Measure Theory Notes

Zachary Glassman

December 3, 2013

Contents

1 9/3/2013

Today is the first day of class. We are going to talk about the problems in Analysis which lead to measure theory. We are going to do the general thing today, so not too much actual notes. Kind of going to get an overview.

1.1 Broad Problems

Here are some problems that we will try to solve.

1. **Fourier Series Completion**
   f is Riemann integrable on $[-\pi, \pi]$. Its four series is defined,
   $$f = \sum a_n e^{inx} \tag{1.1}$$
   where
   $$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$
   We have the Parseval identity
   $$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx. \tag{1.2}$$
   The problem is that f is a Riemann integrable function and this space is not space. However, the left side of the Parseval identity does give rise to a complete space. So this is **The Completion Problem**. In other words $l^2(\mathbb{Z})$ is complete, but the norm on Riemann integrable functions is not complete. So we need to restrict to certain functions.

2. **Limits of Continuous Functions**
   Let $\{f_n\}$ be a sequence of continuous functions on $[0, 1]$ and assume that
   $$\lim_{n\to\infty} f_n(x) = f(x)$$
   exists for every x. The question is what can we say about f? If the convergence is uniform then, then the function f is continuous. However, we can find a sequence of functions that converges everywhere, but the limit is not a continuous function. So when is
   $$\int_0^1 f(x)dx = \lim_{n\to\infty} f_n(x)dx$$
   a true statement. In other words, when can the limit and the integral be interchanged.

3. **Length of Curves**
   Say that C is a continuous curved defined by some parametric representation
   $$C = \{(x(t), y(t)) : a \le t \le b\},$$
   where $x(t), y(t)$ are continuously differentiable functions. Then C is rectifiable if length of C, L is finite. We have sen before that
   $$L = \int_a^b \left[(x'(t))^2 + (y'(t))^2\right]^{1/2} dt. \tag{1.3}$$
   We will seen that if the functions $x(t)$ and $y(t)$ are of bounded variation then (1.3) is meaningful.

We know the fundamental theorem of calculus given as

$$F(b) - F(a) = \int_a^b F'(x)dx. \tag{1.4}$$

Yet there are situations where $F'(x)$ exists for ever x, but is not integrable. We want to establish the following identity

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{1.5}$$

for general classes of functions.

1.2 Notation

1. For d-dimensional Euclidean space.

$$\mathbb{R}^d = \{(x_{1_1}, x_{1_2}, \ldots, x_{1_d}), \ldots (x_{d_1}, x_{d_2}, \ldots, x_{d_d})$$

2. A vector has length or norm

$$|((x_1, x_1, \ldots, x_d)| = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$$

   note that he only uses one bar, usually we use two bars.

3. Let $E \subset \mathbb{R}^d$. Then we denote the complement

$$E^c = \{x \in R^d : x \notin E\}$$

4. We denote the complement of F in E

$$E - F = \{x \in \mathbb{R}^d : x \in E \textit{ and } x \notin F\}$$

5. The distance between two sets E and F.

$$d(E, F) = \inf_{x \in E, y \in F}\{|x - y|\}$$

2 9/5/13

2.1 Basic Analysis Definitions

Definition 2.1 (Open Set). *An open ball in $\mathbb{R}^d$ with center x, radius r is given as*

$$B_r(x) = \{y \in \mathbb{R}^d \ : \ |x - y| < r\}$$

A subset $E \subset \mathbb{R}^d$ if for all $x \in E$, there exists $r > 0$ such that

$$B_r(x) \subset E.$$

Another way of saying this is that there is an open neighborhood around every point in the set.

Theorem 2.2. *A subset $E \subset \mathbb{R}^d$ is closed if E^c is open.*

This is also true if we switch closed and open.

Theorem 2.3. *Consider unions and intersections of open and closed sets.*

1. *Any union of open sets is open and the intersection of finitely many open sets is open.*
2. *Any intersection of closed sets is closed and the union of finitely many closed sets is closed.*

Definition 2.4. *$E \subset \mathbb{R}^d$ is bounded if there exists $r > 0$ finite such that*

$$E \subset B_r(0)$$

Definition 2.5 (Compactness). *If*

$$E \subset \bigcup_{\alpha} U_\alpha$$

and each U_α is open, then

$$E \subset \bigcup_{j=1}^{N} U_{\alpha_j} \tag{2.1}$$

This is the same as saying every open cover admits a finite subcover.

Theorem 2.6 (Heine-Borel). *$E \subset \mathbb{R}^d$ is compact if and only if it is closed and bounded.*

Definition 2.7 (Limit Point). *$x \in \mathbb{R}^d$ is a limit point of E if for every $r > 0$*

$$B_r(x) \bigcap E \neq \phi$$

Essentially this means that there are point in E arbitrarily close to x.

Definition 2.8 (Isolated Point). *An isolated point of E is a point $x \in E$ such that there exists $r > 0$ with*

$$B_r(x) \bigcap E = \{x\}$$

Definition 2.9 (Boundary of a Set). *The boundary of E, denoted ∂E is defined as*

$$\partial E = \overline{E} \setminus E^o$$

Definition 2.10 (Perfect Set). *A closed set E is perfect if E does not have isolate points.*

2.2 Useful Definitions for Measure

We will now talk about the extended non-negative real number system. We define it as

$$[0, \infty) = \{x \in \mathbb{R} : x \geq 0\} \tag{2.2}$$

Note that $+\infty$ is not a real number. We include to obtain non-negative real axis as

$$[0, \infty]$$

We need to extend concepts of multiplication and addition and order structures of $[0, \infty)$ to $[0\infty]$ by declaring

a) $+\infty + x = x + \infty = +\infty \quad \forall x \in [0, \infty]$

b) $+\infty \cdot x = x \cdot (+\infty) = +\infty \quad \forall x \in (0, \infty]$

Laws of cancellation do not apply because we need to be mindful of indeterminate forms so

$$\infty + x = \infty + y \not\Longrightarrow x = y$$
$$\infty \cdot x = \infty \cdot y \not\Longrightarrow x = y$$

If we want to keep all the useful properties for $[0, \infty]$ and add negative numbers, we run into difficulties. So we have a tradeoff where we consider two theories.

- **Non-negative Theory** $[0, +\infty]$
  ex. Monotone Convergence Theorem

- **Absolute integrable theory**$(-\infty, \infty)$, $\mathbb{C}$.
  ex. Dominated Convergence Theorem

Given $x_1, x_2, \dots, x_n \in [0, \infty]$, we can always form

$$\sum_{i=1}^{\infty} x_i$$

We define this in the following way

Definition 2.11.

$$\sum_{n=1}^{\infty} x_n = \sup_{F \subset \mathbb{N}} \sum_{n \in F} x_n \tag{2.3}$$

where F is finite. In other words, it is the supremum of all finite subsums.

We now look at another theorem that will be without proof for now.

Theorem 2.12 (Tonelli's Theorem for Series)**.** *Let $\{x_{n,m}\}$ with $n, m \in \mathbb{N}$ be a doubly infinite sequence of extended non-negative reals $x_{n,m} \in [0, +\infty]$. Then*

$$\sum_{(n,m) \in \mathbb{N}^2} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{n,m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{n,m} \tag{2.4}$$

Theorem 2.13. *Every open set O in $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Theorem 2.14. *Every open set O in $\mathbb{R}^d$ $d \geq 1$ can be written as a countable union of almost disjoint closed cubes.*

Note there is no claim about uniqueness here.

2.3 Rectangles

Definition 2.15 (Closed Rectangle)**.** *A closed rectangle R in $\mathbb{R}^d$ is*

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$$

the product of one dimensional closed and bounded intervals where $a_j \leq b_j$, $j = 1, 2, \dots, d$.

Note that this is a closed set and the sides are parallel to the coordinates axis.

Definition 2.16 (Volume of a Rectangle)**.** *We will denote the volume of a Rectangle as $|R|$ where*

$$|R| = (b_1 - a_1) \cdot (b_2 - a_2) \cdot \cdots \cdot (b_d - a_d). \tag{2.5}$$

Note that a closed and open rectangle have the same volume. So we can write that

$$|R| = |\overline{R}| = |R^o|. \tag{2.6}$$

Definition 2.17 (Cube)**.** *A cube is a rectangle for which*

$$b_i - a_i = b_j - a_j$$

for all $i, j \leq d$.

So we know that if $b_j - a_j = l$, then

$$|Q| = l^d. \tag{2.7}$$

In some ways we are going to use these cubes as the building blocks of our space.

Definition 2.18 (Almost disjoint)**.** *A union of rectangles is said to be almost disjoint if the interiors of the rectangles are disjoint.*

Note that this implies the boundaries of the rectangles can overlap.

Lemma 2.19. *If a rectangle is almost disjoint union of finitely many other rectangles, say*

$$R = \bigcup_{k=1}^{n} R_k, \tag{2.8}$$

then

$$|R| = \sum_{k=1}^{N} |R_k| \tag{2.9}$$

Lemma 2.20. *If $R_1, R_2, \ldots, R_N$ are rectangles and $R \subset \bigcup_{k=1}^{N} R_k$, then*

$$|R| \leq \sum_{k=1}^{N} |R_k| \tag{2.10}$$

2.4 Cantor set

Definition 2.21 (The Cantor Set)**.** *There is a inductive construction*

1. $C_0 = [0, 1]$
2. $C_1 = [0, \frac{1}{3} \cup [\frac{2}{3}, 1]$
3. $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
4. $\ldots$

Note that for each interval C_k is made up of 2^k intervals with each of these intervals having length $\frac{1}{3^k}$. So we have Cantor set C as

$$C = \bigcap_{k=1}^{\infty} C_k \tag{2.11}$$

Some properties are that

1. C is compact
2. C is totally disconnected
   Given $x, y \in C$, there exists non intersecting open balls around any two points of C.
3. C is perfect, $C^o = \phi$.
4. C has cardinality uncountably infinite.
5. Length of the deleted intervals can be written as a geometric series equal to 1.

2.5 Exterior Measure

Definition 2.22 (Exterior Measure)**.** *Let E be any subset of $\mathbb{R}^d$, the exterior measure of E (sometimes called outer measure) is defined as*

$$m_* E = \inf\{\sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j\} \tag{2.12}$$

Note this infimum is taken over countable covering over E by closed cubes Q_j.

3 9/10/13

We begin with some remarks on what we covered last class.

1. It is not sufficient to allow finite sum in the definition of $m_*(E)$.
2. We can take instead of closed cubes closed rectangles.
3. $m_*(\{a\}) = 0$ because a single point can be viewed as a closed cube with volume 0.
4. By definition we can see that $0 \leq m_*(E) \leq \infty$.

Proposition 3.1. *Let Q be a closed cube,*

$$m_*(Q) = |Q|$$

Proof. Since $Q \subset Q$, we know that

$$m_*(Q) \leq |Q|.$$

We know that

$$m_* = \inf\{|Q| : Q \subset Q\}. \tag{3.1}$$

So now we will consider an arbitrary covering of Q

$$Q \subset \bigcup_{j=1}^{\infty} Q_j.$$

Recall that $K_1 \subset K_2 \implies \inf K_1 \geq \inf K_2$. It is sufficient to prove that

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j| \implies |Q| \leq m_*(Q) \tag{3.2}$$

Combining this with the statement about a cube being its own cover, we will then have the proof.
So let $\epsilon > 0$, chose j and open cube S_j such that $Q_j \subset S_j$ with

$$|S_j| \leq (1+\epsilon)|Q_j|$$

We know that

$$Q \subset \bigcup_{j=1}^{\infty} Q_j \subset \bigcup_{j=1}^{\infty} S_j \tag{3.3}$$

which implies that $S = \bigcup_{j=1}^{\infty} S_j$ is an open cover for Q. Since Q is a compact set, we know there exists a finite cover of S_j and so we have that

$$Q \subset \bigcup_{j=1}^{N} S_j \tag{3.4}$$

for some N. Taking the closure of S_j, we know that

$$|Q| \leq \sum_{j=1}^{N} |S_j| \leq \sum_{j=1}^{N} (1+\epsilon)|Q_j| = (1+\epsilon)\sum_{j=1}^{N} |Q_j| \leq (1+\epsilon)\sum_{j=1}^{\infty} |Q_j|. \tag{3.5}$$

□

Claim 3.2. *Q is an open cube, we still have that $m_*(Q) = |Q|$.*

Proof. Note that

$$m_*(E) = \inf\{\sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j\} = \inf\{\sum_{j=1}^{\infty} m_*(Q_j) : E \subset \bigcup_{j=1}^{\infty} Q_j\} \tag{3.6}$$

Let Q_0 be a closed cube contained in Q. Then we know that $Q_0 \subset Q$ and so we know that

$$Q_0 \subset Q \subset \bigcup_{j=1}^{\infty} Q_j. \tag{3.7}$$

So we have the following

$$m_*(Q) = \inf\{\sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j\} = \inf K_1 \tag{3.8}$$

$$m_*(Q) = \inf\{\sum_{j=1}^{\infty} |F_j| : E \subset \bigcup_{j=1}^{\infty} F_j\} = \inf K_2 \tag{3.9}$$

And so we know that $r \in K_1 \implies r \in K_2$ which implies that $K_1 \subset K_2$ and therefore $\inf K_1 \geq \inf K_2$. So we have that

$$m_*(Q_0) \leq m_*(Q) \implies |Q_0| \leq m_*(Q). \tag{3.10}$$

Since we can choose Q_0 with volume as close to $|Q|$, we can use the proof in the previous claim to finish. □

We now offer some other properties.

Claim 3.3.

$$m_*(R) = |R| \tag{3.11}$$

where R is a rectangle.

Claim 3.4.

$$m_*(\mathbb{R}^d) = \infty \tag{3.12}$$

Proof. Let Q be any closed cube $Q \subset \mathbb{R}^d$. Any cover of $\mathbb{R}^d$ is also a covering of Q. This implies that

$$|Q| \leq m_*(\mathbb{R}^d) \tag{3.13}$$

Since this volume is arbitrarily large, we have that $m_*(\mathbb{R}^d) = \infty$. □

3.1 Properties of the Exterior Measure

We can say that from the definition of infimum we can say for all $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with

$$\sum_{j=1}^{\infty} |Q_j| = \sum_{j=1}^{n} m_*(Q_j) \leq m_*(E) + \epsilon. \tag{3.14}$$

1. If $E_1 \subset E_2$, then we have that $m_*(E_1) \leq m_*(E_2)$. This property is called monotonicity.
2. If $E = \bigcup_{j=1}^{\infty} E_j$ then

$$m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j) \tag{3.15}$$

This property is called countable sub-additivity.

Proof. Without loss of generality assume that

$$m_*(E_j) < \infty$$

For each E_j take a covering

$$E_j \subset \bigcup_{k=1}^{\infty} Q_{k,j} \tag{3.16}$$

and we know that

$$\sum_{k=1}^{\infty} |Q_{k,j}| \leq m_*(E_j) + \frac{\epsilon}{2^j}. \tag{3.17}$$

We know that

$$E = \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{k,j=1}^{\infty} Q_{k,j}, \tag{3.18}$$

the covering of E by closed cubes. So we have that

$$m_*(E) \leq \sum_{j,k} |Q_{k,j}| = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |Q_{k,j}| \leq \sum_{j=1}^{\infty} m_*(E_j) + \frac{\epsilon}{2^j} = \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \tag{3.19}$$

for all $\epsilon > 0$. which implies (3.15). $\square$

3. If $E \subset \mathbb{R}^d$, then we know that

$$m_*(E) = \inf m_*(\mathcal{O}) \tag{3.20}$$

where $E \subset \mathcal{O}$, where $\mathcal{O}$ is all open sets containing E. This property is called outer regularity.

Proof. Since we know that $E \subset Q$ and outer measure is monotonic, we have for each Q that

$$M_*(E) \leq m_*(Q) \implies m_*(E) \leq \inf m_*(Q) \tag{3.21}$$

Let $\epsilon > 0$ and choose Q_j cubes such that $E \subset \bigcup_{j=1}^{\infty} Q_j$ with

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \frac{\epsilon}{2}. \tag{3.22}$$

Let Q_j^o be an open cube such that $Q_j \subset Q_j^o$ with

$$|Q_j^o| \leq |Q_j| + \frac{\epsilon}{2^j + 1} \tag{3.23}$$

We know that $\mathcal{O} = \bigcup_{j=1}^{\infty} Q_j^o$ is open and we use countable subadditivity to show that

$$m_*(Q) \leq \sum_{j=1}^{\infty} m_*(Q_j^o) = \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2^j + 1} \tag{3.24}$$

So this implies the following

$$m_*(\mathcal{O}) \leq m_*(E) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = m_*(E) + \epsilon \tag{3.25}$$

for all $\epsilon > 0$ which implies (3.20). $\square$

4. If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then we have that

$$m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2) \tag{3.26}$$

5. If E is a countable union of almost disjoint cubes then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|. \tag{3.27}$$

Proof. From countable sub-additivity, we already know that

$$m_*(E) \leq \sum_{j=1}^{\infty} m_*(Q_j)$$

So what we want to show is

$$m_*(E) \geq \sum_{j=1}^{\infty} m_*(Q_j) \tag{3.28}$$

Consider $Q'_j \subset Q_j$. So we know that

$$|Q_j| = |Q'_j| + \frac{\epsilon}{2^j} \tag{3.29}$$

for every n , the cubes $Q'_1, Q'_2, Q'_3, \ldots, Q'_N$ are disjoint and positive distance from each other. So we know that

$$D(Q'_i, Q'_j) > 0 \quad i, j = 1, \ldots, N \tag{3.30}$$

From property 4 we know that

$$m_* \left(\bigcup_{j=1}^{N} Q'_j \right) = \sum_{=1}^{n} m_*(Q'_j), \tag{3.31}$$

and since we know that $Q'_j \subset Q_j$ we know that

$$\bigcup_{j=1}^{N} Q'_j \subset \bigcup_{j=1}^{\infty} Q_j = E \tag{3.32}$$

So we then have that

$$m_*(E) \geq \sum_{j=1}^{n} |Q'_j| \geq \sum_{j=1}^{N} |Q_j| - \epsilon \tag{3.33}$$

for all ϵ so in the limit at $N \to \infty$ we have what we are looking for. $\square$

4 9/12/13

We are now going to try to translate knowledge of exterior measure to the more general kind of measure. First we need some definitions.

4.1 Measurable Sets and Lebesgue Measure

Definition 4.1 (Measurable Set)**.** *$E \subset \mathbb{R}^d$ is measurable if there exists and open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O} \setminus E) \leq \epsilon$.*

We want some ideas about the existence of Lebesgue measurable sets. We will list some here and prove some of them as well.

1. All open sets and all closed sets are measurable

2. $m_*(E) = 0$ implies that E is measurable.
3. If E is measurable then E^c is measurable.
4. Countable union and countable intersection of measurable sets is measurable.
5. If $E_1, E_2, \dots E_n$ are all measurable, then

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j). \tag{4.1}$$

   If the sets are disjoint then we have that

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j) \tag{4.2}$$

6. If E_j's are measurable and $E_1 \subset E_2 \subset E_3 \subset \dots$, then

$$m\left(\bigcup_{j=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n). \tag{4.3}$$

7. If E_j's are measurable and $E_1 \supset E_2 \supset E_3 \supset \dots$ and $m(E_1) < \infty$, then

$$m\left(\bigcap_{j=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n). \tag{4.4}$$

8. Translation invariance,

$$m(E) = m(E + h) \tag{4.5}$$

We will offer proofs of some of these properties.

Claim 4.2. *$m_*(E) = 0$ implies that E is measurable.*

Proof. Let $E \subset \mathbb{R}^d$. We take the definition of exterior measure to say

$$m_*(E) = \inf m_*(\mathcal{O}) \tag{4.6}$$

taken of all open sets $\mathcal{O}$. So we know that

$$m_*(\mathcal{O}) \leq m_*(E) + \epsilon \implies m_*(\mathcal{O}) \leq m_*(E) + \epsilon, \tag{4.7}$$

Since we know that $\mathcal{O} \setminus E \subset \mathcal{O}$, we know that

$$m_*(\mathcal{O} \setminus E) \leq m_*(\mathcal{O}) \leq \epsilon. \tag{4.8}$$

□

Claim 4.3. *Every open set in $\mathbb{R}^d$ is measurable.*

Proof. Let E be open. So take $E = \mathcal{O}$, our open set. So we know that

$$E \setminus \mathcal{O} = \mathcal{O} \setminus \mathcal{O} = \phi \tag{4.9}$$

So we know that $m(E \setminus \mathcal{O}) = 0 \leq \epsilon$ for all ϵ. □

Claim 4.4. *Countable union of measurable sets is measurable.*

Proof. Let $E = \bigcup_{j=1}^{\infty} E_j$ with each E_j measurable. So given $\epsilon > 0$ we can choose for each j an open set $\mathcal{O}_j$ such that $E_j \subset \mathcal{O}_j$ and

$$m_*(\mathcal{O}_j \setminus E_j) \leq \frac{\epsilon}{2^j}. \tag{4.10}$$

Set

$$\mathcal{O} = \bigcup_{j=1}^{\infty} \mathcal{O}_j, \tag{4.11}$$

which is an open set and $E \subset \mathcal{O}$. So we know that

$$\mathcal{O} \setminus E \subset \bigcup_{j=1}^{\infty} (\mathcal{O}_j \setminus E_j). \tag{4.12}$$

Then we know that

$$m_*(\mathcal{O} \setminus E) \leq m_* \left(\bigcup_{j=1}^{\infty} \mathcal{O}_j \setminus E_j \right) \leq \sum_{j=1}^{\infty} m_*(\mathcal{O}_j \setminus E_j), \tag{4.13}$$

where we have used the monotonicity and sub-additivity of exterior measure. which implies that

$$m_*(\mathcal{O} \setminus E) \leq \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon. \tag{4.14}$$

□

It is reasonable to ask whether all subset of $\mathbb{R}^d$ are measurable. The answer is no and we can give an example due to Vitali in $\mathbb{R}$. First we need to recall two definitions.

Definition 4.5 (Axiom of Choice)**.** *Let $\mathcal{A}$ be a collection of set $\mathcal{A} = \{A, B, C, \dots\}$ and all of the set in $\mathcal{A}$ are non-empty, then there exists a set $\mathcal{Z}$ consisting of precisely one element from A, one element from B and so on.*

Definition 4.6 (Equivalence Relation)**.** *Let X be a set. An equivalence relation $\sim$ is a relation with*

1. *$a \sim a$ (reflexivity)*
2. *$a \sim b \implies b \sim a$ (symmetric)*
3. *$a \sim b$ and $b \sim c$ implies $a \sim c$ (transitivity)*

Example 4.7 (Vitali's)**.** *This uses axiom of choice and a simple equivalence relation on $[0, 1]$. Take $x, y \in [0, 1]$ and define $x \sim y \Leftrightarrow x - y \in Q$. $[0, 1]$ is the disjoint union of equivalence classes. So we know that*

$$[0, 1] = \bigcap_{\alpha} \epsilon_\alpha,$$

where ϵ_α are the equivalence classes of $\alpha \in \mathbb{R}$. Construct N by choosing one element x_α from each equivalence class. which means that $N \subset [0, 1]$ and N intersects each equivalence class at only one point. So $N = \{x_\alpha\}$. The claim is that this N is not measurable.

Proof. Let $r_1, r_2, \dots, r_k$ be an enumeration of rationals in $[-1, 1]$ and consider

$$N_k = N + r_k, \tag{4.15}$$

Observe that

- $N_k \subset [-1, 2]$ for each k.

- $N_k \bigcap N_m = \phi$ for $k \neq m$

  Proof. If this intersection was non-empty then there exists rationals r_k and r_n where $r_k \neq r_n$, but $x_\alpha + r_k = x_\beta + r_m \implies x_\alpha - x_\beta = r_m - r_n \in \mathbb{Q}$. This implies that $x_\alpha = x_\beta$ which contradicts N_k represents one element from each equivalence class. ☐

- $[0,1] \subset \bigcup N_k \subset [-1,2]$.

  Proof. $x \in [0,1]$ which implies that $x \sim x_\alpha$ for some $x - x_\alpha = r_k$ for some k. So we know that

  $$x = x_\alpha + r_k \implies x \in N_k \tag{4.16}$$

  ☐

- If N were measurable then $m(N) = m(N_k)$ by the translation property.

