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Limits and Continuity of Functions

Definition 1.1. *Let (X, d_x) and (Y, d_y) be metric spaces. Let $E \subseteq X$ and $f : E \to Y$ and p be a limit point of E. Then $f(x) \to q$ as $x \to p$ or $\lim_{x\to p} f(x) = q$ if $\exists\ q \in Y$ such that $\forall \epsilon > 0$, there is a $\delta > 0$ such that $0 < d_x(x, p) < \delta$ and $x \in E$ implies $d_y(f(x), q) < \epsilon$.*

Remark 1.2. *p may not be in E.*

Theorem 1.3. *Let $E \subseteq X$, Y, f,p be as above. Then, $\lim_{x\to p} f(x) = q \Leftrightarrow$ for every sequence $\{p_n\}$ in E with $p_n \neq p$ and $p_n \to p$, we have $\lim_{n\to\infty} f(p_n) = q$.*

Proof. $\Rightarrow$
Suppose that $\lim_{x\to p} f(x) = q$. Let $p_n \to p$ where $p_n \neq p$ for all $n \in \mathbb{N}$. We want to show $\lim_{n\to\infty} f(p_n) = q$. So let $\epsilon > 0$. Then we know that $\exists\delta > 0$ such that $\forall x \in E$, $0 < d_x(x, p) < \delta$ implies that $d_y(f(x), q) < \epsilon$ by supposition. Also, $p_n \to p$, so $\exists N \in \mathbb{N}$ such that $\forall n > N$, $0 < d_x(p_n, p) < \delta$. Thus, $\forall n > N$, $d_y(f(p_n), q) < \epsilon$. So $f(p_n) \to q$.
$\Leftarrow$
We prove contrapositive. So suppose $\lim_{x\to p} f(x) \neq q$. This implies that $\exists\epsilon > 0$ such that $\forall\delta > 0$, $\exists x \in E$ with $d_y(f(x), q) \geq \epsilon$, but $0 < d_x(x, p) < \delta$. So choose $\delta_n = \frac{1}{n}$. So $\exists x_n \in E$ such that $d_y(f(x_n, q) \geq \epsilon$ but $0 < d_x(x_n, p) < \frac{1}{n}$. Thus $x_n \to p$, but $d_y(f(x_n), q) \geq \epsilon$ implies $f(x_n) \not\to q$. So we have the left direction. $\square$

Note that $\lim_{x\to p} f(\alpha)$ is unique if it exists. Similarly, we also know that the limit laws also hold since we have proved a relation between these continuity equations and limits of sequences.

We might not want to compute the limit of a function in terms of its definition every time. So we need a notion of continuity on general metric spaces.

Definition 1.4. *Let X, Y be metric spaces, $p, x \in E \supseteq X$, and $f : E \to Y$. Then f is continuous at p if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d_x(x, p) < \delta$ implies $d_y(f(x), f(p)) < \epsilon$.*

Note that f must be defined at p. Also, f is continuous at all isolated points of E as isolated points trivially satisfy the requirements.

Also note that if f is continuous on E if for all convergent $\{x_n\}$ in E, we have

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

And now for Professor Karp's favorite theorem in undergraduate mathematics.

Theorem 1.5. *$f : X \to Y$ is continuous (on X) $\Leftrightarrow$ for every open set $U \subseteq Y$, $f^{-1}(U)$ is open in X.*

Proof. To start we will make a key observation. $\{d_x(x, p) < \delta\}$ is the neighborhood of radius δ centered at p, $B_\delta(p)$.
$\Rightarrow$

Suppose f is continuous. Let $p \in f^{-1}(u)$. Want to show p is an interior point. Note that $f(p) \in U$. Since we know that U is open, $f(p)$ is an interior point. So $\exists \epsilon > 0$ such that $B_\epsilon(f(p)) \subseteq U$. This ball is equal to $\{y \in Y | d(y, f(p)) < \epsilon\}$. But f is continuous so $\exists \delta > 0$ such that $x \in B_\delta(p)$. This implies $d_y(f(x), f(p) < \epsilon$. This means that $f(x) \in B_\epsilon(f(p))$. So we have that $B_\delta(p) \subseteq f^{-1}(U)$. So p is an interior point.
$\Leftarrow$
Use key observation to prove the other direction in a similar manner. $\square$

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First we recall the definition of a continuous function especially Theorem 1.5. Note that this definition does not tell you about definition at a single point, rather as a set.

