135 Notes

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1 Basics of Complex Numbers

Complex numbers are based on the solution to

$$x^2 = -1$$

which has no solution in $\mathbb{R}$. We define $i = \sqrt(-1)$ and for all $z \in \mathbb{C}$, $z = x + iy$, where $x, y \in \mathbb{R}$, so we have $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. By De Moivre's formula, we have,

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

Definition 1.1 (Complex Conjugation)**.** *Given a complex number $z = x + iy$, the complex conjugate is $\overline{z} = x - iy$.*

Theorem 1.2. *Let $z \in \mathbb{C}$, then z is real if and only if, $z = \overline{z}$.*

Proof. $\Rightarrow$ If $z \in \mathbb{R}$, then $z = x + iy = x$ since $y = 0$. Similarly, $\overline{z} = x - iy = x$. Therefore, $z = \overline{z}$.
$\Leftarrow$ If $z = \overline{z}$. Then we know that $x + iy = x - iy$. This implies that $y = 0$ and therefore, $z \in \mathbb{R}$. $\square$

We now present theorems about complex numbers whose proofs are trivial and some will be omitted.

Theorem 1.3 (Properties of Conjugation)**.** *Let $z_1, z_2, z \in \mathbb{C}$.*

a) $\overline{z_1 * z_2} = \overline{z_1} * \overline{z_2}$

b) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

c) $\overline{\overline{z}} = z$

d) $Re(z) = \dfrac{z + \overline{z}}{2}$

e) $Im(z) = \dfrac{z - \overline{z}}{2i}$

Theorem 1.4 (Multiplication of Complex Numbers)**.** *Let $z_1, z_2 \in \mathbb{C}$. Then we define multiplication as*

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + (x_1y_2 + x_2y_1)i - y_1y_2$$

Corollary 1.5. *Let $z \in \mathbb{C}$, then,*

$$z\overline{z} = x^2 + y^2$$

Proof.

$$z\overline{z} = x^2 + (-xy + xy)i - i^2y^2 = x^2 + y^2$$

$\square$

Definition 1.6 (Modulus)**.** *The modulus of a complex number $z = x + iy$ is denoted $|z|$ and defined as*

$$|z| = \sqrt{x^2 + y^2}.$$

Corollary 1.7. *Let $z \in \mathbb{C}$. Then,*

$$z\overline{z} = |z|^2$$

Theorem 1.8 (Division of Complex Numbers)**.** *Let $z_1 = x + iy, z_2 = a + ib$ be elements of $\mathbb{C}$.*

Then assuming that $z_2 \neq 0$ we have that

$$\frac{z_1}{z_2} = \frac{x+iy}{a+bi} = \frac{x+iy}{a+bi}\frac{a-bi}{a-bi} = \frac{xa + (ay - xb)i + by}{a^2 + b^2}.$$

Theorem 1.9 (Properties of Modulus)**.** *Let $z_1 z_2 \in \mathbb{C}$.*

1) $|z_1 z_2| = |z_1||z_2|$

2) $|z_1 + z_2| \leq |z_1| + |z_2|$

3) $|z| = |\overline{z}|$

4) $|z| = |-z|$

5) $Re(z) \leq |z|$

6) $Im(z) \leq |z|$

$\mathbb{C}$ as a Field
We can represent every complex number $z = x + iy$ as an ordered pair (x, y) and we have the following properties

1) $(x_1 + y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

2) $(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_2 x_1)$

Theorem 1.10. *$\mathbb{C}$ is a field with the operations 1 and 2 defined above.*

Proof. We will sketch a proof and leave it to be checked. Both operations are commutative, associative and have identity. Each element has an additive inverse and each element besides 0 has multiplicative inverse. Lastly, multiplication distributes over addition. ☐

Theorem 1.11. *$\mathbb{C}$ contains a subfield isomorphic to $\mathbb{R}$.*

Proof. Define $\mathbb{R}^* := \{(x, 0) | (x, 0) \in \mathbb{C}\}$. We know that $\mathbb{R}^*$ satisfies the field properties and therefore $\mathbb{R}^*$ is a subfield of $\mathbb{C}$. Next we define $f : \mathbb{R} \to \mathbb{R}^*$ by $f(x) = (x, 0)$. Note that this is injective and surjective and preserves field operations. Therefore it is an isomorphism and we have found a subfield of $\mathbb{C}$ isomorphic to $\mathbb{R}$. ☐

Theorem 1.12 (Triangle Inequality)**.** *Let $z, w \in \mathbb{C}$. Then*

$$|z + w| \leq |z| + |w|$$

$$||z| - |w|| \leq |z \pm w|.$$

Proof.

$$|z + w|^2 = (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w})$$

$$= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = |z|^2 + 2Re(z\overline{w}) + |w|^2.$$

Note that $Re(z) \leq |z|$ which implies that

$$|z + w|^2 \leq |z|^2 + 2|z\overline{w}| + |w|^2$$

Since we have that

$$|z\overline{w}| = |z||\overline{w}| = |z||w|$$

we know that

$$|z+w|^2 \leq (|z|+|w|)^2.$$

We take the square root of both sides and have

$$|z+w| \leq |z|+|w|.$$

The second inequality is a direct consequence of this proof. $\square$

Polar Form of a Complex Number
We can express any complex number $z = x + iy$ in polar form. We know from De Moivre's identity that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

So letting $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

This implies that $z = re^{i\theta}$. We have that $r = |z|$ and θ is the argument of z. Note that θ can be an infinite amount of values and therefore, polar representations are not unique.

