132 Notes

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This is the first class. Basically just doing general first day things. We are going to start by going over uniform convergence and other topics covered in analysis I. Look at the handout. First homework is a review of 131. We are going to look at space of continuous functions for the start. We now begin
In Linear Algebra be have seen the space of continuous functions.

Definition 1.1 (Space of Continuous Functions $\mathcal{C}$)**.**

$$\mathcal{C}[a,b] = \{f | f : [a,b] \to \mathbb{R}\}.$$

We have some properties. Let $f, g \in \mathcal{C}$. Then we know that

1) Zero element $z(x) = 0$.

2) Addition- $(f + g)(x) = f(x) + g(x)$

3) Scalar multiplication $cf(x) = f(cx)$

This means that $\mathcal{C}$ is a vector space. We know that the domain of $f, [a,b]$ is closed and bounded set in $\mathbb{R}$ and therefore, we know that $[a,b]$ is compact by the Heine-Borel theorem. So we know that $\mathbb{R}$ is complete when it is target space of x. There are several metrics we can put on $\mathcal{C}$. One is,

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|.$$

This is referred to as the uniform metric. Another metric we can put on this space is

$$d(f,g) = \int_a^b |f(x) - g(x)| dx.$$

In the first metric we have a complete space in the second metric we do not.
Now we will look at functions continuous between two metrics, not just $\mathbb{R}$. So consider,

$$f : A \subset (M, d) \to (N, \rho)$$

where M, N are metric spaces with respective metrics, d, ρ. So suppose that f is continuous. Also suppose that A is a compact subset of (M, d) and (N, ρ) is complete.

Definition 1.2 ($\mathcal{C}_b(A, N)$)**.** *The space of continuous functions defined on a compact set A whose target space is complete.*

We want to look at the relations of continuous functions and compact sets.

Theorem 1.3. *Let $f : A \to N$ be continuous and A be compact, then*

1. *f is bounded.*

2. *$\sup f$ and $\inf f$ exist and f assumes its max and min.*

3. *$f(A)$ is a compact subset of N.*
4. *f is uniformly continuous on some subset B of A.*

Example 1.4. *Show that $f(x) = x^2$ is continuous at some x_0.*

Proof. I need to show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - x_o| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. We have

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0| < \epsilon$$

We know that $\delta(1 + 2x_0) < \epsilon$ implies that

$$\delta = \frac{\epsilon}{1 + 2x_0}.$$

So we can take

$$-1 < x - x_0 < 1$$

which implies that $x + x_0 < 1 + 2x_0$. So we know that δ is a function of ϵ only and therefore it is uniformly continuous on any compact set, however it is not uniformly continuous on all of $\mathbb{R}$. □

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Definition 2.1 (Uniform Convergence)**.** *Let $B \subset A$ and $f : A \to N$. f is uniformly continuous on the set B if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B$, $d(x, y) < \delta$ implies that $D(f(x), f(y)) < \epsilon$. where d, D are the respective metrics.*

Note that δ works for all $x, y \in B$ and only depends upon which ϵ is given.

Example 2.2. $f(x) = x^2$.

Proof. Note that this function has an unbounded slope meaning that the distance between successive points in the range cannot be bounded. So there is no δ that works and therefore this function is not uniformly continuous. □

Example 2.3. *Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and suppose that there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that f is uniformly continuous on (a, b).*

Proof. By the mean value theorem, we know that

$$f(x) - f(y) = f'(x_0)(x - y)$$

where $x_0 \in (x, y)$. So we have that

$$|f(x) - f(y)| = |f'(x_0)||x - y| \leq M|x - y|.$$

So if we let $\delta = \dfrac{\epsilon}{M}$ then we have uniform continuity. □

So this shows that a bounded slope implies uniform continuity under proper conditions. Note this is not an if and only if statement.

Example 2.4. $f(x) = \sqrt{x}$ *is uniformly continuous on* $[0, 1]$.

Note that $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ which is not bounded on $[0, 1]$, yet it is still uniformly continuous.

Theorem 2.5. *Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial function on $\mathbb{R}$. Then p is uniformly continuous if and only if $deg(p) < 2$.*

Definition 2.6 (Pointwise Convergence)**.** *Let $f_n : A \to N$, $n = 1, 2, \ldots$. We say (f_n) converges pointwise (or simply) to a function $f : A \to N$ if for each $x \in A$, $f_n(x)$ converges to $f(x)$ as a sequence in N.*

Example 2.7. $f_n : \mathbb{R} \to \mathbb{R}$. $f_n(x) = \dfrac{x}{n}$.

Proof. Note that these functions are just lines with decreasing slope. So they converge to 0 at every x.

$$\lim_{n\to\infty} f_n(x) = 0$$

so $f_n(x)$ converges pointwise to $f(x) = 0$. $\square$

Example 2.8. $f_n : \mathbb{R} \to \mathbb{R}$. $f_n(x) = x^n$.

Proof. Note that for $x > 1$,

$$\lim_{n\to\infty} x^n = DNE$$

and therefore for $x > 1$, $f_n(x)$ does not converge pointwise. However, if we restrict the domain to $[0, 1]$, then

$$\lim_{n\to\infty}^{n} = \begin{cases} 0 & if\ x \in [0,1) \\ 1 & if\ x = 1 \end{cases}.$$

So we have pointwise convergence to $f(x)$, however, $f(x)$ is not continuous. $\square$

Definition 2.9 (Uniform Convergence)**.** *A sequence of functions $\{f_n\}$ converge uniformly to f on some domain D if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$.*

Theorem 2.10. *A uniform limit of continuous functions is continuous.*

Proof. Let $\epsilon > 0$. Choose N such that for all $x \in D$, $n < N$ implies that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ by uniform continuity. Also choose $\delta > 0$ such that $|x - c| < \delta$ implies $|f_{N+1}(x) - f_{N+1}(c)| < \frac{\epsilon}{3}$. So if $|x - c| < \delta$ we have that

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_{N+1}(x) - f_{N+1}(x) - f_{N+1}(c) - f_{N+1}(c) - f(c) \\ &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(c)| + |f_{N+1}(c) - f(c)| < 3\frac{\epsilon}{3} = \epsilon \end{aligned}$$

where the middle term is by continuity and the outer terms are by uniform convergence. $\square$

Theorem 2.11 (Test for Uniform Converge of Sequences)**.** *A sequence of function of function with domain D converges uniformly to a function f if there is a real valued sequence $\{a_n\}$ such that $a_n \to 0$ and*

$$|f_n(x) - f(x)| < |a_n|$$

for all $x \in D$ and for all $n \in N$.

Proof. We know that $a_n \to 0$. So given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $n > N$, then $|a_n| < \epsilon$. This N is independent of $x \in D$ and so

$$|f_n(x) - f(x)| \leq |a_n| < \epsilon$$

for all $x \in D$ whenever $n > N$. This implies that $\{f_n\}$ converges uniformly. $\square$

Theorem 2.12. *Let $\{f_n\}$ be a sequence of functions defined on A and $f_n \to f$ uniformly on $B \subset A \Leftrightarrow$ the sequence of numbers*

$$d_n := \sup\{|f_n(x) - f(x)| : x \in B, n \in N\}$$

converge to 0.

Proof. $\Rightarrow$. Suppose that $f_n \to f$ uniformly. Then given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n > N$ and for all $x \in B$. So for $n > N$, $d_n < \epsilon$. Therefore, $d_n \to 0$.
$\Leftarrow$. Suppose $d_n \to 0$. Then we know that for sufficiently large n and for all $x \in B$,

$$|f_n(x) - f(x)| < d_n < \epsilon.$$

So $f_n \to f$ uniformly. $\square$

Example 2.13. $f_n(x) = \dfrac{nx^2}{1+n}$. $A = [0, 1]$. *Show that $\{f_n(x)\}$ converges uniformly.*

Proof. Note that by applying l'hopitals law,

$$\lim_{n\to\infty} f_n(x) = lim_{n\to\infty} \frac{nx^2}{1+nx} = x.$$

So we have that

$$|f_n(x) - f(x)| = \left|\frac{nx^2}{1+nx} - x\right| = \left|\frac{x}{1+nx}\right|.$$

So, $d_n = \frac{1}{1+n}$ which goes to 0 as $n \to \infty$. So, we have uniform convergence. □

Theorem 2.14. *Let $\{f_n\}$ be a sequence of continuous of functions on $[a, b]$ that converge pointwise to $f(x)$. The limit of the integrals is equal to the integral of the limit if and only if $\{f_n\}$ is uniformly convergent.*

Note probably will restate this later in a better version.

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So we will prove previous theorem.

Proof. We know for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|f_n(x) - f(x) < \epsilon$. So we have

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b f_n - f\right| \le \int_a^b |f_n - f| \le \epsilon(b-a).$$

□

Note that uniform limits do not in general preserve continuity.

