

Lab Topic 2

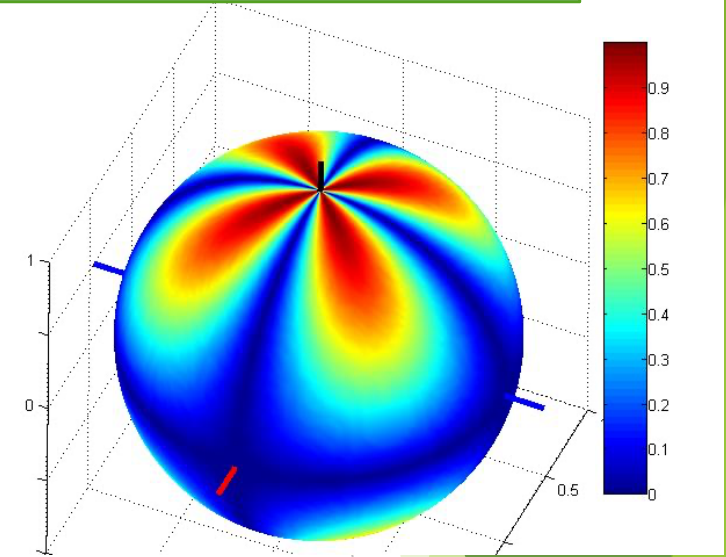
Learning objectives

- ▶ Learn how to calculate the response of a GW detector to a plane GW
 - ▶ Response calculations for both LIGO and LISA
 - ▶ Long wavelength approximation
 - ▶ Detector rotation and motion included for LISA

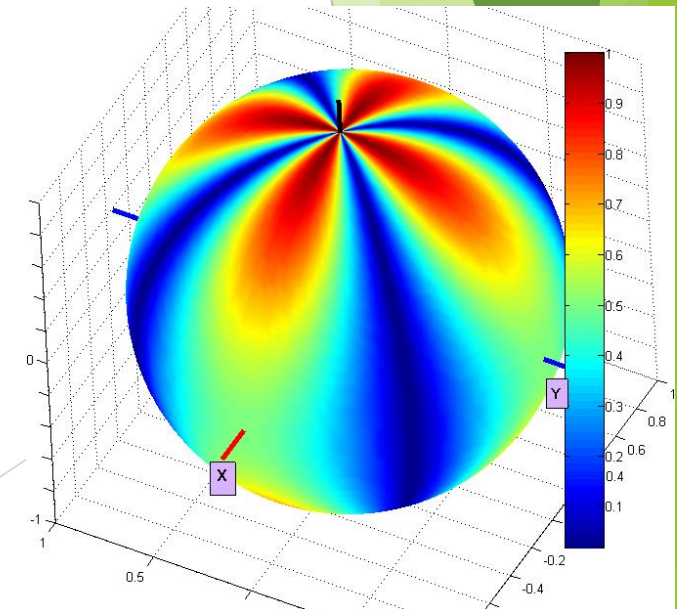
Antenna Patterns: Local frame ← Analytical forms

- ▶ Long wavelength and static detector approximation throughout this Lab
- ▶ Write a code to calculate F_+ and F_\times in an L-shaped interferometer's local frame from their analytical formulae
 - ▶ Source direction: (θ, ϕ) in detector frame
 - ▶ Plot them on a sphere using the GWSC/GW/skyplot.m function
 - ▶ F_+ Demo code: GWSC/GW/formulafp.m, GWSC/GW/testskyplot.m
- ▶ Plots should agree with the pictures in the lecture slides

