ASEN 5070

Statistical Orbit Determination I

Fall 2012

Professor Jeffrey S. Parker

Professor George H. Born

Lecture 23: Process Noise

Announcements

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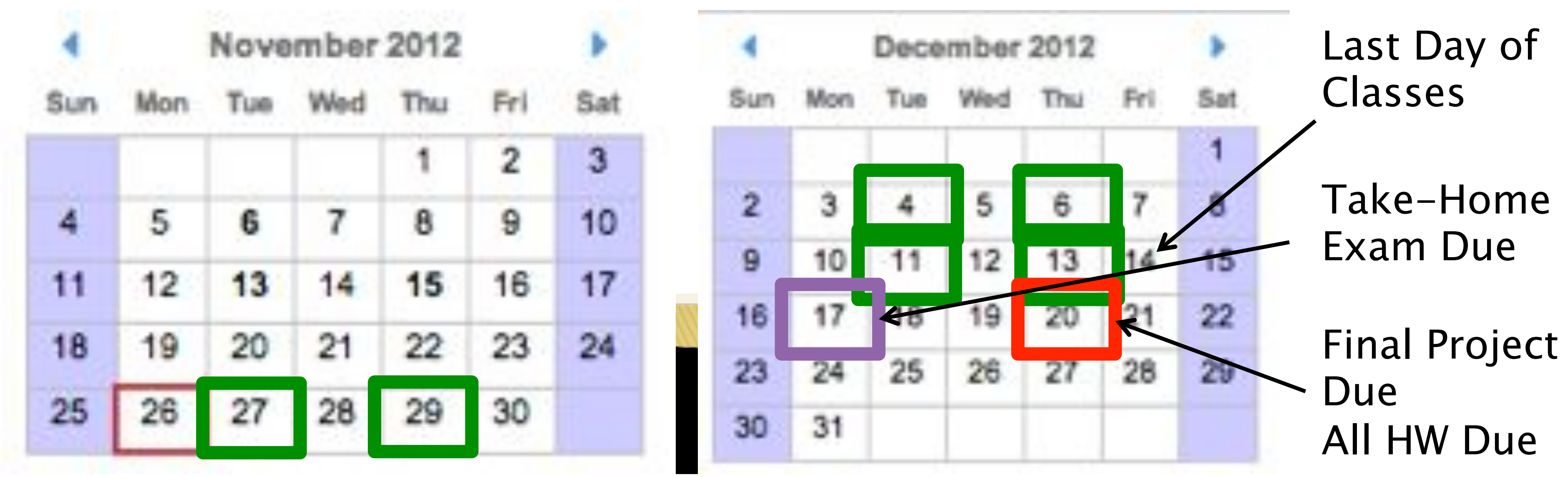
- How was break?

- HW 10 due Thursday.
- HW 11 due next week.

- Grading

- You have the tools needed to finish the project. Only things missing are bonus pieces.

- I may be out tomorrow; if you need office-hour help, check the TAs or visit me Thursday.



Quiz 19 Review

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Information

This week we'll be talking about process noise compensation! So it will be useful to review some matrix math.

Question 1 (1 point)

Let's say A is a matrix, A^T = the transpose of A, and A^-1 is the inverse of A (if it exists).

The relationship (A^T)^-1 = (A^-1)^T holds for which of the following square matrices? Select all answers that are true.

- [ ] All square matrices
- [ ] All invertible square matrices
- [ ] All symmetric positive definite square matrices
- [ ] All invertible diagonal square matrices

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Quiz 19 Review

Question 2 (1 point)

Let's say that A and B are both square matrices, T is the transpose operator, and the matrices A, B, and A^T*B are all invertible.

What is another expression for (A^T*B)^-1 ?

- A*B
- B*A
- B^-1*A^-T
- A^-T*B^-1

Quiz 19 Review

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Question 3 (1 point)

Let's say A, B, and C are matrices and they are the proper size to make A*B*C make sense. Each element of A, B, and C may be a constant or a time-varying value. What is the time derivative of (ABC)? i.e., what is d(ABC). (for simplicity, "dM" is shorthand here for "dM/dt" or "M-dot")

- ○ dA * B * C
- ○ dA * dB *dC
- ○ dC * dB *dA
- ○ dA*B*C + A*dB*C + A*B*dC

Quiz 19 Review

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Quiz 19 Review

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Question 4 (1 point)

Let's say we have n state parameters and m total observations, where m>n. The state estimate covariance matrix is P.

The conventional sequential time update for P looks like

Pbar_(k+1) = Phi*P_k*Phi^T.

We're working on deriving a new time update where we add a Q matrix like this:

Pbar_(k+1) = Phi*P_k*Phi^T + Q_k.

What size should Q_k be?

- ○ nxn
- ○ mxm
- ○ nxm
- ○ mxn

Quiz 19 Review

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What size should Q_k be?

- nxn
- mxm
- nxm
- mxn

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Question 5 (1 point)

If Q is an mxm matrix (completely independent of Q_k in the last question) and we want L*Q*L^T to be an nxn matrix, what should be the size of L?

- nxn
- mxm
- nxm
- mxn

Quiz 19 Review

Question 5 (1 point)

If Q is an mxm matrix (completely independent of Q_k in the last question) and we want L*Q*L^T to be an nxn matrix, what should be the size of L?

- nxn
- mxm
- nxm
- mxn

HW#11

- Due the Thursday after HW10

ASEN 5070 **HW #11**

1. Calculate $\Phi(18340,0)$ using the equation $\Phi(t_k,0)=\Phi(t_k,t_n)\Phi(t_n,0)$

2. Report the state deviation after one pass through the sequential filter.

3. Compare the result from question 2 to your batch results.

4. Using the convention Kalman filter covariance measurement update equation, plot the trace of the position covariance on a semilog y scale (at each measurement time).

5. Using the Joseph formulation of the covariance measurement update equation, plot the trace of the position covariance on a semilog y scale (at each measurement time).

6. Plot the position error ellipsoid for your batch state error covariance. (You may use the function plotEllipsoid.m)

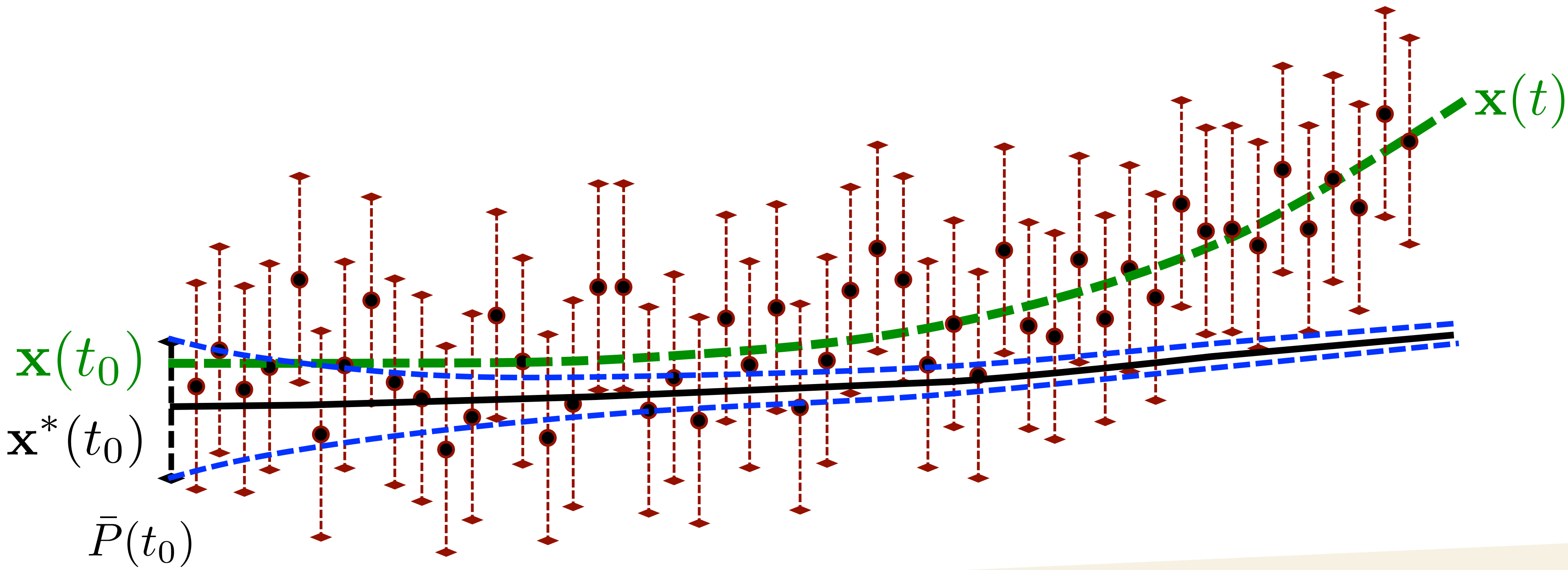
7. Work problem 4-41. You can use this data file hw11.dat, where time is in the first column and the observations are in the second column.

Contents

- Process Noise

Issue: Filter Saturation

- Applying the sequential algorithm to a large amount of data will cause the covariance to shrink down too far.



Filter Saturation

- Sequential filter measurement update:

$$K_k = \bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1}$$

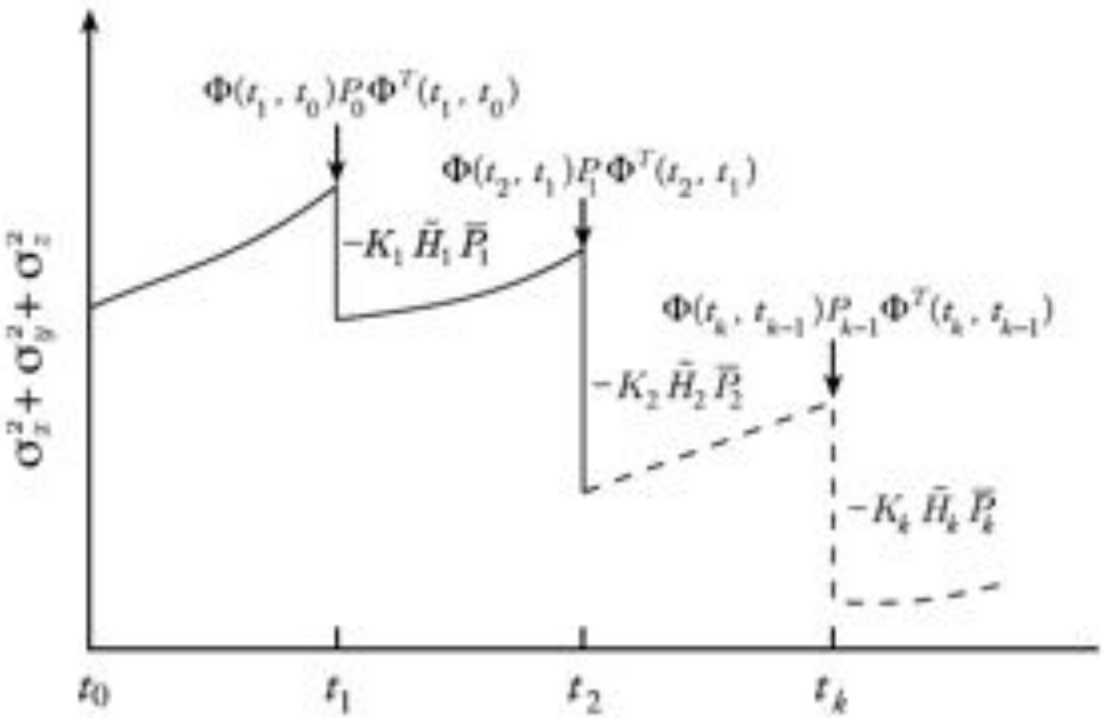
$$\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + K_k \left[\mathbf{y}_k - \tilde{H}_k \bar{\mathbf{x}}_k \right]$$

$$P_k = \left[I - K_k \tilde{H}_k^T \right] \bar{P}_k$$

- Large number of observations will drive the covariance to zero.
  - (which is great if everything is perfectly modeled! But that's never quite so.)