Example 3.1. $f_n(x) = |x|^{1+\frac{1}{n}}$

Proof. Note that $f_n(x) \to f(x) = |x|$. Also observe that $f_n \to f$ uniformly on $[-1, 1]$ since

$$|f_n - f| < \frac{1}{n}.$$

Also note that each f_n is differentiable, however, f is not differentiable at zero. So while continuity is preserved, differentiability is not preserved under uniform convergence. □

Note the Weierstrass Example which is everywhere continuous but nowhere differentiable.

Theorem 3.2. *If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.*

Theorem 3.3 (Dini's Theorem)**.** *Let $A \subset \mathbb{R}$ compact. Suppose $\{f_k\} : A \to \mathbb{R}$ is a sequence of continuous functions such that*

a) $f_k(x) \ge 0$ for each $x \in A$

b) $f_k(x) \le f_l(x)$ whenever $k \ge l$

c) $f_k(x) \to 0$ pointwise for each $x \in A$.

Then $f_k(x) \to 0$ uniformly on $[a, b]$.

What this theorem really tells us is the following. Let $A = [a, b]$. If a sequence of continuous functions $\{f_n(x)\}$ converges monotonically to $f(x)$ for each $x \in [a, b]$ and if $f(x)$ is continuous then the convergence is uniform.
Note that if $f_n \to f$ uniformly we cannot conclude that $f_n' \to f'$ uniformly. An example of this is

$$f(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Definition 3.4 (Convergence of Series). *We say*

$$\sum_{k=1}^{\infty} g_k = g$$

or the series converges to g if the sequence of partial sums $\{s_n\}$ where

$$s_n = \sum_{k=1}^{n} g_k$$

converges to g pointwise, $s_n \to g$. If the convergence is uniform, then the pointwise convergence must be uniform.

Lemma 3.5. *Let $g_k : A \to \mathbb{N}$ be continuous and*

$$\sum_{k=1}^{\infty} g_k = g$$

uniformly. Then g is continuous.

This is true because continuity is preserved under uniform convergence and each g_k is continuous.

Theorem 3.6 (Weierstrass M-test). *Let $\{f_k\}$ be a sequence of functions with domain A and let there exist constants, M_k with*

$$|f_k(x)| \le M_k$$

for all $x \in A$ and

$$\sum_{k=1}^{\infty} M_k$$

converges. Then the series,

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly and absolutely.

Proof. Since I have $|f_k(x)| \le M_k$ for all $x \in A$, I know that

$$\sum_{k=1}^{\infty} |f_k(x)| \le \sum_{k=1}^{\infty} M_k.$$

So by comparison test this series converges absolutely and therefore it converges for each $x \in A$. Thus $\sum f_k$ converges pointwise on A to a point which we will call s. So we will look $s_n = f_1 + f_2 + \cdots + f_n$ and $t_n = M_1 + M_2 + \cdots + M_n$ where each s_n is a function and each t_n is a real number. Say that $s_n \to s$ and $t_n \to t$. So for each x, if $n > M$ we have

$$|s_n(x) - s_m(x)| = \left| \sum_{k=m+1}^{n} f_k(x) \right| \le \sum_{k=m+1}^{n} |f_k(x)| < \sum_{k=m+1}^{n} M_k = t_n - t_m.$$

Thus we have that the sequence

$$\{|s_n(x) - s_m(x)|\}_{n=1}^{\infty}$$

and $\{t_n - t_m\}$ are both convergent. If we take the limit as $n \to \infty$, then we know that $|s(x) - s_m(x)| \le t - t_m$ for all $m \in \mathbb{N}$ and $x \in A$. Since when $m \to \infty$, $t - t_m \to 0$ this gives uniform convergence. $\square$

Example 3.7. *Show that the series*

$$\sum_{n=1}^{\infty} \frac{(sin(nx))^2}{n^2}$$

converges uniformly.

Proof. We know that
$$\left|\frac{(sin(nx))^2}{n^2}\right| \le \frac{1}{n^2}.$$
So let $M_n = \frac{1}{n^2}$. So we know that $\sum \frac{1}{n^2}$ converges and so by Weierstrass M-test, the series converges uniformly. □

Example 3.8. *Show that*
$$\sum_{n=1}^{\infty} e^{-nx}$$
converges uniformly on any closed subinterval of $(0, \infty)$*.*

Proof. Let $I = [a, \infty)$. If $x \in I$, then $e^-nx \le e^{-na}$. We know that
$$\sum_{n=0}^{\infty} e^{-na}$$
is a geometric series with ratio which converges. Therefore be Weierstrass M-test, the series converges uniformly. □

Theorem 3.9 (Cauchy Criterion for $\mathbb{R}$)**.** *Suppose* $\{f_k\} : S \subset \mathbb{R} \to \mathbb{R}$*. be a set of functions. Then* $f_n \to f$ *uniformly if and only if for every* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $n, m > N$ *implies that*
$$|f_n(x) - f_m(x)| < \epsilon$$
for all $x \in S$*.*

Proof. $\Leftarrow$ Fix n and take limit as $m \to \infty$. Since $f_m(x) \to f(x)$ as $m \to \infty$, we obtain that
$$|f_n(x) - f(x) < \epsilon$$
for all $x \in S$ which implies that $f_n \to f$ uniformly. no prove of other direction given. □

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Definition 4.1 (Dense)**.** $G \subset \mathbb{R}$ *is called dense if* $\overline{G} = \mathbb{R}$

Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, but $\mathbb{Z}$ is not dense in $\mathbb{R}$. We say that $\mathbb{Z}$ is nowhere dense.

Definition 4.2 (Nowhere Dense)**.** *A set E is called nowhere-dense if* $\overline{E}$ *contains no non-empty open intervals.*

Theorem 4.3. E *is nowhere dense in* $\mathbb{R}$ *if and only if* $(\overline{E})^c$ *is dense in* $\mathbb{R}$*.*

Theorem 4.4 (Baire's Theorem)**.** *The set of real number* $\mathbb{R}$ *cannot be written as a countable unit of nowhere dense sets.*

Proof. Suppose that one can write $\mathbb{R}$ is a union of nowhere dense sets and get contradiction. □

This theorem is about the size of $\mathbb{R}$. Note that $\mathbb{R}$ is "fat" in a certain sense. Baire's theorem is true for complete metric spaces in general. For example $(\mathcal{C}[a, b], d(f, g))$ is a complete metric space. This is the space of continuous function with the metric
$$d(f, g) = max_{a \le x \le b} |f(x) - g(x)|.$$
In this space the collection of continuous functions that are differentiable at even one point cannot be written as the countable union of nowhere dense sets. So Baire's Theorem tells us that most continuous functions do not have derivatives at all.

Theorem 4.5 (Fundamental Theorem of Calculus)**.** *Given a continuous function f, the function*

$$F(x) = \int_a^x f(x')dx'$$

satisfies

$$F'(x) = f(x).$$

Theorem 4.6 (Cauchy Criterion for Complete Metric Space)**.** *Let (N, ζ) be a complete metric space and $A \subset (N, \zeta)$. Then define $f_k : A \to (N, \zeta)$. So we know that $f_k \to f$ uniformly if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $(\zeta(f_n(x), f_m(x)) < \epsilon$.*

Theorem 4.7 (Cauchy's Criterion for Series)**.** *$\sum_{i=1}^{\infty} g_k$ converges uniformly on A if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k > N$,*

$$||g_k(x) + \cdots + g_{k+p}(x)|| < \epsilon$$

for all $x \in A$ and all integers $p \geq 0$.

Theorem 4.8. *Let $\sum_{n=0}^{\infty} f_n$ be a series of function defined on $[a, b]$ where each f_n is continuous on $[a, b]$. If $\sum f_n$ conerges uniformly to f on $[a, b]$ then*

$$\int_a^b \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} \int_a^b f_n.$$

Proof. Let $s_n = \sum_{k=1}^{\infty} f_k$. Then we write that

$$\int_a^b s_n(x)dx = \int_a^b \sum_{k=1}^{n} f_k(x)dx = \sum_{k=1}^{n} \int_a^b f_k(x)dx.$$

Since we know that $s_n \to f$ on $[a, b]$, we also have

$$\lim_{n\to\infty} \int_a^b = \int_a^b \lim_{n\to\infty} S_n = \int_a^b f = \int_a^b \sum_{k=1}^{\infty} f_k.$$

□

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Today we will look at the space of continuous functions. We will also look at some fixed point theorems which play a major role in mathematics especially in differential equations such as the picard iterates theorem. We want to look at the space of continuous functions with regard to its completeness in terms of certain metrics.