Example 2.1. *$f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$.*

Proof. We will look at three representative sets which will cover all the cases.

1. $(1, 2)$. Now we will look at the pre-image. $f^{-1}(1, 2) = (-\sqrt{2}, -1) \cup (1, \sqrt{2}$ which is an open set since unions of open sets are open.
2. $(-2, -1)$. Now we will look at the pre-image. $f^{-1}(-2. -1) = \phi$. The null set is open.
3. $(-1, 1)$.Now we will look at the pre-image. $f^{-1}(-1, 1) = (-1, 1)$ which is open.

$\square$

Now we will have a corollary to **Theorem 1.5**.

Corollary 2.2. *A function $f : X \to Y$ is continuous for all closed sets $K \subseteq Y$, $f^{-1}(K)$ is closed in X.*

Proof. First we claim $X - f^{-1}(K) = f^{-1}(Y - K)$. Indeed we know that

$$f^{-1}(Y - K) = \{x \in X | f(x) \in Y - k\} = \{x \in X | f(x) \notin K\} = \{x \in X | x \notin f^{-1}(K)\} = X - f^{-1}(K).$$

Let $f : X \to Y$ be continuous and $K \subseteq Y$ be closed. Then $Y - K$ is open, and $f^{-1}(Y - K)$ is open. Thus $X - F^{-1}(K)$ is open by the claim. Therefore $f^{-1}(K)$ is closed. $\square$

Theorem 2.3. *Let $f : X \to Y$ be continuous. If X is compact, then $f(x)$ is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of $f(x)$. Then $f^{-1} \bigcup_\alpha U_\alpha = X$. We know that

$$f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$$

So $\{f^{-1}(U_\alpha)\}$ is an open cover of X. but X is compact so there exists a finite subcover. So

$$X = f^{-1}U_{\alpha_1} \cup \cdots \cup f^{-1}(U_{\alpha_n}).$$

So we know that

$$f(x) = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Thus $\{U_{\alpha_1}, \dots U_{\alpha_2}\}$. $\square$

Corollary 2.4. *Let $f : X \to \mathbb{R}^k$ be continuous. If X is compact, then $f(x)$ is closed and bounded.*

Proof. Heine-Borel Theorem. $\square$

Corollary 2.5. *Let $f : X \to \mathbb{R}$ be continuous. If X is compact, then $f(x)$ achieves both its maximum and minimum values.*

Proof. Let $M = \sup\{f(x)|x \in X\}$ and $m = \inf\{f(x)|x \in X\}$. We know that $M \in \overline{f(x)} = f(x)$. Also, $m \in \overline{f(x)} = f(x)$. $\square$

Note: We say $f : X \to Y$ is bounded if its image $f(x)$ is bounded. This implies that $f : X \to \mathbb{R}^K$ is bounded if X is compact and f is continuous.

Question: Let $f : Y \to X$ be bijective (1-1 and onto) and continuous. Then $f^{-1} : X \to Y$ is defined. Is f^{-1} continuous?
Answer: Not necessarily. We will now define an example. Look at $f : \mathbb{R} to \mathbb{R}$ where the first $\mathbb{R}$ is under the discrete metric and the second under the euclidean metric.

Theorem 2.6. *Let $f : X \to Y$ be bijective and continuous. If X is compact, then $f^{-1} : Y \to X$ is continuous.*

Proof. We must show for any open $U \subseteq X$, $(f^{-1})^{-1}(U)$ is open. Note that $(f^{-1})^{-1} = f$ because f is bijective. Therefore, we want to show that $f(U)$ is open. But U is open, hence $X - U$ is closed. So we now have a closed subset of a compact set, which implies that $X - U$ is compact. Since f is continuous, we know that $f(X - U) = Y - f(U)$ is compact since continuous functions map compact sets to compact sets. Hence $Y - f(U)$ is closed. Therefore $f(U)$ is open. $\square$

Definition 2.7. *A map $f : X \to Y$ is called a homeomorphism if f is bijective and continuous and f^{-1} is continuous.*

Notice that this does not require any notion of metric.

