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ASEN 5070
Statistical Orbit Determination I
Fall 2012

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Professor Jeffrey S. Parker
Professor George H. Born

Lecture 19: Numerical Compensations

Announcements

- Homework 8 due next week.
  - Make sure you spend time studying for the exam

- Exam 2 in one week (Thursday).
- Review on Tuesday.

- Exam 2 will cover:
  - Batch vs. CKF vs. EKF
  - Probability and statistics (good to keep this up!)
    - Haven't settled on a question yet, but it will probably be a conditional probability question. I.e., what's the probability of X given that Y occurs?
  - Observability
  - Numerical compensation techniques, such as the Joseph and Potter formulation.
  - No calculators should be necessary
  - Open Book, Open Notes

Quiz 15 Review

Question 1 (1 point)

Let's say we're using the Batch processor, we have n state parameters and p observation data types. Hence, H-tilde(t) is a pxn matrix. Let's assume that p>1 and n>p (our final project is consistent with this, where n=18 and p=2). W(t) is the weighting matrix between the two data types and is a pxp diagonal matrix and for simplicity it doesn't change with time.

True or False: setting one of the diagonal elements of the weighting matrix to zero is equivalent to not processing that observation data type.

- True
- False

Quiz 15 Review

Question 1 (1 point)

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True or False: setting one of the diagonal elements of the weighting matrix to zero is equivalent to not processing that observation data type.

- True
- False

$$\hat{\mathbf{x}}_0 = \left(H^T W H + \bar{P}_0^{-1}\right)^{-1} \left(H^T W \mathbf{y} + \bar{P}_0^{-1} \bar{\mathbf{x}}_0\right)$$

Use Matlab and try it out

Quiz 15 Review

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True or False: setting one of the diagonal elements of the weighting matrix to zero is equivalent to not processing that observation data type.

- True
- False

This is TRUE for the Batch, but you may run into a problem with the Kalman filters.

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$

$$\hat{\mathbf{x}}_k = \overline{\mathbf{x}}_k + K_k [\mathbf{y}_k - \widetilde{H}_k \overline{\mathbf{x}}_k]$$

$$P_k = [I - K_k \widetilde{H}_k] \overline{P}_k .$$

Quiz 15 Review

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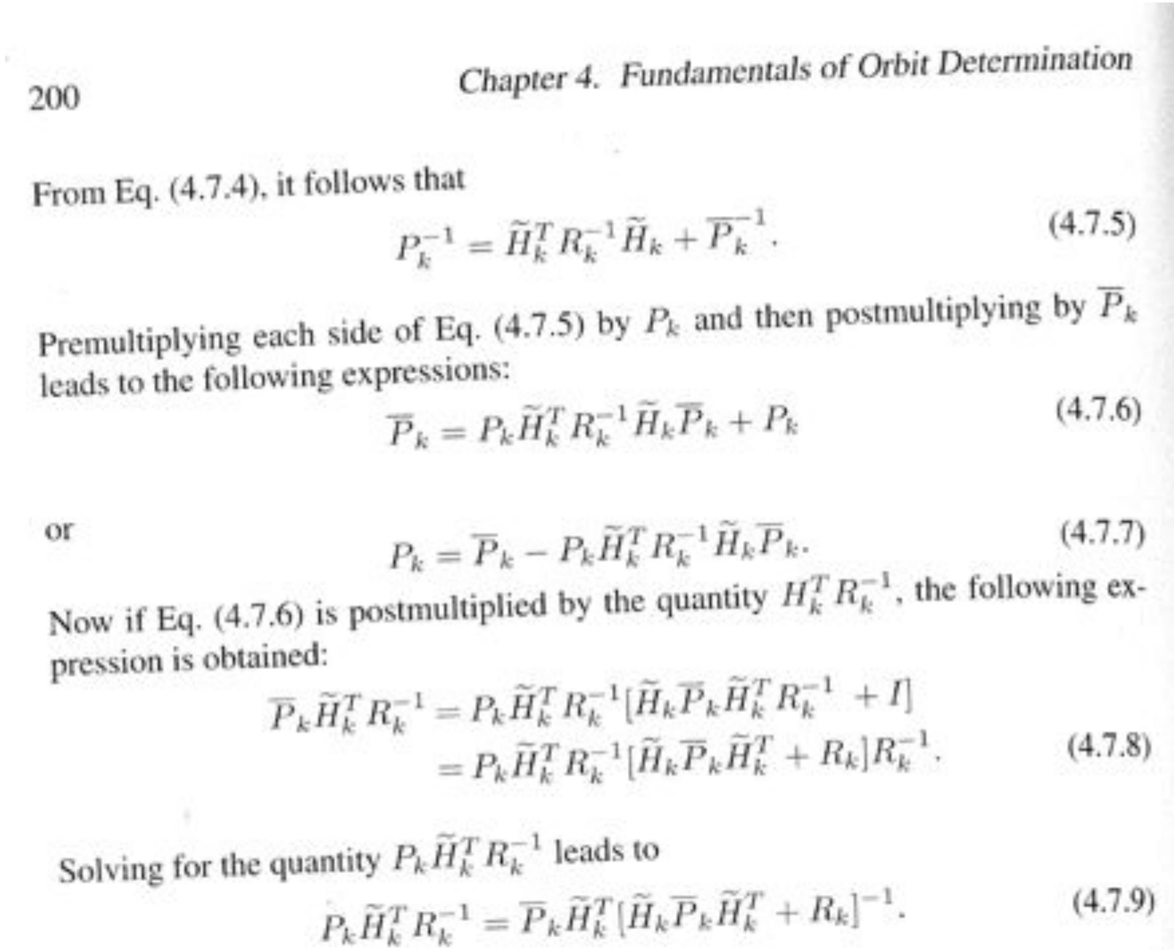
The best scan of the book ever:



