Math Methods Notes

Zachary Glassman

May 30, 2015

I am starting notes a few weeks into the class. We are almost done with Complex Analysis, refer to those notes on paper or from 135 in college for information. The textbook is Arfkin for this class and it is quite good.

Contents

1 Approximation of Integrals

Example 1.1 (Hankel Function). *Define the Hankel function as $s > 0$,*

$$H_\nu^{(1)}(s) = \frac{1}{\pi i} \int_{\to 0}^{-\infty} dz e^{\frac{s}{z}(z - \frac{1}{z})} \frac{1}{z^{\nu+1}} \tag{1.1}$$

Fin $H_\nu^{(1)}$ as $s \to \infty$ and ν is large. Use the Contour given in 1

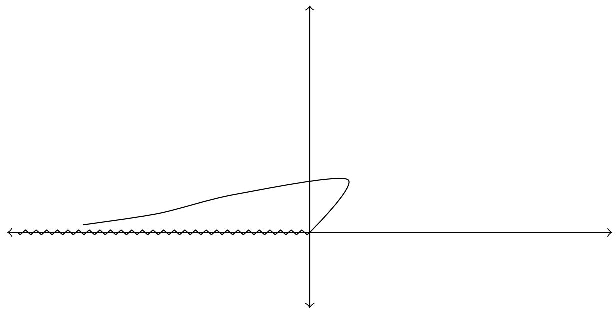

Figure 1: Path for Hankel Function

Proof. First we note that as $z \to 0$, the function goes to zero. We need to expand near the sadle points. Note that

$$\begin{aligned} f(z) &= \frac{s}{z}\left(z - \frac{1}{z}\right) \\ f'(z) &= \frac{s}{z}\left(1 + \frac{1}{z^2}\right) = 0 \\ \implies z_0 &= \pm i \qquad (1.2) \\ f''(z) &= \frac{-s}{z} \qquad (1.3) \end{aligned}$$

We want to find the behavior near the saddle points, so first try $z_0 = i$. We have that

$$f(z_0) = \frac{s}{z}\left(i - \frac{1}{i}\right) = is \tag{1.4}$$

and also that $f''(z_0) = -is$. So this implies that

$$f(z \to z_0) = is - \frac{is(\Delta z)^2}{2}, \tag{1.5}$$

where $\Delta z = z - i$. Wee need to find out the orientation of the saddle points and the path of steepest descent. Let $\Delta z = re^{i\theta}$. which implies that the second term is

$$\frac{-s}{2} r^2 \left(e^{i\pi/2} e^{2i\theta}\right) \tag{1.6}$$

The point is that the saddle point is oriented in one direction and you want the saddle point to be at the maximum. So you need the overall sign of the second term $(\Delta z)^2$ in the expansion to be negative which speaks to the concavity of the function at this point.
So we want

$$\frac{\pi}{2} + 2\theta = 0 \implies \theta = \frac{-\pi}{4} \tag{1.7}$$

To be really careful, plot the full $(\Delta z)^2$ term and find maximum.
This will allow us to use Laplace's method. We will keep to leading order so we have that

$$\left.\frac{1}{z^{\nu+1}}\right|_{z=z_0} =] \frac{1}{\left(e^{i\pi/2}\right)^{\nu+1}}, \tag{1.8}$$

which implies that

$$H_\nu^{(1)}(s) \sim \frac{1}{\pi i}\int_{-\epsilon}^{\epsilon} dr \frac{e^{3\pi i/4} e^{is} e^{-s/2r^2}}{e^{i\pi/2(\nu+1)}} \tag{1.9}$$

where have used the fact that $d(\Delta z) = dz$ and since $\Delta z = re^{3\pi/4}$. Note that we are keeping terms to order by looking at what is large and not. So our integral becomes according to Laplace's method

$$H_\nu^{(1)}(s) = \frac{1}{\pi i}\frac{e^{3\pi i}/4}{e^{i\pi/2(\nu+1)}} e^{is}\int_{-\infty}^{\infty} dr e^{-s/2r^2}. \tag{1.10}$$

This is a Gaussian integral and can be easily done. □

Now we will examine the topography of the real part of this function. Note that

$$f(z) = \left(z - \frac{1}{z}\right) = \left(z - \frac{z^*}{zz^*}\right) = \left(z - \frac{z^*}{|z|}\right) \tag{1.11}$$

which implies that

$$Re(f(z)) = x\left(1 - \frac{1}{r^2}\right) \tag{1.12}$$

We can see this as a plane where outside the circle, the function is positive for $x > 0$ and and negative for $x < 0$. This switches for inside the circle.
Note that to be extra care, we can rewrite

$$H(s) = \frac{1}{\pi i}\oint_C dz e^{s/2(z-1/z)} e^{-(\nu+1)\ln z} \tag{1.13}$$

which implies that

$$\begin{aligned} F(z) &= \frac{s}{2}\left(z - \frac{1}{z}\right) - (\nu+1)\ln z \\ F'(z) &= \frac{s}{2}\left(1 + \frac{1}{z^2}\right) - \frac{\nu+1}{z} \end{aligned} \tag{1.14}$$

which implies that the points where $F'(z) = 0$ can be found by pertubation. Let

$$z = z_0 + z_1 + z_2 + \dots \tag{1.15}$$

which implies that

$$(z_0 + z_1 + \dots)^2 + 1 = \frac{2(\nu+1)}{s}(z_0 + z_1 + \dots) \tag{1.16}$$

since $s \to \infty$, we have that to lowest order

$$z_0^2 + 1 = 0 \tag{1.17}$$

To first order

$$zz_0 z_1 = z(\nu+1)\frac{z_0}{z} \tag{1.18}$$

To second order

$$z_1^2 + zz_0 z_2 = \frac{2(\nu+1)}{s} z_1 \tag{1.19}$$

This implies that

$$z_0 = \pm i \tag{1.20}$$

$$z_1 = \frac{\nu + 1)}{s} \tag{1.21}$$

$$z_2 = \frac{1}{zz_0}\left(\frac{\nu + 1}{s}\right)^2 \tag{1.22}$$

Note: We have a midterm on Thursday October 23rd. Two sheets of paper of notes. Need to check my scores.

2 First Order ODE's

We are moving onto ODE's. The list of stuff is given as

- 1st order ode's
  1. Separation of variables
  2. Integrating factors
- Nth order
  1. const coefficentions
  2. equidimensional
- Energy Method
  1. exact differentiation
- Reduction of order
  1. integral transforms
- Frobenius' method
  1. series solutions

In general, we cannot solve differential equations exactly, so we use numerical analysis of asymptotics.

2.1 Separation of Variables

We have the equations

$$m\dot{v} = mg - m\gamma v \tag{2.1}$$

We want to find $v(t)$ and we know that $v(0) = 0$. So we have that

$$\begin{aligned}
\frac{dv}{dt} &= g - \gamma v \\
\implies \int_0^v \frac{dv'}{g - \gamma v'} &= \int_0^t dt' \\
\implies \ln(g - \gamma v')\Big|_0^v &= -t\gamma \\
\implies ln(g - \gamma v) - \ln g &= -t\gamma \\
\implies v &= \frac{g}{\gamma}(1 - e^{-\gamma t})
\end{aligned} \tag{2.2}$$

We are now going to do an approximation of

$$\frac{dv}{dt} + \delta v = g \tag{2.3}$$

Suppose we want to $v(t)$ as $t \to \infty$. We need to SCALE the equation. By scaling, we note that as $\gamma t \to \infty$, we can discard the $\frac{dv}{dt}$ term and let $v = v_0 + v_1 + v_2 + \dots$. where $|v_n| \gg |v_{n+1}|$. So we get that

$$\dot{v}_0 + \dot{v}_1 + \cdots + \delta(v_0 + v_1 + \dots) = g \tag{2.4}$$

To lowest order solution we have that

$$\gamma v_0 = g \implies v_0 = \frac{g}{\lambda} \tag{2.5}$$

To first order, we have that

$$\dot{v}_0 + \gamma v_1 = 0 \implies \gamma v_1 = 0, \tag{2.6}$$

so this is relatively uninteresting because its a singular perturbation So we are going to change g to $g(t)$. So we have that the lowest order is

$$v_0 = \frac{g(t)}{\lambda} \tag{2.7}$$

Now the first order term is

$$\gamma v_1 = \frac{-\dot{g}}{\gamma} \tag{2.8}$$

To second order, we have that

$$\dot{v}_1 + \gamma v_2 = 0 \implies v_2 = -\frac{\dot{v}_1}{\gamma} = \frac{\ddot{g}}{\gamma^3} \tag{2.9}$$

So we have that

$$v(t) = \frac{g(t)}{\gamma} - \frac{\dot{g}}{\gamma^2} + \frac{\ddot{g}}{\gamma^3} \tag{2.10}$$

This is the asymptotic series for this differential equation. However, by introducing g as a function of t, we cannot use separable variables. We need another method. We need to make sure that this is self-consistent with our requirement that that $|v_n| \gg |v_{n+1}|$. So in this case, we need g to vary slowly compared to γ for our approximation to be accurate.

This general method of separation of variables could apply to non-linear equations.

2.2 Integrating Factor

This only works for linear equations. We start with the equation we had before

$$\frac{dv}{dt} + \gamma v = g(t) \tag{2.11}$$

We solve this as follows

1. Take the factor multiplying the linear term and integrate it

$$\int \gamma dt = \gamma t \tag{2.12}$$

2. Now we exponentiate it to get the integrating factor

$$e^{\int \gamma dt} = e^{\gamma t} \tag{2.13}$$

3. Now we multipy both sides by the integrating factor

$$e^{\gamma t}\left(\frac{dv}{dt} + \gamma v\right) = g e^{\gamma t} \tag{2.14}$$

4. Notice that this is an exact differential and then we can integrate.

$$e^{\gamma t}\left(\frac{dv}{dt} + \gamma v\right) = \frac{d}{dt}(v e^{\gamma t}) \tag{2.15}$$

5. So we can now solve by

$$\int_0^v d(v' e^{\gamma t} = \int_0^t g(t) e^{\gamma t} dt + C \tag{2.16}$$

which implies that

$$v e^{\gamma t} = \int_C^t g(t') e^{\gamma t'} dt', \tag{2.17}$$

where C is set by the initial condition on the problem, at this case $v(0) = 0$, which implies $C = 0$. So

$$v(t) = e^{-\gamma t} \int_0^t dt'\, g(t') e^{\gamma t'} \tag{2.18}$$

Example 2.1 (Laplace's Method Applied to Solution)**.** *We can can now use Laplace's method to approximate this integral. So consider*

$$I = \int_0^t dt' g(t') e^{\gamma t} \tag{2.19}$$

for $\gamma \to \infty$.