Example 5.1. *Consider the geometric series*

$$\sum_{n=0}^{\infty} (it)^n.$$

Proof. Suppose $0 < r < 1$, then $t \in [-r, r]$. We know that this series converges to $\frac{1}{1+t}$ since $|-t|^n < r^n$. $\sum r^n$ converges by Weierstrass M-test, we have uniform convergence on any interval $[-r, r]$ where $0 < r < 1$. Thus if we take $x \in (-1, 1)$ we can integrate this series term by term from 0 to x.

$$\int_0^x \frac{1}{1+t} = \int_0^x \sum_{n=0}^{\infty} (-1)^n (t)^n = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

So we can see that

$$ln(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

for any $x \in (-1, 1)$. So we have obtained the Taylor expansion of $ln(x)$. □

Theorem 5.2. *Consider* $\mathcal{C}[a,b]$, $d(f,g) = \max_{a\leq x\leq b}\{|f(x)-g(x)|\}$, $\{f_n\} \in \mathcal{C}[a,b]$. *Then,* $\{f_n\}$ *converges in this metric if and only if* $f_n \to f$ *uniformly.*

Proof. $\Leftarrow$ trivial
$\Rightarrow$ Suppose that $f_n \to f$ pointwise in the given metric. This means that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(f_n, f) < \epsilon$. In other words

$$d(f_n, f) = \max_{a\leq x\leq b}\{|f_n(x) - f(x)|\} < \epsilon.$$

So certainly, $|f_n(x) - f(x)| < \epsilon$ which implies that $f_n \to f$ uniformly. □

We will now examine the completeness of the space of continuous functions.

Theorem 5.3. $f_n \to f$ *uniformly if and only if* $f_n \to f$ *in* $\mathcal{C}_b(= \mathcal{C}(A, N))$ *which is the space of bounded continuous functions on a compact set.*

Theorem 5.4 (Completeness of function space)**.** *The function space* $\mathcal{C}[a,b]$ *with "sup" metric is a complete linear space.*

Proof. Let $\{f_n\}$ be a Cauchy sequences in $\mathcal{C}[a,b]$. Then we know that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$, we have

$$d(f_n, f_m) = \max_{a\leq x\leq b}\{|f_n(x) - f_m(x)|\} < \epsilon.$$

Hence for any fixed $x_o \in [a,b]$ we know that $|f_n(x_0) - f_m(x_0)| < \epsilon$ whenever $n, m > N$. This shows that

$$(f_1(x_0), f_2(x_0), \dots, f_m(x_0), \dots)$$

is a cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $f_m(x_0) \to f(x_0$ as $m \to \infty$. in this way we can associate for each $x \in [a,b]$ a unique number $f(x)$. This defines a pointwise function. $f \in \mathcal{C}[a,b]$. Looking at the metric in the space of continuous functions we have that

$$\max_{x\in[a,b]}\{|f_m(x) - f(x)|\}$$

when $n > N$. Hence for every $x \in [a,b]$ we have that $f_m(x) - f(x)| < \epsilon$. for $m > N_0$. This shows that $f_m \to f$ uniformly. Since al of the f_m are continuous f is continuous. Therefore we know that $f_m \to f \in \mathcal{C}[a,b]$. Therefore, the space is complete. □

This proof is applicable to spaces $\mathcal{C}(A, N)$ where A is compact and N is complete. For example, $\mathcal{C}_b$ is a complete metric space.

Theorem 5.5 (Heine-Borel for $\mathbb{R}^n$.)**.** $A \in \mathbb{R}^n$ *is compact if and only if* A *is closed and bounded.*

Theorem 5.6 (Arzeld-Ascoli for space of continuous functions.)**.** $A \in \mathcal{C}[a,b]$ *is compact if and only if* A *is closed bounded and equicontinuous.*

We need to clarify

1. $\mathcal{C}[a,b]$ is complete metric space. What is characterization of compactness for a complete metric space or for just a metric space.

2. What is equicontinuous and what does it mean? How does it relate to other types of continuity that we have already seen.

So first we will examine compactness in an arbitrary metric space.

Definition 5.7 (Totally Bounded)**.** *We say* $A \subset (M,d)$ *is totally bounded if given* $\epsilon > 0$ *there exists finitely many points* $x_1, x_2 \dots x_n \in M$ *such that*

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(x_i).$$

We say this set $\{x_1, x_2, \dots, x_n\}$ *is an* ϵ*-net for* A.

Definition 5.8 (Alternate Definition of Totally Bounded). *A is totally bounded if and only if for all $\epsilon > 0$ there exists finite number of sets $A_1, A_1 \dots A_n \subset A$ such that $diam(A_i) < \epsilon$ for all i such that*

$$A \subset \bigcup_{i=1}^{n} A_i.$$

Theorem 5.9. *A metric space (M, d) is compact if and only if it is both complete and totally bounded.*

This is basically because closed subset of a complete metric space are also complete.

Theorem 5.10. *(M, d) is compact if and only if ever sequence in M has a subsequence that converges to a point in M.*

Proof. We want to show that totally bounded + completeness is equivalent to every sequence in the metric space has a Cauchy subsequence that must converge since the space is complete. □

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When we write $\lim_{n\to\infty} f_n(x)$ we can control the x-dependence if we are given info the limit is uniform in x. We want to go on step further. We want the limit to be uniform in n.

Definition 6.1 (Equicontinuity). *Let $F \subset \mathcal{C}((M, d), (N, \rho))$. We say F is equicontinuous family if for all $\epsilon > 0$, there exists $\delta > 0$ such that for $x \in M$ and all $f \in F$, $d(x, x') < \delta$ implies $\rho(f(x), f(x')) < \epsilon$.*

Equicontinuity is the same as uniform continuity except that we demand that δ can be chosen independent of n as well as x. So this is a stronger statement.

Example 6.2. *$F \subset \mathcal{C}([a, b], \mathbb{R})$ where*

$$F = \{f_n : f_n(x) = x + n\}$$

Proof. F is equicontinuous because given $\epsilon > 0$ we can choose δ.

$$|f_n(x) - f_n(y)| = |(x + n) - (y + n)| = |x - y| < \delta < \epsilon.$$

□

We want to find ways to find equicontinuous families.

Example 6.3. *$F = \{f_n : f_n' s$ have the same Lipshitz Constant$\}$.*

Proof. This is the set of all functions with derivatives bounded by the same constant. This is equicontinuous because if we suppose that f_n is differentiable for all n, we can apply mean value theorem. So we have

$$|f_n(x) - f_n(y)| \leq k|x - y|$$

where $k = \sup |f_n(x)|$. This implies that k is independent of n. So we can let $\delta = \frac{\epsilon}{k}$. □

The reason that this equicontinuous property is important is because it will help us characterize compactness in the space of continuous functions.

Theorem 6.4 (Ascoli-Arzela). *Let $E \subseteq \mathcal{C}([a, b]; \mathbb{R})$. Then E is compact if and only if E is closed,bounded and equicontinuous.*

We now saw presentations on the previous theorems proof and on the Cantor set and its applications and it took all class.

7 2/12/13

Today we are going to do some Fixed Point Theorems.

Definition 7.1 (Fixed Point)**.** *Let $T : X \to X$, we say $x \in X$ is a fixed point of T if $T(x) = x$.*

For an example of something that has no fixed points look at

Example 7.2. *$T(x) = x + a$ where $a \neq 0$*

There are also things that fix only some points.

Example 7.3. *$T(x) = x^2$*
Note that 0 and 1 are fixed.

So we need to consider questions of existence and uniqueness. First we will give a fixed point theorem for a closed interval.

Theorem 7.4. *Let f be a continuous function defined on $[a, b]$ whose range is also contained in $[a, b]$. So $f : [a, b] \to [a, b]$. Then there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.*

Proof. This is a consequence of the intermediate value theorem. We assume that $f(a) \neq a$ and $f(b) \neq b$ or we are done. Without loss of generality assume that $f(a) > a$ and $f(b) < b$. Define

$$F(x) = f(x) - x.$$

Then we know that F is continuous. So look at $F(a) = f(a) - a > 0$ and $F(b) = f(b) - b < 0$. So we use the intermediate value theorem to $F(x)$ to say that there exists x_0 such that $F(x_0) = 0$ which implies that $f(x_0) = x_0$. □

Now we are going to look at a special class of functions called contractions.

Definition 7.5 (Contraction)**.** *Let $X = (X, d)$ be a metric space. $T : X \to X$ is called a contraction on X if there is a positive constant $k < 1$ such that for all $x, y \in X$, $d(T(x), T(y)) \leq (k)d(x, y)$.*

Note that a contraction is a continuous map.

Definition 7.6 (Sequential Continuity)**.** *$f : X \to Y$ is continuous at a if $x_n \to a$ implies that $f(x_n) \to f(a)$.*

Since we know that f is a contraction, if $x_n \to a$ we have that if $d(x_n, a) \leq \frac{\epsilon}{k}$ for all $n > N$ we know that $d(T(x_n), T(a)) \leq kd(x_n, a)$.

