ASEN 5070
Statistical Orbit determination I
Fall 2012

CCAR
Colorado Center for
Astrodynamics Research

Professor George H. Born
Professor Jeffrey S. Parker

Lecture 9: Least Squares Estimate

Announcements

- Homework 2 Graded
  - Comments included on D2L
  - Any questions, talk with us this week
  - Homework 3 Graded by Thursday EOD
- Homework 4 due Thursday
  - Solutions online now.
  - The solutions show a lot of information – be sure to demonstrate that you know the answers and are not just copying the solutions!
- Homework 5 out today

Homework 5

- Problem 1:
  - Simulate the Stat OD process on a 2-dimensional, point-mass orbit.
- Problem 2:
  - Simple Batch Process exercise.
- Problem 3:
  - Process noisy observations of a spring-mass system.

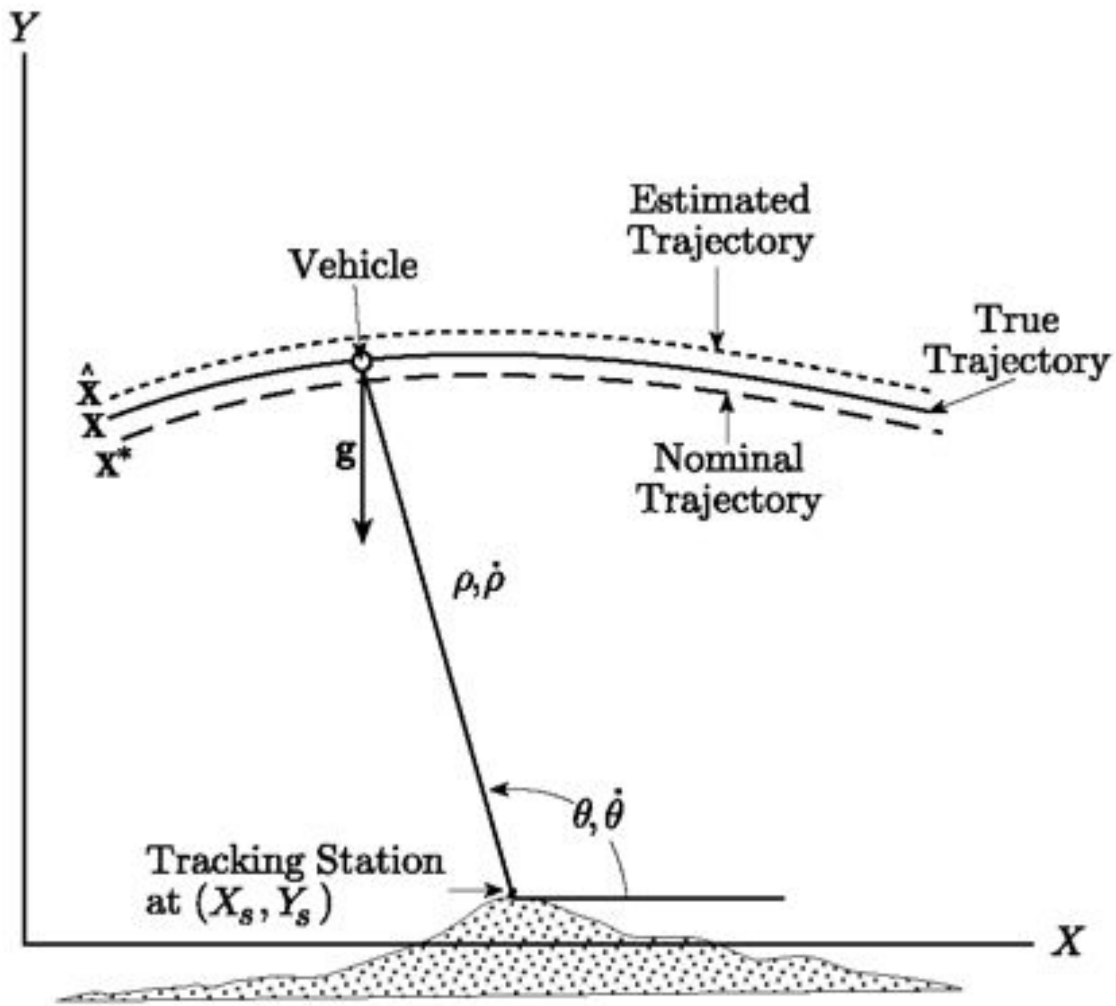
Homework 5

3. Problem 15 from Chapter 4. Please note the following!
   - **Please use the observation data file on the website**, so that you don't have to re-type each value from the book.
   - The problem states that you can build your own observation data set, but for the purpose of this homework use the file on the website.
   - Provide your answer after **FOUR** iterations rather than three.
   - Refer to section 4.8.2 for the algorithm.

Topics Today

- Review of Stat OD Concepts
- Least Squares Estimator
- Quiz: Observability
  - ~10% of the class got 4/4
  - Majority of the class got 2/4
  - We'll go over the topic more closely toward the end of the lecture!

Review of Stat OD Concepts



Observation Residuals

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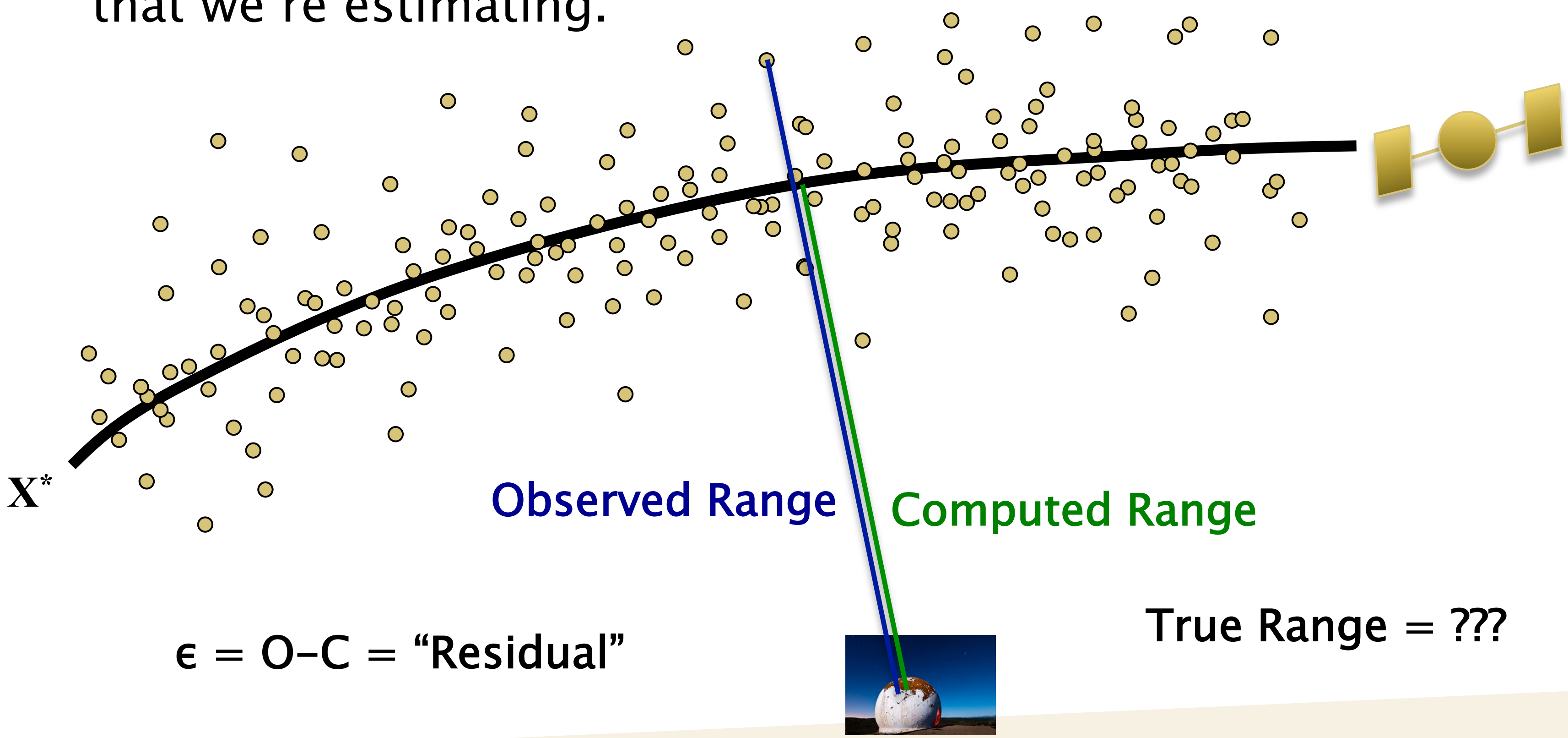
- We have noisy observations of certain aspects of the system.
- We need some way to relate each observation to the trajectory that we're estimating.