Since the t' function looks like Figure 2

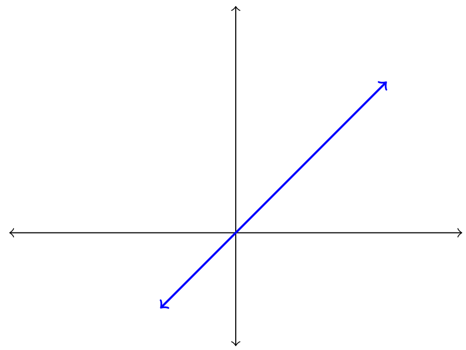

Figure 2: t' graph

We want to expand about $t' = t$. So we have that

$$t' = t'(t) + \frac{dt'}{dt}\Delta t + \dots \tag{2.20}$$

where $t' = t + \Delta t$. Note t' is NOT derivative. We also have that

$$g(t') = g(t) + \dot{g}\Delta t + \ddot{g}\frac{(\Delta t)^2}{2} + \dots \tag{2.21}$$

So we have that

$$I \sim \int_{\epsilon}^{0} d(\Delta t) e^{\gamma t} e^{\gamma \Delta t}[g(t) + \dot{g}\Delta t + \ddot{g}\frac{(\Delta t)^2}{2}] \tag{2.22}$$

We can now let $\epsilon \to \infty$. We will get

$$I \sim e^{\gamma t}\int_0^{\infty} e^{-\gamma \Delta t} d(\Delta t)(g(t) - g'(t)\Delta t + \dots) \tag{2.23}$$

wihch evaluates to

$$\sim \left[\frac{g(t)}{\gamma} - \frac{\dot{g}(t)}{\gamma^2} + \dots\right] \tag{2.24}$$

So we have that the pertubative and asypmtotic solution are equivalent.

3 n'th order Differential Equations

We now move ot higher order differential equations.

3.1 Constant Coefficients

We start with the common example

$$m\ddot{x} = -kx - m\nu\dot{x} \tag{3.1}$$

where we have two initial conditions, $x(t) = 0$, $\dot{x}(t) = v_0$. We can rewrite this as

$$\ddot{x} + 2\nu\dot{x} + \omega_0^2 x = 0 \tag{3.2}$$

This is a second order homogeneous linear differential equation where $\nu and \omega_2 = k/m$ are constant. This equation can be solved analytically. The general method to find the solution is to try $x = e^{\alpha t}$ where α is a constant. This will work in this case since there are constant coefficients. So

$$\begin{aligned} \dot{x} &= \alpha x \\ \ddot{x} &= \alpha^2 x \end{aligned} \tag{3.3}$$

This gives us that

$$\alpha^2 + 2\nu\alpha + \omega_2 = 0 \tag{3.4}$$

This is a quadratic equation for α which gives two solutions solved as

$$\alpha_\pm = \nu \pm \sqrt{\nu^2 - \omega_0^2}. \tag{3.5}$$

So in general x is a linear combination of these solutions

$$x = A_+ e^{\alpha_+ t} + A_- e^{-\alpha_- t}, \tag{3.6}$$

where the intial conditions give the amplitudes $A_\pm$. We can do this exactly, but we can also approximate this when $\nu \ll \omega_0$ by pertubation
To lowest order we can say that

$$\alpha_0^2 = -\omega_0^2 \tag{3.7}$$

To first order we have that

$$2\alpha_0\alpha_1 = -2\nu\alpha_0 \tag{3.8}$$

So this implies that $\alpha_1 = -\nu$, which implies that the solutions are given as

$$e^{i\omega_0 t}e^{-\nu t} \; ; \; e^{-i\omega_0 t}e^{-\nu t}, \tag{3.9}$$

correct to first order in the perturbation series. So our solution is approximatly equal to

$$x(t) \approx e^{-\nu t}\left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right) \tag{3.10}$$

and

$$\dot{x} \approx e^{-\nu t}\left[Ae^{i\omega_0 t}(i\omega_0 - \nu) + Be^{-i\omega_0 t}(-i\omega_0 - \nu)\right] \tag{3.11}$$

We can us our initial conditions to get that $A + B = 0$ and

$$v_0 = A(i\omega_0 - \nu) - B(i\omega_0 + \nu) \implies v_0 = 2Ai\omega_0 \tag{3.12}$$

Which gives A and B. So we have

$$X(t) \approx v_0 \frac{e^{-\nu t}}{2i\omega_0} 2i\sin(\omega_0 t) = \frac{v_0}{\omega_0} e^{-\nu t}\sin(\omega_0 t) \tag{3.13}$$

4 Notes for Exam

These are some notes for the exam in two days. Look at last years midterm which will give a good idea of the type of stuff. Need to know:

- Cauchy-Riemann conditions and what they mean
- What it means to be differentiable
- Branch cuts and they matter to be single valued
- how to make single valued
- Integrals (three or four of them) probably not by brute force (DEFORM)
- Know the meaningful definitions of integrals

$$\int_{1+i}^{3+i} dz\, z^{1/2}, \tag{4.1}$$

  is not a defined integral because there is no branch cut. Now it is defined.

- Residue's and how to find them
- Integrating over 1/0 is not defined, need residues.
- Asymptotics- Laplace's method.
- Know standard form and how to deal with it.

Now back to differential equations.

5 Some Review Examples

5.1 Integrating Factor

We now have an example of integrating factors with equation

$$\dot{x} + \frac{1}{t}x = f(t) \tag{5.1}$$

So we integrate

$$\int \frac{dt}{t} = \ln(t) \tag{5.2}$$

So the integrating factor is

$$e^{\ln t} = t \tag{5.3}$$

By multiplying both sides by the integrating factor, we get

$$\dot{x}t + x = tf(t) \implies \frac{d}{dt}(xt) = tf(t) \tag{5.4}$$

and we can integrate to get the solution.

5.2 Asymptotic Evaluation

Start with the equation

$$\frac{dv}{dt} + \gamma v = g(t), \tag{5.5}$$

where γ is a constant. In the case that $\gamma \to \infty$, we can do the following steps:

1. Scale the equation and look at the relative size of

$$\frac{v}{t} : \gamma v : g \tag{5.6}$$

   and see which one is large.

2. Order the terms with some epsilon (prescription only, not rigorous), in the same order as above as

$$\epsilon : 1 : 1, \tag{5.7}$$

3. Perturbation theory, write

$$v = v_0 + v_1 + v_2 + \dots \tag{5.8}$$

   where $|v_i| \gg |v_{i+1}|$.

4. Self-consistency check.

In this particular problem, there is a subtlety which is referred to as WKB approximation. We will see this later

6 Further ODE Methods

6.1 Equidimensional (Euler Method

We start with the equation

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{y}{x^2}y(x) \tag{6.1}$$

which we can write as

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 0. \tag{6.2}$$

Note that this equation is linear which is the case here. Additionally, the dimension of x is the same for each term. In this case because of the differentials, all of the terms are $1/x^2$, which implies that there is a very simple solution.

$$y = x^\alpha \tag{6.3}$$

works in all cases with these conditions. We can see this by noting that

$$y = x^\alpha \tag{6.4}$$

$$y' = \alpha x^{\alpha-1} \tag{6.5}$$

$$y'' = \alpha(\alpha-1)x^{\alpha-2}, \tag{6.6}$$

we can plug this in to get

$$\alpha(\alpha-1)x^{\alpha-2}+\frac{\alpha x^{\alpha-1}}{x}-\frac{x^{\alpha}}{x^{2}}=0, \tag{6.7}$$

which leads to

$$\alpha(\alpha-1)+\alpha-1=0 \tag{6.8}$$

which we can solve as $\alpha=\pm 1$ so solution set is

$$y=x;\frac{1}{x} \tag{6.9}$$

So the general solution is

$$y=Ax+\frac{B}{x}. \tag{6.10}$$

Notice that the equidimensional could also be found from the initial equation before expanding it.

6.2 Energy Method

Called the energy method because it comes up in energy problems in physics. It has a very specific second order form.

$$\frac{d^2x}{dt^2}=F(x), \tag{6.11}$$

where the function $F(x)$ is explicitly independent of the independent variable, t, in this case and could be nonlinear. So we let

$$F(x)=-\frac{dU}{dx} \tag{6.12}$$

which implies that

$$U(x)=-\int_0^x dx' F(x'). \tag{6.13}$$

We then insert this back into the original equation and get

$$\frac{d^2x}{dt^2}=-\frac{dU}{dx}. \tag{6.14}$$

The trick is to multiply through by $\frac{dx}{dt}$ which leads to

$$\begin{aligned}\frac{dx}{dt}\frac{d^2x}{dt^2}&=-\frac{dx}{dt}\frac{dU}{dx}\\&=-\frac{dU}{dt}\\ \Longrightarrow\quad \frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right]&=-\frac{dU}{dt}.\end{aligned} \tag{6.15}$$

We can integrate both sides since they are both exact differentials in t and get that

$$\int dt\,\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right]=-\int dt\,\frac{dU}{dt}+E \tag{6.16}$$

which we can solve to get that

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + U(x) = E. \tag{6.17}$$

This is a statement of the conservation of energy. We still need to solve this equation though. We have

$$\frac{dx}{dt} = \pm\sqrt{2[E - U(x)]}. \tag{6.18}$$

We can solve this by separation of variables as

$$\int \frac{dx}{\sqrt{2[E - U(x)]}} = \pm \int dt \tag{6.19}$$

We can find the two solutions from this equation.

6.2.1 Example of Energy Method

We will solve the harmonic oscillator in this way.

$$\ddot{x} = -x \tag{6.20}$$

where $x(0) = 0$, $\dot{x}(0) = 1$. By constant coefficients, we can see immediately the solution is $x(t) = \sin(t)$. However by the energy method we can also do this. Let

$$F = -\frac{dU}{dx} \tag{6.21}$$

which implies that

$$U = \frac{1}{2}x^2 \tag{6.22}$$

So we have that

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d}{dt}x\right)^2\right] = -\frac{d}{dt}U \tag{6.23}$$

If we solve this we get that

$$\frac{\dot{x}^2}{2} = U + E \tag{6.24}$$

which gives us that

$$\begin{aligned}
\frac{\dot{x}^2}{2} &= -\frac{1}{2}x^2 + \frac{1}{2} \\
\implies \dot{x} &= \pm\sqrt{1 - x^2} \\
\implies \int_0^x \frac{dx'}{\sqrt{1 - x'^2}} &= \int_0^t dt' \\
\implies \arcsin(x')\Big|_0^x &= t \\
\implies x &= \sin(t)
\end{aligned} \tag{6.25}$$

where the sign is chosen from initial conditions. Need to be really careful about this.

6.3 Reduction of Order

This method applies to linear equations of a specific form. Applies when highest absolute power of d/dx is greater than the highest absolute power of x. Need to be integer powers. For example if we look at the equation

$$\frac{d^2y}{dx^2} = x^2 y \tag{6.26}$$

so highest power of d/dx is 2 and highest power of x is also two, so this method will not work. However, for

$$\frac{d^2y}{dx^2} = xy, \tag{6.27}$$

the method will work as will

$$x\frac{d^2y}{dx^2} = y. \tag{6.28}$$

This method is very useful for integral representations.

6.3.1 Airy Equation with RoO

We start with the Airy equation which is given as

$$\frac{d^2y}{dx^2} = xy, \tag{6.29}$$

which we have been unable to solve before this point. We are going to use and integral transform to solve this. Let

$$y(x) = \int_C dk\, e^{kx} f(k), \tag{6.30}$$

be the integral representation in complex phase space where the endpoints and contour do not depend on x. The endpoints and the contour need to be specified in order for this to work. Now we find the derivative with the goal of eventually plugging the integral representation into the partial differential equation.

$$\frac{dy}{dx} = \int_C dk\, e^{kx} k f(k) \tag{6.31}$$

$$\frac{d^2y}{dx^2} = \int_C dk\, e^{kx} k^2 f(k). \tag{6.32}$$

This implies that

$$\begin{aligned} xy &= \int_C dk (e^{kx}) f(k) \\ &= \int_C dk\, \frac{d}{dk}(e^{kx}) f(k) \\ &= \left[e^{kx} f(k)\right]_{endpoints} - \int_C dk e^{kx} \frac{df}{dk}, \end{aligned} \tag{6.33}$$

where we have used integration by parts. So we can plug in and assume that endpoints vanish. We will verify this is correct later. So we have

$$\int_C dk\, e^{kx} \left[k^2 f(k) + \frac{df}{dk}\right] = 0 \tag{6.34}$$

This implies that

$$\frac{df}{dk} + k^2 f(k) = 0. \tag{6.35}$$

So we have reduced to a first order equation which we can solve. This solution is given as

$$f(k) = De^{-k^3/3}. \tag{6.36}$$

We now must demand a self consistency check that

$$D\left[e^{kx}e^{-k^3/3}\right] = 0, \tag{6.37}$$

and pick endpoints such that it is. We want $R(k^3) > 0$, then the integrand goes to zero at infinity. We can see that going to ∞ along the real axis will work. So let $k = re^{i\theta}$, then

$$r^3 Re(e^{3i\theta}) > 0, \tag{6.38}$$

which implies that $\cos(3\theta) > 0$ if $\theta \in (-\pi/6, \pi/6)$. So we can go to a couple different points. So we choose the correct paths and we have an integral representation of the solution. These are called Airy functions. Other path is called B Airy function.

