Dynamics of a System

ASEN 5010

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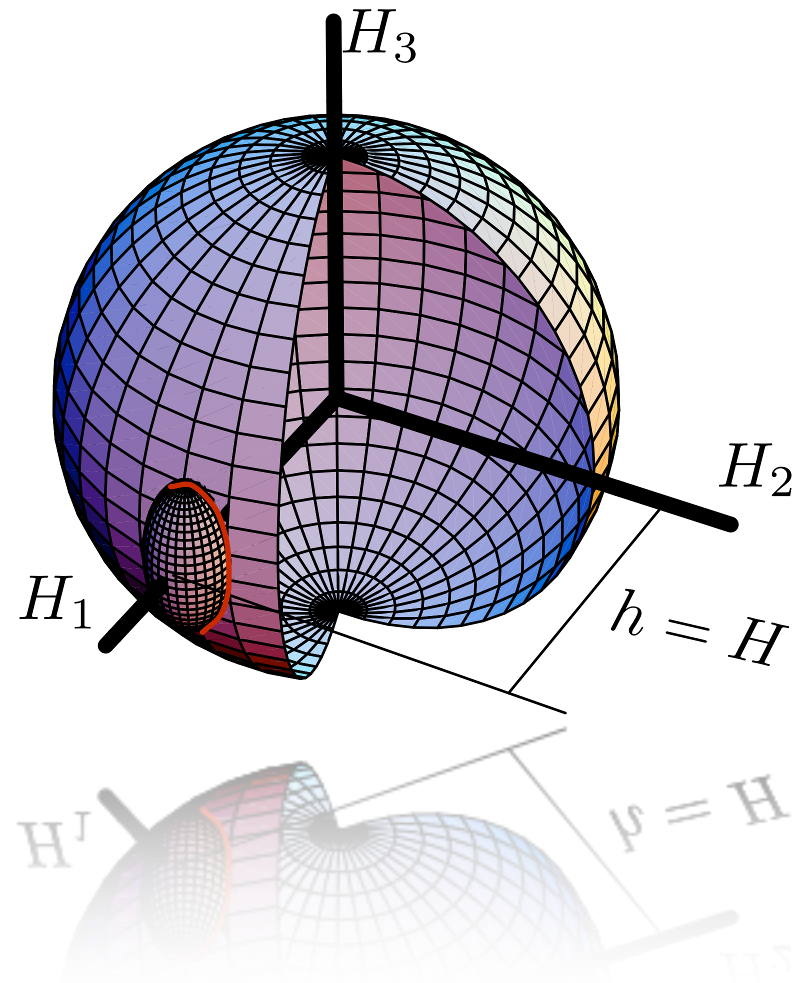
Outline

- Continuous System
- Rigid Body
  - Inertia matrix
  - Energy
  - Equations of motion
  - Spin Stability
- Dual Spin Spacecraft
- Gravity Gradient Torque
  - zero GG Torque attitudes
  - Equilibrium stabilities



Continuous System

What does jello look like in space?

Equations of Motion

Newton's Law: $\mathrm{d}\boldsymbol{F} = \ddot{\boldsymbol{R}}\mathrm{d}m$

Force Vector: $\mathrm{d}\boldsymbol{F} = \mathrm{d}\boldsymbol{F}_E + \mathrm{d}\boldsymbol{F}_I$

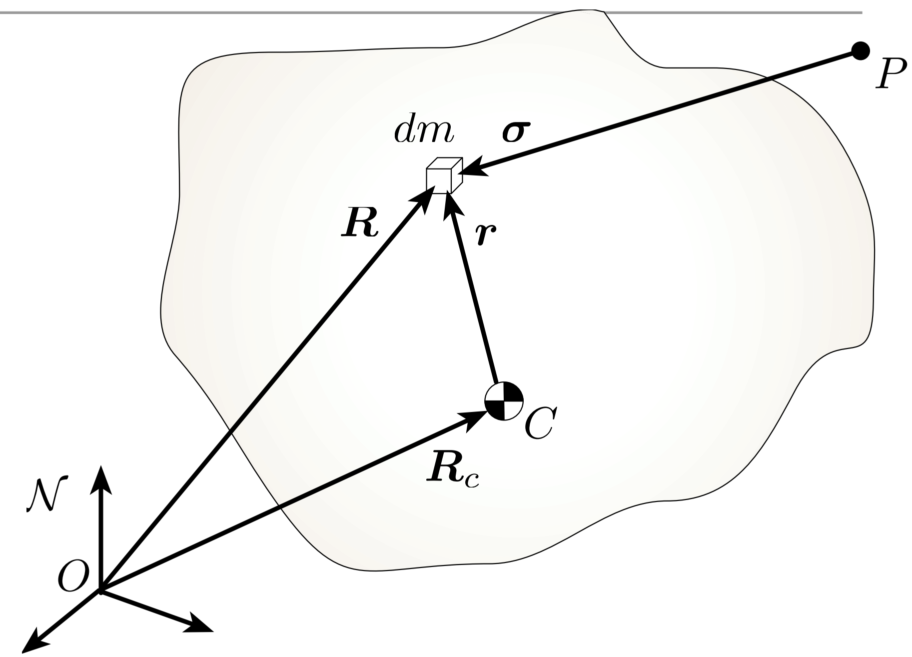
Total Force acting on System: $\boldsymbol{F} = \int_B \mathrm{d}\boldsymbol{F} = \int_B \mathrm{d}\boldsymbol{F}_E$

Center of Mass: $M\boldsymbol{R}_c = \int_B \boldsymbol{R}\mathrm{d}m$

$$M\ddot{\boldsymbol{R}}_c = \int_B \ddot{\boldsymbol{R}}\mathrm{d}m = \int_B \mathrm{d}\boldsymbol{F}$$

$$\boxed{M\ddot{\boldsymbol{R}}_c = \boldsymbol{F}}$$

Super Particle Theorem

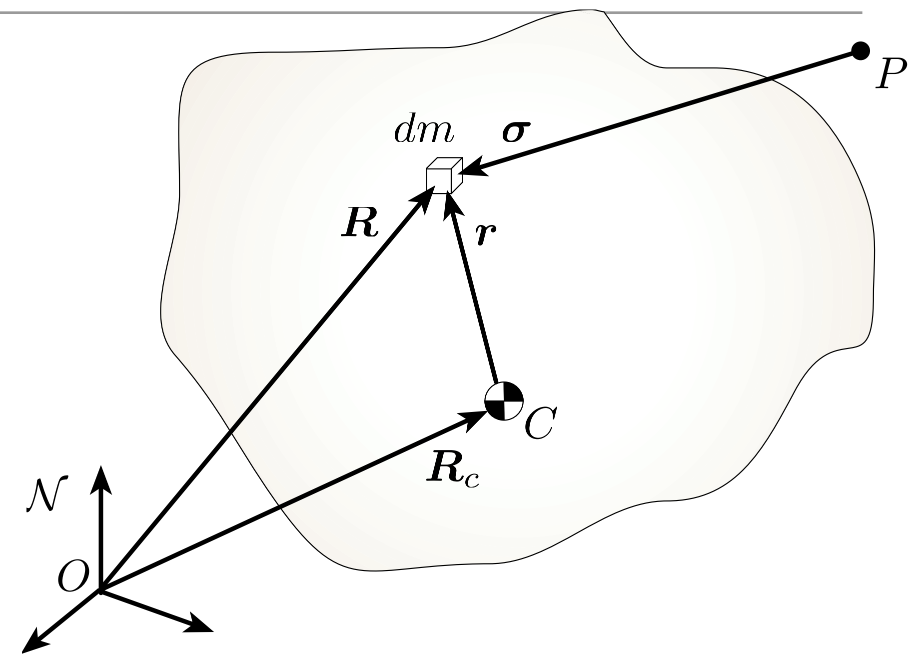


Kinetic Energy

Definition: $$T = \frac{1}{2}\int_B \dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{R}} \mathrm{d}m$$

$$\dot{\boldsymbol{R}} = \dot{\boldsymbol{R}}_c + \dot{\boldsymbol{r}}$$

$$T = \frac{1}{2}\left(\int_B \mathrm{d}m\right)\dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \dot{\boldsymbol{R}}_c \cdot \cancel{\int_B \dot{\boldsymbol{r}} \mathrm{d}m}$$
$$+\frac{1}{2}\int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \mathrm{d}m$$



$$\boxed{T = \frac{1}{2}M\dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \frac{1}{2}\int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \mathrm{d}m}$$

Energy of CM Energy about CM

Work/Energy Principle

Differentiate Energy:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = M\ddot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \int_B \dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} \mathrm{d}m$$

$M\ddot{\boldsymbol{R}}_c = \boldsymbol{F}$ $\quad$ $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} - \ddot{\boldsymbol{R}}_c$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c + \int_B \left(\ddot{\boldsymbol{R}}\mathrm{d}m\right) \cdot \dot{\boldsymbol{r}} - \ddot{\boldsymbol{R}}_c \cdot \underbrace{\cancel{\int_B \dot{\boldsymbol{r}}\mathrm{d}m}}_{\text{C.M.}}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c + \int_B \mathrm{d}\boldsymbol{F} \cdot \dot{\boldsymbol{r}}$$

$$\boxed{T(t_2) - T(t_1) = \int_{\boldsymbol{R}(t_1)}^{\boldsymbol{R}(t_2)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R}_c + \int_B \int_{\boldsymbol{r}(t_1)}^{\boldsymbol{r}(t_2)} \mathrm{d}\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}}$$

Work energy/principle for system of particles

Linear Momentum

Definition:

$$\mathrm{d}\boldsymbol{p} = \dot{\boldsymbol{R}}\mathrm{d}m$$

$$\boldsymbol{p} = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{p} = \int_{\mathcal{B}} \dot{\boldsymbol{R}}\mathrm{d}m = \int_{\mathcal{B}} (\dot{\boldsymbol{R}}_c + \dot{\boldsymbol{r}})\mathrm{d}m = \left(\int_{\mathcal{B}} \mathrm{d}m\right)\dot{\boldsymbol{R}}_c + \underset{\text{C.M.}}{\cancel{\int_{\mathcal{B}} \dot{\boldsymbol{r}}\mathrm{d}m}}$$

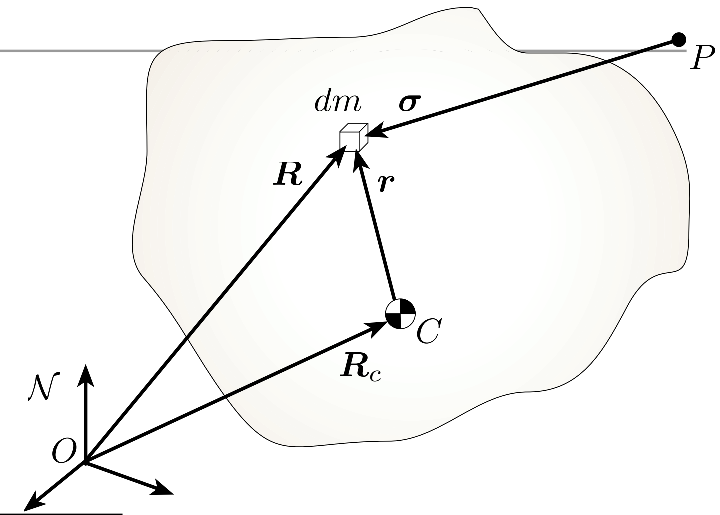
$$\boldsymbol{p} = M\dot{\boldsymbol{R}}_c$$

Linear Momentum Rate:

$$\dot{\boldsymbol{p}} = \int_{\mathcal{B}} \ddot{\boldsymbol{R}}\mathrm{d}m = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{F} = \boldsymbol{F}$$

$$\boldsymbol{F} = \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p})$$

Angular Momentum



Ang. Momentum about *P*:

$$\boldsymbol{H}_P = \int_B \boldsymbol{\sigma} \times \dot{\boldsymbol{\sigma}} \mathrm{d}m$$

$$\boldsymbol{\sigma} = \boldsymbol{R} - \boldsymbol{R}_P$$

Inertial Time Derivative:

$$\dot{\boldsymbol{H}}_P = \cancel{\int_B \dot{\boldsymbol{\sigma}} \times \dot{\boldsymbol{\sigma}} \mathrm{d}m} + \int_B \boldsymbol{\sigma} \times \boxed{\ddot{\boldsymbol{\sigma}}} \mathrm{d}m$$

$$\dot{\boldsymbol{H}}_P = \boxed{\int_B \boldsymbol{\sigma} \times \ddot{\boldsymbol{R}} \mathrm{d}m} - \boxed{\left(\int_B \boldsymbol{\sigma} \mathrm{d}m\right)} \boxed{\times \ddot{\boldsymbol{R}}_P}$$

$$\int_B \boldsymbol{\sigma} \mathrm{d}m = \int_B \boldsymbol{R} \mathrm{d}m - \left(\int_B \mathrm{d}m\right) \boldsymbol{R}_P = \boxed{M(\boldsymbol{R}_c - \boldsymbol{R}_P)}$$

Torque about *P*:

$$\boldsymbol{L}_P = \boxed{\int_B \boldsymbol{\sigma} \times \ddot{\boldsymbol{R}} \mathrm{d}m} = \int_B \boldsymbol{\sigma} \times \mathrm{d}\boldsymbol{F}$$

$$\dot{\boldsymbol{H}}_P = \boldsymbol{L}_P + M \ddot{\boldsymbol{R}}_P \times (\boldsymbol{R}_c - \boldsymbol{R}_P)$$

$$\boxed{\dot{\boldsymbol{H}}_P = \boldsymbol{L}_P}$$

If *P* is CM or Inertial

Rigid Body Dynamics

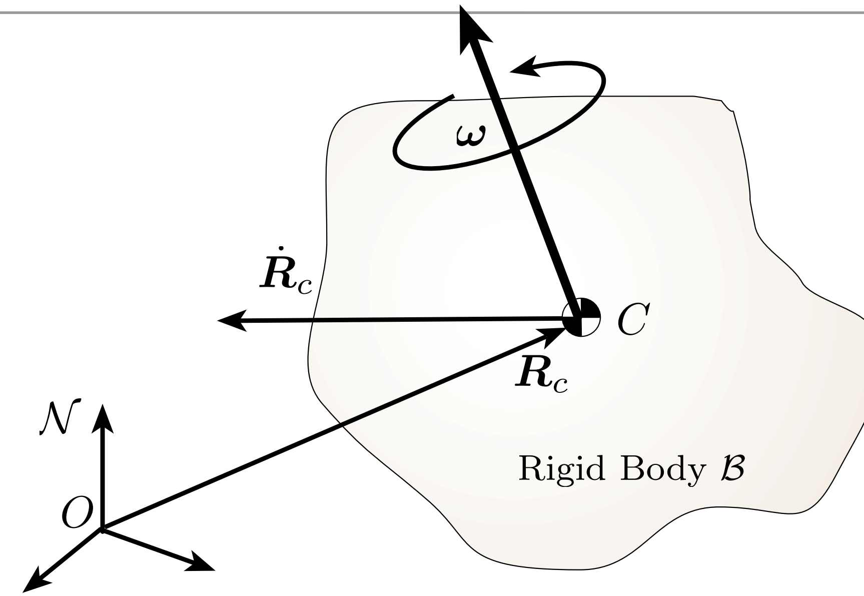
The 101 of spacecraft dynamics...