Theorem 1.13. *Let $\theta\phi \in \mathbb{R}$. Then*

$$e^{i\theta+i\phi} = e^{i\theta}e^{i\phi}.$$

Proof.

$$e^{i\theta+i\phi} = e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi).$$

Now using trig identities, we have our result. $\square$

Corollary 1.14. *Let $z = x + iy$,*

$$e^z = e^x e^{iy}.$$

Corollary 1.15. *Let $\alpha, \beta \in \mathbb{C}$,*

$$e^{\alpha+\beta} = e^\alpha e^\beta.$$

Theorem 1.16. *For any complex numbers z and w, $|zw| = |z||w|$ and $arg(zw) = arg(z)+arg(w) \pmod{2\pi}$.*

Theorem 1.17 (De Moivre's Formula)**.**

$$z^n = (re^{i\theta})^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Note that in the case, $r = 1$, we have used this formula in prior sections.

Theorem 1.18 (Fundamental Theorem of Algebra)**.** *A polynomial of degree n with coefficients in $\mathbb{C}$ has n roots.*

Note: all root are in $\mathbb{C}$, which is not true for a polynomial with coefficients in $\mathbb{R}$.

Example 1.19. *Distinct roots of $z^n = 1$ are*

$$e^{i\frac{2k\pi}{n}}$$

where $k \in \mathbb{N}$.

Note also the useful identity derived from the geometric series,

$$(1-z)(1+z+z^2+\cdots+z^n) = 1 - z^{n+1}.$$

Now we have many facts about complex numbers, it is time to examine functions of these complex numbers. In general, complex functions have very different behaviors from their real counterparts, however, in many ways, they behave better.

2 Complex Functions

Definition 2.1 (Complex Function)**.** *Let $w = f(z)$ where $z, w \in \mathbb{C}$, we say that f is a complex function of a complex variable.*

Example 2.2.

$$f(z) = e^{z+2\pi i}.$$

We see that $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$. This means that e^z is periodic. This is very different that e^x, $x\ in \mathbb{R}$ which is not periodic is monotonically increasing over $\mathbb{R}$.

Definition 2.3 (Parametric Curves)**.** *if $x(t), y(t)$ are real valued functions of a real variable t then $S = \{z(t) | z(t) = x(t) + iy(t)\ a < t < b\}$ is called a parametric curve or a complex parametric curve.*

Example 2.4. *Parametrize a line between z and w, where $z, w \in \mathbb{C}$.*

$z(t) = z(1-t) + wt$, $t \in [0, 1]$, is the line going from z to w.

Example 2.5. *Parametrize a circle at point a with radius r,*

$z(t) = a + re^{it}$, $t \in [0, 2\pi]$.

We now have a need to extend the complex plane to include infinity.

Definition 2.6. *We extend the complex plane to include ∞ and view lines as generalized circles with $r = \infty$.*

Example 2.7. *Find the image of $x = 1$ under the mapping, $w = \dfrac{1}{z}$.*

Proof. We know that $w = \frac{1}{z}$ and so we let $z = x + iy$ and we have

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

When $x = 1$, we have that

$$\frac{1}{1+y^2} - i\frac{y}{1+y^2}$$

Which by letting $u = \frac{1}{1+y^2}$ and $v + \frac{-y}{1+y^2}$, we have that $v = -yu$. So when $u \neq 0$, we can write

$$y = \frac{-v}{u}.$$

Substituting this into the expression for u¡ we have that

$$u^2 - u + v^2 = 0.$$

We complete the square to get

$$(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}.$$

So this is a circle with center at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. We can also express this as

$$|1 - \frac{1}{2}| = \frac{1}{2}$$

where $w \neq 0$. □

Definition 2.8 (Linear Fractional Transformation (Mobius Transfrmation))**.** *$F : \mathbb{C} \to \mathbb{C}$ defined as*

$$F(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Observation 2.9 (Properties of Linear Fractional Transformations)**.**

1) Some important points.

i) When $az+b=0$ or $z=\frac{-b}{a}$, then $F(z)=0$.

ii) When $cz+d=0$ or $z=\frac{-d}{c}$, then $F(z)=\infty$.

iii) $\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{az+b}{cz+b} = \lim z \to \infty \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c}$. So $F(\infty)=\frac{a}{c}$.

iv) $F(0)=\frac{b}{d}$.