7 Perturbation Theory

There are some equations which cannot be solved by any of the methods listed above. We will develop some new ways of pertubation to accomplish this.

7.1 Bessel's Equation

One example is Bessel's equation which is given as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0. \tag{7.1}$$

In this case, we use Frobenius method, which uses power series. It can work for many of the equations we have already studied and in general under certain conditions for linear equations.

7.1.1 Asymptotic Behavior of Bessel Functions

We can rewrite the equation as

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - \nu^2)y = 0. \tag{7.2}$$

We can note that $x = 0$ is an important point and so we want to study it. So consider $x \to 0$. We will now find an asymptotic solution.

1. Scale the equation. We are interested when x is small, so we can see that for small x, x^2 is very small.

2. Order the terms, which is saying that x^2 is very small.

3. Let $= y_0 + y_1 + y_2 + \dots$ and assume $|y_{n+1}| \ll |y_n|$. So we have that

$$\begin{aligned} x^2(y_0 + y_1 {+} y_2 + \dots)'' + x(y_0 + y_1 + y_2 + \dots)' \\ + x^2(y_0 + y_1 + y_2 + \dots) - \nu^2(y_0 + y_1 + y_2 + \dots) \end{aligned} \tag{7.3}$$

So to lowest order

$$x^2 y_0'' + x y_0' - \nu^2 y_0 = 0, \tag{7.4}$$

to first order

$$x^2 y_1'' + x y_1' + x^2 y_0 - \nu^2 y_1 = 0, \tag{7.5}$$

and further on.

4. We can solve these equation to get the pertubative solution. We can solve the lowest order equation with the equidimensional method. So this means we try $y_0 = x^\alpha$ and we have that

$$\alpha(\alpha - 1) + \alpha - \nu^2 = 0 \implies \alpha = \pm\nu \tag{7.6}$$

So

$$y_0 = \begin{cases} x^\nu \\ x^{-\nu} \end{cases}, \tag{7.7}$$

with a special case of $\nu = 0$, we can write the equation as

$$x\frac{d}{dx}\left(x\frac{dy_0}{dx}\right) = 0 \tag{7.8}$$

which implies that

$$y_0 = \begin{cases} \ln x \\ 1 \end{cases} \tag{7.9}$$

So we have found the lowest order terms. We will continue by assuming that $\nu = 0$. Then we can find the first order term as

$$x(xy_1')' = -x^2 \begin{cases} 1 \\ \ln x \end{cases} \tag{7.10}$$

So if take the solution of y_0 as 0 and integrate

$$xy_1' = -\frac{x^2}{2} \implies y_1 = -\frac{x^2}{4} \tag{7.11}$$

Now for $\ln x$, we have that

$$(xy_1)' = -x \ln x. \tag{7.12}$$

We can do this integral through integration by parts and come to the final answer of

$$y_1 = \frac{-x^2}{4}(\ln x - 1) \tag{7.13}$$

So first order, our solutions for this case by regular perturbation theory are

$$y = \begin{cases} 1 - \frac{x^2}{4} \\ \ln x - \frac{x^2}{4}(\ln x - 1) \end{cases} \tag{7.14}$$

Now check our asymptotic cases. In the limit as $x << 1$, the lowest order terms are much larger than the first order terms, so we have generated valid series. This self consistency check is **CRITICAL**.
We now go to $\nu \neq 0$, but its very messy, however the techniques are the same and solutions can be found.

We want to examine some of these solutions. Lets look at the pertubative $y = \ln x - \frac{x^2}{4}(\ln x - 1)$ as compared to the power series solutions which are Bessel's functions $J_0(x), Y_0(x)$. In this case, we have the Y_0 function. We can plot these as and these essentially exactly the same for small x. As x grows there is a divergence.

7.2 Singular Perturbation Theory (WKB)

As we take $x \to \infty$, we cannot use this method because regular perturbation theory fails. We will use singular Perturbation theory which is also known as the WKB method for ODE. The method prescribes to try

$$y(x) = e^{S(x)}, \tag{7.15}$$

with the ansatz that $|S| \gg 1$. So we have that

$$y = e^S \tag{7.16}$$
$$y' = S'e^S \tag{7.17}$$
$$y'' = S'^2 e^S + S'' e^S. \tag{7.18}$$

We can apply this to Bessel's function and we get

$$x^2(S'^2 + S'')e^S + xS'e^S + x^2 e^S - \nu^2 e^S = 0 \tag{7.19}$$

where we can cancel the the exponentials to get

$$x^2 S'^2 + x^2 S'' + xS' + x^2 - \nu^2 = 0 \tag{7.20}$$

We can use the same scaling methods as before with the assumptions now that $x \to \infty$ and $|S| \to \infty$. So to scale the terms we have

$$\begin{array}{rrrrr} \frac{x^2 S^2}{x^2} : & \frac{x^2 S}{x^2} : & \frac{xS}{x} : & x^2 : & S^2 \\ S^2 : & S : & S : & x^2 : & \nu^2 \\ 1 : & \epsilon : & \epsilon : & A : & \delta A, \end{array}$$

where we have assumed ν is of order one. It is important to note here that if there is no way to determine the smallness between two terms relative to each other, you must keep both terms.

This is known as optimal ordering.
Let $S = S_0 + S_1 + S_2 + \dots$ So to lowest order, we have

$$x^2 S_0'^2 + x^2 = 0 \implies S_0'^2 + 1 = 0. \tag{7.21}$$

To the first order, we have that

$$x^2(2S_0'S_1') + x^2 S_0'' + xS_0' = 0 \tag{7.22}$$

If we solve these equations, we get that

$$S_0' = \pm i \implies S_0 = \pm ix \tag{7.23}$$

To find the solution to S_1, we plug in the solution to S_0, which leads to

$$2x^2 S_0'S_1' + xS_0' \implies 2xS_1' = 0 \tag{7.24}$$

which we can solve as

$$S_1 = -\frac{1}{2}\ln x \tag{7.25}$$

So we have that S asymptotes as

$$S \sim \pm ix - \frac{1}{2}\ln x \tag{7.26}$$

which implies that our asymptotic behavior for y is

$$y \sim \frac{e^{\pm ix}}{\sqrt{x}} \tag{7.27}$$

We now check self consistency, which we can see is true since $|ix| \gg |\frac{1}{2}\ln x|$ as $x \to \infty$.

7.3 Airy Equation

We start with the Airy equation given as

$$y'' = xy, \tag{7.28}$$

and evaluate it with the same techniques. First we evaluate behavior as $x \to 0$. As $x \to 0$, xy is very small compared to y''. So we expand $y = y_0 + y_1 + y_2 + \dots$ which gives to lowest order

$$y_0'' = 0 \implies y_0 = \begin{cases} 1 \\ x \end{cases} \tag{7.29}$$

and to first order we have that

$$y_1'' = y_0 x \implies y_1'' = \begin{cases} x \\ x^2 \end{cases} \tag{7.30}$$

Which we can solve as

$$y_1 = \begin{cases} \frac{x^3}{6} \\ \frac{x^4}{12} \end{cases}, \tag{7.31}$$

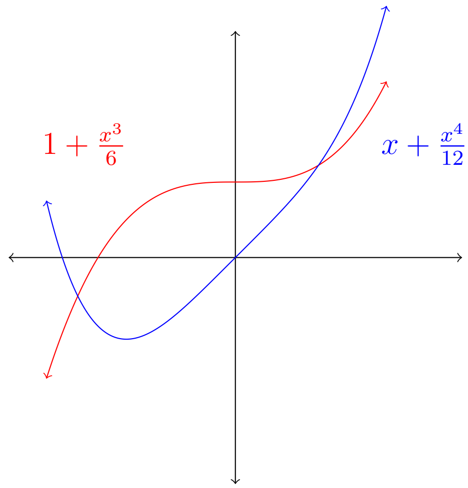

Figure 3: Airy function asypmtotics

which leads to the solutions

$$y = \begin{cases} 1 + \frac{x^3}{6} + \dots \\ x + \frac{x^4}{12} + \dots . \end{cases} \tag{7.32}$$

we can generate higher order terms if necessary, but this is good enough and we can see the terms are properly sized relative to each other as $x \to 0$. We plot these solutions in 3 We know how to evaluate the exact solution for this function and it turns out to be a linear combination of the airy functions Ai and Bi. So we can compare (will do on homework).
We can now do the asymptotic as $x \to \infty$. We will need to use WKB to do this. So we try e^S and solve the equation

$$S'^2 + S'' = x \tag{7.33}$$

NEED TO FILL IN NOTES LATER

8 Frobenius' Method

We will illustrate by example of the Harmonic Oscillator method whose solution we already know,

$$\frac{d^2y}{dx^2} = -y. \tag{8.1}$$

We will try

$$y(x) = x^\nu \sum_{n=0}^{\infty} a_n x^n, \tag{8.2}$$

with $a_0 \neq 0$ and compute derivatives as

$$y'(x) = \sum_{n=0}^{\infty} (\nu + n) a_n x^{\nu+n-1} \tag{8.3}$$

$$y''(x) = \sum_{n=0}^{\infty} (\nu + n)(\nu + n - 1) a_n x^{\nu+n-2}, \tag{8.4}$$

, So we can plug into our equation as

$$\sum_{n=0}^{\infty}(\nu+n)(\nu+n-1)a_n x^{\nu+n-2} = -x^{\nu}\sum_{n=0}^{\infty} a_n x^n \tag{8.5}$$

which we can simplify as

$$\sum_{n=0}^{\infty}(\nu+n)*(\nu+n-1)a_n x^{\nu-2} = -\sum_{n=0}^{\infty} a_n x^n \tag{8.6}$$

If we match up all of the terms, in these series, we see that the second derivative has terms of x^{-2} and x^{-1} and we need both of those to be zero for equality. So we have that

$$\nu(\nu-1)a_0 = 0 \tag{8.7}$$

$$(\nu+1)\nu a_1 = 0. \tag{8.8}$$

Thus, we see there are two cases since $a_0 \neq 0$. If $\nu = 0$, then a_1 is arbitrary, if $\nu = 1$, then $a_1 = 0$.
Now we redefine $m = n-2$ and have that

$$\sum_{m=0}^{\infty}(\nu+m+2)(\nu+m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} a_m x^m, \tag{8.9}$$

where we have changed the dummy variable on the right hand side as well. So we know that

$$(\nu+m+2)(\nu+m+1)a_{m+2} = -a_m, \tag{8.10}$$

which implies that

$$a_{m+2} = \frac{-a_m}{(\nu+m+2)(\nu+m+1)}. \tag{8.11}$$

which is the recursion relation we are looking for. We can plug in our solutions for ν and compute the series. Plugging in $\nu = 1$, we get the series

$$\frac{y^{(2)}}{a_0} = x - \frac{x^3}{3} + \frac{x^5}{5!} + \cdots = \sin(x) \tag{8.12}$$

for the second solution, we have that a slightly more complicated case since a_1 is arbitrary. The simple way to do it is to say that $a_1 = 0$ which will give the pure solution and not a linear combination with the first term. So this will give

$$\frac{y^{(1)}}{a_0} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \cos(x), \tag{8.13}$$

and so we have recovered the correct solutions.