True Range = ???

ϵ = O–C = "Residual"

Fitting the Data

- How do we best fit the data?

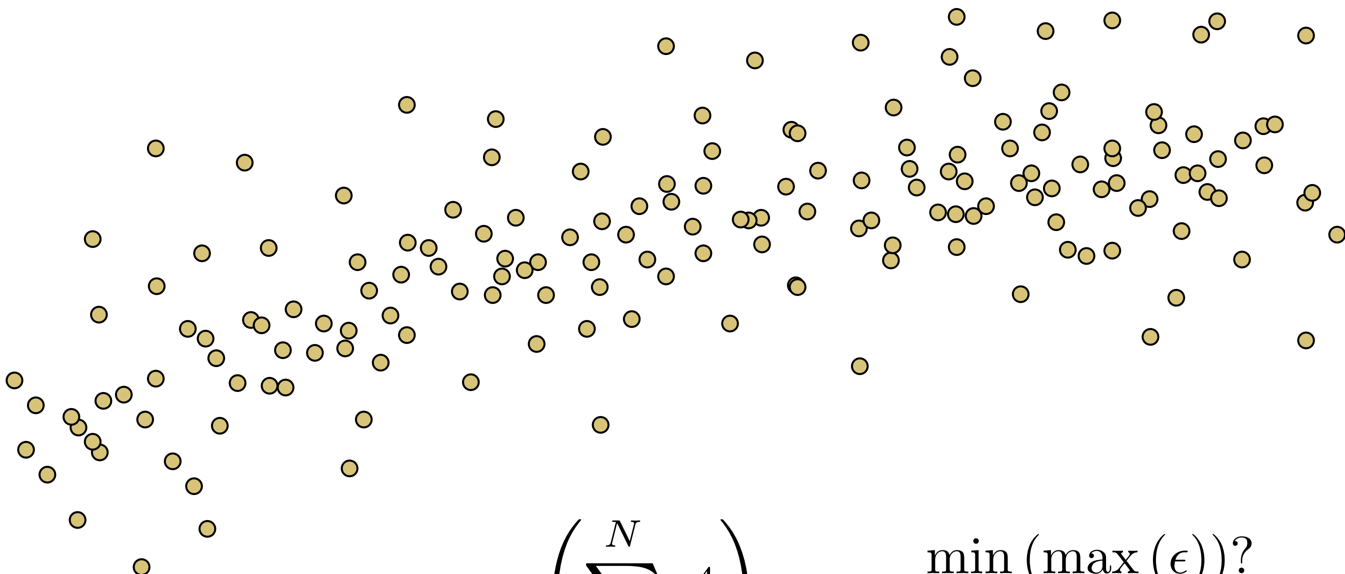
Residuals = ϵ = O–C

$$\min\left(\sum_{i=1}^{N}\epsilon_i\right)?$$ No

$$\min\left(\sum_{i=1}^{N}\epsilon_i\right)^2 ?$$ No

$$\min\left(\sum_{i=1}^{N}|\epsilon_i|\right)?$$ Not bad

$$\min\left(\sum_{i=1}^{N}\epsilon_i^2\right)?$$ Good



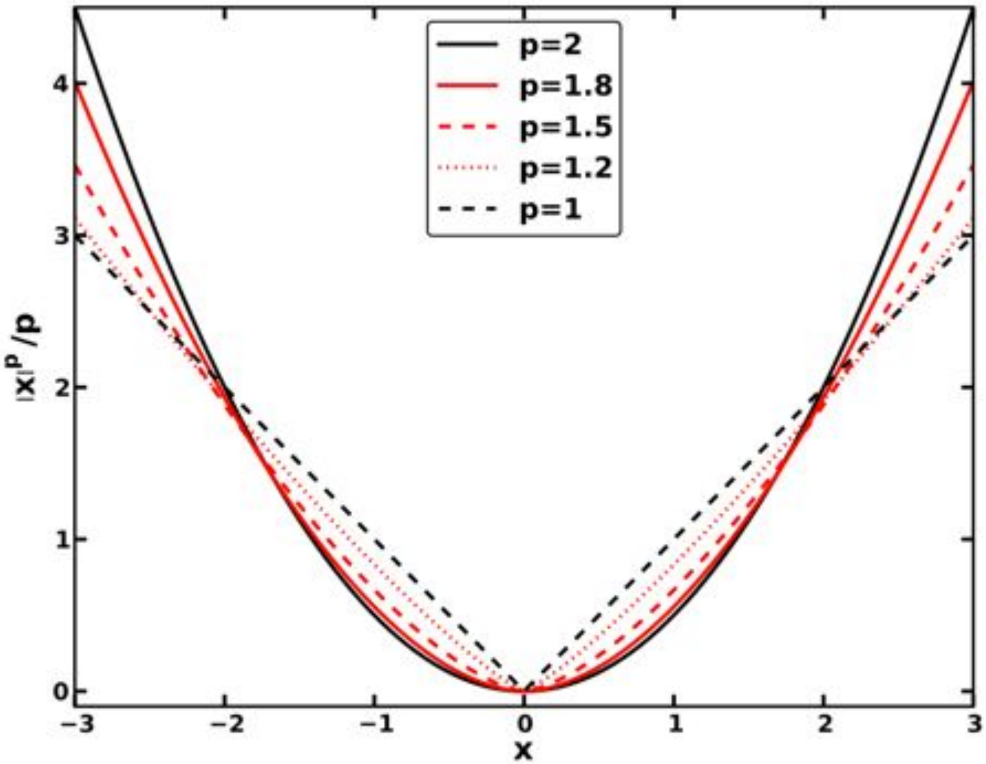
$$\min\left(\sum_{i=1}^{N}\epsilon_i^4\right)?$$

$$\min\left(\max\left(\epsilon\right)\right)?$$

Fitting the Data

- Lp norm estimators

$$\min\left(\frac{1}{p}\sum_{i=1}^{N}\epsilon_i^p\right)$$



$p = 1$: even weights, but discontinuous

$p = 2$: favors outliers

Interior options: optimal for certain scenarios that involve outliers.