We know since the union of all N_k is disjoint we have that

$$1 \leq \sum_{k=1}^{\infty} m(N_k) \leq 3 \implies 1 \leq \sum_{k=1}^{\infty} m(N) \leq 3, \tag{4.17}$$

But we know that

$$\sum_{k=1}^{\infty}(N) = \begin{cases} 0 & if \quad m(N) = 0 \\ \infty & if \quad m(N) = \infty \end{cases} \tag{4.18}$$

So we reach the following conclusion from (4.17)

$$\begin{cases} 1 \leq 0 \\ \infty \leq 3 \end{cases} \tag{4.19}$$

So we have a contradiction and therefore this set is not measurable. ☐

5 9/17/13

We are going to go over the out measure axioms.

i) $m_*(\phi) = 0$

ii) If $E \subset F \subset \mathbb{R}^d$ then $m_*(E) \leq m_*(F)$

iii) $E_1, E_2, \dots \subset \mathbb{R}^d$ countable sequence of sets then

$$m_*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m_*(E_i). \tag{5.1}$$

Note that if we combine $i)$ and $iii)$, we have finite sub-additivity. These sub-additivity properties are useful in establishing upper bounds for outer measure. We have proved a couple of properties already

1. Every open set is measurable.
2. $m_*(E) = 0 \implies$ E is measurable.
3. $E_1, E_2, \dots$ seq of measurable sets then the countable union is measurable.

Refer back to the proof of 4.4. We are now going to look at why every closed set is measurable.

Claim 5.1. *Every closed set is measurable.*

Proof. Recall that every closed set is the countable union of closed and bounded sets. So if E is closed then

$$E = \bigcup_{i=n}^{\infty} E \bigcap B_n \tag{5.2}$$

By Heine-Borel, we know that E is compact.

Also now that we know that

$$m_*(\mathcal{O}) \leq m_*(E) + \epsilon. \tag{5.3}$$

for a set open set $\mathcal{O}$ such that $E \subset \mathcal{O}$. Note that we want to show

$$m_*(\mathcal{O} \setminus E) \leq \epsilon \tag{5.4}$$

Since E is closed we know that $\mathcal{O} \setminus E$ is open. So we have that

$$\mathcal{O} \setminus E = \bigcup_{j=1}^{\infty} Q_j \tag{5.5}$$

where Q_j are a set of almost disjoint cubes. So we as we have shown we have that

$$m_*(\mathcal{O} \setminus E) = \sum_{j=1}^{\infty} |Q_j|. \tag{5.6}$$

Note that we have now reduced the problem to showing that

$$\sum_{j=1}^{N} |Q_j| < \epsilon. \tag{5.7}$$

for all finite N. since we know that a finite union of closed cubes is closed. We also know that the distance between a compact and a closed set is positive,

$$d(E, \bigcup_{j=1}^{N} Q_j) > 0. \tag{5.8}$$

So we have that

$$m_* \left(E \bigcup \left[\bigcup_{j=1}^{N} Q_j \right] \right) = m_*(E) + m_* \left(\bigcup_{j=1}^{N} Q_j \right) \tag{5.9}$$

We know that both of these sets, E and $\bigcup_{j=1}^{N} Q_j$ are subset of $\mathcal{O}$. So we know that

$$m_* \left(E \bigcup \left[\bigcup_{j=1}^{N} Q_j \right] \right) \subset \mathcal{O} \implies m_* \left(E \bigcup \left[\bigcup_{j=1}^{N} Q_j \right] \right) \leq m_*(\mathcal{O}) \tag{5.10}$$

So we have that

$$\begin{aligned} m_*(E) + m_* \left(\bigcup_{j=1}^{N} Q_j \right) &\leq m_*(E) + \epsilon \\ \implies m_* \left(\bigcup_{j=1}^{N} Q_j \right) &= \sum_{j=}^{N} |Q_j| \leq \epsilon \end{aligned} \tag{5.11}$$

So we have shown that all closed sets are measurable. $\square$

Claim 5.2. *If $E \subset \mathbb{R}^d$ is measurable, then so is E^c.*

Proof. E is measurable. This implies that for each n there is an open set $\mathcal{O}_n$ where $E \subset \mathcal{O}_n$ such that

$$m_*(\mathcal{O}_n \setminus E) \leq \frac{1}{n}. \tag{5.12}$$

Since these are open note that $\mathcal{O}_n^c$ is closed. By 5.1 we know that $\mathcal{O}_n^c$ is measurable. We also know that for all n

$$E \subset \mathcal{O}_n \implies \mathcal{O}_n^c \subset E^c. \tag{5.13}$$

Let $S = \bigcup_{n=1}^{\infty} \mathcal{O}_n^c$. Since this is a countable union of measurable sets, S is measurable. We now wish to show that

$$E^c \setminus S \subset \mathcal{O}_n \setminus E. \tag{5.14}$$

We know that for all n

$$\begin{aligned} x \in E^c \setminus S &\implies X \in E^c, x \notin S \\ &\implies x \notin E, x \notin \mathcal{O}_n^c \\ & x \in \mathcal{O}_n \setminus E \end{aligned} \tag{5.15}$$

So we know that

$$m_*(E^c \setminus S) \leq m_*(\mathcal{O}_n \setminus E) = \frac{1}{n} \quad \forall n. \tag{5.16}$$

This shows that

$$m_*(E^c \setminus S) = 0 \implies E^c \setminus S \text{ is measurable.} \tag{5.17}$$

Then we have since the union of two measurable set is measurable,

$$E^c = (E^c \setminus S) \bigcup S, \tag{5.18}$$

is measurable. ☐

Claim 5.3. *If $E_1, E_2 \cdots \subset \mathbb{R}^d$ is a sequence of measurable sets, then their intersection*

$$\bigcap_{n=1}^{\infty} E_n$$

is measurable.

Proof. Recall DeMorgan laws which says that

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c \tag{5.19}$$

So we have that

$$\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{\infty} E_i^c \right)^c = \bigcap_{i=1}^{\infty} (E_i^c)^c \tag{5.20}$$

This completes the proof. ☐

5.1 Lebesgue Measure

Theorem 5.4 (The Measure Axioms)**.** *Here are the axioms for measure (Lebesgue Measure).*

i) $m(\phi) = 0$

ii) If $E_1, E_2, \cdots \subseteq \mathbb{R}^d$ *is a countable sequence of disjoint measurable sets then we have that*

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i) \tag{5.21}$$

Furthermore, one can conclude finite additivity also holds.

We now do some important monotone convergence theorems.

Theorem 5.5 (Monotone convergence Theorem for Measurable Sets)**.** .

1. *Let* $E_1 \subset E_2 \subset \cdots \subset \mathbb{R}^d$ *be a countable non-decreasing Lebesgue measurable sets. Then we have that*

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n) \tag{5.22}$$

2. *Let* $E_1 \supset E_2 \supset \cdots \supset \mathbb{R}^d$ *be a countable non-increasing sequence of Lebesgue measurable sets. If at least one of the* $m(E_k) < \infty$*, then we have*

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n) \tag{5.23}$$

Proof. 1. Set $G_1 = E_1$ and $G_2 = E_2 \subset E_1$. Note that G_2 is measurable as $G_2 = E_1 \bigcap E_1^c$. Furthermoe, $E_1 \bigcup E_2 = G_1 \bigcup G_2$. We wish to write it this way because we now have separated sets. So in general, let

$$G_k = E_k \setminus E_{k-1} \quad k \geq 2. \tag{5.24}$$

By construction we have that

(a) G_k is measurable for all k.

(b) G_k are disjoint for all k.

(c) $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} G_k$

So we know that

$$m(E) = \sum_{k=1}^{\infty} m(G_k) = \lim_{N\to\infty} \sum_{k=1}^{N} m(G_k) = \lim_{N\to\infty} m\left(\bigcup_{k=1}^{N} G_k\right) = \lim_{n\to\infty} m(E_n). \tag{5.25}$$

2. It is enough to assume that $m(E_1)$ is finite since we have monotonicity that this will force the rest of $m(E_n)$ to be finite. So we can write E_1 as

$$E_1 = E \bigcup \left(\bigcup_{k=1}^{\infty} G_k\right), \tag{5.26}$$

where $G_k = E_k \setminus E_{k-1}$ We know from a similiar argument to above that G_k are disjoint and measurable. So we know that

$$
\begin{aligned}
m(E_1) = m(E) + \lim_{N\to\infty} \sum_{k=1}^{N} m(G_k) = m(E) + \lim_{N\to\infty} \sum_{k=1}^{N} m(E_k \setminus E_{k-1}) \\
= m(E) + \lim_{N\to\infty} [(m(E_1) - m(E_2)) + (m(E_2) - m(E_3) + \cdots + (m(E_{N-1}) - m(E_N))] \\
= m(E_1) + \lim_{N\to\infty} [m(E_1) - m(E_n)]
\end{aligned} \tag{5.27}
$$

where the last step comes from a telescoping series.

□

Theorem 5.6 (3.4(p21))**.** *Suppose $E \subset \mathbb{R}^d$ is measurable. Then for all $\epsilon > 0$ we have*

1. *Outer approximation by open set*
   There exists an open set $\mathcal{O}$, $E \subset \mathcal{O})$ and $m(\mathcal{O} \setminus E) < \epsilon$.
2. *Inner approximation by closed set*
   There exists a closed set $\mathcal{F}$, $\mathcal{F} \subset E$ and $m(E \setminus F) < \epsilon$.
3. *Inner approximation by compact set for finite measures.*
   If $m(E)$ is finite, there exists a compact set $K \subset E$ such that $m(E \setminus K) < \epsilon$.
4. *Approximation by symmetric difference for finite measures*
   If $m(E)$ is finite, there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that $m(E\Delta F) < \epsilon$.
   Here Δ denotes the symmetric difference and

$$
E\Delta F = (E \setminus F) \bigcup (F \setminus E) \tag{5.28}
$$

5.2 Proof by Mike

We now have a proof by Micheal. He will prove a lemma.

Lemma 5.7. *Let $E \subset \mathbb{R}^d$ where E is open. Then we have that*

$$
E \bigcup_{i=1}^{\infty} Q_j
$$

where Q_j are closed cubes.

Proof. Begin by defining the cubes as

$$
Q_j = \left[\frac{i_1}{2^n}, \frac{i_1 + 1}{2^n}\right] \times \cdots \times \left[\frac{i_d}{2^n}, \frac{i_d + 1}{2^n}\right], \tag{5.29}
$$

where $i_j \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. Note that as n increases, we have smaller and smaller cubes, which leads to better and better approximations. So we have closed cubes in an open set E. So we know that if we take the union of all the cubes, we can approximate E, but we need to show that they are almost disjoint.
I am going to read the proof myself, I am not exactly following what he is saying. □

6 Notes On Sigma Algebras and 9/19/13

Definition 6.1 (σ-algebra)**.** *Let $\mathcal{A}$ be a collection of subset of $\mathbb{R}^d$. $\mathcal{A}$ is called an σ-algebra if $\mathcal{A}$ is closed under countable unions, intersections and complements.*

Example 6.2. *1. $\mathcal{P}(\mathbb{R}^d)$, the collection of all subset of $\mathbb{R}^d$ is a σ-algebra.*

2. $\mathcal{M}_{\mathbb{R}^d}$, the set of all measurable subsets of $\mathbb{R}^d$.

Definition 6.3 (Borel σ-algebra)**.** *The smallest σ-algebra containing all open subsets of $\mathbb{R}^d$, which we denote by $\mathcal{B}_{\mathbb{R}^d}$. Elements of $\mathcal{B}_{\mathbb{R}^d}$ are called Borel sets.*

We can ask what do we mean by the smallest and how we know such a σ-algebra exists or is unique. Smallest means if S is any σ-algebra that contains all open subsets of $\mathbb{R}^d$, then we must have

$$\mathcal{B}_{\mathbb{R}^d} \subset S \qquad \mathcal{B}_{\mathbb{R}^d} = \bigcap S_i. \tag{6.1}$$

Since all open sets are measurable we know that $\mathcal{B}_{\mathbb{R}^d} \subset \mathcal{M}_{\mathbb{R}^d}$
We know there exists Lebesgue measurable sets which are not Borel sets

$$\mathcal{B}_{\mathbb{R}^d} \neq \mathcal{M}_{\mathbb{R}^d}. \tag{6.2}$$

Some examples are the Cantor-Lebesgue function and problem 35 in the book.

6.1 Some Borel Sets

We define two notations

$$G_\delta = \text{countable intersection of open sets} \tag{6.3}$$
$$F_\sigma = \text{countable union of closed sets} \tag{6.4}$$

Note: The nature of $\mathbb{R}$ could be elusive. The structure of *open* sets is fairly straightforward. Every open set in $\mathbb{R}$ is either finite or a countable union of open intervals.
Standing in opposition to this description of *open* sets is the Cantor set. The Cantor set is closed, uncountable and contains no intervals of any kind.
We know that an arbitrary union of open sets is open and an arbitrary intersection of closed sets is closed.

Definition 6.4 (F_σ, G_δ sets)**.** *A set $A \subseteq \mathbb{R}$ is called an F_σ set if it can be written as the countable union of closed sets. A set $B \subseteq \mathbb{R}$ is a G_δ set if it can be written as a countable intersection of open sets.*

Note the following fact

$$A \text{ is } G_\delta \Leftrightarrow A^c \text{ is } F_\sigma \tag{6.5}$$

F_σ and G_δ are relatively simple types of Borel sets, one can also consider
$F_{\sigma\delta}$ = intersection of countable collection of sets each of which F_σ,

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, d\ldots, G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, \ldots$$

are all classes of Borel sets (all are measurable). For further theory of Borel sets, consult C. Kuratowski "Topologie I".

Remark 6.5. *If $\mathcal{A}$ is a σ-algebra of sets on X, then $\sigma \in \mathcal{A}$ and $X \in \mathcal{A}$.*

This is because $E \in \mathcal{A}$

$$E \cap E^c = \phi \in \mathcal{A} \qquad E \cup E^c = X \in \mathcal{A} \tag{6.6}$$

One can replace a sequence of sets in a σ-algebra by a disjoint sequence of sets.

Proposition 6.6. *Let $\mathcal{A}$ be an σ-algebra and $\{E_1\}$ be a sequence of sets in $\mathcal{A}$. There exists a sequence of set $\{F_i\}$ in $\mathcal{A}$ such that*

i) $F_n \cap F_m = \phi$ for $n \neq m$.

ii) $\displaystyle\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$

Proof. Let $E_1 = F_1$, for each natural number $n > 1$, define

$$F_n = E_n \setminus \left[\bigcup_{j=1}^{n-1} E_j\right]$$

$$= E_n \bigcap \left[\bigcup_{j=1}^{n-1} E_j\right]^c = E_n \bigcap \left[\bigcup_{j=1}^{n-1} E_j^c\right]$$

$$F_n = E_n \cap E_1^c \cap \cdots \cap E_{n-1}^c \tag{6.7}$$

Not that since complements and intersections of sets in $\mathcal{A}$ are in $\mathcal{A}$, $F_n \in \mathcal{A}$ and $F_n \subset E_n$ for all n. Suppose that F_n and F_m are two sets with $m \leq n$. Then

$$F_m \cap F_n \subset E_\cap F_n = E_m \cap E_n \cap E_1^c \cap \cdots \cap E_m^c \cap E_{m+1}^c \cap \cdots \cap E_{n-1}^c = \phi. \tag{6.8}$$

Since $F_i \subset E_i$ for all i we have that $\displaystyle\bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} E_e$. Now let $x \in \cup E_i$, then $x \in E_i$ for some i¿ LEt N be the smallest value of i such that $x \in E_i$. So we have that $x \in E_n$ and $x \in E_1^c, E_2^c, \ldots, E_{n-1}^c$. This implies that $x \in F_n \implies x \in \displaystyle\bigcup_{n=1}^{\infty} F_n$. So we have that

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n. \tag{6.9}$$

□

Definition 6.7 (Null Set). *If $A \subseteq \mathbb{R}^d$ is measurable and $m_*(A) = 0$, we say A is a null set (or a set of measure 0).*

Observe that A is null set if and only if $m_*(A) = 0$. This is because if $m_*(A) = 0 \implies A$ is measurable and $m(A) = m_*(A)$.
Notice also that countable union of null sets are again null sets.

Theorem 6.8. *Suppose A is a measurable set in $\mathbb{R}^d$¿ Then A can be decomposed in the following manner:*

$A = E \cup N$

E and N are disjoint

E is a Borel set

N is a null set.

Proof. By the approximation property of measurable sets for every $k \in \mathbb{N}$ there exists a closed set $F_k \subset A$ such that

$$m(A \setminus F_k) < \frac{1}{K}. \tag{6.10}$$

let $E = \displaystyle\bigcup_{k=1}^{\infty} F_k$, then we know that $E \subset A$. and each F_k is Borel measurable since

$$F_k \text{ closed} \implies F^c \text{ open} \implies F_k^c \in \mathbb{B}_{\mathbb{R}^c} \implies F_k \in \mathbb{B}_{\mathbb{R}^d},$$

which implies that E is Borel measurable. For any k,

$$m(A \setminus E) \leq m(A \setminus F_k) < \frac{1}{k} \implies m(A \setminus E) = 0. \tag{6.11}$$

This implies that $N = A \setminus E$ is a null set. □

Note that the abve theorem says measurable sets are close to being Borel sets- they differ from Borel sets only by being null sets.

From the point of view of the Borl sets, the Lebesgue sets arise as the completion of the σ-algebra of Borel sets, that is, by adjoining all subsets of Borel sets of measure zero.

Corollary 6.9. *We have the following two statements:*

i) $E \subset \mathbb{R}^d$ *is measurable if and only if* E *differs from a* G_δ *by a set of measure zero.*

ii) $E \subset \mathbb{R}^d$ *is measurable if and only if* E *differs from an* F_σ *by a set of measure zero.*

Proof. i) $\Leftarrow$. Clear because of $m(\mathcal{O} \setminus E) = 0$ where $\mathcal{O} \in G_\delta$ and $E \subset \mathcal{O}$, then

$$m(\mathcal{O}) = m(\mathcal{O} \setminus E) + m(E) \implies m(E) = m(\mathcal{O}) - m(\mathcal{O} \setminus E) \tag{6.12}$$

We can conclude that $\mathcal{O}$ is measurable which implies that $\mathcal{O} \setminus E$ is measurable. So E is measurable. $\Rightarrow$. Suppose E is measurable, then for all $n \geq 1$, there exists $\mathcal{O}_n$ open such that $E \subset \mathcal{O}_n$ and

$$m(\mathcal{O} \setminus E) \leq \frac{1}{n} \tag{6.13}$$

. We define $S = \bigcup_{n=1}^{\infty} \mathcal{O}_n$ which implies that $S \in G_\delta$ and $E \subset S$. This implies that

$$S \setminus E \subset \mathcal{O}_n \setminus E \implies m(S \setminus E) \leq \frac{1}{n}, \tag{6.14}$$

for all n. This leads us to

$$m(S \setminus E) = m_*(S \setminus E) = 0 \tag{6.15}$$

which implies that $S \setminus E$ is measurable.

ii) Let $Z := \{p_1, p_2, \dots\}$ be a countable subset of $\mathbb{R}^d$. We will show this is null. Let $p_i \in I_i$ with $m(I_1) < \frac{\epsilon}{2^i}$. Then we know that

$$0 \leq m(Z) \leq \sum_{i=1}^{\infty} m(I_i) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon. \tag{6.16}$$

□

Note there are Lebesgue null sets of $\mathbb{R}^d$ that contain uncountably many points. For example the Cantor set. We know have the following theorem.

Theorem 6.10. *This is about the cardinality of two sets.*

1. $card(\mathcal{M}_{\mathbb{R}}) = 2^c$.
2. $card(\mathcal{B}_{\mathbb{R}}) = c$.

Proof. 1. C=cantor set is Lebesgue measurable because $m_*(C) = 0$. m_* is a complete msausre which implies that $c \in (M)_{\mathbb{R}}$. So every subset of a Cantor set belongs to $\mathcal{M}_{\mathbb{R}}$. This implies that $\mathcal{P}(C) \leq \mathcal{M}_{\mathbb{R}}$. This implies what we want to show.

2. Consider a single point in $\mathbb{R}$, every single point is a closed set. Closed sets belong to

$$\mathbb{B}_{\mathbb{R}} \implies card(\mathcal{B}_{\mathbb{R}}) \geq c. \tag{6.17}$$

We get the other way by the following theorem which will complete the proof.

□

Theorem 6.11 (Hausdorff)**.** *Let* $\epsilon \subset \mathcal{P}(x)$*. If* $\mathcal{M}(\epsilon$ *is a* σ*-algebra generate by* ϵ *and if* $card(\epsilon) \leq c$ *this implies that* $card\mathcal{M}(\epsilon) \leq c$*.*

6.2 Lipschitz Trans of $\mathbb{R}^n$ and Measurable Sets.

Definition 6.12 (Lipshitz). *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\vec{y} = T\vec{x}$. We call this Lipschitz if there is a constant c such that*

$$|Tx - Tx'| \leq c|x - x'| \tag{6.18}$$

Recall that

$$T\vec{x} = (y_1, y_2, \ldots, y_n) = (f_1(x), f_2(x), \ldots, f_n(x)) \tag{6.19}$$

. So we know that $y_j = f_j(x)$ are coordinate functions. T is Lipschitz if and only if each coordinate function f_j satisfies a Lipschitz condition.
Note that a linear transformation on $\mathbb{R}^n$ is a Lipschitz function.

Claim 6.13. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz, then T maps measurable sets to measurable sets.*

Proof. First we show that continuous transformation sends sets of type F_σ into set of type F_σ. This is because a continuous map T maps compact sets into compact sets. Since any closed set can be written as a countable union of compact sets, we have that T maps closed sets into sets of type F_σ. where we have used

$$T\left(\bigcup E_k\right) = \bigcup T(E_k) \tag{6.20}$$

Next we claim that T sends sets of measure zero into sets of measure zero. This is directly from the definition of Lipschitz. Reason being image of a set of diameter δ is at most $c\delta$. so there exists c' such that

$$m(TQ_j) \leq c' m(Q_j), \tag{6.21}$$

for any cubes. So if $m(E) = 0$, we choose closed cubes $\{Q_j\}$ suc that $E \subset \bigcup Q_j$ such that

$$\sum_{j=1}^{\infty} |Q_j| \leq \epsilon. \tag{6.22}$$

This gives us what we need. So by 6.8, E measurable implies that $E = H \cup Z$ where H is F_σ and Z is null. So then we have that

$$T(E) = T(H \cup Z) = T(H) \cup T(Z), \tag{6.23}$$

so we have that E is sum of two measurable sets and is therefore measurable. ☐

6.3 Measurable Functions

Let $E \subset \mathbb{R}^d$ and let $f :\to \mathbb{R}$. We have the following properties

1. $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$
2. $f^{-1}\left(\bigcup_{n=1}^{\infty} A)n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$
3. $f^{-1}\left(\bigcap_{n=1}^{\infty} A)n\right) = \bigcap_{n=1}^{\infty} f^{-1}(A_n)$

Now look at $f^{-1}([-\infty, a)) = \{x \in E : f(x) \in [-\infty, a)\}$. We are going to use the shorthand

$$f^{-1}([-\infty, a)) = \{f < a\} \tag{6.24}$$

Definition 6.14 (Measurable Function). *A function f defined on a measurable set E is called measurable if for all $a \in \mathbb{R}$, the set $\{f < a\}$ is measurable.*

The question is why are we using this set? The answer is we have the following equivalent definitions which we will state as a theorem.

Theorem 6.15 (Equivalent Definitions). *Each of the following four conditions implies the other three.*

i) $\{x : f(x) > a\}$ *is measurable for all real* a.

ii) $\{x : f(x) \geq a\}$ *is measurable for all real* a.

iii) $\{x : f(x) < a\}$ *is measurable for all real* a.

iv) $\{x : f(x) \leq a\}$ *is measurable for all real* a.

7 9/24/13

7.1 Cantor-Lebesgue Function

We are starting off with some things about the Cantor-Lebesgue function, also known as the Cantor function or the devil's staircase function.
We are now going to set up a function f_k which is continuous on $[0, 1]$ which satisfies the following properties

1. $f_k(0) = 0$
2. $f_k(1) = 1$
3. $f_k(x) = \dfrac{j}{2^k}$ on I_j^k where $j = 1, 2, \ldots, 2^k - 1$.
4. f_k is linear on each subinterval of C_k where C_k is as defined in the Cantor set.