Theorem 7.7. *Let $I = [a, b]$ and $f : I \to I$. If*

$$\alpha = \sup_{x \in I} |f'(x)| < 1$$

then f is a contraction.

Proof. For any $x, y \in I$ by the mean value theorem we have

$$f(x) - f(y) = f'(\zeta)(x - y)$$

for some ζ between x and y. So we have that

$$|f(x) - f(y)| = |f'(\zeta)||x - y| \leq \sup_{x \in I} |f; (\zeta||x - y| < \alpha|x - y|.$$

□

Theorem 7.8 (Banach Contraction Mapping Theorem)**.** *Let $X = (X, d)$ be a complete metric space. Then any contraction $f : X \to X$ has a unique fixed point.*

Proof. We will do this proof in steps.

1. Choose an arbitrary point $x_0 \in X$. Define a sequence $\{x_n\}$ inductively as follows.
$$x_{n+1} = f(x_n).$$
So we have that
$$\begin{gathered} x_1 = f(x_0) \\ x_2 = f(x_1) = f(f(x_0)) = f^2(x_0) \\ \cdots \\ \cdots \\ x_n = f^n(x_0) \end{gathered}$$
where f^n is the nth composition of f.

2. We will show that this sequence is Cauchy. Observe that
$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq kd(x_n, x_{n-1}) = kd(f(x_{n-1}), f(x_{n-2})) \leq k^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq k^n d(x_1, x_0)$$
Where we have used the fact that f is a contraction. Thus if $n > m$. we have that
$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m=1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq (k^m + k^{m+1} + \cdots + k^{n-1}) d(x_1, x_0) = [k^m(1 + k + \cdots + k^{n-m-1})] d(x_1, x_0). \end{aligned}$$
We now use the identity
$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$
Since we know that $n > m$ we have that
$$0 < k < 1 \implies 1 - k^{n-m} < 1$$
So we have that
$$d(x_m, x_n) = \frac{k^m}{1-k} d(x_0, x_1).$$
Then it is clear that $d(x_m, x_n) \to 0$ as $n, m \to \infty$. Since $\{x_n\}$ is Cauchy in a complete metric space, we know that $x_n \to x^* \in X$.

3. Since f is continuous as contractions imply continuity we know that $f(x_n) \to f(x^*)$. So we have that
$$f(x^*) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^* \implies f(x^*) = x^*$$
So we have found a fixed point.

4. Now we will prove uniqueness. Suppose that there is another fixed point, y such that $f(y) = y$. Observe
$$d(y, x^*) = d(f(y), f(x^*)) \leq kd(y, x^*) = 0$$
since we have a contraction. Therefore we have uniqueness.

□

BCMT is extremely powerful because it allows us to construct a fixed point for a mapping by computing the limit of the sequence of iterates of that map. So it is constructive. It is also extremely dependent on the fact that $k < 1$. The class of functions when $k = 1$, $d(f(x), f(y)) < d(x, y)$ are known as non-expansive maps. In fact, a restrictive geometry needs to be placed on the domain of the mapping before the corresponding fixed point theorem are given.
Note, in the proof we had
$$d(x_m, x_n) \leq \frac{k^m}{1-k} d(x_1, x_0).$$
So if we take
$$\lim_{n\to\infty} d(x_m, x) \leq \frac{k^m}{1-k} d(x_1, x_0)$$
we have an error estimate.

8 2/21/13

The last few classes we have been studying fixed point theorems which are contained in the note packet. Today we started with a presentation on an example.

Example 8.1. *Let X be the set of continuous functions on $[a,b]$ and*

$$d(x,y) = \int_0^1 |x(t) - y(t)|dt$$

Then we know that (X,d) is not complete.

We know that this is true because we can find a Cauchy sequence that does not converge.
We are going to look at the **Weierstrass Approximation Theorem**. We know that polynomials are continuous and we know that uniform continuity preserves continuity. The Weierstrass approximation theorem is the converse of this. We now want to motivate this theorem a bit. We have been studying the space of continuous functions with the sup norm.
Note that for all practical purposes, instead of looking at $\mathcal{C}[a,b]$ we can look at $\mathcal{C}[0,1]$. We will now show why we can do this.

Lemma 8.2. *There is a linear isometry from $\mathcal{C}[0,1]$ onto $\mathcal{C}[a,b]$ that maps polynomials to polynomials.*

Proof. Define $g[a,g] \to [0,1]$ by

$$g(x) = \frac{x-a}{b-a}$$

for $a \leq x \leq b$. Then g is a homeomorphism. Rest of proof in notes. □

Proposition 8.3. $\mathcal{C}[0,1]$ *is separable.*

Proof. Complicated but idea is in note handout. Kind of proof by picture. □

Theorem 8.4 (Weierstrass Approximation Theorem)**.** *If a function $f(x)$ is continuous on a closed and bounded interval $[a,b]$, then for each $\epsilon > 0$ there exists a polynomial $P(x)$ such that $||f - P||_\infty < \epsilon$. Hence there is a sequence of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly on $[a,b]$.*

Note that there is a difference between this and Taylor expansions, mainly the locality of the approximation.

Definition 8.5 (Bernstein Polynomial)**.** *Let $f \in \mathcal{C}[0,1]$, the polynomial $\{B_n(f)\}_{n=1}^{\infty}$*

$$B_n(x,f) = (B_n(f)(x) = \sum_{k=0}^{n} f(\frac{k}{n})(\binom{n}{k} x^k (1-x)^{n-k}$$

is called the nth Bernstein polynomial of f¿ Note that $B_n(f)$ is a polynomial of degree at most n.

Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We also have

Theorem 8.6 (Binomial Theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

There are examples in the notes.

9 2/26/13

For test on Thursday:

1. Know Banach Contraction Mapping Theorem and its proof for exam on Thursday.
   The method of the proof is important
2. Weierstrass M-test
3. Point-wise and uniform convergence
4. Advantages of uniform convergence
5. State of Arezela-Ascoli Theorem
6. One homework question.
7. Guided thinking question.

Last class we were talking about a proof of Bernstein's Theorem. We have a new sheet of notes for this class as well so I will only fill in gaps in **TeX**.

Definition 9.1 (Algebra)**.** *Let $\beta \subset \mathcal{C}(A, \mathbb{R})$. β is called an algebra if*

1. *if $f, g \in \beta$, then $f + g \in \beta$ and $fg \in \beta$.*
2. *If $f \in \beta$, then $\alpha f \in \beta$ for all $\alpha \in \mathbb{R}$.*

Definition 9.2 (Separations)**.** *M is a metric space. A is a compact set in M. Let $\beta \in \mathcal{C}(A, \mathbb{R})$ be an algebra. We say that β separates the points of A if for $x \neq y \in A$ there is a function $f \in \beta$ such that $fx) \neq f(y)$.*

Example 9.3. *Let P_ϵ be the set of all even polynomials on $[-1, 1]$. This does not separate the points on A since*

$$f(-1) = f(1)$$

We know that Weierstrass Approximation theorem states that polynomials are dense in $\mathcal{C}[a, b]$. This means that the real linear combinations of the functions, $1, x, x^2, \ldots x^n \ldots$ are dense in $\mathcal{C}[a, b]$. Since P is generated by 1 and x, we can not omit 1 or x from the generating set. If we omit 1, then all of the terms vanish at zero and therefore the non-zero constant function would not be in the algebra. If this is the case, the closure will not be the space of continuous functions. If we only have 1, then the algebra only has constant functions and the closure will not be the space of continuous functions. So we must have both 1 and x as generators.

Theorem 9.4 (Stone Weierstrass)**.** *Let M be a metric space and $A \subset M$ is compact. Let $B \in \mathcal{C}[a, b]$ satisfying the following:*

1. *B is an algebra*
2. *B separates the points in A.*
3. *The constant function lies in B*

Then we have that $\overline{B} = \mathcal{C}[a, b]$ and therefore B is dense in $\mathcal{C}[a, b]$.

Notice

$$(f \wedge g)(x) = \min\{f(x), g(x)\} = \frac{1}{2}\{f + g\} - \frac{|f - g|}{2}$$
$$(f \vee g)(x) = \max\{f(x), g(x)\} = \frac{1}{2}\{f + g\} + \frac{|f - g|}{2}$$

This will be useful in the proof of the theorem.