From Eq. (4.7.4), it follows that

$$P_k^{-1} = \widetilde{H}_k^T R_k^{-1} \widetilde{H}_k + \overline{P}_k^{-1}. \quad (4.7.5)$$

Premultiplying each side of Eq. (4.7.5) by P_k and then postmultiplying by $\overline{P}_k$ leads to the following expressions:

$$\overline{P}_k = P_k \widetilde{H}_k^T R_k^{-1} \widetilde{H}_k \overline{P}_k + P_k \quad (4.7.6)$$

or

$$P_k = \overline{P}_k - P_k \widetilde{H}_k^T R_k^{-1} \widetilde{H}_k \overline{P}_k. \quad (4.7.7)$$

Now if Eq. (4.7.6) is postmultiplied by the quantity $H_k^T R_k^{-1}$, the following expression is obtained:

$$\begin{aligned} \overline{P}_k \widetilde{H}_k^T R_k^{-1} &= P_k \widetilde{H}_k^T R_k^{-1} [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T R_k^{-1} + I] \\ &= P_k \widetilde{H}_k^T R_k^{-1} [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k] R_k^{-1}. \end{aligned} \quad (4.7.8)$$

Solving for the quantity $P_k \widetilde{H}_k^T R_k^{-1}$ leads to

$$P_k \widetilde{H}_k^T R_k^{-1} = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}. \quad (4.7.9)$$

$$K_k = \overline{P}_k \widetilde{H}_k^T [\widetilde{H}_k \overline{P}_k \widetilde{H}_k^T + R_k]^{-1}$$

Quiz 15 Review

Question 2 (1 point)

Same setup as (1).

True or False: Assuming we still have enough observations remaining to yield a solution, this will yield the same covariance matrix as not processing the observation type.

- True
- False

Quiz 15 Review

Question 2 (1 point)

Same setup as (1).

True or False: Assuming we still have enough observations remaining to yield a solution, this will yield the same covariance matrix as not processing the observation type.

- True
- False

$$P_k = \left(H^T W H\right)^{-1}$$

```
H =
    0.6020    0.6541    0.7482    0.0838
    0.2630    0.6892    0.4505    0.2290
W =
         1         0
         0         0
>> H'*w*H
ans =
    0.3624    0.3937    0.4504    0.0505
    0.3937    0.4278    0.4894    0.0548
    0.4504    0.4894    0.5597    0.0627
    0.0505    0.0548    0.0627    0.0070
```

```
>> H2 = H(1,:)
H2 =
    0.6020    0.6541    0.7482    0.0838
>> w=1
w =
     1
>> H2'*w*H2
ans =
    0.3624    0.3937    0.4504    0.0505
    0.3937    0.4278    0.4894    0.0548
    0.4504    0.4894    0.5597    0.0627
    0.0505    0.0548    0.0627    0.0070
```

Quiz 15 Review

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Question 3 (1 point)

This question is courtesy of Dr. Born. He's quite interested in estimating the height of a table using radiometric data!