For f_1, we look at

$$I_1^1 = (\frac{1}{3}, \frac{2}{3}) \implies f_1(x) = \frac{1}{2} \text{ on } I_1^1 \tag{7.1}$$

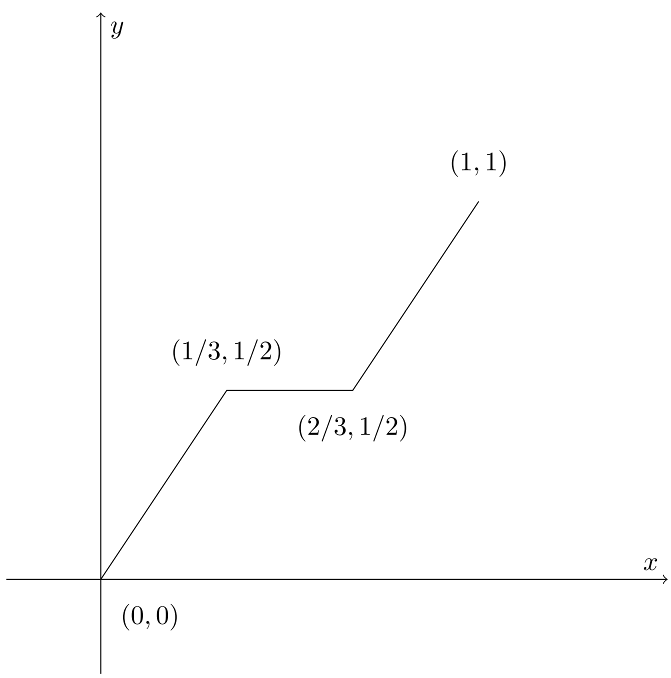

Figure 1: f_1

for f_2 we have removed intervals I_1^2, I_2^2, I_3^2 and it is given by

$$f_2(x) = \begin{cases} \frac{1}{2^2} \text{ on } I_1^2 \\ \frac{2}{2^2} \text{ on } I_2^2 \\ \frac{3}{2^2} \text{ on } I_3^2 \end{cases} \tag{7.2}$$

One can show that

$$|f_k - f_{k+1}| < \frac{1}{2^{k+1}} \tag{7.3}$$

We can show that $\sum_{n=1}^{\infty}(f_k - f_{k+1})$ converges uniformly. Can check uing Weierstrass M-test. And so we know that $f = \lim_{k\to\infty} f_k$. So we can see that

1. $f(0) = 0$
2. $f(1) = 1$
3. f is monotone increasing on $[0, 1]$.
4. f is constant on every interval removed in constructing the Cantor set.

This function is called Cantor-Lebesgue function.

7.2 Measurable Functions

We are going to continue our study of measurable functions which we have already defined in 6.14. We are now going to prove some of 6.15

Proof. We know that

$$\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty}\{x : f(a) > a - \frac{1}{n}\} \tag{7.4}$$

which is a countable intersection of measurable sets. So we have that

$$\{x : f(x) < a\} = \{x : f(x) \geq a\}^c \tag{7.5}$$

$$\{x : f(x) \leq a\} = \bigcap_{n=1}^{\infty}\{x : f(x) < a - \frac{1}{n}\} \tag{7.6}$$

$$\{x : f(x) > a\} = \{x : f(x) \leq a\}^c \tag{7.7}$$

So we can see that these things are all equivalent to our original definition. □

Theorem 7.1. *If f is measurable then $|f|$ is measurable.*

Proof. We know that

$$\{x : |f(x)| < a\} = \{x : f(x) < a\} \bigcap \{x : f(x) > -a\} \tag{7.8}$$

which is an intersection of measurable sets. So we are done. □

Note that f is finite valued if $-\infty < f(x) < \infty$ for all x.

Lemma 7.2. *if f is measurable, then $\{x : f(x) = \infty\}$ is a measurable set.*

Proof. This follows from

$$\{x : f(x) = \infty\} = \bigcap_{k=1}^{\infty}\{f(x) \geq k\} \tag{7.9}$$

and the fact that countable intersections are measurable. □

Note that if f is measurable then $\bigcup_{k=1}^{\infty}\{f > -k\}$ is measurable and in addition, the set $\{x : f = -\infty\}$ is measurable then

$$E = \{f = -\infty\} \bigcup \left(\bigcup_{k=1}^{\infty} \{f > k\} \right) \tag{7.10}$$

is a measurable set.

Theorem 7.3. *Let $E \in \mathbb{R}^d$ and $: E \to \mathbb{R}$. E is measurable if and only if every for open set $\mathcal{O} \subset \mathbb{R}$ the inverse image $f^{-1}(\mathcal{O})$ is a measurable subset of $\mathbb{R}^d$.*

Proof. $\Leftarrow$. Suppose that for open set $\mathcal{O} \subset \mathbb{R}$ the inverse image $f^{-1}(\mathcal{O})$ is a measurable subset of $\mathbb{R}^d$. We have if $\mathcal{O} = (a, \infty)$, then

$$f^{-1}(\mathcal{O}) = \{x : f(x) > a\} \tag{7.11}$$

Since $f^{-1}(\mathcal{O})$ is measurable for every open $\mathcal{O}$, f is measurable.
$\Rightarrow$. Suppose f is measurable and $\mathcal{O} \subset \mathbb{R}$ is an open set. Then

$$\mathcal{O} = \bigcup_{k=1}^{\infty} (a_k, b_k) \tag{7.12}$$

$$f^{-1}(\mathcal{O}) = f^{-1} \left(\bigcup_{k=1}^{\infty} (a_k, b_k) \right) = \bigcup_{k=1}^{\infty} f^{-1}(a_k, b_k) \tag{7.13}$$

But by assumption $f^{-1} = \{x : a_k < f(x) < b_k\}$ is measurable. So this implies that $f^{-1}(\mathcal{O})$ is a countable union of measurable sets and is therefore, measurable. □

We we have the following properties

1. If f is continuous on $\mathbb{R}^d$ then f is measurable. This follows directly from definition.
2. If $-\infty < f(x) < \infty$ and f is measurable and Φ is a continuous, then

   $$\Phi \circ f \tag{7.14}$$

   is measurable.
   So look at

   $$\begin{aligned} (\Phi \circ f)^{-1}((-\infty, a)) &= (f^{-1} \circ (\Phi^{-1})(-\infty, a)) \\ &= f^{-1}(\Phi^{-1}(-\infty, a)) = f^{-1}(\mathcal{O}) \end{aligned} \tag{7.15}$$

   where $\mathcal{O} = \Phi^{-1}(-\infty, a)$. This is measurable since f is finite valued.
   Note that definition of measurable functions depends on particular σ-algebra. Thus if we have

   $$(\mathbb{R}, \mathcal{M}_{\mathbb{R}}) \xrightarrow{\Phi} (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \xrightarrow{f} (\mathcal{B}_{\mathbb{R}}) \tag{7.16}$$

   with $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_{\mathbb{R}}$ and $A \in \mathcal{B}_{\mathbb{R}}$, then for a measurable function f we have

   $$f^{-1}(A) \in \mathcal{M}_{\mathbb{R}}. \tag{7.17}$$

   But, unless $f^{-1}(A) \in \mathcal{B}_{\mathbb{R}}$, there is no guarantee that

   $$(f \circ \Phi)^{-1}(A) = \Phi^{-1}(f^{-1}(A)) \in \mathcal{M}_{\mathbb{R}} \tag{7.18}$$

   However if f is Borel measurable then $f \circ \Phi$ is measurable.
3. Suppose $\{f_n\}$ is a sequence of measurable functions. Then $\sup f_n(x), \inf f_n(x), \lim_{n\to\infty} \sup f_n(x), \lim_{n\to\infty} \inf f_n(x)$ are measurable.

Proof. Lt $g(x) = \sup f_n(x)$ and $h(x) = \lim_{n\to\infty} \sup f_n(x)$. Then we know that

$$\{x : g(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\} \tag{7.19}$$

$$\limsup f_n(x) = \inf_k \{\sup_{n\geq k} f_n(x)\} = \inf_k g_k(x) = h(x) \tag{7.20}$$

Note the first is a countable union of measurable sets and we can define infinum by negative supremum of negative $f_n(x)$, so we are done. □

We now define some notation. Denote

$$f^+ = \max\{f(x), 0\} \tag{7.21}$$

$$f^- = -\min\{f(x), 0\} = \sup\{-f(x), 0\} \tag{7.22}$$

for all x. This gives us the following properties $f = f^+ - f^-$ and $|f| = f^+ + f^-$. So we have some facts.

1. If f and g are measurable, then $\max\{f, g\}, \min\{f.g\}$ are measurable.
2. The limit of a convergent sequence of measurable functions is measurable. So if $\{f_n\}$ is a collection of measurable functions with
$$\lim_{n\to\infty} f_n(x) = f(x), \tag{7.23}$$
then $f(x)$ is measurable.
3. If f and g are measurable
   i) The integer powers f^k with $k \geq 1$ are measurable.
   ii) $f + g$ and fg are measurable if both f and g are finite volumes.

Proof. i) k is odd implies
$$\{f^k > a\} = \{f > a^{1/k}\} \tag{7.24}$$
If k is even and $a \geq 0$, we have
$$\{f^k > a\} = \{f > a^{1/k}\} \bigcup \{f < -a^{1/k}\} \tag{7.25}$$

ii) We start with a lemma

Lemma 7.4. *If f and g are measurable, then $\{x : f(x) > g(x)\}$ is measurable.*

Proof. Since between any two real numbers there is a rational number, if $\{r_k\}$ is a sequence of rationals, then $g(x) < r_k < f(x)$. So we have

$$\begin{aligned}\{x : f(x) > g(x)\} &= \bigcup_k \{x : g(x) < r_k f(x)\} \\ &= \bigcup_k \left[\{x : g(x) < r_k\} \bigcap \{x : f(x) > r_k\}\right]\end{aligned} \tag{7.26}$$

This is a countable union of an intersection of countable measurable sets, so this is measurable and the claim follows. □

To show that $f + g$ is measurable we see that

$$\begin{aligned}\{x : f(x) + g(x) > a\} &= \{x : f(x) > a - g(x)\} \\ &\bigcup_k \left[\{f > r_k\} \bigcap \{g > a - r_k\}\right]\end{aligned} \tag{7.27}$$

We can also see that

$$fg = \frac{1}{4}\left[(f + g)^2 - (f - g)^2\right] \tag{7.28}$$

□

8 9/26/12

First we give a definition of almost everywhere. Note that this is the same as "almost all" or "almost always", abbreviated "a.a". In probability they generally use "almost surely". In some older books, they usually use "p.p" which is the from the original French "Presque prtout" We have seen some examples of this

1. Cantor function f satisfies $f'(x) = 0$ a.e
2. Characteristic function of $\mathbb{Q}$ defined as

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \tag{8.1}$$

   Here $\chi_{\mathbb{Q}} = 0$ a.e since $\chi_{\mathbb{Q}} = 1$ when $x \in \mathbb{Q}$ and $m(\mathbb{Q}) = 0$.

Definition 8.1 (Almost Everywhere (a.e))**.** *Let $f, g : E \subset \mathbb{R}^d \to \mathbb{R}$. We say $f(x) = g(x)$ a.e on E if*

$$m\left(\{x : f(x) \neq g(x)\}\right) = 0 \tag{8.2}$$

Proposition 8.2. *Suppose f is measurable and $f(x) = g(x)$ a.e. Then $g(x)$ is measurable.*

Proof. Let $A = \{x : f(x) \neq g(x)\}$. Then from our hypothesis, $m(A) = 0$. What we want to show is that if we take $\mathcal{O} \in \mathbb{R}$, then $g^{-1}(\mathcal{O})$ is measurable. Since f is measurable we know that $f^{-1}(\mathcal{O})$ is measurable. Look at the set

$$A^c \cap g^{-1}(\mathcal{O}) = A^c \cap f^{-1}(\mathcal{O}). \tag{8.3}$$

Since the right side is measurable, the left side is also measurable. Now look

$$A \cap g^{-1}(\mathcal{O}) \subset A \tag{8.4}$$

Since $m(A) = 0$, then we know that $A \cap g^{-1}(\mathcal{O})$ is also measurable. So if we write $g^{-1}(\mathcal{O})$ as

$$g^{-1}(\mathcal{O}) = \left[A \cap g^{-1}(\mathcal{O})\right] \bigcup \left[A^c \cap g^{-1}(\mathcal{O})\right] \tag{8.5}$$

Since this is a union of measurable sets, this set is measurable. □

Corollary 8.3. *If $\{f_n(x)\}$ is a sequence of measurable functions and if*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad a.e. \tag{8.6}$$

Then $f(x)$ is also measurable.

Proof. Since we know that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e we can set

$$B = \{x : \lim_{n\to\infty} f_n(x) \neq f(x)\} \tag{8.7}$$

then $m(B) = 0$. So we can then look at

$$\{x : f > a\} = [\{x : f(x) > a\} \cap B] \bigcup [\{x : f(x) > a\} \cap B^c] \tag{8.8}$$

and get the proof. □

We want to now give an example of a function which is not measurable. Note we are going to use the notation χ_E for the characteristic or indicator function which is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases} \tag{8.9}$$

This function is not measurable because we can see that

$$f^{-1}\left(\{1\}\right) = E \tag{8.10}$$

So we can see that if E is not measurable then its characteristic function is not measurable. So we have some facts about the characteristic function. We will not prove these here.

1. $\chi_\phi = 0$
2. If $A \subset B$, then $\chi_A \leq \chi B$
3. $\chi_{A \cap B} = \chi_A \cdot \chi_B$
4. $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$

So we can see that if $A = \bigcup_{i=1}^{\infty} A_i$ is disjoint. Then

$$\chi_A = \sum_{i=1}^{\infty} \chi_A \tag{8.11}$$

Now suppose that E is measurable. Then there exists a set $\{R_j\}$ disjoint such that

$$m\left(E \Delta \bigcup_{j=1}^{N} R_j\right) \leq 2\epsilon. \tag{8.12}$$

We can also see that we can write

$$\chi_E = \chi_{(E \setminus \bigcup_{j=1}^N R_j) \bigcup (\bigcup_{j=1}^N R_j)} = \chi_{E \setminus \bigcup_{j=1}^N R_j} + \sum_{j=1}^{N} \chi_{R_j} \tag{8.13}$$

8.1 Beginning Integration

In Riemann integration we looked at area of rectangles, which really reduces to looking at the characteristic function of rectangles. We want to develop a theory that takes characteristic function of not rectangles, but measurable sets. We will define a notion of simple functions and then we be able to build Lebesgue integration.
We are now going to do an example.

Example 8.4. *Suppose*

$$f(x) = \begin{cases} 0 & x < 0 \ or \ x > 2 \\ 1 & 0 \leq x \leq 1 \\ 2 & 1 \leq x \leq 2 \end{cases} \tag{8.14}$$

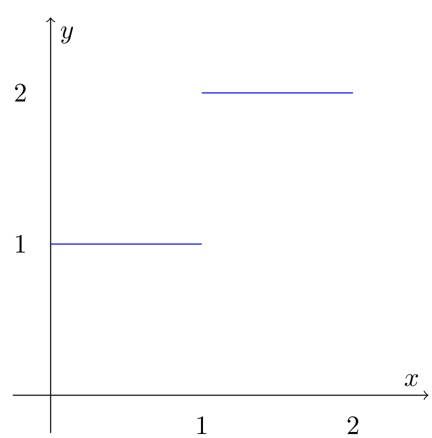

Figure 2: f

Note we can also write this as

$$f = \chi_{[0,1]} + 2\chi_{(1,2]} + 0\chi_{(-\infty,0)\cup(2,\infty)} \tag{8.15}$$

Note that the last part does not effect that value of f since it is always 0 times something.

Example 8.5 (Greatest Integer Function). *Let $f(x) = [|x|]$*

This function looks like

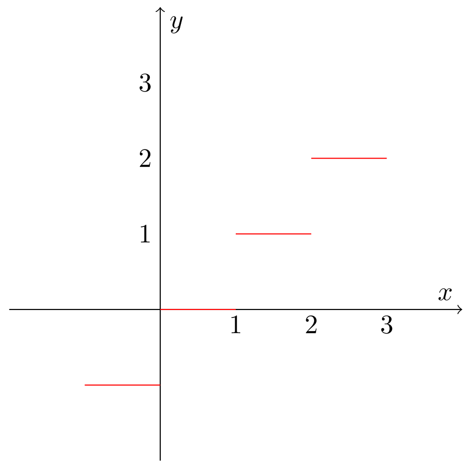

Figure 3: f

Example 8.6 (Dirichlet Function).

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \tag{8.16}$$

We now will define a simple function.

Definition 8.7 (Simple Functions). *A finite linear combination of characteristic functions of measurable sets E_k, $1 \leq k \leq N$ with $m(E_k) < \infty$ is called a simple function.*

A simple function $|phi$ has the form

$$\phi = \sum_{k=1}^{N} a_k \chi_{E_k} \tag{8.17}$$

Observe that the range of $\phi = \{a_1, a_2, \ldots, a_N\}$. We can also see that

$$E_k = \phi^{-1}(\{a_k\}) = \{x : \phi(x) = a_k\} \tag{8.18}$$

Which basically says that

$$\chi_{E_k}(x) = \begin{cases} 1 & \text{if } x \in E_k \\ 0 & \text{otherwise} \end{cases} \tag{8.19}$$

This basically sas that $x \in E_k \implies \phi(x) = a_k$.
So we want to know how to approximate a measurable function with a nice set of functions. We can do this using simple functions.

The basic construction is the following theorem

Theorem 8.8. *Let $f : E \subset \mathbb{R}^d \to [0, \infty]$ be a non-negative measurable function. Then we can find an increasing sequence of non-negative simple functions $\{\phi_n\}$ with $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$. Such that $\{\phi_n\}$ converges pointwise to f a.e. on E. Furthermore, if f is bounded then the sequence of simple functions will converge uniformly.*

tis now leads to the following theorem

Theorem 8.9 (Foundation Theorem for Measurable Functions)**.** *We have the following three statements*

i) Every function f can be written as a limit of sequence $\{f_k\}$ of simple functions.

ii) If $f \geq 0$, the sequence can be chosen to increase to f, that is chosen such that for every k

$$f_k \leq f_{k+1}$$

iii) If the function in either (i) or (ii) is measurable, then f_k can be chosen to be measurable.

Proof. We first prove the second statement. Suppose $f \geq 0$. For each k, subdivide the values of f which fall in $[0, k]$ by partitioning $[0, k]$ into subintervals

$$\left[\frac{j-1}{2^k}, \frac{j}{2^k}\right] \tag{8.20}$$

where $j = 1, 2, \ldots, k2^k$¿ Let

$$f_k(x) = \begin{cases} \frac{j-1}{2^k} & \text{if} \frac{j-1}{2^k} \leq f(x) < \frac{j}{2^k}, \quad j = 1, 2, \ldots, k2^k \\ k & \text{if} \quad f(x) \geq k \end{cases} \tag{8.21}$$

Observe the following

1. Each f_k is a simple function defined everywhere in the domain of f.
2. $f_k \leq f_{k+1}$ since in passing from f_k to f_{k+1} each subinterval is divided by half.
3. $f_k \to f$. This is because $0 \leq f - f_k \leq 2^{-k}$ for sufficiently large k whenever f is finite and $f_k = k \to \infty$ whenever $f = \infty$.

This shows (ii).

To show (i), recall $f + f^+ - f^-$ where both f^+ and f^- are non-negative. Thus apply (ii) to each of the non-negative functions f^+ and f^-, obtaining sequences $\{f'_k\}$ and $\{f''_k\}$ of simple functions such that

$$f'_k \to f^+ \quad \text{and} \quad f''_k \to f^- \tag{8.22}$$

Then $f'_k - f''_k$ is simple and

$$f'_k - f''_k \to f^+ - f^- = f \tag{8.23}$$

It is enough to prove (iii) for $f \geq 0$. We can write f_k as

$$f_k = \sum_{j=1}^{k2^k} \frac{j-1}{2^k} \chi_{\{\frac{j-1}{2^k} \leq f \leq \frac{j}{2^k}\}} + k\chi_{f \geq k} \tag{8.24}$$

If f is measurable, then all of the sets involved in the above expression are measurable which implies f_k is measurable for each k.

Note: If f is bounded, the simple functions will converge uniformly to f. $\square$

We can now look at functions of the form

$$\phi = \sum_{k=1}^{N} a_k \phi_{R_k} \tag{8.25}$$

where R_k are rectangles. We will call these *Step functions*

We now state the following very important theorem.

Theorem 8.10 (pg. 32). *Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges pointwise to $f(x)$ for a.e. x.*

Proof. It is enough to show that if E is a measurable set with finite measure, then $f = \chi_E$ can be approximated by step functions. Since E is measurable, then there exists $\epsilon > 0$ such that there exists $Q_1, Q_2, \ldots, Q_N$ cubes such that

$$m\left(E\Delta \bigcup_{i=1}^{N} Q_i\right) \leq \epsilon. \tag{8.26}$$

Extending the sides of the gird formed by Q_j, we can obtain almost disjoint rectangles, $R_1', R_2', \ldots R_M'$ such that

$$\bigcup_{j=1}^{N} Q_j = \bigcup_{j=1}^{N} R_j' \tag{8.27}$$

Now take $R_j \subset R_j'$ which are slightly smaller in size. So we have that $\displaystyle\bigcup_{j=1}^{N} R_j$ are disjoint such that

$$m\left(E\Delta \bigcup_{j=1}^{M} R_j\right) \leq 2\epsilon \tag{8.28}$$

□

9 10/1/13

There are basically three ways in which a sequence of functions $\{f_n\} : \mathbb{R}^d \to [0, \infty]$ can converge.

1. Pointwise convergence. $f_n(x) \to f(x)$ for every $x \in \mathbb{R}^d$.
2. Pointwise almost everywhere convergence. $f_n(x) \to f(x)$ for almost all $x \in \mathbb{R}^d$.
3. Uniform convergence. For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}^n$.

It is easy to see that 3 implies 1 implies 2.
Recall that $\{f_n\} \to f$ uniformly if and only if $\lim_{n\to\infty} a_n = 0$ where

$$a_n := \sup\{|f_n(x) - f(x)| : x \in E\} \tag{9.1}$$

We will show this with an example.

Example 9.1. *Suppose*

$$f_n(x) = \frac{nx}{1 + nx} \tag{9.2}$$

for $x \geq 0$.

1. *Find $f(x) = \lim_{n\to\infty} f_n(x)$*
2. *Show that for $a > 0$, $f_n \to f$ uniformly on $[a, \infty)$*
3. *Show that $\{f_n\}$ does not converge uniformly to f on $[0, \infty)$.*

Proof. 1. Observe that

$$\lim_{n\to\infty} \frac{nx}{1 + nx} = \lim_{n\to\infty} \frac{1}{\frac{1}{nx} + 1} = 1 \tag{9.3}$$

so $f(x) = 1$ for $x > 0$.

2. For $x \geq a$ we have that

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} \tag{9.4}$$

So we have that

$$\frac{1}{1+nx} \leq \frac{1}{1+na} \implies \lim_{n\to\infty} \frac{1}{1+na} = 0 \tag{9.5}$$

So we have uniform convergence.

3. If $0 < x < \frac{1}{n}$. Let $n \geq 1$. Then we know that $1 + nx \leq 2$. So we have that

$$\left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} > \frac{1}{2} \tag{9.6}$$

This implies we do not have uniform convergence.

□

We can now make a strong statement about uniform convergence and measurable sets. This is similar to Littlewoods first principle.