Fitting the data

- A good solution, and one easy to code up, is the least-squares solution

$$\text{Minimize } J = \frac{1}{2}\epsilon^T \epsilon \qquad \epsilon_i = O_i - C_i$$

$$\frac{\partial J}{\partial X} = 0$$

$$\frac{\partial^2 J}{\partial^2 X} \text{ is positive definite}$$

State Deviation and Linearization

- Linearization
- Introduce the state deviation vector

$$\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t)$$

- If the reference/nominal trajectory is close to the truth trajectory, then a linear approximation is reasonable.

State Deviation and Linearization

- Goal of the Stat OD process:
- The best fit trajectory

$$\hat{\mathbf{X}}(t) = \mathbf{X}^*(t) + \hat{\mathbf{x}}(t)$$

is represented by

$$\hat{\mathbf{X}}(t_0) = \mathbf{X}^*(t_0) + \hat{\mathbf{x}}(t_0)$$

This is what we want

State Deviation Mapping

- How do we map the state deviation vector from one time to another?
- The state transition matrix.
  - It tracks how a perturbation grows/shrinks/changes over time along the trajectory.

$$\mathbf{x}(t_2) = \Phi(t_2, t_1)\mathbf{x}(t_1)$$

$$\Phi(t_2, t_1) = \Phi(t_1, t_2)^{-1}$$

State Transition Matrix

- The state transition matrix maps a deviation in the state from one epoch to another.

$$\mathbf{x}(t_2) = \Phi(t_2, t_1)\mathbf{x}(t_1)$$

$$\Phi(t_2, t_1) = \Phi(t_1, t_2)^{-1}$$

- It is constructed via numerical integration, in parallel with the trajectory itself.

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I$$

The A Matrix

- The "A" Matrix:

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I$$

$$A(t) = \frac{\partial \boldsymbol{f}(t, \mathbf{X}(t))}{\partial \mathbf{X}(t)} = \frac{\partial \dot{\mathbf{X}}(t)}{\partial \mathbf{X}(t)}$$

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \qquad A(t) = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \cdots & \frac{\partial \dot{x}}{\partial \dot{z}} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \cdots & \frac{\partial \dot{y}}{\partial \dot{z}} \\ \vdots & & \ddots & \vdots \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial y} & \cdots & \frac{\partial \ddot{x}}{\partial \dot{z}} \end{bmatrix}$$

The Typical Stat OD A Matrix

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} [\mathbf{R}]_{3\times1} \\ \left[\dot{\mathbf{R}}\right]_{3\times1} \\ [\mathbf{C}]_{2\times1} \end{bmatrix}_{8\times1} = \begin{bmatrix} [\mathbf{R}]_{3\times1} \\ [\mathbf{V}]_{3\times1} \\ [\mathbf{C}]_{2\times1} \end{bmatrix}_{8\times1}$$

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \left[\dot{\mathbf{R}}\right]_{3\times1} \\ \left[\ddot{\mathbf{R}}\right]_{3\times1} \\ \left[\dot{\mathbf{C}}\right]_{2\times1} \end{bmatrix}_{8\times1} = \begin{bmatrix} [\mathbf{V}]_{3\times1} \\ [\mathbf{A}]_{3\times1} \\ [\mathbf{0}]_{2\times1} \end{bmatrix}_{8\times1}$$

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$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} \rightarrow \begin{bmatrix} [\dot{\mathbf{R}}]_{3\times1} \\ [\ddot{\mathbf{R}}]_{3\times1} \\ [\dot{\mathbf{C}}]_{2\times1} \end{bmatrix}_{8\times1} = \begin{bmatrix} [\mathbf{V}]_{3\times1} \\ [\mathbf{A}]_{3\times1} \\ [\mathbf{0}]_{2\times1} \end{bmatrix}_{8\times1}$$

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} \left[\frac{\partial \mathbf{V}}{\partial \mathbf{R}}\right]_{3\times3} & \left[\frac{\partial \mathbf{V}}{\partial \mathbf{V}}\right]_{3\times3} & \left[\frac{\partial \mathbf{V}}{\partial \mathbf{C}}\right]_{3\times2} \\ \left[\frac{\partial \mathbf{A}}{\partial \mathbf{R}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{V}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{C}}\right]_{3\times2} \\ \left[\frac{\partial \dot{\mathbf{C}}}{\partial \mathbf{R}}\right]_{2\times3} & \left[\frac{\partial \dot{\mathbf{C}}}{\partial \mathbf{V}}\right]_{2\times3} & \left[\frac{\partial \dot{\mathbf{C}}}{\partial \mathbf{C}}\right]_{2\times2} \end{bmatrix}_{8\times8}$$

The Typical Stat OD A Matrix

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} [\mathbf{R}]_{3\times1} \\ [\dot{\mathbf{R}}]_{3\times1} \\ [\mathbf{C}]_{2\times1} \end{bmatrix}_{8\times1} = \begin{bmatrix} [\mathbf{R}]_{3\times1} \\ [\mathbf{V}]_{3\times1} \\ [\mathbf{C}]_{2\times1} \end{bmatrix}_{8\times1}$$

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} \rightarrow \begin{bmatrix} [\dot{\mathbf{R}}]_{3\times1} \\ [\ddot{\mathbf{R}}]_{3\times1} \\ [\dot{\mathbf{C}}]_{2\times1} \end{bmatrix}_{8\times1} = \begin{bmatrix} [\mathbf{V}]_{3\times1} \\ [\mathbf{A}]_{3\times1} \\ [\mathbf{0}]_{2\times1} \end{bmatrix}_{8\times1}$$

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} [\mathbf{0}]_{3\times3} & [\mathbf{I}]_{3\times3} & [\mathbf{0}]_{3\times2} \\ \left[\frac{\partial \mathbf{A}}{\partial \mathbf{R}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{V}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{C}}\right]_{3\times2} \\ [\mathbf{0}]_{2\times3} & [\mathbf{0}]_{2\times3} & [\mathbf{0}]_{2\times2} \end{bmatrix}_{8\times8}$$

Process

- Step 1: Define your state and your dynamics.

Process

- Step 1: Define your state and your dynamics.

