Max-flow min-cut

LEO and BOSS

June 2022

Contents

1 Definitions: Directed graphs, flows and cuts

1.1 Directed graph

Definition 1.1. A pair of the form (V, E), where

- V is a set, and
- $E \subset V \times V$, such that $(u, v) \in E \implies (v, u) \notin E$

is called a *directed graph*.

1.2 Flows

Definition 1.2. Suppose $G = (V, E)$ is a directed graph and $f : V \times V \to \mathbf{R}$ a function that vanishes on $(V \times V) \setminus E$. Then, we define the functions $\mathtt{In}_f : \mathscr{P}(V) \to \mathbf{R}$ and $\mathtt{Out}_f : \mathscr{P}(V) \to \mathbf{R}$ as follows:

$$\mathtt{In}_f(S) = \sum \{f[(u, v)] : (u, v) \in (V \setminus S) \times S\}, \text{ and}$$
$$\mathtt{Out}_f(S) = \sum \{f[(u, v)] : (u, v) \in S \times (V \setminus S)\}.$$

Definition 1.3. Let $G = (V, E)$ be a directed graph. A function $c : V \times V \to \mathbf{R}_{\geq 0}$ that satisfies

- $c(e) > 0$ if $e \in E$, and
- $c(e) = 0$ otherwise

is called a *capacity function on* G.

Definition 1.4. The quintuple (V, E, c, s, t) is called a *flow network* if $G = (V, E)$ is a directed graph, $c : V \times V \to \mathbf{R}_{\geq 0}$ is a capacity function on G and $s, t \in V$ are distinct.

Definition 1.5. Suppose $N = (V, E, c, s, t)$ is a flow network. If $f : E \to \mathbf{R}$ is such that

- $e \in E \implies f(e) \geq 0$,
- $e \in E \implies f(e) \leq c(e)$, and
- $v \in V \setminus \{s, t\} \implies \mathtt{In}_f(\{v\}) = \mathtt{Out}_f(\{v\})$

it is called a *flow on* N.

Definition 1.6. The sextuple $A = (V, E, c, s, t, f)$ is called an *active flow network* if $N = (V, E, c, s, t)$ is a flow network and f is a flow on N.

Definition 1.7. Let $\mathtt{ActiveFlows}_N$ denote the set of active flows on $N = (V, E, c, s, t)$. Define the function $\mathtt{F\text{-}value} : \mathtt{ActiveFlows}_N \to \mathbf{R}$ as $\mathtt{F\text{-}value}(A) = \mathtt{Out}_f(s) - \mathtt{In}_f(s)$, where $A = (V, E, c, s, t, f)$ is an active flow.

1.3 Cuts

Definition 1.8. Let $N = (V, E, c, s, t)$ be a flow network. A tuple (S, T) is called an *s-t-cut of N* if

- $S \cup T = V$,
- $S \cap T = \emptyset$,

- $s \in S$ and $t \in T$.

Definition 1.9. Let $\mathtt{Cuts}_N$ denote the set of s-t-cuts of $N = (V, E, c, s, t)$. The function $c_{\mathtt{Cuts}} : \mathtt{Cuts}_N \to \mathbf{R}_{\geq 0}$ is defined as

$$c_{\mathtt{Cuts}}[(S, T)] = \mathtt{Out}_c(S).$$

The *value* of an s-t-cut (S, T) is defined as $c_{\mathtt{Cuts}}[(S, T)]$.

2 Lemma 77

Lemma 1. *For an active flow $A = (V, E, c, s, t, f)$ and $S \subset V \backslash \{s, t\}$, we have that*

$$\mathtt{In}_f(S) - \mathtt{Out}_f(S) = \sum \{\mathtt{In}_f(\{u\}) - \mathtt{Out}_f(\{u\}) : u \in S\}.$$