2) Dependence on c.
If $c=0$, then $F(z)=\frac{az+b}{d}=\frac{a}{d}z+\frac{b}{d}$. This is a linear transformation.
If $c\neq 0$, then one can write

$$F(z) = \frac{a}{c} + \frac{bc-ad}{c} + \frac{1}{cz+d}.$$

This is also a linear map. It is a composition of a linear mapping, followed by an inversion, followed by a rotation and a translation.
3) Inverse.
Linear fractional transformations are bijective and therefore have an inverse. We know that $ad-bc\neq 0$. We can think of this as a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $det(A)=0$. We know that $\forall\lambda\in\mathbb{C}$ such that $\lambda\neq 0$ an equivalent map is given by:

$$B = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$

So we know that this matrix is invertible and the inverse is given by:

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Therefore we know that the inverse linear fractional transformation, $F^{-1}(w)=z$ is given by

$$F^{-1}(w) = \frac{dw-b}{-cw+a}.$$

Note that a composition of a linear fractional transformation produces a linear fractional transformation.
4) Fixed Points.

Definition 2.10 (Fixed Point)**.** *A fixed point for F is a point where $f(z_0)=z_0$.*

Example 2.11. *Find a fixed point for $f(z)=az+b$.*

Note that ∞ is a fixed point.

Lemma 2.12. *Let F be a linear fractional transformation. If ∞ is a fixed point of F, then there exists $a,b\in\mathbb{C}$ such that $f(z)=az+b$.*

Proof. We know that $F(z)=\frac{az+b}{cz+d}$ where $c\neq 0$. We know that $F(\infty)=\frac{a}{c}\neq\infty$, but by hypothesis, $c=0$. This implies

$$F(c) = \frac{a}{d}z + \frac{b}{d} = Az + B.$$

□

Lemma 2.13. *Let F be a linear fractional transformation. If F has three fixed points, then F is an identity.*

Proof. We have two cases.

Case 1 Suppose ∞ is one of these fixed points. Then we are done by previous lemma.

Case 2 Supose $z_1 \in \mathbb{C}$ and $F(z_1) = z_1$. Then we know that $az_1 + b = z_1$ which implies that $z_1 = \dfrac{b}{1-a}$. If $a = 1$, then $z_1 = \infty$. So $F(z) = z + b$, $F(z_2) = z_2 + b$ which implies that $b = 0$. Therefore, $F(z) = z$.

$\square$

Theorem 2.14. *Given any three distinct points $z_1.z_2, z_3$ on the Riemann sphere and three distinct points w_1, w_2, w_3, there exists a unique linear fractional transformation such that $z_i = w_i$, $i = 1, 2, 3$.*

Theorem 2.15 (Cross Ratio Formula)**.** *We can find the unique representation of the previous result by using the cross ratio formula:*

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

We now introduce some definitions and notion.

Definition 2.16 (Disk)**.** *An open disk centered at z_0 with radius $r > 0$ is given by*

$$D(z_0; r) = \{z \in \mathbb{C} | |z - z_0| < r\}.$$

A closed disk centered at z_0 with radius $r > 0$ is given by

$$\overline{D}(z_0; r) = \{z \in \mathbb{C} | |z - z_0| \leq r\}.$$

Note that an open disk is an open set, and closed disk is not.

Theorem 2.17. *A finite intersection of open sets is open. Arbitrary union of open sets is open.*

Definition 2.18 (Limits of Functions)**.** *Let $f : S \subset \mathbb{C} \to \mathbb{C}$ be an arbitrary complex function. Let $a \in S$. Then $\lim_{z\to a} f(z) = w$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |z - a| < \delta$ implies $|f(z) - f(w)| < \epsilon$.*

Note that $f(a)$ does not have to equal w.

Corollary 2.19 (Limits at ∞)**.** *$\lim_{z\to\infty} f(z) = w$ if for all $\epsilon > 0$, there exists M, such that $|z| \geq M$ implies $|f(z) - w| < \epsilon$.*

3 Differentiability in $\mathbb{C}$

Several words are used to describe functions which are differentiable in the complex sense:
regular, holomorphic, analytic.
Note that if G is an open set, for all $z \in G$, there exists $\epsilon > 0$ such that $D(z, \epsilon) \subset G$ and if $z \in G$, then $z + h \in G$ for h sufficiently small.

Definition 3.1 (Differentiable)**.** *Let f be a complex valued function defined on an open set G. f is said to be differentiable at $w \in G$ with derivative $f'(w)$ or $\dfrac{df}{dw}$ at $w \in G$ if*

$$\lim_{h\to 0} \frac{f(w+h) - f(w)}{h}$$

exists.

Definition 3.2 (Holomorphic)**.** *A function which is differentiable at every point of G is holomorphic in G. $H(G)$ is the set of functions that are holomorphic in G.*
Let S be any subset of $\mathbb{C}$. We say f is holomorphic on S if $f \in H(G)$ for some open set G with $S \subseteq G$.

Note that h in the definition of differentiability is not necessarily real. $h \to 0$ on the complex plain in general.