Theorem 9.2 (Egorov's Theorem)**.** *Suppose $\{f_n\}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$. Suppose $\{f_n\} \to f$ a.e on E and let $\epsilon > 0$. Then there exists a Lebesgue measurable set A with $m(A) < \epsilon$ such that $f_n \to f$ uniformly on A^c.*

Proof. By "modifying" f_n and f on a set of measure 0, we may assume $f_n \to f$ pointwise everywhere. This means that for all $x \in \mathbb{R}^d$ and $m > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{1}{m}$ for all $n \geq N$.
So fix m and define the set

$$E_N^m := \{x \in \mathbb{R}^d : |f_n(x) - f(x)| < \frac{1}{m} \quad \text{for some} \quad n \geq N\} \tag{9.7}$$

We can write the condition of pointwise convergence set-theoretically as

$$\bigcap_{N=0}^{\infty} E_N^m = \phi \quad \text{for each} \quad m \tag{9.8}$$

We can see that each E_N^m is measurable and

$$E_{N+1}^m \subset E_N^m \tag{9.9}$$

which means that E_N^m is decreasing in N. So we can apply the downward convergence theorem to get that

$$0 = m\left(\bigcap_{N=1}^{\infty} E_n^m\right) = \lim_{N\to\infty} m\left(E_N^m\right) \tag{9.10}$$

What this means is that for any $m \geq 1$ we can find some N_m such that

$$m\left(E_{N_m}^m\right) < \frac{\epsilon}{2^m} \tag{9.11}$$

for all $N \geq N_m$. So we now have a natural candidate for A. Let

$$A := \bigcup_{m=1}^{\infty} E_{N_m}^m \tag{9.12}$$

We have already shown that this is a union of measurable sets and therefore A is measurable and by countable subadditivity, $m(A) < \epsilon$.
Now we show that $f_n \to f$ uniformly on A^c. Note that

$$A^c = \bigcap_{m=1}^{\infty} \left(E_{N_m}^m\right)^c \tag{9.13}$$

By construction we have

$$|f_n(x) - f(x)| < \frac{1}{m} \tag{9.14}$$

whenever $n \geq 1$ $x \in A^c$ with $n \geq N_m$. Also note that $A^c \subset \left(E^m_{N_m}\right)^c$. So look at the supremum

$$\sup_{x \in A^c} |f_n(x) - f(x)| \leq \sup_{x \in \left(E^m_{N_m}\right)^c} |f_n(x) - f(x)| < \frac{1}{m} \tag{9.15}$$

This implies that $f_n \to f$ uniformly. $\square$

So now we will show an example where this theorem is not true for sets with measure ∞. Basically one can easily see this by noting we used the monotone decreasing convergence theorem.

Example 9.3. *We will look at the "Moving bump function". This defined as $f_n := \chi_{[n,n+1]}$.*

People often say this escapes to infinity. This function converges pointwise to the zero function $f = 0$. But if we choose $0 < \epsilon < 1$, then we have for any N

$$|f_n(x) - f(x)| > \epsilon \tag{9.16}$$

on a set of measure 1. $m([n, n+1]) = 1$. So if we want $f_n \to f$ uniformly on A^c, then the set A should have sets of measure 1. So A must contain the intervals $[n, n+1]$ for sufficiently large n. This must have infinite measure.

Theorem 9.4 (Lusin's Theorem)**.** *Suppose that f is measurable and finite values on E with $m(E) < \infty$. Then for all $\epsilon > 0$ there exists a closed set F_ϵ such that $F_\epsilon \subset E$ and $m(E \setminus F_\epsilon) < \epsilon$ and f restricted to F_ϵ is continuous.*

Proof. The idea of the proof is to let $\{f_n\}$ be a sequence of step functions such that $f_n \to f$ a.e. This means we may find set E_n with $m(E_n) < \frac{1}{2^n}$ and f_n is continuous on E_n^c. By Egorv's theorem, find a set $A_{\epsilon/3}$ on which $f_n \to f$ uniformly and $m(E \setminus A_{\epsilon/3})$. Then set

$$F' = A_{\epsilon/3} \setminus \bigcup_{n \geq N} E_n \tag{9.17}$$

Recall that $\displaystyle\sum_{n \geq N} \frac{1}{2^n} < \frac{\epsilon}{3}$. Thus for every $n \geq N$ we have that f_n is continuous on F'. Thus f being the uniform limit of continuous functions $\{f_n\}$ is continuous on F'.
Now approximate F' by F_ϵ so that $F_\epsilon \subset F'$. Then we have that

$$m(F' \setminus F) < \frac{\epsilon}{3} \tag{9.18}$$

Observe that

$$E \setminus F_\epsilon = (E \setminus A_{\epsilon/3}) \cup (A_{\epsilon/3} \setminus F') \cup (F' \setminus F_\epsilon) \tag{9.19}$$

So this implies that

$$F' = A_{\epsilon/3} \setminus \bigcup_{n \geq N} E_n \tag{9.20}$$

$$A_{\epsilon/3} = (F') \cup (\bigcup_{n \geq N} E_n) \tag{9.21}$$

$$\implies A_{\epsilon/3} \setminus F' = \bigcup_{n \geq N} E_n \tag{9.22}$$

$$\implies m(A_{\epsilon_3} \setminus F') \leq \sum_{n \geq N} \frac{1}{2^n} < \frac{\epsilon}{3} \tag{9.23}$$

$\square$

Theorem 9.5 (Littlewood's Three Principles)**.** *These principles are really a collection of observations form theorems we have proven*

1. *(Egerov's Theorem) Every convergent sequence of functions is nearly uniformly convergent.*
2. *(Lusin's Theorem) Every measurable functions is almost continuous.*
3. *Every measurable set is almost an open set. This means for all $\epsilon > 0$ there exists $\mathcal{O}$, and open set and $E \subset \mathcal{O}$ such that*
$$m(\mathcal{O} \setminus E) < \epsilon$$

9.1 Back to Simple Functions

A simple function ϕ looks like

$$\phi = \sum_{k=1}^{n} a_k \chi_{E_k} \tag{9.24}$$

So we have two questions.

1. What if some of the $a_k = 0$ or $a_i = a_j$?
2. What happens if E_k are **not** disjoint.

Note for question 1. the following case and assume that E_k are disjoint sets.

$$\phi = a_1 \chi_{E_1} + a_2 \chi_{E_2} + a_2 \chi_{E_3} + a_3 \chi_{E_4} \tag{9.25}$$

Let $a = a_2$, then we have that $E_2 \cup E_3 = E'_a$ and $m(E_2) + m(E_3) = m(E'_a)$. The set E'_a are disjoint. So in general we have

$$m(E'_a) = \sum m(E_k) \tag{9.26}$$
$$\phi = \sum a \chi_{E'_a} \tag{9.27}$$

where the sum is over distinct non zero values of $\{a_k\}$.

10 10/3/13

We start with a lemma.

Lemma 10.1. *Let $\mathcal{A}$ be a $\sigma-$algebra and $\{E_i\}$ be a sequence of sets in $\mathcal{A}$. Then there is a seuqence of sets $\{F_i\}$ in $\mathcal{A}$ such that*

i) $F_n \cap F_m = \phi$ *for* $n \neq m$

ii) $\displaystyle\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$

Proof. Let $F_1 = E_1 +$. For each $n > 1$ we have that

$$F_n = E_n \setminus \left[\bigcup_{j=1}^{n-1} E_j \right] \tag{10.1}$$
$$F_n = E_n \cap E_1^c \cap E_2^c \cap \cdots \cap E_{n-1}^c \tag{10.2}$$

since the F_n are measurable, we know that $F_n \subset E_n$ for all n.
Now suppose that $m < n$ and then notice

$$F_m \cap F_n \subset E_m \cap F_n = E_m \cap E_n \cap E_1^c \cap \cdots \cap E_m^c \cap \cdots \cap E_{n-1}^c \tag{10.3}$$

Since $F_i \subset E_i$ for all i, we know that $\cup F_i \subset E_i$. So let $x \in \bigcup_{i=1}^{\infty} E_i$, then $x \in E_i$ for some i. So let n be the smallest value of i such that $x \in E_i$. This implies that

$$x \notin E_1, E_2, \ldots, E_{n-1} \quad \text{but} \quad x \in E_n \tag{10.4}$$

Note this is inverted for the complements. So this implies that $x \in F_n$ for some n. This implies that $\cup E_i \subset \cup F_i$. □

10.1 The Lebesgue Integral for Simple Functions

We now start integration.

Definition 10.2 (Canonical Form)**.** *The canonical form of a simple function ϕ is the unique decomposition*

$$\phi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x) \tag{10.5}$$

where for all k, a_k is disjoint and non-zero and E_k is disjoint.

So now we can define the Lebesgue integral.

Definition 10.3 (Lebesgue Integral)**.** *The Lebesgue integral of ϕ is given as*

$$\int \phi := \sum_{k=1}^{N} a_k m(F_k) \tag{10.6}$$

If E is a measurable set $m(E) < \infty$, then $\phi(x)\chi_E(x)$ is a simple function. So this allows us to define

$$\int_E \phi = \int_{\mathbb{R}^d} \phi \chi_i := \sum_{k=1}^{N} a_k m(E \cap F_k). \tag{10.7}$$

Lemma 10.4. *Let $k, k' \geq 0$ be natural numbers, $c_1, \ldots, c_k, c_1', \ldots, c_k' \in [0, \infty]$. And let $E_1, \ldots, E_k, E_1', \ldots, E_k' \subset \mathbb{R}^d$ be Lebesgue measurable sets such that the identity*

$$c_1 \chi_{E_1} + \ldots c_k \chi_{E_k} = c_1' \chi_{E_1} + \ldots c_k' \chi_{E_k} \tag{10.8}$$

Then one has

$$\sum_{i=1}^{k} k c_i m(E_i) = \sum_{i=1}^{k'} c_i' m(E_i') \tag{10.9}$$

Proof. Given in packet □

Theorem 10.5. *The integral of a simple function satisfies the following*

i) (Linearity) If ϕ and ψ are simple, $a, b \in \mathbb{R}$, then

$$\int [a\phi + b\psi] = a \int \phi + b \int \psi \tag{10.10}$$

ii) (Additivity) If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure then

$$\int_{E \cup F} \phi = \int_E \phi + \int_F \phi \tag{10.11}$$

iii) (Monotonicity) If $\phi \leq \psi$ then

$$\int \phi \leq \int \psi \tag{10.12}$$

iv) (Triangle Inequality) If ϕ is a simple function, the $|\phi|$ is a simple function and

$$\left|\int \phi\right| \leq \int |\phi| \tag{10.13}$$

Proof. We will prove the additivity. Let $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ and $\psi = \sum_{k=1}^{m} b_k \chi_{F_k}$. Then we know that $\phi + \psi$ can assume at most $a_j + b_k$ for $j = 1, \ldots, n$ and $k = 1, \ldots, m$. It assumes this value if $E_j \cap F_k$ is not an empty intersection. So we have that

$$\phi + \psi = \sum_{i=1}^{p} c_i \chi_A \tag{10.14}$$

where $A := \cup\{E_j \cap F_k : a_j + b_k = c_i\}$. and p is some integer equal to at most $mn¿$ By the definition of the integral we have

$$\int \phi + \psi = \sum_{i=1}^{p} c_i m\left(\cup\{E_j \cap F_k : a_j + b_k = c_i\}\right) = \sum_{i=1}^{p} c_i \{\sum m(E_j \cap F_k) : a_j + b_k = c_i\} \tag{10.15}$$

where we have used the finite additivity of measure. If we add zero terms whenever $E_j \cap F_k = \phi$ we can make the last sum encompass al value $m(E_j \cap F_k)$ and

$$\int \phi + \psi = \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) m(E_j \cap F_k) \tag{10.16}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} \left[a_j m(E_j \cap F_k) + b_k m(E_j \cap F_k)\right] \tag{10.17}$$

$$= \sum_{j=1}^{n} a_j \sum_{k=1}^{m} m(E_j \cap F_k) + \sum_{k=1}^{m} b_k \sum_{j=1}^{n} m(E_j \cap F_k) \tag{10.18}$$

Since $E_j = \bigcup_{k=1}^{m}(E_j \cap F_k)$ and $F_k = \bigcup_{j=1}^{n}(E_j \cap F_k)$. where the unions are disjoint and using finite additivity of m implies that

$$\int \phi + \psi = \sum_{j=1}^{n} a_j m(E_j) + \sum_{k=1}^{m} b_k m(F_k) \tag{10.19}$$

$\square$

11 For Midterm on 10/10/13

1. exterior measure, definition and properties
2. measure definition and properties
3. sigma and borel sets
4. relationship between measure and and G_δ and F_σ
5. Construction of a non-measurable sets
6. construction of a counter Lebesgue function
7. measurable functions and their properties
8. littlewoods three principles
9. 3.2

10. corollary 3.3
11. 3.4
12. structure theorem for measurable functions
13. Basic properties of integrals of simple functions

12 10/8/13

We are going to look at bounded functions first that are supported on a set of finite measure.

12.1 Bounded Functions

Definition 12.1 (Support)**.** *If f is measurable, then the support of f is*

$$supp(f) = \{x \in E : f(x) \neq 0\} \tag{12.1}$$

We say that f is supported on E if $f(x) = 0$ when $x \notin E$.

We can see that $supp(f)$ is a measurable set. We are considering f to be bounded by some M and supported on E, with $m(supp(f)) < \infty$. Then we know that there exist a sequence of simple functions $\{\phi_n\}$ such that

1. Each ϕ_n is bounded by M
2. Each ϕ_n is supported on E
3. $\phi_n(x) \to f(x)$ for all x

This comes from the structure theorem for simple functions.

Lemma 12.2. *Let f be a bounded functions supported on a set E with $m(E) < \infty$. If $\{\phi_n\}$ is a sequence of simple functions bounded by M supported on E and with $\phi_n(x) \to f(x)$ almost everywhere on x. Then*

1. $\lim\limits_{n\to\infty} \phi_n$ *exist*
2. *If* $f = 0 \implies \lim\limits_{n\to\infty} \phi_n = 0$

Proof. Set $I_n := \int \phi_n$. We first claim $\{I_n\}$ is a Cauchy sequence. We are going to use Egorov's theorem, (9.2). So we can let $E = E \setminus A_\epsilon \cup A_\epsilon$ as a disjoint union. So we have that

$$\begin{aligned} |I_n - I_m| = |\int \phi_n - \phi_m| \leq \int |\phi_n - \phi_m| \leq \int_E |\phi_n - \phi_m| \\ = \int_{E\setminus A_\epsilon} |\phi_n - \phi_m| + \int_{A_\epsilon} |\phi_n - \phi_m| \end{aligned} \tag{12.2}$$

Recall that $m(E \setminus A_\epsilon) \leq \epsilon$ and $|\phi_n(x) - \phi_m(x)| \leq 2M$ the first by Egorov and so we have that

$$|I_n - I_m| \leq 2M\epsilon + \int_{A_\epsilon} |\phi_n(x) - \phi_m(x)| \tag{12.3}$$

Since uniform convergence on the set A_ϵ, we know that

$$|\phi_N(x) - \phi_m(x)| < \epsilon \tag{12.4}$$

for a $x \in A_\epsilon$, for sufficiently large n and m. This is the Cauchy criterion for uniform convergence. So we have that

$$|I_n - I_m| < 2M\epsilon + m(A_\epsilon)\epsilon \tag{12.5}$$

Since $m(A_\epsilon) \leq m(E) < \infty$. So we have that $\{I_n\}$ is a Cauchy sequence and hence convergent in this complete space. So we know that $\lim_{n\to\infty} I_n$ exists.

To prove the second part, if $f = 0$ we can repeat the argument above. We know that

$$|I_n| = |\int \phi_n| \leq \int_{A_\epsilon} |\phi_n| + \int_{E\setminus A_\epsilon} |\phi_n| \leq m(E)\epsilon + Mm(E \setminus A_\epsilon) \tag{12.6}$$

So we know that $\lim\limits_{n\to\infty} I_n = 0$. □

So now we can define integration.

Definition 12.3 (Integration)**.** *Let f be bounded and supported on E with $m(E) < \infty$. We each such f we define*

$$\int f(x)dx = \lim_{n\to\infty} \int \phi_n(x)dx \tag{12.7}$$

where

1. $|\phi_n| \leq M$
2. $supp(f) = supp(\phi_n)$
3. $\phi_n(x) \to f(x)$ *a.e as* $n \to \infty$.

We now must show this is well defined.

Claim 12.4. $\int f$ *is independent of the limiting sequence* $\{\phi_n\}$.

Proof. Suppose that there exists $\{\psi_n\}$ such that $|\psi_n| \leq M$ with $supp(\psi_n) = supp(f)$ and $\psi_n(x) \to f(x)$ a.e. Note that if we set

$$\zeta_n = \phi_n - \psi_n \tag{12.8}$$

Then we have that $\{\zeta_n\}$ is a sequence of simple functions with

- $|\zeta_n| \leq 2M$
- $m(supp(\zeta_n)) < \infty$
- $\zeta_n \to 0$ a.e as $n \to \infty$.

By the above lemma $\lim\limits_{n\to\infty} \int \zeta_n = 0$ which means we have that

$$\begin{aligned}\lim_{n\to\infty} \int (\phi_n - \psi_n) &= \lim_{n\to\infty} \int \phi_n - \lim_{n\to\infty} \int \psi_n = 0 \\ \implies \lim_{n\to\infty} \int \psi_n &= \lim_{n\to\infty} \int \phi_n \end{aligned} \tag{12.9}$$

which implies we have a well define integral. □

Theorem 12.5 (Bounded Convergence Theorem)**.** *Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M supported on E with a finite measure. Further, suppose that $f_n(x) \to f(x)$ a.e. Then f is measurable, f is bounded, f is supported on E a.e for all x, and*

$$\lim_{n\to\infty} \int |f_n - f| = 0 \tag{12.10}$$

Note that this implies that

$$\lim_{n\to\infty} \int f_n = \int f \tag{12.11}$$

Proof. The proof of this is like the proof of the above lemma using Egorov's theorem. We know from assumption that $|f| \leq M$ a.e and that $f(x) \neq 0$ for all $x \in E$ with E having finite measure. So we have that

$$\begin{aligned}\int |f_n(x) - f(x)| &\leq \int_{A_\epsilon} |f_n(x) - f(x)| + \int_{E \setminus A_\epsilon} |f_n(x) - f(x)| \\ &\leq \epsilon m(E) + 2Mm(E \setminus A_\epsilon) \end{aligned} \tag{12.12}$$

□

We now have a useful lemma.

Lemma 12.6. *Suppose $f \geq 0$ is supported on a set of finite measure and if $\int f = 0$, then $f = 0$ a.e.*

Proof. For each $k \geq 1$, let $E_k = \{x \in E : f(x) \geq \frac{1}{k}\}$. So we know that $\frac{1}{k} \leq f(x)$ for $x \in E$, which can be written as

$$\frac{1}{k}\chi_{E_k} \leq f(x) \implies \frac{1}{k}m(E_k) \leq \int f(x) = 0 \implies m(E_k) = 0 \tag{12.13}$$

for all K. Now look at $\{x : f(x) > 0\} = \bigcup_{k=1}^{\infty} E_k$. So we have that

$$m\left(\{x : f(x) \geq 0\}\right) = \sum_{k=1}^{\infty} m(E_k) = 0 \implies f = 0 \ a.e \tag{12.14}$$

□

Note the above lemma is sometimes called the vanishing property.

12.2 Non-Bounded

Now we have the case where f is measurable but not necessarily bounded, although f is still non-negative. Recall that if f is unbounded then the supremum of f is infinity.

Definition 12.7 ((Extended) Lebesgue Integral)**.** *If f is measurable and non-negative, then we define the extended Lebesgue integral of f as*

$$\int f(x)dx = \sup_g \left\{ \int g(x)dx \right\} \tag{12.15}$$

where the supremum is taken over all measurable functions g such that

- $0 \leq g \leq f$
- *g is bounded*
- *g is supported on a set of finite measure.*

Note that $\sup_g$ can be ∞ or finite. If this supremum is finite, then we can see that $\int f(x)dx < \infty$ and we say f is Lebesgue integrable or just integrable.

Proposition 12.8. *The integral of non-negative measurable functions have the following properties*

***Property** 1. Linearity.*
If $f, g \geq 0$ and a, b are positive real numbers then

$$\int af + gb = a\int f + b\int g \tag{12.16}$$

***Property** 2. Additivity*
If $E \cap F = \phi$ and $f \geq 0$, then

$$\int_{E \cup F} f = \int_E f + \int_F f \tag{12.17}$$

***Property** 3. If g is integrable and $0 \leq f \leq g$, then f is integrable.*

***Property** 4. Monotonicity*
If $0 \leq f \leq g$, and g is integrable then we have that

$$\int f \leq \int g \tag{12.18}$$

***Property** 5. Finiteness*
If f is integrable, $f \geq 0$, then

$$f(x) < \infty \implies \{x : f(x) = \infty\} \tag{12.19}$$

is a null set.

***Property** 6. Vanishing Property*

$$\int f = 0 \implies \{x : f(x) > 0\} \tag{12.20}$$

is a null set.

We now offer a proof of the Finiteness property.

Proof. Let us set $E_k = \{x : f(x) \geq k\}$ and $E_\infty = \{x : f(x) = \infty\}$. Ten we know that $k \leq f(x)$ for $x \in E_k$. We also know that

$$k\chi_{E_k}(x) \leq f(x) \implies m(E_k) \leq \frac{1}{k}\int f \tag{12.21}$$

which implies that $m(E_k) \to 0$ as $k \to \infty$. So we know that $E_{k+1} \subset E_k$ which implies that

$$m(E_\infty) = \lim_{n\to\infty} = m(E_k) = 0 \tag{12.22}$$

Which means that if f is integrable then $f(x) < \infty$ almost everywhere since the set of points where $f(x) = \infty$ is a set of measure zero. ☐

So we ant to know that if $f \geq 0$ and $f_n(x) \to f(x)$ for almost every x. is it true that $\int f_n(x)dx \to \int f(x)dx$? The answer is no and there are two rather classic counterexamples.

Example 12.9. *Look at the sequence of functions defined by when $0 \leq x \leq \frac{1}{n}$ f_n is the sides of the isosceles triangle with altitude n and base $\left[0, \frac{1}{n}\right]$ and when $\frac{1}{n} \leq x \leq 1$ we have that $f_n(x) = 0$.*

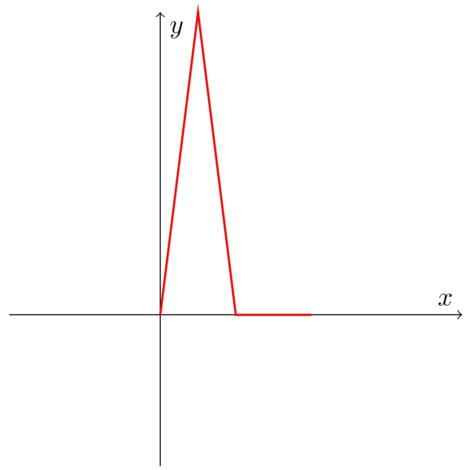

Figure 4: f_n

Proof. We can see that $\{f_n(x)\} \to 0$ as $n \to \infty$ on $[0,1]$. But

$$\int_0^1 f_n = \frac{1}{2}\frac{1}{n}n = \frac{1}{2} \tag{12.23}$$

for all n. So this does not work. hence

$$\int f_n(x)dx \neq \int f(x)dx \tag{12.24}$$

□

Example 12.10. *Define a function*

$$f_n(x) = \begin{cases} n & \text{if} \quad 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} \tag{12.25}$$

Proof. Note that $f_n(x) \to 0$ as $n \to \infty$ for all x which implies that

$$\int f_n(x)dx = \int_0^{\frac{1}{n}} ndx = 1 \tag{12.26}$$

for all n. □

13 10/15/13

We have defined for $f \geq 0$ the integral as

$$f := \sup_{0 \leq g \leq f} \int g \tag{13.1}$$

for g a bounded function with support on a set of finite measure.

Claim 13.1. $\int_E f$ *is well behaved on the set E in the following sense.*

1. *If $m(E) = 0 \implies \int_E f = 0$.*
2. *If $f \geq 0$, and $E \subset F$ with F measurable, then*

$$\int_E f \leq \int_F f \tag{13.2}$$

3. *If $f \geq 0$ is bounded above on E, say $0 \leq f \leq k$ on E then*

$$\int_E f \leq km(E) \tag{13.3}$$

4. *If $f \geq 0$ and f is integrable for $\alpha > 0$ and*

$$f \geq \alpha \chi_{\{x:f>\alpha\}} \tag{13.4}$$

then

$$\int f \geq \alpha m(\{x : f > a\}) \tag{13.5}$$

This is called Chebyshev inequality. We will present another version of it below.