8.1 Bessel Equation Frobenius

Given the Bessel equation

$$x(xy')' + x^2 y = 0, \tag{8.14}$$

which is the $\alpha = 0$ case. So we again try

$$y(x) = x^\nu \sum_{n=0}^{\infty} a_n x^n, \tag{8.15}$$

with $a_0 \neq 0$ and compute derivatives as

$$y'(x) = \sum_{n=0}^{\infty} (\nu + n) a_n x^{\nu+n-1} \tag{8.16}$$

$$y''(x) = \sum_{n=0}^{\infty} (\nu + n)(\nu + n - 1) a_n x^{\nu+n-2}, \tag{8.17}$$

which leads to

$$x(xy')' = x^\nu \sum_{n=1}^{\infty} (\nu + n)^2 a_n x^n \tag{8.18}$$

which we can equate to $-x^2 y$ and get

$$\sum_{n=0}^{\infty} (\nu + n)^2 a_n x^n = -\sum_{n=0}^{\infty} a_n x^{n+2}. \tag{8.19}$$

So we can find the initial equation by noting the left hand side has terms which have no equivalent in the right hand side and so

$$\nu^2 a_0 = 0 \tag{8.20}$$

$$(\nu + 1)^2 a_1 = 0, \tag{8.21}$$

for the x^0 and x^1 coefficient respectively. Since $a_0 \neq 0$, $\nu = 0$ and $a_1 = 0$. Now we let $n = m + 2 \implies m = n - 2$. So we have

$$\sum_{m=0}^{\infty} (m + 2)^2 a_{m+2} x^{m+2} = -\sum_{n=0}^{\infty} a_n x^{n+2}, \tag{8.22}$$

by switching dummy variables, we get that

$$a_{m+2} = \frac{-a_0}{(m + 2)^2}. \tag{8.23}$$

We can now compute our solution as

$$J_0(x) = \frac{y(x)}{a_0} = 1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 \cdot 2^2} - \frac{x^6}{6^2 \cdot 4^2 \cdot 2^2} + \dots \tag{8.24}$$

8.2 Fuch's Theroem

This theorem gives information about existence of solutions and is based on singular points of a linear ODE. This theorem focuses on singular points of a second order ODE given as

$$y'' + P(x)y' + Q(x)y = 0, \tag{8.25}$$

where P, Q are given. Singular points are where $P(x) \to \infty$ or $Q(x) \to \infty$. For example, for the Harmonic oscillator written as

$$y'' + y = 0, \tag{8.26}$$

there are no singular points. For the Bessel zero equation

$$y'' + \frac{y'}{x} + y = 0, \tag{8.27}$$

there is a singular points at 0 and this called regular singular since $P(x)$ does not blow up worse than $1/x$. Another example would be a Bessel α

$$y'' + \frac{y'}{x} + \frac{1-\alpha^2}{x^2}y = 0, \tag{8.28}$$

since the Q function can blow up no worse than $1/x^2$. So

$$y'' = \frac{y}{x^3}, \tag{8.29}$$

would be irregular singular. So we have

Theorem 8.1 (Fuch's)**.** *If a second order linear differential equation regular singular (or "better"), there exists at least one series solution.*

8.3 Non Regular Singular Point Example

We start with the equation

$$y'' = \frac{y}{x^3} \tag{8.30}$$

We apply the WKB approximation to guess that $y = e^S$ which gives that

$$S'^2 + S'' = \frac{1}{x^3}. \tag{8.31}$$

We can scale this equation to get that S'' is the small term which allows us to write S as a series and write that

$$S_0'^2 = \frac{1}{x^3} \tag{8.32}$$

which implies that $S_0 = \pm\frac{2}{\sqrt{x}}$. So hte lowest order solution y_0 is given as

$$y_0 = \begin{cases} e^{2/\sqrt{x}} \\ e^{-2/\sqrt{x}} \end{cases}, \tag{8.33}$$

which since there is a branch cut and since there is a branch point at 0, Frobenius will **NOT** work. Now we will try

$$y'' = \frac{y}{x^4} \tag{8.34}$$

We apply the WKB approximation to guess that $y = e^S$ which gives that

$$S'^2 + S'' = \frac{1}{x^4}. \tag{8.35}$$

which yields first order solutions of

$$y_0 = \begin{cases} e^{1/x} \\ e^{-1/x} \end{cases}, \tag{8.36}$$

which means there is no branch cut. We can expand this as a Lorentz series, but there is an essential singularity. So it does not look like Frobenius' series and it won't come out.

8.4 Legendre Equation

The singular point does not have to be at $x = 0$. An example is the Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] = 1 - \nu(\nu+1)y, \tag{8.37}$$

which we can write in standard form as

$$y'' - \frac{2x}{1-x^2}y' + \frac{\nu(\nu+1)}{1-x^2}y = 0. \tag{8.38}$$

Note that the singularities here are not too bad and ± 1 are singular poles and Fuch's theorem guarantees at least one solution. This is on the homework. There should only be one solution unless certain conditions are satisfied. At the origin there are two nicely behaved solutions and there will be a connection through the full series to one of the asymptotic series (the divergent one). The other solution will be nice on the positive x axis and blow up to negative infinity on the negative x axis. These are called P_ν solutions and Q_ν solutions. This is for $\nu \notin \mathbb{Z}$, then things change and things don't blow up. Here we get the Legendre Polynomials.

9 List of Important Equations

We have studied many of these equations, we will list them here for convenience as we are now moving to Green's functions for inhomogeneous ODE's and Sturm-Liouville Theory and PDE's. These are some of the most important homogeneous ODE's for physics.

- Airy

$$y'' = xy \tag{9.1}$$

- Bessel

$$x(xy')' + (x^2 - \nu^2)y = 0 \tag{9.2}$$

- Harmonic
$$y'' = -y \tag{9.3}$$
- Legendre
$$[(1-x^2)y']' + \nu(\nu+1)y = 0 \tag{9.4}$$
- Harmonic Oscillator
$$y'' + (E - x^2)y = 0 \tag{9.5}$$

10 Green's Functions

We will follow Morse and Feshbach.

10.1 Introduction

Green's functions help solve, inhomogeneous, linear, ODE's. The standard form for these equations (in second order) is
$$y'' + P(x)y' + Q(x)y = S(x), \tag{10.1}$$
where $S(x)$ is a source function which is given and P, Q are given. Find $y(x)$ with some boundary conditions. We can also write this as
$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = s(x). \tag{10.2}$$
Usually we are given boundary conditions which could be $\pm\infty$. To solve this conveniently, we define a Green's function $G(x';x)$ which satisfies
$$\frac{d}{dx'}\left(p(x')\frac{dG}{dx'}\right) + q(x')G = \delta(x'-x) \tag{10.3}$$
so G is a function of x' variable and x. We also need information from boundary conditions to solve.

10.2 Green's Functions Solutions

To find $y(x)$, we consider the trivial statement
$$y(x) = \int_a^b dx'\delta(x'-x)y(x'), \tag{10.4}$$
and then substitute the Green's function for the δ function and so we have
$$\begin{aligned} y(x) &= \int_a^b dx' y(x')\left[\frac{d}{dx'}\left(p(x')\frac{dG}{dx'}\right) + q(x')G\right] \\ &= \int_a^b dx'\, G\left[\frac{d}{dx}\left(p\frac{dy}{dx'}\right) + qy\right] + \left[p\left(y(x')\frac{dG}{dx'} - G\frac{dy}{dx'}G\right)\right]_a^b \\ &= \int_a^b dx'\, G(x';x)S(x') + \left[p\left(y(x')\frac{dG}{dx'} - G\frac{dy}{dx'}G\right)\right]_a^b, \end{aligned} \tag{10.5}$$

where we have integration by parts twice. We can simplify the boundary term and then we need to solve the Green's function, which is generally easier to solve since it has a simple source function. Usually the boundary term can be simplified by choosing boundary conditions on G.

10.2.1 Dirichlet Boundary Term

Suppose we have Dirichlet boundary conditions which means that $y(a), y(b)$ are given. Then the boundary term is given as

$$\left[p\left(y(x')\frac{dG}{dx'} - G\frac{dy}{dx'}G\right)\right]_a^b, \tag{10.6}$$

all of which are known except for $\frac{dy}{dx'}$. So if we set $G = 0$, we have no problems. So we let $G = 0$ on the boundaries which gives us boundary conditions for the Green's function. Now we can solve this equation by solving the Green's equation and then evaluating the integrals. This is simpler problem since the Green's function has a simple source function and homogeneous boundary conditions, in addition, there is guarenteed to be a solution.

10.2.2 Neumann Boundary Conditions

Suppose we have Neumann Boundary Conditions which means that $y'(a), y'(b)$ are given, we set $G'(a) = G'(b) = 0$, where the derivatives are taken with respect to x'. So again, we can solve the Green's function.

10.3 Example Problem

We start with the problem

$$y'' - k^2 y = S(x), \tag{10.7}$$

where k, S are given with the mixed boundary conditions

$$y'(0) = 0 \tag{10.8}$$

$$y(b) = 1, \tag{10.9}$$

where we will solve on the region $x \in [0, b]$. Can also do this by Fourier transforms and such, but we will use Green's functions. The Green's Function is given as

$$\frac{d^2G}{dx'^2} - k^2 G = \delta(x - x'), \tag{10.10}$$

where

$$\left.\frac{dG}{dx'}\right|_0 = 0\,;\; G\Big|_b = b. \tag{10.11}$$

So our solution is given as

$$\begin{aligned} y(x) &= \int_0^1 dx' G(x';x)S(x') + \left[\frac{dG}{dx'}y\right]_{x'=b} \pm \left[G\frac{dy}{dx}\right]_{x'=0} \\ &= \int_0^1 dx' G(x';x)S(x') + \left[\frac{dG}{dx'}\right]_{x'=b} \end{aligned} \tag{10.12}$$

We now solve the Green's function. Anytime there is a δ function, solve the equation for $x' < x$ and $x' > x$ and then patch the solutions together with the boundary conditions and jump conditions. The jump conditions are

$$\int_{x-\epsilon}^{x+\epsilon} dx' \, \frac{d^2G}{dx'^2} - k^2 G = 1 \tag{10.13}$$

which leads to

$$\lim_{\epsilon \to 0} \left[\frac{dG}{dx'} \right]_{x-\epsilon}^{x+\epsilon} = 1. \tag{10.14}$$