Let's assume we have a scenario where we want to estimate the position and velocity of a satellite AND the height of a table for a total of n=7 state parameters. Let's assume we have m>7 independent observations of the range and range-rate of the satellite (p=2). Let's process these observations with a Batch processor, though any processor should work roughly the same. In this case, the H-tilde matrix is 2x7 and there are a lot of observations. Recall that the Batch processor accumulates the sum of H^T*H for each observation. There is no a priori information available.

What conclusion can we draw from this scenario?

- ○ The system is observable and the accumulated information matrix (H^T*H) is rank n.
- ○ The system is unobservable and the accumulated information matrix (H^T*H) is rank n.
- ○ The system is unobservable and the accumulated information matrix (H^T*H) is rank p-1.
- ○ The system is unobservable and the accumulated information matrix (H^T*H) is rank n-1.

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- ○ The system is unobservable and the accumulated information matrix (H^T*H) is rank p-1.
- ○ The system is unobservable and the accumulated information matrix (H^T*H) is rank n-1.

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Question 4 (1 point)

This question is also courtesy of Dr. Born. In addition to estimating table heights, he's also quite interested in observing their heights.

Let's assume we have a scenario where we want to estimate the position and velocity of a satellite for a total of n=6 state parameters. Let's assume we have m=6 independent observations of the range and range-rate of the satellite AND the height of a table (p=3) (NOTE: we're now observing the height of the table). Let's process these observations with a Batch processor, though any processor should work roughly the same. In this case, the H-tilde matrix is 3x6 and there are a lot of observations. Recall that the Batch processor accumulates the sum of H^T*H for each observation. There is no a priori information available.

What conclusion can we draw from this scenario?

- The system is observable and the accumulated information matrix (H^T*H) is rank n.
- The system is unobservable and the accumulated information matrix (H^T*H) is rank n.
- The system is unobservable and the accumulated information matrix (H^T*H) is rank p-1.
- The system is unobservable and the accumulated information matrix (H^T*H) is rank n-1.

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What conclusion can we draw from this scenario?

- The system is observable and the accumulated information matrix (H^T*H) is rank n.
- The system is unobservable and the accumulated information matrix (H^T*H) is rank n.
- The system is unobservable and the accumulated information matrix (H^T*H) is rank p-1.
- The system is unobservable and the accumulated information matrix (H^T*H) is rank n-1.

HW#8

- Due in 7 days

1. Beginning with Equation 4.7.20 from the text and the *a priori* information given in the text, write a program (Matlab, Python, etc) to compute the following:

   1-a) The exact value of P2 using Equation 4.7.24.

   1-b) P2 using a conventional Kalman algorithm.

   1-c) P2 using the Joseph formulation.

   1-d) P2 using the Potter algorithm (Equation 5.7.17).

   1-e) P2 using the Batch processor.

Problem 1 does not require any written response, but be sure to include your code in your submission (D2L accepts .txt, .rtf, .doc, and .pdf formats, but not .m, .py, .c, or any other useful formats). You may copy and paste the code into the same file or upload many files. We expect your code to have at least a minimal header stating that you wrote it, etc., as well as a few comments throughout making it clear the purpose for each part of the code. (points will be awarded accordingly).

HW#8

2. Plot the trace of the exact value of P2 minus the trace of P2 vs ε for each of the following:

   2-a) Conventional Kalman

   2-b) Joseph formulation

   2-c) Potter algorithm

   2-d) Batch processor

Generate the plot for ε values ranging from 1×10^{-6} to 1×10^{-15} in logarithmic steps (1×10^{-6}, 1×10^{-7},…,1×10^{-15} – include smaller steps if you have interest). The plots will most likely be more informative if you plot the logarithm of the absolute value of the difference. Explore Matlab's "semilogx" and "loglog" plotting functions. However you plot them, make sure you make it clear what is being plotted! Full credit will be awarded to submissions that include at least a brief description as well as clear plots that are well labeled. It's not necessary to include the code for the plots.

3. Compare and contrast the behavior and stability of the various filters. Which was the most accurate for the smallest ε values?