Proof. We will prove the first three and leave the fourth as a separate lemma.

1. . If ϕ is a bounded measurable function supported on a set of finite measure with $0 \leq \phi \leq f\chi_E$. Then we have that

$$\phi(x) \leq f\chi_E(x) = f(1) \tag{13.6}$$

but $m(E) = 0$ so we know that $\phi(x) = 0$ a.e and hence the integral is still 0.

2. This follows from the fact that

$$f\chi_E \leq f\chi_F \tag{13.7}$$

3. We know that

$$f\chi_E \leq k\chi_E \tag{13.8}$$

which implies that

$$\int_E f = \int f\chi_E \leq k \int \chi_E = km(E) \tag{13.9}$$

$\square$

Lemma 13.2 (Chebyshev Inequality)**.** *If $f \geq 0$ and is integrable on E, then*

$$\int_E f \geq \alpha m(\{x : f(x) > \alpha\}) \tag{13.10}$$

for all $\alpha > 0$.

We start with an immediate application

Example 13.3. *If $f \geq 0$ and integrable then f is finite almost everywhere.*

Proof. Look at

$$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\} \tag{13.11}$$

and notice that the set $\{f \geq n\}$ decreases as n increases. So by the Chebysheve inequality we have that

$$m(\{x : f(x) > n\}) \leq \frac{1}{n} \int f \tag{13.12}$$

In the limit as $n \to \infty$ we have that this goes to 0. So we have that

$$m(\{x : f(x) = \infty\}) = 0 \tag{13.13}$$

$\square$

Claim 13.4. *We calim that*

$$\mu(E) = \int_E f \tag{13.14}$$

is a measure

Proof. We need to show that

- μ is nonnegative
- μ is monotone
- $\mu(\phi) = 0$
- If $E = \bigcup_{j=1}^{\infty} E_j$ disjoint, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) \tag{13.15}$$

□

We now we have a question about exchanging the limit and integral for a sequence of nonnegative functions converges pointwise almost everywhere. In general this answer is no. However, if our functions were bounded, we can do this. So the next thing that we can have is an inequality. So we have the following lemma

Lemma 13.5. *Suppose $\{f_n\}$ is a sequence of measurable functions $f_n \geq 0$. If $\lim\limits_{n\to\infty} f_n(x) = f(x)$ a.e x. Then*

$$\int f \leq \lim \inf_{n\to\infty} \int f_n \tag{13.16}$$

So the conclusion of Fatou's lemma is that we can rewrite conditions as

$$f_n \int L^+(f \geq 0) \implies \int \liminf f_n \leq \liminf \int f_n \tag{13.17}$$

We also know that if $\lim \int f_n$ exists then by Fatou's lemma

$$\int \lim_{n\to\infty} f_n \leq \lim_{n\to\infty} \int f_n \tag{13.18}$$

We also know that if $f_n(x) \to f(x)$ almost everywhere, then this implies that $f_n(x) \to f(x)$ almost everywhere on E and $m(E^c) = 0$. This implies that

$$\int f = \int_E f + \int_{E^c} f = \int_E f \leq \liminf \int f_n \chi_E \tag{13.19}$$

Proof. Suppose that $0 \leq g \leq f$. This implies that g is bounded and supported on a set of finite measure E. So define a sequence of functions

$$g_n = \min(g(x), f_n(x)) \tag{13.20}$$

This certainly implies that $g_n \leq f_n$. So what we know about g_n is that g_n is measurable, it is supported on E, $\int g_n(x) \leq \int f_n(x)$ and $\lim_{n\to\infty} g_n(x) = g(x)$ almost everywhere. So we now use the bounded convergence theorem. So by the bounded convergence theorem

$$\lim \int g_n = \int \lim g_n = \int g \tag{13.21}$$

So by construction we know that

$$\lim_{n\to\infty} \int g_n \leq \liminf \int f_n \tag{13.22}$$

So if we take sup of g we have that

$$\sup_{0 \le g \le f} \int g \le \liminf \int f_n \implies \int f \le \liminf \int f_n \tag{13.23}$$

$\square$

So we have some results following from this.

Corollary 13.6. *Suppose that $f \ge 0$ is measurable and $f_n(x) \le f(x)$ almost everywhere on x. If $f_n(x) \ge 0$ and $f_n(x) \to f(x)$ almost everywhere on x then*

$$\lim_{n\to\infty} \int f_n = \int f \tag{13.24}$$

Proof. We know that $f_n(x) \le f(x)$ for a.e x which implies that $\int f_n(x) \le \int f(x)$. So we know that

$$\limsup_{n\to\infty} \int f_n(x) \le \int f(x) \tag{13.25}$$

so by application of Fatou we have that

$$\int f \le \liminf \int f_n \tag{13.26}$$

since we know that $\liminf \int f_n \le \limsup \int f_n$ and so we have equality by comparing (13.25) and (13.26), implying

$$\inf f = \lim_{n\to\infty} \int f_n \tag{13.27}$$

$\square$

We now have a big theorem that is a consequence of Fatou's lemma.

Theorem 13.7 (Monotone Convergence Theorem (MCT))**.** *Suppose that $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$. Then,*

$$\lim_{n\to\infty} \int f_n = \int f \tag{13.28}$$

which implies that f and $\int f$ are completely determined by the sequence $\{f_n\}$.

Proof. Since $f_n \nearrow f$ we know that

$$f = \lim_{n\to\infty} f_n(x) = \sup_n f_n(x). \tag{13.29}$$

So this further implies that

$$\begin{aligned} 0 \le \int f_n \le \int f_{n+1} \le \int f \\ \implies \limsup \int f_n \le f \end{aligned} \tag{13.30}$$

So by Fatuous we have that

$$\int f \le \liminf \int f_n \implies \limsup \int f_n \le \int f \le \liminf \int f_n \tag{13.31}$$

Using the definition of lim inf and lim sup we see that this implies

$$\lim_{n\to\infty} \int f_n = \int f \tag{13.32}$$

$\square$

We can use the MCT to prove linearity property for integrals.
Take $f, g \in L^+$. We know that $\int f+g = \int f + \int g$ because we can choose a sequence of $\phi_n \to f$ and $\psi_n \to g$, where these are bounded, nonegative and supported ona set of finite measure. Then then add these two sequence together and see that $(\phi_n + \psi_n) \nearrow f+g$. This implies that

$$\int f+g = \int \lim_{n\to\infty} (\phi_n + \psi_n) = \lim_{n\to\infty} \left[\int \phi_n + \int \psi_n\right] = \lim_{n\to\infty} \int \phi_n + \lim_{n\to\infty} \int \psi_n \tag{13.33}$$

We can then use the bounded convergence theorem and the result is proved. We now have a theorem about series.

Corollary 13.8 (Beppo-Levi Theorem). *Consider a series* $\int_{k=1}^{\infty} a_k(x)$ *where* $a_k(x) \geq 0$ *and measurable for every* $k \geq 1$. *Then we have that*

$$\int \sum_{k=1}^{\infty} a_k(x) = \sum_{k=1}^{\infty} \int a_k(x). \tag{13.34}$$

Furthermore if $\sum_{k=1}^{\infty} \int a_k(x)$ *is finite, then the series* $\sum_{k=1}^{\infty} a_k(x)$ *converges for almost every* x.

Proof. Let $f_n(x) = \sum_{k=1}^{n} a_k(x)$ and $f(x) = \sum_{k=1}^{\infty} a_k(x)$. Then the functions $f_n(x)$ are measurable and increasing in the sense that $f_n(x) \leq f_{n+1}(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. By linearity of finite sums we have

$$\int f_n = \int \sum_{k=1}^{n} a_k(x) = \sum_{k=1}^{n} \int a_k(x) \tag{13.35}$$

So we have the following two equations

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \sum_{k=1}^{n} \int a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x)dx \tag{13.36}$$

$$\int \lim_{n\to\infty} f_n = \int \sum_{k=1}^{\infty} a_k(x)dx \tag{13.37}$$

and we can see that (13.36) and (13.37) must be equal by MCT. We can slo see that if $\sum_{k=1}^{\infty} \in a_k(x)dx$ is finite then since

$$\int \sum_{k=1}^{\infty} a_k(x)dx < \infty \implies \sum_{k=1}^{\infty} a_k(x) \tag{13.38}$$

is integrable. Recall that if $f \geq 0$ and integrable then f is finite almost everywhere. Thus we can say that $\sum_{k=1}^{\infty} a_k(x)$ is finite almost everywhere. □

14 10/17/13

We are going to start with an example.

Example 14.1 (Escape to Vertical Infinity). *Let* $X = [0,1]$. *Then define*

$$f_n = n\chi_{\left[\frac{1}{n}, \frac{2}{n}\right]} \tag{14.1}$$

Notice that this function escapes to infinity as we approach 0.

We can show that $f_n(x) \to f(x)$ pointwise and $\int_{[0,1]} f_n(x)dx = 1$, yet $\int_{[0,1]} f = 0$.

14.1 Connection to Borel-Cantelli Lemma

If $\{E_k\}_{k=1}^{\infty}$ is a collection of measurable subsets of $\mathbb{R}^d$ such that $\sum_{k=1}^{\infty} m(E_k) < \infty$, then the set of points that belong to infinitely many E_k's has measure zero. In short, this says that if $\sum_{k=1}^{\infty} m(E_k) < \infty$ then $m(E) = 0$. To connect this to the Beppo-Levi theorem, we need to make the connection that

$$a_k(x) = \chi_{E_k}(x). \tag{14.2}$$

So x belongs to infinitly many E_k's if and only if

$$\sum_{k=1}^{\infty} a_k(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x) = \infty. \tag{14.3}$$

Notice our assumption implies

$$\sum_{k=1}^{\infty} \int a_k(x) = \sum_{k=1}^{\infty} \int \chi_{E_k}(x) < \infty. \tag{14.4}$$

By the Beppo-Levi theorem we know that

$$\sum_{k=1}^{\infty} \int a_k = \int \sum_{k=1}^{\infty} a_k < \infty \implies \sum_{k=1}^{\infty} a_k(x) < \infty \text{ a.e} \tag{14.5}$$

So this is finite except on a set of measure zero.

14.2 Integrability

So we know that the integrability of any real valued measurable function f on $\mathbb{R}^d$ can be defined.

Definition 14.2 (Integrable)**.** *We say f is integrable if $|f|$ is integrable in the sense of the previous section*

We can now talk about integration of arbitrary functions in terms of the non-negative ones. Remember that we know that $f = f^+ - f^-$ and further note that all of these functions are non-negative and that $f^{\pm} \leq |f| \implies f^+, f^-$ are integrable if f is integrable. So we can define the integral of f as

Definition 14.3 (Lebesgue Integral)**.** *The Lebesgue Integral of a measurable function f is given as*

$$\int f = \int f^+ - \int f^- \tag{14.6}$$

Notice that both the integrability of f and the value of its integral are unchanged with if we modify f arbitrarily on a set of measure zero. So we have the following:

i) We can add two integrable functions $f + g$ and $f(x) + g(x) < \infty$ for almost every x.

ii) When we say f is integrable we are talking about a collection of functions g such that $f = g$ a.e.

Proposition 14.4. *The Lebesgue integrable function is*

- *Linear*
- *Additive*
- *Monotone*
- *Satisfies triangle inequality*

Proposition 14.5. *Suppose f is integrable on $\mathbb{R}^d$, then for every $\epsilon > 0$*

i) there exists a set of finite measureB such that

$$\int_{B^c} f < \epsilon \tag{14.7}$$

ii) There is a $\delta > 0$ *such that* $\int_E f < \epsilon$ *whenever* $m(E) < \delta$.

Note that ii) is called absolute continuity of the integral.

Proof. i) Without loss of generality assume that $f \geq 0$ and look at $B_N(0)$. Then look at

$$f_N = f\chi_{B_N(0)}. \tag{14.8}$$

Note that $f_n \geq 0$ implies that f_N's are measurable and $f_N(x) \leq f_{N+1}(x)$ and $\lim_{n\to\infty} f_N(x) = f(x)$. So by the MCT we know that

$$\lim_{n\to\infty} \int f_N = \int f \implies \lim_{n\to\infty} \int f\chi_{B_N(0)} = \int f \tag{14.9}$$

Thus for large N

$$0 \leq \int f - \int f\chi_{B_N(0)} < \epsilon \tag{14.10}$$

So we have that

$$0 \leq f(1 - \chi_{B_N(0)}) = \int f\chi_{B_N^c(0)} = \int_{B^c} f < \epsilon \tag{14.11}$$

ii) Assume that $f \geq 0$ and let

$$f_n(x) = f(x)\chi_{E_N}(x). \tag{14.12}$$

We know that $f_N \geq 0$ which implies that it is measurable and that $f_N(x) \leq f_{N+1}(x)$. Recall that by MCT we know that

$$\lim_{N\to\infty} \int f_N(x) = \int f(x). \tag{14.13}$$

Given $\epsilon > 0$ we know that there exists an integer N such that

$$\int f - F_n < \frac{\epsilon}{2} \tag{14.14}$$

Pick $\delta > 0$ such that $N\delta < \frac{\epsilon}{2}$. The we know that if $m(E) < \infty$ we have that

$$\int_E f = \inf_N f - f_N + \int_N f_N \leq \int f - f_N + Nm(E) \leq \epsilon \tag{14.15}$$

□

14.3 Dominated Convergence

Theorem 14.6 (Lebesgue Dominated Convergence Theorem (DCT))**.** *Let* $\{f_n\}$ *be a sequence of measurable functions such that* $f_n(x) \to f(x)$ *almost everywhere as* $n \to \infty$. *If:*

1. $|f_n(x)| \leq g(x)$ *(Domination condition)*
2. g *is integrable*

Then $\int |f_n - f| \to 0$ *as* $n \to \infty$. *Consequently, we have that* $\int f_n \to \int f$ *as* $n \to \infty$.

Proof. For each $N \geq 0$ define a set

$$E_n = \{x : |x| \leq N, g(x) \leq N\}. \tag{14.16}$$

Given $\epsilon > 0$ by the previous lemma, there exists $N \in \mathbb{N}$ such that

$$\int_{E_N^c} g < \epsilon \tag{14.17}$$

Then the function $f_n \chi_{E_N} \leq N$ and supported on a set of finite measure. So by the bounded convergence theorem

$$\int_{E_N} |f_n - f| < \epsilon \tag{14.18}$$

for N large enough. So look at

$$\int |f_n - f| = \int_{E_n} |f_n - f| + \int_{E_N^c} |f_n - f| \leq \epsilon + 2\int_{E_N^c} g \leq 3\epsilon \tag{14.19}$$

Since ϵ is arbitrary, this completes the proof. □

Example 14.7. *Suppose f is integrable and $\{E_n\}$ is a sequence of disjoint measurable subsets of X. If $E = \bigcup_{i=1}^{\infty} E_i$. Then*

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f \tag{14.20}$$

Proof. Idea is set $F_n = \bigcup_{i=1}^{n} E_i$. Then we know that $|f\chi_{F_n}| \leq |f|$ for each n. Then we have that

$$f\chi_{F_n} \to f\chi_E \text{ a.e.} \tag{14.21}$$

Then apply DCT. □

Example 14.8. *Show that*

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1 \tag{14.22}$$

Proof. Recall the following from calculus

$$\int_0^{\infty} e^{-x} dx = \lim_{t\to\infty} \int_0^t e^{-x} dx = \lim_{t\to\infty} (1 - e^{-t}) = 1 \tag{14.23}$$

and therefore the function e^{-x} is Lebesgue integrable over $[0, \infty)$.So let

$$g_n(x) = \left(1 + \frac{x}{n}\right)^n e^{-2x} \chi_{[0,1]}(x) \tag{14.24}$$

So we know that each g_n is Lebesgue integrable over $[0, \infty]$. From calculus we know that $\left(1 + \frac{x}{n}\right)^n \nearrow e^x$ for each $x \geq 0$. Therefore we know that $g_n(x) \nearrow e^{-x}$ for each $x \geq 0$. So we know that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = \lim_{n\to\infty} \int g_n(x) = \int \lim_{n\to\infty} g_n(x) = \int_0^{\infty} e^{-x} = 1 \tag{14.25}$$

where we have use MCT. □

Example 14.9. *Look at the Fresnel Integral*

$$\int_0^{\infty} \sin\left(x^2\right) dx \tag{14.26}$$

Show that the improper Riemann integral exists but not the Lebesgue integral over $[0, \infty)$.

Proof. The idea of the proof is that if $0 < s < t$ and we take $w = x^2$. Then $dw = 2xdx$ and $dx = \frac{dw}{2\sqrt{2}}$. so we have that

$$\int_s^t \sin x^2 dx = \frac{1}{2}\int_{s^2}^{t^2} \sin(w)\frac{dw}{\sqrt{w}} \tag{14.27}$$

Then we can apply integration by parts. Also note that

$$\int_0^{\sqrt{n\pi}} |\sin(x^2)|dx = \sum_{k=1}^{n}\int_{\sqrt{k\pi-\pi}}^{\sqrt{k\pi}} |\sin x^2|dx \geq \frac{1}{\sqrt{\pi}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \tag{14.28}$$

Which shows that the integral does not exists since the series does not converge which means that $\sin x^2$ is not Lebesgue integral. So look at

$$\int_{\sqrt{k\pi-\pi}}^{\sqrt{k\pi}} |\sin x^2|dx = \frac{1}{2}\int_{k\pi-\pi}^{k\pi}\frac{\sin w}{\sqrt{w}}dw \geq \frac{1}{2}\frac{1}{\sqrt{k\pi}}\int_{k\pi-\pi}^{k\pi}|\sin w|dw = \frac{1}{\sqrt{k\pi}} \tag{14.29}$$

□

Example 14.10. *Show that the improper Riemann integral of*

$$\int_0^\infty \frac{\sin x}{x}dx \tag{14.30}$$

exists while the Lebesgue integral of it does not.

Proof. Note for the improper Riemann integral, we can write this as

$$\int_0^\infty \frac{\sin x}{x}dx = \sum_{n=1}^{\infty}\int_{(n-1)\pi}^{n\pi}\frac{\sin x}{x}dx = \sum_{n=1}^{\infty}(-1)^{n-1}\int_{(n-1)\pi}^{n\pi}\frac{|\sin x|}{x}dx \tag{14.31}$$

We use a change of variables, $t = x - [(n-1)\pi]$ which implies that $\dfrac{|\sin t|}{t+(n-1)\pi}$ decreases as n increases for a fixed t. So we have that

$$\int_0^\pi \frac{|\sin t|}{t+(n-1)\pi}dt \leq \frac{1}{n-1} \to 0 \tag{14.32}$$

For the Lebesgue integral notice that

$$\int_0^\infty \frac{|\sin x|}{x} = \sum_{n=1}^{\infty}\int_{(n-1)\pi}^{n\pi}\frac{|\sin x|}{x}dx \geq \sum_{n=1}^{\infty}\frac{1}{n\pi}\int_0^\pi |\sin x|dx = \infty \tag{14.33}$$

□

Note that this example illustrates the difference between the improper and the Lebesgue integral is roughly the same difference as between conditionally convergent series and absolutely convergent series. The improper Riemann integral may exists due to the effect of cancellations while the Lebesgue integral does not permit such issues to arise. Also note there is no such thing as an improper Lebesgue integral because we have made no special assumptions about the boundedness of our integrand or the boundedness of the set over which we are integrating.

15 10/24/13

We are going to talk a little bit about an abstract measure space. Suppose I have a set X with a σ-algebra $\mathcal{M}$ and a measure μ. Then $(X, \mathcal{M}, \mu)$ is an abstract measure space. We want to look at an example of how to define an abstract measure.

Example 15.1. *Let X be a set and $\mathcal{M} = P(X)$. Define*

$$\mu : \mathcal{M} \to [0, \infty] \tag{15.1}$$

as $\mu(A) = \infty$ if A is an ∞ set and $\mu(A)$ is the cardinality of A if A is a finite set. The μ is called a counting measure. We proved stuff about it in the homework.

Now we have the following claim.

Claim 15.2. *Let μ be a counting measure on $\mathbb{N}$. Then the function $f : \mathbb{N} \to \mathbb{R}$ is integrable if and only if $\sum_{n=1}^{\infty} |f(n)| < \infty$. In this case,*

$$\int f d\mu = \sum_{n=1}^{\infty} |f(n)| \tag{15.2}$$

Proof. Let $f : \mathbb{N} \to \mathbb{R}$. we know that f is ingetrable if and only if both f^+, f^- are integrable. Without loss of generality, assume that $f(k) \geq 0$ hold for all k. Then we have that

$$\phi_n(x) = \sum_{k=1}^{n} f(k)\chi_{\{k\}} \tag{15.3}$$

then ϕ_n is a sequence of simple functions and

$$\begin{aligned} \int \phi_n &= \sum_{k=1}^{n} f(k)\mu(\{k\}) \\ \implies \lim_{n\to\infty} \int \phi_n &= \sum_{k=1}^{\infty} f(k) \end{aligned} \tag{15.4}$$

and $\phi_n \to f(k)$ as $n \to \infty$. Then we have that

$$\int f = \sum_{k=1}^{\infty} f(k) \tag{15.5}$$

and f is integrable if and only if $\sum_{k=1}^{\infty} f(k) < \infty$. □

Claim 15.3. *If f is integrable and $F(x) = \int_{-\infty}^{\infty} f$, then F is continuous.*

Proof. Let $x_n \to x$. We know that this implies that

$$f\chi_{(-\infty,x_n]} \to f\chi_{(-\infty,x]} \tag{15.6}$$

almost everywhere. But we know that

$$\left|f\chi_{(-\infty,x_n]}\right| \leq |f| \tag{15.7}$$

since f is L^1 (integrable. So by the DCT, we knw that

$$\int f_{[(-\infty,x_n]} \to \int f\chi_{-\infty,x]} \tag{15.8}$$

$$\int_{-\infty}^{\infty} f \to \int_{-\infty}^{x} f \tag{15.9}$$

$$F(x_n) \to F(x) \tag{15.10}$$

□

15.1 Complex Valued Functions

Consider $f : \mathbb{R}^d \to \mathbb{C}$. We can write

$$f(x) = u(x) + iv(x) \tag{15.11}$$

where u and v are real valued functions.
So we know the following

1. f is measurable if and only if u and v are measurable.
2. f is Lebesgue integrablle if

$$|f(x)| = \left[u(x)^2 + v(x)^2\right]^{1/2} \tag{15.12}$$

   is integrable.

3.

$$|u(x)| \leq |f(x)| \tag{15.13}$$

$$|v(x)| \leq |f(x)| \tag{15.14}$$

4. Also for $a, b \geq 0$, we know that

$$(a+b)^{1/2} \leq a^{1/2} + b^{1/2} \implies |f(x)| \leq |u(x)| + |v(x)| \tag{15.15}$$

   So we know that a complex valued function is integrable if and only if its real and imaginary parts are integrable.

5.

$$\int f(x)dx = \int u(x)dx + i\int v(x)dx \tag{15.16}$$

6. $E \subset \mathbb{R}^d$ is measurable

$$\int_E f = \int f\chi_E \tag{15.17}$$

Now we will state the Dominated convergence theorem for an abstract measure space.

Theorem 15.4 (Dominated Convergence Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of measurable functions where $f_n : X \to \mathbb{C}$. Suppose that $f_n \to f$ a.e where limit of $f : X \to \mathbb{C}$ is a measurable function. Suppose $g : X \to [0, \infty]$ such that $|f_n| \leq g$ pointwise a.e for each n. Then we have that*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int f d\mu \tag{15.18}$$

Note that the moving jump functions are examples where DCT fails if there is no absolutely integrable dominated function g.

15.2 Space of L^1 of Integrable Functions

Integrable functions form a vector space $f \in L^1$. with norm

$$\|f\|_{L^1} = \|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|dx \tag{15.19}$$

We now must check to ensure this satisfies all the properties of a norm.

Lemma 15.5. *$\|f\|_{L^1}$ is a norm.*

Proof. Note the first two are simple to prove.

1. $\|f\|_{L^1} = 0 \Leftrightarrow f = 0$ a.e

2. $\|af\|_{L^1} = |a|\|f\|_{L^1}$ for all scalars a.

3. $\|f+g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}$.
   Note that the metric on $L^1(\mathbb{R}^d)$ is given as

$$d(f,g) = \|f-g\|_{L^1(\mathbb{R}^d)} \tag{15.20}$$

It is easy to check the properties of this metric space. and therefore the triangle inequality holds.