The other jump conditions is that

$$G\Big|_{-\epsilon}^{\epsilon} = 0. \tag{10.15}$$

First we solve to the left. We have the equation

$$\frac{d^2G}{dx'^2} = k^2 G, \tag{10.16}$$

which we know how to solve as

$$G \sim \begin{cases} e^{kx'} \\ e^{-kx'} \end{cases} \tag{10.17}$$

which can also be written in the basis of

$$G \sim \begin{cases} \sinh(kx') \\ \cosh(kx') \end{cases} \tag{10.18}$$

by taking linear combinations of the first case. So we let

$$G_{\leftarrow} = A \cosh(kx'), \tag{10.19}$$

which easily satisfies the boundary conditions.
For the right and side, we have the same equation, but we want the function to vanish at $x' = b$ and so we let

$$G_{\rightarrow} = B \sinh[k(x' - b)]. \tag{10.20}$$

One way to make this easier is to symmetrize the equations. So we can write the equations as

$$G_{\leftarrow} = A \cosh(kx') \sinh[k(x - b)] \tag{10.21}$$

$$G_{\rightarrow} = B \sinh[k(x' - b)] \cosh(kx), \tag{10.22}$$

where we have just multiplied by a constant (differential equation is in x' not x. So now when $x' = x$, we know $G_{\leftarrow} = G_{\rightarrow}$ which means we can let $A = B$ and get

$$G_{\leftarrow} = A \cosh(kx') \sinh[k(x - b)] \tag{10.23}$$

$$G_{\rightarrow} = A \sinh[k(x' - b)] \cosh(kx), \tag{10.24}$$

Now we will evaluate the jump conditions, so we have that

$$\left. \frac{dG_{\rightarrow}}{dx'} \right|_{x'=x} = Ak \cosh[k(x - b)] \cosh(kx) \tag{10.25}$$

$$\left. \frac{dG_{\leftarrow}}{dx'} \right|_{x'=x} = Ak \sinh(kx) \sinh[k(x - b)]. \tag{10.26}$$

For the jump conditions, we have that

$$Ak\cosh[k(x-b)]\cosh(kx) - Ak\sinh(kx)\sinh[k(x-b)] = 1, \tag{10.27}$$

which implies that

$$kA = -\frac{1}{\cosh(kb)}. \tag{10.28}$$

To completely solve, we need

$$\left.\frac{dG_{\rightarrow}}{dx'}\right|_{x'=b} = Ak\cosh(kx). \tag{10.29}$$

We can now plug into our solution as

$$y(x) = \int_0^b dx'\, S(x')G(x';x) + \frac{\cosh(kx)}{\cosh(kb)}, \tag{10.30}$$

where

$$G(x', x) = \frac{-1}{k\cosh(kb)}\left[\cosh(kx_{\leftarrow})\sinh[k(x_{\rightarrow}-b)]\right]. \tag{10.31}$$

11 Sturm-Liouville Theory

The reason we are doing this is to generate complete sets of functions using linear algebra. For example we will make complete eigenbasis out of quantized Fourier eigenfunctions, Bessel eigenfunctions, Legendre, etc. So

$$\left\{\begin{array}{c}\sin(k_n x)\\ \cos(k_n x)\end{array}\right\};\left\{\begin{array}{c}J_\nu(k_n x)\\ N_\nu(k_n x)\end{array}\right\};\left\{\begin{array}{c}P_{\nu_n}(x)\\ Q_{\nu_n}(x)\end{array}\right\} \tag{11.1}$$

11.1 Review of Finite Dimensional Vector Spaces

We define vectors as usual, for example

$$x = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{11.2}$$

where $a, b \in \mathbb{C}$. We can define a vector space as closed under addition and multiplication. So if $y = \begin{pmatrix} c \\ d \end{pmatrix}$, then we have

$$x + y = \begin{pmatrix} a+c \\ b+c \end{pmatrix}, \tag{11.3}$$

must also be in the space. We define the multiplication by a complex number α as

$$\alpha x = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}, \tag{11.4}$$

which also must be in the vector space. Now we define an inner product as

$$\langle x, y\rangle := (a, b)^* \begin{pmatrix} c \\ d \end{pmatrix} = a^*c + b^*d \tag{11.5}$$

where * is complex conjugation. So we can notice that

$$\langle x, y\rangle^* = \langle y, x\rangle, \tag{11.6}$$

as well as

$$\langle x, x\rangle^* = \langle x, x\rangle = a^*a + b^*b = |a|^2 + |b|^2, \tag{11.7}$$

which is real and positive. So if $\langle x_1, x_2\rangle = 0$, we we call x_1 and x_2 orthogonal. This is equivalent to perpendicular lines in the euclidean space subset.
Now we must introduce operators we are anything which takes arbitrary vector x to some vector y, so

$$Ax = y. \tag{11.8}$$

In this case an operator is a 2×2 matrix, for example if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{11.9}$$

then

$$A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}. \tag{11.10}$$

So now we define the eigenproblem. Suppose that

$$A\phi = \lambda\phi, \tag{11.11}$$

where A is an operator and λ is a complex number. Does ϕ exist? So given an operator can we find a set of eigenfunctions. The answer is that ϕ may exist for particular values of λ, it only exists for certain eigenvalues. This means that we can speak of ϕ_n where the n labels eigenfunctions (or eigenvectors) corresponding to n eigenvalues, so

$$A\phi_n = \lambda_n\phi_n. \tag{11.12}$$

Example 11.1. *Find the eigenvalues for*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.13}$$

Solution: We want to solve the equation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}, \tag{11.14}$$

which is true only if

$$det \left[\begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \right], \tag{11.15}$$

which gives rise to the following equation,

$$(1-\lambda)(1+\lambda) = 0, \tag{11.16}$$

which is true for $\lambda = \pm 1$ and with corresponding eigenvectors

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; a_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11.17}$$

11.1.1 Self-Adjoint(Hermitean) Operator

Self- adjoint operators can be define as any operator A which for arbitrary vectors x, y satisfies the condition

$$\langle y, Ax\rangle = \langle Ay, x\rangle \tag{11.18}$$

We can prove we can write all self-adjoint operators as

$$A = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}, \tag{11.19}$$

where $\alpha, \beta \in \mathbb{R}$. Now we have the main theorem,

Theorem 11.1. *Suppose H is a hermitean operator with eigenvectors ϕ_n and corresponding eigenvalues λ_n satisfying*

$$H\phi_n = \lambda_n\phi_n, \tag{11.20}$$

then

$$\lambda_m^* - \lambda_n)\langle\phi_m, \phi_n\rangle = 0. \tag{11.21}$$

Proof. We take the following operatoion

$$\langle\phi_m, H\phi_n\rangle = \lambda_n\langle\phi_m\phi_n, \tag{11.22}$$

and take the complex conjugate which gives

$$\langle H\phi_n, \phi_m\rangle = \lambda_n^*\langle\phi_n, \rangle\phi_m\rangle, \tag{11.23}$$

and after switching m and n, we use the fact that H is self adjoint to get that

$$\langle\phi_m, H\phi_n\rangle = \lambda_m^*\langle\phi_m, \phi_n\rangle. \tag{11.24}$$

By comparing our first and last expression, we have proved the theorem. □

Implications

1. We can first look at the case where $m = n$, then we get that by the positive definiteness of the inner product, λ_m are real (nice for quantum mechanics).

2. $m \neq n$ and we assume the eigenvalues are not degenerate and show that ϕ_n, ϕ_m are orthogonal.

3. The $\{\phi_m\}$ form a complete sense in the sense that for any x in the space, x can be written as a linear combination of the ϕ_m,

$$x = \sum a_n\phi_n. \tag{11.25}$$

11.2 Infinite Dimensional Spaces

Now we define an infinite dimensional vector in analogy to the finite case as

$$\begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} f(a) \\ f(a+x) \\ \vdots f(b) \end{pmatrix}, \tag{11.26}$$

where f is a complex function on the interval $[a, b]$. Obviously, there are infinitely many points and so we have an infinite dimensional space. We define addition between vectors as the additions of the corresponding functions. So we need the addition of the functions themselves to be closed under addition. So we are really doing functional analysis. This means that boundary conditions are important. We restrict our focus to the space of square integrable functions which says that we only allow functions which satisfy

$$\int_a^b dx\, |f(x)|^2 < \infty. \tag{11.27}$$

We can now define an inner product as

$$\langle f, g \rangle = \int_a^b dx\, f^*(x) g(x), \tag{11.28}$$

which can be generalized with inclusion of a weight function $w(x)$ which is real and positive definite in the domain. Suppose we take an operator A such that

$$A = \frac{d^2}{dx^2}, \tag{11.29}$$

and space of $f(x)$ where $f(0) = 0$ and $f(b) = 0$. Then we know that

$$\frac{d^2}{dx^2} \phi_n = \lambda_n \phi_n, \tag{11.30}$$

so we can let $\lambda_n = -k_n^2$, which implies that

$$\phi'' = -k^2 \phi, \tag{11.31}$$

which has the solution

$$\phi = \begin{cases} \sin(kx) \\ \cos(kx) \end{cases} \tag{11.32}$$

and we choose the sin solution as our trial solution since it satisfies the boundary conditions. Since $\phi(b) = \sin(kb) = 0$, we need that

$$k_n = \frac{\pi n}{b}, \tag{11.33}$$

which implies that

$$\phi_n = \sin\left(\frac{n\pi}{b} x\right). \tag{11.34}$$

11.3 Hermitian Differential Operators

In order for differential operators to be Hermitian, they need to restricted. So in the case as before where $A = d^2/dx^2$, with boundary conditions as before $(f(0) = 0; f(b) = 0$. We want to know if A is Hermitian, which requires definition of an inner product as

$$\int_0^b dx\, f * g \tag{11.35}$$

To show this is Hermitian, note that

$$\begin{aligned}
\langle f, AG\rangle &= \int_0^b f^* \frac{d^2}{g} dx^2 \\
&= \left[f^* \frac{dg}{dx} \right]_0^b - \int_0^b dx\, \frac{f^* dx}{dx} \frac{dg}{dx} \\
&= \left[f^* \frac{dg}{dx} - \frac{df^*}{dx} g \right]_0^b + \int_0^b dx\, \frac{d^2 f^*}{dx^2} g \\
&= \int_0^b dx\, \frac{d^2 f^*}{dx^2} g \\
&= \langle Af, G\rangle,
\end{aligned} \tag{11.36}$$

so the operator is Hermitian. We have already solved the eigenproblem and so by the theorem know that we have a complete and orthogonal basis set for any $f(x)$ in the space. So if $f(x)$ is in the space,

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x). \tag{11.37}$$

Yet, there is some subtlety to this.

11.3.1 Completeness Theorem For Hilbert Space

In order to define completeness, we will speak of "completeness in the mean". So suppose we have some function which is given by a power series which we truncate as

$$f_N = \sum_{n=1}^{N} a_n \phi_n, \tag{11.38}$$

then there are three types of convergence. There is uniform convergence which states that given $\epsilon > 0$ there exists N such that

$$|f_N - f| < \epsilon \tag{11.39}$$

for all x. This is not guarenteed by the completeness theorem. We will not even get pointwise convergence which is when N depends on x. So we will instead have a much weaker form called completeness in the ean. Here for $\epsilon > 0$, there exists N such that

$$\int_a^b dx\, |f_N(x) - f(x)|^2 < \epsilon. \tag{11.40}$$

We will illustrate this by example. I am bit bothered why this isn't uniform convergence in the norm of the space. Maybe its because the norm requires a square root? Still doesn't really fix it for me.

We can run into the Gibb's phenomena which causes narrow ripples in the solutions near the edges. The results is that we can never match the boundary conditions. This says that you need to choose the right basis. This means that there are tons of eigenfunctions which allows one to make good choices.