HW#8 Solutions to (2)

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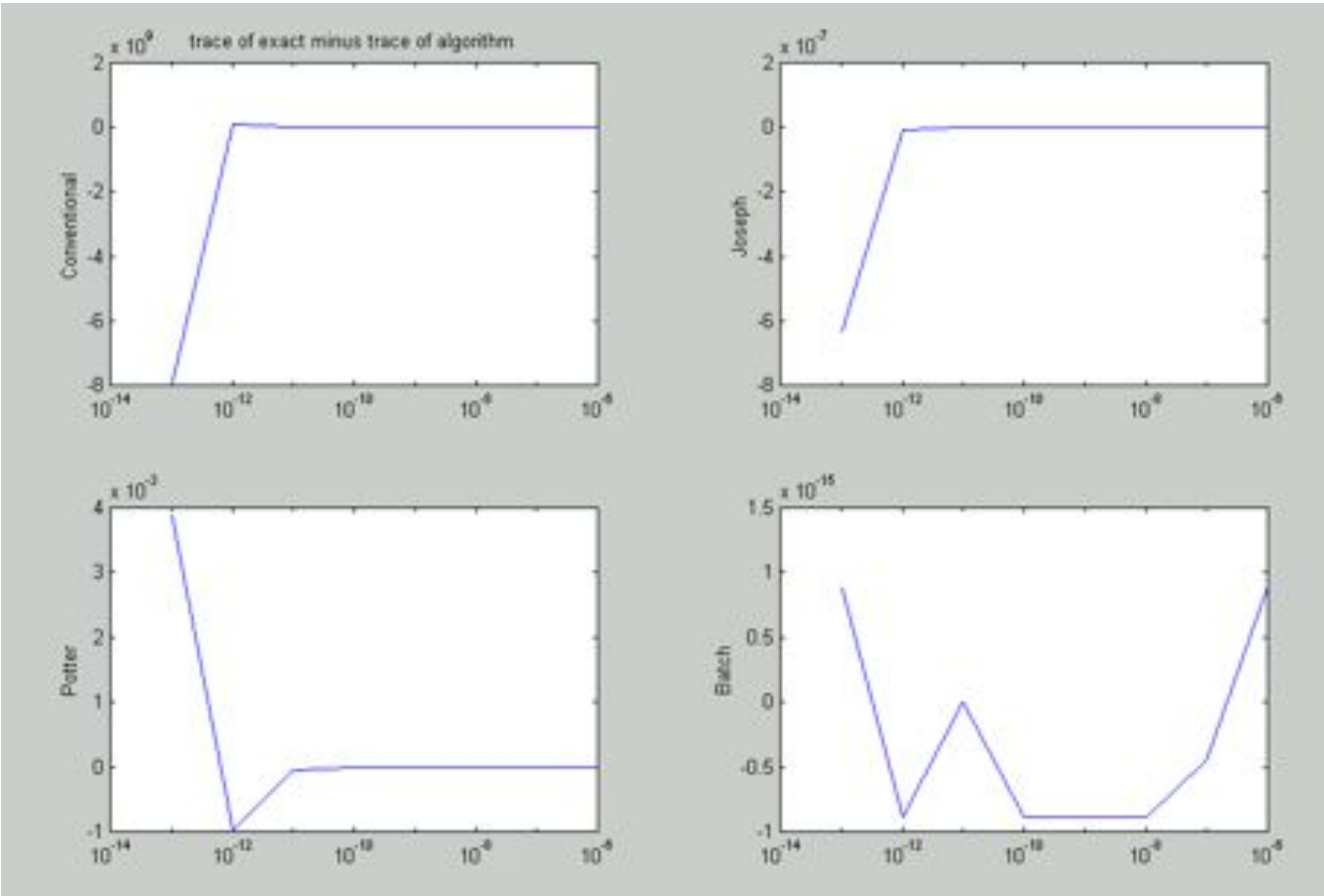

Note that these plots aren’t 100% well-labeled!

HW#8

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4. Assume that two observations, Y_1 and Y_2, are made of two constants X_1 and X_2. Observations are of the form

$$Y = \begin{bmatrix} X_1 + 2\epsilon X_2 \\ X_1 + 3\epsilon X_2 \end{bmatrix} \tag{1}$$

where ϵ is a constant such that $0 < \epsilon \ll 1$ and

$$\begin{gathered} 1 + \epsilon \neq 1 \\ 1 + \epsilon^2 = 1 \end{gathered}$$

when limited by the finite precision of a computer. Observations have zero mean and unit variance. The constants do not change with time, so $\Phi(t, t_0) = I$. The *a priori* estimated state vector is

$$\bar{X} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

with covariance $\bar{P}_0 = \sigma^2 I$ where $\sigma^2 = 1/\epsilon^2$ and I is the 2x2 identity matrix. The true state is

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \tag{2}$$

- The biggest pitfall

- When processing the 2nd observation, set

$$\bar{P}_2 = P_1$$

that is, use the most current covariance you have as the a priori! Not the original one.