Proof. First, we note the following:

$$\begin{aligned}
&\mathtt{In}_f(S) + \sum \{f(u,v) : (u,v) \in S \times S\} \\
&= \sum \{f(u,v) : (u,v) \in (V \backslash S) \times S\} + \sum \{f(u,v) : (u,v) \in S \times S\} \\
&= \sum \{f(u,v) : (u,v) \in (V \backslash S) \times S\} \sqcup \{f(u,v) : (u,v) \in S \times S\} \\
&= \sum \{f(u,v) : (u,v) \in (V \backslash S) \times S \cup (S \times S)\} \\
&= \sum \{f(u,v) : (u,v) \in V \times S\}
\end{aligned}$$

and similarly:

$$\begin{aligned}
&\mathtt{Out}_f(S) + \sum \{f(u,v) : (u,v) \in S \times S\} \\
&= \sum \{f(u,v) : (u,v) \in S \times (V \backslash S)\} + \sum \{f(u,v) : (u,v) \in S \times S\} \\
&= \sum \{f(u,v) : (u,v) \in S \times (V \backslash S)\} \sqcup \{f(u,v) : (u,v) \in S \times S\} \\
&= \sum \{f(u,v) : (u,v) \in S \times (V \backslash S) \cup (S \times S)\} \\
&= \sum \{f(u,v) : (u,v) \in S \times V\} \\
&= \sum \{f(v,u) : (v,u) \in S \times V\}.
\end{aligned}$$

We thus obtain that

$$\begin{aligned}
\mathtt{In}_f(S) - \mathtt{Out}_f(S) =& \mathtt{In}_f(S) - \mathtt{Out}_f(S) + \sum \{f(u,v) : (u,v) \in S \times S\} \\
&- \sum \{f(u,v) : (u,v) \in S \times S\} \\
&= \sum \{f(u,v) : (u,v) \in V \times S\} - \sum \{f(v,u) : (v,u) \in S \times V\} \\
&= \sum_{v \in S} \left(\sum \{f(u,v) : u \in V\} - \sum \{f(v,u) : u \in V\} \right) \\
&= \sum_{v \in S} (\mathtt{In}_f(\{v\}) - \mathtt{Out}_f(\{v\})) \\
&= \sum_{v \in S} 0.
\end{aligned}$$

////

Lemma 2. *Suppose that $A = (V, E, c, s, t, f)$ is an active flow. Let $S \subset V \setminus \{s, t\}$. Then $\mathtt{In}_f(S) = \mathtt{Out}_f(S)$.*

Not so formal proof. Consider the difference $\mathtt{In}_f(S) - \mathtt{Out}_f(S)$ and notice that it may equivalently be rewritten as

$$\sum\{\mathtt{In}_f(\{u\}) - \mathtt{Out}_f(\{u\}) : u \in S\}; \tag{1}$$

indeed, if $u, v \in S$ and $(u, v) \in E$, then, in (1), the terms $f(u, v)$ and $f(v, u)$ cancel, and so we are left with $\mathtt{In}_f(S) - \mathtt{Out}_f(S)$. Since (1) evaluates to 0, per definition, the statement follows. ////

Lemma 3. *Suppose that $A = (V, E, c, s, t, f)$ is an active flow and let $S, T \subset V$ be disjoint. Then the following equality holds:*

$$\mathtt{Out}_f(S \cup T) - \mathtt{In}_f(S \cup T) = \mathtt{Out}_f(S) + \mathtt{Out}(T) - \mathtt{In}_f(S) - \mathtt{In}_f(T).$$

Proof. We have the following chain of equalities:

$$\begin{aligned}
&\mathtt{Out}_f(S \cup T) + \sum\{f(u, v) : (u, v) \in S \times T\} + \sum\{f(u, v) : (u, v) \in T \times S\}\\
=&\sum\{f(u, v) : (u, v) \in (S \cup T) \times V \setminus (S \cup T)\} + \sum\{f(u, v) : (u, v) \in S \times T\}\\
&+ \sum\{f(u, v) : (u, v) \in T \times S\}\\
=&\sum\{f(u, v) : (u, v) \in (S \times V \setminus (S \cup T)) \cup (T \times V \setminus (S \cup T))\}\\
&+ \sum\{f(u, v) : (u, v) \in S \times T\} + \sum\{f(u, v) : (u, v) \in T \times S\}\\
=&\sum\{f(u, v) : (u, v) \in S \times V \setminus (S \cup T)\} + \sum\{f(u, v) : (u, v) \in T \times V \setminus (S \cup T)\}\\
&+ \sum\{f(u, v) : (u, v) \in S \times T\} + \sum\{f(u, v) : (u, v) \in T \times S\}\\
=&\sum\{f(u, v) : (u, v) \in (S \times V \setminus (S \cup T)) \cup (S \times T)\}\\
&+ \sum\{f(u, v) : (u, v) \in (T \times V \setminus (S \cup T) \cup T \times S\}\\
=&\sum\{f(u, v) : (u, v) \in S \times V \setminus S\} + \sum\{f(u, v) : (u, v) \in T \times V \setminus T\}\\
=&\mathtt{Out}_f(S) + \mathtt{Out}_f(T).
\end{aligned}$$