The coefficients a_n may be found by inversion which means that if

$$f = \sum a_n \phi_n, \tag{11.41}$$

then

$$\langle \phi_m, f \rangle = \sum_n a_n \langle \phi_m, \phi_n \rangle = a_m \langle \phi_m, \phi_m \rangle, \tag{11.42}$$

which implies that

$$a_m = \frac{\langle \phi_m, f \rangle}{\langle \phi_m, \phi_m \rangle}. \tag{11.43}$$

11.3.2 General Operator - Sturm-Liouville Systems

We now move to the general operator. The objective is to find general differential operators and define an appropriate space so we have the Sturm Liouville System. Then we will be guaranteed the complete set of eigenfunctions that we want. So we can write the general operator

$$L = p(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + q(x), \tag{11.44}$$

in the space with boundary conditions at a and b. We define this on the vector space of square integrable functions. We want to force this to be a Hermitian operator, which we can do by defining a suitable inner product. Define the inner product as

$$\langle f, g \rangle = \int_a^b dx\, w(x) f^* g, \tag{11.45}$$

where $w(x)$ is a positive weight function. We need the weight function so what

$$\langle f, Lg \rangle = \langle Lf, g \rangle, \tag{11.46}$$

for all f, g. Now we can see in general, that the following will not be true in general,

$$\int_a^b dx\, f^* Lg = \int_a^b dx\, g(LF)^*, \tag{11.47}$$

which is why we need the weight function which leads to

$$\int_a^b dx\, w f^* Lg = \int_a^b dx\, w g(Lf)^*, \tag{11.48}$$

where we can find a w to force this to be true (Two integration by parts). So for a correct choice of w, L is Hermitian and looks like

$$Lf = \frac{1}{w}\frac{d}{dx}\left[pw\frac{df}{dx}\right] + qf \tag{11.49}$$

which we can also write as

$$Lf = p\frac{d^2 f}{dx^2} + \frac{(pw)'}{2}\frac{df}{dx} + qf, \tag{11.50}$$

and by comparing to the initial equation, we can see that

$$\frac{(pw)'}{2} = p_1, \tag{11.51}$$

which gives $w(x)$. So we can solve this by noting that

$$p\frac{w'}{w} + p' = p_1 \implies \frac{w'}{w} + \frac{p'}{p} = \frac{p_1}{p}, \tag{11.52}$$

noting that we have divided by p, so it cannot be zero. This implies that

$$(\ln(pw))' = \frac{p_1}{p}, \tag{11.53}$$

which implies that

$$\begin{aligned} \ln(pw) &= \int \frac{p_1}{p}\,dx \\ \implies w &= \frac{1}{p}\exp\left[\int \frac{p_1}{p}\,dx\right], \end{aligned} \tag{11.54}$$

and then we have a Hermitian operator. Now we need to get the boundary conditions right. If we demand that

$$\langle g, Lf\rangle = \langle Lg, f\rangle, \tag{11.55}$$

then we have that

$$\begin{aligned} &\int dx\, wg^* \left[\frac{1}{w}\frac{d}{dx}\left(pw\frac{df}{dx}\right) + qf\right] \\ &= [pwf'g^*]_a^b - \int dx\,\left[\frac{dg^*}{dx}(pw)\frac{df}{dx} + qfg^*w\right] \\ &= [pwf'g^*]_a^b - [(g^*)'pwf]_a^b + \underbrace{\int dx\,\left[f\frac{d}{dx}\left[pw\frac{dg^*}{dx}\right] + gfg^*w\right]}_{\langle Lg,f\rangle}, \end{aligned} \tag{11.56}$$

so we need the boundary conditions to make those first two terms 0. So we have the boundary conditions as

$$[pw(f'g^* - f(g^*)']_a^b = 0. \tag{11.57}$$

If we do all this then the eigenfunctions we get are guaranteed to be a complete set.

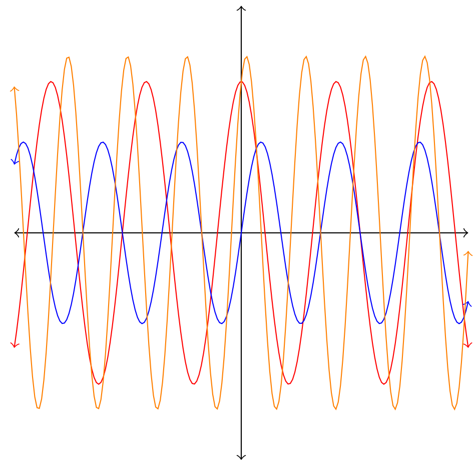

Figure 4: Periodic functions

11.4 Examples

(1) Suppose f is periodic in b an example of which is shown in Figure 4 Now let $A = d/dx$. This is not Hermitian which can be shown quite easily and so it is not applicable to the Sturm-Louiville framework we have developed. In this case, we have a complex error, we need the operator to be something like id/dx which is the momentum operator (missing some $-\hbar$ factor).

(2) Suppose we have the operator

$$\frac{d^2}{dx^2}, \tag{11.58}$$

and the boundary conditions are $f'(0) = 0$ and $f(b) = 0$. So we want the solutions

$$\frac{d^2\phi}{dx^2} = \lambda\phi, \tag{11.59}$$

which we know how to solve. If λ is positive and real we get exponential equations and if λ is negative and real we get sin and cos functions. These are known as evanescent and oscillatory respectively. It will be very hard to satisfy these boundary conditions with evanescent functions, so often it is convenient to use oscillatory functions. So for convenience, we let

$$\lambda = -k^2, \tag{11.60}$$

so our equation becomes

$$\frac{d^2\phi}{dx^2} = -k^2\phi, \tag{11.61}$$

which means are solution is given as linear combination of the solutions which is

$$\phi = A\sin(k\phi) + B\cos(k\phi), \tag{11.62}$$

and since $\phi'(0) = 0$, $A = 0$, and since $\phi(b) = 0$,

$$\cos(kb) = 0, \tag{11.63}$$

which quantizes k as

$$k_n b = \frac{\pi}{2}n, \tag{11.64}$$

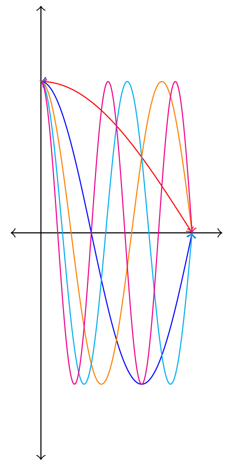

Figure 5: Complete set for $b = 1$. These are known as accordian since we keep turning up the period.

where n odd. So

$$\phi_n = \cos(k_n x). \tag{11.65}$$

We can plot these as in 5

(3) We now have the Bessel operator

$$L_B(f) = \frac{1}{x}\frac{d}{dx}\left(x\frac{df}{dx}\right), \tag{11.66}$$

and by comparison, we have that $q = 0, p = 1, w = x$. We demand that

$$[x(f'g^* - f(g^*)')]_{a=0}^{b>0} \tag{11.67}$$

If we suppose that $f(0)$ and $f'(0)$ are not specified, we can try to solve the eigenproblem given as

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) = \lambda\phi, \tag{11.68}$$

with boundary conditions $\phi(b) = 0$ and $\phi(a)$ unspecified, but we want $xf'g \to 0$ at $x = 0$ "well behaved". We will switch notation to the cylindrical coordinate ρ which is usually used for Bessel functions and get the following eigenvalue problem,

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi}{d\rho}\right) = \lambda\phi. \tag{11.69}$$

Then we let $\lambda = -k^2$ for the same reason as before and let

$$k\rho = x, \tag{11.70}$$

and we get that the solutions are $J_0(k\rho)$ and $N_0(k\rho)$. So we know that

$$\phi(x) = AJ_0(k\rho) + BN_0(k\rho), \tag{11.71}$$

where we can set $B = 0$ since it is not well behaved at $x = 0$, where well behaved is defined as $xf'g \to 0$ at $x = 0$. We need $\phi(b) = 0$ and so this is possible for J_0, although they are not periodic with

$$J(x_{0n}) = 0, \tag{11.72}$$

for specific numbers x_{0n}. So we have the quantization condition that

$$k_n b = x_{0n}, \tag{11.73}$$

which implies that

$$\phi_n = J_0\left(x_{0n}\frac{\rho}{b}\right), \tag{11.74}$$

which is a complete set of functions. We can define any function in the space

$$f(\rho) = \sum a_n J_0\left(x_{0n}\frac{\rho}{b}\right), \tag{11.75}$$

where

$$a_n = \frac{\int_0^b d\rho\, \rho J_0\left(x_{0n}\frac{\rho}{b}\right) f(\rho)}{\int_0^b d\rho\, \rho J_0\left(x_{0n}\frac{\rho}{b}\right) J_0\left(x_{0n}\frac{\rho}{b}\right)}. \tag{11.76}$$

Now we return to the question of why we threw away N_0. Note that as $x \to 0$ this function asymptotes to $\ln(x)$. There are two possible reasons for this:

- Not square integrable (not a problem here)
- Boundary condition $[x(f'g - fg')] \to 0$ as $x \to 0$ for any f, g in the space.

The boundary conditions are not satisfied for the N_0 so it cannot be part of the solution set. However, there are other boundary conditions where N_0 is a perfectly valid solution. So we can define the orthonormal condition as

$$\nu \int_0^b d\rho\rho J_\nu(k_n\rho) J_\nu(k_m\rho) = \delta_{mn}\frac{b^2}{2}[J_{\nu+1}(x\nu_n)]. \tag{11.77}$$

(4) Nowe we have the Legendre operator which is given as

$$Lf = \frac{d}{dx}\left[(1-x^2)\frac{df}{dx}\right], \tag{11.78}$$

so $q = 0, w = 1, p = 1 - x^2$. We need p to not be zero on the domain. So the domain must be in $[-1, 1]$ or outside on either side. The standard domain to take here is $x \in [-1, 1]$ since this is very useful for spherical coordinates. The domain outside of this is a nasty differential equation, but could be solved asymptotically. With this domain, we must have

$$[(1-x^2)((f^*)'g - f^*g')]_{-1}^1, \tag{11.79}$$

which works as long as f, g don't blow up terribly at end points (see homework 11). So now we want to solve the eigenvalue problem which is

$$L\phi_n = \lambda_n \phi_n$$

$$\implies \frac{d}{dx}\left[(1-x^2)\frac{d\phi}{dx}\right] = -\nu(\nu+1)\phi, \tag{11.80}$$

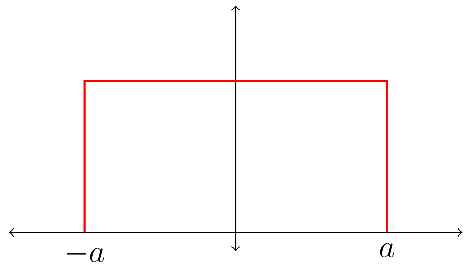

Figure 6: Box function

where we have chosen the minus sign by looking at the asymptotics. To solve this equation, we might first want to look at the asymptotic solutions (done on a homework). We can use Frobenius to solve this equation and find one solution which is the P_ν solution (blows up at $x = 1$). We can also find one that blows up at $x = -1$ which is the Q_ν solution. So in principlee

$$\phi = AP_\nu(x) + BQ_\nu(x), \tag{11.81}$$

but we need to make sure the functions are square integrable (these are) and check the boundary condition which fails in this case. The P_ν are well behaved if $\nu \in \mathbb{Z}$, but the Q_ν do not. So the solution must be that P_n are the eigenfunctions which are the Legendre polynomials. If we have different endpoints, then we need more restrictions on our functions since $1 - x^2 \neq 0$ and we still need the boundary condition satisfied for Sturm-Liouville theory to work.