Example 3.3. *$f(z) = \overline{z}$ is not differentiable ant any point z.*

Proof. We start by checking

$$\lim_{h \to 0} \frac{f(w+h) - f(w)}{h} = \lim_{h \to 0} \frac{\overline{w+h} - \overline{w}}{h} = \lim_{h \to 0} \frac{\overline{w} + \overline{h} - \overline{w}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}.$$

Since this limit does not exist, we know that f is not differentiable. □

Given any complex valued function $f(z)$ we can write

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are real valued function of x and y. So we can also talk about the partial derivative to these functions.

Theorem 3.4 (Cauchy-Riemann Equations)**.** *If a function $f(z) = u(x, y) + iv(x, y)$ is holomorphic than the following equations are satisfied:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The contrapositive of this theorem is a powerful tool for proving a function is not holomorphic.
Note that in general the converse is not true, however, with an added condition we have the following theorem,

Theorem 3.5. *If f satisfies the Cauchy-Riemann equations and its partials are all continuous, then f is holomorphic.*

4 Complex Sequences

Definition 4.1 (Convergence of a Sequence)**.** *A complex sequence $\{z_n\}$ converges to z if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $|z_n - z| < \epsilon$.*

Theorem 4.2. *Let $\{z_n\} = \{x_n\} = I\{y_n\}$. Let $z = x + iy$. Then*

$$\lim_{n \to \infty} z_n = z \Leftrightarrow \lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y.$$

There are several extensions to this theorem.

1) If $z_n \to z$, then $|z_n| \to |z|$ and $\overline{z_n} \to \overline{z}$.

2) $\lim_{z \to a} f(z)$ exists $\Leftrightarrow \lim_{z \to a} Re(f(z))$ and $\lim_{z \to a} Im(f(z))$ exist.

3) $f(z) \to w \Leftrightarrow Re(f(z)) \to Re(w)$ and $Im(f(z)) \to Im(w)$.

4) f is continuous at $a \in S \subset \mathbb{C} \Leftrightarrow Re(f)$ and $Im(f)$ are continuous at a.

Lemma 4.3. *Failure of Cauchy Riemann equations signals non-differentiability. Thus, if we have a non-constant real valued function in an open disk or in a region, we have a non-holomorphic function.*

Note that we have continuous functions whose real and imaginary parts are continuous and even differentiable, yet not have holomorphic functions.
I had my concussion about here, so I missed a day or two of notes. It would be nice to get them. I should remember that.

5 Complex Series

Definition 5.1. *Suppose $\{a_n\}$ is a sequence of complex numbers. We say*

$$\sum_{n=0}^{\infty} a_n$$

is convergent $\Leftrightarrow$

$$\sum_{n=0}^{\infty} Re(a_n) \text{ and } \sum_{n=0}^{\infty} Im(a_n)$$

both converge.

Some basic facts about complex series, many proved in real analysis class.

1) Suppose $\sum a_n$ converges, then

$$\lim_{n\to\infty} (a_n) = 0$$

and there exists a ream constant M such that $|a_n| < M$ for all n. In other words $\{a_n\}$ is bounded.

2) If $\sum a_n$ and $\sum b_n$ both converge, then

$$\sum a_n + kb_n = \sum a_n + k \sum b_n$$

and are convergent for any k.

3) Absolute convergence implies convergence.

Tests for Convergence

1) Comparison Test
Let $\sum b_n$ be convergent and $b_n \geq 0$ for all n. Suppose there exists $k > 0$ uch that $|a_n| \leq kb_n$ for all n. Then $\sum |a_n|$ is absolutely convergent, which implies that $\sum a_n$ is convergent.

2) Ratio Test
Assume $\{a_n\}$ is such that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $l < 1$, then $\sum |a_n|$ is convergent. If $l > 1$, $\sum |a_n|$ diverges. If $l = 1$, we have no information.

3) Root Test
Assume $\{a_n\}$ is such that

$$l = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

If $l < 1$, then $\sum |a_n|$ is convergent. If $l > 1$, $\sum |a_n|$ diverges. If $l = 1$, we have no information.

Geometric Series

Definition 5.2 (Geometric Series)**.** *A geometric series is a series of the form $\sum_{n=0}^{\infty} z^n$ where $|z| < 1$.*

Note that if $|z| < 1$,

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots + z^n = \frac{1}{1-z}.$$

Many similar series can be manipulated into a geometric series. Remember that they must start with $n = 0$ and the modulus must be less than 1 or the series will diverge.

Power Series

Definition 5.3 (Power Seres)**.** *A power series in* $(z-a)$ *is*

$$\sum_{n=0}^{\infty} c_n(c-a)^n$$

where $a \in \mathbb{C}$*,* $c_n \in \mathbb{C}$ *and* $n \geq 0$*. If* $a = 0$*, then we have*

$$\sum_{n=0}^{\infty} c_n z^n.$$

Definition 5.4 (Radius of Convergence)**.** *The radius of convergence is defined as*

$$R := \sup\{|z|| \sum_{n=0}^{\infty} |c_n(z-a)^n| \text{ converges } \}.$$

Generally we find the radius of convergence by applying the ratio or the root test to the series. Some series have infinite radii of convergence.
The following lemma implies that every power series

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

with a radius of convergence $R > 0$ has a disk of convergence $D(a; R$. We will later see that this power series is holomorphic in this disk.
The series diverges when $|z-a| > R$ and any behavior is possible for the boundary $|z-a| = R$, the series could even converge and diverge for different points on the boundary.