Analagously we obtain

$$\begin{aligned}
&\mathtt{In}_f(S \cup T) + \sum\{f(u, v) : (u, v) \in S \times T\} + \sum\{f(u, v) : (u, v) \in T \times S\}\\
=&\mathtt{In}_f(S) + \mathtt{In}_f(T).
\end{aligned}$$

We thus obtain that

$$\mathtt{Out}_f(S \cup T) - \mathtt{In}_f(S \cup T) = \mathtt{Out}_f(S) + \mathtt{Out}_f(T) - \mathtt{In}_f(S) - \mathtt{In}_f(T).$$

////

Lemma 4. *Suppose that $A = (V, E, c, s, t, f)$ is an active flow and let $S \subset V$. Then $\texttt{Out}_f(S) = \texttt{In}_f(V\backslash S)$.*

Proof. By definition we have that

$$\begin{aligned}\texttt{Out}_f(S) &= \sum\{f(u,v) : (u,v) \in S \times V\backslash S\} \\ &= \sum\{f(v,u) : (u,v) \in V\backslash S \times S\} \\ &= \texttt{In}_f(V\backslash S).\end{aligned}$$

////

Följdsats 2.1. Let A, S be as in the above lemma. Then $\texttt{In}_f(S) = \texttt{Out}_f(V\backslash T)$.

Lemma 5. *Suppose that $A = (V, E, c, s, t, f)$ is an active flow and let (S, T) be an s-t-cut thereof. Then the following equality holds:*

$$\texttt{Out}_f(\{s\}) - \texttt{In}_f(\{s\}) = \texttt{Out}_f(S) - \texttt{In}_f(S).$$

Proof. By lemma 4 we have that

$$\texttt{Out}_f(S\backslash\{s\}) = \texttt{In}_f(T \cup \{s\}) \tag{2}$$

and by följdsats 2.1 we have that

$$\texttt{In}_f(S\backslash\{s\}) = \texttt{Out}_f(T \cup \{s\}). \tag{3}$$

By considering 2 - 3, using lemma 1, lemma 3 (since $\{s\} \cap T = \emptyset$ as $s \in S$, with $S \cap T = \emptyset$) and lemma 4 (in that order) we obtain the following chain of equalities:

$$\begin{aligned}0 &= \texttt{Out}_f(S\backslash\{s\}) - \texttt{In}_f(S\backslash\{s\}) \\ &= \texttt{In}_f(T \cup \{s\}) - \texttt{Out}_f(T \cup \{s\}) \\ &= \texttt{In}_f(T) + \texttt{In}_f(\{s\}) - \texttt{Out}_f(T) - \texttt{Out}_f(\{s\}) \\ &= \texttt{Out}_f(S) + \texttt{In}_f(\{s\}) - \texttt{In}_f(S) - \texttt{Out}_f(\{s\})\end{aligned}$$

which rearranges to the desired expression. ////

Lemma 6. *Suppose that $A = (V, E, c, s, t, f)$ is an active flow and $S \subset T$. Then $\texttt{Out}_f(S) \leq \texttt{Out}_c(S)$.*

Proof. By definition we have the following:

$$\begin{aligned}\texttt{Out}_f(S) &= \sum\{f(u,v) : (u,v) \in S \times (V\backslash S)\} \\ &\leq \sum\{c(u,v) : (u,v) \in S \times (V\backslash S)\} \\ &= \texttt{Out}_c(S).\end{aligned}$$

////

Lemma 7. *Suppose that $A = (V, E, c, s, t, f)$ is an active flow and $C = (S, T)$ an s-t-cut. Then $\mathtt{F\text{-}value}(A) \leq c_{\mathtt{Cuts}}(C)$.*