Definition 9.5 (Trigonometric Polynomial)**.** *A trigonometric Polynomial is a finite linear combination of the functions* $\cos(kx)$ *and* $\sin(kx)$*,* $k = 1, 2 \ldots, n$ *which we write as*

$$T(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

where $a_k, b_k \in \mathbb{R}$*. The* ***degree*** *of the polynomial is the order of the highest non-zero coefficient.*

Let $\mathcal{C}_{2\pi}$ denote the set of 2π-periodic continuous functions $f : \mathbb{R} \to \mathbb{R}$. Let T_n be the collection of trig polynomials of degree at most n. Then T_n is a subspace of $\mathcal{C}_{2\pi}$.
Note that we can inductively show that $\cos(nx)$ can be written as a trig function of $\cos(x)$. For example

$$\cos(2x) = 2\cos^2(x) - 1$$

$$cos(3x) = 4\cos^3(x) - 3\cos(x)$$

$$\cos(4x) = 8\cos^2(x) - 8\cos^2(x) + 1$$

Note that there is a difference between Bernstein polynomials and regular polynomials and trig polynomials. However, similar results are true about uniform convergence of functions. Also note that Fourier series are all trig polynomials.
We can strengthen Weierstrass Approximation theorem by taking an arbitrary set of constants and finding a polynomial that will still work. **This is cool**.

10 3/5/2013

First we saw a proof the of the Stone-Wierstrass theorem. We have a packet on it and now we will move to differentiation. We have handwritten notes on this.

11 3/7/2013

Today I finished texing the book.

Theorem 11.1. *Let* X *and* Y *be normed linear spaces and* T *a linear transformation of* X *into* Y*. Then the following are equivalent:*

1. T *is continuous.*
2. T *is continuous at the origin.*
3. T *is bounded.*
4. *if* $B = \{x : ||x|| \leq 1\}$ *is the closed unit ball in* X*, then its image* $T(B)$ *is bounded in* Y*.*

Proof. We know that $1 \implies 2$ since if T is continuous everywhere, it is continuous at the origin.
Now we show $2 \implies 1$. Suppose T is continuous at the origin, then there exists $x_n \to 0$ so, want to show $Tx_n \to 0$. So we have

$$x_n \to x \Leftrightarrow x_n - x \to 0 \implies t(x_n - x) \to 0 \Leftrightarrow T(x_n) \to T(x)$$

$3 \implies 2$ is clear because boundedness.
$2 \implies 3$. Assume that T is not bounded. So assume no such M exists. So there is a sequence that goes to infinity. This is equivalent to saying that

$$\frac{||Tx_n||}{n||x_n||} < 1 \implies \left|\left|T\left(\frac{x_n}{n||x_n||}\right)\right|\right| > 1.$$

Set

$$y_n = \frac{x_n}{n||_n||} \to 0$$

but

$$T(y_n) \not\to 0.$$

So T is not continuous at the origin.
$3 \implies 4$. Since any non-empty subset of numbers is bounded if and only if it is contained in some closed sphere centered at the origin which implies what we want.
$4 \implies 3$. Suppose that $T(B)$ is contained in some closed ball of radius M centered at the origin.

Case 1 If $x = 0$, $Tx = 0$ clearly $||tx|| \leq M||x||$.

Case 2 If $x \neq 0$, $\frac{x}{||x||} \in B$ by 4. Since we know that $T(\frac{x}{||x||})$ is a bounded subset of Y. We can say that there exists $M \geq 0$ such that

$$\left|\left|T\left(\frac{x}{||x||}\right)\right|\right| \leq M \implies ||Tx|| \leq M||x||.$$

□

We now look at different types of norms.

$$||T|| = \sup\{||Tx|| : ||x|| \leq 1\}$$

$$= \sup\{||Tx|| : ||x|| = 1\}$$

With these we can get

$$||T|| = \inf\{M : M \geq 0, \text{and } ||Tx|| \leq M||x|| \quad \forall x\}.$$

So we have the following inequality

$$Tx|| \leq ||T||||x||$$

and

$$||Tx|| \leq \lambda||x|| \implies ||T|| \leq \lambda.$$

We call $||T||$, the operator norm.

Example 11.2. $C[a, b] \to \mathbb{R}$.

$$T(f) = \int_a^b f(x)dx$$

Show that T is linear and that $||T|| = b - a$.

We have that

$$|T(f)| = \left|\int_a^b f(x)dx\right| \leq \max_{a \leq t \leq b} |f(t)(b - a).$$

We know that this means

$$|T(f)| \leq (b - a)||f||_\infty.$$

So this implies that

$$||T|| = \sup_{||f||=1} |T(f)| \leq b - a.$$

To prove $||T|| \geq b - a$ we choose

$$f = f_0 = 1.$$

Then we have that

$$|T(f_0)| \leq ||T|| \; ||f_0||.$$

Since we know that $|T(f_0)| = b - a$ and therefore

$$||T|| \geq \frac{|T(f_0)|}{||f_0||} = b - a.$$

Theorem 11.3. *If $T, S \in L(\mathbb{R}^n, \mathbb{R}^m)$ and α is a scalar. Then*

1. $||T + S|| \leq ||T|| + ||S||$
2. $||\alpha T|| = |\alpha| ||T||$

and if $T \in L(R^n, R^m)$ and $S \in R^m \to R^k$, then

$$||ST|| \leq ||S|| ||T||.$$

Proof. proof is in notes and is mostly algebra with some properties of norms applies. □

We can induce a metric by a norm,

$$d(T, S) = ||T - S||.$$

Note that in general not all metrics cannot be induced by a norm. Frechet was the inventor of the metric, so we are going to look at the Frechet Metric on the set of all sequences.

$$S = \{x : \{x_n\} : \{s_n\} \text{ a}sequence\}.$$

Then we can define the Frechet Metric

$$d(x, y) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}.$$

This metric is not induced from a norm. Norms are defined by inner products and this is a stronger condition than a metric.

Theorem 11.4. *$B \subset L(R^n)$ be the set of all invertible operators on R^n. B is an open subset of $L(\mathbb{R}^n)$.*

Proof. we know that for all $T \in B$ there exists $\epsilon > 0$ such that

$$\{S \in L(\mathbb{R}^n) : ||S - T|| < \epsilon\} \subset B.$$

so we want to show

$$||S - T|| ||T^{-1}|| < 1$$

then $S \in B$. Let

$$||T^{-1}|| = \frac{1}{\epsilon}.$$

and $||S - T|| = \beta$. Then we have that $\beta < \epsilon$. So this gives

$$\epsilon ||x|| = \epsilon ||T^{-1} T x|| \leq \epsilon ||T^{-1}|| ||Tx|| \leq ||(T - S)x|| + ||Sx||$$
$$\leq \beta ||x|| + ||Sx||$$

So we have that

$$(\epsilon - \beta) ||x|| \leq ||Sx|| \implies Sx \neq 0 \implies x \neq \implies S \in B.$$

□

We are now going to talk about Frobenius Norms.

Definition 11.5 (The Frobenius Norm)**.** *The Frobenius norm $||A||_F$ of a matrix $A = (a_{ij})$ is obtained as*

$$||A||_F = \sqrt{\sum_{ij} a_{if}^2}.$$

We now did some examples of Frobenius norms, but my life texing of matrices will probably not be adequate here. Need to get faster.

Theorem 11.6. *If we denote the euclidean norm as $||x||_E$, then we have that*

$$||Ax||_E \leq ||A||_F ||x||_E.$$

When we found the row sum criterion earlier, we were actually doing a matrix norm.

12 3/12/2013

Definition 12.1. *The norm of the linear linear transformation $T : V \to W$, $||T||$ is*

$$||T|| = \sup_{||v|| \leq 1} ||Tv||.$$

We can use this norm to induce a metric $d(T, U) = ||T - U||$. We know that the linear transformations from $\mathbb{R} \to \mathbb{R}$ are all lines that pass through the origin since we know that $T(0) = 0$. Today we are going to talk about the derivative. We will give multiple definitions.

Definition 12.2 (The Derivative in one variable)**.**

1. *Let $f : \mathbb{R} \to \mathbb{R}$, $x \in R$. If*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists we say that f is differentiable and call that limit $f'(x)$.

2. *if there exists a number $f'(x)$ such that*

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0$$

then we saw that f is differentiable at x and $f'(x)$ is the derivative of f at x.

3. *If there exists a number $f'(x)$ such that*

$$r(h) = f(x+h) - f(x) - f'(x)h$$

and one has

$$\lim_{h \to 0} \frac{|r(h)|}{|h|} = 0$$

then f is differentiable at x and $f'(x)$ is the derivative of f at x.

Note that $r(h)$ can be interpreted as the error. So this gives that

$$f(x+h) \approx f(x) + f'(x)h$$

since $r(h)$ goes to zero faster than h does.

Definition 12.3 (Differentiability in Higher Dimensions)**.** *$f : \mathbb{R}^n \to \mathbb{R}^m$ and $x \in \mathbb{R}^n$. f is differentiable at $x \in \mathbb{R}^n$ if there exists a linear transformation $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that*

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - Ah||}{||h||} = 0$$

In that case $f'(x) = A$.

Say $n = m$, then $f : \mathbb{R}^n \to R^n$. We need to check uniqueness of this definition. It turns out it is unique. Suppose that $f(x+ = Bx$, for some $B \in (\mathbb{R}^n, \mathbb{R}^m)$.

$$f'(x) = Bh$$

13 3/28/13

Today we are going to do more about differentiability and get into Taylor's theorem for several variables. The differentiability stuff was covered in 131.