(5) Now we want to do the infinite domain. So now $x \in (-\infty, \infty)$. The best way to do this is to take

$$x \in \left[-\frac{L}{2}, \frac{L}{2}\right], \tag{11.82}$$

and then let $L \to \infty$. We want to represent a local function (or periodic) with an approximate scale (where it is significant) which we will call $a \ll L$. So we will look at the momentum operator which gives

$$i\frac{d}{dx}\phi = -k\phi, \tag{11.83}$$

and choose periodic boundary conditions as $\phi(x + L) = \phi(x)$. We can solve this equation as

$$\phi = e^{ikx}, \tag{11.84}$$

and by satisfying the boundary condition as $k_n L = 2\pi n$, we have that

$$\phi_n = e^{2i\pi xn/L}, \tag{11.85}$$

and we can now take $L \to \infty$. We will illustrate this by example. Suppose that $f(x)$ is a box function as shown in figure 6 We can expand

$$f(x) = \sum A_n\phi_n, \tag{11.86}$$

and we know that

$$\langle\phi_n f(x)\rangle = A_n\langle\phi_n, \phi_m\rangle, \tag{11.87}$$

and solving for A_n, we get

$$A_n = \frac{2\sin(k_n a)}{k_n L}, \tag{11.88}$$

which we have gotten by evaluating the function at the endpoints and computing the inner product as L. Now we can write our funciotn as

$$f(x) = \sum_n A_n e^{2\pi i x/L}. \tag{11.89}$$

This summation can be written as an integral by letting

$$f(x) = \sum_{n=-\infty}^{\infty} A(n)\Delta n e^{ik(n)x} = \int_{-\infty}^{\infty} dn A(n) e^{ik(n)x}, \tag{11.90}$$

by calculating dk and n, we can find the integral as

$$f(x) = \frac{L}{2\pi}\int_{-\infty}^{\infty} dk\, A(k) e^{ikx}, \tag{11.91}$$

and we can absorb the constant into A if we want.

11.4.1 Comments for Midterm

There will be no Sturm-Liouville theory on the exam. There is a typo in solution to 10.1. It should be

$$y(x) = s^{\alpha}\sum_{n=0}^{\infty} a_n s^n, \tag{11.92}$$

instead of x where the first s is located. Some topics are

- Regular Pertubation Theory
- Singular Pertubation Theory
- Frobenious Method
- Green's functions
- Integral Transform

11.4.2 Summary of Sturm-Liouville

Given an operator

$$L(f) = \frac{1}{2}\frac{d}{dx}\left(pw\frac{df}{dx}\right) + qf, \tag{11.93}$$

want a complete set of eigenfunctions for a defined $x \in [a, b]$. So we make the system Sturm-Liouville.

1. let

$$\langle f, g\rangle := \int_a^b dx\, w f^* g. \tag{11.94}$$

2. Pick boundary conditions such that

$$[pw((f^*)'g - f^*g')]_a^b = 0, \tag{11.95}$$

where $w > 0$ and $p \neq 0$ on domain.

3. We can now find eignefunctions

$$L\phi_n = \lambda_n \phi_n, \tag{11.96}$$

where $\langle \phi_n, \phi_m \propto \delta_{m,n}$.

Then we are guarented that

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \tag{11.97}$$

with completeness in the mean. And then we can calculate the coefficients.

12 PDE's of Mathematical Physics

The most important partial differential equations of physics are

- Laplace's equations

$$\nabla^2 \phi(\vec{x}) = 0. \tag{12.1}$$

- Diffusion equation

$$\frac{\partial \psi(\vec{x}, t)}{\partial t} = D\nabla^2 \psi(\vec{x}, t) \tag{12.2}$$

- Shrödinger equation

$$-i\frac{\partial \psi(\vec{x}, t)}{\partial t} = \frac{1}{2}\nabla^2 \psi(\vec{x}, t) - V(x)\psi(\vec{x}, t) \tag{12.3}$$

- Wave equation

$$\frac{\partial^2 \psi(\vec{x}, t)}{\partial t^2} - c^2\nabla^2 \psi(\vec{x}, t) \tag{12.4}$$

Notice that all of these equations include the ∇^2 operator, and all are homogeneous, but could be made inhomogeneous. Also notice that the diffusion equation and Shrödinger's equations are somewhat similar. Perhaps most importantly, all of the equation are linear. An example of a non-linear system is the Navier-Stokes equations which looks like

$$\frac{\partial}{\partial t} n + \vec{D} \cdot (n\vec{u}) = 0 \tag{12.5}$$

$$n\left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}\right)\vec{u} = -T\vec{\nabla} n, \tag{12.6}$$

where T is a constant. Note that both $n, \vec{u}$ are functions are $\vec{x}, t$. This equation is much more complicated because there are non-linear terms. This models an isothermal fluid, it can be made more general.

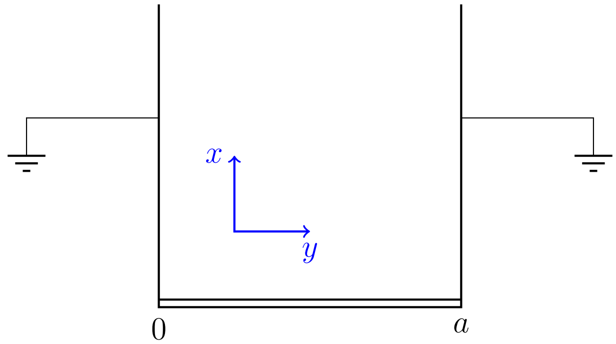

Figure 7: Conduction field for Laplace's eqution problem.

12.1 Laplace's Equations

Our first example will be a charged plate on an insulator surrounded by grounded walls as shown in Figure 7 So we ant to solve

$$\nabla^2\phi = 0 \tag{12.7}$$

Subject to the boundary conditions $\phi(x \to \infty) = 0$ and $\phi = 0$ on the grounded surfaces. On the induction place $\phi = V(y)$ is given. We are going to assume it is long that there is no z dependence and ϕ is a function of x, y. So we can write that

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2, \tag{12.8}$$

but we know that $\partial_z = 0$, so we have our equation as

$$\partial_x^2\phi + \partial_y^2\phi = 0. \tag{12.9}$$

We can separate the variables as

$$\phi(x, y) = P(x)Q(x), \tag{12.10}$$

which leads to

$$\frac{1}{P}\frac{d^2P}{dx^2} + \frac{1}{Q}\frac{d^2Q}{dy^2} = 0. \tag{12.11}$$

This is only possible is

$$\frac{1}{P}\frac{d^2P}{dx^2} = -\alpha\,;\ \frac{1}{Q}\frac{d^2Q}{dy^2} = \alpha \tag{12.12}$$

which we can solve by previous methods. We will get four solutions of the form

$$P \propto e^{\pm i\sqrt{\alpha}x}\,;\, Q \propto e^{\pm\sqrt{\alpha}y}, \tag{12.13}$$

and now we need to satisfy the boundary conditions. Once we do this, by the uniquess theorem, we have the solution. The better way to do this is to anticipate the physics and pick solutions which are physically reasonable in terms of boundary conditions.
Approach with boundary conditions in mind. The PDE given in Equation (12.9) has constant coefficients. So we should try

$$e^{iky} \to \begin{cases}\sin(ky)\\ \cos(ky)\end{cases} \to \sin(ky), \tag{12.14}$$

where we have picked $\sin(ky)$ to match the boundary condition at zero. So we will try

$$\phi(x, y) = f(x)\sin(ky). \tag{12.15}$$

Plugging this back into the equation, we get that

$$\sin(ky)\left(\frac{d^2 f}{dx^2} - k^2 f\right) = 0, \tag{12.16}$$

so we can set the term in parenthesis to zero since $\sin(ky)$ not in general zero. The solutions will be an exponential or a linear combination of $\sinh(kx), \cosh(ky)$, but we cannot use either of them since they diverge at infinity and so we must choose $f(x) = e^{-kx}$. So we have that

$$\phi(x, y) = e^{-kx}\sin(ky). \tag{12.17}$$

Now we look at the boundary condition on the right by choosing $k \in \mathbb{R}_+$. We need $\sin(ka) = 0$ so $ka = n\pi$. Therefore

$$k_n = \frac{n\pi}{a}. \tag{12.18}$$

So in general the solution is given as

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n e^{-k_n x}\sin(k_n y), \tag{12.19}$$

where the A_n is determined by the bottom boundary condition. So at $x = 0$

$$V(y) = \sum_{n=1}^{\infty} A_n \sin(k_n x). \tag{12.20}$$

So we know that

$$\int_0^a \sin(k_n y)V(y) = A_n \int_0^a dy\,[\sin(k_n y)]^2 \tag{12.21}$$

which implies that

$$A_n = \frac{2}{a}\int_0^a dy\,\sin(k_n y)V(y). \tag{12.22}$$

So we have found the solution.

12.2 Diffusion Equation

In this problem, we have a room with width L and some temperature as a function of x, $E(x)$. The walls are at $T = 0$. If there are no heaters, the temperature will diffuse to 0 since the walls are being maintained at zero. The equation governing this is

$$\frac{\partial T(x,t)}{\partial t} = D\frac{\partial^2 T(x,t)}{\partial x^2}, \tag{12.23}$$

where T is temperature. We will do this using the physics of the problem and noting that we want oscillatory solutions due to the boundary conditions. So we will choose $\sin(kx)$ since we need zero at 0. We can also quantize k as

$$\sin(k_n x), \tag{12.24}$$

where $k_n = n\pi/L$. So we can let

$$T(x,t) \propto u(t)\sin(k_n x), \tag{12.25}$$

which leads to

$$\frac{du}{dt}\sin(kx) = -k_n^2 D u(t)\sin(kx) \implies \frac{du}{dt} = -k_n^2 D u(t) \tag{12.26}$$

which we can solve as

$$u(t) = e^{-k_n^2 D t}, \tag{12.27}$$

which implies that

$$T(x,t) = \sum_{n=1}^{\infty} A_n e^{-k_n^2 D t}\sin(k_n x). \tag{12.28}$$

So now we can invert the initial condition to get the A_n as done in the previous problem which will give the complete solution.