□

We will see later two major results about this space.

1. L^1 is complete. (Riess-Fisher Theorem)

2. The continuous functions with compact support is dense in $L^1(\mathbb{R}^d)$. In particular $C[a,b]$ is dense $L^1[a,b]$. This implies that $L^1[a,b]$ is separable.

Definition 15.6 (Convergent in Measure)**.** *We say $\{f_n\}$ converge to f in measure (m) and we write $f_n \xrightarrow{m} f$ if for all $\epsilon > 0$*

$$\lim_{n\to\infty} m\left(\{x : |f_n(x) - f(x)| \geq \epsilon\right) = 0 \tag{15.21}$$

or for all $\epsilon > 0$ there exists N such that for all $n \geq N$

$$m\left(\{x : f_n(x) - f(x)| \geq \epsilon\}\right) < \epsilon \tag{15.22}$$

Definition 15.7 ($f_n \to f \in L^1$)**.** *When we say $f_n \to f \in L^1$, we mean for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ suc that*

$$\|f_n - f\|_{L^1} < \epsilon \tag{15.23}$$

for all $n \geq N$.

Note that if $f_n \to f$ in L^1, then $f_n \xrightarrow{m} f$. We can see this by looking at the sets

$$E_{n,\epsilon} = \{x : |f_n(x) - f(x)| \geq \epsilon\} \tag{15.24}$$

We know that

$$\begin{aligned}\epsilon m(E_{n,\epsilon}) &\leq \int_{E_{n,\epsilon}} |f_n(x) - f(x)| \leq \int |f_n(x) - f(x)| \\ \implies m(E_{n,\epsilon}) &\leq \frac{1}{\epsilon} \int |f_n(x) - f(x)| \to 0\end{aligned} \tag{15.25}$$

Theorem 15.8 (Riess-Fisher)**.** *The vector space L^1 is complete.*

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in L^1. This means that $\|f_n - f_m\| \to 0$ as $n, m \to \infty$. We want to show that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\|f_{n_k} - f\| \to 0$ as $k \to \infty$ and $f \in L^1$.
Note since $\{f_n\}$ is Cauchy, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \frac{\epsilon}{1}$ whenever $n, m > N$. Thus if n_k is chosen such that $n_k > N$, then we have that $\|f_{n_k} - f\| < \frac{\epsilon}{2}$. Then we have that

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \epsilon \tag{15.26}$$

when $n_k > N$. Thus $\{f_n\}$ has a limit f in L^1. Now to the main part of the proof.

Consider a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ with

$$\|f_{n_{k+1}} - f_{n_k}\| \leq \frac{1}{2^k} \tag{15.27}$$

Note the existence of the subsequence is guaranteed by the fact that we have a Cauchy sequence. Now define

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)) \tag{15.28}$$

$$g(x) = \left|f_{n_1}(x)\right| + \sum_{k=1}^{\infty} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right| \tag{15.29}$$

Observe several things

1. $|f| \leq g$

2. Observe that
$$\sum \int |f_{n_{k+1}} - f_{n_k}| = \sum \|f_{n_{k+1}} - f_{n_k}\|_1 \leq \sum \frac{1}{2^k} < \infty \tag{15.30}$$

3. If we define
$$g_m(x) = |f_{n_1}(x)| + \sum_{k=1}^{m} |f_{n_{k+1}}(x) - f_{n_k}(x)| \tag{15.31}$$
then $g_m(x) \nearrow g(x)$

We can see that g is integrable by noting the following

$$\int \lim_{m\to\infty} g_m(x) = \lim_{m\to\infty} \int g_m(x)dx$$
$$\int g(x)dx = \lim_{m\to\infty} \left[\int |f_{n_1}(x)| + \sum_{k=1}^{m} \frac{1}{2^k}\right] < \infty \tag{15.32}$$

to see that $f_{n_k} \to f(x)$ for almost every x we need at the partial sums.

$$\begin{aligned} f_{n_1}(x) + \sum_{i=1}^{k-1} f_{n_{i+1}}(x) - f_{n_i}(x) &= f_{n_1}(x) + (f_{n_2}(x) - f_{n_1}(x)) + \cdots + (f_{n_k} - f_{n_{k-1}}) \\ &= f_{n_k} \end{aligned} \tag{15.33}$$

And now we know that

$$\lim_{n\to\infty} |f_{n_k}(x)| = |f_1(x)| + \lim_{k\to\infty} \sum_{i=1}^{k-1} |f_{n_{k+1}}(x) - f_{n_k}| = f(x). \tag{15.34}$$

where we have used the fact that both terms are finite.

Now to see that f_{n_k} in L^1, note that we have

$$\begin{aligned} |f - f_{n_k}| &= \left| f_{n_1} + \sum_{i=1}^{\infty} f_{n_{i+1}}(x) - f_{n_i}(x) - \left[f_{n_1} + \sum_{i=1}^{k-1} f_{n_{i+1}}(x) - f_{n_i}(x) \right] \right| \\ &= \left| \sum_{i=k-1}^{\infty} f_{n_{i+1}}(x) - f_{n_i}(x) \right| \leq \sum_{i=k-1}^{\infty} |f_{n_{i+1}}(x) - f_{n_i}(x)| \leq \sum_{i=1k-1}^{\infty} \frac{1}{2^k} < \epsilon \end{aligned} \tag{15.35}$$

□

Corollary 15.9. *If $\{f_n\}$ converges to f in L^1 the there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f(x)$ a.e x.*

Proof. L^1 is complete and so $\{f_n\}$ converges in L^1 if and only if $\{f_n\}$ is a Cauchy sequence. using the above argument we get our corollary. □

16 10/29/13

We are going to do more about convergence today. We have seen that

$$f_n \xrightarrow{m} f \implies f_n \xrightarrow{L^1} f \tag{16.1}$$

We have proved Chebechev's theorem among other things. The first example is the typewriter sequence. It is in the book. We now proceed with some lemmas.

Lemma 16.1. *For $f, g \in L^1$, the following are equivalent:*

i) $f = g$ *a.e*

ii) $\int |f-g| = 0$

iii) $\int_E f = \int_E g$ *for every measurable set* E*.*

Proof. We will prove in a cyclic manner. We can see that $i) \Leftrightarrow ii)$ as follows

$$\int |f-g| \Leftrightarrow |f-g| = 0 \text{ a.e} \Leftrightarrow f = g \text{ a.e} \tag{16.2}$$

We can see that $ii) \implies iii)$ as

$$\left|\int_E f - \int_E g\right| \leq \int_E |f-g| = 0 \implies \int_E f = \int_E g \tag{16.3}$$

We know that $iii) \implies ii)$ because if we let $E = \{x : f - g \geq 0\}$, then we have that

$$\int |f-g| = \int_E (f-g) + \int_{E^c} g - f = 0 \tag{16.4}$$

Note that we are really looking at equivalence classes of f. □

16.1 Approximation of Integrable Functions

The following families of functions are dense in $L^1(\mathbb{R}^d)$.

1. The Simple functions. This means that if f is integrable, then given $\epsilon > 0$ there exists a simple function ϕ such that
$$\int |f-\phi| < \epsilon. \tag{16.5}$$
2. Step functions.
3. The continuous functions of compact support.

So we know that $C[a,b]$ where $[a,b]$ is a compact subset of $\mathbb{R}$. In particular, we know that $C[a,b]$ is dense in $L^1([a,b])$ which implies that $L^1([a,b])$ is separable.

17 11/5/13

Today we will do Fubini's Theorem.

Theorem 17.1 (Fubini)**.** *Given* $f(x,y)$*, a function of two variables,* $f(x,y)$ *is continuous on a rectangle* $R = [a,b] \times [c,d]$*. Then*

$$\int_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy \tag{17.1}$$

Although this works for most functions defined on R, there are exceptions.

Example 17.2.

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} dydx \tag{17.2}$$

Note that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dydx = \frac{\pi}{4} \neq -\frac{\pi}{4} = \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy \tag{17.3}$$

where we have sued the fact that $x = y \tan\theta$. Fubini's theorem does not apply here since there is a discontinuity at 0.

So now suppose that $\vec{x} = (x_1, x_2, \ldots, x_n)$ is a point. Consider an n-dimensional interval

$$I_1 = \{\vec{x} : a_i \leq x_i \leq b_i \quad i = 1, 2, \ldots, n\}. \tag{17.4}$$

Let $\vec{y} = (y_1, y_2, \ldots, y_m\}$ be a point of an m-dimensional interval

$$I_2 = \{\vec{y} : c_j \leq y_j \leq d_j \quad j = 1, 2, \ldots, m\}. \tag{17.5}$$

The Cartesian product $I = I_1 \times I_2$ is an $(n+m)$-dimensional interval consisting of points $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = (x, y)$. Then the question is when does the following integral exist and when is it equal to iterated integrals.

$$\int_I f = \int_I f(\vec{x}, \vec{y}) d\vec{x} d\vec{y} \tag{17.6}$$

Note that in Stein's notation, he says $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $(x, y) \in \mathbb{R}^d$ means that $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. We now offer some motivating examples.

17.1 Motivation for Fubini's Theorem

1. Suppose that m is the Lebesgue measure on $\mathbb{R}$. Let I, J be intervals in $\mathbb{R}$. Then the $I \times J \in \mathbb{R}^2$ is a rectangle and we know that

$$m(I \times J) = length(I) \times length(J) = m(I) \cdot m(J) \tag{17.7}$$

2. Suppose that Γ and Ψ are two collections of finite sets and μ and ν are two counting measure on Γ and Ψ. Note that a counting measure in the number of elements in a set. So if $A \subset \Gamma$ and $B \subset \Psi$, then

$$N(A \times B) = \mu(A) \cdot \nu(B) \tag{17.8}$$

   This is known as fundamental principle of counting.

So we really need to ask how we can properly define an area element. Consider two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. Let $X \times Y = \{(x, y) : x \in X, y \in Y\}$. The question is how do we know the existance of a σ-algebra, $\mathcal{A}$ of the subset $X \times Y$ that contains the sets of the form

$$\{S \times T : \ S \in \mathcal{M} \ \ , T \in \mathcal{N}\} \tag{17.9}$$

and a measure w on $\mathcal{A}$ such that

$$w(S \times T) = \mu(S)\nu(T) \tag{17.10}$$

17.2 Slices of Functions and Sets

Assume here we are in $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Definition 17.3 (Slices of Functions). *Fix $y \in \mathbb{R}^{d_2}$ and define f^y as a function of $x \in \mathbb{R}^{d_1}$ such that*

$$f^y(x) = f(x, y) \tag{17.11}$$

Fix $x \in \mathbb{R}^{d_1}$ and define f_x as a function of $y \in \mathbb{R}^{d_2}$ such that

$$f_x(y) = f(x, y) \tag{17.12}$$

Definition 17.4 (Slices of Sets)**.** *Given a set $E \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, slices of E are defined as follows.*

$$E^y = \{x \in \mathbb{R}^{d_1} : (x, y) \in E\} \tag{17.13}$$
$$E_x = \{y \in \mathbb{R}^{d_2} : (x, y) \in E\} \tag{17.14}$$

were E^y means y is fixed and E_x means x is fixed.

Example 17.5. *Let $E = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 4\}$. Find E_x and E^y.*

We simply need to solve and we find that

$$E_x = \begin{cases} \left[-\frac{1}{2}\sqrt{4 - x^2}, \frac{1}{2}\sqrt{4 - x^2}\right] & \text{for } |x \leq 2 \\ \phi & \text{otherwise} \end{cases} \tag{17.15}$$

$$E^y = \begin{cases} \left[-2(1 - y^2)^{1/2}, 2(1 - y^2)^{1/2}\right] & \text{for } |y| \leq 1 \\ \phi & \text{otherwise} \end{cases} \tag{17.16}$$

So we have a question. IF f is measurable does that imply that f_x, f^y are measurable and if E is measurable then are E_x, E^y measurable. The answer to this question is no in general.

Example 17.6. *Let $A \subset \mathbb{R}$ be non-measurable. Suppose that $E \subset \mathbb{R}^2$ be a measurable set with $m(E) = 0$. Then if we look at E^y for $y = 0$ on the real line such that $E^0 = A$, we can see this isn't measurable.*

17.3 Fubini's Theorem

Theorem 17.7 (Fubini's Theorem)**.** *Suppose $f \in L^1\left(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\right)$. Then for almost every $y \in \mathbb{R}^{d_2}$:*

i) The slice $f^y \in L^1(\mathbb{R}^{d_1})$.

ii) The function $\displaystyle\int_{\mathbb{R}^{d_1}} f^y(x) \in L^1(\mathbb{R}^{d_2})$

iii) $\displaystyle\int_{\mathbb{R}^{d_2}} \left[\int_{\mathbb{R}^{d_1}} f(x, y)dx\right] dy = \int_{\mathbb{R}^d} f(x, y)$

Some remarks

1. The above theorem is symmetric in x and y, so we can conclude the following almost all $x \in \mathbb{R}d_1$
   - i) The slice $f_x \in L^1(\mathbb{R}^{d_2})$
   - ii) The function $\displaystyle\int_{\mathbb{R}^{d_1}} f_x(y) \in L^1(\mathbb{R}^{d_1})$
   - iii) $\displaystyle\int_{\mathbb{R}^{d_1}} \left[\int_{\mathbb{R}^{d_2}} f(x, y)dy\right] dx = \int_{\mathbb{R}^d} f(x, y)$
2. So we can compute the integral of $f(x, y)$ on $\mathbb{R}^d$ can be computed by integrating lower dimensional integrals and the iteration can be taken in any order.
$$\int_{\mathbb{R}^{d_2}} \left[\int_{\mathbb{R}^{d_1}} f(x, y)dx\right] dy = \int_{\mathbb{R}^{d_1}} \left[\int_{\mathbb{R}^{d_2}} f(x, y)dy\right] dx = \int_{\mathbb{R}^d} f(x, y) \tag{17.17}$$
3. We can assume that f is real valued and since the theorem applies to the real and imaginary parts of the complex valued function.

Proof. We are going to give an idea of the proof. Define $\mathcal{F}$ as the set of integrable functions on $\mathbb{R}^d$ which satisfies $i), ii), iii)$ of (17.7). He wants to show that $L^1(\mathbb{R}^d) \subset \mathcal{F}$. So Stein considers a sequence of special cases.

1. Any finite linear combination of functions in $\mathcal{F}$ also belongs to $\mathcal{F}$.
2. If $\{f_k\}$ is a sequence of measurable functions in $\mathcal{F}$ and f_k increases or decreases to $f \in L^1(\mathbb{R}^d)$. Then $f \in \mathcal{F}$.
3. see 5 (technical step)
4. see 5 (technical step)
5. Let $E \subset \mathbb{R}^d$ be a set with finite measure. Then χ_E has the property of $\mathcal{F}$.
6. $f = f^+ - f^-$. Then there exists $\{\phi_n\}$ that increases to f¿ Each ϕ_n is a linear combination of characteristic functions of measurable sets with finite measure. So this shows that $\phi_n \in \mathcal{F}$. Then he uses previous properties to get the result we need.

□

18 11/7/13

Given a sheet for the exam review.

Example 18.1. *Look at the function*

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \tag{18.1}$$

One can show that

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = \int_{-1}^{1}\int_{-1}^{1} f(x,y)dydx = 0 \tag{18.2}$$

but

$$\int_{-1}^{1}\int_{-1}^{1} |f(x,y)|dxdy = +\infty \tag{18.3}$$

18.1 Tonelli's Theorem

Theorem 18.2 (Tonelli's Theorem)**.** *Let $f(x,y)$ be a nonnegative and measurable function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. We have that*

i) f^y *is measurable on* $\mathbb{R}^{d_1}$

ii) $\int_{\mathbb{R}^{d_1}} f^y$ *is measurable on* $\mathbb{R}^{d_2}$.

iii) $\int_{\mathbb{R}^{d_2}}\int_{\mathbb{R}^{d_1}} f(x,y) = \int_{\mathbb{R}^d} f(x,y)$

Since the role of x and y can be interchanged in the statement of Tonelli's theorem, it folows that if $f \geq 0$ and measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then

$$\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}^{d_2}} f(x,y) = \int_{\mathbb{R}^d} f(x,y) \tag{18.4}$$

This conditions hold even if $\int_{\mathbb{R}^d} f = \pm\infty$. In fact ,if $\int_{\mathbb{R}^d} f = \infty$, then $\int_{\mathbb{R}^d} f^+ = \infty$ and $\int_{\mathbb{R}^d} f^- < \infty$. So by Tonelli's theorem we have

$$\int_{\mathbb{R}^d} f^+ = \int_{\mathbb{R}^{d_1}} \left[\int_{\mathbb{R}^{d_2}} f^+ dy\right] dx \tag{18.5}$$

$$\int_{\mathbb{R}^d} f^- = \int_{\mathbb{R}^{d_1}} \left[\int_{\mathbb{R}^{d_2}} f^- dy\right] dx < \infty \tag{18.6}$$

Note that we can use these theorems in tandem. Suppose that we are given a measurable function f on $\mathbb{R}^d$ and asked to compute $\int_{\mathbb{R}^d} f$. To justify the use of iterated integrals, we can use a system of steps.

1. Apply Tonelli's theorem to $|f|$. Try to show that $\int_{\mathbb{R}^d} |f| < \infty$.
2. Apply Fuini's theorem to conclude

$$\int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} f(x,y) = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} f(x,y) \tag{18.7}$$

Example 18.3. *Evaluate* $\int_E y\sin(x)e^{-xy}dxdy$ *where* $E = \{(x,y) : 0 < x < \infty; 0 < y < 1\}$.

Proof. Since the function $f(x,y) = y\sin(x)e^{-xy}$ is continuous, it is Lebesgue measurable. If we performed the iterated integration doing the x integration first, then

$$F(y) = \int_0^\infty y\sin(x)e^{-xy}dy = \left[\frac{y}{y^2+1}\right]_0^\infty = \frac{y}{y^2+1} \tag{18.8}$$

We then look at

$$\int_0^1 F(y)dy = \int_0^1 \frac{y}{1+y^2} = \frac{1}{2}\ln(y^2+1)\Big|_0^1 = \frac{1}{2}\ln(2) \tag{18.9}$$

But the question is can we assert that $\int_E f(x,y)dxdy = \frac{1}{2}\ln(2)$. If we knew the function f to be integrable, this would be a consequence of the Fubini's theorem. To show $f(x,y)$ is actually integrable, we use Tonelli's theorem for non-negative measurable functions.

$$|f(x,y)| = |y\sin(x)e^{-xy}| \leq ye^{-xy}$$

$$\implies \int_E |f(x,y)| \leq \int_E ye^{-xy}dxdy = \int_0^1 dy \int_0^\infty ye^{-xy}dx = \int_0^1 dy = 1 \tag{18.10}$$

□

So if we integrated with respect to y first then

$$\int_0^1 y\sin(x)e^{-xy}dy = \frac{\sin(x)}{x}\left(\frac{1-e^{-x}}{x} - e^{-x}\right) \tag{18.11}$$

So we have that

$$\int_0^\infty \frac{\sin(x)}{x}\left(\frac{1-e^{-x}}{x} - e^{-x}\right)dx = \frac{1}{2}\ln(2) \tag{18.12}$$

Corollary 18.4. *If E is a measurable set in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ then for almost all $y \in \mathbb{R}^{d_2}$*

i) The slice $E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$ is a measurable subset of $\mathbb{R}^{d_1}$.

ii) $m(E^y)$ is a measurable function of y.

iii) $m(E) = \int_{\mathbb{R}^{d_2}} m(E^y)$.

Proof. Apply Tonelli by taking $f = \chi_E$, $f^y = \chi_{E^y}$. If E is measurable $\Leftrightarrow \chi_E$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then we have

i) $(\chi_E)^y = \chi_{E^y}$ is measurable in $\mathbb{R}^{d_1}$ which implies that E^y is measurable on $\mathbb{R}^{d_1}$.

ii) $\int_{\mathbb{R}^{d_1}} \chi_{E^y}(x)dx$ is measurable in $\mathbb{R}^{d_2}$.

iii) $\int_{\mathbb{R}^{d2}} \int_{\mathbb{R}^{d_1}} \chi_{E^y} = \int_{\mathbb{R}^d} \chi_E = m(E)$

$\square$

Note there are similar statements for the x sections by symmetry. Now let us consider the converse. This does not exist and we will prove this by counterexample.

Example 18.5. *Consider a non-measurable set N and the set*

$$E = [0,1] \times N \subseteq \mathbb{R} \times \mathbb{R} \tag{18.13}$$

For a fixed $y \in N$ defined

$$E^y = \begin{cases} [0,1] & \text{if} \quad y \in N \\ \phi & \text{if} \quad y \notin N \end{cases} \tag{18.14}$$

Note that $E^y \subset [0,1]$. E^y is measurable for every Y. However, if E were measurable then by the above corrolary, E_x is measurable, but $E_x = N$ which is nonmeasurable.

18.2 Convolution

Definition 18.6 (Convolution of Two Integrable Functions)**.** *Let f,g be integrable complex valued functions. Let*

$$h(x) = (f * g)(x) = \int f(x-y)g(y)dy \tag{18.15}$$

Proposition 18.7. *Suppose that given $f, g \in L^1(\mathbb{R})$. Then*

i) $\int |f(x-y)g(y)|dy < \infty$ *for almost all x.*

ii) Define $h(x) = \int f(x-y)g(y)dy$*. Show that*

- *a)* $h \in L^1(\mathbb{R})$
- *b)* $\|h\|_1 \leq \|f\|_1 \|g\|_1$

Proof. First notice that $F(x,y) = f(x-y)g(y)$ is measurable since f and g are measurable. To see that $f(x-y)$ is measurable, let $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $\phi(x,y) = x - y$ and then we can the composition of measurable functions to see that $f(\phi)$ is measurable.

First we claim that $F(x,y) \in L^1(\mathbb{R}^2)$.

$$\iint |F(x,y)|dxdy \leq \iint |f(x-y)| \cdot |g(y)|dxdy \tag{18.16}$$

$$\int |f(x-y)| \leq \int |f(x)|dx = \|f\|_1 \tag{18.17}$$

$$\implies \iint |F(x,y)|dxdy \leq \|f\|_1 \int |g(y)|dy = \|f\|_1 \cdot \|g\|_1 < \infty \tag{18.18}$$

Since $F(x,y) \in L^1(\mathbb{R}^2)$, we can apply Fubini's theorem. This gives us that

$$\iint |F(x,y)|dxdy = \iint |F(x,y)|dydx < \infty \tag{18.19}$$

This implies that $|F(x,y)|dy < \infty$ for almost all $x \in \mathbb{R}$. This means that h is defined for almost all x.

Next we show $h \in L^(\mathbb{R})$. We can see that

$$\begin{aligned} \|h\|_1 = \int |h(x)|dx &= \int \left| \int f(x-y)g(y)dy \right| dx \\ &\leq \iint |f(x-y||g(y)|dydx \leq \|f\|_1 \|g\|_1 \end{aligned} \tag{18.20}$$

$\square$

We have proved that $h(x)$ is finite for almost all x. Note that the product of two L^1 functions is not necessarily in L^1. An example is

$$f(x) = g(x) = \begin{cases} \frac{1}{\sqrt{x}} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{18.21}$$

We have shown that if $f, g \in L^1$, then $f * g \in L^1$. So here are the properties of convolutions.

1. $f, g \in L^1 \implies f * g \in L^1$
2. $f * g = g * f$
3. $f * (g * h) = (f * g) * h$
4. $f * (g + h) = (f * g) + (f * h)$

18.3 Convolution and Fourier Series

Suppose that $f, g \in L^1[0,1]$, periodic. Consider

$$F(x) = \int_0^1 f(x-t)g(t)dt = \int_0^1 f(t)g(x-t)dt \tag{18.22}$$

We have shown that F exists for almost all x. We also showed that it is integrable on $[0,1]$.

$$\begin{aligned} \int_0^1 F(x)e^{-2\pi inx}dx &= \int_0^1 \left[\int_0^1 f(x-t)g(t)dt\right] e^{-2\pi inx}dx \\ &= \int_0^1 g(t)e^{-2\pi int} \int_0^1 f(x-t)e^{-2\pi in(x-t)}dx \end{aligned} \tag{18.23}$$

Eventually we will show that the Fourier transform of F is the Fourier transform of f multiplied by the Fourier transform of g, which is the Fourier transformation of the convolution of f and g.