Theorem 2.8. *Let $f : X \to Y$ be continuous. If X is connected then $f(X)$ is connected.*

Proof. Recall that X is connected $\Leftrightarrow X \neq U \bigcup V$ where $U, V \neq \phi$ are open and $U \bigcap V = \phi$. We will prove the contrapositive of this statement. Suppose that $f(X)$ is disconnected. So there exists $U, V \subseteq f(X)$ such that $U, V \neq \phi$ are open and $f(X) = U \bigcup V$. Then we know that

$$f^{-1}(U \cup V) = X.$$

But we also now that

$$X = f^{-1}(U) \cup f^{-1}(V).$$

Suppose that $x \in U \bigcap V$. Then $f(x) \in U \bigcap V = \phi$. thus $f^{-1}(U)$ and $f^{-1}(V)$ are separate. $\square$

Corollary 2.9 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If c is between $f(a)$ and $f(b)$ then there exists $x \in (a, b)$ such that $f(x) = c$.*

Proof. Without loss of generality $f(a) \leq c \leq f(b)$. If no such x exists the $(f(a), c), (c, f(b)$ separate $f([a, b])$. $\square$

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Definition 3.1. *A function $f : X \to Y$ is uniformly continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_x(x, p) < \delta$ implies $d_y(f(x), f(p)) < \epsilon$.*

Example 3.2. *Let $f(x) = \frac{1}{x}$. Prove f is not uniformly continuous on $\mathbb{R} - 0$ (the function's domain).*

Proof. Let $\epsilon > 0$. Choose any $\delta > 0$. Then there exists $N \in \mathbb{N}$ such that $\forall\, n > N$, $\frac{1}{n} < \delta$. Choose $p \in (\frac{-1}{2n}, 0)$ $q \in (0, \frac{1}{2n})$. Then we know

$$d_x(p, q) < \frac{1}{2n} - \frac{-1}{2n} = \frac{1}{n} < \delta.$$

But we also know that

$$d_y(f(p), f(q)) = \frac{1}{q} - \frac{1}{p} > 4n.$$

We also know there exists $N' \in \mathbb{N}$ such that $\forall n > N'$, $4n > \epsilon$. So choose $n > max\{N, N'\}$. Then we know that $d_x(p, q) < \delta$, but $d_y(f(p), f(q)) \geq \epsilon$. So this function is not uniformly continuous. $\square$

Theorem 3.3. *If f is uniformly continuous then f is continuous.*

Proof. This proof is in the book, it is very straightforward. $\square$

So we know that if a function is not continuous then it is not uniformly continuous by contrapositive of the previous theorem.

Theorem 3.4. *Let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.*

Proof. Let $\epsilon > 0$ and $p \in X$. We want to show that there exists $\delta > 0$ such that

$$d_x(x,p) < \delta \implies d_y(f(x), f(p)) < \epsilon$$

for all $p \in X$. Since f is continuous on X, f is continuous at each $p \in X$. So $\delta_p > 0$ such that

$$d_x(x,p) < \delta_p \implies d(f(x), d(p)) < \frac{\epsilon}{2}.$$

Note that this collection of open balls of radius δ_p centered at p is an open cover of X. Since X is compact, there exists points $p_1, \ldots, p_n$ such that

$$X = B_{\delta_1}(p_1) \cup \cdots \cup B_{\delta_n}(p_n).$$

Let $\delta = min\{\delta_1, \ldots \delta_n\}$. Suppose $d_x(x,p) < \delta$. If we can show that there exists i such that $x, p \in B_{\delta_i}(p_i)$, then

$$d_y(f(x), f(p)) \leq d_y(f(x), f(p_i)) + d_y(f(p), f(p_i)).$$

We will show this with the following lemma.