Previous Lecture

- Processing an observation vector one element at a time.
- Whitening
- Cholesky
- Joseph

- Today
  - Positive Definiteness
  - Conditioning Number
  - Potter
  - Householder

Positive Definite Matrices

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- Definition

Positive Definite Matrices

- Properties of PD matrices

Positive Definite Matrices

- Properties of PD matrices

- Quick Break

Example Illustrating Numerical Instability of Sequential (Kalman) Filter (see 4.7.1)

Summary of P_2 Results

Exact to order ε

$$\begin{bmatrix} 1+2\varepsilon & -(1+3\varepsilon) \\ -(1+3\varepsilon) & 2+4\varepsilon \end{bmatrix}$$

Conventional Kalman

$$\frac{1}{1-2\varepsilon}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Joseph

$$\begin{bmatrix} 1+2\varepsilon & -(1+3\varepsilon) \\ -(1+3\varepsilon) & 2+\varepsilon \end{bmatrix}$$

Batch

$$\begin{bmatrix} 1+2\varepsilon & -(1+3\varepsilon) \\ -(1+3\varepsilon) & 2+4\varepsilon \end{bmatrix}$$

Conditioning Number

- Conditioning Number of matrix A: C(A)

$$C(A) = \gamma_{\max}/\gamma_{\min}$$

- where the gammas are the min/max eigenvalue of the matrix.

- Inverting A with *p* digits of precision becomes error-prone as C(A) $\rightarrow 10^{p}$.
- If we invert the square root of A:

$$WW^T = A$$

- then C(W)=sqrt(C(A)).
- Numerical difficulties will arise as C(W) $\rightarrow 10^{2p}$.

- Square Root Filter Algorithms

Square Root Filter Algorithms

- Motivation:
  - Loss of significant digits that occurs in computing the measurement update of the state error covariance matrix (P) at the observation epoch (Kaminski et al., 1971)
  - If eigenvalues spread a wide range, then the numerical errors can destroy the symmetry and PD-ness of the P matrix. Filter divergence can occur.

Square Root Filter Algorithms

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- Define *W*, the square root of P:

$$WW^T = P$$

- Observe that if we have *W*, then computing *P* in this manner will always result in a symmetric PD matrix.

- Note: Square root filters are typically derived to process one observation at a time. Hence,

$$R = \sigma^2$$

5.7.1 The Square Root Measurement Update Algorithms

Using these ideas, the covariance measurement update equation, Eq. (4.7.10), can be expressed in square root form as follows:

$$P = \bar{P} - \bar{P}H^T \left[H\bar{P}H^T + R\right]^{-1} H\bar{P}. \tag{5.7.6}$$

Now, let $P = WW^T$ and make this substitution in Eq. (5.7.6) to obtain

$$WW^T = \overline{W}\,\overline{W}^T - \overline{W}\,\overline{W}^T H^T \left[H\overline{W}\,\overline{W}^T H^T + R\right]^{-1} H\overline{W}\,\overline{W}^T. \tag{5.7.7}$$

(Sorry for just scanning the text, but it's a pretty concise description!)

5.7.1 The Square Root Measurement Update Algorithms

Using these ideas, the covariance measurement update equation, Eq. (4.7.10), can be expressed in square root form as follows:

$$P = \overline{P} - \overline{P}H^T\left[H\overline{P}H^T + R\right]^{-1}H\overline{P}. \tag{5.7.6}$$

Now, let $P = WW^T$ and make this substitution in Eq. (5.7.6) to obtain

$$WW^T = \overline{W}\,\overline{W}^T - \overline{W}\,\overline{W}^T H^T\left[H\overline{W}\,\overline{W}^T H^T + R\right]^{-1} H\overline{W}\,\overline{W}^T. \tag{5.7.7}$$

Using the following definitions

$$\tilde{F} = \overline{W}^T H^T,\ \alpha = (\tilde{F}^T\tilde{F} + R)^{-1}, \tag{5.7.8}$$

(Sorry for just scanning the text, but it's a pretty concise description!)