Lemma 5.5. *Let* $\sum a_n z^n$ *be a power series with a positive radius of convergence, R.*

i) $\sum a_n z^n$ *converges absolutely for all* z *such that* $|z| < R$*.*

ii) $\sum a_n z^n$ *fails to converge for all* z *such that* $|z| > R$*.*

Theorem 5.6. *Let*

$$\sum_{n=0}^{\infty} c_n z^n$$

have a radius of convergence $R > 0$*. Then the following are true:*

1)

$$\sum_{n=0}^{\infty} c_n z^{n-1}$$

has a radius of convergence R.

2) f *is continuous on* $D(0; R)$*.*

3) f *is holomorphic in* $D(0; R)$*.*

4) f' *is obtained by term by term differentiation.*

5) f *has derivative of all orders in this disk.*

Note that this theorem can be extended readily to series centered around an arbitrary value a.

6 Complex Exponents and Logarithms

Theorem 6.1. *Let n be a positive integer, $z \neq 0$, then $w^n = z$ has n solutions given in terms of polar representations.*

$$z = re^{i\theta}$$
$$w = r^{\frac{1}{n}} e^{\frac{\theta+2\pi k}{n} i}$$

for $k = 0, 1, 2, \ldots, n-1$.

Now let α be a complex number. Then we know that

$$z^\alpha = e^{lnz^\alpha}.$$

Theorem 6.2.

$$z^\alpha = e^{lnz^\alpha} = [|z^\alpha|] = \{e^{\alpha(ln|z|+i\theta)} | \theta \in [|logz|]\}.$$

Example 6.3. *What is $(-i)^i$?*

Proof. Choose principal branch $[0, 2\pi)$. Then we have that

$$(-i)^i = e^{ilog(-i)} = e^{ilog|-i|+i\frac{3\pi}{2}} = e^{-\frac{3\pi}{2}}.$$

$\square$

7 Integration

Overall question is what is

$$\int_\gamma f(z)dz.$$

Definition 7.1 (Complex Curves)**.** *Let $[\alpha, \beta]$ be a compact interval in $\mathbb{R}$. A curve γ is a continuous complex valued function $[\alpha, \beta] \to \mathbb{C}$ where the initial point is $\gamma(\alpha)$ and the terminal point is $\gamma(\beta)$. It is called simple if it does not intersect itself. γ is closed if $\gamma(\alpha) = \gamma(\beta)$.*

Theorem 7.2. *If γ is a curve with parameter interval $[\alpha, \beta]$, then $\gamma^* = \gamma([\alpha, \beta])$ is a compact subset of $\mathbb{C}$.*

Definition 7.3 (Smooth Curve)**.** *A parameterized curve is smooth if $\gamma'(t)$ exists and is continuous for $t \in [\alpha, \beta]$.*

Given $\gamma : [\alpha, \beta] \to \mathbb{C}$. Then

$$\int_\gamma f(z)dz$$

is the integral along γ. If γ is closed, than it is the integral around γ.

Theorem 7.4. *If a curve is parametrized by $\gamma(t) : [\alpha, \beta] \to \mathbb{C}$, then*

$$\int_\gamma f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

Definition 7.5 (The Fundamental Integral)**.** *For a complex number a and $r > 0$,*

$$\int_{\gamma(a;r)} (z-a)^n dz = \begin{cases} 0 & if\ n \neq -1 \\ 2\pi i & if\ n = -1 \end{cases}$$

Definition 7.6 (Circular Contour)**.** *A circular contour is a circle centered at a with radius r.*

$$\gamma(a; r) = a + re^{it}$$

where $t \in [0, 2\pi)$. Note that $dz = rie^{it}$.

Theorem 7.7 (Fundamental Theorem of Calculus)**.**

$$F(x) = \int_a^t f(t)dt \implies F'(x) = f(x)$$

$$\int_a^b f(x)dx = G(x)|_a^b = G(b) - G(a)$$

where $G'(x) = f(x)$*.*

Remark 7.8.

It is useful to know how to parametrize certain common curves.
For a line $z_0 \to z_1$ we have the parametrization,

$$z(t) = tz_1 + (1-t)z_0$$

where $t \in [0, 1]$.
For a circle with orientation counterclockwise (positive direction), we have $z(t) = \gamma(t) = \gamma(z_0; r)$ and

$$z(t) = z_0 + re^{it}$$

where $t \in [0, 2\pi]$.