Lemma 3.5 (Lebesgue Covering Lemma)**.** *If $\{U_\alpha\}$ is an open cover of a compact metric space, then there exists $\delta > 0$ such that for all $x \in X$, $B_\delta(x) \subseteq U_\alpha$ for some α.*

Note that δ is called the Lebesgue number of the open cover.

Proof. Since X is compact, there exists a finite subcover $\{U_{\alpha_1}, \ldots U_{\alpha_n}\}$. For any $K \subseteq X$ define $d(x,k) = \inf\{d(x,y) | y \in K\}$. We claim that $d(x,k)$ is a continuous function. We will leave this without proof. Intuitively, this map goes from $X \to \mathbb{R}$. We know that if it maps to an open interval in $\mathbb{R}$, the pre-image is also open. Therefore, it is continuous. Let

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, X - U_{\alpha_i}).$$

Then f is continuous on X because d is. Since we know that f is continuous on a compact set X it must achieve its minimum value. Call this minimum value δ. For all $x \in X$, $f(x) \geq \delta$ at least one $d(x, X - U_{\alpha_i}) \geq \delta$. Thus $B_\delta) \subseteq U_{\alpha_i}$. $\square$

This completes the proof of Theorem 3.4. $\square$

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Discontinuities There is more than one way in which a function can fail to be continuous. The first way involved left hand and right hand limits.

Definition 4.1. *Let $f : (a,b) \to \mathbb{R}$, If, for all sequences $\{t_n\}$ in $(a,x)$where $a \leq x \leq b$ with $t_n \to x$ and we have $f(t_n) \to q$. Then we write $\lim_{t \to x^-} f(t) = q$ or $f(x-) = q$.*
Similarly if $f(s_n) \to p$ for all $\{s_n\}$ in (x,b) with $s_n \to x$ we write $\lim_{s \to x^+} f(s)$ or $f(x^+) = p$.

Definition 4.2. *If $f(x)$ is discontinuous at x_0 but $f(x_0^-)$ and $f(x_0^+)$ exist, then f has a discontinuity of the first kind, or a simple discontinuity.*

Theorem 4.3. $\lim_{x\to x_o} f(x)$ *exists if and only if* $f(x_0^+) = f(x_0^-)$.
Note: this implies that both the left hand and right hand limits exist.

To prove continuity we would need more information, but this does show the existence of the limit.

Example 4.4 (The Topologists Sine Curve)**.**

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Note that this function has a frequency approaching infinity as x approaches 0. This is not continuous and it is a discontinuity of the second kind at 0 because the left and the right hand limits don't exist.

Example 4.5.

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Around 0, the amplitude of this function goes to 0 as the amplitude is controlled by x. This function is bounded by x and so we can do a sort of squeeze theorem to show that this is continuous. This means there is no discontinuity at 0.

Example 4.6 (Dirichlet's Function)**.**

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

There is a discontinuity of the second kind at every point.

Example 4.7.

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Continuous at $x = 0$, but not anywhere else. Discontinuity of the second kind at all other points.

Example 4.8.

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \\ 0 & \text{otherwise} \end{cases}$$

Note that $f(0) = 0$

This is left as a challenge problem to think about when I have nothing to do. This probably won't happen though. I think that it is continuous at 0, but it seems to have other points of interest.

Theorem 4.9. *If* $f : \mathbb{R} \to \mathbb{R}$ *is monotonic, then* f *has no discontinuities of the second kind.*

Proof. Obvious. How Harvey Mudd. Proof probably in book. □

Sequences and Series of Functions

Definition 4.10. *For each* $n \in \mathbb{N}$ *let* $f_n : X \to Y$*. Then* $\{f_n\}$ *is a sequence of functions.*

We want to think about the convergence of a sequence of functions. Note that for any $a \in X$, $\{f_n(a)\}$ is a sequence in Y.

Definition 4.11. *If* $\{f_n(a)\}$ *converges in* Y *for all* $a \in X$*, we say the sequence of functions converges pointwise to* $f(x)$*. We write*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

This is a pointwise construction of a new function.