Eq. (5.7.7) can be expressed as

$$W W^T = \overline{W}[I - \tilde{F} \alpha \tilde{F}^T]\overline{W}^T. \tag{5.7.9}$$

Eq. (5.7.7) can be expressed as

$$WW^T = \overline{W}[I - \tilde{F}\alpha\tilde{F}^T]\overline{W}^T. \tag{5.7.9}$$

If a matrix $\tilde{A}$ can be found such that

$$\tilde{A}\tilde{A}^T = I - \tilde{F}\alpha\tilde{F}^T, \tag{5.7.10}$$

This is a key

then Eq. (5.7.9) can be expressed as

$$WW^T = \overline{W}\tilde{A}\tilde{A}^T\overline{W}^T. \tag{5.7.11}$$

Hence,

$$W = \overline{W}\tilde{A}. \tag{5.7.12}$$

5.7.1 The Square Root Measurement Update Algorithms

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The square root measurement update algorithm can be expressed as follows:

$$\begin{aligned}
\tilde{F} &= \overline{W}^T H^T \\
\alpha &= (R + \tilde{F}^T \tilde{F})^{-1} \\
K &= \overline{W} \tilde{F} \alpha \\
W &= \overline{W} \tilde{A} \\
\hat{\mathbf{x}} &= \overline{\mathbf{x}} + K(\mathbf{y} - H\overline{\mathbf{x}}),
\end{aligned}$$

where $\tilde{A} = [I - \tilde{F}\alpha\tilde{F}^T]^{1/2}$. The primary differences in the various algorithms for computing the measurement update in square root form lie in the manner in which the matrix $\tilde{A}$ is computed. The method first used in practice is that given by Potter (Battin, 1999).

Potter Square Root Filter

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Introduction

In general, square root filters are more numerically stable than the conventional Kalman filter. Note that the condition number for the square root of a covariance matrix is the square root of the condition number of the covariance matrix. Hence, the square root filter will be less affected by numerical problems.

The first square root filter development is due to Potter who developed an algorithm for the limited case of uncorrelated scalar observations with no process noise. This filter was used in the Lunar Excursion Module (LEM) for the Apollo Program.

Potter Square Root Filter

Potter Algorithm Derivation

Begin with the time update equation for the estimation error covariance (we will assume that the time update is from t_{k-1} to t_k and drop the indices)

$$\bar{P} = \Phi P \Phi^T \tag{1}$$

Define the square root of P

$$WW^T = P. \tag{2}$$

Potter Square Root Filter

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Potter Algorithm Derivation

Begin with the time update equation for the estimation error covariance (we will assume that the time update is from t_{k-1} to t_k and drop the indices)

$$\bar{P} = \Phi P \Phi^T \quad (1)$$

Define the square root of P

$$WW^T = P . \quad (2)$$

From Eq. (1) and (2)

$$\bar{P} = \Phi W W^T \Phi^T \equiv \bar{W}\bar{W}^T \quad (3)$$

where

$$\bar{W} = \Phi W . \quad (4)$$

Potter Square Root Filter

Next write the expression for the Kalman gain and the measurement update for P in terms of W. (Note that observations are processed one at a time and are assumed to have uncorrelated errors.)

$$
\begin{aligned}
K &= \bar{P}\tilde{H}^T\left(\tilde{H}\bar{P}\tilde{H}^T + \sigma^2\right)^{-1} \\
&= \bar{W}\bar{W}^T\tilde{H}^T\left(\tilde{H}\bar{W}\bar{W}^T\tilde{H}^T + \sigma^2\right)^{-1}
\end{aligned}
\qquad (5)
$$

where σ^2 is the variance of the observation error.

Potter Square Root Filter

Next write the expression for the Kalman gain and the measurement update for P in terms of W.
(Note that observations are processed one at a time and are assumed to have uncorrelated errors.)

$$K = \bar{P}\tilde{H}^T\left(\tilde{H}\bar{P}\tilde{H}^T + \sigma^2\right)^{-1}$$
$$= \bar{W}\underbrace{\bar{W}^T\tilde{H}^T}_{\tilde{F} = \bar{W}^T\tilde{H}^T}\underbrace{\left(\tilde{H}\bar{W}\bar{W}^T\tilde{H}^T + \sigma^2\right)^{-1}}_{\alpha} \qquad (5)$$

$$\tilde{F} = \bar{W}^T\tilde{H}^T$$

$$\alpha = \left(\tilde{H}\bar{W}\underbrace{\bar{W}^T\tilde{H}^T}_{\tilde{F} = \bar{W}^T\tilde{H}^T} + \sigma^2\right)^{-1}$$