Not so formal proof. From lemma 5, we realize that $\mathtt{F\text{-}value}(A) = \mathtt{Out}_f(S) - \mathtt{In}_f(S)$. By trivial inequality and then lemma 6, we obtain

$$\mathtt{Out}_f(S) - \mathtt{In}_f(S) \leq \mathtt{Out}_f(S) \leq \mathtt{Out}_c(S) = c(S, T)$$

as desired. ////

Definition 2.1. $A : \mathtt{ActiveFlows}$ is *maximal* if $\forall A' : \mathtt{ActiveFlows}(A'.\text{Flow network} = A.Network \longrightarrow \mathtt{F\text{-}value}(A') \leq \mathtt{F\text{-}value}(A))$.

Definition 2.2. $C : \mathtt{Cuts}$ is *minimal* if $\forall C' : \mathtt{Cuts}(C'.Network = C.Network \longrightarrow c_{\mathtt{Cuts}}(C') \geq c_{\mathtt{Cuts}}(C))$.

Superlemma 1. *Let $A = (V, E, c, s, t, f)$ be an active flow and C be an s-t-cut of $N = (V, E, c, s, t)$. If $C_{\mathtt{Cuts}}(C) = \mathtt{F\text{-}value}(A)$, then A is maximal[1] and C is minimal.*

Not so formal proof. Let A' be an active flow. By lemma 7, we have that $\mathtt{F\text{-}value}(A') \leq c_{\mathtt{Cuts}}(C)$, which can be rewritten as $\mathtt{F\text{-}value}(A') \leq \mathtt{F\text{-}value}(A)$. Next, if we let C' be some cut, then lemma 7 yield $\mathtt{F\text{-}value}(A) \leq c_{\mathtt{Cuts}}(C')$, which can be rewritten as $c_{\mathtt{Cuts}}(C) \leq c_{\mathtt{Cuts}}(C')$. ////

3 Lemma 81

3.1 Definitions: Residual network

Definition 3.1. *Directed multigraph*

Definition 3.2. *Path in a multigraph*

Definition 3.3. For a path p in a weighted directed multigraph, we define ω_p as the smallest weight in p.

Definition 3.4. Suppose that $A = (V, E, c, s, t, f)$ is an active flow. Define the function $c_f : E \longrightarrow \mathbf{R}$ as follows:

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v), & \text{if } (u, v) \in E \\ f(v, u), & \text{if } (v, u) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Next, form the set

$$E' = \{(u, v) : c_f(u, v) \neq 0\}.$$

The triple $A_f = (V, E', c_f, s, t)$ is called *the residual network of A*.

[1] Some overloading required; a flow A is maximal if for any other active flow $A' = (V, E, c, s, t, f')$, we have $\mathtt{F\text{-}value}(f) \geq \mathtt{F\text{-}value}(f')$.

Definition 3.5. Let $A_f = (V, E', c_f, s, t)$ be the residual network of A, then the tuple (V, E') is a multigraph. A path in (V, E'), from s to t, is called an *augmenting path in A_f*.

Definition 3.6. Suppose that A_f is a residual network and p an augmenting path thereof.

Definition 3.7. Suppose that $A_f = (V, E', c_f, s, t)$ is the residual network of A. Furthermore, let p be an augmenting path in A_f. We then define

$$f_{\text{NEW}}(u,v) = \begin{cases} f(u,v) + \omega_p, & (u,v) \in E \wedge (u,v) \in E(p). \\ f(u,v) - \omega_p, & (u,v) \in E \wedge (v,u) \in E(p) \\ f(u,v), & (u,v) \in E \wedge (v,u), (u,v) \notin E(p) \\ 0, & \text{else.} \end{cases}$$

Lemma 8 (Non-negative). *Let $A = (V, E, c, s, t, f)$ be an active flow. Then $\forall u, v \in V : f_{NEW}(u,v) \geq 0$.*

Proof. Since $\omega_p > 0$, all cases of edges (u,v), but the one saying $(u,v) \in E \wedge (v,u) \in E(p)$, are trivial. We therefore consider such an edge.