Definition 4.12. *Let $f_n : X \to Y$ be a sequence of functions. If $\sum f_n(x)$ converges for all $x \in X$, we say the series converges pointwise to*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

This is another pointwise construction of a new function.
Question:
If $f_n \to f$ pointwise, and each f_n is continuous, is f continuous?

Example 4.13. *Let $f_n(x) = \frac{x}{n}$.*

We know that $\lim_{n\to\infty} f_n(a) = 0$ for all $a \in \mathbb{R}$, so the limiting function is $f(x) = 0$, the zero function.

Example 4.14. *$f_n(x) = x^n$ where $f[0,1] \to \mathbb{R}$.*

So we know that the limiting function is

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

So we have an example where the question is false. So in general we cannot answer this question.

Example 4.15. *$f_n(x) = \frac{1}{n}\sin(n^2 x)$*

Pointwise, this function converges to 0. Note that $f_n'(x) = n\cos(n^2 x)$. This does not converge as a real number. However, $f'(x) = 0$. So derivative is not always preserved.
This implies that pointwise convergence is not really that great because we cannot say much about the limiting behavior. So we will look at other kinds of convergence of sequences of functions.

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So we have that $\lim_{n\to\infty} f_n(x) = f(x)$ if $f_n \to f$ pointwise. Also $f(x)$ is continuous at x if $\lim_{t\to x} f(t) = f(x)$. Thus if f is continuous, and $f_n \to f$ pointwise, we know that

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

So here we can reverse limits which is something that we cannot do in general.
Note that differentiation and integration both involve limits of limits.

Definition 5.1. *We say $f_n \to f$ uniformly on $E \subseteq X$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $||f_n - f|| < \epsilon$ where $||g|| = sup_{x\in E}\{|g(x)|\}$.*

Notice that $||f_n - f|| < \epsilon \Leftrightarrow |f_n(x) - f(x)| < \epsilon$ for all $x \in E$. The set $\{f : E \to Y\}$ is a metric space with metric $||\ ||$. Notation: $C_b(E) = \{f : E \to Y\}$. f is continuous and bounded. Also, $C_b(E)$ is compete.

Theorem 5.2 (Cauchy Criterion)**.** *$f_n \to f$ uniformly $\Leftrightarrow$ for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $|f_n(x) - f(x)| < \epsilon$.*

Theorem 5.3 (Weierstrass 1872)**.**

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$$

where $0 < b < 1$, $a \in 2\mathbb{Z} = 1$ and $ab > 1 + \frac{3\pi}{2}$. Then f is continuous everywhere, but differentiable nowhere.

Theorem 5.4. *Let $\{f_n\}$ be continuous. If $f_n \to f$ uniformly, then f is continuous.*

Proof.
$$|f(x) - f(y)| \leq |f(x) - f_l(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)|$$
Since $f_n \to f$ there exists $N \in \mathbb{N}$ such that for all $n > N$,$\| f_n(x) - f(x)| < \frac{\epsilon}{3}$. Also, f_k is continuous, so we know there exists $\delta > 0$ such that for $|x - y| < \delta$ implies $|f_k(x) - f(y)| < \frac{\epsilon}{3}$. Now let $|x - y| < \delta$ and $k > N$. Then we know that $|f(x) - f(y)| < 3\frac{\epsilon}{3} = \epsilon$. □

Example 5.5. $xy = 1$, $x > 0$. *Is this function uniformly continuous?*

Proof. It is not uniformly continuous because as we get arbitrary close to the y-axis, the image gets further apart. Another way of saying it is that the slop is unbounded. □

So we have function that are not uniformly continuous even though they are continuous.

Example 5.6. $f_n(x) = \frac{x}{n}$. *How does this converge pointwise and uniformly?*

Proof. We see that $f_n(x) \to 0$ pointwise. This does not converge uniformly on $\mathbb{R}$. This is because $|f_n(x) - f(x)| = |\frac{x}{n}|$. Choose $x > n\epsilon$. Intuitively because we always have positive slope, we cannot contain any line in an ϵ neighborhood. However, on a bounded interval, it would be true. □

The Derivative

Definition 5.7 (Differentiability)**.** *A function* $f[a, b] \to \mathbb{R}$ *is differentiable at* $x \in [a, b]$ *if this limit exists:*
$$f'(x) : \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

Note that f continuous on$[a, b]$ does not necessarily imply that f is differentiable on $[a, b]$. However, differentiability on $[a, b]$ implies continuity on $[a, b]$. This proof is simple.
Also, note that if f is differentiable on $[a, b]$, then f' is not necessarily continuous on $[a, b]$.
We know that if f' exists it is not arbitrarily nasty, f' always satisfies the intermediate value property and has no simple discontinuities.