Potter Square Root Filter

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Let

$$\alpha \equiv \left(\tilde{H} \bar{W} \bar{W}^T \tilde{H}^T + \sigma^2 \right)^{-1} \tag{6}$$

where α is a scalar. Define

$$\tilde{F} = W^T \tilde{H}^T \tag{7}$$

then

$$\alpha = \left(\tilde{F}^T \tilde{F} + \sigma^2 \right)^{-1} \tag{8}$$

and

$$K = \alpha \bar{W} \tilde{F} \tag{9}$$

Potter Square Root Filter

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The measurement update for P is

$$
\begin{aligned}
P = WW^T &= \left(I - K\tilde{H}\right)\bar{P} \\
&= \left(I - \alpha\bar{W}\tilde{F}\tilde{H}\right)\bar{W}\bar{W}^T \\
&= \bar{W}\left(I - \alpha\tilde{F}\tilde{H}\bar{W}\right)\bar{W}^T \\
&= \bar{W}\left(I - \alpha\tilde{F}\tilde{F}^T\right)\bar{W}^T \quad .
\end{aligned}
\tag{10}
$$

Potter Square Root Filter

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The measurement update for P is

$$
\begin{aligned}
P = WW^T &= \left(I - K\tilde{H}\right)\bar{P} \\
&= \left(I - \alpha \bar{W}\tilde{F}\tilde{H}\right)\bar{W}\bar{W}^T \\
&= \bar{W}\left(I - \alpha \tilde{F}\tilde{H}\bar{W}\right)\bar{W}^T \\
&= \bar{W}\left(I - \alpha \tilde{F}\tilde{F}^T\right)\bar{W}^T .
\end{aligned} \tag{10}
$$

Potter observed that if a matrix $\bar{A}$ could be found such that

$$\bar{A}\bar{A}^T = \left(I - \alpha \tilde{F}\tilde{F}^T\right) \tag{11}$$

(say, using Cholesky)

then

$$P = \overline{W A A^T} \bar{W}^T = WW^T . \tag{12}$$

Potter Square Root Filter

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To find $\bar{A}$ introduce the scalar γ so that

$$\begin{aligned}\bar{A}\bar{A}^T &= \left(I-\gamma\alpha\tilde{F}\tilde{F}^T\right)\left(I-\gamma\alpha\tilde{F}\tilde{F}^T\right) \\ &= \left(I-\alpha\tilde{F}\tilde{F}^T\right)\end{aligned} \tag{13}$$

Solving for γ,

$$I-\alpha\tilde{F}\tilde{F}^T = I-2\gamma\alpha\tilde{F}\tilde{F}^T+\gamma^2\alpha^2\tilde{F}\tilde{F}^T\tilde{F}\tilde{F}^T. \tag{14}$$

Potter Square Root Filter

Define

$$\beta \equiv \tilde{F}^T_{1\times n} \tilde{F}_{n\times 1} \,. \tag{15}$$

where β is a scalar. Then

$$I - \alpha \tilde{F}\tilde{F}^T = I - 2\gamma\alpha \tilde{F}\tilde{F}^T + \gamma^2 \alpha^2 \beta \tilde{F}\tilde{F}^T$$

Potter Square Root Filter

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Define

$$\beta \equiv \tilde{F}^T_{1\times n}\tilde{F}_{n\times 1}, \tag{15}$$

where β is a scalar. Then

$$I - \alpha\tilde{F}\tilde{F}^T = I - 2\gamma\alpha\tilde{F}\tilde{F}^T + \gamma^2\alpha^2\beta\tilde{F}\tilde{F}^T$$

or

$$\left(\gamma^2\alpha^2\beta - 2\gamma\alpha + \alpha\right)\tilde{F}\tilde{F}^T = 0$$
$$= \left(\alpha\beta\gamma^2 - 2\gamma + 1\right)\alpha\tilde{F}\tilde{F}^T = 0 .$$

$\alpha\tilde{F}\tilde{F}^T = 0$ is a trivial solution.

Potter Square Root Filter

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Thus

$$\alpha\beta\gamma^2 - 2\gamma + 1 = 0\,. \tag{16}$$

Using the solution for a quadratic equation (after some algebra)

$$\gamma = \frac{1}{1 \mp \sqrt{\alpha\sigma^2}}\,, \tag{17}$$

where the + sign is chosen to prevent the possibility that $\gamma = \infty$ when $\alpha\sigma^2 = 1$.