Definition 7.9 (Contour)**.** *A contour is a simple closed path which is a finite join of paths of lines or parts of circles. The image of a contour consists of finitely many line segments and circular arcs, and does not cross itself.*

If I have a piecewise smooth curve I can break it up into a finite amount of section and parameterize each one individually.

Theorem 7.10. *If* $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ *and* γ *is smooth in* $[t_{i-1}, t_i]$*, then*

$$\int_\gamma f(z)dz = \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\gamma'(t)dt.$$

Definition 7.11 (Arc-Length)**.** *The arc-length of a smooth curve* $\gamma(t)$ *is given by*

$$L(\gamma) = \int_\alpha^\beta |\gamma'(t)|dt.$$

If we are given $\gamma(t) = x(t) + iy(t)$*, then we know that* $\gamma'(t) = x'(t) + iy'(t)$ *and*

$$|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

Theorem 7.12. *Complex valued integrals are linear, which means*

$$\int_\gamma [af(z) + bg(z)]dz = a\int_\gamma f(z)dz + b\int_\gamma f(z)dz.$$

Theorem 7.13. *If* $-\gamma$ *is* γ *with reverse direction we have that*

$$\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz.$$

Lemma 7.14 (Estimation Lemma)**.**

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma} |f(z)| L(\gamma).$$

Theorem 7.15. *Let γ be a path with parameter interval $[\alpha, \beta]$, $F(z)$ defined on an open set containing γ^* and that $F'(z)$ exists and continuous at each point of γ^*. then*

$$\int_\gamma F'(z)dz = \begin{cases} F(\gamma(\beta)) - F(\gamma(\alpha)) & in\ general \\ 0 & if\ \gamma\ is\ closed \end{cases}$$

Theorem 7.16. *If*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

converges in $|z - z_0| < R$, then

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$$

also converges for $|z - z_0| < R$ and $F'(z) = f(z)$.

Definition 7.17 (Simply Connected)**.** *A plane region whose boundary consists of one simple closed curve is called simply connected.*

Theorem 7.18 (Greens Theorem)**.** *Let Ω be a simply connected region with boundary γ. Let $P(x, y)$ and $Q(x, y)$, $P : \Omega \to \mathbb{C}$, $Q : \Omega \to \mathbb{C}$ are C^1 functions. Then*

$$\int_\gamma Pdx + Qdy = \int\int_\Omega (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$$

Theorem 7.19 (Complex Form of Green's Theorem)**.** *Let Ω be a simply connected region and γ its boundary. Assume that f has continuous partials on an open set containing Ω and γ. Then*

$$\int_\gamma f(z)dz = i\int_\Omega \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}.$$

Theorem 7.20 (Jordan Curve Theorem for a Contour)**.** *Let γ be a countour, γ^* be its image. Let $I(\gamma)$ be the inside of γ. Let $O(\gamma)$ designate the outside of γ. Denote the complement of x as x^c. Then*

$$(\gamma^*)^c = I(\gamma) \cup O(\gamma)$$

where $I(\gamma)$ and $O(\gamma)$ are disjoint connected open sets. $I(\gamma)$ is bounded and $O(\gamma)$ is unbounded.

Theorem 7.21 (Antiderivative Theorem)**.** *Let G be convex. Then if f is holomorphic in G, there exists a function $F \in H(G)$ such that $F'(z) = f(z)$.*

We will now present Cauchy's theorem which is the most important theorem in complex analysis. It takes many forms and is quite powerful both analytically and computationally.

Theorem 7.22 (Cauchy's Theorem)**.** *1) For a Triangle*

Let f be holomorphic on an open set G containing the triangle T its inside. Then

$$\int_T f(z)dz = 0.$$

2) For a Convex Region

Let G be convex and $f \in H(G)$. Let γ be a closed path in G. Then

$$\int_\gamma f(z)dz = 0.$$

3) For a Contour
Let f be holomorphic inside and on a contour γ. Then

$$\int_\gamma f(z)dz = 0.$$

4) For a Simply Connected Region
Let G be simply connected and $f \in H(G)$. Then for a closed path γ,

$$\int_\gamma f(z)dz = 0.$$

We now want to study regions in which Cauchy's theorem may not apply directly such as a region with a point of discontinuity. We now wish to look at the keyhole contour which consists of two circles connected by a narrow corridor. Note that

1. γ_1 (the outer circle) and γ_2 (the inner circle) are positively oriented.
2. along the corridor we have path γ_0 and $-\gamma_0$.

Theorem 7.23 (Deformation Theorem)**.** *If γ' surrounds the "hole" and is interior to γ, and f is holomorphic at each point interior of γ and exterior to γ'. Then we have*

$$\int_\gamma f(z)dz = \int_{\gamma'} f(z)dz.$$

Corollary 7.24 (Extension of Deformation Theorem)**.** *Suppose $\gamma_1, \ldots, \gamma_n$ are simple closed curve and γ is a simple closed curve with f holomorphic on the region between γ and $\gamma_1, \ldots, \gamma_n$, then*

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz.$$

Theorem 7.25 (Cauchy's Integral Formula)**.** *Let f be holomorphic inside an on a positively oriented contour γ. Then if a is inside γ,*

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a} dw.$$

This formula is remarkable for it says the values of f on γ completely determine the values of f inside γ. In other words, f is determined by its boundary values. Also, Cauchy's integral formula is extremely useful in computations. Cauchy's integral formula can be stregnthened by requiring that f is continuous on γ and holomorphic "inside" γ. This makes little difference for most examples.