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ c_1 \\ c_2 \end{bmatrix} \qquad \dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ a_x = function(\mathbf{X}) \\ a_y = function(\mathbf{X}) \\ a_z = function(\mathbf{X}) \\ 0 \\ 0 \end{bmatrix} \qquad \dot{\Phi}(t, t_0) = A\Phi(t, t_0)$$

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} \begin{bmatrix}\mathbf{0}\end{bmatrix}_{3\times3} & \begin{bmatrix}\mathbf{I}\end{bmatrix}_{3\times3} & \begin{bmatrix}\mathbf{0}\end{bmatrix}_{3\times2} \\ \begin{bmatrix}\frac{\partial \mathbf{A}}{\partial \mathbf{R}}\end{bmatrix}_{3\times3} & \begin{bmatrix}\frac{\partial \mathbf{A}}{\partial \mathbf{V}}\end{bmatrix}_{3\times3} & \begin{bmatrix}\frac{\partial \mathbf{A}}{\partial \mathbf{C}}\end{bmatrix}_{3\times2} \\ \begin{bmatrix}\mathbf{0}\end{bmatrix}_{2\times3} & \begin{bmatrix}\mathbf{0}\end{bmatrix}_{2\times3} & \begin{bmatrix}\mathbf{0}\end{bmatrix}_{2\times2} \end{bmatrix}_{8\times8}$$

Process

- Step 2: Integrate your state and STM.

$$\mathbf{X}(t_0) = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \\ c_1 \\ c_2 \end{bmatrix} \qquad \Phi(t_0, t_0) = [\mathbf{I}]_{8\times 8}$$

Use the "reshape" command:

X0 = [X; reshape(Phi, 8*8, 1)]

$$\mathbf{X}(t_0) = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \\ c_1 \\ c_2 \\ \phi_{11} \\ \phi_{12} \\ \vdots \\ \phi_{18} \\ \phi_{21} \\ \phi_{22} \\ \vdots \\ \phi_{88} \end{bmatrix}_{96\times 1}$$

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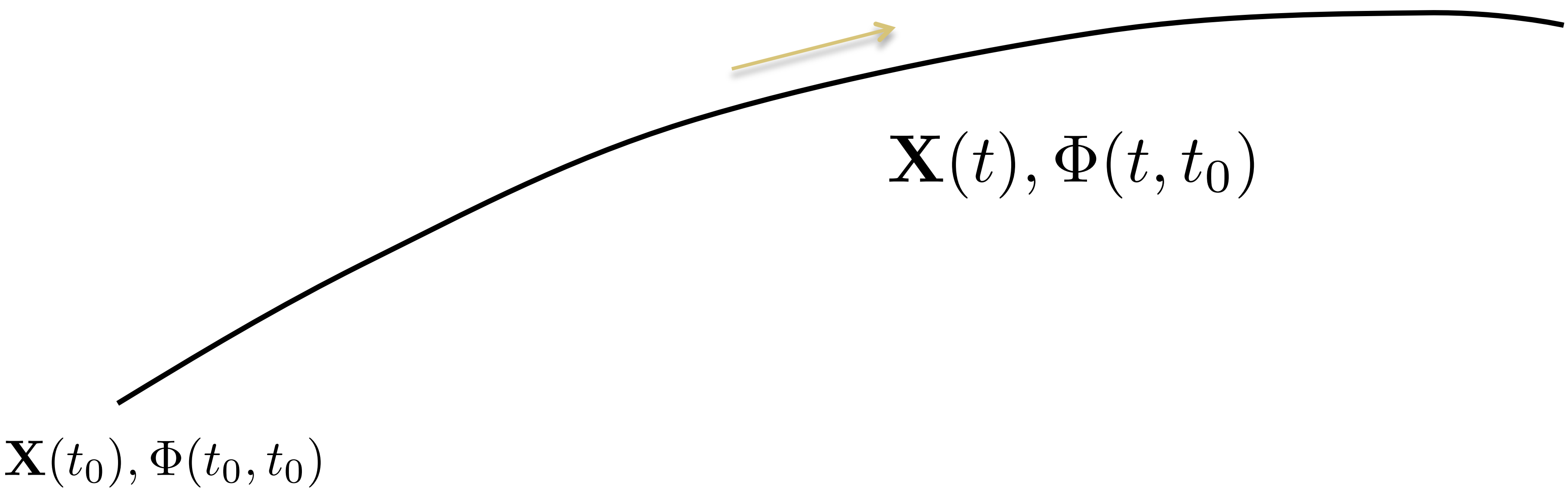
- Step 2: Integrate your state and STM.



- Step 2: Integrate your state and STM.
  - Did we do too much work? Yes, yes, we did.

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{c_1} \\ \dot{c_2} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ a_x = function(\mathbf{X}) \\ a_y = function(\mathbf{X}) \\ a_z = function(\mathbf{X}) \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$$

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} \left[\mathbf{0}\right]_{3\times3} & \left[\mathbf{I}\right]_{3\times3} & \left[\mathbf{0}\right]_{3\times2} \\ \left[\frac{\partial \mathbf{A}}{\partial \mathbf{R}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{V}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{C}}\right]_{3\times2} \\ \left[\mathbf{0}\right]_{2\times3} & \left[\mathbf{0}\right]_{2\times3} & \left[\mathbf{0}\right]_{2\times2} \end{bmatrix}_{8\times8}$$

Process

- Step 2: Integrate your state and STM.
  - Did we do too much work? Yes, yes, we did.

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ a_x = function(\mathbf{X}) \\ a_y = function(\mathbf{X}) \\ a_z = function(\mathbf{X}) \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$$

Don't have to integrate these parameters!

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} [\mathbf{0}]_{3\times3} & [\mathbf{I}]_{3\times3} & [\mathbf{0}]_{3\times2} \\ \left[\frac{\partial \mathbf{A}}{\partial \mathbf{R}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{V}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{C}}\right]_{3\times2} \\ [\mathbf{0}]_{2\times3} & [\mathbf{0}]_{2\times3} & [\mathbf{0}]_{2\times2} \end{bmatrix}_{8\times8}$$

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- Step 2: Integrate your state and STM.
  - Did we do too much work? Yes, yes, we did.

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ a_x = function(\mathbf{X}) \\ a_y = function(\mathbf{X}) \\ a_z = function(\mathbf{X}) \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$$

Don't have to integrate these parameters!

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}} = \begin{bmatrix} [\mathbf{0}]_{3\times3} & [\mathbf{I}]_{3\times3} & [\mathbf{0}]_{3\times2} \\ \left[\frac{\partial \mathbf{A}}{\partial \mathbf{R}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{V}}\right]_{3\times3} & \left[\frac{\partial \mathbf{A}}{\partial \mathbf{C}}\right]_{3\times2} \\ [\mathbf{0}]_{2\times3} & [\mathbf{0}]_{2\times3} & [\mathbf{0}]_{2\times2} \end{bmatrix}_{8\times8}$$

Might not have to integrate these either!

Process

- Step 2: Integrate your state and STM.

$$\mathbf{X}(t_0) = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \\ c_1 \\ c_2 \end{bmatrix}$$

Phi0 = eye(6,8)

X0 = [X; reshape(Phi, 6*8, 1)]

$$\mathbf{X}(t_0) = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \\ c_1 \\ c_2 \\ \phi_{11} \\ \phi_{12} \\ \vdots \\ \phi_{18} \\ \phi_{21} \\ \phi_{22} \\ \vdots \\ \phi_{88} \end{bmatrix}_{96\times 1} \qquad \mathbf{X}(t_0) = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \\ \phi_{11} \\ \phi_{12} \\ \vdots \\ \phi_{18} \\ \phi_{21} \\ \phi_{22} \\ \vdots \\ \phi_{68} \end{bmatrix}_{80\times 1}$$

Process

- Step 2: Integrate your state and STM.
  - This is important for your final project.
  - Rather than integrating 19x18 = 342 equations, Integrate only the 10x6 = 60 equations that change over time.