Potter Square Root Filter

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Recall from Eq. (12) that

$$\begin{aligned} W &= \overline{W}A \\ &= \overline{W}\left(I - \gamma\alpha\tilde{F}\tilde{F}^T\right) \end{aligned} \tag{18}$$

$$\boxed{P = \overline{W}AA^T\overline{W}^T = WW^T}$$

$$\boxed{AA^T = \left(I - \gamma\alpha\tilde{F}\tilde{F}^T\right)\left(I - \gamma\alpha\tilde{F}\tilde{F}^T\right)}$$

Potter Square Root Filter

Recall from Eq. (12) that

$$\begin{aligned} W &= \overline{WA} \\ &= \bar{W}\left(I - \gamma\alpha\tilde{F}\tilde{F}^T\right) \end{aligned} \tag{18}$$

and

$$K = \alpha\bar{W}\tilde{F}. \tag{19}$$

Hence,

$$W = \bar{W} - \gamma K\tilde{F}^T \tag{20}$$

which is the measurement update for W.

Potter Square Root Filter

Given $\bar{W}_k$, $\bar{x}_k$, y_k, $\tilde{H}_k$ where $\bar{P}_k = \bar{W}_k \bar{W}_k^T$.

Compute

1. $\tilde{F}_k = \bar{W}_k^T \tilde{H}_k^T$

2. $\alpha_k = \left(\tilde{F}_k^T \tilde{F}_k + \sigma^2 \right)^{-1}$

3. $K_k = \alpha_k \bar{W}_k \tilde{F}_k$

4. $\hat{x}_k = \bar{x}_k + K_k \left(y_k - \tilde{H}_k \bar{x}_k \right)$

5. $\gamma_k = \dfrac{1}{1 + \sqrt{\alpha_k \sigma^2}}$

6. $W_k = \bar{W}_k - \gamma_k K_k \tilde{F}_k^T$, $P_k = W_k W_k^T$

Note: if you are given an *a priori P* matrix, convert it to *W* using Cholesky or equivalent at start.

Potter Square Root Filter

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7. Integrate the reference orbit and $\dot{\Phi} = A\Phi$ forward to $k+1$

8. Time update W_k and $\hat{x}_k$ to $k+1$

$$\overline{W}_{k+1} = \Phi(t_k, t_{k+1}) W_k$$
$$\overline{x}_{k+1} = \Phi(t_k, t_{k+1}) \hat{x}_k$$

9. Return to step 1 with $k = k+1$

Potter Square Root Filter

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We have assumed here that the observation vector, y_k, contains a single measurement. If y_k contains more than one data type at each observation time, we would return to step 1 after completing step 6 for each element of y_k. After we have processed all elements of y_k, we would proceed to the time update of step 8.

Note that unlike P, W contains n^2 distinct elements (i.e., W is not symmetric). The Carlson algorithm reduces W to an upper (or lower) triangular matrix (see Bierman, 1977).

And page 339 of Stat OD text.

Numerical Instability of Kalman Filter

Summary of P_2 Results

Exact to order ε

$$\begin{bmatrix} 1+2\varepsilon & -(1+3\varepsilon) \\ -(1+3\varepsilon) & 2+4\varepsilon \end{bmatrix}$$

Potter Algorithm

$$\begin{bmatrix} 1-2\varepsilon & -(1-\varepsilon) \\ -(1-\varepsilon) & 2-4\varepsilon \end{bmatrix}$$

Conventional Kalman

$$\frac{1}{1-2\varepsilon}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Joseph

$$\begin{bmatrix} 1+2\varepsilon & -(1+3\varepsilon) \\ -(1+3\varepsilon) & 2+\varepsilon \end{bmatrix}$$

Batch

$$\begin{bmatrix} 1+2\varepsilon & -(1+3\varepsilon) \\ -(1+3\varepsilon) & 2+4\varepsilon \end{bmatrix}$$

The End (unless there's a lot more time)

- Homework 8 due next week.
  - Make sure you spend time studying for the exam

- Exam 2 in one week (Thursday).
- Review on Tuesday.

- Exam 2 will cover:
  - Batch vs. CKF vs. EKF
  - Probability and statistics (good to keep this up!)
    - Haven't settled on a question yet, but it will probably be a conditional probability question. I.e., what's the probability of X given that Y occurs?
  - Observability
  - Numerical compensation techniques, such as the Joseph and Potter formulation.
  - No calculators should be necessary
  - Open Book, Open Notes