Filter Saturation

- Trace of covariance over time.



Filter Saturation

- As $\bar{P}_k \to 0$

$$K_k = \bar{P}_k \tilde{H}_k^T \left[\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \right]^{-1}$$

$$\tilde{H}_k \bar{P}_k \tilde{H}_k^T + R_k \to R_k$$

$$K_k \to \bar{P}_k \tilde{H}_k^T R_k^{-1}$$

$$\bar{P}_k \to 0$$

$$K_k \to 0$$

Filter Saturation

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- As $K_k \to 0$

$$\hat{\mathbf{x}}_k = \bar{\mathbf{x}}_k + K_k \left[\mathbf{y}_k - \tilde{H}_k \bar{\mathbf{x}}_k \right]$$

The filter will begin ignoring observations and the best estimate will remain constant over time.

Example of Filter Saturation

- This is the example given in Appendix F
- I coded it up, and you can too. It's pretty straightforward. Plus this way I could investigate different aspects of the scenario.

- Given:
  - A particle moving along the x-axis in a positive direction.
  - It is nominally moving at a constant 10 m/s velocity.
  - It is *actually* being perturbed by an unknown acceleration – unmodeled!
  - The acceleration is a small-amplitude oscillation.
  - The particle is being tracked by an observer, also on the x-axis at 10 Hz observation frequency.
    - Range with white noise N(0,1) m
    - Range-rate with white noise N(0,0.1) m/s
- Can we estimate the state and even the unmodeled force?

Example of Filter Saturation

- Our system

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

$$\dot{\mathbf{X}}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

- Notice that the acceleration is not in our model.

Example of Filter Saturation

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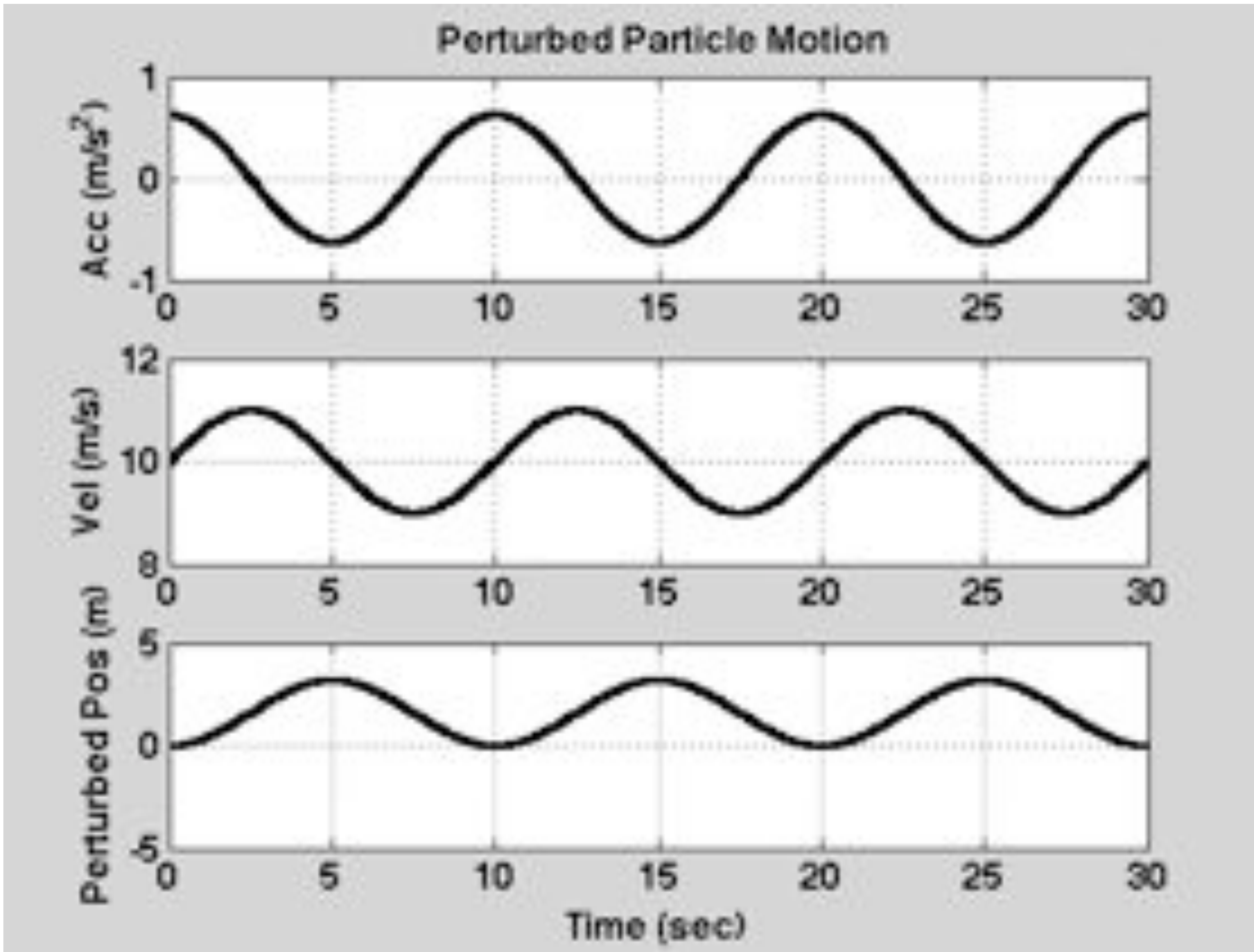

Example of Filter Saturation

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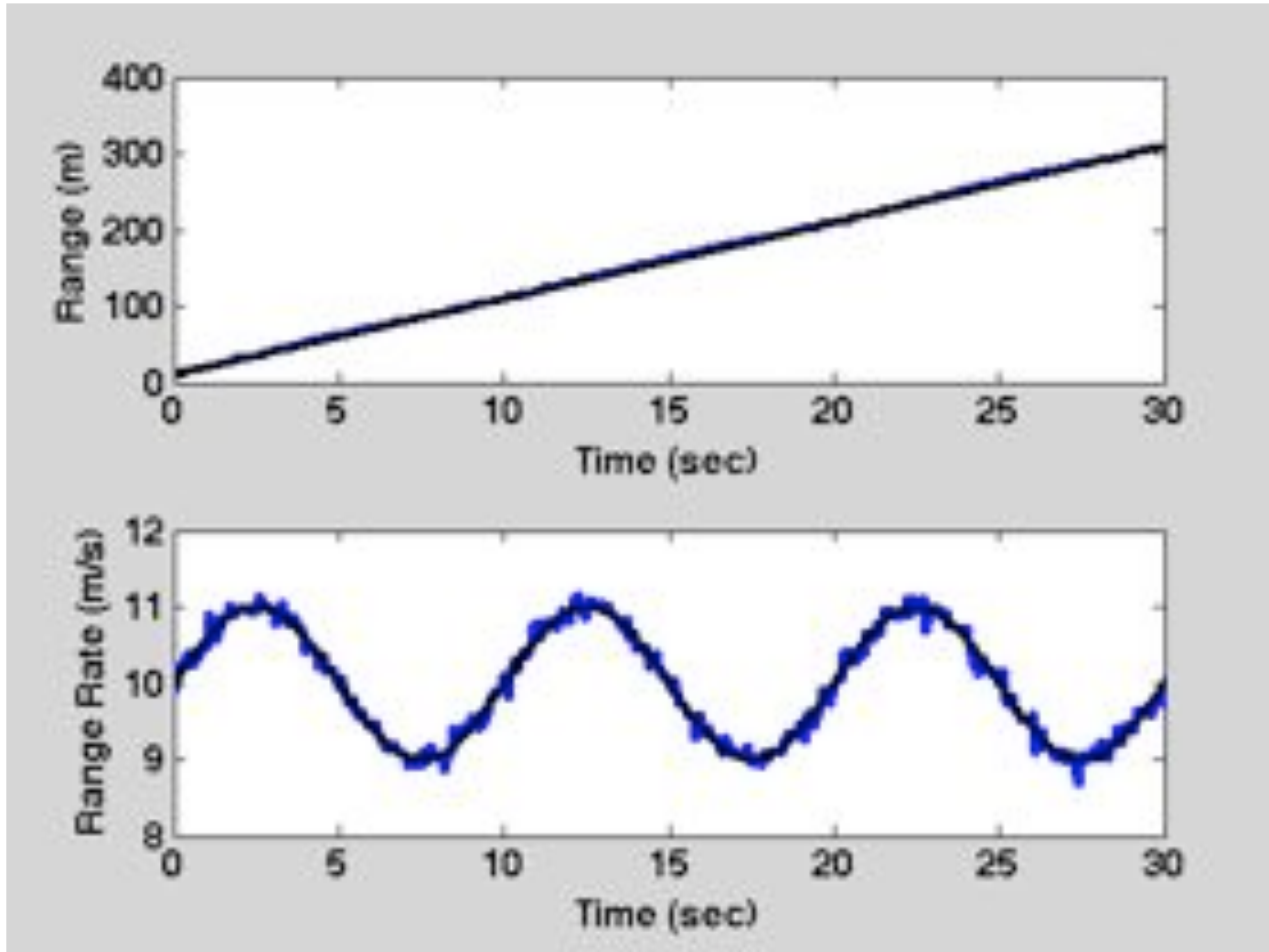
- Observations
  - Range
    N(0,1) meter
  - Range Rate
    N(0,0.1) m/s