Theorem 5.8 (Mean Value Theorem)**.** *If* f *is continuous on* $[a, b]$ *and differentiable on* (a, b)*, then there exists* $c \in (a, b)$ *such that*
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We will prove a stronger theorem by using Rolle's Theorem. □

Theorem 5.9 (Rolle's Theorem)**.** *If* f *is continuous on* $[a, b]$ *and differentiable on* (a, b) *and* $f(a) = f(b) = 0$ *then there exists* $c \in (a, b)$ *such that*
$$f'(c) = 0.$$

Proof. Proved with intermediate value theorem. □

Theorem 5.10 (Generalized Mean Value Theorem)**.** *If* f *and* g *are continuous on* $[a, b]$ *and differentiable on* (a, b)*, then there exists* $c \in (a, b)$ *such that*
$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Note that if $g = x$, we have the regular Mean Value Theorem.

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Integration
Goal of today is to define three different notions of integration that are all related to each other.

Remark 6.1. *The area "under" the graphs of a function. Area is not well defined at this moment, we need a more rigorous approach.*

One answer to this problem may be to squeeze this region between two converging regions. For now we are dealing with real valued functions. Essentially we usually think of integration as breaking up the region into different rectangles and we know that the area of a rectangle is its base times its height.
However, now think of dividing up a function into partitions and then either take inf or sup of these partitions depending on whether we take values that are $\leq$ or $\geq$ the values of the function. The big idea is that if we take the limit as the number of partitions go to ∞, the difference between these sup and inf should converge. We will now make this more rigorous.

Definition 6.2 (Partition)**.** *A partition P of $[a, b]$ is a set*

$$P = \{x_0, \ldots, x_n\}$$

such that $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$.

Definition 6.3 (Mesh)**.** *Let $\Delta x_i = x_i - x_{i-1}$.*
The mesh of P is $Mesh(P) = |P| = Max\{\Delta x_i | i = 1, \ldots, k\}$.

Definition 6.4 (Refinement)**.** *A partition Q is a refinement of P if $P \subseteq Q$.*

Notation:
Let f be bounded on $[a, b]$ and let P be a partition of $[a, b]$. Let

$$M_i = \sup\{f(x) | x \in [x_{i-1}, x_i\}.$$

$$m_i = \inf\{f(x) | x \in [x_{i-1}, x_i\}.$$

Definition 6.5 (Upper and Lower Darboux Sums)**.** *The upper and lower Darboux sums of f on $[a, b]$ with respect to P.*

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

Definition 6.6 (Upper and Lower Darboux Integrals)**.** *The upper and lower Darboux integrals of f over $[a, b]$ are*

$$\int_{a-}^{b} = \sup\{L(P, f) | P \text{ partition of } [a, b]\}$$

$$\int_{a}^{b+} = \inf\{U(P, f) | P \text{ partition of } [a, b]\}$$

We can use a sup and an inf because our sets are bounded by the function.

Definition 6.7 (Darboux Integrable)**.** *If*

$$\int_{a-}^{b} f(x)dx = \int_{a}^{b+} f(x)dx$$

we say f is Darboux integrable and we write

$$\int_{a}^{b} f(x)dx.$$

Example 6.8. *Let $f(x) = 3, a = 2, b = 4$. Integrate this with Darboux integrals.*

Proof. Let $P = \{2 = x_0, \dots, x_n = 4\}$ be a partition.

$$M_i = \sup\{f(x) | x \in [x_{i-1}, x_i\} = 3.$$

So therefore,

$$U(P, f) = \sum_{i=1}^{n} 3\Delta x_i = 3\sum_{i=1}^{n}(x_i - x_{i-1} = 3(x_1 - x_0 + x_2 - x_1 + \dots + x_n - x_{n-1} = 3(x_n - x_0) = 6.$$

We have used the fact we have a telescoping sequence and independence of partition to get

$$\int_a^{b+} f(x)dx = 6.$$

Similarly,

$$\int_{a-}^{b} f(x)dx = 6.$$

Therefore, $\inf_a^b f(x)dx = 6$. □

Now we will move to Riemann integration.