Process

- Step 3: Process observations.
- Step 4: Determine a new best estimate of the state.
- Step 5: Iterate/repeat as needed.

Mapping an observation

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- How do we map an observation to the trajectory?

$$\text{Observed} = \mathbf{Y}(t) = \mathbf{h}(t, \mathbf{X}(t)) + \epsilon = \begin{bmatrix} \rho(t) + \rho_{\text{bias}} + \rho_{\text{noise}} \\ \dot{\rho}(t) + \dot{\rho}_{\text{bias}} + \dot{\rho}_{\text{noise}} \end{bmatrix}$$

$$\rho = \sqrt{(x_h - x_g)^2 + (y_h - y_g)^2 + (z_h - z_g)^2}$$

$$\dot{\rho} = \frac{(x_h - x_g)(\dot{x}_h - \dot{x}_g) + (y_h - y_g)(\dot{y}_h - \dot{y}_g) + (z_h - z_g)(\dot{z}_h - \dot{z}_g)}{\sqrt{(x_h - x_g)^2 + (y_h - y_g)^2 + (z_h - z_g)^2}}$$

Measurement Mapping Matrix

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$$\mathbf{y}(t) = \tilde{H}(t)\mathbf{x}(t) + \epsilon(t) \qquad \mathbf{y}(t) = H(t)\mathbf{x}(t_0) + \epsilon(t)$$

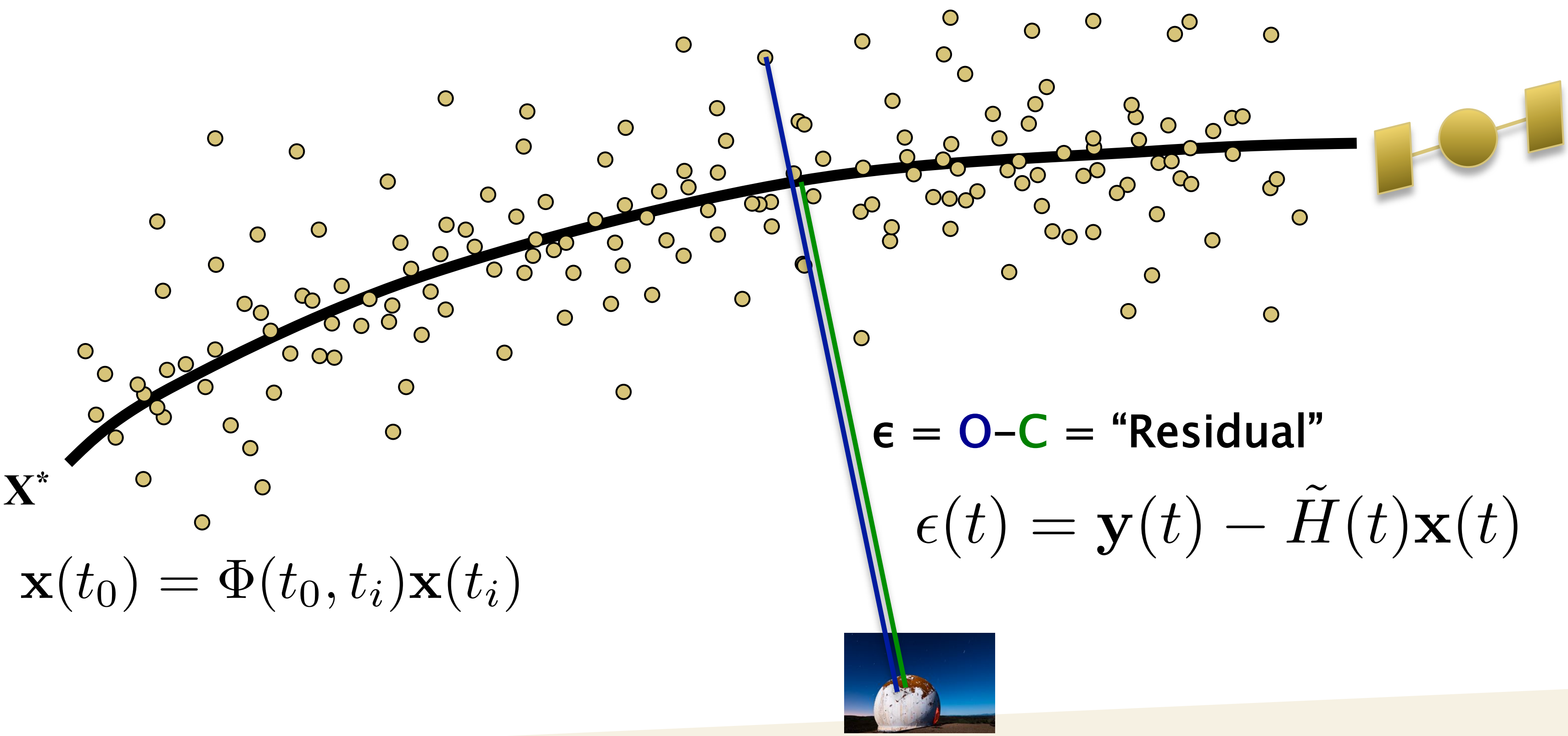
- The Mapping Matrix

$$\tilde{H}(t) = \frac{\partial \boldsymbol{h}(t, \mathbf{X}(t))}{\partial \mathbf{X}(t)}$$

$$\tilde{H}(t) = \begin{bmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \cdots & \frac{\partial \rho}{\partial \dot{z}} \\ \frac{\partial \dot{\rho}}{\partial x} & \frac{\partial \dot{\rho}}{\partial y} & \cdots & \frac{\partial \dot{\rho}}{\partial \dot{z}} \end{bmatrix}$$

Measuring the residuals

- Our state and measurement mappings.



$$\epsilon(t) = \mathbf{y}(t) - \tilde{H}(t)\mathbf{x}(t)$$

$$\mathbf{x}(t_0) = \Phi(t_0, t_i)\mathbf{x}(t_i)$$

Relating the observations to an epoch

- If the state has n unknown parameters and there are l observations at different times
- Then there are a total of $n \times l$ unknowns.
- If we can instead relate each state to an epoch, then we can reduce it to only n unknown state parameters.

$$
\begin{aligned}
\mathbf{y}_1 &= \widetilde{H}_1 \Phi(t_1, t_k)\mathbf{x}_k + \epsilon_1 \\
\mathbf{y}_2 &= \widetilde{H}_2 \Phi(t_2, t_k)\mathbf{x}_k + \epsilon_2 \\
&\vdots \\
\mathbf{y}_\ell &= \widetilde{H}_\ell \Phi(t_\ell, t_k)\mathbf{x}_k + \epsilon_\ell .
\end{aligned}
$$

If the errors are equal to zero, then we can deterministically solve this system using n linearly independent observations.