Example of Filter Saturation

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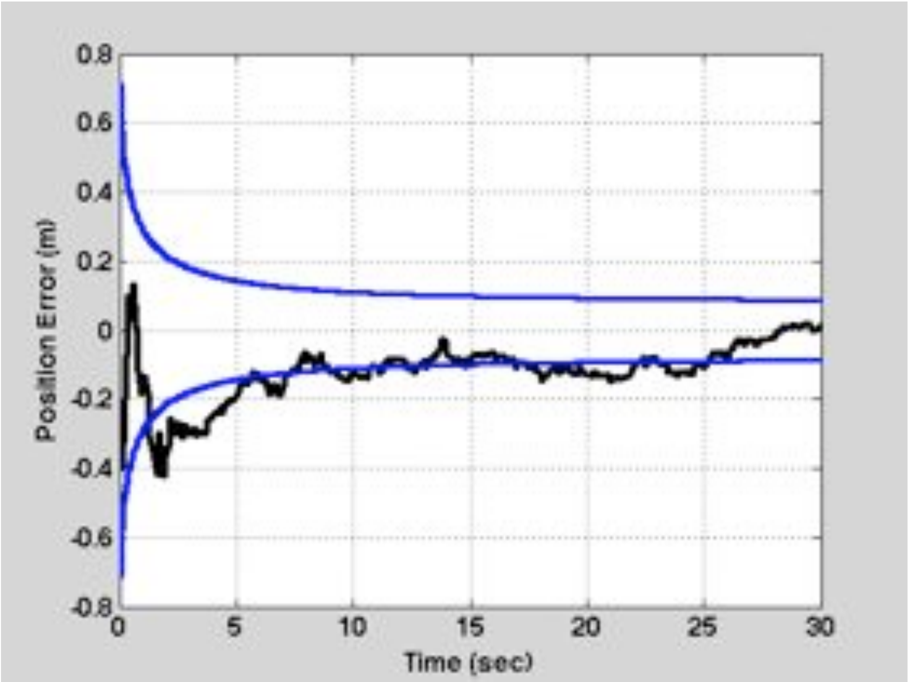
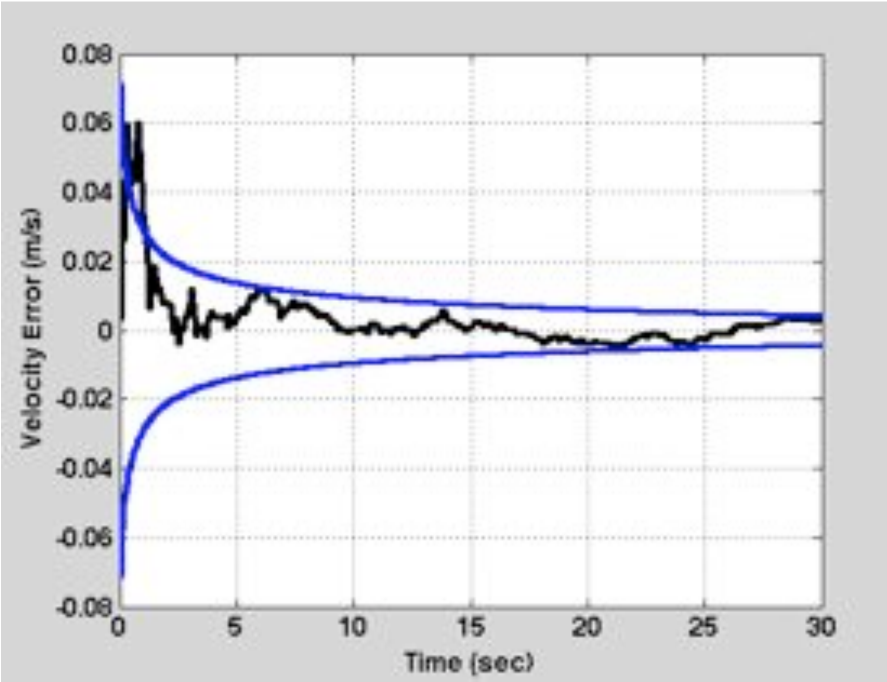
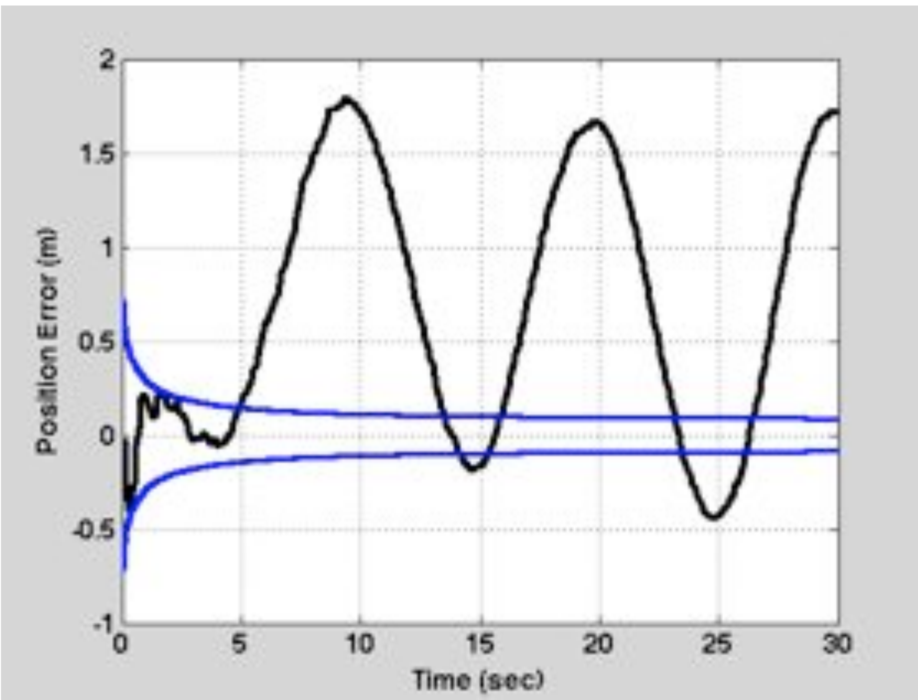
- If you KNEW the acceleration, then this is what you would see using an EKF

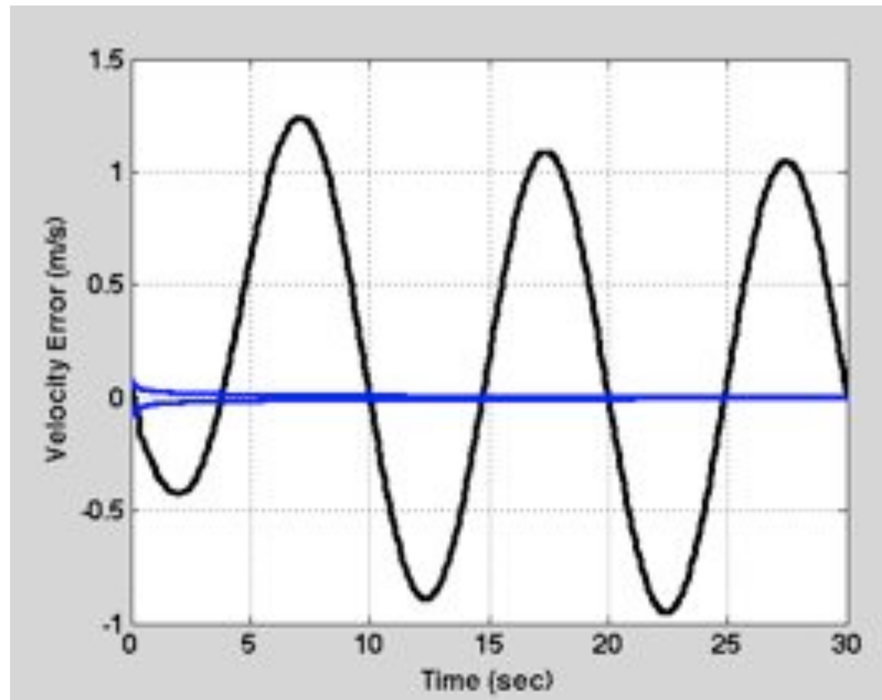




Example of Filter Saturation

- You don't know the acceleration, so this is what you DO see with an EKF





Example of Filter Saturation

- We need a way to add expected process noise to our filter.

- Many ways to do that:

  - State Noise Compensation
    - Constant noise
    - Piecewise constant noise
    - Correlated noise
    - White noise ← Example
  - Dynamic Model Compensation
    - 1^{st} order linear stochastic noise
    - More complex model compensations

Example of SNC

- Let's say that we know that our particle is being perturbed by some acceleration.
- We characterize it by simple white noise
(note: we know that it is far more structured than that!)

$$\ddot{x}(t) = \eta(t) = u(t)$$

$$u(t) \equiv \text{white noise}$$

stationary Gaussian process with a mean of zero and a variance of $\sigma_u^2 \delta(t-\tau)$

(Dirac delta function)

Example of SNC

- Adjust our system accordingly

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

$$\dot{\mathbf{X}}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$ (No perturbation)

$$\dot{\mathbf{X}}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{A}} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{B}} u(t)$$

Example of SNC

- We can show that the time update for the sequential algorithm does not change:
  (we'll show this later)

$$\bar{\mathbf{x}}(t) = \Phi(t, t_{k-1})\hat{\mathbf{x}}_{k-1}$$

- The time update for the covariance *does* change (we'll derive it later)

$$\bar{P}_k = \Phi_{k,k-1} P_{k-1} \Phi_{k,k-1}^T + Q_k$$

Example of SNC

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- Introduce the *Process Noise Covariance Matrix* Q

$$Q_\eta(t) = \sigma_u^2 \int_{t_0}^{t} \Phi(t,\tau) B B^T \Phi^T(t,\tau) d\tau$$

(this equation is specific to this example and we'll derive the general equation later)

- For our problem, we have Phi and B. Perform the integration and we find

$$Q_\eta(t) = \sigma_u^2 \begin{bmatrix} \frac{1}{3}(t-t_0)^3 & \frac{1}{2}(t-t_0)^2 \\ \frac{1}{2}(t-t_0)^2 & t-t_0 \end{bmatrix}$$

Example of SNC

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- Further, since the time update interval is always 0.1 sec (10 Hz observation frequency)

$$Q_\eta(t) = \sigma_u^2 \begin{bmatrix} \frac{1}{3}(t-t_0)^3 & \frac{1}{2}(t-t_0)^2 \\ \frac{1}{2}(t-t_0)^2 & t-t_0 \end{bmatrix}$$

$$Q_\eta(t) = \sigma_u^2 \begin{bmatrix} \frac{0.001}{3}\ \mathrm{sec}^3 & \frac{0.01}{2}\ \mathrm{sec}^2 \\ \frac{0.01}{2}\ \mathrm{sec}^2 & 0.1\ \mathrm{sec} \end{bmatrix}$$

σ_u^2 has units of m^2/s^3

Example of SNC

$$Q_\eta(t) = \sigma_u^2 \begin{bmatrix} \frac{0.001}{3}\ \mathrm{sec}^3 & \frac{0.01}{2}\ \mathrm{sec}^2 \\ \frac{0.01}{2}\ \mathrm{sec}^2 & 0.1\ \mathrm{sec} \end{bmatrix}$$

- Implication: original deterministic constant velocity model of the motion is modified to include
  - Random component
  - Constant-diffusion Brownian motion process
  - σ_u^2 is the *diffusion coefficient* and is a *tuning parameter*
- Remember: this is a very broad, simple process noise compensation. This assumes white noise acceleration.