Definition 6.9 (Riemann Sum)**.** *A Riemann Sum of f over $[a, b]$ with respect to P is*

$$R(P, f) = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where $c_i \in [x_{i-1}, x_i]$.

Definition 6.10 (Riemann Integral)**.** *For $s \in \mathbb{R}$ we say the Riemann integral of f over $[a, b]$ is equal to s if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $Mesh(P) < \delta$ then $|R(P, f) - s| < \epsilon$. In this case, we write*

$$\int_a^b f(x)dx = s$$

and we say that f is Riemann integrable.

Theorem 6.11. *Let f be bounded on $[a, b]$. Then f is Darboux integrable if and only if f is Riemann integrable.*

We will prove this theorem, however, in doing so we will introduce another version of the integral and gather other pertinent results.

Definition 6.12 (Upper and Lower Riemann-Stieltjes Sums)**.** *Let f be a bounded function on $[a, b]$. Let α be a monotonically increasing function on $[a, b]$. Given a partition P, of $[a, b]$ we write,*

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0.$$

The upper and lower Riemann-Stieltjes sums are given by

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

Definition 6.13 (Upper and Lower Riemann-Stieltjes Integrals)**.** *The upper and lower Riemann-Steltjes integrals of f over $[a,b]$ are*

$$\int_{a-}^{b} f d\alpha = \sup\{L(P,f,\alpha) | P \ partition \ of \ [a,b]\}$$

$$\int_{a}^{b+} f d\alpha = \inf\{U(P,f,\alpha) | P \ partition \ of \ [a,b]\}$$

Definition 6.14 (Riemann-Stieltjes Integrable)**.** *If*

$$\int_{a-}^{b} f d\alpha = \int_{a}^{b+} f d\alpha$$

we say f is Riemann-Stieltjes integrable and we write

$$\int_{a}^{b} f\alpha.$$

Theorem 6.15. *If Q is a refinement of P, then*

$$L(P,f,\alpha) \leq L(Q,f,\alpha)$$

$$U(Q,f,\alpha) \leq U(P,f,\alpha)$$

Proof. Lets take the simplest non-trivial case where

$$Q = P \cup x^*$$

with $x^* \notin P$, $x \in (x_{i-1}, x_i)$. Let

$$w_1 = \inf\{f(x) | x \in [x_{i-1}, x^*]\}$$

$$w_2 = \inf\{f(x) | x \in [x^*, x_i]\}$$

Note that $w_1, w_2 \geq m_i$. Thus,

$$L(Q,f,\alpha) - L(P,f,\alpha) = w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) - m_i \Delta\alpha_i.$$

Note that $m_i \Delta\alpha_i = m_i(\alpha(x_i) - \alpha(x_{i-1})) = m_i(\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1}))$. So we have that

$$L(Q,f,\alpha) - L(P,f,\alpha) = (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})) + (w_2 - m_i)(\alpha(x_i) - \alpha(x^*)) \geq 0.$$

□

7 12/5/12

We want to show an example that shows a cool fact about Riemann integrable function.

Example 7.1. *Consider the function $y = x$ if $x > 0$ and 0 if $x \leq 0$.*

We know that if we integrate this function from 0 to 1, then we have

$$\int_0^1 x = x^2 \frac{1}{2}|_0^1 = \frac{1}{2}.$$

Let $\epsilon << 1$.

Case 1 $\delta > 1$.
Does not work.

Case 2 $\delta < 1$.

Mesh of a group of partition between 0 and 1 is greater than δ even though we can make ϵ arbitrarily small. So this shows the one way implication in the definition of Riemann integrability.