Example 13.1.

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ = & \text{if } x \in \mathbb{R} \end{cases}$$

Is f continuous and differentiable at 0?

We know that $f(0) = 0$ and we can see that in the irrational case, it tends to 0 as $x \to 0$. We can check differentiability and see that it is both continuous and differentiable at 0.

Example 13.2. *Does the existence of partial derivatives imply that Df exists?*

This is not true in general and is illustrated by the following example.

Example 13.3.

$$f(x,y) \begin{cases} x & y = 0 \\ y & x = 0 \\ 1 & else \end{cases}$$

We can see that the partials all exist, but this function is not continuous because

$$\lim_{(x,y)} f(x,y) \neq f(0,0).$$

Note that in general if we can find two paths along which the limit is different then we know it is not continuous.

Example 13.4.

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

a) Find $\dfrac{\partial f}{\partial y}$ *and* $\dfrac{\partial f}{\partial y}$

b) Find $\dfrac{\partial^2 f}{\partial x \partial y}$ *and* $\dfrac{\partial^2 f}{\partial y \partial x}$

Note that

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

We can arrive at this by using the definition of the limit. Now for part b, we can use the partial derivatives we found in part a) to evaluate the expression. What we find is that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$$

So mixed partials are not necessarily the same. What went wrong is that$f$ is not a C^2 function.

Definition 13.5 (Directional Derivative)**.** *$f : A \subset \mathbb{R}^n \to \mathbb{R}$. $\vec{v} \in \mathbb{R}^n$ is a unit vector. We write*

$$D_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}.$$

Example 13.6. $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$. $\vec{v} = i = (1,0)$. *Then we know that*

$$\vec{a} + h\vec{v} = (a_1, a_2) + h(1,0) = (a_1 + h, a_2).$$

So we have that

$$D_{\vec{i}} f(\vec{a}) = \lim_{h \to 0} \frac{f(a_1 + h, a_2) - f(a_1, a_2)}{h} = \frac{\partial f}{\partial x}.$$

Also note that

$$D_{\vec{v}} f(\vec{a}) = \vec{\nabla} f \cdot \vec{v} = ||\vec{\nabla} f|| \cdot ||\vec{v}|| cos\theta$$

So the maximum of the directional derivative is the magnitude of the gradient.
We see that the existence of the directional derivative of f at x_0 does not imply the differentiability of f at x_0. We can think of it mostly as a steepest descent. Now we start Taylor's theorem in many variables which is in the handout.

14 4/2/13

Today we are going to consider the mean value theorem for higher dimensions. We are going to generalize from the one dimensional case.

Theorem 14.1 (Mean Value Theorem). *Given $f : [a, b] \to \mathbb{R}$ continuous and $f(a, b) \to \mathbb{R}$ is differentiable, then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

We want to generalize this for $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Example 14.2. $f(0, 1]0) \to \mathbb{R}^2$ *such that*

$$f(x) = (x^2, x^3).$$

We want to find a $c \in (0, 1)$ such that

$$\frac{f(1) - f(0)}{1} = f'(c),$$

which implies that

$$\frac{(1, 1) - (0, 0)}{1} = (2c, 3c^2).$$

This implies that

$$2c = 1$$

$$3c^2 = 1.$$

There are no c that satisfies these equations, so we cannot just use the mean value theorem for single variable functions. So now we will have a more rigorous definition starting with a better definition of c.

Definition 14.3 (Lines in $\mathbb{R}^2$). *c lies on the line segment joining x and y (or c is between x and y) if*

$$c = (1 - \lambda)x + \lambda y,$$

for some $\lambda \in [0, 1]$.

Definition 14.4 (Convex Set). *A set $A \subset \mathbb{R}^n$ is called convex if for each $x, y \in A$, the line segment joining x and y also lies in A.*

Theorem 14.5 (Mean Value Theorems). *There are two mean value theorems.*

1. *Suppose that $f : A \subset \mathbb{R}^n \to R$ is differentiable on an open set A. For $x, y \in A$ where the line segment joining x and y lies in A, there is a point c on the line segment such that*

$$f(y) - f(x) = Df(c) \cdot (y - x).$$

   Note that this is equivalent to saying

$$f(y) - f(x) = \nabla f(c) \cdot (y - x).$$

2. *$f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on an open set A. Suppose the line segment joining x and y lies in A and $f = (f_1, f_2, \ldots, f_m)$. Then there exists points $c_1, c_2, \ldots, c_m)$ on that segment such that*

$$f_j(y) - f_j(x) = Df_j(c_j) \cdot (y - x).$$

   Here $J = 1, 2, \ldots m$. This is basically doing the first law component wise.

Proof. Consider $h : [0, 1] \to \mathbb{R}$ (h is a composition of $f : \mathbb{R}^n \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}^n$.) So define h as

$$h(t) = f((1 - t)x + ty).$$

Note that $g(t) = (1 - t)x + ty$. Since this is a composition of functions, h is differentiable on $(0, 1)$ and so there exists $t_0 \in (0, 1)$ such that

$$h(1) - h(0) = h'(t_0)(1 - 0).$$

This is equivalent to

$$f(y) - f(x) = h'(t_0) = f'(g(t_0)) \cdot g'(t_0) = Df(g(t_0)) \cdot (y - x),$$

by the chain rule. So choose

$$c = g(t_0) = (1 - t_0)x + t_0 y.$$

Then we have

$$f(y) - f(x) = Df(c) \cdot (y - x).$$

This easily generalizes to higher dimensions by a component wise argument. □

Example 14.6. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable on a convex set A¿ Suppose that $||\nabla f|| \leq M$ for $x \in A$. Prove*

$$|f(x) - f(y)| \leq M||x - y||$$

for $x, y \in A$. Is this true if A is not convex?

Proof. $Df(c)$ is $1 \times n$ matrix and therefore $Df(c) = \nabla f(c)$ so I can apply mean value theorem and take the absolute value to get the first result.
The second part is not true. We will prove with counterexample. Suppose that

$$A = \{(x, y) \in \mathbb{R}^2 | x < 0 \text{ or } x > 1\}$$

Define

$$f = \begin{cases} 1 & \text{if} \quad x > 1 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

f is differentiable and $f'(x) = 0$ which implies that $f'(x)$ is bounded by $\frac{1}{10}$. So look at $x = 2$ and $y = -2$. Then I have that

$$|f(x) - f(y)| = |1$$

and

$$||x - y|| = 4$$

the first part would implies then that

$$1 \leq \frac{1}{10} 4$$

so we know that A must be convex for the theorem to hold. □

Theorem 14.7 (Mean-Value Inequality). *Let A be an open convex set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to R^m$ be differentiable with a continuous derivative. suppose that there exists $M > 0$ such that*

$$||Df(x)(y)|| \le M||y||$$

for all $x \in A$, $y \in \mathbb{R}^n$. Then we have

$$||f(x_1) - f(x_2)|| \le M||x_1 - x_2||.$$

Proof. Recall that

$$\left|\left|\int_a^b f(x)dx\right|\right| \le \int_a^b ||f(x)||dx.$$

Idea of the proof is very similar to the proof of the mean value theorem for higher dimensions. The setup is the same. Then we have

$$\frac{d}{dt}\left[h(t)\right] = Df(tx_1 + (1-t)x_2) \cdot (x_1 - x_2).$$

Since f is a C^1 function we can integrate. So we have

$$\int_0^1 \frac{d}{dt}[f(tx_1 + (1-t)x_2)]dt = \int_0^1 Df(tx_1 + (1-t)x_2) \cdot (x_1 - x_2)dt$$

$$= [f(tx_1 + (1-t)x_2]_0^1 = f(x_1) - f(x_2)$$

By taking norms we have that

$$||f(x_1) - f(x_2)|| \le \int_0^1 ||Df(tx_1 + (1-t)x_2) \cdot (x_1 - x_2)||dt \le M \int_0^1 ||x_1 - x_2||dt.$$

This implies that

$$||f(x_1) - f(x_2)|| \le M||x_1 - x_2||.$$

□

15 4/4/13

Definition 15.1 (Non-Decreasing). *We say $h : \mathbb{R} \to \mathbb{R}$ is non-decreasing if*

$$h'(x) \ge 0.$$

This comes from the mean value theorem.

Example 15.2. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and $f(0) = 0$. If f is differentiable on this interval and f' is non-decreasing, prove that*

$$gx = \frac{f(x)}{x}$$

is non decreasing for $x > 0$.

We need to apply mean value theorem to prove this.