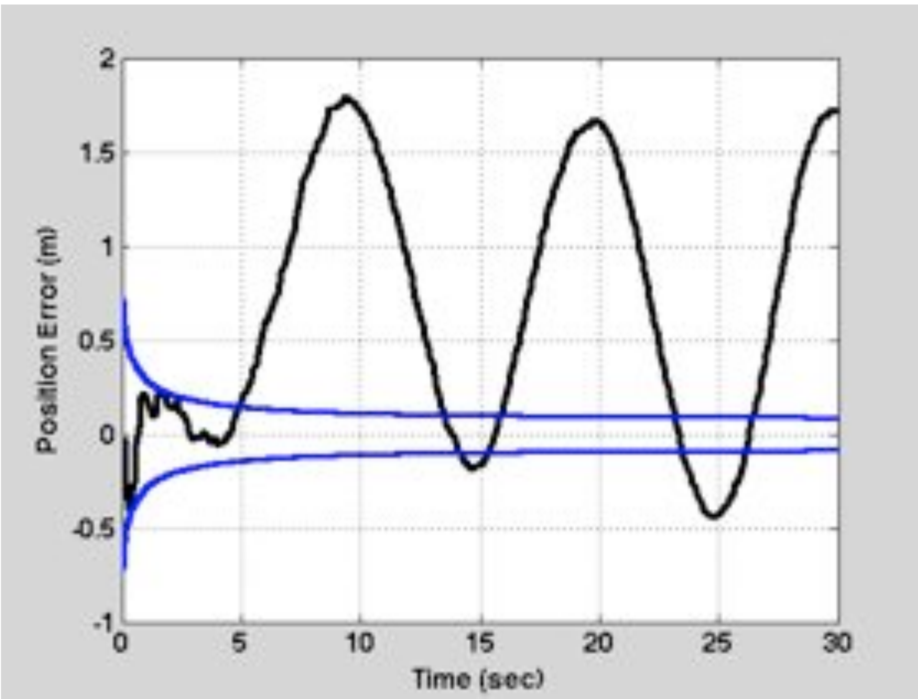
Example of SNC

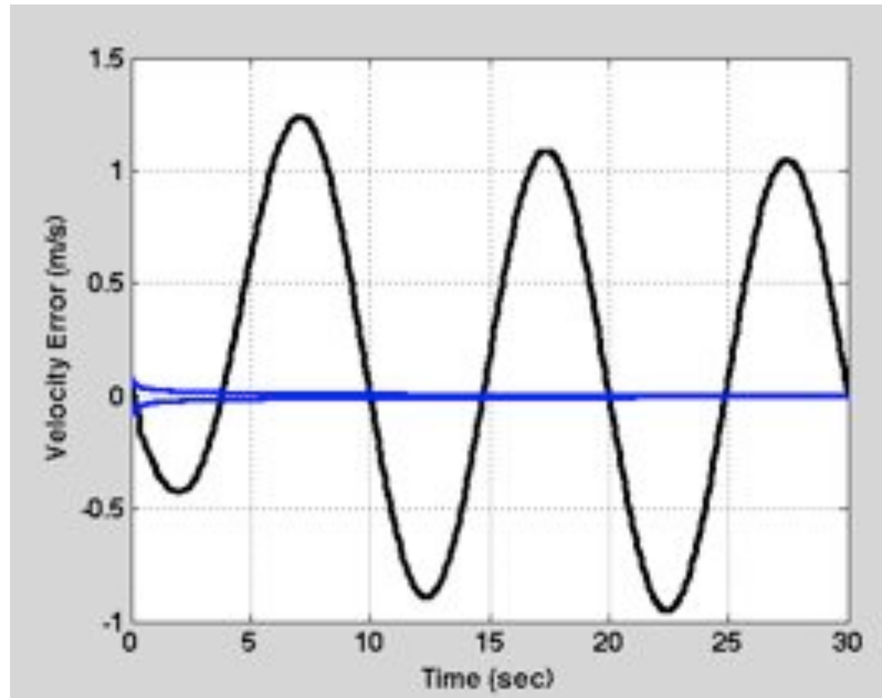
$$Q_\eta(t) = \sigma_u^2 \begin{bmatrix} \frac{0.001}{3} \text{ sec}^3 & \frac{0.01}{2} \text{ sec}^2 \\ \frac{0.01}{2} \text{ sec}^2 & 0.1 \text{ sec} \end{bmatrix}$$

- New time update:
- Old: $\bar{P}_k = \Phi(t_k, t_{k-1}) P_{k-1} \Phi^T(t_k, t_{k-1})$
- New for Example: $\bar{P}_k = \Phi_{k,k-1} P_{k-1} \Phi_{k,k-1}^T + Q_\eta(t_k)$
- General: $\bar{P}_k = \Phi_{k,k-1} P_{k-1} \Phi_{k,k-1}^T + \Gamma_{k,k-1} Q_k \Gamma_{k,k-1}^T$

Example of SNC

- Old Result





Example of SNC

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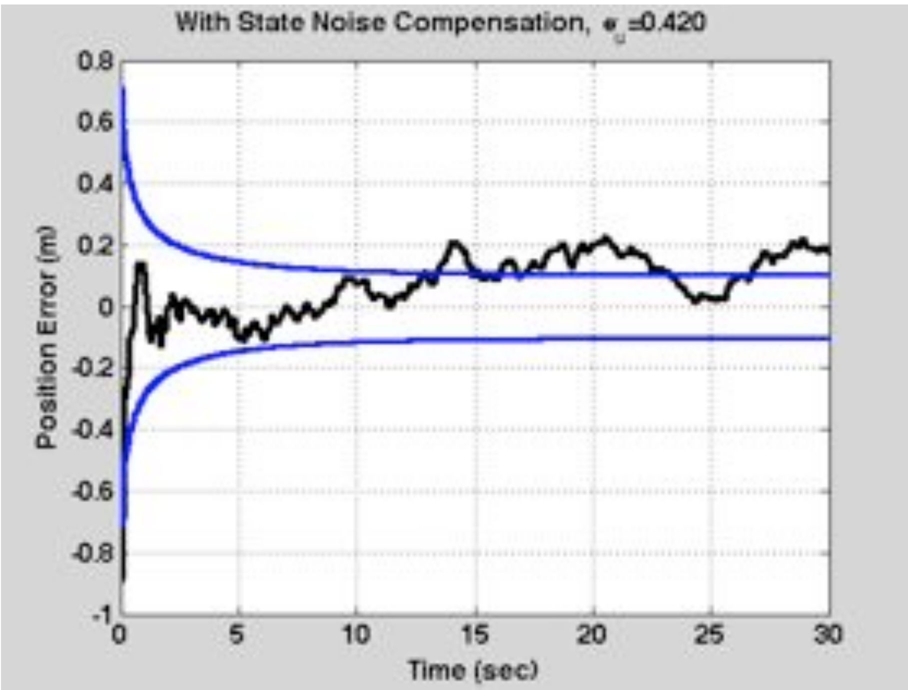
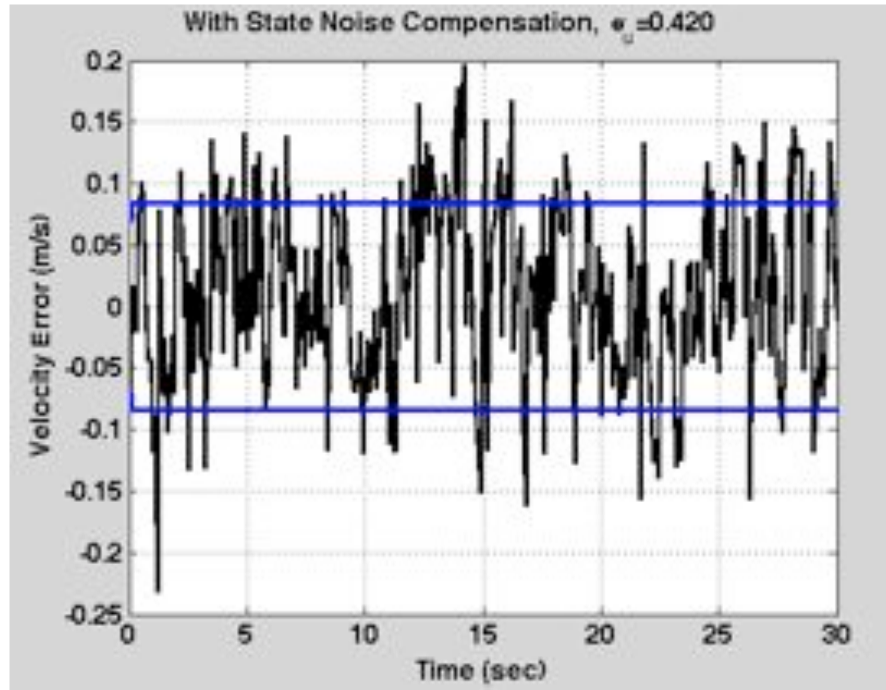
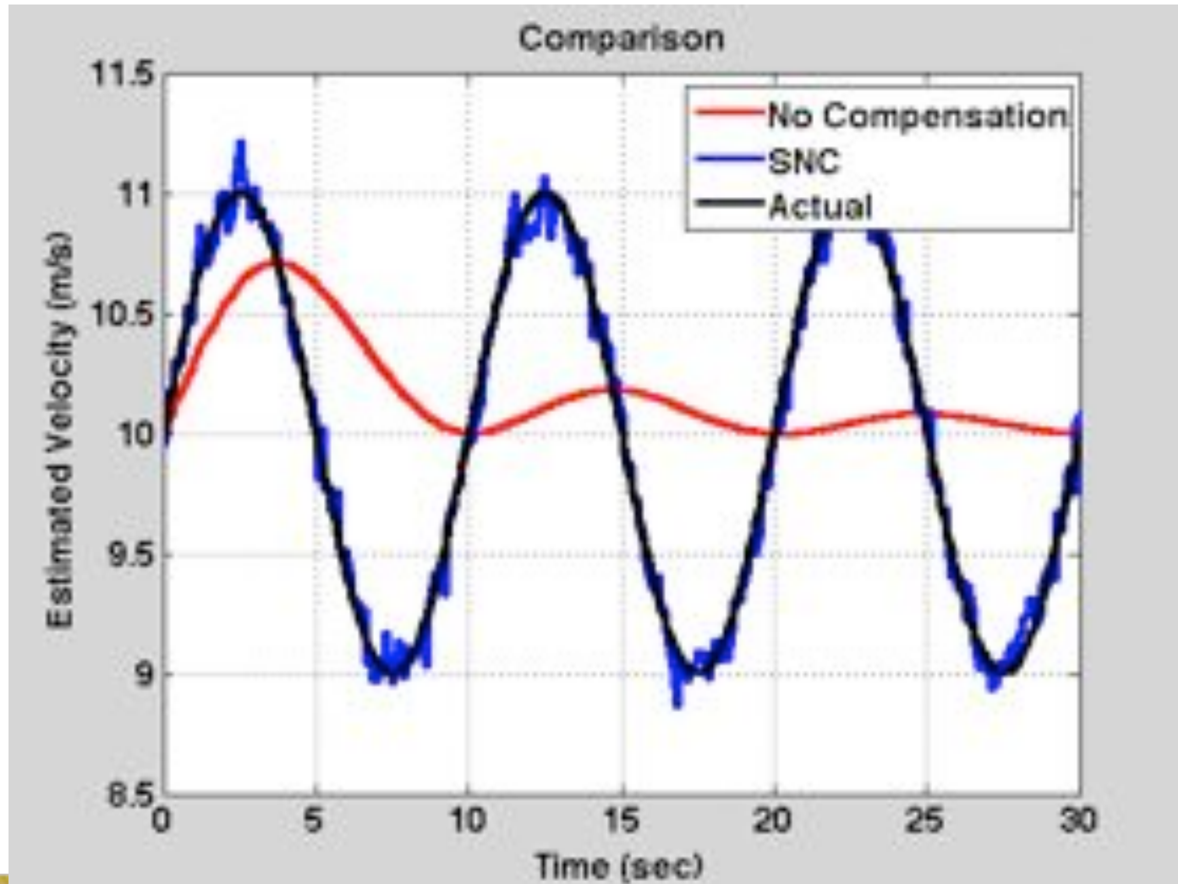
- New Result