Theorem 7.2. *A bounded function is Riemann integrable $\Leftrightarrow$ Darboux integrable.*

Proof. $\Leftarrow$ Note that for all $i = 1, 2, \ldots n$. $m_i \leq f(c_i) \leq M_i$. where c_i is some point in the i^{th} subinterval. Thus

$$\sum m_i \Delta x_i \leq \sum f(c_i)\Delta x_i \leq \sum M_i \Delta x_i$$

in other words,

$$L(P, f) \leq R(P, f) \leq U(P, f).$$

If f is Darboux integrable, we have

$$\lim_{|P|\to 0} L(P, f) = \lim_{|P|\to 0} U(P, f).$$

Thus Darboux integrability implies Riemann integrability.

$\Rightarrow$ Suppose that f is Riemann integrable on $[a, b]$. The big idea is to choose c_i such that $f(c_i)$ is close to m_i or M_i. Let $\epsilon > 0$. Then we know there exists $\delta > 0$ such that $|R(P, f) - S| < \epsilon$ for all P with $Mesh(P) < \delta$.

$$M_i = \sup\{f(x)|x \in [x_{i-1}, x_i]\}.$$

Thus $f(c_i) \leq M_i$. For any $\zeta > 0$, $f(c_i) > M_i - \zeta$ for some choice of c_i. Thus

$$M_i - f(c_i) < \zeta.$$

This bounds

$$U(P, f) - L(P, f) = \sum(M_i - f(c_i))\Delta_i.$$

Similarly, we bound $R(P, f) - L(P, f)$ is bounded by m_i. So Riemann integrability implies Darboux integrability. $\square$

Theorem 7.3.

$$\underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx$$

Proof. Let P_1, P_2 be partitions of $[a, b]$. Then consider the mutual refinement, $P = P_1 \cup P_2$. We have

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(p, f, \alpha) \leq U(P_1, f, \alpha)$$

since we have a refinement. So,

$$L(P_1, f, \alpha \leq U(P_2, f, \alpha).$$

Thus, consider

$$\underline{\int_a^b} f dx \sup\{L(P_1, f, \alpha)|P_1\} \leq U(P_2, f, \alpha.$$

Therefore,

$$\underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx.$$

$\square$

Theorem 7.4. *$f \in \mathscr{R}$ on $[a, b]$ $\Leftrightarrow$, for all $\epsilon > 0$, there exists P such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Theorem 7.5. *If f is continuous on $[a, b]$, then $f \in \mathscr{R}$ on $[a, b]$.*

Theorem 7.6. *If f is monotonic on $[a,b]$, and α is continuous on $[a,b]$, then $f \in \mathscr{R}$.*

Proof. Without loss of generality, f is monotonically increasing. Let $\epsilon > 0$. Fix $n \in \mathbb{N}$. We construct partition P such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$$

Then

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum (M)i - m_i 0 \Delta x_i = \frac{\alpha(b)-\alpha(a)}{n} \sum (M_i - m_i) = \frac{\alpha(b)-\alpha(a)}{n} = \frac{\alpha(b)-\alpha(a)}{n}(f(b) - f(a))$$

Now choose n such that this is less than ϵ. □

Theorem 7.7. *Let f be bounded on$[a,b]$. If f has only finitely many discontinuities and α is continuous at the discontinuities of f. Then $f \in \mathscr{R}$ on $[a,b]$.*

Theorem 7.8. *Suppose $\alpha' \in \mathbb{R}$ on $[a,b]$. If f is bounded, then $f \in R(\alpha)$ if and only if $f\alpha' \in \mathbb{R}$. Moreover,*

$$\int_a^b f dx = \int_a^b f(x)\alpha'(x) dx$$

Example 7.9. *Consider a rod. Compute moment of inertia around middle point. Note, rod might not have even mass distribution.*

Proof. Since we know that $I = \sum m_i r_i^2$, we have

$$I = \int x^2 dm$$

By the theorem above, we have that

$$I = \int_a^b x^2 m'(x) dx$$

We know that $m'(x)$ is the density of the rod. So we have

$$I = \int_z^b x^2 \rho(x) dx.$$

□

This is the end of new material!