12.3 Polar Geometry

We are going to look at a cylinder of radius a and solve the equation

$$\nabla^2\psi = 0, \tag{12.29}$$

with the boundary conditions $\phi(r = a, \phi) = f(\phi)$. So we can use polar coordinates (ρ, ϕ, z) and by symmetry we know that ∂_z. So we can write the Laplacian in polar coordinates

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\psi + \underbrace{\frac{\partial^2\psi}{\partial z^2}}_{=0} = 0. \tag{12.30}$$

We look for the constant coefficient which is ϕ and so we can try $e^{ik\phi}$ where we have included the i since we know it will make it easier. By the peroidicty of boundary we have that

$$e^{ik(\phi+2\pi)} = e^{ik\phi}, \tag{12.31}$$

and also that

$$e^{ik2\pi} = 1 = e^{2\pi i m}, \tag{12.32}$$

so $k = m$ where m is any integer. So we separate variables and try

$$\psi = f(s)e^{im\phi} \tag{12.33}$$

If we plug this into our equation and multiply by ρ^2, we have that

$$\rho\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - m^2 f = 0. \tag{12.34}$$

This is equidimensional and so we can try ϕ^α which gives that $\alpha = \pm m$. So this means we can try any combination of the form

$$\psi = e^{im\phi}\begin{cases}\rho^m \\ \rho^{-m}\end{cases} \tag{12.35}$$

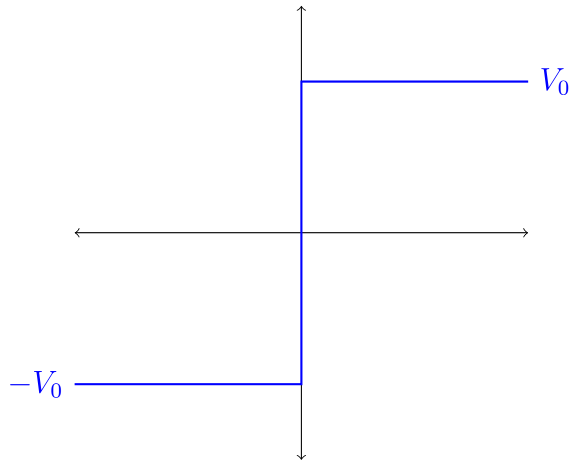

Figure 8: Step function

yet we know that one of these terms blows up for $\rho \to 0$ for $m > 0$ and the other if $m < 0$. So we can write the solution for $m \neq 0$ as

$$\psi = \sum_{m=\infty}^{\infty} \rho^{-m} e^{im\phi} + \sum_{m=1}^{\infty} B_n \rho^m e^{im\phi} \tag{12.36}$$

and for $m = 0$, then we have the solutions are that $f = c$, $f = \ln \rho$. In this case we can write the general solution as

$$\psi = \sum_{m=0}^{\infty} \rho^m (A_m e^{im\phi} + B_m e^{-im\phi}). \tag{12.37}$$

Now suppose that $f(\phi)$ is a step function in ϕ shown in figure 8. Here we ould use the symmetry to try

$$\psi = \sum_{m=1}^{\infty} A_m \rho^m \sin(m\phi). \tag{12.38}$$

Now we will try the case where the semicircle is separated in half with the top at V_0 and the bottom at $-V_0$ with an inner place of radius b at $V = 0$. So we can use the same solution set as before. We can still choose the $\sin(m\phi)$ due to the parity of the problem. So we have that

$$\psi = \sum_{m=1}^{\infty} \frac{A_m \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{\rho}{b}\right)^{-m} \right]}{\left[\left(\frac{a}{b}\right)^m - \left(\frac{a}{b}\right)^{-m} \right]} \sin(m\phi), \tag{12.39}$$

where we have used many tricks of normalization to make it easier. Now if we apply the final boundary condition, we get that the A_m.

12.3.1 Pizza Problem

We have a wedge of side a and angle β. where the far side is positively charged at $f(\phi)$ and the other sides are grounded. The solutions are the same as given in previous problem, except there we have no periodicity an in ϕ we must have $\sin(k\phi) = 0$ for β which we will set to be $\phi/4$ for convenience. So this means that $k_n = 4n$. So our solution set is

$$\psi = \sin(4n\phi) \begin{cases} \rho^m \\ \rho^{-m} \end{cases} \tag{12.40}$$

12.4 Diffusion Polar Geometry

Now we have a ring where $T = 0$ on the outside and T difuesses inside according to

$$\frac{\partial T}{\partial t} = D\nabla^2 T, \tag{12.41}$$

where the initial value is given as a function of ρ, $f(\rho)$. We want to find the solution for all t. For a linear equation, our symmetries will normally remain. Here we have that $\partial_z = 0$,and $\partial_\phi = 0$. So we know that $T = T(\rho, t)$. So we have that

$$\frac{\partial T}{\partial t} = D\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial T}{\partial \rho}\right). \tag{12.42}$$

We can see that t is a constant coefficients. So we can try $e^{-\alpha t}$, which will tranform the equation to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dT}{d\rho}\right) + \frac{\alpha T}{D} = 0, \tag{12.43}$$

which is the Bessel zero equation. So the solutions are $J_0(k\rho)$, $N_0(k\rho)$, where $k^2 = \alpha/D$. So we can write our solution as

$$T(\rho, t) = \sum_{n=1}^{\infty} A_n J_0(k_n\rho)e^{-k^2 Dt}, \tag{12.44}$$

where we know that $J_(ka) = 0$ which quantizes k. We now must put in the intial condition to gt A_na s

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_0(k_n\rho), \tag{12.45}$$

which we can invert to get our answer.

12.5 Spherical Laplaces Equation

So we again deal with

$$\nabla^2\psi = 0 \tag{12.46}$$

where we are working on a sphere of radius a split into a top and bottom where the top is held at V_0 and the bottom and $-V_0$ and there are insulators in between the two sides as shown in Figure 9. We will use (r, θ, ϕ), we can see that $\partial_\phi = 0$ by symmetry. We can work Laplaces equation in spherical coordinates as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\partial r\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{12.47}$$

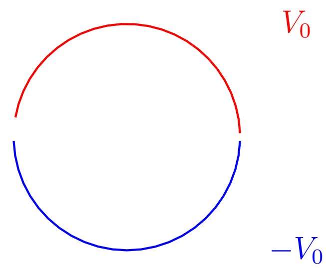

Figure 9: A sphere of radius a split into a top and bottom where the top is held at V_0 and the bottom and $-V_0$ and there are insulators in between the two sides

So our equation works out to be

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)=0 \tag{12.48}$$

We can separate variables as

$$\psi = R(r)T(\theta), \tag{12.49}$$

and so we plug in and divide by ψ and we get that

$$0=\underbrace{\frac{1}{R}\frac{d}{dR}\left(r^2\frac{dR}{dr}\right)}_{-C}+\underbrace{\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)}_{C}. \tag{12.50}$$

We can let $x=\cos\theta$ and so $\sin\theta d\theta=-dx$ which allows us to structure the second part of the equation as

$$C=\frac{1}{T}\frac{d}{dx}\left((1-x^2)\frac{dT}{dx}\right), \tag{12.51}$$

which is just our Legendre operator. Note now that $\theta\in[0\to\pi]$ and so $x\in[-1,1]$. We can let $C<0$ and we can write it as $\nu(\nu+1)$ to bring it into the normal which gives us two equations

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right]=-\nu(\nu+1)T \tag{12.52}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\nu(\nu+1)R, \tag{12.53}$$

where we know that the solutions to the first equation are $P_\nu(x)$ and to the second equation are $Q_\nu(x)$. So we know that

$$T=AP_\nu+BQ_\nu, \tag{12.54}$$

however, both of these blow up unless ν is an integer. So since we must have physical solutions, $\nu=n$, a positve integer which gives the Legendre Polynomials. We can solve the r part for a given n by equidimensional methods and so the solution is r^α where $\alpha=n,-(n+1)$. This second solution gives an issue at $r=0$ and so we must discard that solution and the solution is given as

$$\psi(r,\theta)=\sum_{n=1}^{\infty}A_n\left(\frac{r}{a}\right)^n P_n(\cos(\theta)) \tag{12.55}$$

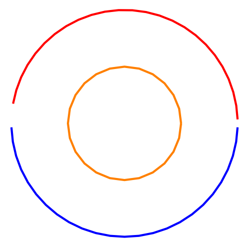

Figure 10: A sphere of radius a split into a top and bottom where the top is held at V_0 and the bottom and $-V_0$ and there are insulators in between the two sides. There is an inner circle at $v = 0$.

We know that if we look at $\phi(a, \theta)$ we have a step function since we are along the edge of the sphere. So we can apply this boundary condition to find the A_n. So if we let this step function be S, then

$$S = \sum_{n=1}^{\infty} A_n P_n(x), \tag{12.56}$$

where we can use the orthagaonality condition of the legendre polynomicals which si given as

$$\int_{-1}^{1} P_l P_m dx = \delta_{lm} \frac{2l+1}{2}. \tag{12.57}$$

We would need to do the integrals for each term in the series and we could get the A_n.

12.5.1 Spherical Laplaces Example 2

If we now add a sphere with radius b with potential at 0, then we can use the same solution set as before with the exception that we cannot throw away the r^{n+1} solution since $r \neq 0$ and this is well behaved. This is shown in Figure 10. So, now we have the solution set

$$\psi(r, \theta) = \sum_{n=1}^{\infty} A_n \frac{\left[\left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^{n+1}\right]}{\left[\left(\frac{a}{b}\right)^n - \left(\frac{b}{a}\right)^{n+1}\right]} P_n(x) \tag{12.58}$$

where we have chosen these constants to make the boundary conditions easier.

12.6 Spherical Laplaces Example 3

Now we will cut a bowtie out of the circle as shown in 11. Here we have to change our quantization conditions on T. So

$$T = AP_\nu(x) + BQ_\nu(x). \tag{12.59}$$

We can get equations from both the top and bottom and bottom, so

$$0 = AP_\nu(-\beta_0) + BQ_\nu(-\beta_0) \tag{12.60}$$
$$0 = AP_\nu(\beta_0) + BQ_\nu(\beta_0), \tag{12.61}$$

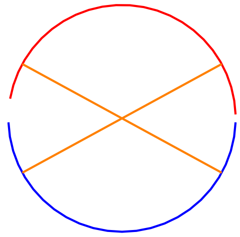

Figure 11: A sphere of radius a split into a top and bottom where the top is held at V_0 and the bottom and $-V_0$ and there are insulators in between the two sides. There is a bowtie potential in the middle at potential 0.

which means that the determinant of the matrix represenation of this equation must vanish and this will give us quantization conditions. We can then continue the problem, but its quite messy.

12.7 Non-Linear PDE

The problem is posed as a flow past a smooth sphere of radius a. The flow has the same velocity u_0 at both $-\infty$ and ∞. The equations are the ideal fluid equations which are derived from the Navier-Stokes equations. So for steady state, we have the equations

$$\vec{\nabla} \cdot (n\vec{u}) = 0 \tag{12.62}$$

$$nM\vec{\nabla}\vec{u} = -\vec{\nabla}p \tag{12.63}$$

$$\vec{u} \cdot \vec{\nabla}p + \frac{5}{3}p\vec{\nabla} \cdot \vec{u} = 0, \tag{12.64}$$

where n is the number density, $\vec{u}$ is the flow,M is the mass of a typical particle, and p is the pressure. This is a very complicated set of PDE's which are non-linear and have many terms. There are no standard techniques and the so we should look at asymptotics, so scale the equations. For the first equation and third equations, both terms are of the same order. However, for the second term if we work out the proper vector identifies, we can see that we approximately have to lowest order

$$-\vec{\nabla}p_0 = 0, \tag{12.65}$$

where $p = p_0 + p_1 + \dots$ and also write that $\vec{u} = \vec{u}_0 + \vec{u}_1$ and $n = n_0 + n_1 + \dots$, which gives the following set of equations

$$\vec{\nabla} \cdot (n_0\vec{u}_0) = 0 \tag{12.66}$$

$$0 = -\vec{\nabla}p \tag{12.67}$$

$$\vec{u}_0 \cdot \vec{\nabla}p_0 + \frac{5}{3}p_0\vec{\nabla} \cdot \vec{u}_0 = 0, \tag{12.68}$$

which says that p_0 is a constant and so we get for the third equation

$$\vec{\nabla} \cdot \vec{u}_0 = 0, \tag{12.69}$$

which is the condition for incompressible flow, and our first equation becomes

$$\vec{u}_0 \cdot \vec{\nabla} n_0 = 0, \tag{12.70}$$

which givs that n_0 is a constant. So we can go to wrist order and look at the equation

$$n_0 M \vec{u}_0 - \vec{\nabla} u_0 = -\vec{\nabla} p_1 \tag{12.71}$$

which if we take the curl of and use some identites, we get the equation

$$\nabla \times [\vec{u}_0 \times (\vec{\nabla} \times \vec{u}_0)] = 0, \tag{12.72}$$

which implies that

$$\vec{\nabla} \times \vec{u} - 0 = 0, \tag{12.73}$$

which implies we can describe this whole equation with some boundary conditions as Laplace's equation to lowest order for a subsonic fluid.