General Angular Momentum

Definition: $$\boldsymbol{H}_O = \int_B \boldsymbol{R} \times \dot{\boldsymbol{R}} \mathrm{d}m$$

or

$$\boldsymbol{H}_O = \underbrace{\boldsymbol{R}_c \times M\dot{\boldsymbol{R}}_c}_{\text{CM}} + \int_B \boldsymbol{r} \times \dot{\boldsymbol{r}} \mathrm{d}m$$



Momentum about CM:

$$\boldsymbol{H}_c = \int_B \boldsymbol{r} \times \dot{\boldsymbol{r}} \mathrm{d}m$$

$$\dot{\boldsymbol{r}} = \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \cancel{\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})} + \boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{H}_c = \int_B \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\, \mathrm{d}m = \left(\int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}] \mathrm{d}m \right) \boldsymbol{\omega}$$

Inertia Matrix Properties

Definition:

$$
{}^{\mathcal{B}}[I_c] = \int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m = \int_B {}^{\mathcal{B}}\begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} \mathrm{d}m
$$

Angular Momentum Expression:

$$
\boldsymbol{H}_c = {}^{\mathcal{B}}\begin{pmatrix} H_{c_1} \\ H_{c_2} \\ H_{c_3} \end{pmatrix} = \int_B {}^{\mathcal{B}}\begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \mathrm{d}m = [I_c]\boldsymbol{\omega}
$$

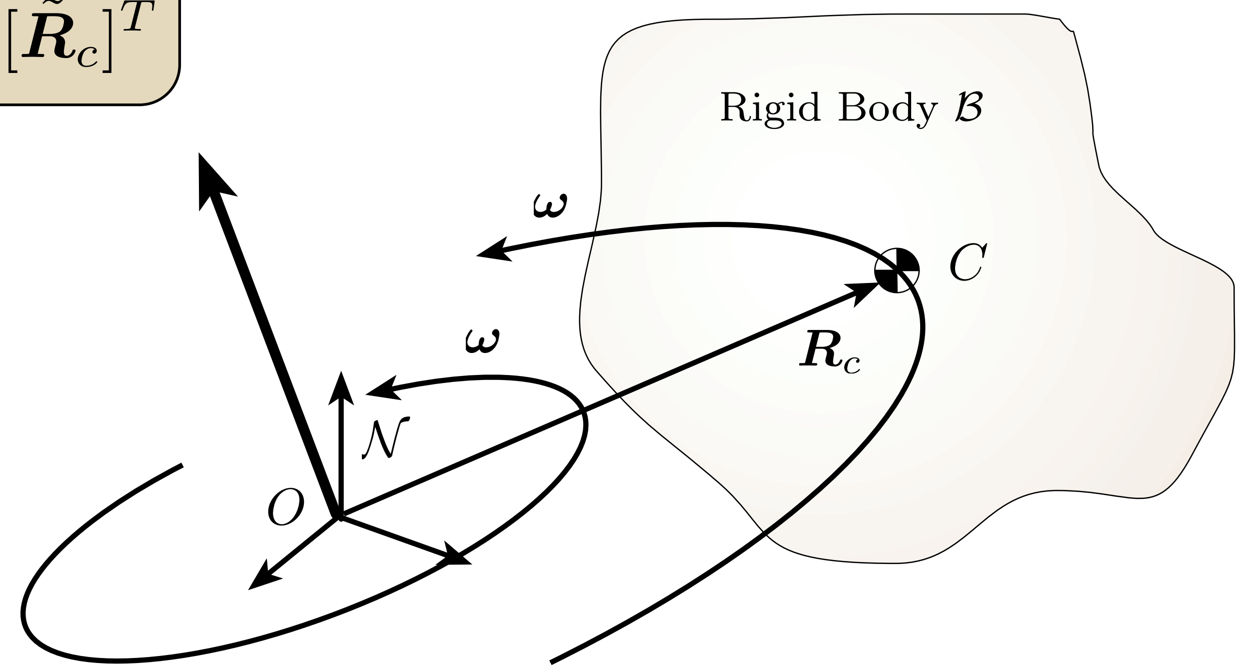
Parallel Axis Theorem

Total Mass

CM Offset

$$[I_O] = [I_c] + M[\tilde{\boldsymbol{R}}_c][\tilde{\boldsymbol{R}}_c]^T$$

Inertia about CM



Coordinate Transformation

$$\boxed{{}^{\mathcal{F}}[I] = [FB]\,{}^{\mathcal{B}}[I]\,[FB]^T}$$

B - Body Frame

F - 2nd Coordinate Frame

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} {}^{\mathcal{B}}\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} = {}^{\mathcal{F}}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Rotation matrix [*C*] contains the eigenvectors of [*I*]

Principal inertia matrix whose diagonal entries are the eigenvalues of [*I*]

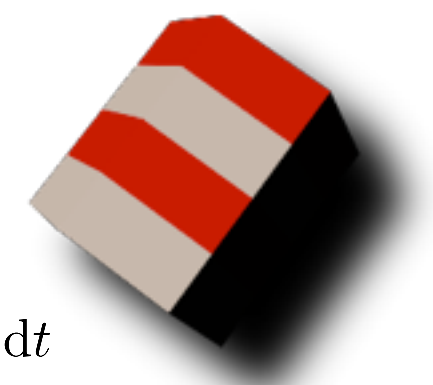
$$[C] = [V]^T = \begin{bmatrix} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \boldsymbol{v}_3^T \end{bmatrix}$$

Kinetic Energy

Total Energy: $$T = \frac{1}{2} M \dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \frac{1}{2} \int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \mathrm{d}m = T_{\text{trans}} + T_{\text{rot}}$$

Rotational Energy: $$T_{\text{rot}} = \frac{1}{2} \int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \mathrm{d}m = \frac{1}{2} \int_B (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}) \, \mathrm{d}m$$

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \int_B \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \, \mathrm{d}m = \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_c = \frac{1}{2} \boldsymbol{\omega}^T [I] \boldsymbol{\omega}$$



Energy Rate: $$\dot{T} = \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c + \boldsymbol{L}_c \cdot \boldsymbol{\omega}$$

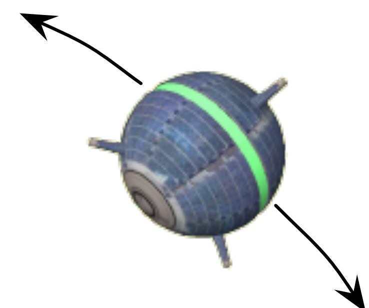
Work/Energy Principle: $$W = T(t_2) - T(t_1) = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c \mathrm{d}t + \int_{t_1}^{t_2} \boldsymbol{L}_c \cdot \boldsymbol{\omega} \mathrm{d}t$$

Equations of Motion

Euler's equation:

$$\dot{\boldsymbol{H}}_c = \boxed{\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{H}_c\right)} + \boldsymbol{\omega} \times \boldsymbol{H}_c = \boldsymbol{L}_c$$

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{H}_c\right) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left([I]\right)\boldsymbol{\omega} + [I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\omega}\right) = [I]\dot{\boldsymbol{\omega}}$$



Euler's rotational equations of motion:

$$\boxed{[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{L}_c}$$

Principal axis version of rotational EOM:

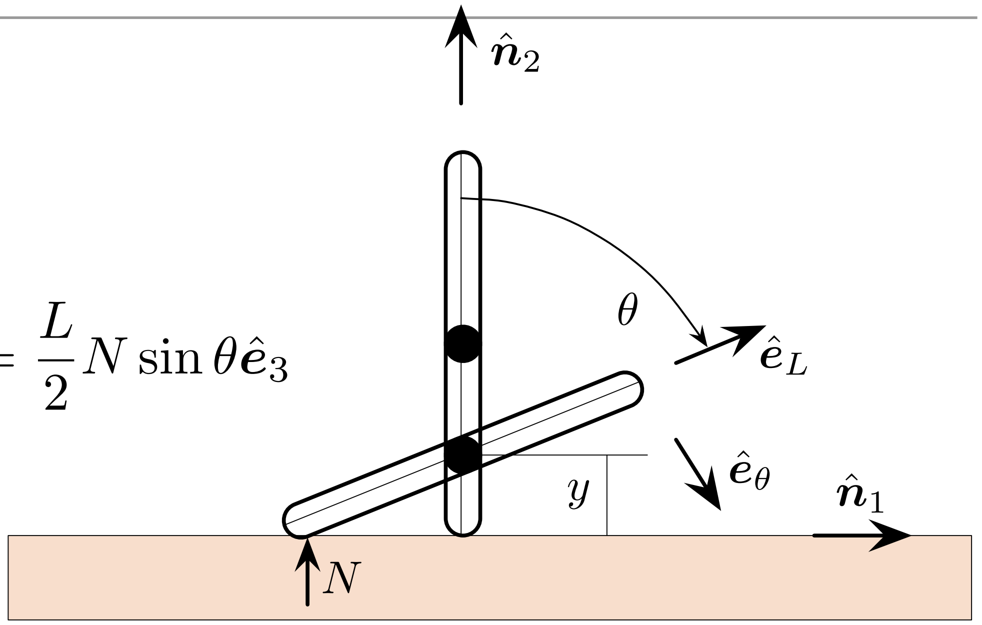
$$\begin{aligned} I_{11}\dot{\omega}_1 &= -(I_{33} - I_{22})\omega_2\omega_3 + L_1 \\ I_{22}\dot{\omega}_2 &= -(I_{11} - I_{33})\omega_3\omega_1 + L_2 \\ I_{33}\dot{\omega}_3 &= -(I_{22} - I_{11})\omega_1\omega_2 + L_3 \end{aligned}$$

Example: Slender Rod Falling

Rod Inertia about CM: $I_c = \frac{m}{12} L^2$

Momentum about CM: $\boldsymbol{H}_c = I_c \dot{\theta} \hat{\boldsymbol{e}}_3$

Torque: $\boldsymbol{L}_c = \left(-\frac{L}{2} \hat{\boldsymbol{e}}_L \right) \times N \hat{\boldsymbol{n}}_2 = \frac{L}{2} N \sin\theta \hat{\boldsymbol{e}}_3$



Euler's Eqn: $\dot{\boldsymbol{H}}_c = \boldsymbol{L}_c$

$$\frac{m}{12} L^2 \ddot{\theta} - \frac{L}{2} N \sin\theta = 0$$

Newton's Eqn: $m \ddot{y} \hat{\boldsymbol{n}}_2 = (N - mg) \hat{\boldsymbol{n}}_2$

$$y = \frac{L}{2} \cos\theta \Rightarrow \ddot{y} = -\frac{L}{2} \ddot{\theta} \sin\theta - \frac{L}{2} \dot{\theta}^2 \cos\theta$$

$$\Rightarrow N = mg - m\frac{L}{2} \ddot{\theta} \sin\theta - m\frac{L}{2} \dot{\theta}^2 \cos\theta$$

EOM:

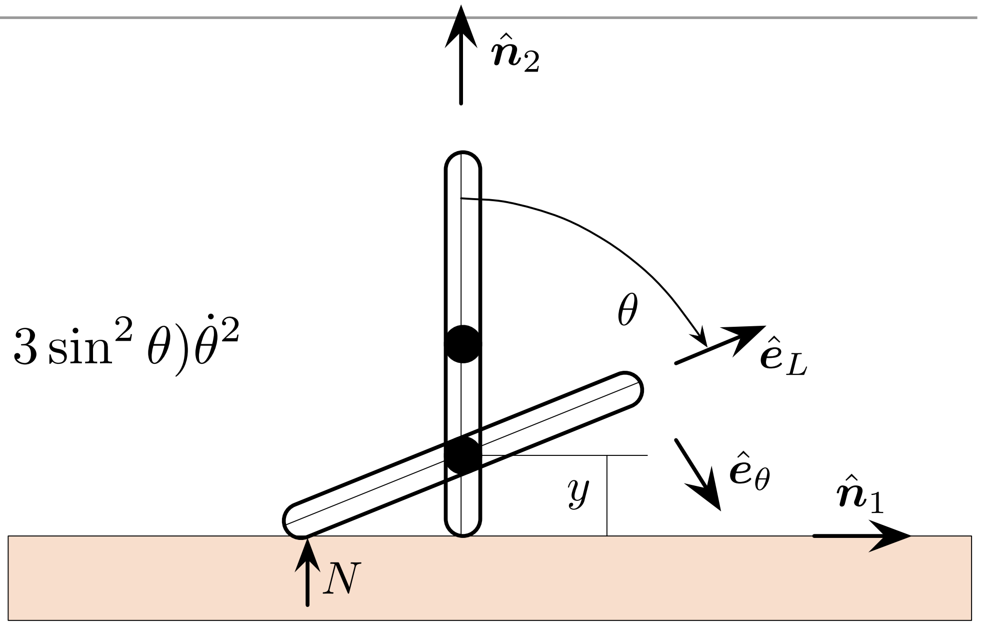
$$\boxed{\frac{m}{12} L^2 \ddot{\theta} (1 + 3 \sin^2\theta) + \frac{m}{4} L^2 \dot{\theta}^2 \sin\theta \cos\theta - \frac{m}{2} L g \sin\theta = 0}$$

Example: Slender Rod Falling

Energy functions:

$$V(\theta) = mgy = mg\frac{L}{2}\cos\theta$$

$$T(\theta,\dot\theta) = \frac{m}{2}\dot y^2 + \frac{I_c}{2}\dot\theta^2 = \frac{mL^2}{24}(1+3\sin^2\theta)\dot\theta^2$$



Initial energy level:

$$E(t_0) = T_0 + V_0 = mg\frac{L}{2}$$

$$T(\theta,\dot\theta) + V(\theta) = E(t_0)$$

Energy Conservation avoids having to solve ODE.

$$\Rightarrow \quad \boxed{\dot\theta^2 = \frac{12g(1-\cos\theta)}{L(1+3\sin^2\theta)}}$$

Momentum/Energy Surfaces

- Let's study a special class of rigid body motion where no external torque is acting on the body.
- In this case the kinetic energy and the angular momentum of the rigid body are conserved!
- In inertial frame vector components, we find

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = [BN]^T \, {}^{\mathcal{B}}\boldsymbol{H}$$

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix} \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

$$\dot{\boldsymbol{H}} = 0 = \boldsymbol{f}(\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3)$$

- Notice that the constant angular momentum condition can become very complicated and difficult to study!