From the assumption, we conclude that $c_f(v,u) = f(u,v)$. By the minimality of ω_p, we have that $m_p \leq f(u,v)$ and so $0 \leq f(u,v) - \omega_p$. ////

Lemma 9 (No-overflow). *Let $A = (V, E, c, s, t, f)$ be an active flow. Then $\forall u, v \in V : f_{NEW}(u,v) \leq c(u,v)$.*

Proof. Since $\omega_p > 0$, all cases of edges (u,v), but the one saying $(u,v) \in E \wedge (u,v) \in E(p)$, are trivial. We therefore consider such an edge.

From the assumption, we conclude that $c_f(u,v) = c(u,v) - f(u,v)$. By the minimality of ω_p, we have that $m_p \leq c(u,v) - f(u,v)$ and so $f(u,v) + \omega_p \leq c(u,v)$. ////

Lemma 10. *Suppose that $v \in V \backslash \{s,t\}$ and $(u,v) \in E(p)$ and $(v,w) \in E(p)$. Then, if $(u,v) \in E \wedge (v,w) \in E$, we have*

- $\texttt{Out}_{f_{NEW}}(v) = \texttt{Out}_f(v) + f_{NEW}(u,v) - f(u,v)$
- $\texttt{In}_{f_{NEW}}(v) = q\texttt{In}_f(v) + f_{NEW}(v,w) - f(v,w)$

Lemma 11. *Suppose that $v \in V \backslash \{s,t\}$ and $(u,v) \in E(p)$ and $(v,w) \in E(p)$. Then, if $(u,v) \notin E \wedge (v,w) \in E$, we have*

- $\texttt{Out}_{f_{NEW}}(v) = \texttt{Out}_f(v) + f_{NEW}(u,v) - f(u,v)$

- $\mathtt{In}_{f_{NEW}}(v) = \mathtt{In}_f(v)$

Lemma 12 (Conservation of flow). *Let $A = (V, E, c, s, t, f)$ be an active flow. Then $\forall u \in V \backslash \{s, t\} : \mathtt{In}_{f_{NEW}}(u) = \mathtt{Out}_{f_{NEW}}(u)$.*

Proof. Suppose that $v \in V \backslash \{s, t\}$ and $(u, v) \in E(p)$ and $(v, w) \in E(p)$. We distinguish and prove the assertion for four cases:

- $(u, v) \in E \wedge (v, w) \in E$: Then

$$\begin{aligned}\mathtt{Out}_{f_{\text{NEW}}}(v) - \mathtt{In}_{f_{\text{NEW}}}(v) =&(\mathtt{Out}_f(v) + f_{\text{NEW}}(u, v) - f(u, v)) \\ &- (\mathtt{In}_f(v) + f_{\text{NEW}}(v, w) - f(v, w)) \\ =&(\mathtt{Out}_f(v) + \omega_p) - (\mathtt{In}_f(v) + \omega_p) \\ &= \mathtt{Out}_f(v) - \mathtt{In}_f(v) \\ &= 0.\end{aligned}$$

- $(u, v) \notin E \wedge (v, w) \in E$: Then

$$\begin{aligned}\mathtt{Out}_{f_{\text{NEW}}}(v) - \mathtt{In}_{f_{\text{NEW}}}(v) =&(\mathtt{Out}_f(v) + f_{\text{NEW}}(v, u) - f(v, u) + f_{\text{NEW}}(v, w) - f(v, w)) - \mathtt{In}_f(v) \\ =&\mathtt{Out}_f(v) - (\mathtt{In}_f(v) - \omega_p + \omega_p) \\ &= \mathtt{Out}_f(v) - \mathtt{In}_f(v) \\ &= 0.\end{aligned}$$

- $(u, v) \in E \wedge (v, w) \notin E$: Then

$$\begin{aligned}\mathtt{Out}_{f_{\text{NEW}}}(v) - \mathtt{In}_{f_{\text{NEW}}}(v) =&\mathtt{Out}_f(v) \\ &- (\mathtt{In}_f(v) + f_{\text{NEW}}(u, v) - f(u, v) + f_{\text{NEW}}(v, w) - f(v, w)) \\ =&\mathtt{Out}_f(v) - (\mathtt{In}_f(v) + \omega_p - \omega_p) \\ =&\mathtt{Out}_f(v) - \mathtt{In}_f(v) \\ =&0.\end{aligned}$$