Example of SNC

- Comparison of estimated velocity



Example of SNC

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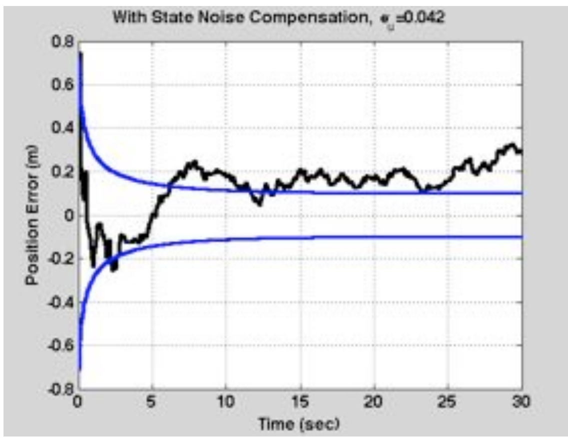
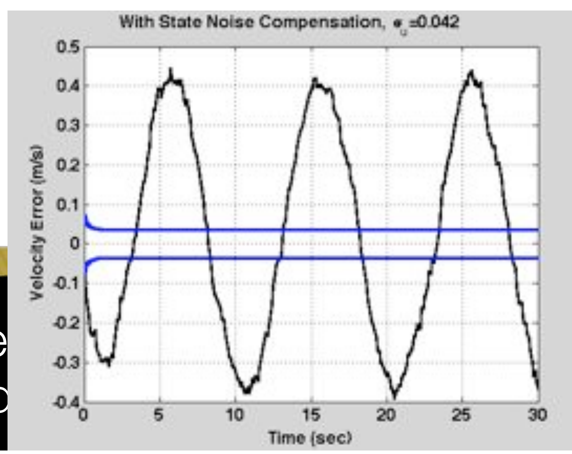
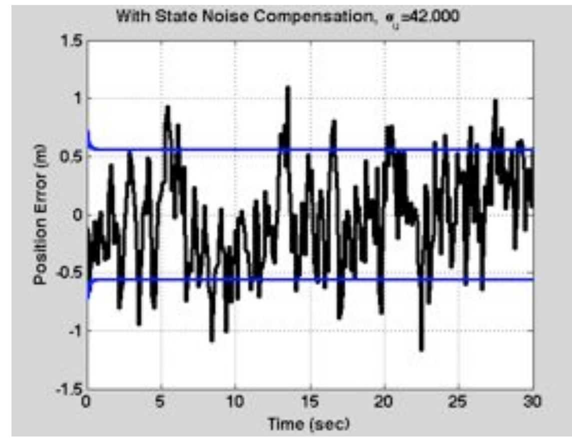
- This does depend on the value of σ_u^2

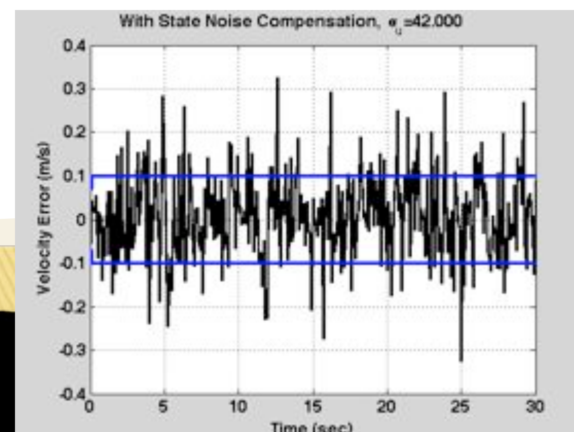
Too low: not enough help





Too high: no carry-over information





Example Compensations Revisited

- We need a way to add expected process noise to our filter.

- Many ways to do that:
  - State Noise Compensation
    - Constant noise
    - Piecewise constant noise
    - Correlated noise
    - White noise ← Example 1
  - Dynamic Model Compensation
    - 1st order linear stochastic noise ← Example 2
    - More complex model compensations

Example DMC

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- Let's reformulate our compensation assumption.

- Rather than assuming that our particle is under the influence of some white noise acceleration, let's assume there's some structure.
- DMC assumes that the unknown acceleration can be characterized as:
  - 1st-order linear stochastic differential equation
  - Known as the Langevin equation

$$\dot{\eta}(t) + \beta\eta(t) = u(t)$$

Example DMC

- What does this expression *mean*?

$$\dot{\eta}(t) + \beta\eta(t) = u(t)$$

- $\eta(t)$ is our estimated acceleration
- β is the inverse of a correlation time (another tuning parameter!) $\beta = \frac{1}{\tau}$
- $u(t)$ is a white zero-mean Gaussian process
- Solution: $\eta(t) = \eta_0 e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-\tau)} u(\tau) d\tau$

Deterministic Stochastic

Example DMC

- We can add the deterministic component of this acceleration to our state.

$$\eta(t) = \eta_0 e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-\tau)} u(\tau) d\tau$$

Deterministic Stochastic

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \eta_D(t) \end{bmatrix}$$

Example DMC

- Integrate this to observe its effects on the dynamical model

$$\eta(t) = \eta_0 e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-\tau)} u(\tau) d\tau$$

Deterministic Stochastic

$$\mathbf{X}(t) = \left[\begin{array}{c} x(t) \\ \dot{x}(t) \\ \eta_D(t) \end{array} \right] = \left[\begin{array}{c} x_0 + \dot{x}_0(t-t_0) + \frac{\eta_0}{\beta}(t-t_0) + \frac{\eta_0}{\beta^2}\left(e^{-\beta(t-t_0)} - 1\right) \\ \dot{x}_0 + \frac{\eta_0}{\beta}\left(1 - e^{-\beta(t-t_0)}\right) \\ \eta_0 e^{-\beta(t-t_0)} \end{array} \right]$$

Example DMC

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- Note: tau (the correlation constant) may also be added to the state to produce a 4-element state.
- This example proceeds by setting tau to be a constant.
    - Users could fiddle with tau to tune the filter.

$$\mathbf{X}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \eta_D(t) \end{bmatrix} = \begin{bmatrix} x_0 + \dot{x}_0(t-t_0) + \frac{\eta_0}{\beta}(t-t_0) + \frac{\eta_0}{\beta^2}\left(e^{-\beta(t-t_0)} - 1\right) \\ \dot{x}_0 + \frac{\eta_0}{\beta}\left(1 - e^{-\beta(t-t_0)}\right) \\ \eta_0 e^{-\beta(t-t_0)} \end{bmatrix}$$

Example DMC

- Re-derive the H-tilde, A, Phi, etc matrices to accommodate the new state parameters.

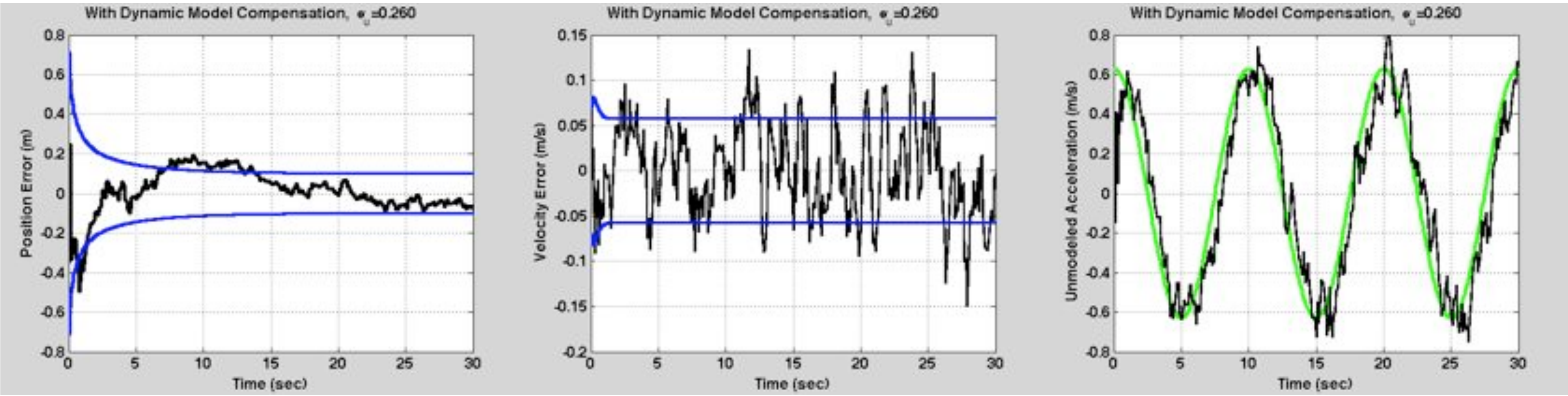
- Results:
  - A great estimate of the state
  - A much wider range of acceptable values for the tuning parameters – which is very useful when we don't know the answer!
  - A good estimate of the unmodeled acceleration profile – which may be used as information to track down missing pieces of one's dynamical model.