- Instead of writing ***H*** in inertial frame components, we chose to write it in the body frame where the inertia matrix is a constant for a rigid body.

- Assume the angular momentum vector ***H*** is written in body frame components, and that principal axes were chosen for the body frame *B*.

$$
\begin{aligned}
\boldsymbol{\omega} &= \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3 \\
\boldsymbol{H} = {}^{\mathcal{B}}\boldsymbol{H} &= H_1 \hat{\boldsymbol{b}}_1 + H_2 \hat{\boldsymbol{b}}_2 + H_3 \hat{\boldsymbol{b}}_3 \\
&= I_1 \omega_1 \hat{\boldsymbol{b}}_1 + I_2 \omega_2 \hat{\boldsymbol{b}}_2 + I_3 \omega_3 \hat{\boldsymbol{b}}_3
\end{aligned}
\qquad
[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}
$$

- This allows us to write ***H*** as:

$$\boldsymbol{H} = {}^{\mathcal{B}}\boldsymbol{H} = {}^{\mathcal{B}}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = {}^{\mathcal{B}}\begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

- Because ***H*** is constant, all possible rigid body angular velocities must lie on the surface of the following momentum ellipsoid:

$$H^2 = \boldsymbol{H}^T\boldsymbol{H} = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{constant}$$

- Similarly, since the kinetic energy is conserved, all possible rigid body angular velocities must also lie on the surface of the following energy ellipsoid:

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

- The final admissible angular velocities will be on the intersection of these two ellipsoids.

- Using the momenta coordinates

$$H_1 = I_1\omega_1 \qquad H_2 = I_2\omega_2 \qquad H_3 = I_3\omega_3$$

- we can write the momentum magnitude constraint as

$$H^2 = H_1^2 + H_2^2 + H_3^2 \Rightarrow$$ Sphere

- and the kinetic energy constraint as

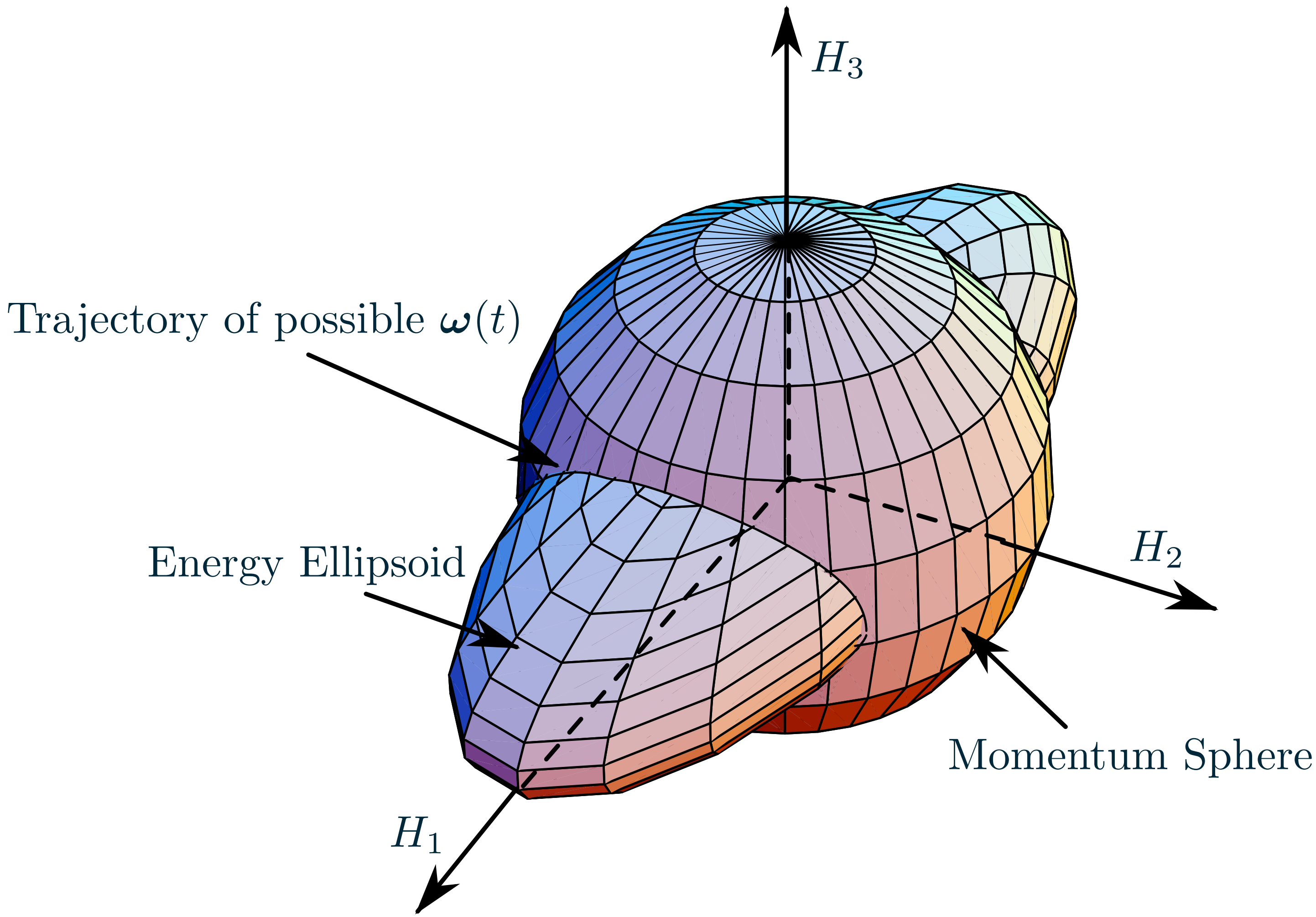
$$1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T}$$

Compare to ellipsoid equation:

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

$$\sqrt{2I_iT} \Rightarrow$$ semi-axes of ellipsoid

- Clearly, for a given *H*, only a certain range of kinetic energies is possible.
- Let's assume the common notation: $I_1 \geq I_2 \geq I_3$



- Let's look at the Minimum Energy Case:

- The surfaces will only intersect at:

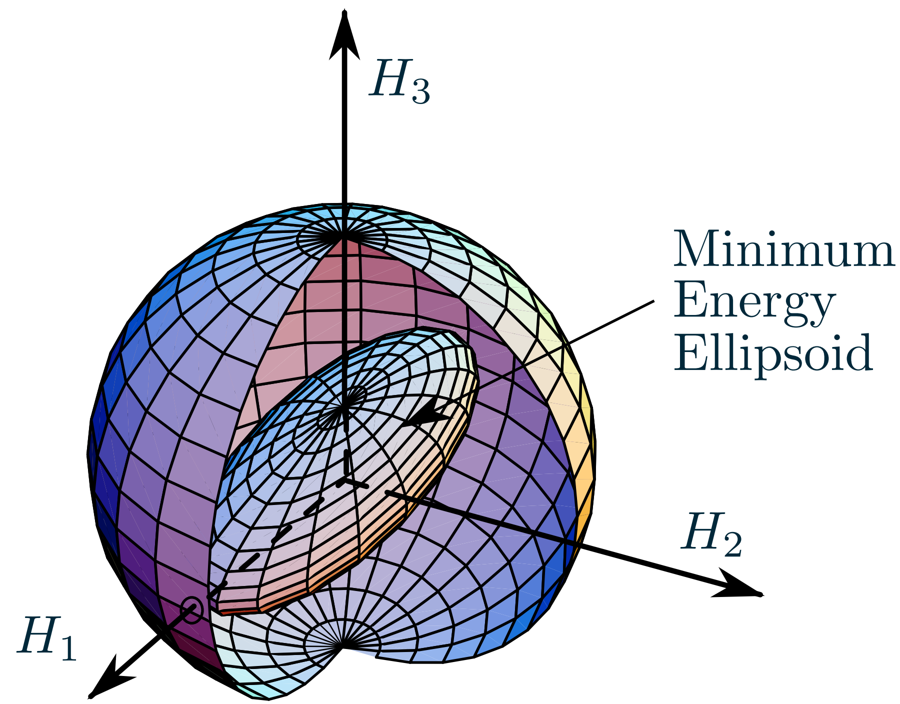
$$^{\mathcal{B}}\boldsymbol{H} = \pm H\hat{\boldsymbol{b}}_1$$

$$H_1 = H \qquad H_2 = H_3 = 0$$

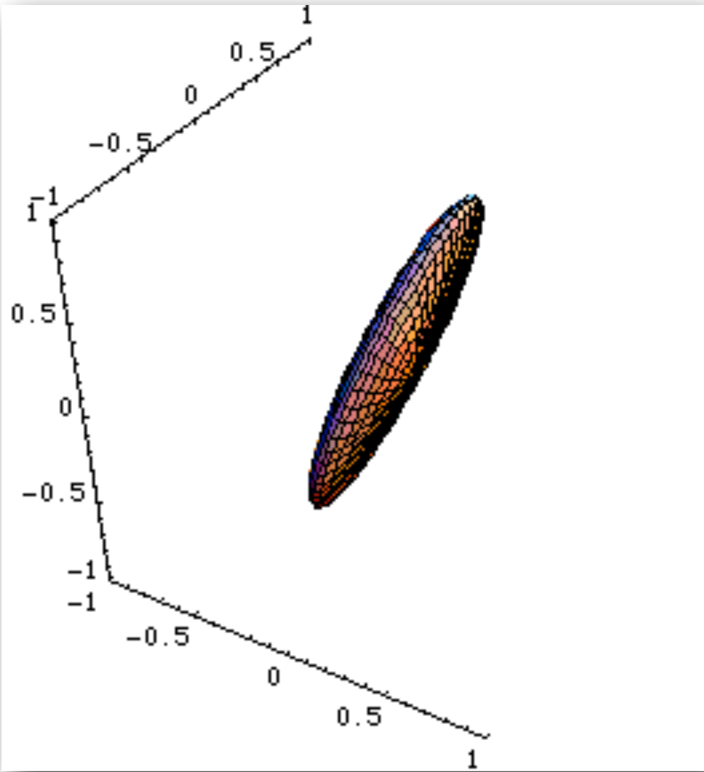
The kinetic energy is:

$$1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T}$$

$$\boxed{T_{\text{min}} = \frac{H^2}{2I_1}}$$

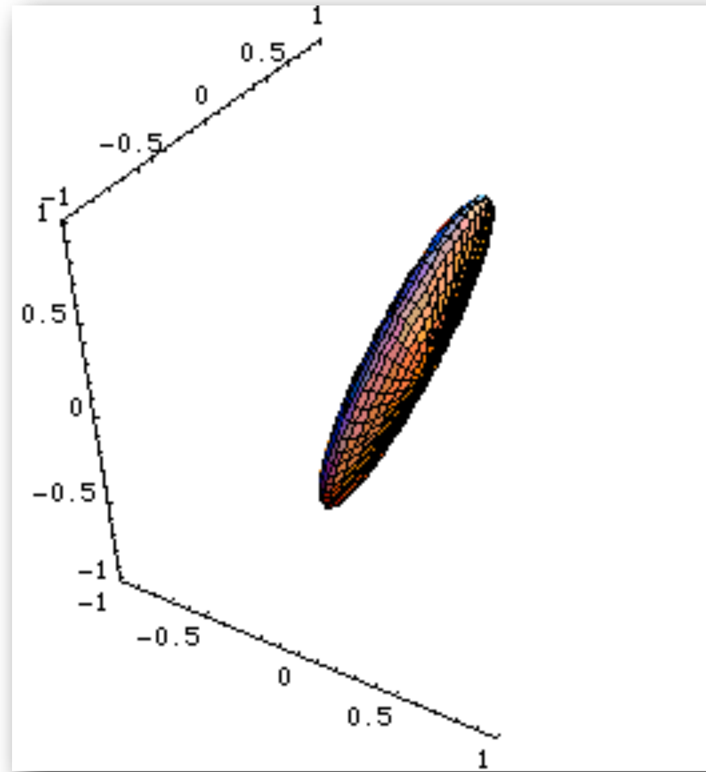


General Rotation of Rigid Body with $I_1 \geq I_2 \geq I_3$



Pure Spin

$$\boldsymbol{\omega}_0 = (10^o, 0^o, 0^o)\ /\mathrm{s}$$



Slightly Off Spin

$$\boldsymbol{\omega}_0 = (10^o, 0.5^o, 0.5^o)\ /\mathrm{s}$$

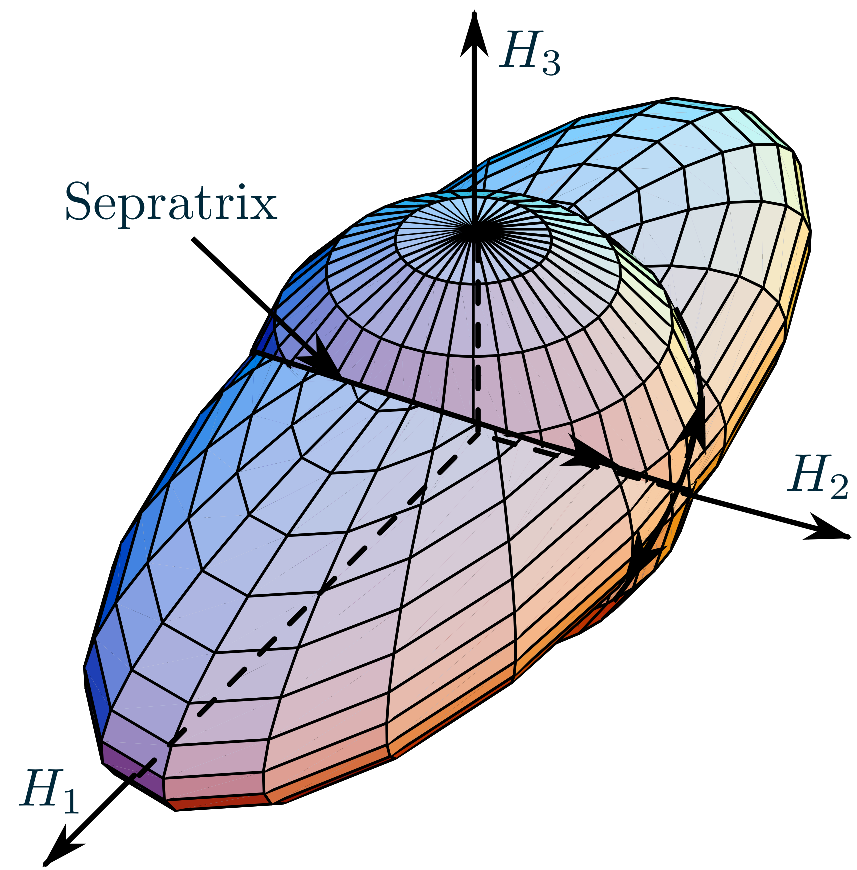
- Let's look at the Intermediate Energy Case:
- The surfaces will only intersect at:

$$^{\mathcal{B}}\boldsymbol{H} = \pm H\hat{\boldsymbol{b}}_2$$

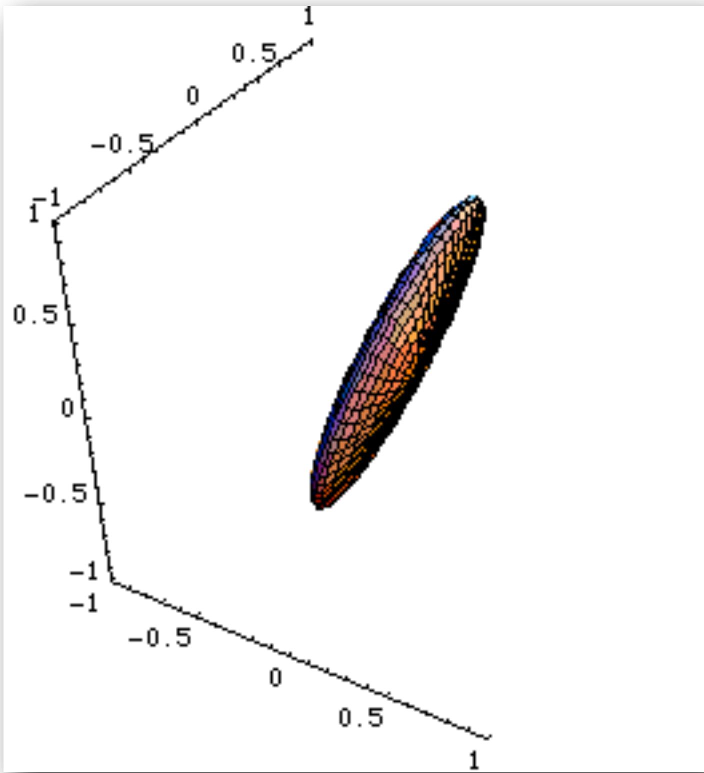
The kinetic energy is:

$$1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T}$$

$$\boxed{T_{\text{int}} = \frac{H^2}{2I_2}}$$

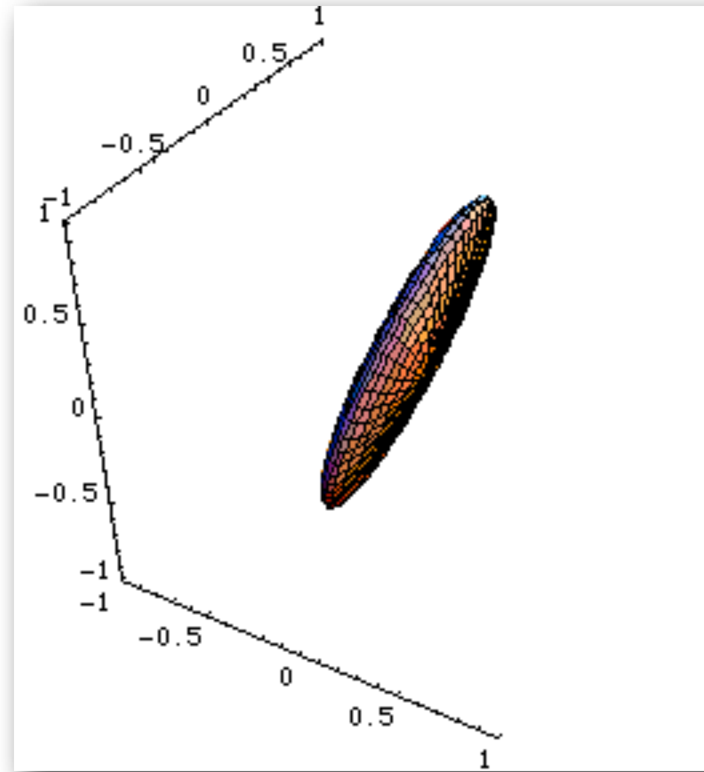


General Rotation of Rigid Body with $I_1 \geq I_2 \geq I_3$



Pure Spin

$\boldsymbol{\omega}_0 = (0^o, 10^o, 0^o)$ /s



Slightly Off Spin

$\boldsymbol{\omega}_0 = (0.5^o, 10^o, 0.5^o)$ /s

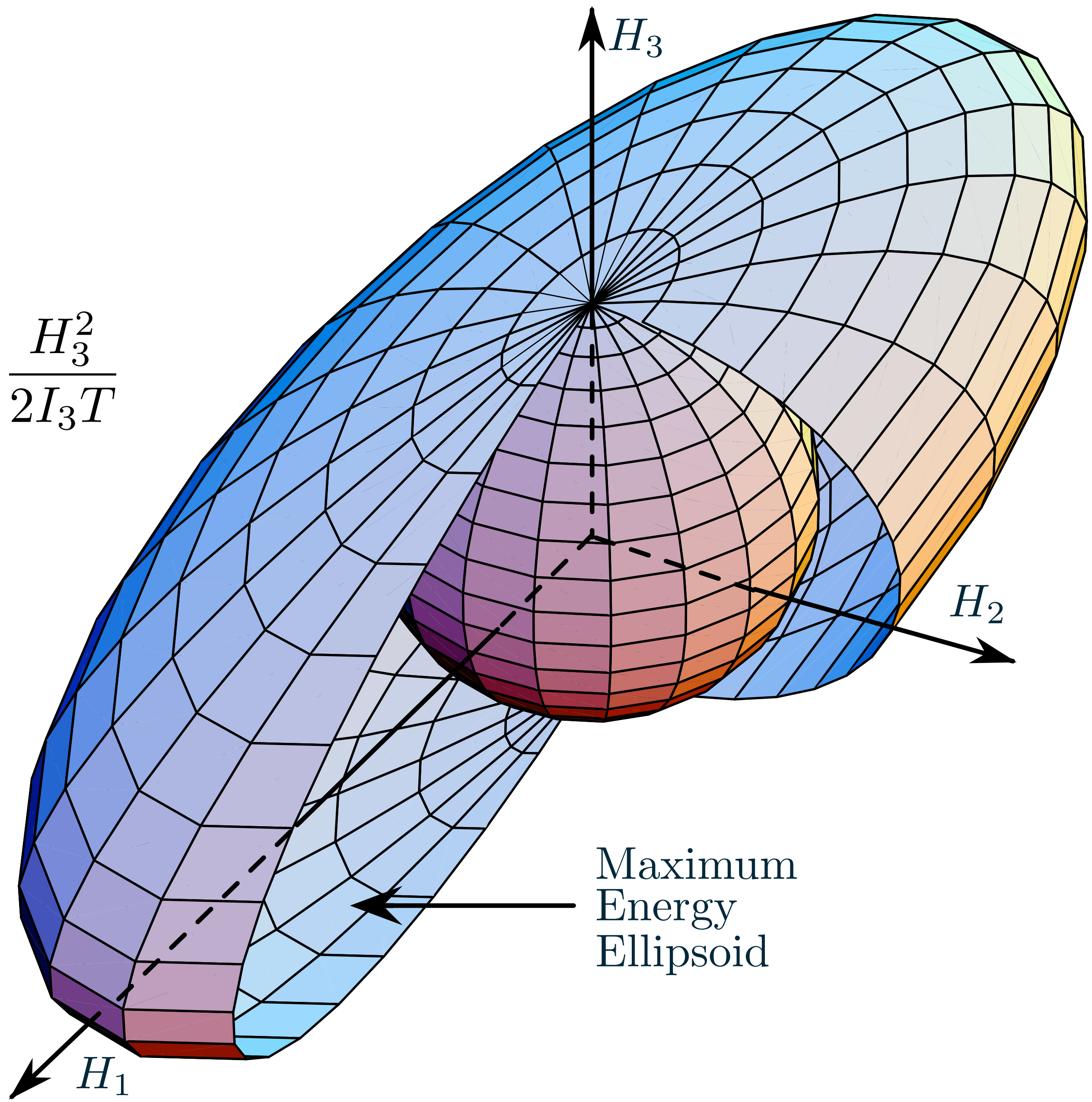
- Maximum Energy Case:

$$ {}^{\mathcal{B}}\boldsymbol{H} = H\hat{\boldsymbol{b}}_3 $$

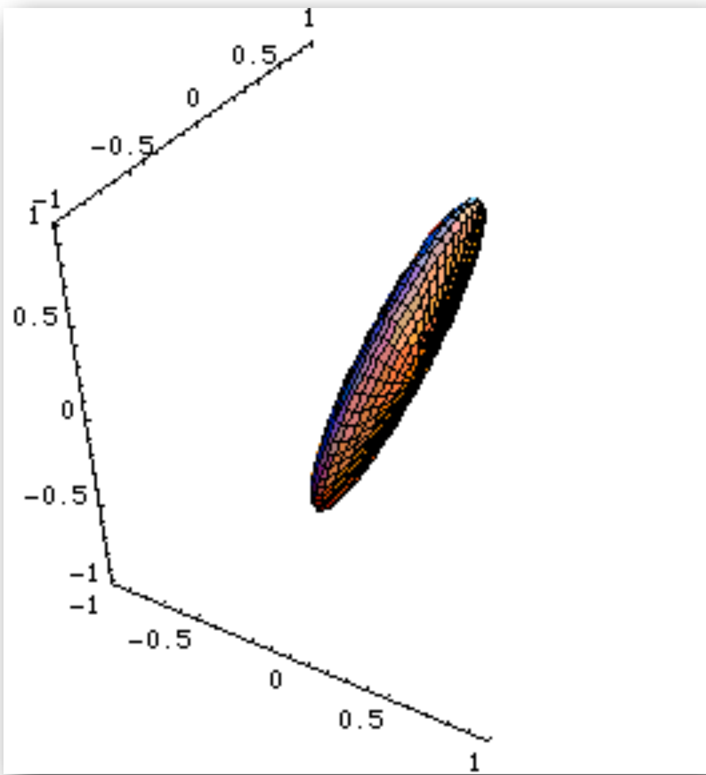
The kinetic energy is:

$$ 1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T} $$

$$ \boxed{T_{\max} = \frac{H^2}{2I_3}} $$

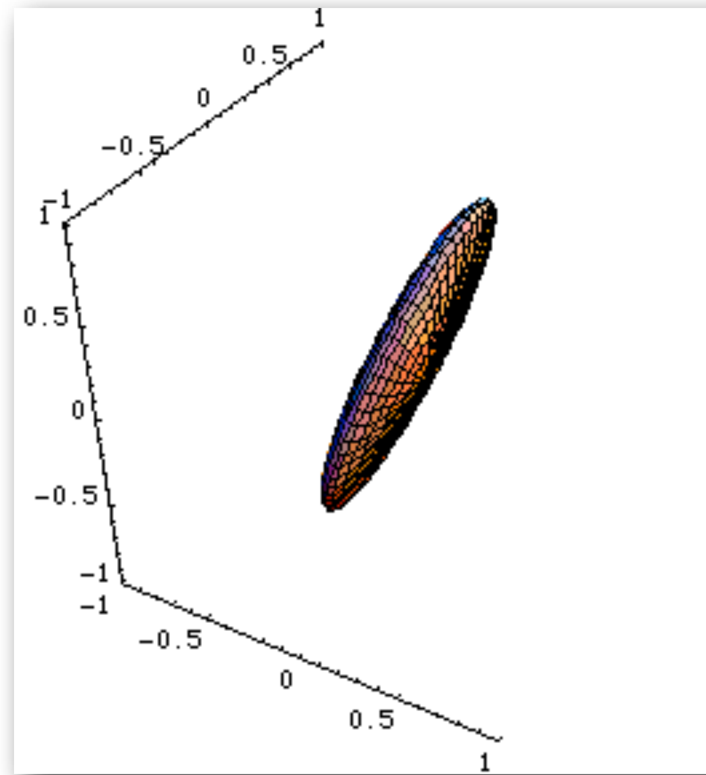


General Rotation of Rigid Body with $I_1 \geq I_2 \geq I_3$



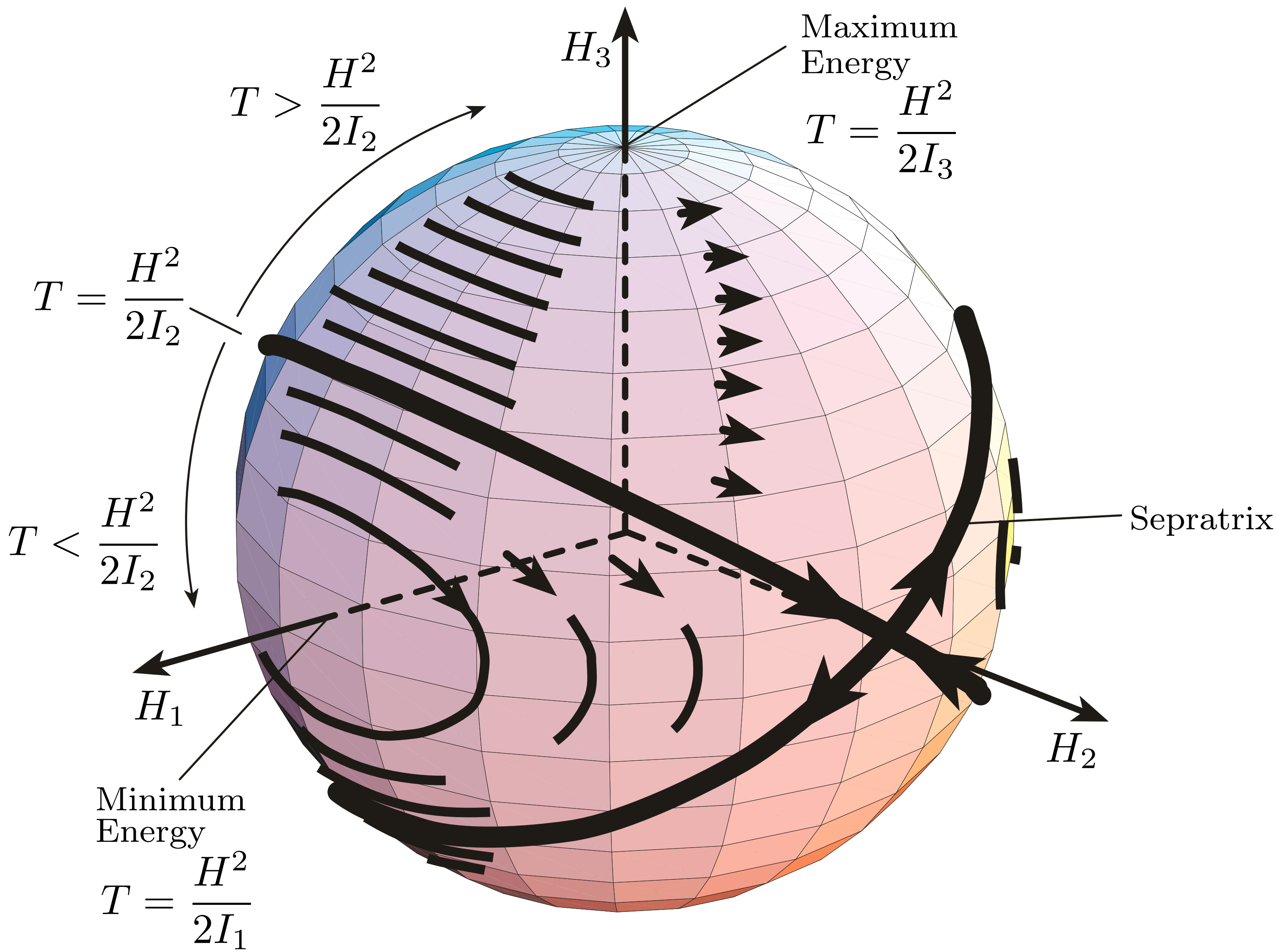
Pure Spin

$$\boldsymbol{\omega}_0 = (0^o, 0^o, 10^o)\ \text{/s}$$

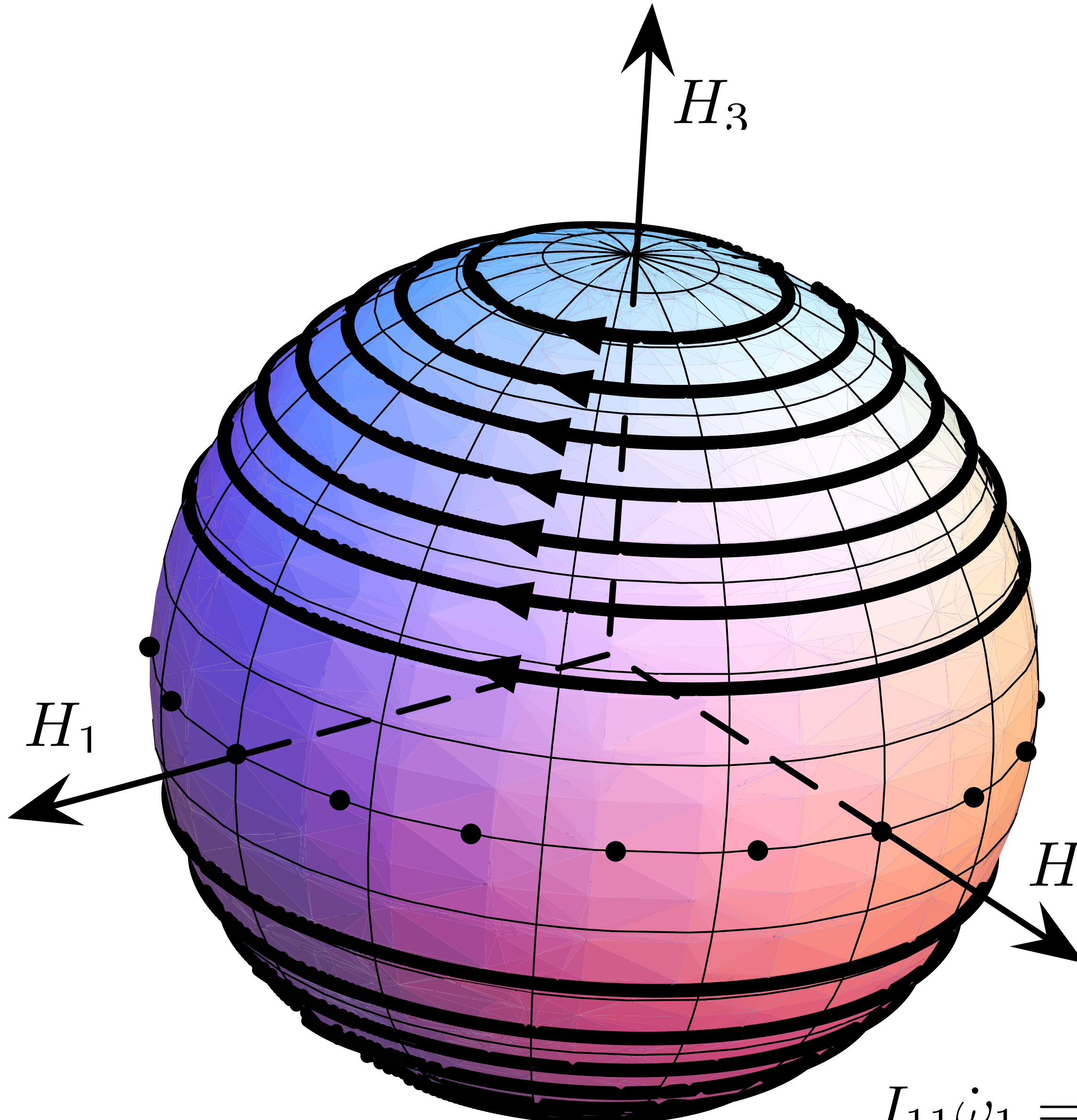


Slightly Off Spin

$$\boldsymbol{\omega}_0 = (0.5^o, 0.5^o, 10^o)\ \text{/s}$$

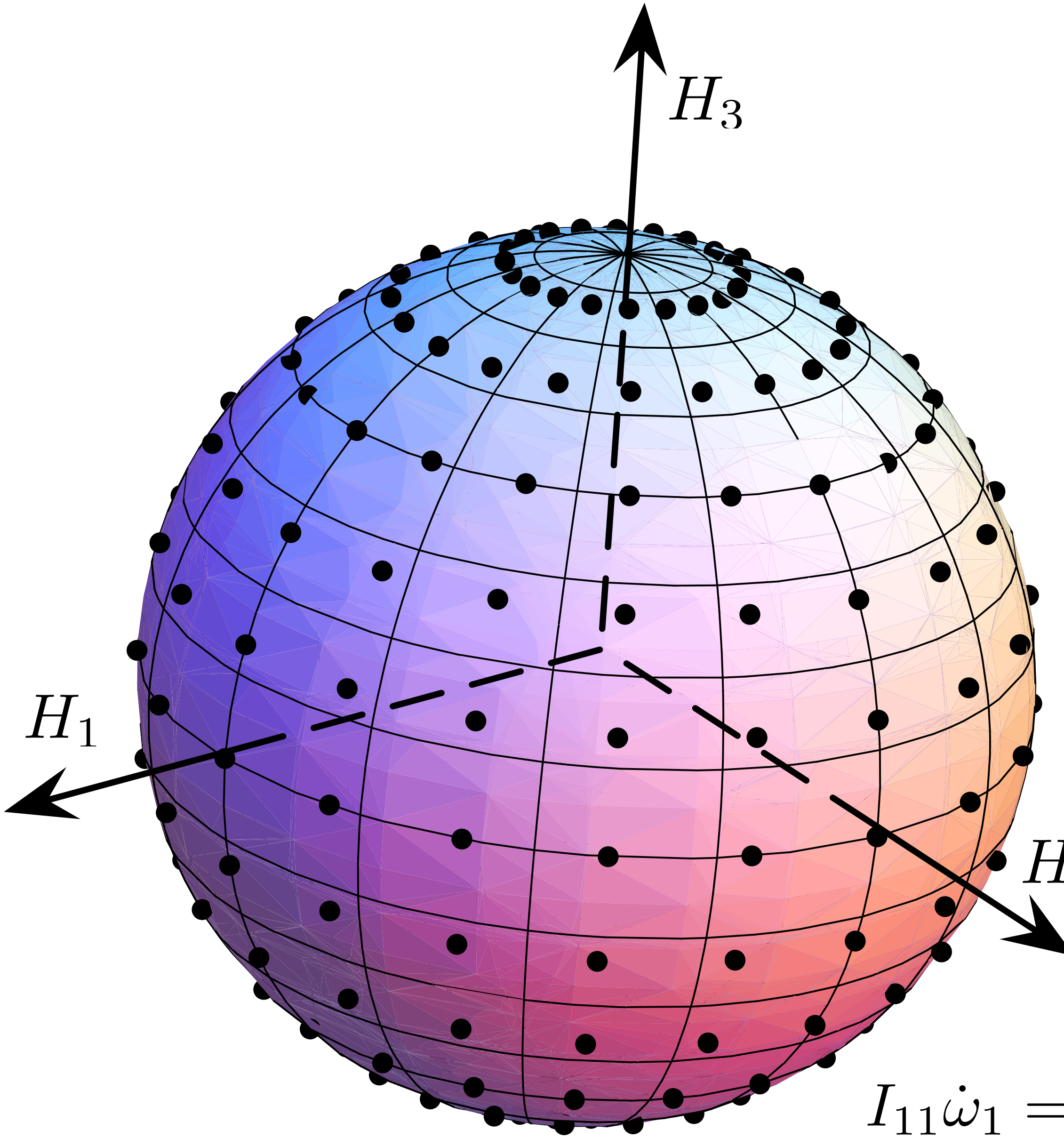


Family of energy ellipsoid and momentum sphere intersections.



What type of spacecraft body would yield this?

$$I_{11}\dot{\omega}_1 = -(I_{33} - I_{22})\omega_2\omega_3$$
$$I_{22}\dot{\omega}_2 = -(I_{11} - I_{33})\omega_3\omega_1$$
$$I_{33}\dot{\omega}_3 = -(I_{22} - I_{11})\omega_1\omega_2$$



What type of spacecraft body would yield this?

$$I_{11}\dot{\omega}_1 = -(I_{33} - I_{22})\omega_2\omega_3$$
$$I_{22}\dot{\omega}_2 = -(I_{11} - I_{33})\omega_3\omega_1$$
$$I_{33}\dot{\omega}_3 = -(I_{22} - I_{11})\omega_1\omega_2$$

Analytical Torque-Free Motion

- Let us assume that there are no external torques acting on the rigid body, and the equations of motion are given by:

$$\begin{aligned} I_{11}\dot{\omega}_1 &= -(I_{33} - I_{22})\omega_2\omega_3 \\ I_{22}\dot{\omega}_2 &= -(I_{11} - I_{33})\omega_3\omega_1 \\ I_{33}\dot{\omega}_3 &= -(I_{22} - I_{11})\omega_1\omega_2 \end{aligned}$$

- We are looking for analytical solutions to the angular motion.
- Assume that the body coordinate frame is a principal frame, and the inertia matrix is diagonal.

General Inertia Case*

$$H^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$$

Momentum and kinetic energy conservation yield two integrals of the torque-free motion.

$$\omega_2^2 = \left(\frac{2I_3T - H^2}{I_2(I_3 - I_2)}\right) - \frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\omega_1^2$$

$$\omega_3^2 = \left(\frac{2I_2T - H^2}{I_3(I_2 - I_3)}\right) - \frac{I_1(I_2 - I_1)}{I_3(I_2 - I_3)}\omega_1^2$$

We can use these two equations to solve for two of the angular rates!

Analogously, we can solve for the two angular velocities in terms of other angular rates.

$$\omega_1^2 = \left(\frac{2I_3T - H^2}{I_1(I_3 - I_1)}\right) - \frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\omega_2^2$$

$$\omega_3^2 = \left(\frac{2I_1T - H^2}{I_3(I_1 - I_3)}\right) - \frac{I_2(I_1 - I_2)}{I_3(I_1 - I_3)}\omega_2^2$$

$$\omega_1^2 = \left(\frac{2I_2T - H^2}{I_1(I_2 - I_1)}\right) - \frac{I_3(I_2 - I_3)}{I_1(I_2 - I_1)}\omega_3^2$$

$$\omega_2^2 = \left(\frac{2I_1T - H^2}{I_2(I_1 - I_2)}\right) - \frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)}\omega_3^2$$

* Junkins, J. L., Jacobson, I. D., and Blanton, J. N., "A Nonlinear Oscillator Analog of Rigid Body Dynamics," *Celestial Mechanics*, Vol. 7, pp. 398 – 407, 1973.