Definition 15.3 (Maxima and Minima). *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ where A is open. Then if $f(x_0) \ge f(x)$ for all x in a neighborhood of x_0 , then $f(x_0)$ is a local maximum. If $f(x_0) \le f(x)$ for all x in a neighborhood of $x_)$, then $f(x_0)$ is a local minimum. A point is called extreme if it is either local maximum or local minimum.*

Theorem 15.4. *If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable, A is open and $x_0 \in A$ is an extreme point of f, then*

$$Df(x_0) = 0,$$

and we call this a critical point.

Note that a critical point is not sufficient to guarantee that this point is extreme.

Example 15.5.

$$z = f(x, y) = x^2 = y^2$$

It is easy to show that this has non-extreme critical points.
Now we did a bunch of stuff with Hessians which we got a packet on Tuesday the 2$^{\text{nd}}$.

Definition 15.6 (Positive Definite)**.** *A bilinear mapping $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called positive definite if*

$$B(x, x) > 0,$$

for all $x \neq 0$ in $\mathbb{R}^n$.

Definition 15.7 (Positive Semi-definite)**.** *A bilinear mapping $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called positive semi-definite if*

$$B(x, x) \geq 0,$$

for all $x \neq 0$ in $\mathbb{R}^n$.

Negative definite and negative semi-definite are defined in similar terms.

Theorem 15.8. *If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is a C^2 function defined on an open set A and if x_0 is a critical point of f such that H_{x_0} is negative definite, then f has a local maximum at x_0.*

We will now talk about inverse functions and the inverse function theorem.

Definition 15.9. *If f is one to one with domain A and range B, then the inverse function f^{-1} with domain B and range A is defined by*

$$f^{-1}(y) = x \Leftrightarrow f(x) = y,$$

for all $y \in B$.

Note that we have identity function by composition. ie

$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(y)) = y$$

$\forall x \in A$ and $\forall y \in B$.
We can compute the derivative of the inverse function using the chain rule.

$$\frac{df^{-1}}{dy}\frac{dy}{dx} = 1 \implies \frac{df^{-1}}{dy} = \frac{1}{\frac{df}{dx}}$$

16 4/9/13

Example 16.1. *Find an example of a function that is infinitely differentiable on an interval $I \subset \mathbb{R}$, maps I onto itself, is one to one, has a continuous inverse, but f^{-1} is not differentiable at some point of I.*

Proof. Consider $f : [-1, 1] \to [-1, 1]$ where

$$f(x) = x^3.$$

We know that this function is infinitely differentiable and has a continuous inverse

$$f^{-1}(x) = x^{1/3}.$$

However we have that

$$\left(f^{-1}(x)\right)' = \frac{1}{3x^{2/3}}.$$

Therefore the derivative does not exists at 0. □

In linear algebra we were given the following system

$$\begin{cases} a_{11}x_1 + 1_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \qquad \ldots \qquad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

We can solve if the determinate of the coefficient matrix is non-zero.
We know that the Determinant of the Jacobian

$$J(f(x)) = det(Df(x))$$

where

$$J(f(x)) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} = \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

If $J(f(x)) \neq 0$ one might hope to solve $f(\vec{x}) = \vec{y}$.

Definition 16.2 (Local Invertible)**.** *$f : A \subset \mathbb{R}^n \to \mathbb{R}^n$ is called **locally invertible** or locally 1-1 on A if for any point $a \in A$ there is a neighborhood in which f is invertible.*

Example 16.3. *$f : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ where*

$$f(r, \theta) = (r\cos(\theta), r\sin(\theta)).$$

Proof. So we have that

$$Jf(r,\theta) = \begin{vmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

So we can invert this map and this is the translation between polar and Cartesian coordinates. □

Recall that equations relating spherical and Cartesian coordinates in $\mathbb{R}^3$. Also note that mathematicians and physicists invert the ϕ and θ.

Example 16.4. *$f \in C^1$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and*

$$f(x, y) = (x^2 - y^2, 2xy).$$

We claim that f is locally invertible at $\mathbb{R}^2 \setminus \{0, 1\}$, but not globally invertible.

Proof. If we make the transition into polar coordinates we have the function

$$f(r\cos\theta, r\sin\theta) = (r^2\cos 2\theta, r^2\sin 2\theta).$$

f maps circle of radius r twice around circle of radius r^2. so f is not one to one. We look at

$$Jf(x) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x(x^2 + y^2) \neq 0$$

at each point of $\mathbb{R}^2 \setminus \{0, 0\}$. So f is not one to one but it is locally invertible. □

Theorem 16.5 (Inverse Function Theorem)**.** *$f : A \subset \mathbb{R}^n \to \mathbb{R}^n$ of class C^1, $x_0 \in A$, $Jf(x_0) \neq 0$. Then there exists an open neighborhood U of x_0 in A and an open neighborhood W of $f(x_0)$ such that*

1. *$f(U) = W$*

2. *Restriction of f to U has C^1 inverse*
$$f^{-1} : W \to U.$$

3. *Take $y \in W$ and $x = f^{-1}(y)$ we have*
$$Df^{-1}(y) = [Df(x)]^{-1},$$
the inverse matrix of $Df(x)$.

4. *If f is of class C^p where $p \geq 1$ then so is f^{-1}.*

Remarks

1. To prove existence and uniqueness of an x such that $f(x) = y$ we use Banach Contraction Mapping Theorem.

2. Although the proof we are going to see only applies to Euclidean spaces, a more general proof applies to Banach spaces.

3. The power of the inverse function stems partly from the fact that $Jf(x_0) \neq 0$ implies invertability of the C^1 mapping f in a neighborhood of x_0 which when it is differentiable or impossible to find the local inverse.

4. Inverse function theorem explains how to differentiate solutions.

Example 16.6. *Suppose that we have $f : \mathbb{R}^2 \to \mathbb{R}^2$*
$$(x, y) \to (u, v) = (f_1(x, y), f_2(x, y)).$$
If
$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$
So suppose that
$$Jf(x_0, y_0) = \begin{vmatrix} \frac{\partial f_1}{\partial x}(x_0, y_0) & \frac{\partial f_1}{\partial y}(x_0, y_0) \\ \frac{\partial f_2}{\partial x}(x_0, y_0) & \frac{\partial f_2}{\partial y}(x_0, y_0) \end{vmatrix} \neq 0$$
Then inverse of f which we will call g is given by
$$Dg(u, v) = [Df(x, y)]^{-1}.$$
and also
$$\frac{\partial x}{\partial u} = \frac{1}{Jf(x_0, y_0)} \frac{\partial f_2}{\partial x}.$$

17 4/16/13

So last time we proved parts of the inverse function theorem and now we will continue the proof. It is in the book so we will not write it here. We also have another version in a handout just given.

Theorem 17.1 (Implicit Function Theorem)**.** *$F : A \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. So $F(x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_n) = (F_1, F_2, \ldots, F_n)$ where A is open. So where we have $F(x_0, y_0) = 0$ where F is of class C^p, we can solve for y as a function of x. If*
$$\Delta(x_0, y_0) = \begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_1} & \cdots & \frac{\partial F_2}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{vmatrix}$$
then there exists an open neighborhood U of x_0 in $\mathbb{R}^n$ and an a neighborhood $V \subseteq \mathbb{R}^m$ of y_0 and a unique function $f : U \to V$ such that $F(x, f(x)) = 0$ for all $x \in U$. Furthermore f is of class C^p.

Note that if we have $m = 1$, we can find $\frac{\partial F}{\partial x_i}$ by the chain rule

$$\frac{\partial}{\partial x_i} F(x, f(x_i)) = \frac{\partial F}{\partial x_i}\frac{dx_i}{dx_i} + \frac{\partial F}{\partial y}\frac{\partial f}{\partial x_i} = 0.$$

This implies that

$$\frac{\partial f}{\partial x_i} = -\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}}.$$

More generally we have that

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

Now we will prove the implicit function theorem.