Example DMC

- Estimated State

$$\tau = 200 \text{ sec}$$

$$\sigma_u = 0.26 \text{ m/sec}^{5/2}$$



Example DMC

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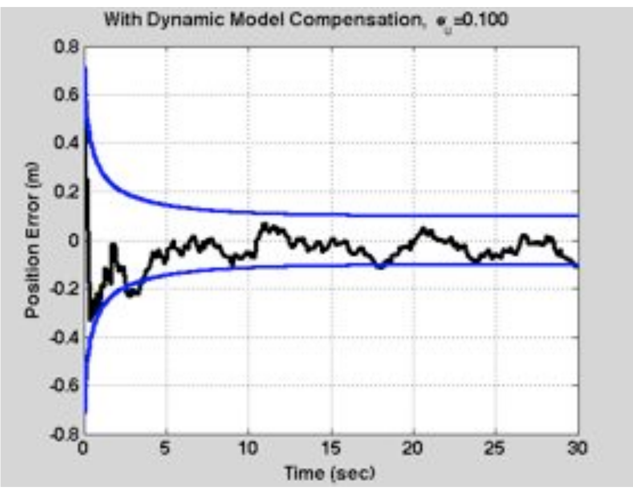
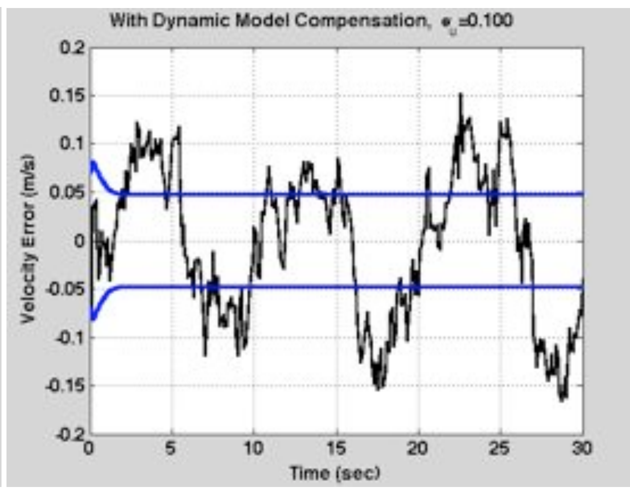
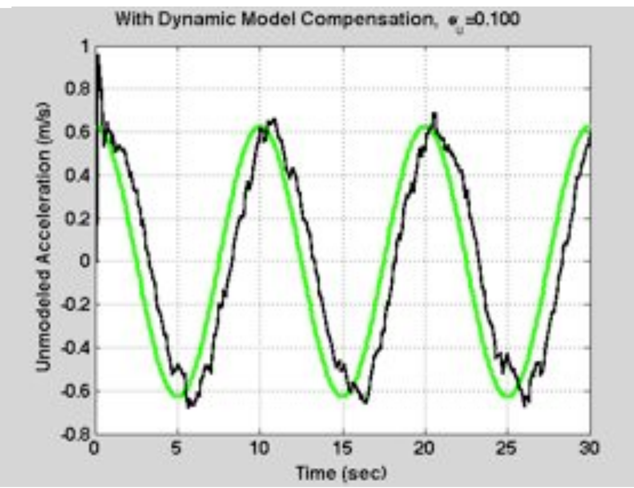
- Estimated State

$$\tau = 200 \text{ sec}$$

$$\sigma_u = 0.1 \text{ m/sec}^{5/2}$$





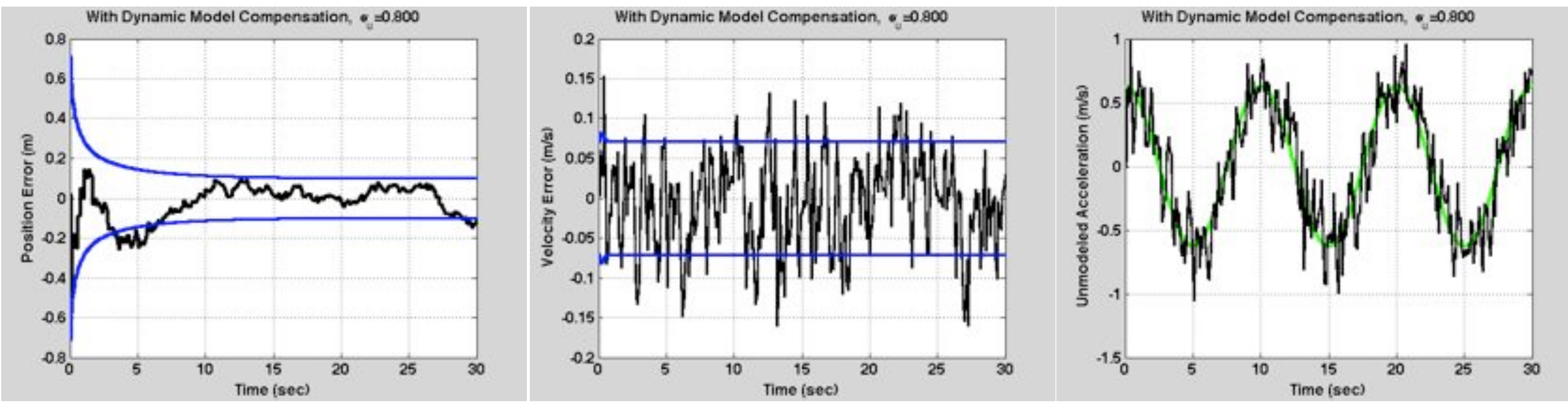


Example DMC

- Estimated State

$$\tau = 200 \text{ sec}$$

$$\sigma_u = 0.8 \text{ m/sec}^{5/2}$$



Example DMC

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- DMC is less sensitive to tuning than SNC

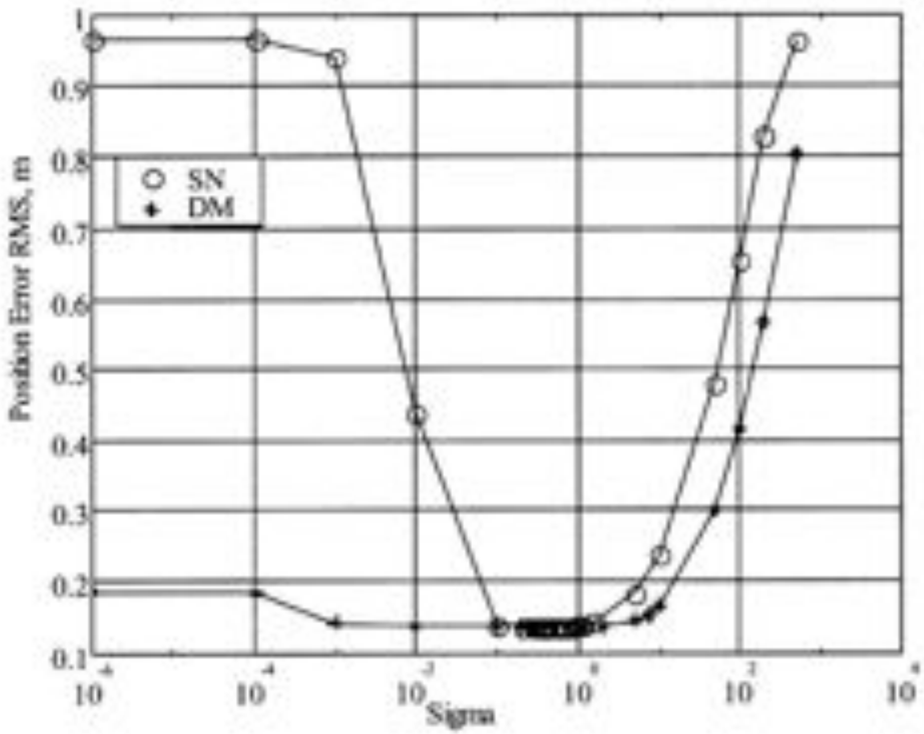


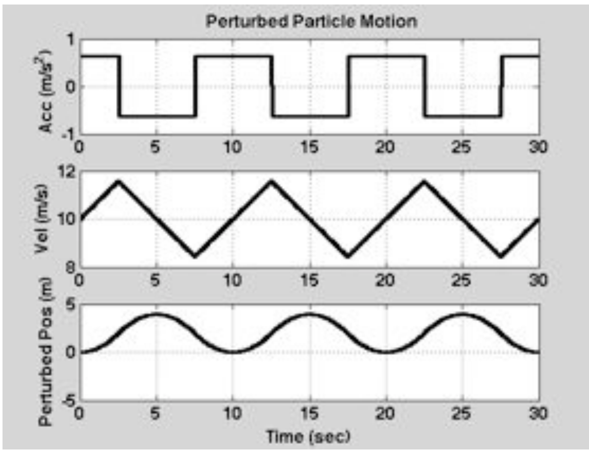
Figure F.3.2: RMS Position error as a function of σ_u for the two-state SNC and three-state DMC filters.

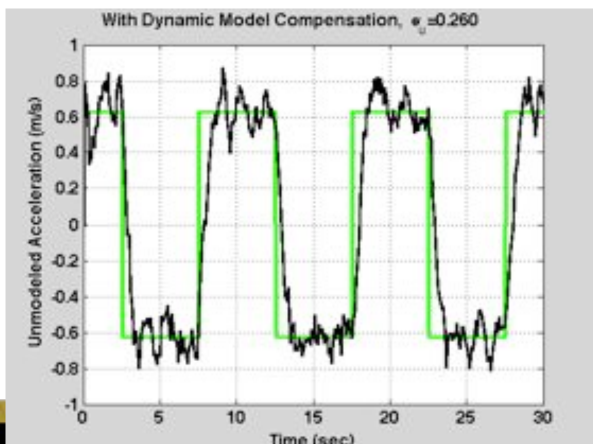
Example SNC / DMC

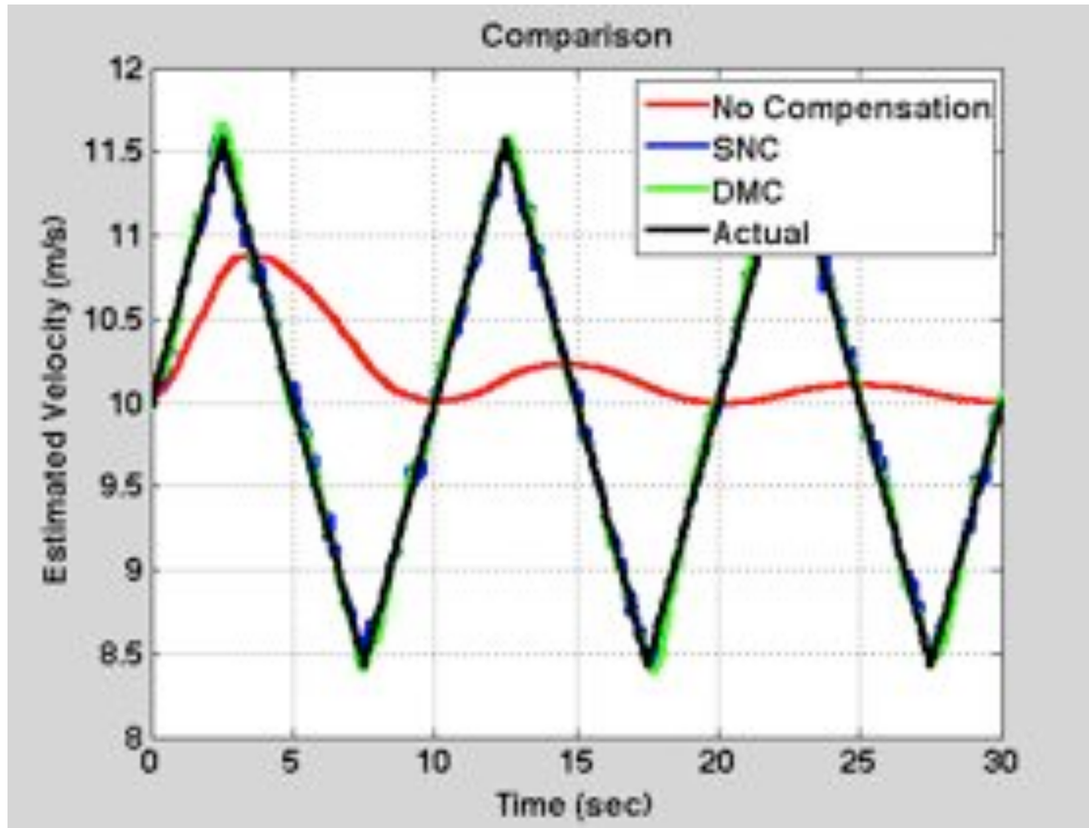
- What if we changed the perturbing acceleration?
  - Triangle wave
  - Square wave
  - Constant acceleration
  - White noise