$$
\boxed{
\begin{aligned}
I_1\dot{\omega}_1 &= -(I_3 - I_2)\omega_2\omega_3 \\
I_2\dot{\omega}_2 &= -(I_1 - I_3)\omega_3\omega_1 \\
I_3\dot{\omega}_3 &= -(I_2 - I_1)\omega_1\omega_2
\end{aligned}
}
\quad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}
$$

$$
\begin{aligned}
\ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\left(\boxed{\dot{\omega}_2}\omega_3 + \omega_2\boxed{\dot{\omega}_3}\right) \\
\ddot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\left(\boxed{\dot{\omega}_3}\omega_1 + \omega_3\boxed{\dot{\omega}_1}\right) \\
\ddot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\left(\boxed{\dot{\omega}_1}\omega_2 + \omega_1\boxed{\dot{\omega}_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\left(\frac{I_1 - I_2}{I_3}\omega_1\omega_2^2 + \frac{I_3 - I_1}{I_2}\omega_1\omega_3^2\right) \\
\ddot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\omega_2\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_2\omega_3^2\right) \\
\ddot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\left(\frac{I_3 - I_1}{I_2}\omega_3\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_3\omega_2^2\right)
\end{aligned}
$$

$$\ddot{\omega}_1 = \frac{I_2 - I_3}{I_1}\left(\frac{I_1 - I_2}{I_3}\omega_1\boxed{\omega_2^2} + \frac{I_3 - I_1}{I_2}\omega_1\boxed{\omega_3^2}\right)$$

$$\ddot{\omega}_2 = \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\omega_2\boxed{\omega_1^2} + \frac{I_2 - I_3}{I_1}\omega_2\boxed{\omega_3^2}\right)$$

$$\ddot{\omega}_3 = \frac{I_1 - I_2}{I_3}\left(\frac{I_3 - I_1}{I_2}\omega_3\boxed{\omega_1^2} + \frac{I_2 - I_3}{I_1}\omega_3\boxed{\omega_2^2}\right)$$

$$\omega_2^2 = \left(\frac{2I_3T - H^2}{I_2(I_3 - I_2)}\right) - \frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\omega_1^2$$

$$\omega_3^2 = \left(\frac{2I_2T - H^2}{I_3(I_2 - I_3)}\right) - \frac{I_1(I_2 - I_1)}{I_3(I_2 - I_3)}\omega_1^2$$

$$\omega_1^2 = \left(\frac{2I_3T - H^2}{I_1(I_3 - I_1)}\right) - \frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\omega_2^2$$

$$\omega_3^2 = \left(\frac{2I_1T - H^2}{I_3(I_1 - I_3)}\right) - \frac{I_2(I_1 - I_2)}{I_3(I_1 - I_3)}\omega_2^2$$

$$\omega_1^2 = \left(\frac{2I_2T - H^2}{I_1(I_2 - I_1)}\right) - \frac{I_3(I_2 - I_3)}{I_1(I_2 - I_1)}\omega_3^2$$

$$\omega_2^2 = \left(\frac{2I_1T - H^2}{I_2(I_1 - I_2)}\right) - \frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)}\omega_3^2$$

$$\boxed{\ddot{\omega}_i + A_i\omega_i + B_i\omega_i^3 = 0 \qquad \text{for } i = 1, 2, 3}$$

homogenous, undamped Duffing equation

Duffing equations are often found studying nonlinear mechanical oscillations, where the cubic “stiffness” term arises to approximately account for nonlinear departure from Hooke’s law. For the torque-free motion, this equation is the *exact differential equation*!

$$\ddot{\omega}_i + A_i \omega_i + B_i \omega_i^3 = 0 \qquad \text{for } i = 1, 2, 3$$

- These equations form three *uncoupled nonlinear oscilllators*.

- Notice that while the oscillators are *uncoupled*, they are not *independent*! The six spring constants are all uniquely determined from initially evaluated inertia, energy and momentum constants.

i	A_i	B_i
1	$\frac{(I_1 - I_2)(2I_3T - H^2) + (I_1 - I_3)(2I_2T - H^2)}{I_1I_2I_3}$	$\frac{2(I_1 - I_2)(I_1 - I_3)}{I_2I_3}$
2	$\frac{(I_2 - I_3)(2I_1T - H^2) + (I_2 - I_1)(2I_3T - H^2)}{I_1I_2I_3}$	$\frac{2(I_2 - I_1)(I_2 - I_3)}{I_1I_3}$
3	$\frac{(I_3 - I_1)(2I_2T - H^2) + (I_3 - I_2)(2I_1T - H^2)}{I_1I_2I_3}$	$\frac{2(I_3 - I_1)(I_3 - I_2)}{I_1I_2}$

- The oscillator differential equations have three immediate integrals of the form

$$\dot{\omega}_i^2 + A_i \omega_i^2 + \frac{B_i}{2}\omega_i^4 = K_i \qquad \text{for } i = 1, 2, 3$$

- Here K_1, K_2 and K_3 are the three oscillator "energy-type" integral constants of the motion.

$$K_1 = \frac{(2I_2T - H^2)(H^2 - 2I_3T)}{I_1^2 I_2 I_3}$$

$$K_2 = \frac{(2I_3T - H^2)(H^2 - 2I_1T)}{I_1 I_2^2 I_3}$$

$$K_3 = \frac{(2I_1T - H^2)(H^2 - 2I_2T)}{I_1 I_2 I_3^2}$$

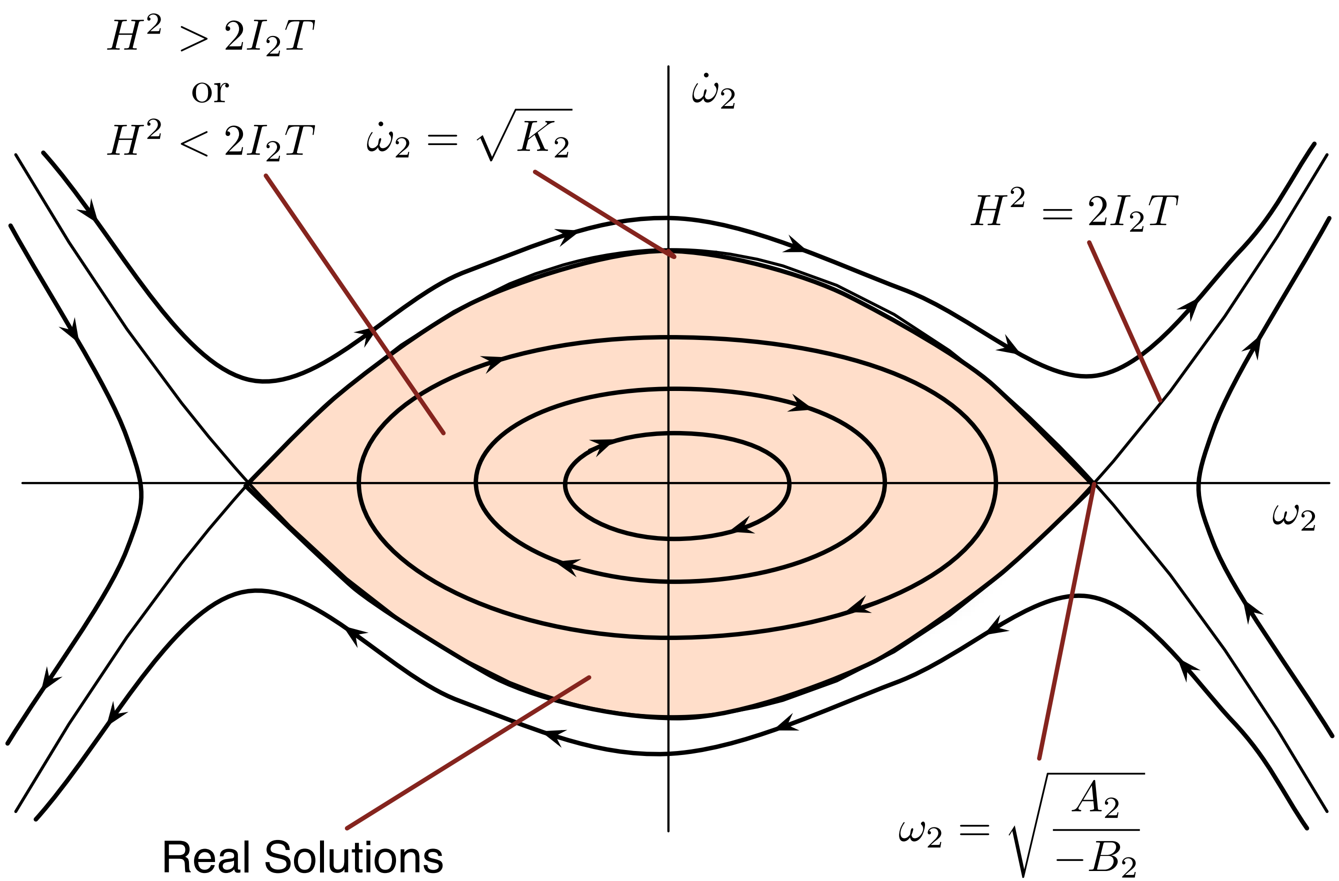
Assume: $I_1 \geq I_2 \geq I_3$

i	A_i	B_i
1	not defined	>0
2	>0	<0
3	not defined	>0

- The linear "spring constants" A_1 and A_3 can produce de-stabilizing spring forces (negative spring effect).

- The positive cubic "spring constants" B_1 and B_3 always produce restoring forces and are therefore hard springs. Because cubic springs will override linear springs for sufficiently large displacements, all trajectories of the 1st and 3rd phase planes must be closed.

- The cubic spring constant B_2 produces a de-stabilizing force (soft spring), and will eventually override the stabilizing linear spring force.



- Only solutions with $K_2 \geq 0$ are physically possible
- The limiting trajectory occurs if
  - $I_1 \rightarrow I_3$
  - $H^2 \rightarrow 2\, I_2 T$ (pure spin about intermediate inertia axis)
  - $I_1 I_2 I_3 \rightarrow \infty$

- Next, let's study the phase plane behavior for the 1st and 3rd oscillator equations.

- Due to the strong cubic spring term, all trajectories will be closed. However, to they encircle the origin?

Intercepts:

$$\dot{\omega}_i^2(\text{for } \omega_i = 0) = K_i$$

$$\omega_i^2(\text{for } \dot{\omega}_i = 0) = \frac{-A_i \pm \sqrt{A_i^2 + 2B_iK_i}}{B_i}$$

$$K_1, \quad A_1^2 + 2B_1^2K_1 > 0$$

$$K_3, \quad A_3^2 + 2B_3^2K_3 > 0$$

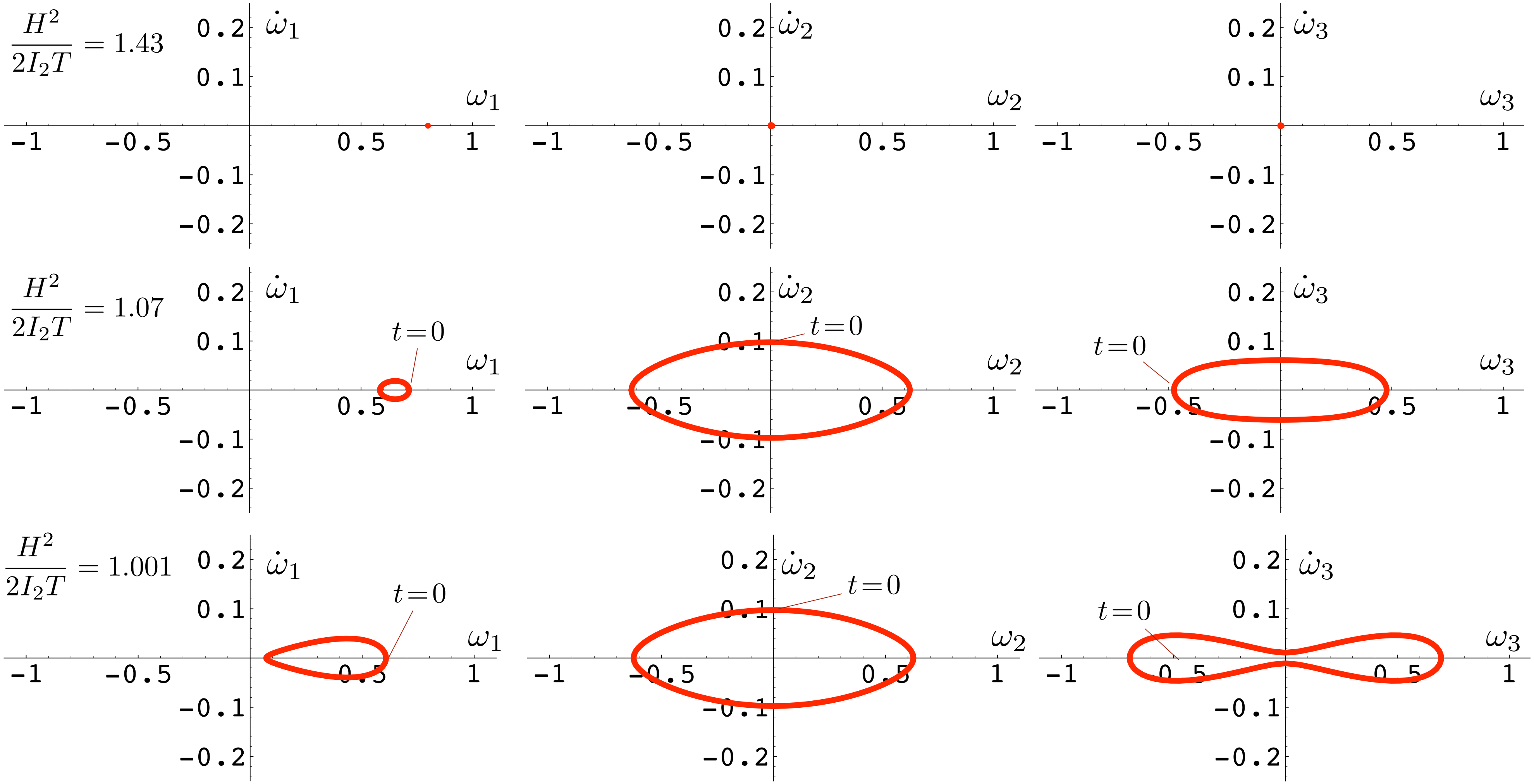
To have real intercepts, and have the trajectories circle the origin.