Proof. Define $G : A \subseteq \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ such that G is of class C^p and

$$G(x, y) = (x, F(x, y)).$$

We now examine the Jacobian

$$JG(x_0, y_0) = \begin{vmatrix} 1 & 0 & \dots & 0 & | & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & | & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_1}{\partial x_1} & \dots & \dots & \frac{\partial F_1}{\partial x_n} & | & \frac{\partial F_1}{\partial y_1} & \dots & \dots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \ddots & \vdots & | & \vdots & \ddots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \dots & \dots & \frac{\partial F_m}{\partial x_n} & | & \frac{\partial F_m}{\partial y_1} & \dots & \dots & \frac{\partial F_m}{\partial y_n} \end{vmatrix} = \Delta(x_0, y_0)$$

Since $\Delta(x_0, y_0) \neq 0$, we can use inverse function theorem. So by the inverse function theorem there is an open set W containing

$$(x_0, 0) = (x_0, F(x_0, y_0))$$

and open open set S containing (x_0, y_0) such that

$$G(x) = W$$

G has a C^p inverse $G^{-1} : W \to S$. S is open and therefore there exists $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^m$ both open and $x_0 \in U$, $y_0 \in V$ such that $U \times V \subset S$.

$$G(U \times V) = Y \subset G(S) = W.$$

$G : U \times V \to Y$ is a C^p diffeomorphism which means that G is of class C^p and has an inverse $G^{-1} : Y \to U \times V$. Since

$$G^{-1}(x, w) = (x, H(x, w)),$$

where $H(x, w)$ is a C^p function from $Y \to V$. Now let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\pi(x, y) = y.$$

So we have that

$$\pi \circ G = F.$$

So we now have that

$$F(x, H(x, w)) = (\pi \circ G)(x, H(x, w)) = \pi \circ G \circ G^{-1}(x, w) = \pi(x, w) = w.$$

So since

$$G^{-1}(x, H(x,w)),$$

we have that $x \in U$ if $(x,w) \in Y$.
Now define $f : U \to V$ such that

$$f(x) = H(x,0).$$

So if we look at

$$F(x, H(x,0)) = 0 \implies F(x, f(x)) = 0.$$

Since we know that H is of class C^p, we know that f is of class C^p. So by the inverse function theorem H is uniquely determined and therefore f is uniquely determined. $\square$

Example 17.2. *Let $F : D \to \mathbb{R}$ where $D = \{(x,y) \in \mathbb{R}^2 : \ a \leq x \leq b\}$. where partial derivatives with respect to x and y exist and there exist $m, M > 0$ such that*

$$0 < m < \frac{\partial F}{\partial y} \leq M$$

for all $(x,y) \in D$. Prove that there is a unique continuous function $y(x)$ on $[a,b]$ such that

$$F(x, y(x)) = 0.$$

Proof. The equation $F(x,y) = 0$ does implicitly define a unique continuous function in terms of x. We will now define $T : ([a,b], ||\cdot||_\infty) \to (C[a,b], ||\cdot||_\infty)$ defined to be

$$Ty(x) = y(x) - \frac{1}{M}F(x, y(x)).$$

We claim that T is a contraction on $C[a,b]$. So we look at

$$Ty_1(x) - Ty_2(x) = \left[y_1(x) - \frac{1}{M}F(x,y_1)\right] - \left[y_2(x) - \frac{1}{M}F(x,y_2)\right] = [y_1(x) - y_2(x)] - \frac{1}{M}\left[F(x,y_1) - F(x,y_2)\right].$$

$$= [y_1(x) - y_2(x)] - \frac{1}{M}\frac{\partial F}{\partial y}|_{(x,c)}\,[y_1(x) - y_2(x)]$$

where c is between y_1 and y_2. Therefore with $||\cdot||_\infty$ on $C[a,b]$ we have that

$$||Ty_1 - Ty_2||_\infty = \max_{a \leq x \leq b} |Ty_1(x) - Ty_2(x)| \leq \max_{a \leq x \leq b} \left(1 - \frac{m}{M}\right)|y_1(x) - y_2(x)|$$

$$\leq \left|1 - \frac{m}{M}\right| \cdot ||y_1(x) - y_2(x)||_\infty$$

since we know that

$$1 - \frac{m}{M} < 1,$$

it follows that T is a contraction on $C[a,b]$ therefore it has a unique fixed point, call it y. Thus for all $x \in [a,b]$ we have that

$$y(x) = y(x) - \frac{1}{M}F(x, y(x)).$$

Since we know that $M \neq 0$ we have that $F(x, y(x)) = 0$. $\square$

18 4/18/13

Today we start with a proof on the board that is given in the handouts. We will now start talking about integration.

Example 18.1. *Let $f[0,1] \to \mathbb{R}$ be a C^1 function. Find*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_{o}^{1} f(x)dx.$$

Proof. We will use the fact that

$$\int_0^1 f(x)dx = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx.$$

So we have that

$$\sum_{k=1}^{n} f(k/n) - n\int_0^1 f(x)dx = n\sum_{k=1}^{n} \frac{1}{n} f(k/n) - n\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx$$

$$= n\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left[f(k/n) - f(x)\right] dx = n\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f'(\theta_k(x))(k/n - n)dx$$

for any $k = 1, 2 \ldots, n$ by the mean value theorem. Because we know that $f'(x)$ is continuous on $[0,1]$, it is bounded. So if we set

$$\begin{cases} m_k = \inf\{f'(x) : \frac{k-1}{n} \le x \le \frac{k}{n}\} \\ M_k = \sup\{f'(x) : \frac{k-1}{n} \le x \le \frac{k}{n}\} \end{cases}$$

So we have

$$\int_{\frac{k-1}{n}}^{k/n} m_k(k/n - x)dx \le \int_{\frac{k-1}{n}}^{k/n} \left[f(k/n) - f(x)\right] dx \le \int_{\frac{k-1}{n}}^{k/n} M_k(k/n - x)dx.$$

Since we know that

$$\int_{\frac{k-1}{n}}^{k/n} (k/n - x)dx = \frac{2}{n^2}$$

we have that

$$\sum_{k=1}^{n} \frac{m_k}{2n^2} \le \int_{\frac{k-1}{n}}^{k/n} \left[f(k/n) - f(x)\right] dx \le \sum_{k=1}^{n} \frac{M_k}{2n^2}.$$

Since we know that these two sums are the upper and alower sums associated with $f'(x)$ and the partition

$$P = \{k/n : k = 1, 2, \ldots, n\}$$

of $[0,1]$ we have that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{m_k}{n} = \int_0^1 f'(x)dx = f(1) - f(0) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{M_k}{n}.$$

Then by the squeeze theorem we have that

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_{o}^{1} f(x)dx = \frac{f(1) - f(0)}{2}.$$

$\square$

19 4/25/13

20 4/30/13

21 5/7/2013

This is the last class of the semester. Hopefully we will figure out some stuff about the exam. We are doing a review today. We are going to consider the fundamental theorem of calculus.

Theorem 21.1 (Fundamental Theorem of Calculus)**.** *f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^b f(t)dt$$

where $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b) with

$$g'(x) = f(x).$$

Proof. If x and $x + h$ are in $(, b)$, then we have that

$$g(x + h) - g(x) = \int_a^{x+b} f(t)dt - \int_0^x f(t)dt$$

which implies that

$$\frac{g(x + h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt.$$

Assume that $h > 0$¿ We know that f is continuous on $[x, x + h]$ and by the extreme value theorem we have that there exists u and v in $[x, x + h]$ such that $f(u) = m$, $f(v) = M$ where n and M are the absolute minimum and absolute maximum values of f on $[x, x + h]$ respectively. So we have that

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh.$$

This implies that

$$f(u) \leq \frac{1}{h} \int_x^{x+h} f(t)dt \leq f(v).$$

For $h < 0$ we can get this same iniequality. If we let $h \to 0$ then we have that $u \to x$ and $v \to x$. since f is a continuous function we have that

$$\begin{cases} \lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \\ \lim_{h\to 0} f(v) = \lim_{v\to x} fvu) = f(x) \end{cases}$$

So we have from the squeeze theorem that

$$g'(x) = \lim_{h\to 0} \frac{g(x + h) - g(x)}{h} = f(x).$$

□

Example 21.2.

$$g(x) = \int_0^{x^3} f(y)dy$$

find $g'(x)$.

Proof. Let $u = x^3$ then we have that

$$g(u) = \int_0^u f(y)dy$$

which implies that

$$g'(u) = f(u)$$

and we have that

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx} = f(u)3x^2 = f(x^3)3x^2.$$

$\square$

Review for Final

The very first thing we did was consider the space of continuous functions with different metrics. One metric we put on this space was

$$d(f,g) = \max_{a \leq x \leq b} |f(x) - g(x)|.$$

$C[a.b]$ under this metric is a complete metric space. Some facts about this.

1. $f_n \to f$ in this metric, $d(f,g)$, (uniform norm) implies that $f_n \to f$ uniformly. This is why its called uniform norm.

2. If all the f_n are continuous and $f_n \to f$ uniformly then f is continuous.

3. Uniform convergence allows us to switch limits and integrals and sums and limits and integrals and sums.

4. Cauchy Criterion for uniform convergence which says for $f_k : A \subset M \to N$, if $f_k \to f$ if $\forall \epsilon > 0$ there exists N such that

$$d(f_n(x), f_m) \leq \epsilon.$$

for all $x \in D$ and $\forall n \in N$.

5. Weierstrass M-test.

6. Suppose $g_k : [a,b] \to \mathbb{R}$ are differentiable and g_k' are continuous and

$$\sum_{k=1}^{\infty}$$

converges point wise and

$$\sum_{k=1}^{\infty} g_k'$$

converges uniformly then we can interchange derivative and sum.

We have seen some big theorems for the spaces of continuous functions.

i) Arzela-Ascoli Theorem

ii) Weierstrass Approximation Theorem

iii) Banach Contraction Mapping Theorem

We then discussed the concept of differentiability. Which takes us up to the last exam.