19 11/12/13

We are first going to talk about Fourier transforms.

19.1 Fourier Transform

We can give the Fourier transform as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi ix\xi}dx \tag{19.1}$$

We can get the function back from the Fourier Inversion Formula given as

$$\hat{f}(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{-2\pi ix\xi}d\xi \tag{19.2}$$

Proposition 19.1. *IF $f \in L^1(\mathbb{R}^d)$, then $\hat{f}$ is continuous and bounded on $\mathbb{R}^d$.*

Proof. We can see that

$$\left|f(x)e^{-2\pi x\xi}\right| = |f(x)| \tag{19.3}$$

which implies that

$$\sup_{\xi \in \mathbb{R}^d} |\hat{f}(\xi)| \le \int_{\mathbb{R}^d} |f(x)|dx = \|f\|_1 < \infty \tag{19.4}$$

So we will now show continuity. Suppose we have a sequence $\xi_n \to \xi_0$, where $\xi_n \in \mathbb{R}^d$. Then for every x, we have that

$$f(x)e^{-2\pi i x\xi_n} \to f(x)e^{-2\pi i x \xi_0} \tag{19.5}$$

Then by DCT, we have that $\hat{f}(\xi_n) \to \hat{f}(\xi_0)$ and this implies that $\hat{f}$ is continuous. $\square$

Example 19.2 (Problem 22 pg 94)**.** *Prove that if $f \in L^1(\mathbb{R}^d)$ and $\hat{f}(\xi)$ is defined as usual. Then show that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. This is also known as Riemann-Lebesgue lemma or at least very similar.*
Hint: Write $\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} [f(x) - f(x-\xi']e^{-2\pi i x\xi}dx$ where $\xi' = \frac{1}{2}\frac{\xi}{|\xi|^2}$ and use proposition 2.5 from book.

Note that the Riemann Lebesgue lemma is usually formulated as $\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)dx = 0$.
Proposition 2.5 says that

Proposition 19.3 (Continuity in Ł^1 norm)**.** *If $f \in L^1$, then $\|f_h - f\|_1 \to 0$ as $h \to 0$.*

Proof. Write

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}dx = \int_{\mathbb{R}^d} f(x-\xi')e^{-2\pi i(x-\xi')\xi}dx \tag{19.6}$$

which we can do by translational invariance where $\xi' = \frac{1}{2}\frac{\xi}{|\xi|^2}$ So now note that if we rewrite the exponential with the proper substitution we have that

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)e^{2\pi i x\xi}dx &= \int_{\mathbb{R}^d}\left[f(x) - \frac{\xi}{2|\xi|^2}\right]e^{-2\pi i x\xi}(-1) \\
\implies \hat{f}(\xi) &= \frac{1}{2}\int_{\mathbb{R}^d}[f(x) - f(x-\xi']e^{-2\pi i x\xi}dx \qquad (19.7)\\
\implies |\hat{f}(\xi)| &\le \frac{1}{2}\int_{\mathbb{R}^d} |f(x) - f(x-\xi']|\,dx \\
&\le \frac{1}{2}\left\|\left(f(x) - \frac{\xi}{2|\xi|^2}\right)\right\|_1 \qquad (19.8)
\end{aligned}$$

So in the limit as $|\xi| \to \infty$ we have by proposition 2.5 that $|\hat{f}(\xi)| \to 0$. $\square$

In general if $f \in L^1(\mathbb{R}^d)$, we do not know $\hat{f} \in L^1(\mathbb{R}^d)$. But we do have the following theorem.

Theorem 19.4. *Suppose that $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$. Then the inversion formula*

$$\hat{f}(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{-2\pi i x\xi}d\xi \tag{19.9}$$

holds a.e.

19.2 Motivation for study of Differentiation

We want to have some idea that differentiation and integration are inverse operations. So we have some questions

1. For which F on$[a, b]$ does the formula

$$F(b) - F(a) = \int_a^b F'(x)dx \tag{19.10}$$

hold? If $F'(x)$ integrable, then at least we will need $F'(x)$ to exists a.e in $[a, b]$. It turns out that this alone is not enough. We can see this by recalling the Cantor-Lebesgue function which has $f'(x) = 0$ a.e and $f(1) = 1$, $f(0) = 0$. Thus we have that

$$\int_0^1 f'(x)dx = 0 \tag{19.11}$$

$$f(1) - f(0) = 1 \tag{19.12}$$

So we need more conditions. To answer this question, we need to introduce the concept of bounded variation. and we will prove that if we have this concept then the derivative exists almost everywhere.

2. Suppose f is differentiable on $[a, b]$ and let F be its definite integral, meaning that

$$F(x) = \int_a^x f(y)dy \tag{19.13}$$

The questions we need to ask are, if the function F differentiable (at least a.e x)? Furthermore, if F is differentiable, is $F'(x) = f(x)$? These questiosn will be resolved with Hardy-Littlewood maximal function and Lebesgue differentiation theorem.

19.3 Differentiation of an Integral

Given $F(x) = \displaystyle\int_a^x f(y)dy$ where $a \le x \le b$. To find $F'(x)$ we must first form the quotient

$$\frac{F(x+h) - F(x)}{h} \tag{19.14}$$

and then take the limit as $h \to 0$.

So we can see that if h positive,

$$F(x+h) - F(x) = \int_a^{x+h} f(y)dy - \int_a^x f(y)dy = \int_x^{x+h} f(y)dy \tag{19.15}$$

So let $I = (x, x+h)$ and note that $|I| = h$. Then we can see that

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)dy = \frac{1}{|I|}\int_I f(y)dy \tag{19.16}$$

which is an average value. So we have that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{|I|\to 0} \frac{1}{|I|}\int_I f(y)dy = f(x) \quad a.e\ x \tag{19.17}$$

To study this question in higher dimensions, we will replace I with balls B containing x and we will replace $|I|$ with $m(B)$.

19.3.1 In $\mathbb{R}^d$

The question is really an averaging problem. Suppose f is integrable in $\mathbb{R}^d$. When does the following equality hold for a.e x

$$\lim_{m(B)\to 0} \frac{1}{m(B)}\int_B f(y)dy = f(x) \tag{19.18}$$

where the limit is taken over the measure of all open balls containing x.

Claim 19.5. *If f is continuous at x, then the previous question is answered in the affirmative.*

Proof. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$. So we look at

$$\begin{aligned} f(x) - \frac{1}{m(B)} \int_B f(y)dy &= \frac{1}{m(B)} \int_B [f(x) - f(y)]dy \\ &\leq \frac{1}{m(B)} \int_B |f(x) - f(y)|dy \end{aligned} \tag{19.19}$$

So whenever B is a ball with $r < \frac{\delta}{2}$ that contains x, then

$$\frac{1}{m(B)} \int_B |f(x) - f(y)|dy < \epsilon \tag{19.20}$$

□

Definition 19.6 (The Hardy-Littlewood Maximal Function)**.** *Let $f \in L^1(\mathbb{R}^d)$. We define its maximal function f^* as*

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)|dy \tag{19.21}$$

where $x \in \mathbb{R}^d$ and the sup *is taken over all balls containing x.*

Theorem 19.7. *Suppose f is integrable on $\mathbb{R}^d$, then*

i) f^* is measurable.

ii) $f^*(x) < \infty$ a.e x.

iii) f^* satisfies the following inequality called the Hardy-Littlewood inequality

$$m(\{x \in \mathbb{R}^d : f(x) > \alpha\}) \leq \frac{A}{\alpha} \|f\|_{L^1(\mathbb{R}^d)} \tag{19.22}$$

This inequality holds for all $\alpha > 0$ where $A = 3^d$.

Proof. We prove each part.

i) Look at the set $E_\alpha = \{x : f^*(x) > \alpha\}$. If we can show that E_α is open, we are done. Note that $\bar{x} \in E_\alpha$ and for $x \in B$,

$$\frac{1}{m(B)} \int_B |f(y)|dy > \alpha \tag{19.23}$$

If we take $\bar{x}$ close to x¡ then $\bar{x} \in B \subset E_\alpha$ which implies that E_α is an open set.

ii) Note that $\{x : f^*(x) = \infty\} \subset \{x : f^*(x) > \alpha\}$ for all α. By part 3),

$$m\big(\{x : f^*(x) > \alpha\}\big) < \frac{A}{\alpha} \|f\|_1 \tag{19.24}$$

when $\alpha \to \infty$, we get required result.

iii) Set again $E_\alpha = \{x : f^*(x) > \alpha\}$, $x \in E_\alpha$. Then there exists B_x such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|dy > \alpha \tag{19.25}$$

It is enough to prove this for $K \subset E_\alpha$ compact since E_α is measurable and therefore there exists K such that $m(E \setminus K) \leq \epsilon$. So since K is compact we know that

$$k \subset \bigcup_{x \in E_\alpha} B_x \implies k \subset \bigcup_{l=1}^{N} B_l \tag{19.26}$$

By Vitali's covering lemma, there exists disjoint ball $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ such that

$$K \subset \bigcup_{l=1}^{N} B_l \subset \bigcup_{j=1}^{k} 3B_{i_j} \tag{19.27}$$

This implies that

$$m(K) \leq m\left(\bigcup_{j=1}^{k} 3B_{i_j}\right) = 3\sum_{j=1}^{k} m(B_{i_j}) < \frac{3^d}{\alpha}\sum_{j=1}^{n}\int_{B_{i_j}} |f(y)|dy < \frac{3^d}{\alpha}\int_{\mathbb{R}^d} |f(y)|dy \tag{19.28}$$

□

The Hardy-Littlewood inequality is also called the weak type inequality because it is weaker then the Tchebychev inequality.
Also note that the constant A is independent of α and f. This also implies that

$$f^*(x) \geq |f(x)| \tag{19.29}$$

for a.e x, and we will see this later. However, it is not that much larger.

20 1/19/13

Received a packet regarding the Lebesgue Differentiation Theorem. We continue with this subject.

20.1 Lebesgue Differentiation Theorem

We need some tools to prove this.

Theorem 20.1 (Vitali's Covering Argument)**.** *Suppose $\mathcal{B} = \{B_1, B_2, \ldots, B_3\}$ is a finite collection of open balls in $\mathbb{R}^d$. Then there exists a finite disjoint subcollection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of $\mathcal{B}$ that satisfies*

$$\bigcup_{l=1}^{N} B_l \subset \bigcup_{j=1}^{k} 3B_{i_j} \tag{20.1}$$

In particular by finite subadditivity

$$m\left(\bigcup_{l=1}^{N} B_l\right) \leq e^d \sum_{j=1}^{k} m(B_{i_j}) \tag{20.2}$$

Note that the proof of this is done by a greedy algorithm argument, selecting the balls B_{i_j} to be as large as possible while remaining disjoint. Proof debends on the following observation:
If B and B' are a pair of balls that intersect with the radius of B' not greater than the radius of B, then B' is contained in $\tilde{B}$ where $\tilde{B}$ is concentric with B and the radius of $\tilde{B}$ is three times the radius of B.
Note that given an open ball $B = B(x, r)$ in $\mathbb{R}^d$, and a real number $c > 0$, we write

$$cB := B(x, cr) \tag{20.3}$$

for the ball with the same center as B but c times the radius. Moreover,

$$m(cB) = c^d m(B) \tag{20.4}$$

There is a family of related covering lemmas which are useful for a variety of tasks in harmonic analysis. The estimate obtained in part c) of the properties of f^* leads to a solution of the averaging problem which is the Lebesgue differentiation theorem in $\mathbb{R}^d$.

Theorem 20.2 (Lebesgue Differentiation Theorem)**.** *If* $f \in L^1(\mathbb{R}^d)$*, then*

$$\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B f(y)dy = f(x) \quad a.x\ x \in B \tag{20.5}$$

Proof. Step 1. For each $\alpha > 0$ and $x \in B$, set

$$E_\alpha = \left\{ x : \limsup m(B) \to 0 \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| > 2\alpha \right\} \tag{20.6}$$

We claim that $m(E_\alpha) = 0$ because this assertion them implies that the set $E = \bigcup_{n=1}^{\infty} E_{1/n}$ has $m(E) = 0$. then the conclusion of LDT holds on E^c.

Step 2. Fix α and recall the approximation theorem for L^1 function which states that for each $\epsilon > 0$ we can select a continuous function g with compact support such that $\|f - g\|_1 < \epsilon$. Furthermore, since g is a continuous function, the averaging problem has an affirmative answer for g which means that

$$\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B g(y)dy = g(x) \quad a.e\ \ x \in B \tag{20.7}$$

Step 3. Rewrite the difference $\frac{1}{m(B)} \int_B f(y)dy - f(x)$ as follows:

$$\frac{1}{m(B)} \int_B [f(y) - f(y)]dy + \frac{1}{m(B)} \int_B g(y)dy - g(x) + g(x) - f(x) \tag{20.8}$$

Taking absolute value of both sides and taking lim sup as $m(B) \to 0$ and using the triangle inequality we have for $x \in B$

$$\begin{aligned} &\limsup_{m(B)\to 0} \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| \\ \le &\limsup_{m(B)\to 0} \frac{1}{m(B)} \int_B |f(y) - g(y)|dy * \left| \limsup_{m(B)\to 0} \frac{1}{m(B)} \int_B g(y)dy - g(x) \right| + |g(x) - f(x)| \end{aligned} \tag{20.9}$$

Note the first part is $(f - g)^*(x)$ and the second part is 0 since g is continuous. So we have that

$$\limsup_{m(B)\to 0} \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| \le (f - g)^*(x) + |g(x) - f(x)| \tag{20.10}$$

where by $(f - g)^*$ we mean the maximal function of $(f - g)$.

Step 4. Setting $F_\alpha = \{x : (f - g)^*(x) > \alpha\}$ and $G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$, we find the measure of these sets. For $m(F_\alpha)$, we use the weak estimate and we get that since $\|f - g\|_1 < \epsilon$

$$m(F_\alpha) \le \frac{A}{\alpha} \|f - g\|_1 \le \frac{A}{\alpha} \epsilon \tag{20.11}$$

For $m(G_\alpha)$, we can use Tchebychev's inequality to get that

$$m(G_\alpha) \le \frac{1}{\alpha} \|f - g\|_1 \le \frac{1}{\alpha} \epsilon \tag{20.12}$$

Step 5. Observe that $E_\alpha \subset F_\alpha \cup G_\alpha$. This is because if k_1, k_2 are positive and if $k_i > \alpha$, then $k_1 + k_2 > 2\alpha$. So we have that

$$m(E_\alpha) \le m(F_\alpha \cup G_\alpha) \le \frac{A}{\alpha}\epsilon + \frac{\epsilon}{\alpha} \tag{20.13}$$

Since ϵ was arbitrary, we must have that $m(E\alpha) = 0$ and the proof is complete.

□

Corollary 20.3. *If $f \in L^1(\mathbb{R}^d)$, then $f^*(x) \geq |f(x)|$ for a.e x.*

Proof. Apply LDT to $|f|$ to conclude for $x \in B$

$$\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B |f(y)|dy = |f(x)| \tag{20.14}$$

a.e x. This implies that $f^*(x) \geq |f(x)|$ a.e x. □

20.1.1 Local Integrability

LDT assumes that $f \in L^1(\mathbb{R}^d)$ which is a global assumption, however, differentiation is a local notion. Thus assuming $f \in L^1(\mathbb{R}^d)$ might not be necessary. Furthermore, the limit in LDT is over balls B that shrink to the point x, thus the behavior of f far from x could be irrelevant. To answer these concerns we define f to be locally integrable.

Definition 20.4 (Locally Integrable)**.** *A measurable function f on $\mathbb{R}^d$ is said to be locally integrable if for every ball B¡ the function $f(x)\chi_B(x)$ is integrable. In particular, if f is a measurable function of $\mathbb{R}$, we say f is locally integrable if $\int_a^b |f| < \infty$ for every bounded interval $[a, b]$.*

We will denote $L^1_{loc}(\mathbb{R}^d)$ the space of all locally integrable functions on $\mathbb{R}^d$.
The conclusion of LDT holds under the weaker assumption that f is locally integrable.

Theorem 20.5. *If $f \in L^1_{loc}(\mathbb{R}^d)$, then for $x \in B$,*

$$\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B f(y)dy = f(x) \tag{20.15}$$

for a.e x.

20.1.2 Lebesgue Differentiation Theorem in $\mathbb{R}$

We now consider the simpler case where we are in $\mathbb{R}$.

Theorem 20.6 (Lebesgue Differentiation Theorem in $\mathbb{R}$)**.** *If $f \in L^1(\mathbb{R})$, with $F(x) = \int_0^x f(t)dt$, then $F'(x) = f(x)$ a.e x.*

Proof. We have to show that the set of x such that

$$\lim_{y\to x} \left| \frac{F(y) - F(x)}{y - x} - f(x) \right| > 0 \tag{20.16}$$

has measure 0. It is enough to show that for each $\alpha > 0$ and $|epsilon > 0$ the set

$$E = \left\{ x : \lim_{y\to x} \left| \frac{F(y) - F(x)}{y - x} - f(x) \right| > \alpha \right\} \tag{20.17}$$

has measure 0. Since $f \in L^1(\mathbb{R})$, we can choose a continuous function g with $\|f - g\|_1 < \epsilon$. If we set $G(x) = \int_a^x g(y)dy$, since g is continuous, the fundamental theorem of calculus with cold and we can assert that $G'(x) = g(x)$.
Now given x, y with $x \neq y$, let I be the interval end points x and Y, then using the triangle inequality we get that

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| \leq \underbrace{\left| \frac{F(y) - F(x)}{y - x} - \frac{G(y) - G(x)}{y - x} \right|}_{E_1} + \underbrace{\left| \frac{G(y) - G(x)}{y - x} - g(x) \right|}_{E_2} + \underbrace{|g(x) - f(x)|}_{E_3} \tag{20.18}$$

Note that the second term goes to 0 in the limit of $y \to x$. For the third term we have by Tchebychev's inequality

$$m(E_3) = m(\{x \in \mathbb{R} : |g(x) - f(x)| > \alpha\}) \leq \frac{1}{\alpha}\|f-g\|_1 \leq \frac{\epsilon}{\alpha} \tag{20.19}$$

Now note the first term

$$\frac{F(y)-F(x)}{y-x} = \frac{\int_0^y f(t)dt - \int_0^x f(t)dt}{|I|} = \frac{1}{|I|}\int_I f(t)dt \leq (f-g)^* \tag{20.20}$$

Applying the weak inequality we have that

$$m(E_1) = m(\{x \in \mathbb{R} : (f-g)^* > \alpha\}) \leq \frac{A}{\alpha}\|f-g\|_1 \leq \frac{A}{\alpha}\epsilon \tag{20.21}$$

Now observe $E \subset E_1 \cup E_3$ and the conclusion follows. □

21 11/21/13

We are going to do things about density.

21.1 Lebesgue Density Theorem

Definition 21.1 (Point of Density). *Given a Lebesgue measurable set $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ is called a point of density for E if*

$$\frac{m(E \cap B(x,r))}{m(B(x,r))} \to 1 \quad \textit{as} \quad r \to 0 \tag{21.1}$$

Example 21.2. $E = [-1,1] \setminus \{0\}$. *Then every point in $(-1,1)$ including 0 is a point of density for E. However, end points $-1, 1$ as well as the exterior points of E are not points of density.*

Note that points of density are almost interior points. We cannot necessarily fit the ball in the set, but we almost can.

Theorem 21.3 (Lebesgue Density Theorem). *Let $E \subset \mathbb{R}^d$ be measurable.*

i) almost all $x \in E$ is a point of density of E.

ii) Almost all $x \notin E$ is not a point of density of E.

Proof. Take $f = \chi_E$ in the sense of the proof of the Lebesgue Differentiation Theorem. Then for $x \in B$

$$\lim_{m(B)\to 0} \frac{1}{m(B)}\int_B f(y)dy = f(x) \tag{21.2}$$

a.e. So we have that

$$\frac{1}{m(B)} \to 0 \int_B \chi_E = \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \tag{21.3}$$

□

21.2 Almost Everywhere Differentiability

We know that not every continuous function $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

Example 21.4. $f(x) = |x|$ *is not differentiable at $x = 0$. However, it is differentiable everywhere else and is therefore a.e differentiable.*

We can construct continuous funtions that are nowhere differentiable.

Example 21.5 (Weierstrass Function). *Let $F : \mathbb{R} \to \mathbb{R}$. where*

$$F(x) = \sum_{n=1}^{\infty} 4^{-n} \sin(8^n \pi x) \tag{21.4}$$

This function is well defined, bounded, continuous, but not differentiable at any point $x \in \mathbb{R}$.

The point is that continuous functions can contain a large amount of oscillation which can lead to a breakdown of differentiability. However, if we limit the amount of oscillation, we can often recover differentiability.

Theorem 21.6 (Monotone Differentiation Theorem). *Any function $F : \mathbb{R} \to \mathbb{R}$ which is monotone in either sense is differentiable a.e.*

The proof needs Dini Derivatives, and we will not actually prove this.

21.3 Functions of Bounded Variation

While this will relate to the previous part of almost everywhere differentiation, we are almost motivated by an example of length of curves.

Example 21.7. *Suppose $f(t) = (x(t), y(t))$ for $a \leq t \leq b$. what is the length of this curve?*

We may consider a polygonal approximation of f with nodes $a = t_0 < t_1, \cdots < t_n = b$, find the length of the approximating polygon.

$$\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\| \tag{21.5}$$

If we let $\gamma = f(t)$, then we take a refinement, and we get that

$$L(\gamma) = \sup_{P} \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\| \tag{21.6}$$

where P is a partition. If $x(t)$ and $y(t)$ are "nice" then this definition will work fine.

Now suppose that $F(t)$ is a complex valued function defined on $[a, b]$ where $a = t_0 < t_1 < \cdots < t_N < b$ is a partition of $[a, b]$. Let $V_a^b(F, P)$ denote the variation of F on this partition P.

$$V_a^b(F, P) = \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| \tag{21.7}$$

Definition 21.8 (Bounded Variation). *F is said to be a function of Bounded Variation if the variation over all partitions is bounded. So, there exists $M < \infty$ such that*

$$V_a^b(F, P) = \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| < M \tag{21.8}$$

for all partitions P, or we can also define

$$V_a^b(F) = \sup_{P} V_a^b(F, P) < M \tag{21.9}$$

these are equivalent. When we write $BV[a, b]$ mean the collection of all functions of bounded variation on $[a, b]$.

We will assume that in this case

$$F(t) = z(t) = (x(t), y(t)) = x(t) + iy(t) \tag{21.10}$$

Definition 21.9 (Rectifiable Curve). *A curve $\gamma = z(t)$ is called rectifiable if there exists $M < \infty$ so that for any partition $a = t_0 < t_1 < \cdots < t_N = b$ of $[a, b]$*

$$\sum_{j=1}^{N} |z(t_j) - z(t_{j-1})| \leq M \tag{21.11}$$

Theorem 21.10. *A parametrized curve $\gamma(t)$ is rectifiable if and only if both $x(t)$ and $y(t)$ are of bounded variation.*

Proof. Let $F(t) = x(t) + iy(t)$ which implies that

$$F(t_j) - F(t_{j-1}) = [x(t_j) - x(t_{j-1})] + i[y(t_j) - y(t_{j-1})] \tag{21.12}$$

and recall that the inequalities

$$|a + ib| \leq |a| + |b| \leq 2|a_i b| \tag{21.13}$$

with a, b real implies that

$$|x(t_j)| - x(t_{j-1})| + |y(t_j) - y(t_{j-1})| \leq 2|F(t_j) - F(t_{j-1})| \tag{21.14}$$

□

Theorem 21.11. *Suppose F is differentiable at every point, and F^1 is bounded then F is of BV.*

Proof. Suppose $|F^1(x)| \leq M$ for every $x \in [a, b]$. By the mean-value theorem

$$|F(x) - (y)| \leq M|x - y| \tag{21.15}$$

for all $x, y \in [a, b]$. Thus, take any partition of $[a, b]$, say $a_0 = t_0 < t_1 < \cdots < t_N = b$. Then

$$\begin{aligned} \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| &= \sum_{j=1}^{N} |F'(c)(t_j - t_{j-1}| \\ &\leq M \sum_{j=1}^{N} (t_j - t_{j-1}) = M(b - a) \end{aligned} \tag{21.16}$$

Thus $F \in BV[a, b]$. □

Note that not even bounded continuous functions are BV.