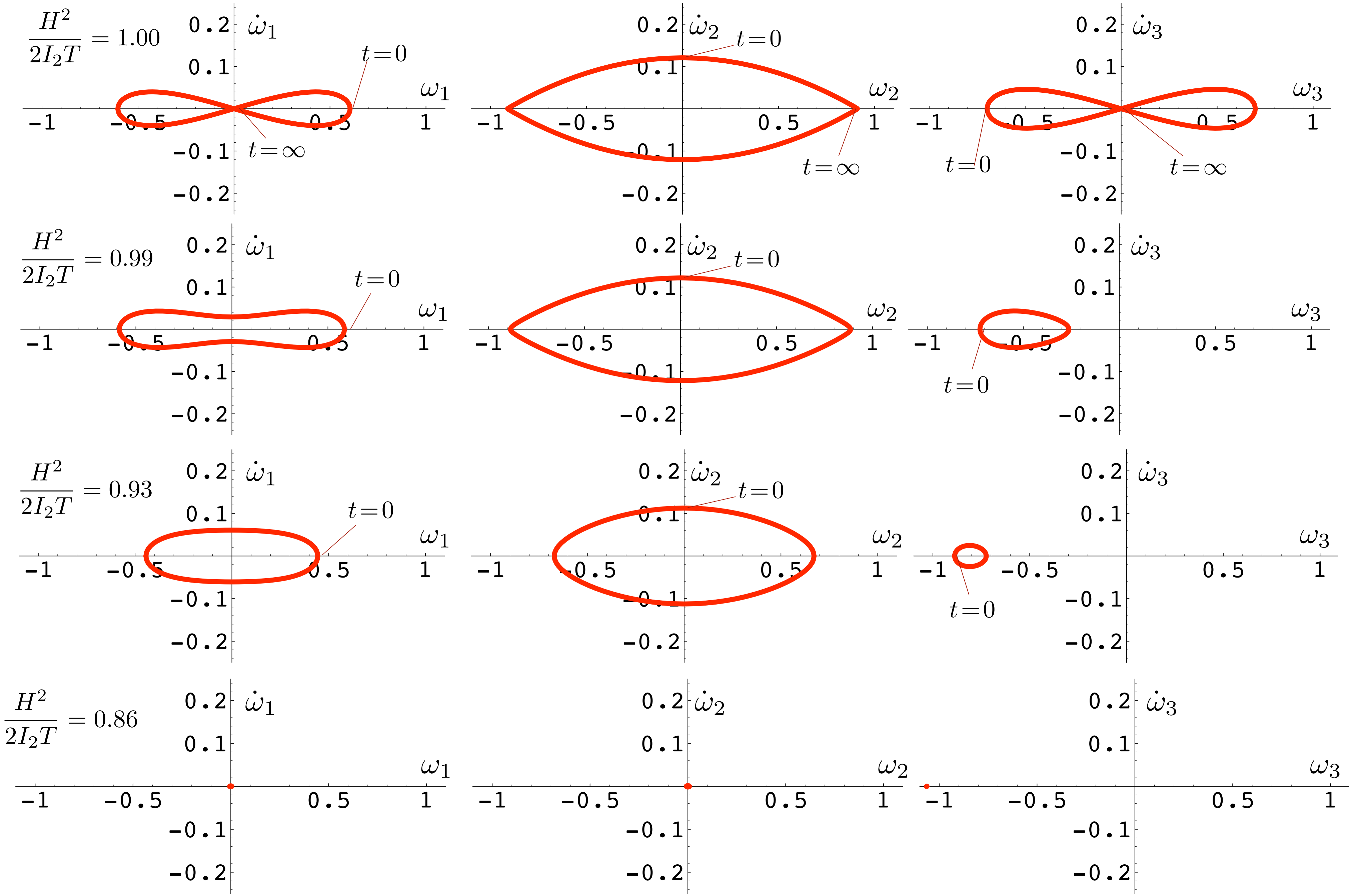
Note that

$H^2 > 2I_2T \quad \rightarrow \quad K_1 < 0, \quad K_3 > 0$ towards minimum energy case

$H^2 < 2I_2T \quad \rightarrow \quad K_1 > 0, \quad K_3 < 0$ towards maximum energy case

Let's sweep through cases from a maximum energy case to a minimum energy case. The momentum is held constant here.

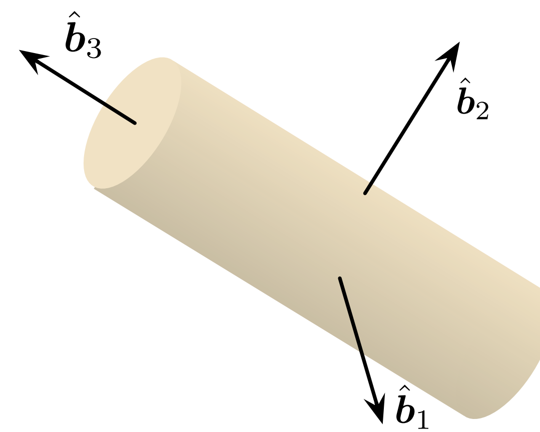



$\frac{H^2}{2I_2T} = 1.00$
$\frac{H^2}{2I_2T} = 0.99$
$\frac{H^2}{2I_2T} = 0.93$
$\frac{H^2}{2I_2T} = 0.86$
$\dot{\omega}_1$
ω_1
$\dot{\omega}_2$
ω_2
$\dot{\omega}_3$
ω_3
$t=0$
$t=\infty$
0.2
0.1
-0.1
-0.2
-1
-0.5
0.5
1

Axi-Symmetric Case

- Let the external torque be zero. Consider the special principal inertia case where

$$I_T = I_{11} = I_{22}$$



- Here the EOM are given by

$$\begin{aligned} I_T\dot{\omega}_1 &= -(I_{33} - I_T)\omega_2\omega_3 \\ I_T\dot{\omega}_2 &= (I_{33} - I_T)\omega_3\omega_1 \\ I_{33}\dot{\omega}_3 &= 0 \end{aligned}$$

- From this equation it is clear that the third angular velocity component will be constant.

$$\omega_3(t) = \omega_3(t_0) = \text{constant}$$

- Let's examine the remaining two differential equations more carefully. Taking the derivative of the first one we find

$$I_t\dot{\omega}_1 = -(I_{33} - I_T)\omega_2\omega_3$$

$$I_T\ddot{\boldsymbol{\omega}}_1 = -(I_{33} - I_T)\dot{\omega}_2\omega_3$$

$$\dot{\omega}_2 = \frac{1}{I_T}\left((I_{33} - I_T)\omega_3\omega_1\right)$$

$$\ddot{\omega}_1 + (\frac{I_{33}}{I_T} - 1)^2\omega_3^2\omega_1 = 0$$

Mathematically equivalent to simple Spring-Mass Systems!

Similarly, we can find:

$$\ddot{\omega}_2 + (\frac{I_{33}}{I_T} - 1)^2\omega_3^2\omega_2 = 0$$

- The analytical solution to a spring-mass dynamical system is the simple oscillator equation

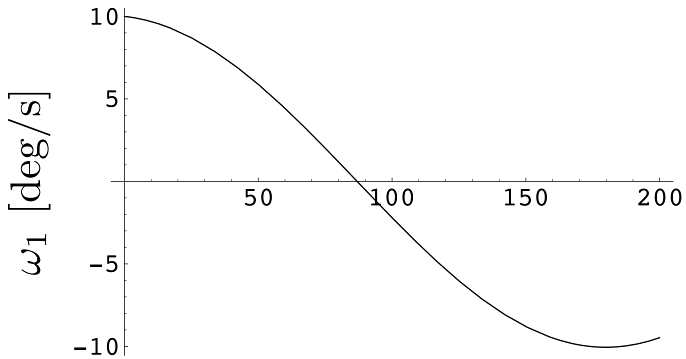
$$\omega_1(t) = A_1 \cos\omega_p t + B_1 \sin\omega_p t$$
$$\omega_2(t) = A_2 \cos\omega_p t + B_2 \sin\omega_p t$$

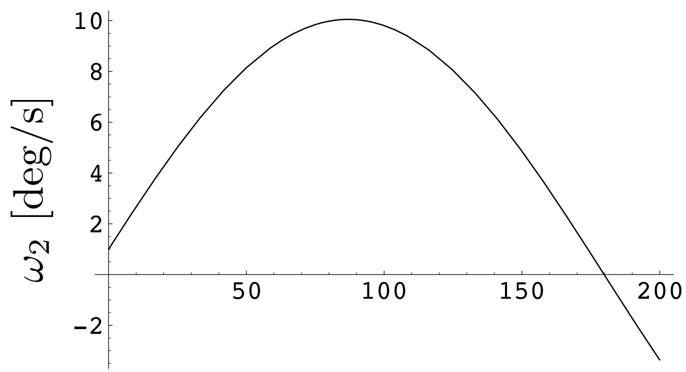
- Using the initial conditions, we find the analytical solution of the body angular velocity components for the axi-symmetric spacecraft case:

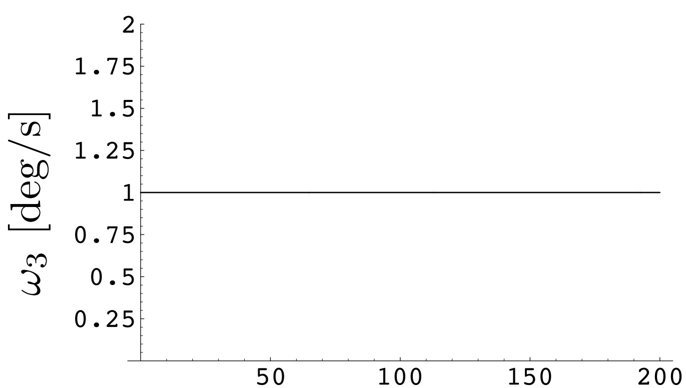
$$\omega_p = \left(\frac{I_{33}}{I_T} - 1\right)\omega_3$$

where

$$\omega_1(t) = \omega_{1_0} \cos\omega_p t - \omega_{2_0} \sin\omega_p t$$
$$\omega_2(t) = \omega_{2_0} \cos\omega_p t + \omega_{1_0} \sin\omega_p t$$
$$\omega_3(t) = \omega_{3_0}$$

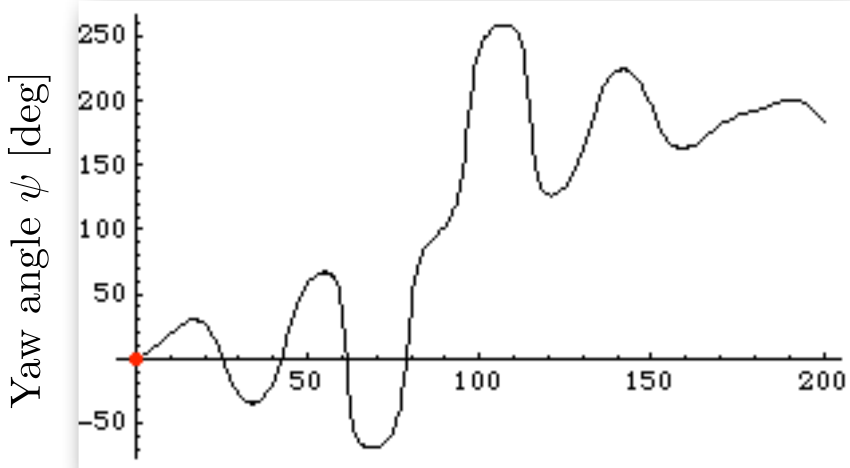




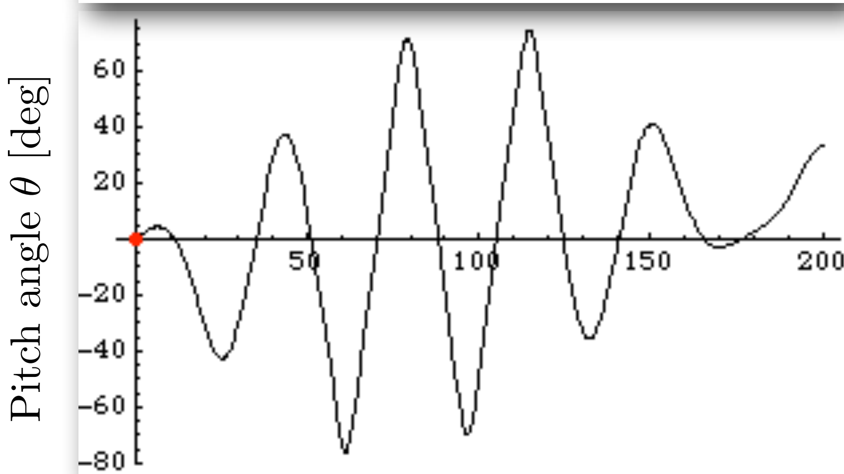


The first and second body angular velocity components are sinusoidal in nature.

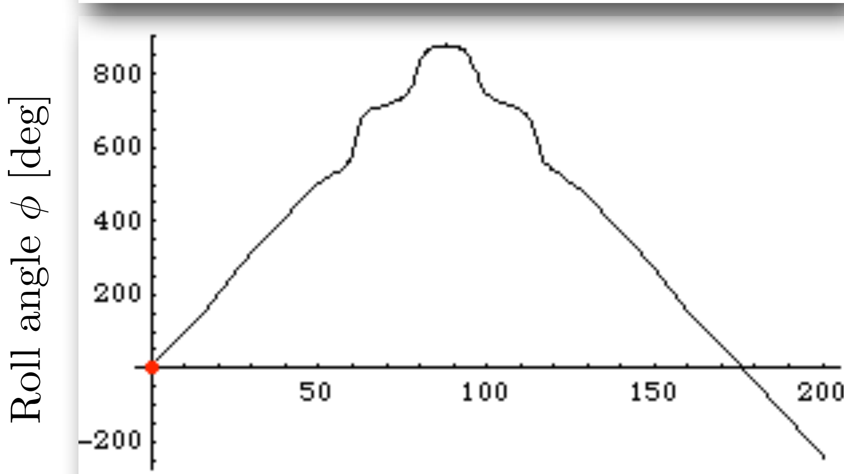
As predicted, the third body angular velocity component remains constant here.