Theorem 7.26 (Cauchy's Integral Formula for Derivatives)**.** *Suppose f is holomorphic inside and on a positively oriented contour γ. Then $f^{(n)}(a)$ exists for $n \in \mathbb{N}$ and*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw.$$

Theorem 7.27 (Cauchy's Inequality)**.** *Suppose f is holomorphic on a simply connected domain D and let c be the circle defined as $|z-a| = R$ that lies entirely in D. Then if $|f(z)| \leq M$ for all z on c, then*

$$|f^n(a)| \leq \frac{n!M}{R^n}.$$

Theorem 7.28 (Liouville's Theorem)**.** *The only bounded entire functions are constants.*

Theorem 7.29 (Fundamental Theorem of Algebra)**.** *If $p(z)$ is a non-constant polynomial than the equation $p(z) = 0$ has at least one root.*

Theorem 7.30. *If f is holomorphic in an open set, then f has derivative of all orders.*

Theorem 7.31 (Moerera's Theorem)**.** *Let f be continuous on a region G and suppose*

$$\int_{\gamma} f(z)dz = 0$$

for every closed simple curve γ in G. Then f is holomorphic.

Theorem 7.32 (Maximum Modulus Theorem)**.** *Suppose f is holomorphic and non-constant on a closed region D. D is bounded by a simple closed curve γ, the the max of the modulus is attained at the boundary.*

8 Complex Power Series

We want to study complex power series in order to be able to understand Residue Theory. So we first want to determine uniform convergence of a complex series. So we have

Theorem 8.1 (Weierstrass M-test)**.** *Suppose a sequence of functions f_k is defined on a set E and $M_k \geq 0$ satisfies the relation*

$$\sum_{k=1}^{\infty} M_k < \infty.$$

If $|f_k(x)| < M_k$ for $k \in \mathbb{N}$, $x \in E$. Then the series formed by the functions converges uniformly.

Remember if a power series has a radius of convergence R and is equal to a function f. The f is holomorphic in a disk of that radius around the point that the power series is expanded around. This has a converse.

Theorem 8.2 (Taylor's Theorem)**.** *Let $f \in H(D(a;R))$. Then there exists c_n such that*

$$f(z) = \sum_{n=1}^{\infty} c_n (z-a)^n$$

$z \in D(a;R)$ where

$$c_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw = \frac{f^{(n)}(a)}{n!}$$

where γ is any circle centered at a with radius $r < R$.

Theorem 8.3. *If f is holomorphic in $\mathbb{C}$ and there exist positive constants M and a positive integer k such that*

$$|f| \leq M|z|^k$$

for $|z| > k$ then f is a polynomial of at most degree k.

Theorem 8.4 (Multiplication of Power Series)**.** *Suppose $f(z) = \sum a_n z^n$ has radius of convergent r_1 and $g(z) = \sum b_n z^n$ has radius of convergence r_2. Then $h(z) = \sum c_n z^n$ where*

$$c_n = \sum_{r=0}^{n} a_r b_{n-r}.$$

$h(z)$ has radius of convergence $R \geq min\{r_1, r_2\}$. $h(z) = f(z)g(z)$ for $|z| < R$.

Definition 8.5 (Zeros and their orders)**.** *Suppose $f \in D(a;r)$ for some $r > 0$. We say a is a zero of f if $f(a) = 0$.*
We say a zero of f is of order m if

$$0 = f(a) = f'(a) = f^{(m-1)}(a)$$

but $f^{(m)}(a) \neq 0$.

Theorem 8.6 (Characterization for zeros of order m)**.** *Let $f \in H(D(a;R))$ and suppose Taylor series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

in $D(a;R)$ then the following are equivalent:

1. $0 = f(a) = f'(a) = f^{(m-1)}(a)$, *but* $f^{(m)}(a) \neq 0$.
2. $$f(z) = \sum_{n=m}^{\infty} c_n (z-a)^n$$ *where $c_m \neq 0$.*
3. $f(z) = (z-a)^m g(z)$ *where g is also holomorphic on the same disk and $g(a) \neq 0$.*
4. *There exists non-zero constant $c \in \mathbb{C}$ such that* $$\lim_{z \to a} (z-a)^{-m} f(z) = c \neq 0.$$

Definition 8.7 (Punctured Disk)**.** *A punctured disk centered at a with radius r is $\{z \in \mathbb{C} | 0 < |z-a| < r\}$.*

Theorem 8.8. *Suppose $f \in H(D(a;R))$ and $f(a) = 0$ then either*

1. *f is always 0 on the disk*
2. *a is an isolated point.*

Theorem 8.9 (Identity Theorem)**.** *Let G be a region and $f \in H(G)$. If the center of f has a limit point in G, then $f = 0$ in G.*