Relating the observations to an epoch

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$$
\begin{aligned}
\mathbf{y}_1 &= \widetilde{H}_1 \Phi(t_1, t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_1 \\
\mathbf{y}_2 &= \widetilde{H}_2 \Phi(t_2, t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_2 \\
&\vdots \\
\mathbf{y}_\ell &= \widetilde{H}_\ell \Phi(t_\ell, t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_\ell .
\end{aligned}
$$

$$
\mathbf{y} \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix}; \quad H \equiv \begin{bmatrix} \widetilde{H}_1 \Phi(t_1, t_k) \\ \vdots \\ \widetilde{H}_\ell \Phi(t_\ell, t_k) \end{bmatrix}; \quad \boldsymbol{\epsilon} \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_\ell \end{bmatrix}
$$

Relating the observations to an epoch

$$\mathbf{y} \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix}; \quad H \equiv \begin{bmatrix} \widetilde{H}_1 \Phi(t_1, t_k) \\ \vdots \\ \widetilde{H}_\ell \Phi(t_\ell, t_k) \end{bmatrix}; \quad \epsilon \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_\ell \end{bmatrix}$$

If the errors are not zero, then we now have n unknown state parameters and l unknown errors.

If we have multiple observation types (say p types), then we have $m = p \times l$ observations and a total of m unknown observation errors.

Total unknowns: $m+n$

Total equations: m

Least squares criterion provides a best estimate solution in this scenario.

How do we solve the problem?

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- Least squares estimate

$$\text{Minimize } J = \frac{1}{2}\epsilon^T\epsilon$$

$$\epsilon_i = O_i - C_i = \left(\mathbf{y}_i - \tilde{H}\mathbf{x}_i\right)$$

$$\epsilon \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_l \end{bmatrix} \qquad \mathbf{y} \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix} H \equiv \begin{bmatrix} \tilde{H}_1\Phi(t_1, t_k) \\ \vdots \\ \tilde{H}_l\Phi(t_l, t_k) \end{bmatrix}$$

$$J = \frac{1}{2}\left(\mathbf{y} - H\mathbf{x}\right)^T\left(\mathbf{y} - H\mathbf{x}\right)$$

How do we solve the problem?

- Least squares estimate

$$\text{Minimize } J = \frac{1}{2}\epsilon^T\epsilon$$

$$J = \frac{1}{2}\left(\mathbf{y} - H\mathbf{x}\right)^T\left(\mathbf{y} - H\mathbf{x}\right)$$

$$\frac{\partial J}{\partial X} = 0 \qquad \left(\mathbf{y} - H\hat{\mathbf{x}}_k\right)^T H = 0$$

$$H^T H\hat{\mathbf{x}}_k = H^T\mathbf{y}$$

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T\mathbf{y}$$

Best estimate

- The state deviation vector that minimizes the least-squares cost function:

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

- Some observations of this solution:
  - What is the k?
  - Are all data points equally weighted?
  - Does this solution consider correlated data or statistical information in the observations?
  - What happens if we acquire more data?

Best estimate

- The state deviation vector that minimizes the least-squares cost function:

$$\hat{\mathbf{x}}_k = \left(H^T H\right)^{-1} H^T \mathbf{y}$$

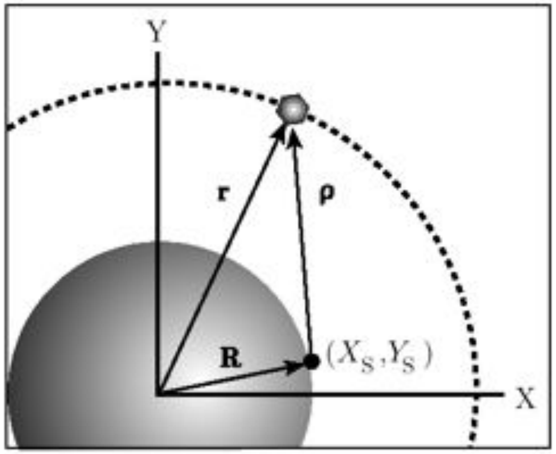
- Additional Details:
  - $H^T H$ is called the *normal matrix*
  - If *H* is full rank, then this will be positive definite.
    - If it's not then we don't have a least squares estimate!
  - The best estimate of the observation errors:
  - $\hat{\epsilon} = \mathbf{y} - H\hat{\mathbf{x}}$

Example 4.2.1

Example 4.2.1

Compute the A matrix and the $\widetilde{H}$ matrix for a satellite in a plane under the influence of only a *central force*. Assume that the satellite is being tracked with range observations, ρ, from a single ground station. Assume that the station coordinates, (X_S, Y_S), and the gravitational parameter are unknown. Then, the state vector, $\mathbf{X}$, is given by

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ U \\ V \\ \mu \\ X_S \\ Y_S \end{bmatrix}$$



where U and V are velocity components and X_S and Y_S are coordinates of the tracking station. From Newton's Second Law and the law of gravitation,

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}$$

Or in component form:

$$\ddot{X} = -\frac{\mu X}{r^3} \qquad \ddot{Y} = -\frac{\mu Y}{r^3}$$

Expressed in first order form:

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{U} \\ \dot{V} \\ \dot{\mu} \\ \dot{X}_S \\ \dot{Y}_S \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} U \\ V \\ -\frac{\mu X}{r^3} \\ -\frac{\mu Y}{r^3} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example 4.2.1

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$$F_1 = \dot{X} = U$$
$$F_2 = \dot{Y} = V$$
$$F_3 = \dot{U} = -\frac{\mu X}{r^3}$$
$$F_4 = \dot{V} = -\frac{\mu Y}{r^3}$$
$$F_5 = \dot{\mu} = 0$$
$$F_6 = \dot{X}_S = 0$$
$$F_7 = \dot{Y}_S = 0$$

$$A(t) = \frac{\partial F(\mathbf{X}^*, t)}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial U} & \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial \mu} & \frac{\partial F_1}{\partial X_S} & \frac{\partial F_1}{\partial Y_S} \\ \frac{\partial F_2}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_2}{\partial Y_S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_7}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_7}{\partial Y_S} \end{bmatrix}^*$$

Example 4.2.1

$$
= \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{\mu}{r^3} + \frac{3\mu X^2}{r^5} & \frac{3\mu XY}{r^5} & 0 & 0 & -\frac{X}{r^3} & 0 & 0 \\
\frac{3\mu XY}{r^5} & -\frac{\mu}{r^3} + \frac{3\mu Y^2}{r^5} & 0 & 0 & -\frac{Y}{r^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^*
$$

Example 4.2.1

The $\widetilde{H}$ matrix is given by

$$\widetilde{H} = \frac{\partial \rho}{\partial \mathbf{X}} = \left[\begin{array}{ccccccc} \frac{\partial \rho}{\partial X} & \frac{\partial \rho}{\partial Y} & \frac{\partial \rho}{\partial U} & \frac{\partial \rho}{\partial V} & \frac{\partial \rho}{\partial \mu} & \frac{\partial \rho}{\partial X_{\mathrm{S}}} & \frac{\partial \rho}{\partial Y_{\mathrm{S}}} \end{array} \right]^{*}$$

where

$$\rho = \left[(X - X_{\mathrm{S}})^2 + (Y - Y_{\mathrm{S}})^2 \right]^{1/2}.$$

Example 4.2.1

The $\widetilde{H}$ matrix is given by

$$\widetilde{H} = \frac{\partial \rho}{\partial \mathbf{X}} = \left[\begin{array}{ccccccc} \dfrac{\partial \rho}{\partial X} & \dfrac{\partial \rho}{\partial Y} & \dfrac{\partial \rho}{\partial U} & \dfrac{\partial \rho}{\partial V} & \dfrac{\partial \rho}{\partial \mu} & \dfrac{\partial \rho}{\partial X_S} & \dfrac{\partial \rho}{\partial Y_S} \end{array} \right]^*$$

where

$$\rho = \left[(X - X_S)^2 + (Y - Y_S)^2 \right]^{1/2}.$$

It follows then that

$$\widetilde{H} = \left[\begin{array}{ccccccc} \dfrac{X - X_S}{\rho} & \dfrac{Y - Y_S}{\rho} & 0 & 0 & 0 & -\dfrac{(X - X_S)}{\rho} & -\dfrac{(Y - Y_S)}{\rho} \end{array} \right]^*$$