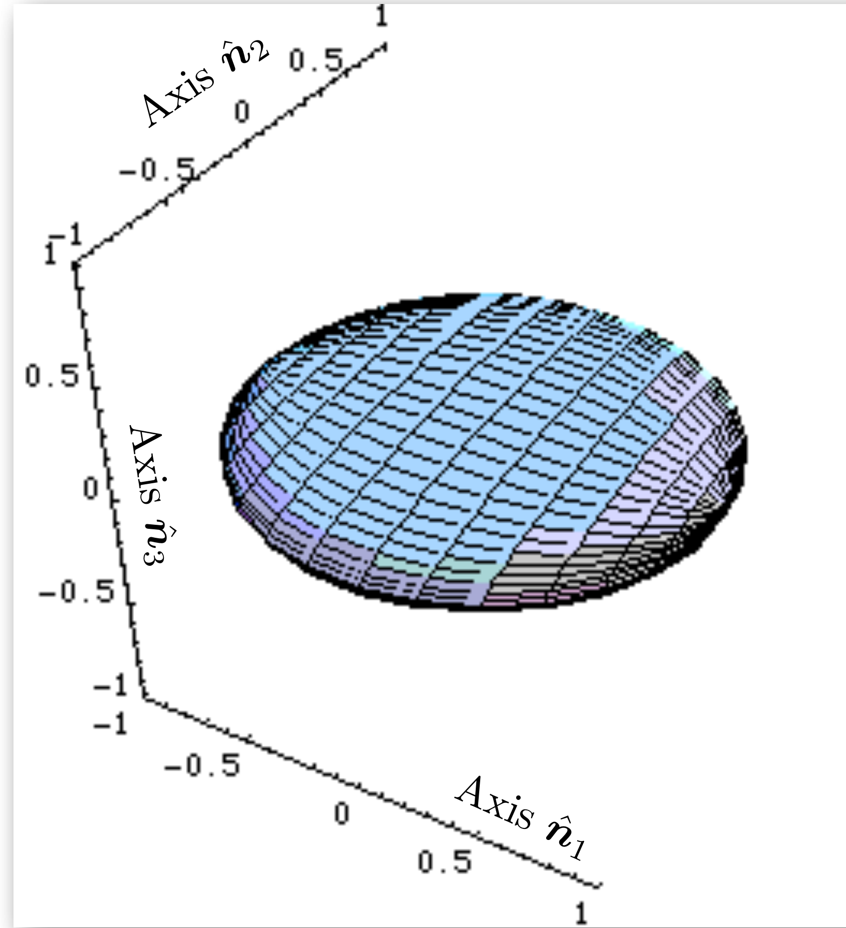
Yaw angle ψ [deg]
250
200
150
100
50
-50
50
100
150
200



Pitch angle θ [deg]
60
40
20
-20
-40
-60
-80
50
100
150
200



Roll angle ϕ [deg]
800
600
400
200
-200
50
100
150
200



Axis $\hat{n}_2$
Axis $\hat{n}_3$
Axis $\hat{n}_1$
1
0.5
0
-0.5
-1

General Free Rotation

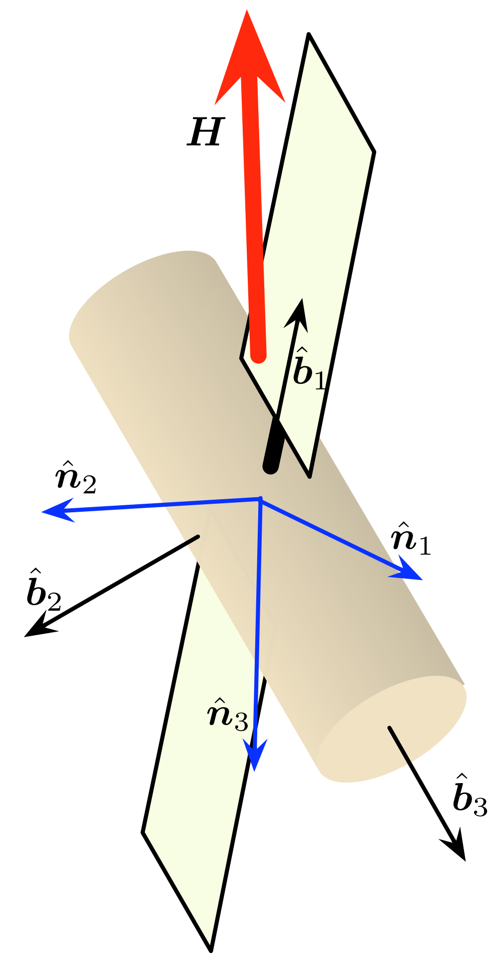
- We would like to study the general free rotation of a rigid body using the 3-2-1 Euler angles.

- Because the inertial angular momentum vector ***H*** is constant as seen by the inertial frame, we can always align our inertial frame such that

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = -H\hat{\boldsymbol{n}}_3 = {}^{\mathcal{N}}\!\begin{pmatrix} 0 \\ 0 \\ -H \end{pmatrix}$$

- Using the rotation matrix [*BN*], we find

$$^{\mathcal{B}}\boldsymbol{H} = [BN]\,{}^{\mathcal{N}}\boldsymbol{H}$$



- Recall the mapping between the rotation matrix [BN] and the 3-2-1 Euler angles:

$$[BN] = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}$$

This leads to

$$^{\mathcal{B}}\boldsymbol{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = [BN] \, ^{\mathcal{N}}\boldsymbol{H} = \begin{pmatrix} H \sin\theta \\ -H \sin\phi \cos\theta \\ -H \cos\phi \cos\theta \end{pmatrix} = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}$$

Which can be solved for the rigid body angular velocity.

$$\begin{pmatrix} \frac{H}{I_1} \sin\theta \\ -\frac{H}{I_2} \sin\phi \cos\theta \\ -\frac{H}{I_3} \cos\phi \cos\theta \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- Recall the 3-2-1 Euler angle differential kinematic equation:

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Solving these equations for the Euler angle rates, we obtain:

$$\dot{\psi} = -H\left(\frac{\sin^2\phi}{I_2} + \frac{\cos^2\phi}{I_3}\right) \Rightarrow \text{cannot be positive}$$

$$\dot{\theta} = \frac{H}{2}\left(\frac{1}{I_3} - \frac{1}{I_2}\right)\sin 2\phi\cos\theta$$

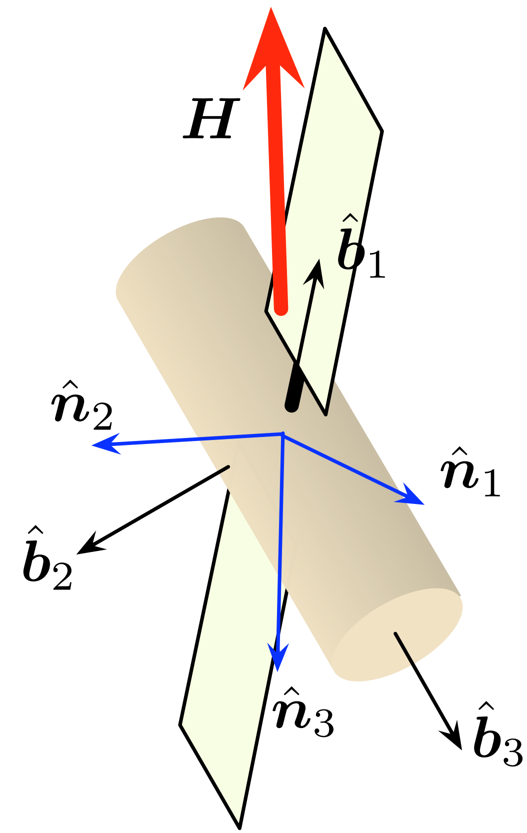
$$\dot{\phi} = H\left(\frac{1}{I_1} - \frac{\sin^2\phi}{I_2} - \frac{\cos^2\phi}{I_3}\right)\sin\theta$$

These are the spinning top equations of motion.

Axi-Symmetric Coning Motion

- Assume the spacecraft is axi-symmetric with $I_2 = I_3$, and align the inertial frame such that

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = -H\hat{\boldsymbol{n}}_3 = {}^{\mathcal{N}}\!\begin{pmatrix} 0 \\ 0 \\ -H \end{pmatrix}$$

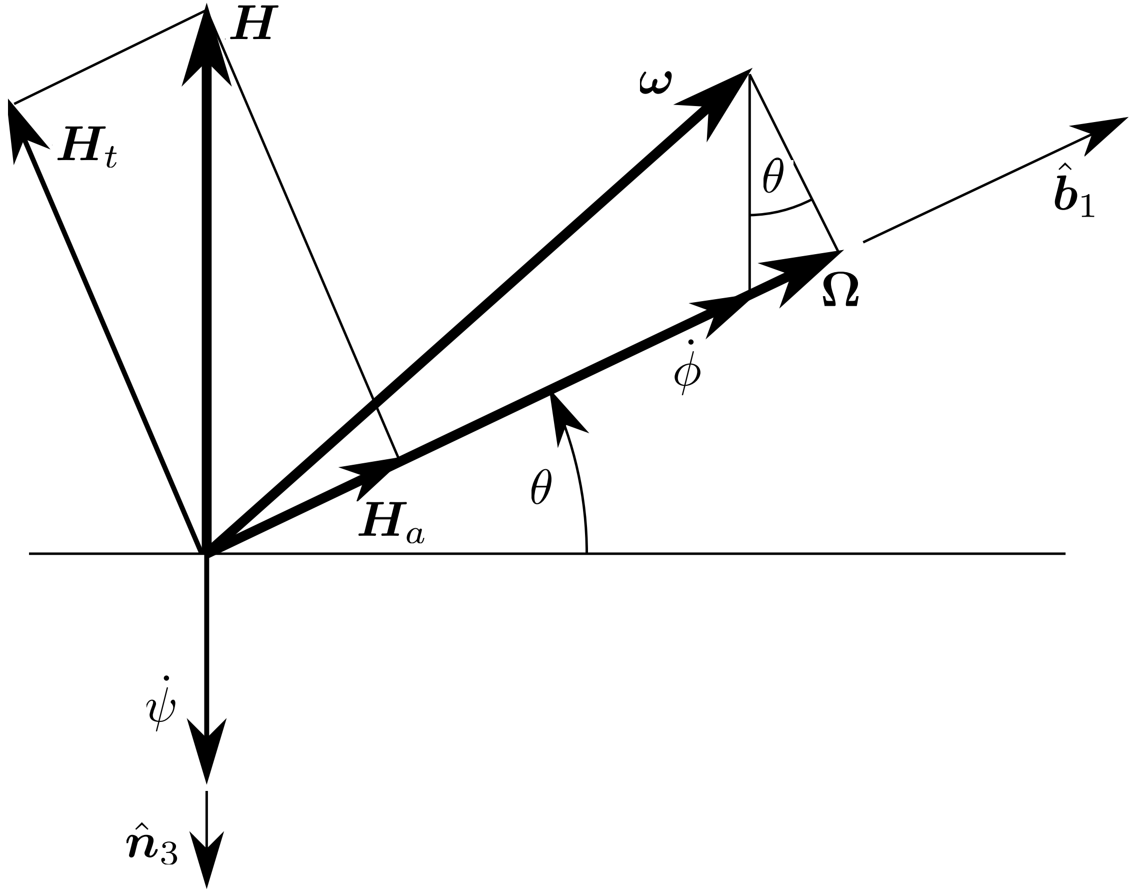


- The 3-2-1 Euler angle differential equation are then given by:

$$\dot{\psi} = -\frac{H}{I_2}$$

$$\dot{\theta} = 0$$

$$\dot{\phi} = H\left(\frac{I_2 - I_1}{I_1 I_2}\right)\sin\theta$$



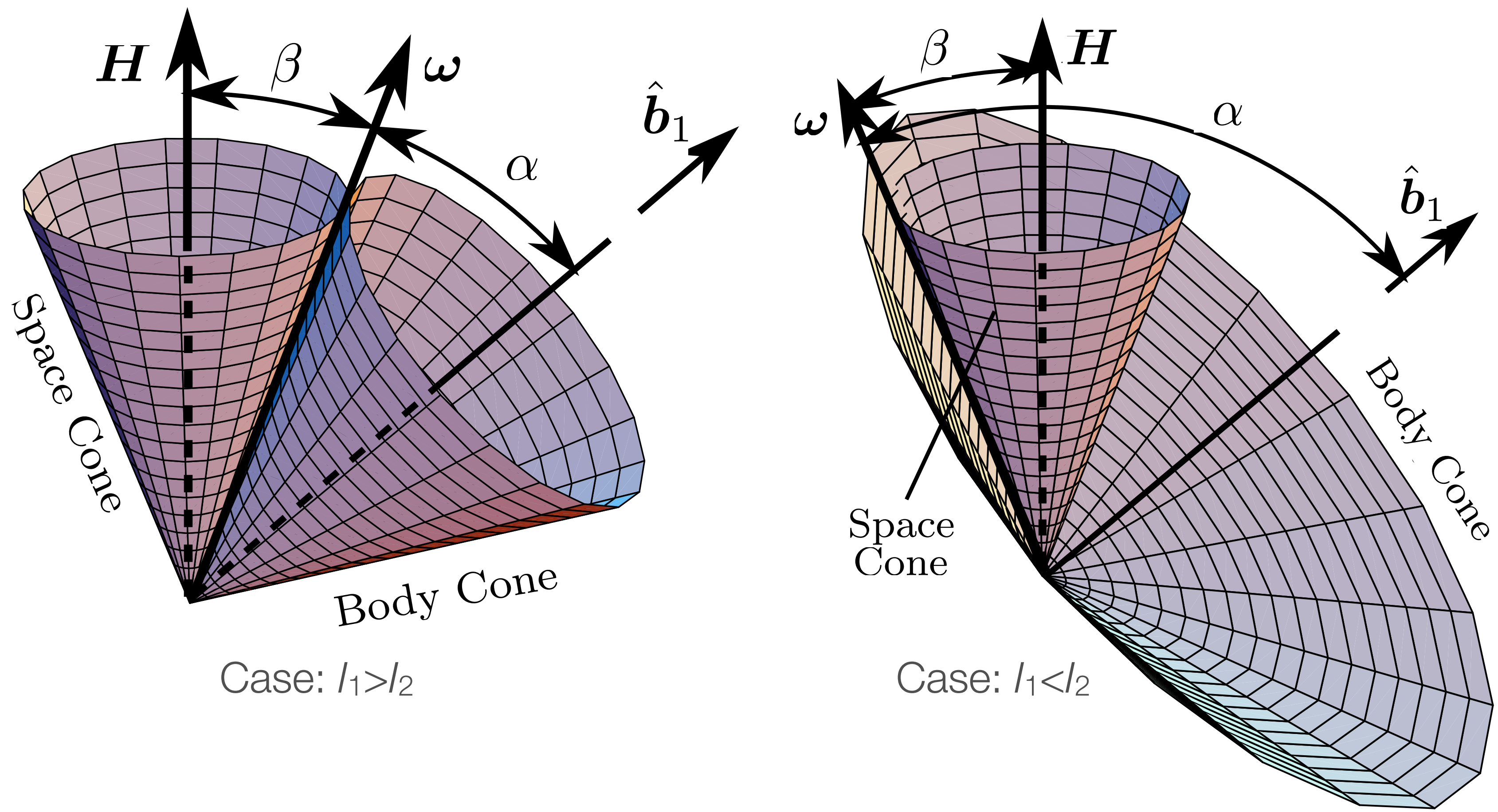
Let $\Omega = \omega_1 \longrightarrow \Omega = \dfrac{H}{I_1}\sin\theta$

Note that for $0 \leq \theta \leq \pi/2$

we find that $\Omega > 0$

The EOM can be written as

$$\dot{\psi} = -\frac{I_1}{I_2}\frac{\Omega}{\sin\theta} \qquad \dot{\phi} = \frac{I_2 - I_1}{I_2}\Omega$$



Since the pitch angle θ is shown to remain constant during this torque-free rotation, the resulting motion can be visualized by two cones rolling on each other. The *space cone* is fixed in space and its cone axis is always aligned with the angular momentum vector ***H***. The cone angle β is defined as the angle between the vectors ***H*** and **ω**. The *body cone* axis is aligned with the first body axis and has the cone angle α which is the angle between **ω** and first body axis.