Example SNC / DMC

- Square Wave

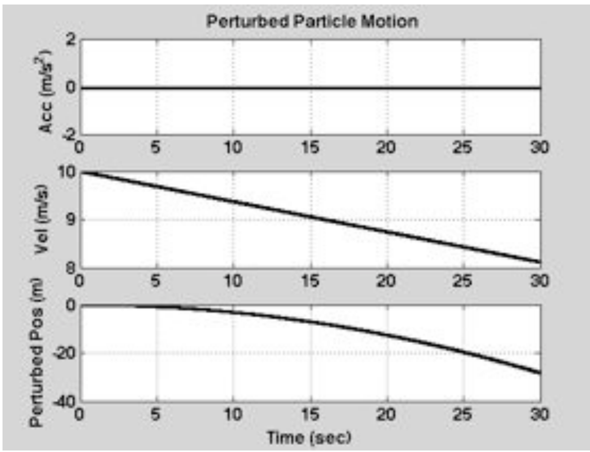


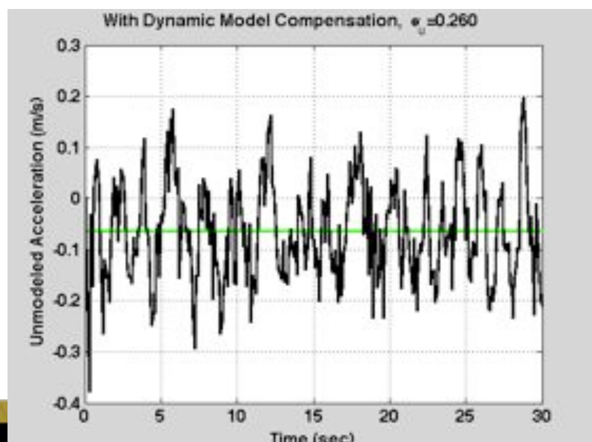


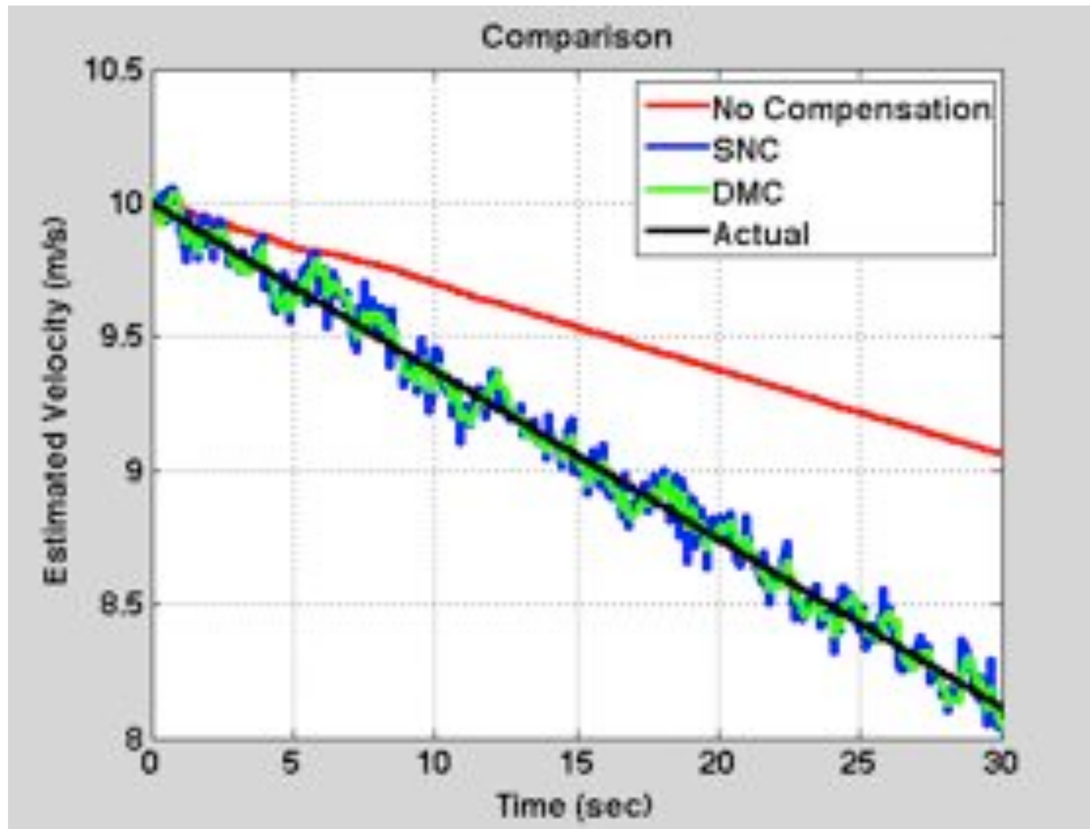


Example SNC / DMC

- Constant Acceleration

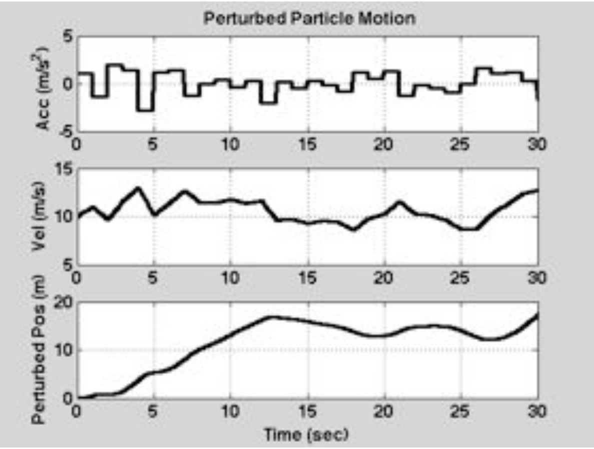


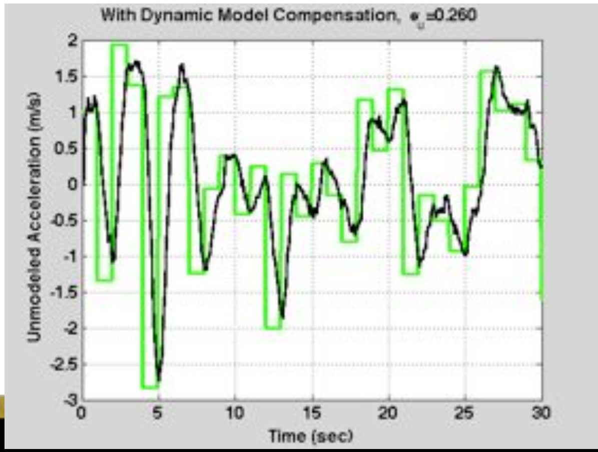


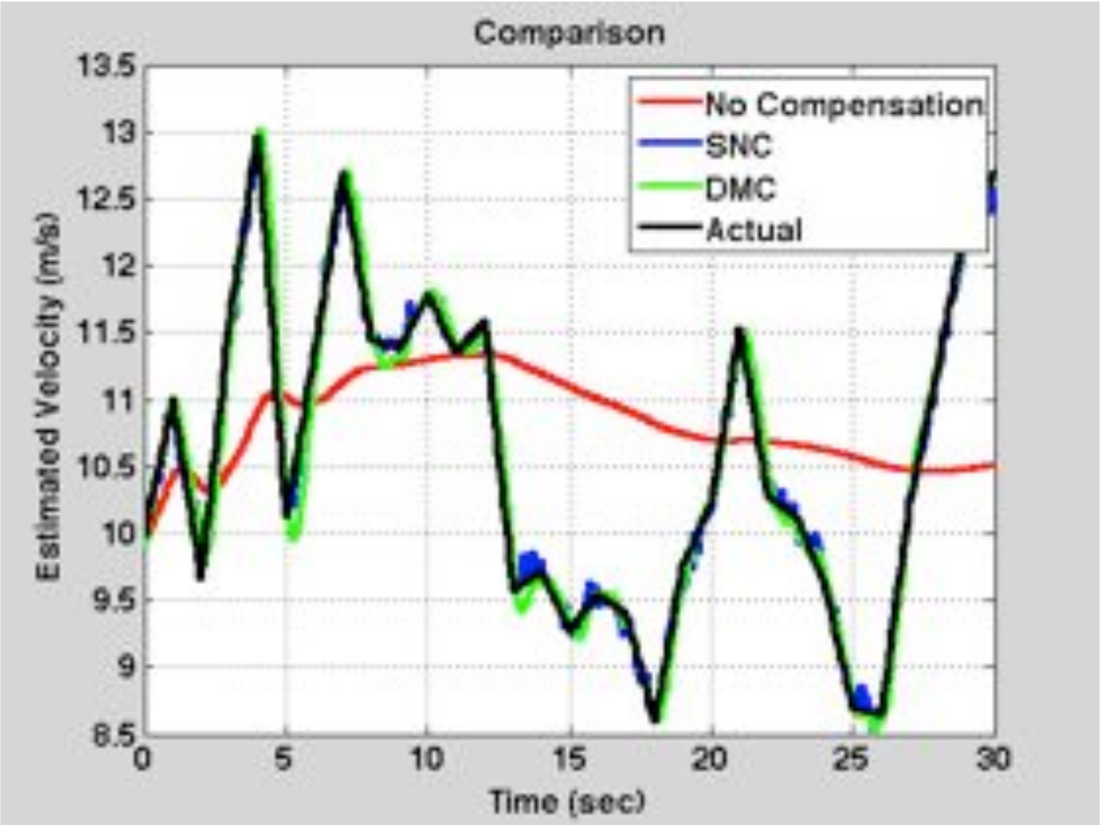


Example SNC / DMC

- ~White Noise

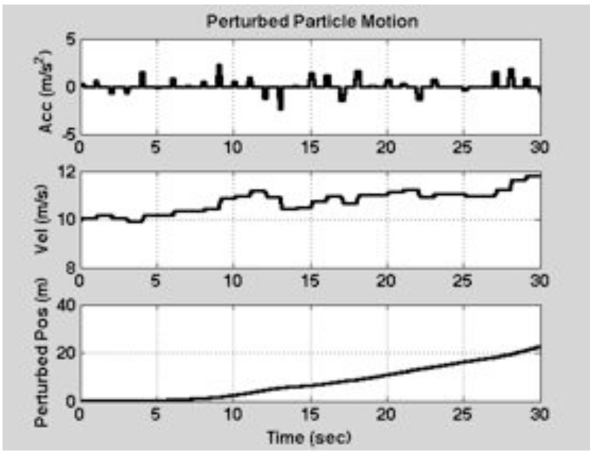


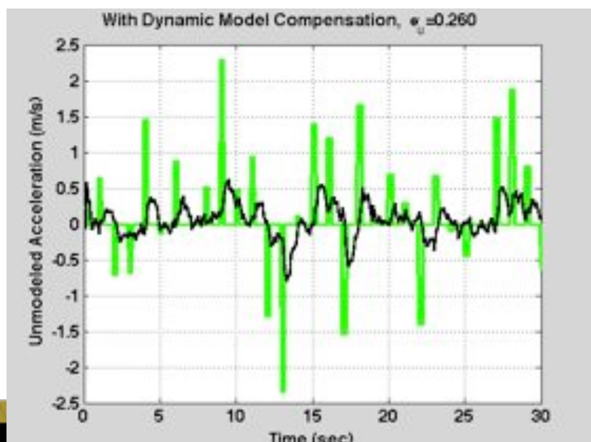


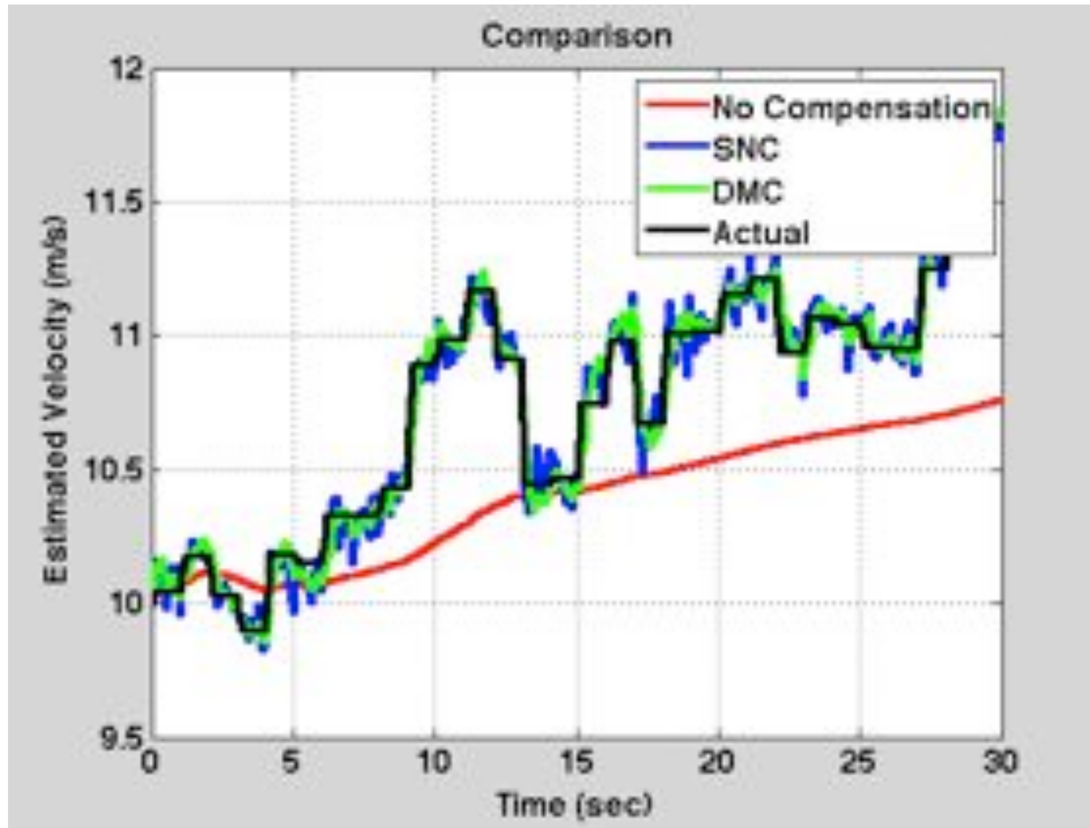


Example SNC / DMC

- ~Impulses







Status

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- Shown an example
- Now we'll derive the SNC and DMC formulations
- Then we'll apply them to our satellite state estimation problem.
- But first! A quick break.

- Our setting:
  - We are not modeling everything perfectly – and we either suspect it or know it.

- State dynamics of a linear system under the influence of process noise:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

  - A(t) and B(t) are known.
  - u(t) is mx1
  - B(t) is nxm

Derivation of State Noise Compensation

- State dynamics of a linear system under the influence of process noise:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

  - u(t) can include all kinds of processes.
    - White noise, constant noise, piecewise constant, correlated, etc.

  - The standard *State Noise Compensation* (SNC) algorithm assumes that u(t) is white noise
  - Dynamic Model Compensation assumes that u(t) is more complex, as we saw earlier.

  - SNC: $$E[\mathbf{u}(t)] = 0$$

$$E[\mathbf{u}(t)\mathbf{u}^T(t)] = Q(t)\delta(t-\tau)$$

Derivation of State Noise Compensation

- Solve the linear system via a method of variation of parameters.

- Homogeneous equation: $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$

- Solution of the form: $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}_0$

- Select C_0 as a function of time so that

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}(t)$$

Derivation of State Noise Compensation

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}(t)$$

- Then we can take the derivative to find

$$\dot{\mathbf{x}}(t) = \dot{\Phi}(t, t_0)\mathbf{C}(t) + \Phi(t, t_0)\dot{\mathbf{C}}(t)$$

Derivation of State Noise Compensation

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}(t)$$

- Then we can take the derivative to find

$$\dot{\mathbf{x}}(t) = \dot{\Phi}(t, t_0)\mathbf{C}(t) + \Phi(t, t_0)\dot{\mathbf{C}}(t)$$

- Plugging this into $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$

$$\dot{\Phi}(t, t_0)\mathbf{C}(t) + \Phi(t, t_0)\dot{\mathbf{C}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

Derivation of State Noise Compensation

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$$\dot{\Phi}(t, t_0)\mathbf{C}(t) + \Phi(t, t_0)\dot{\mathbf{C}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

- Recall our old friend: $\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0)$

$$A(t)\Phi(t, t_0)\mathbf{C}(t) + \Phi(t, t_0)\dot{\mathbf{C}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

Derivation of State Noise Compensation

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$$A(t)\Phi(t,t_0)\mathbf{C}(t) + \Phi(t,t_0)\dot{\mathbf{C}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

- Substitute the solution from above:

$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{C}(t)$$

$$A(t)\Phi(\cancel{t,t_0})\mathbf{C}(t) + \Phi(t,t_0)\dot{\mathbf{C}}(t) = A(t)\Phi(\cancel{t,t_0})\mathbf{C}(t) + B(t)\mathbf{u}(t)$$