Theorem 8.10. *Let p be a polynomial with no zeros on the simple closed positively oriented contour γ. The the number of $0's$ of p counting multiplicity that lie inside γ is given by*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{p'(z)}{p(z)} dz.$$

Theorem 8.11 (Uniqueness Theorem)**.** *G is a region. $f, g \in H(G)$. Suppose $f(z) = g(z)$ for all $z \in S$. If S has a limit point in G, then $f(z) = g(z)$ for all $z \in G$.*

We now move into Laurent expansions which are usefull when Taylor series is limited for example $f(z) = \frac{1}{z}$ around $z = 0$. Also, for non-converging Taylor series such as $e^{\frac{-1}{z^2}}$.

Definition 8.12 (Annulus)**.** *An annulus is a region defined as $A = \{|a < |z| < b\}$ where a can be 0 and b can be ∞.*

Some notation.

$$\sum_{-\infty}^{\infty} c_n (z-a)^n = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}.$$

We call the first sum on the right side the holomorphic part and if both parts converge then this equality is true.

Theorem 8.13 (Laurent's Theorem)**.** *Let $G = \{z \in \mathbb{C} | R < |z-a| < S\}$. and $f \in H(G)$. Then*

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$

and $\gamma = \gamma(a;r)$, $R < r < S$.

We set $c_n = a_n$ for positive n and $c_n = b_n$ for negative n.

Lemma 8.14. *Let γ_1 and γ_2 be two cocentric circles around a of radii R_1 and R_2 with $R_1 < R_2$. If z lies between the circles and is holomorphic on the region containing γ_1 and γ_2 and the region between them, then*

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{(w-z)}dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(w-z)}.$$

9 Singularities and Residue Theory

Definition 9.1 (Isolated Singularity)**.** *Consider the Laurent expansion of f when $R_1 = 0$. In this case f is holomorphic in $\{z|0 < |z-a| < R_2\}$ which is a deleted neighborhood of a and we say a is an isolated singularity. In this case*

$$f(z) = \cdots + \frac{b_n}{(z-a)^n} + \cdots + \frac{b_1}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

is valid for $0 < |z-a| < R$.

Definition 9.2 (Removable Singularity)**.** *a is called a removable singularity if $b_n = 0$ for all n.*

Definition 9.3 (Pole of order m)**.** *a is called a pole of order m if $b_m \neq 0$ but $b_n = 0$ for all $n > m$.*

Definition 9.4 (Essential Singularity)**.** *a is called an essential singularity if $b_n \neq 0$ for infinitely many n.*

Theorem 9.5. *Suppose $z = a$ is an isolated singularity of the function $f(z)$. Then:*

1) a is a removable singularity if and only if

$$\lim_{z\to a}(z-a)f(z) = 0$$

or

$$\lim_{z\to 0} f(z) \quad finite.$$

2) a is a pole of order m if and only if

$$\lim_{z\to a}(z-a)^m f(z) = l \neq 0$$

3) a is a pole of order 1, a simple pole, if and only if

$$\lim_{z\to a} f(z) = \infty.$$

4) a is an essential singularity if and only if

$$\lim_{z\to a} f(z) \quad DNE.$$

Definition 9.6. *The coefficient c_{-1} of $(z-a)^{-1}$ is called residue.*

$$c_{-1} = Res(f(z), a).$$

Calculation of Residues

I) Residue at a Removable Singularity:
If $f(z)$ has a removable singularity at a, then $Res(f(z), a) = 0$.

II) Residue at Simple Pole
If a is a simple pol of $f(z)$, then $Res(f(z), a) = \lim_{z\to a}(z-a)f(z)$.

III) Residue at a pole of order $k \geq 2$.
If a is a pole of order k for $f(z)$, then

$$Res(f(z), a) = \frac{1}{(k-1)!} \lim_{z \to a} \{ \frac{d^{k-1}}{dz^{k-1}} [(z-a)^k f(z)] \}$$

Sometimes with non-rational functions we have easier ways to calculate residues.

Theorem 9.7. *Suppose* $f(z) = \dfrac{g(z)}{h(z)}$ *where* g, h *holomorphic at* $z = a$. *If* $g(a) \neq 0$ *and if the function* h *has a zero of order* 1 *at* a *then* $f(z)$ *has a simple pole at* $z = a$ *and*

$$Res(f(z), a) = \frac{g(a)}{h'(a)}.$$

Theorem 9.8 (Cauchy's Residue Theorem)**.** *Let* D *be a simply connected domain and* γ *a simple closed contour lying entirely within* D. *If* f *is holomorphic except at a finite number of isolated singularities* $z_1, \ldots, z_n$ *within* γ, *then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} Res(f(z), z_i).$$

Cauchy's Residue Theorem has many applications which are in the notes handed out by the Professor in class. I will try to update some of this to include more examples, but not sure if that will actually happen.