- $(u, v) \notin E \wedge (v, w) \notin E$: Then

$$\begin{aligned}\mathtt{Out}_{f_{\text{NEW}}}(v) - \mathtt{In}_{f_{\text{NEW}}}(v) =&(\mathtt{Out}_f(v) + f_{\text{NEW}}(u, v) - f(u, v)) \\ &- (\mathtt{In}_f(v) + f_{\text{NEW}}(v, w) - f(v, w)) \\ =&(\mathtt{Out}_f(v) - \omega_p) - (\mathtt{In}_f(v) - \omega_p) \\ &= \mathtt{Out}_f(v) - \mathtt{In}_f(v) \\ &= 0.\end{aligned}$$

////

Superlemma 2. *Let A be an active flow and A_f its residual network. If there is an augmenting path in A_f, then A is not maximal*[2].

Följdsats 3.1. Let A be an active flow and A_f its residual network. If A is maximal, then A_f contains no augmenting path.

Proof. This is the contraposition of superlemma 2. ////

4 Lemma 'extra'

Lemma 13. *Let $A = (V, E, c, s, t, f)$ be an active flow that does not contain an augmenting path. If $S := \{u \in V : \exists \text{path from } s \text{ to } u\}$ and $T := V \setminus S$. Then (S, T) is an s-t-cut.*

Lemma 14. *Let A, S and T be as in lemma 13. Then $(u, v) \notin A_f(E)$ (here $A_f(E)$ denotes the edge set of the residual network of A).*

Lemma 15. *Let $A = (V, E, c, s, t, f)$ be an active flow that does not contain an augmenting path and (S, T) a cut thereof such that $(u, v) \in S \times T \implies (u, v) \notin A_f$. Then $(u, v) \in S \times T \implies c_f(u, v) = 0$.*

Lemma 16. *Let $A = (V, E, c, s, t, f)$ be an active flow that does not contain an augmenting path. Suppose that $A, B \subset V$ are disjoint and satisfy $(u, v) \in A \times B \implies c_f(u, v) = 0$. Then the following holds*

(a) $(u, v) \in A \times B \implies f(u, v) = c(u, v)$, *and*

(b) $(u, v) \in B \times A \implies f(u, v) = 0$.

Lemma 17. *Let $A = (V, E, c, s, t, f)$ be an active flow that does not contain an augmenting path and (S, T) a cut thereof that satisfies $(u, v) \in T \times S \implies f(u, v) = 0$. Then $\texttt{F-value}(A) = \texttt{Out}_f(S)$.*

Superlemma 3. *Let $A = (V, E, c, s, t, f)$ be an active flow that does not contain an augmenting path. Then there is an s-t-cut C such that $\texttt{F-value}(A) = c(C)$.*

Not so formal-proof. Define $S = \{u : \exists \text{augmenting path from } s \text{ to } u\}$ and $T = V \setminus S$. Since A contains no augmenting path, T must be non-empty. By the choice of S, we must have that $(u, v) \in S \times T \implies f(u, v) = c(u, v)$. By lemma 2 we have that

$$\texttt{F-value}(A) = \sum \{f(u, v) : (u, v) \in S \times T\},$$

which can then be rewritten as

$$\begin{aligned} \texttt{F-value}(A) &= \sum \{c(u, v) : (u, v) \in S \times T\} \\ &= c_{\texttt{Cuts}}(S, T). \end{aligned}$$

////

[2] Maximal with respect to $\texttt{F-value}$.

5 Ford-Fulkerson

Sats 1. *Let $A = (V, E, c, s, t, f)$ be an active flow. Then the following are equivalent:*

(i) A is maximal.

(ii) A_f contains no augmenting path.

(iii) There is an s−t-cut on $N = (V, E, c, s, t)$ such that $c_{\texttt{cuts}}([(S, T)]) = \texttt{F-value}(A)$.

Proof. The statement $(i) \implies (ii)$ is the contraposition of superlemma (2).

$(ii) \implies (iii)$ follows from superlemma 3 and $(iii) \implies (i)$ from superlemma 1. ////