$$\Phi(t,t_0)\dot{\mathbf{C}}(t) = B(t)\mathbf{u}(t)$$

Derivation of State Noise Compensation

$$\Phi(t,t_0)\dot{\mathbf{C}}(t) = B(t)\mathbf{u}(t)$$

- Hence $\dot{\mathbf{C}}(t) = \Phi^{-1}(t,t_0)B(t)\mathbf{u}(t)$

- Integrating yields

$$\mathbf{C}(t) = \mathbf{C}_0 + \int_{t_0}^{t} \Phi^{-1}(\tau,t_0)B(\tau)\mathbf{u}(\tau)d\tau$$

Derivation of State Noise Compensation

- Substituting $\mathbf{C}(t) = \mathbf{C}_0 + \int_{t_0}^{t} \Phi^{-1}(\tau, t_0) B(\tau) \mathbf{u}(\tau) d\tau$

into $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}(t)$

yields $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}_0 + \int_{t_0}^{t} \underbrace{\Phi(t, t_0)\Phi^{-1}(\tau, t_0)}_{\Phi(t,\tau)} B(\tau)\mathbf{u}(\tau) d\tau$

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{C}_0 + \int_{t_0}^{t} \Phi(t, \tau) B(\tau)\mathbf{u}(\tau) d\tau$$

Derivation of State Noise Compensation

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$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{C}_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\mathbf{u}(\tau)d\tau$$

- Initial conditions $t = t_0, \quad \mathbf{x}(t_0) = \mathbf{x}_0$

We can show that $C_0 = x_0$

$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\mathbf{u}(\tau)d\tau$$

Derivation of State Noise Compensation

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathbf{u}(\tau)d\tau$$

- This equation is the general solution for the inhomogeneous equation

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

- It indicates how the true state propagates under the influence of process noise.

Derivation of State Noise Compensation

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Next, we wish to understand how the <u>estimate of the state</u> $\mathbf{x}(t)$ propagates in the presence of process noise.

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathbf{u}(\tau)d\tau$$

is a stochastic integral and cannot be evaluated in a deterministic sense.

$$\bar{\mathbf{x}}(t) = E\left[\mathbf{x}(t) \middle| \mathbf{y}_{k-1}\right] \qquad \text{for} \qquad t \geq t_{k-1}$$

$$\dot{\bar{\mathbf{x}}}(t) = E\left[\dot{\mathbf{x}}(t) \middle| \mathbf{y}_{k-1}\right] = E\left[\left\{A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)\right\} \middle| \mathbf{y}_{k-1}\right]$$

but $$E\left[\mathbf{u}(t) \middle| \mathbf{y}_{k-1}\right] = 0$$

Derivation of State Noise Compensation

so

$$\dot{\bar{\mathbf{x}}}(t) = A(t)E\left[\mathbf{x}(t) \middle| \mathbf{y}_{k-1}\right] = A(t)\bar{\mathbf{x}}(t)$$

$$\dot{\bar{\mathbf{x}}}(t) = A(t)\bar{\mathbf{x}}(t)$$

We have our familiar solution

$$\bar{\mathbf{x}}(t) = \Phi(t, t_{k-1})\hat{\mathbf{x}}_{k-1}$$

This is good news! It means that we don't have to change our methods to update x-bar in our sequential algorithm.

Derivation of State Noise Compensation

- Since $E[\mathbf{u}(t)] = 0$

  we have shown that the time update of the state is unchanged

$$\bar{\mathbf{x}}(t_k) = \Phi(t_k, t_{k-1})\hat{\mathbf{x}}(t_{k-1})$$

- If the mean of the process noise is nonzero

$$E[\mathbf{u}(t)] = \bar{\mathbf{u}}$$

$$\bar{\mathbf{x}}(t_k) = \Phi(t_k, t_{k-1})\hat{\mathbf{x}}(t_{k-1}) + \Gamma(t_k, t_{k-1})\bar{\mathbf{u}}$$

(defined later)

The End

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- How was break?
- HW 10 due Thursday.
- HW 11 due next week.
- Grading
- You have the tools needed to finish the project. Only things missing are bonus pieces.
- I may be out tomorrow; if you need office-hour help, check the TAs or visit me Thursday.