  - Q: What would cause it to shrink? To grow?

Kalman Filter

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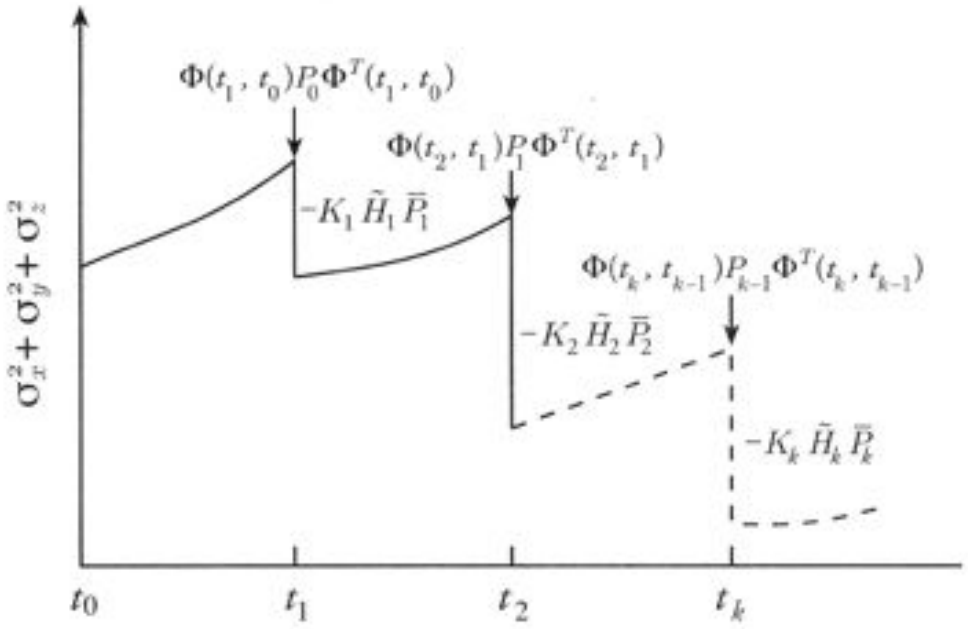
- Evolution of the covariance matrix as observations are processed.



Kalman Filter History

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- Least squares estimation began with Gauss
- 1963: Kalman's sequential approach
  - Introduced minimum variance
  - Introduced process noise
  - Permitted covariance analyses without data
- Schmidt proposed a linearization method that would work for OD problems
  - Supposed that linearizing around the best estimate trajectory is better than linearizing around the nominal trajectory
- 1970: Extended Kalman Filter
- Gradually, researchers identified problems.
  - (a) Divergence due to the use of incorrect *a priori* statistics and unmodeled parameters.
  - (b) Divergence due to the presence of nonlinearities.
  - (c) Divergence due to the effects of computer round-off.

Kalman Filter History

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- Numerical issues cause the covariance matrix to lose their symmetry and nonnegativity
- Possible corrections:
  - (a) Compute only the upper (or lower) triangular entries and force symmetry
  - (b) Compute the entire matrix and then average the upper and lower fields
  - (c) Periodically test and reset the matrix
  - (d) Replace the optimal Kalman measurement update by other expressions (Joseph, Potter, etc)
  - (e) Use larger process noise and measurement noise covariances.

Kalman Filter History

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- Potter is credited with introducing square root factorization.
  - Worked for the Apollo missions!
- 1968: Andrews extended Potter's algorithms to include process noise and correlated measurements.
- 1965 – 1969: Development of the Householder transformation
  - Worked for Mariner 9 in 1971!
- 1969: Dyer-McReynolds filter added additional process noise effects.
  - Worked for Mariner 10 in 1973 for Venus and Mercury!

Final Statements

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- Homework 5 due Today
- Exam on 10/11.
  - Eduardo and/or Paul will be reviewing subjects on Tuesday – send them emails with questions/subjects that you'd like them to cover.
  - 1 hour, open book, open notes.
  - Topics: Definitions of variables, Probability/Statistics Observability, Linearization Least squares, Batch processor

$$
\begin{array}{ccc}
\mathbf{X}(t_i), & \mathbf{X}^*(t_i), & \hat{\mathbf{X}}(t_i) \\
\mathbf{x}(t_i), & \mathbf{x}^*(t_i), & \hat{\mathbf{x}}(t_i) \\
\mathbf{Y}(t_i), & \epsilon(t_i), & \hat{\epsilon}(t_i) \\
\Phi(t_i, t_j), & H(t_i), & \tilde{H}(t_i) \\
A(t_i), & P(t_i), & W(t_i) \\
\bar{\mathbf{x}}(t_i), & \bar{P}(t_i), & \bar{W}(t_i)
\end{array}
$$