Example 4.2.1: Observability

- Notice that all variables end up in either the A or $\widetilde{H}$ matrix, but not necessarily both.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\mu}{r^3} + \frac{3\mu X^2}{r^5} & \frac{3\mu XY}{r^5} & 0 & 0 & -\frac{X}{r^3} & 0 & 0 \\ \frac{3\mu XY}{r^5} & -\frac{\mu}{r^3} + \frac{3\mu Y^2}{r^5} & 0 & 0 & -\frac{Y}{r^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^*$$

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}}$$

$$\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$$

$$\mathbf{y}(t) = \widetilde{H}(t)\mathbf{x}(t) + \epsilon(t)$$

$$\widetilde{H} = \left[\frac{X - X_S}{\rho} \quad \frac{Y - Y_S}{\rho} \quad 0 \quad 0 \quad 0 \quad -\frac{(X - X_S)}{\rho} \quad -\frac{(Y - Y_S)}{\rho} \right]^*$$

Example 4.2.1: Integration Efficiency

- Which equations of motion do we need to integrate?

$$A(t) = \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{X}}$$

$$\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$$

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{U} \\ \dot{V} \\ \dot{\mu} \\ \dot{X}_S \\ \dot{Y}_S \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} U \\ V \\ -\frac{\mu X}{r^3} \\ -\frac{\mu Y}{r^3} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\mu}{r^3} + \frac{3\mu X^2}{r^5} & \frac{3\mu XY}{r^5} & 0 & 0 & -\frac{X}{r^3} & 0 & 0 \\ \frac{3\mu XY}{r^5} & -\frac{\mu}{r^3} + \frac{3\mu Y^2}{r^5} & 0 & 0 & -\frac{Y}{r^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^*$$

Example 2

- The previous example (4.2.1 from the book) left us off at the definitions of the matrices.
- Next example takes us through the least squares estimate.

Least Squares Example

Example of least squares

Let

$$Y_i = \alpha + \beta t_i + \varepsilon_i$$

(Note that this is a linear system)

Assume we wish to estimate

$$X = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \longrightarrow Y_i = \begin{bmatrix} 1 & t_i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \varepsilon_i$$

$$Y_i = H_i \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \varepsilon_i$$

Least Squares Example

$$\hat{X} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \left(H^T H\right)^{-1} H^T Y$$

where

$$H = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ & \vdots \\ 1 & t_\ell \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_\ell \end{bmatrix}$$

$$H^T H = \begin{bmatrix} \ell & \sum_{i=1}^{\ell} t_i \\ \sum_{i=1}^{\ell} t_i & \sum_{i=1}^{\ell} t_i^2 \end{bmatrix}, \quad H^T Y = \begin{bmatrix} \sum_{i=1}^{\ell} Y_i \\ \sum_{i=1}^{\ell} t_i Y_i \end{bmatrix}$$

Note that $H^T H$ is always a symmetric matrix

Least Squares Example

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Assume:

$$\ell = 3, \quad t = 1,2,3 \ \& \ Y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Then

$$\hat{X} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 32 \end{bmatrix} = \begin{bmatrix} 7/3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 32 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Least Squares Example

Assume:

$$\ell = 3, \quad t = 1, 2, 3 \;\; \& \;\; Y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Then

$$\hat{X} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 32 \end{bmatrix} = \begin{bmatrix} 7/3 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 32 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \hat{\varepsilon} = Y - H\hat{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. we have chosen perfect observations

Quick Break

- Quick Break

- Next subject: observability

Quiz Results

Information

There are many prerequisites for a parameter to be observable in a Stat OD filter. This concept quiz will consider several of them. When answering these questions, consider the following state that includes 11 parameters:

X = [x, y, z, vx, vy, vz, mu, Cd, Sx, Sy, Sz],

Where

x, y, & z = position coordinates

vx, vy, & vz = velocity coordinates

mu = gravitational parameter of the Earth

Cd = coefficient of drag

Sx, Sy, & Sz = position coordinates of the tracking station.

Quiz Results

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Question 1 (1 point)

For a state parameter to be observable, the parameter must have a non-zero partial derivative in:

- The A matrix
- The H-tilde matrix
- Both the A and H-tilde matrices
- Either the A or the H-tilde matrices
- Neither the A nor the H-tilde matrices

Quiz Results

Question 1 (1 point)

For a state parameter to be observable, the parameter must have a non-zero partial derivative in:

- The A matrix
- The H-tilde matrix
- Both the A and H-tilde matrices
- Either the A or the H-tilde matrices
- Neither the A nor the H-tilde matrices

Quiz Results

Question 2 (1 point)

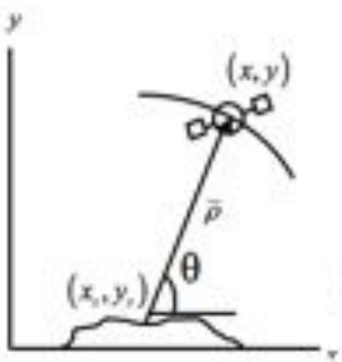
For a state parameter to be observable, the parameter must:

- Be linearly dependent with one or more other state parameters.
- Not be linearly dependent with one or more other state parameters.
- Doesn't matter.

Example

2. Given observations of range rate $\dot{\rho}_j$, j=1----5, all elements of the following state vectors can be solved for (indicate T or F for case A and B)

$$\dot{\rho}_j = \frac{1}{\rho_j}\left[\left(x_0 - x_s + \dot{x}_0 t_j\right)\dot{x}_0 + \left(y_0 - y_s + \dot{y}_0 t_j - \frac{g t_j^2}{2}\right)\left(\dot{y}_0 - g t_j\right)\right]$$



Case A: $X = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ y_0 \\ \dot{y}_0 \\ y_s \end{bmatrix}$ T or F______, Case B: $X = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ y_s \\ g \end{bmatrix}$ T or F___

Example

2. Given observations of range rate $\dot{\rho}_j$, j=1----5, all elements of the following state vectors can be solved for (indicate T or F for case A and B)

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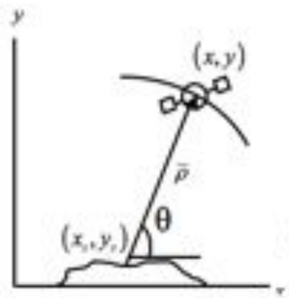