Example 21.12. *Let*

$$F(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \textit{for } \ 0 < x \leq 1 \\ 0 & \textit{for } \ x = 0 \end{cases} \tag{21.17}$$

F is not of bounded variation, even though it is continuous.

Proof. Fix n and pick a partition P on $[0, 1]$ containing the points $t_k = \dfrac{1}{(2k+1)\pi}$ for $k = 0, 1, \ldots, n$. Notice that

$$F(t_k) = \frac{2}{(2k+1)\pi} \sin \frac{(2k+1)\pi}{2} = (-1)^k t_k \tag{21.18}$$

Then we can see that

$$|F(t_{k+1}) - F(t_k)| = t_{k+1} + t_k \geq 2t_{k+1} = 2\frac{2}{[2(k+1)+1]\pi} \geq \frac{3}{3\pi}\frac{1}{k+1} \tag{21.19}$$

Consequently $V_a^b(F, P) \geq \dfrac{4}{3\pi} \displaystyle\sum_{k=1}^{N-1} \frac{1}{k+1} \to 0$ as $N \to \infty$. □

21.3.1 Some observations about $V_a^b(F,P)$

We have the following observations

1. $F \in BV[a,b]$ and if $[a_0,b_0] \subset [a,b]$, then $F \in BV[a,b]$. And
$$V_{a_0}^{b_0}(F,P) \leq V_a^b(F,P) \tag{21.20}$$

2. Suppose Q is a refinement of the partition P, then
$$|F(b)-F(a)| \underbrace{\leq}_{\{a,b\}} V_a^b(F,P) \leq V_a^b(F,Q) \tag{21.21}$$

3. Suppose that $f \in BV[a,c]$ and $f \in BV[c,b]$ where $a<c<b$. Then $f \in BV[a,b]$ and
$$V_a^b(F) = V_a^c(F) + V_c^b(F) \tag{21.22}$$

22 11/26/13

We continue our study of almost everywhere differentiability, particularly bounded variation and total variation. First note the following

1. Continuity of F does not imply f is of bounded variation.
2. If F is real valued, monotonic and bounded, then F is of bounded variation.
3. If F is differentiable at every point and F' is bounded, then F is of bounded variation.

We now have the following theorem.

Theorem 22.1. *A real valued function F on $[a,b]$ is of bounded variation if and only if F is the difference of two increasing bounded functions.*

in order to prove this we need some concepts of different types of variation.

22.1 Different Variations

We will now define some concepts of variation.

Definition 22.2 (Total Variation)**.** *For $x \in [a,b]$, the* total variation *of F over $[a,b]$ is given as*
$$T_F(a,x) = \sup \sum_{j=1}^{N} |f(t_j) - F(t_{j-1})| \tag{22.1}$$
where the supremum is taken over all partitions of $[a,x]$.

This leads to the following properties.

1. $T_F(a,x)$ is an increasing function of x.
2. $T_F(a,x)$ is continuous at x if and only if F is continuous at x.
The idea of the proof depends on the observation that if we let $x<y$, then
$$|F(x)-F(y)| \leq T_F(x,y) = T_F(a,y) - T_F(a,x) \tag{22.2}$$

We now define positive and negative variation.

Definition 22.3 (Positive Variation)**.** *Assume F is real valued function. The the* positive variation *of F on $[a, x]$ is*

$$P_F(a, x) = \sup \sum_{(+)} F(t_j) - F(t_{j-1}) \tag{22.3}$$

where the sum is taken over all j with $F(t_j) \geq F(t_{j-1})$

Note that $P_F(a, x)$ is also an increasing function of x and it is bounded.

Definition 22.4 (Negative Variation)**.** *Assume F is a real valued function, then the* negative variation *of F on $[a, x]$ is defined by*

$$N_F(a, x) = \sup \sum_{(-)} - [F(t_j) - F(t_{j-1})] \tag{22.4}$$

where the supremum is taken over all j with $F(t_j) \leq F(t_{j-1})$.

Note that $N_F(a, x)$ is an increasing function of x and it is bounded. The following lemma gives the relationship between the three theorems.

Lemma 22.5. *Suppose F is real valued and of bounded variation on $[a, b]$. Then for all $a \leq x \leq b$, we have*

1. *$F(x) - F(a) = P_F(a, x) - N_F(a, x)$*
2. *$T_F(a, x) = P_F(a, x) + N_F(a, x)$*

We can now prove the larger theorem which we express here with its more common name.

Theorem 22.6 (Jordan Decomposition Theorem)**.** *A real valued function F on $[a, b]$ is of bounded variation if and only if F is the difference of two increasing bounded functions.*

Proof. $\Leftarrow$ If $F = F_1 - F_2$ where both F_1 and F_2 are bounded and increasing, then $F \in BV[a, b]$.
$\Rightarrow$ Suppose that F is of bounded variation. Then we let

$$F_1(x) = P_F(a, x) + F(a) \tag{22.5}$$

$$F_2(x) = N_F(a, x) \tag{22.6}$$

Both $F_1(x)$ and $F_2(x)$ are increasing and of bounded variation and

$$\begin{aligned} F_1(x) - F_2(x) &= P_F(a, x) + F(a) - N_F(a, x) \\ &= P_F(a, x) - N_F(a, x) + F(a) \\ &= F(x) - F(a) + F(a) \\ &= F(x) \end{aligned} \tag{22.7}$$

$\square$

So we can now prove the main theorem we are trying to prove which we will restate.

Theorem 22.7. *If F is of bounded variation on $[a, b]$, then F is differentiable a.e x. In other words, F' exists a.e. x*

First note note the following remarks.

1. F is differentiable a.e x means the quotient

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \tag{22.8}$$

   exists a.e. $x \in [a, b]$¿

2. By theorem 3.3 every $F \in BV$ can be written as $F = F_1 - F_2$ where F_1 and F_2 are bounded increasing functions, thus it is enough to prove (22.7) for increasing functions.

3. So we will assume that F is increasing and continuous. The continuity of F makes the argument simpler, then we turn to the question if $F \in BV$ what kind of discontinuities can it possibly have. The answer is that it can have at most countable many discontinuities.

4. When we focus on the case of F is continuous , this allows us to the use the *rising sun lemma*. To understand differentiability of F, one needs to introduce the four *Dini Derivatives* of F at x.

If F is both increasing and continuous, then we have the following corollary.

Corollary 22.8.

1. F' exists a.e

2. F' is measurable and non-negative.

3. $\int_a^b F'(x)dx \leq F(b) - F(a)$.

4. If F is bounded on $\mathbb{R}$, then F' is integrable.

Proof. For $n \geq 1$, set

$$G_n(x) = \frac{F\left(1 + \frac{1}{n}\right) - F(x)}{\frac{1}{n}} \tag{22.9}$$

We know that $\lim_{n\to\infty} G_n(x) = F'(x)$ a.e. And so F' is also measurable and non-negative. Now extend F to a continuous function on all of $\mathbb{R}$. Applying Fatou's lemma, we have that

$$\int_a^b F'(x)dx \leq \liminf \underbrace{\int_a^b G_n(x)dx} \tag{22.10}$$

We now look at the delineated term.

$$\begin{aligned}
\int_a^b G_n(x)dx = \int_a^b \frac{F\left(x + \frac{1}{n}\right)}{\frac{1}{n}}dx = \frac{1}{\frac{1}{n}} \int_a^b F\left(x + \frac{1}{n}\right) - \frac{1}{\frac{1}{n}} \int_a^b F(x)dx \\
= \frac{1}{\frac{1}{n}} \int_{a+1/n}^{b+1/n} F(u)du - \frac{1}{\frac{1}{n}} \int_a^b F(x)dx \\
= \frac{1}{\frac{1}{n}} \int_b^{b+1/n} F(x)dx - \frac{1}{\frac{1}{n}} \int_a^{a+1/n} F(x)dx
\end{aligned} \tag{22.11}$$

Since F is continuous, the first and second term converges to $F(b)$ and $F(a)$ respectively which completes the proof. We will show this for the first

$$\frac{1}{\frac{1}{n}} \int_b^{b+1/n} F(x)dx \leq \frac{1}{\frac{1}{n}} \left[\max_{x\in[b,b+1/n]} F(x)\right] \left[\left(b + \frac{1}{n}\right) - b\right] \tag{22.12}$$

Since F is increasing we have that $\max F(x) = F\left(b + \frac{1}{n}\right)$, and since F is continuous, $\lim_{n\to\infty} F\left(b + \frac{1}{n}\right) = F(b)$. ∎

Now we have the rising sun lemma

Lemma 22.9 (Rising Sun)**.** *Suppose G is real valued and continuous on $\mathbb{R}$. Let*

$$E = \{x \in \mathbb{R} : G(x + h) > G(x) \ \ \textit{for some } h = h_x > 0\} \tag{22.13}$$

If E is non-empty, then it must be open, hence

$$E = \bigcup (a_k, b_k) \tag{22.14}$$

a disjoint union of open intervals. If (a_k, b_k) is a finite interval in this union, then $G(a_k) = G(b_k)$.

The name can be explained by imagining the graph of $\{(x, G(x)) : x \in E\}$ as depicting a hilly landscape with sun shining horizontally from the rightward infinity. Sun is rising form the east with lights parallel to the x-axis. Those x for which $G(y) \leq G(x)$ are the locations on the landscape which are illuminated by the sun. the interval $I_k = (a_k, b_k)$, then represents the portions of the landscape that are in shadow.

Corollary 22.10. *Suppose that G is continuos and real valued on $[a, b]$. Let $E = \{x \in (a, b) : G(x + h) > G(x)\}$ then E is either empty or open. When E is open it is a disjoint union of countably many intervals (a_k, b_k) and $G(a_k) = G(b_k)$, except possibly when $a = a_k$, in this case $G(a_k) \leq G(b_k)$.*

We can now offer a restaed version of the Rising sun lemma.

Lemma 22.11 (Rising Sun-version 2)**.** *Let $[a, b]$ be a compact interval and $G : [a, b] \to \mathbb{R}$ is continuous. The one can find an at most countable family of disjoint non-empty open intervals $I_n = (a_n, b_n)$ in $[a, b]$ with the following properties*

i) For each n¡ either $G(a_n) = G(b_n)$ or else $a = a_n$ and $G(b_n) \geq G(a_n)$.

ii) If $x \in [a, b]$ does not lie in any of the intervals I_n, then one must have $G(y) \leq G(x)$ for all $x \leq y \leq b$.

Definition 22.12 (Dini Derivatives)**.** *We first define*

$$\Delta_h F(x) = \frac{F(x + h) - F(x)}{h} \tag{22.15}$$

We now define the four Dini Derivatives of F at x.

$$D^+F(x) = \limsup_{h \to 0^+} \Delta_h F(x) \tag{22.16}$$

$$D_+F(x) = \liminf_{h \to 0^+} \Delta_h F(x) \tag{22.17}$$

$$D^-F(x) = \limsup_{h \to 0^-} \Delta_h F(x) \tag{22.18}$$

$$D_-F(x) = \liminf_{h \to 0^-} \Delta_h F(x) \tag{22.19}$$

We have the following properties.

1. Regardless of whether F is differentiable or not or even whether it is continuous, the four Dini derivatives always exist and takes values in the extended real line. Note that if F is defined only on interval $[a, b]$, rather than on the endpoints, then some of the Dini derivatives amy not exist as the endpoints, but this is a measure zero set and will not impact our analysis.

2. $D_+ \leq D^+$ and $D_- \leq D^-$.

We can now prove (22.7).

Proof. We assume that F is increasing, bounded, and continuous on $[a, b]$.
To show that F is differentiable on a.e x, we need to show that all four Dini derivatives are finite and equal a.e x. To show that all four Dini derivatives are equal a.e. x and finite, it suffices to show that

i) $D^+F(x) < \infty$ a.e x

ii) $D^+F(x) \leq D_-F(x)$ a.e x

To see this claim, if we apply $ii)$ to $-F(-x)$ instead of $F(x)$ and recalling that

$$\limsup(-a_n) = -\liminf(a_n) \tag{22.20}$$

, then we get that $D^-F(x) \leq D_+F(x)$ a.e, thus

$$D^+ \leq D_- \leq D^- \leq D_+ \leq D^+ < \infty \tag{22.21}$$

a.e.
For a fixed $\gamma > 0$, let $E_\gamma = \{x : D^+F(x) > \gamma\}$. Observe E is measurable, then apply rising sun lemma to the function

$$G(x) = F(x) - \gamma(x) \tag{22.22}$$

We then have $E_\gamma \subset \cup_k (a_k, b_k)$, where

$$\begin{aligned} F(b_k) - F(a_k) &\geq \gamma(b_k - a_k) \\ \implies m(E_\gamma) \leq \sum_k m(a_k, b_k) &\leq \frac{1}{\gamma} \sum_k F(b_k) - F(a_k) \leq \frac{1}{\gamma}(F(b) - F(a)) \\ \implies \lim_{n\to\infty} m(E_\gamma) &= 0 \end{aligned} \tag{22.23}$$

Since $\{x : D^+F(x) < \infty\} \subset E_\gamma$ for all γ, we know that $D^+F(x) < \infty$ for a.e x.
Now fix r and R such that $R > r$ and let

$$E = \{x \in [a, b] : D^+F(x) > R \text{ and } r > D_-F(x)\} \tag{22.24}$$

If we prove $m(E) = 0$, we assume $m(E) > 0$ and arrive at a contradiction.
proof has a few more steps, but they are not too bad and we don't have time. Look in Stein. □

23 12/3/13

Today we are going to look at absolutely continuous and singular functions. The motivation is we would want to find conditions under which the second form of the fundamental theorem of calculus

$$\int_a^b F'(x)dx = F(b) - F(a) \tag{23.1}$$

is true. We know it is not enough for a function to be continuous as we have seen for the Cantor-Lebesgue function which is increasing and continuous on $[0, 1]$, but

$$\int_0^1 F'(x)dx \neq F(1) - F(0) \tag{23.2}$$

Furthermore, if $F \in BV[0, 1]$, then $F'(x)$ exists a.e, but we don't know anything about (23.1).

23.1 Absolutely Continuous Functions

Definition 23.1 (Absolutely Continuous)**.** *A function F defined on $[a, b]$ is absolutely continuous if for any $\epsilon > 0$, there exists $\delta > 0$, such that*

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \epsilon \quad \textit{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta \tag{23.3}$$

and the intervals (a_k, b_k) are disjoint for $k = 1, 2, \ldots N$.

We will use $AC[a, b]$ for the set of absolutely continuous function on $[a, b]$.
Note that the requirement that the open intervals $\{(a_k, b_k)\}_{k=1}^N$ be disjoint is sometimes stated as non-overlapping. However, definition requires that

$$m\left(\bigcup_{k=1}^{N} (a_k, b_k)\right) < \delta \tag{23.4}$$

Example 23.2. *If f is integrable on $[a, b]$ and $F = \displaystyle\int_a^b f(y)dy$, then F is absolutely continuous*

This means that the indefinite integral is an absolutely continuous function. Note the main result we want to prove is the converse, however, we do offer a proof of the above.

Proof.

$$F(b_k) - F(a_k) = \int_a^{b_k} f(y)dy - \int_a^{a_k} f(y)dy = \int_{a_k}^{b_k} f(y)dy$$
$$\implies |F(b_k) - F(a_k)| \leq \int_{a_k}^{b_k} |f(y)|dy \leq \int_{\cup_{k=1}^N (a_k, b_k)} |f(y)|dy \tag{23.5}$$

Recall that if f is integrable on $\mathbb{R}^d$, then for all $\epsilon > 0$, there exists $\delta > 0$ such that $\int_E |f| < \epsilon$ whenever $m(E) < \delta$. So this shows that F is absolutely continuous. □

A result of this is that any Lipschitz function F is absolutely continuous. However, the family of absolutely continuous functions is large than the family of Lipschitz functions in the sense that functions can be absolutely continuous and no Lipschitz, for example $F(x) = \sqrt{x}$. Also, it follows directly from the definition that the sum or product of two absolutely continuous functions on $[a, b]$ remains absolutely continuous.

23.1.1 Properties of Absolutely Continuous Functions

1. Every absolutely continuous function is uniformly continuous and therefore continuous.
2. If $F \in AC[a, b]$, then $F \in BV[a, b]$.

Proof. Choose δ such that $\sum |F(b_i) - F(a_i)| \leq 1$ for any collection of non-overlapping interval with $\sum(b_i - a_i) \leq \delta$. Then the variation of F over any subinterval of $[a, b$ with length less than δ is at most 1. Hence if we split $[a, b]$ into N intervals, each with length less than δ, then $V_F[a, b] \leq N$. □

Now notice that

$$a = T_0 < t_1 < \cdots < t_n = b \tag{23.6}$$

such that $t_i - t_{j-1} < \delta$ for all i, then $V_F(t_{j-1}, t_i) \leq 1$, thus total variation of F

$$V_F[a, b] = \sum_{i=1}^N V_F(t_{j-1}, t_j) \leq N < \infty \tag{23.7}$$

We have also established that

$$Lip[a, b] \subset AC[a, b] \subset C[a, b] \tag{23.8}$$
$$AC[a, b] \subset BV[a, b] \tag{23.9}$$

23.2 Singular Function

Definition 23.3 (Singular Function)**.** *A function F for which $F'(x) = 0$ a.e in $[a, b]$ is called singular on $[a, b]$.*

Note that the Cantor-Lebesgue function is an example of non-constant singular function.
We can now combine the singular property with the absolutely continuous property to get the following theorem.

Theorem 23.4. *If F is both absolutely continuous and singular on $[a, b]$, then it is constant on $[a, b]$*

Proof. The proof is quite difficult, however, we offer an outline. It is enough to prove that $F(a) = F(b)$ since this result applied to any subinterval proves that F is constant. Since every absolutely continuous function is a function of bounded variation, it is therefore the difference of two continuous monotonic functions. For those functions, we already know $F'(x)$ exists a.e. x. Since F is also singular $F'(x) = 0$ a.e. To show that F is constant requires a more detailed version of Vitali's covering argument. □

23.3 Fundamental Theorem of Calculus

We can now rework the Fundamental Theorem of Calculus.

Theorem 23.5. *A function F is abosolutely continuous on $[a, b]$ if and only if $F'(x)$ exsts a.e $x \in (a, b)$, F' is integralbe on (a, b) and*

$$F(x) - F(a) = \int_a^x F' \quad (a \leq x \leq b) \tag{23.10}$$

By selecting $x = b$, we get

$$F(b) - F(a) = \int_a^b F'(y)dy \tag{23.11}$$

Proof. $\Leftarrow$ we have already observed that any function $G(x) = \displaystyle\int_a^x F'(y)dy$ where $F' \in L^1(a, b)$ is absolutely continuous. hence the sufficiency of the condition follows.
$\Rightarrow$ Suppose F is absolutely continuous and let $G(x) = \displaystyle\int_a^x F'(y)dy$. G is wee defined because absolutely continuous functions are of bounded variation and if $F \in BV[a, b]$, then F' exist a.e and $F' \in L[a, b]$. By Lebesgue differentiation theorem, $G' = F'$ a.e in $[a, b]$. It follows that $G(x) - F(x)$ both absolutely continuous and singular on $[a, b]$ and thus $G(x) - F(x)$ is constant on $[a, b]$. So we have that

$$G(x) - F(x) = c \tag{23.12}$$

$$\int_a^x F'(y) - F(x) = c \tag{23.13}$$

$$\implies \int_x^x F'(y)dy = F(x) + c$$

$$\text{at } x = a \implies 0 = F(a) + c$$

$$\implies c = -F(a)$$

$$\implies \int_a^x F'(y)dy = F(x) - F(a) \tag{23.14}$$

□

Theorem 23.6. *If $F \in BV[a, b]$, then F can be written as $F = G + H$, where G is absolutely continuous on $[a, b]$ and H is singular on $[a, b]$. Moreover, G and H are unique up to additive constants.*

Proof. Let $G(x) = \displaystyle\int_a^x F'(y)dy$ and $H = F - G$. Then $H' = F' - G' = F' - F' = 0$ a.e in $[a, b]$ which implies that H is singular on $[a, b]$ and the formula $F = G + H$ gives the desired decomposition.
If $F = G_1 + H_1$ is another such decomposition, then $G - G_1 = H_1 - H$ since $G - G_1$ is absolutely continuous and $H_1 - H$ is singular, it follows that

$$G - G_1 = H_1 - H = constant \tag{23.15}$$

□

23.4 Integration by Parts

We have the normal theorem from calculus which says that

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu \tag{23.16}$$

In the case that v is also a function of x we have the following theorem.

Theorem 23.7 (Integration by Parts)**.** *If both f and g are absolutely continuous on $[a, b]$, then*

$$\int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx \tag{23.17}$$

Proof. Since product of absolutely continuous functions is continuous $f \cdot g$ is absolutely continuous, thus

$$\int_a^b (f \cdot g)'dx = f \cdot g\big|_a^b = f(b)g(b) - f(a)g(a) \tag{23.18}$$

If we then apply the product rule to $(f \cdot g)' = f'g + g'f$ we have

$$\int_a^b (f'g + g'f) = \int_a^b f'gdx + \int_a^b g'fdx = f(b)g(b) - f(a)g(a) \tag{23.19}$$

$$\implies \int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx \tag{23.20}$$

□

Theorem 23.8 (Integration by Parts v2)**.** *Let $f \in L^1[a, b]$ and $g \in AC[a, b]$, then*

$$\int_a^b fgdx = F(b)g(b) - F(a)g(a) - \int_a^b Fg'dx \tag{23.21}$$

where $F(x) = \int_a^x f(t)dt$

The proof of this is similar to the previous proof.

23.5 Differentiability of Jump Functions

We have made assumptions before that F is a continuous function. We want to work on reducing the continuity hypothesis. If we try to run a density argument as we have done before, the argument does not work as the space of monotone continuous functions are not necessarily dense in the space of monotone functions in the sense of total variation. So to bridge the gap, we introduce another class of monotone functions known as jump functions.

Definition 23.9 (Jump Function)**.** *A basic jump function J_n is a function of one of the following two forms.*

$$J(x) = \begin{cases} 0 & \text{if } x < x_0 \\ \phi & \text{if } x = x_0 \\ 1 & \text{if } x > x_0 \end{cases} \tag{23.22}$$

$$J_n(x) = \begin{cases} 0 & \text{if } x < x_n \\ \phi & \text{if } x = x_n \\ 1 & \text{if } x > x_n \end{cases} \tag{23.23}$$

for some real number $x_0 \in \mathbb{R}$ and $0 \le \theta \le 1$. We will work with the first definition and call x_0 the point of discontinuity for J_n and θ a fraction. Observe that

i) J_n is monotone and non-decreasing

ii) J_n has discontinuity at one point

The jump function associated with an increaseing and bounded function F is given as

$$J_F(x) = \sum_{n=1}^{\infty} \alpha_n J_n(x) \tag{23.24}$$

where n rangers over an at most countable set and each J_n is a basic jump function with $\alpha_n \in \mathbb{R}_+$ and $\sum_n \alpha_n < \infty$. If a function F is bounded ,then we must have that

$$\sum_{n=1}^{\infty} \alpha_n \leq F(b) - F(a) < \infty \tag{23.25}$$

hence the series defining J converges absolutely and uniformly.

Note that a bounded increasing function F on $[a, b]$ has at most countable many discontinuities.

Example 23.10. *If $q_1, q_2, q_3, \ldots$ is any enumerate of the rationals, then*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{[a_n, \infty)} \tag{23.26}$$

is a jump function.

23.5.1 Properties of Jump Functions

We have some properties.

1. All jump functions are monotone and non-decreasing
2. $\sum_{n=1}^{\infty} \alpha_n J_n$ is the uniform limit of $\sum_{n=1}^{N} \alpha_n J_n$. This follows from the absolute convergence of the α_n.
3. The points of discontinuity of a jump function $\sum_{n=1}^{\infty} \alpha_n J_n$ are precisely those of the individual summands $\alpha_n J_n$.
4. The key fact is that these functions, together with the continuous monotone functions, essentially generate all monotone functions in the bounded case.