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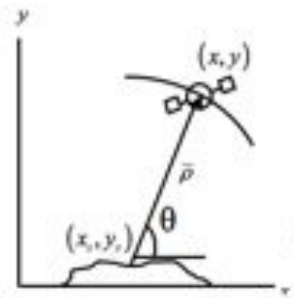
$$\frac{\partial \dot{\rho}}{\partial x_0} =$$

$$\frac{\partial \dot{\rho}}{\partial \dot{x}_0} =$$

Example

2. Given observations of range rate $\dot{\rho}_j$, j=1----5, all elements of the following state vectors can be solved for (indicate T or F for case A and B)

$$\dot{\rho}_j = \frac{1}{\rho_j}\left[\left(x_0 - x_s + \dot{x}_0 t_j\right)\dot{x}_0 + \left(y_0 - y_s + \dot{y}_0 t_j - \frac{g t_j^2}{2}\right)\left(\dot{y}_0 - g t_j\right)\right]$$



Case A: $X = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ y_0 \\ \dot{y}_0 \\ y_s \end{bmatrix}$ T or F______, Case B: $X = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ y_s \\ g \end{bmatrix}$ T or F___

$$\frac{\partial \dot{\rho}}{\partial y_0} =$$

$$\frac{\partial \dot{\rho}}{\partial y_s} =$$

Quiz Results

Question 2 (1 point)

For a state parameter to be observable, the parameter must:

- ○ Be linearly dependent with one or more other state parameters.
- ○ Not be linearly dependent with one or more other state parameters.
- ○ Doesn't matter.

Quiz Results

Question 3 (1 point)

Observation noise can increase a parameter's estimation variance, or even make the parameter unobservable.

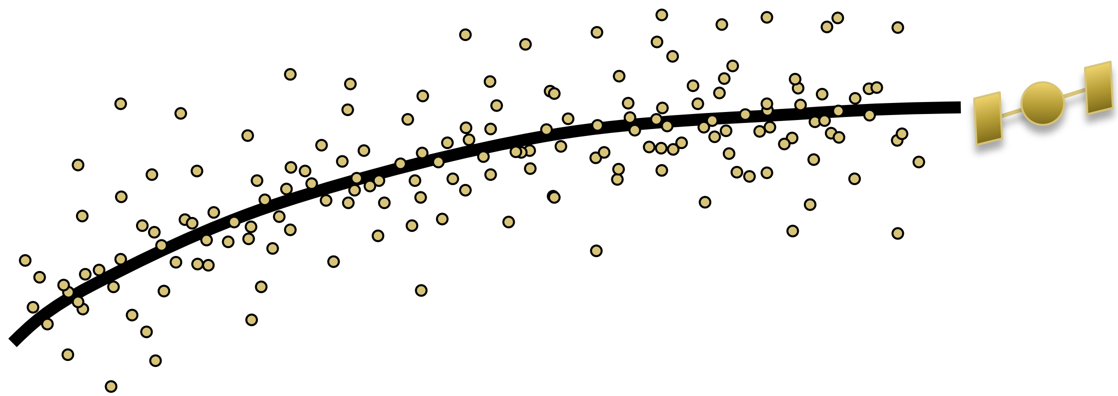
- True
- False

Quiz Results

Question 3 (1 point)

Observation noise can increase a parameter's estimation variance, or even make the parameter unobservable.

- True
- False



Quiz Results

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Question 4 (1 point)

Which of the following improves the estimation of a state parameter?

- Observations in different geometries.
- Different measurement types.
- Observations over time.
- A priori information with small variance.
- Depending on the system, any of the above.

Quiz Results

Question 4 (1 point)

Which of the following improves the estimation of a state parameter?

- Observations in different geometries.
- Different measurement types.
- Observations over time.
- A priori information with small variance.
- Depending on the system, any of the above.

Least Squares Shortcomings

Three major shortcomings of simple least squares solution:

1. Each observation error is weighted equally even though the accuracy of observations may differ

2. The observation errors may be correlated (not independent), and the simple least squares solution makes no allowance for this.

3. The method does not consider that the errors are samples from a random process and makes no attempt to utilize statistical information

Addressing these shortcomings

Weighted Least Squares

–includes weighting matrix for observations

Minimum Variance

–considers statistical characteristics of measurement errors

Minimum Variance w/ A Priori Information

Weighted Least Squares Solution

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$$
\begin{aligned}
\mathbf{y}_1 &= H_1\mathbf{x}_k + \epsilon_1; \qquad w_1\\
\mathbf{y}_2 &= H_2\mathbf{x}_k + \epsilon_2; \qquad w_2\\
&\vdots \quad \vdots \quad \vdots\\
\mathbf{y}_\ell &= H_\ell\mathbf{x}_k + \epsilon_\ell; \qquad w_\ell
\end{aligned}
$$

$$H_i = \widetilde{H}_i\Phi(t_i, t_k).$$

Weighted Least Squares Solution

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$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_\ell \end{bmatrix}; \quad H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_\ell \end{bmatrix};$$

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_\ell \end{bmatrix}; \quad W = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & w_\ell \end{bmatrix}$$

Weighted Least Squares Solution

$$\mathbf{y} = H\mathbf{x}_k + \boldsymbol{\epsilon}; \quad W.$$

$$J(\mathbf{x}_k) = 1/2\boldsymbol{\epsilon}^T W \boldsymbol{\epsilon} = \sum_{i=1}^{\ell} 1/2\boldsymbol{\epsilon}_i^T w_i \boldsymbol{\epsilon}_i.$$

$$J(\mathbf{x}_k) = 1/2(\mathbf{y} - H\mathbf{x}_k)^T W(\mathbf{y} - H\mathbf{x}_k).$$

Weighted Least Squares Solution

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$$\frac{\partial J}{\partial \mathbf{x}_k} = 0 = -(y - H\mathbf{x}_k)^T W H = -H^T W(\mathbf{y} - H\mathbf{x}_k)$$

$$(H^T W H)\mathbf{x}_k = H^T W \mathbf{y}.$$

$$\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}.$$

Choosing the Weighting Matrix

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The Weighting Matrix may be chosen by using the RMS of the observation residuals.

$$\hat{\varepsilon} = y - H\hat{x}$$

Compute the RMS of the observation errors for each type of observation

$$[RMS]_i = \sqrt{\frac{\hat{\varepsilon}_1^2 + \hat{\varepsilon}_2^2 + \ldots \hat{\varepsilon}_\ell^2}{\ell}}$$

Choosing the Weighting Matrix

Let *i* represent the observation type–say

$$i = 1 \Rightarrow \text{range}$$

$$i = 2 \Rightarrow \text{range rate}$$

so for two observation types let

$$W = \begin{bmatrix} \dfrac{1}{(RMS)_1^2} & 0 \\ 0 & \dfrac{1}{(RMS)_2^2} \end{bmatrix}$$

We use the mean square (MS) so that $J(x) = (y - Hx)^T W (y - Hx)$ will be dimensionless. This can enhance numerical stability of the normal equations.

Final Statements

- Homework 2 Graded
  - Comments included on D2L
  - Any questions, talk with us this week
  - Homework 3 Graded by Thursday EOD
- Homework 4 due Thursday
  - Solutions online now.
  - The solutions show a lot of information – be sure to demonstrate that you know the answers and are not just copying the solutions!
- Homework 5 out today