$$F_\times(\theta, \phi) = \cos \theta \sin 2\phi$$

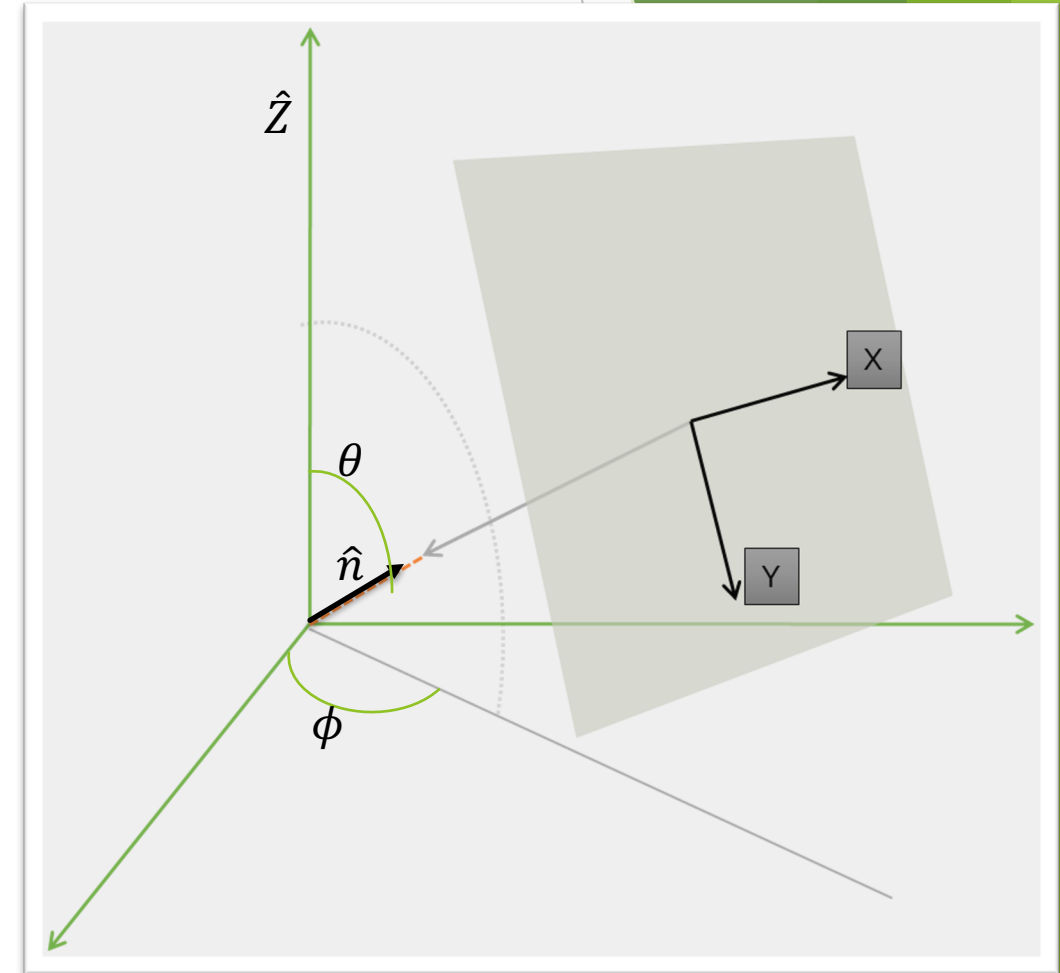


$$F_+(\theta, \phi) = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi$$



Antenna Patterns: Local frame ← using tensors

- ▶ Use the expression for (a) polarization tensors, (b) Detector tensor, and (c) Contraction of polarization and detector tensors to obtain $F_{+, \times}$
- ▶ All tensor components must be expressed in a common frame before the tensors are contracted → express all unit vector components in a common frame
- ▶ We will use the detector frame as the common one:
 - ▶ Detector arm unit vectors and their components in the detector frame: $\hat{n}_X = (1, 0, 0)$, $\hat{n}_Y = (0, 1, 0)$
 - ▶ Detector frame Z vector: $\hat{Z} = (0, 0, 1)$,
 - ▶ Source direction vector in detector frame (for polar angles θ and ϕ): \hat{n}
- ▶ Wave frame unit vector components for polarization tensor calculation (burst GW convention):
 - ▶ Wave frame $\hat{x} \propto \hat{Z} \times \hat{n}$ (Note: must normalize)
 - ▶ Wave frame $\hat{y} = \hat{x} \times \hat{n}$
 - ▶ (Use GWSC/GW/vcrossprod.m to obtain vector cross product components numerically)



Strain signal

- Detector tensor:

$$\vec{D} = \frac{1}{2}(\hat{n}_X \otimes \hat{n}_X - \hat{n}_Y \otimes \hat{n}_Y)$$

- Wave tensor:

$$\vec{W} = h_+(t) \vec{e}_+ + h_\times(t) \vec{e}_\times$$

- Polarization tensors: $\vec{e}_+ = \hat{x} \otimes \hat{x} - \hat{y} \otimes \hat{y}$; $\vec{e}_\times = \hat{x} \otimes \hat{y} + \hat{y} \otimes \hat{x}$

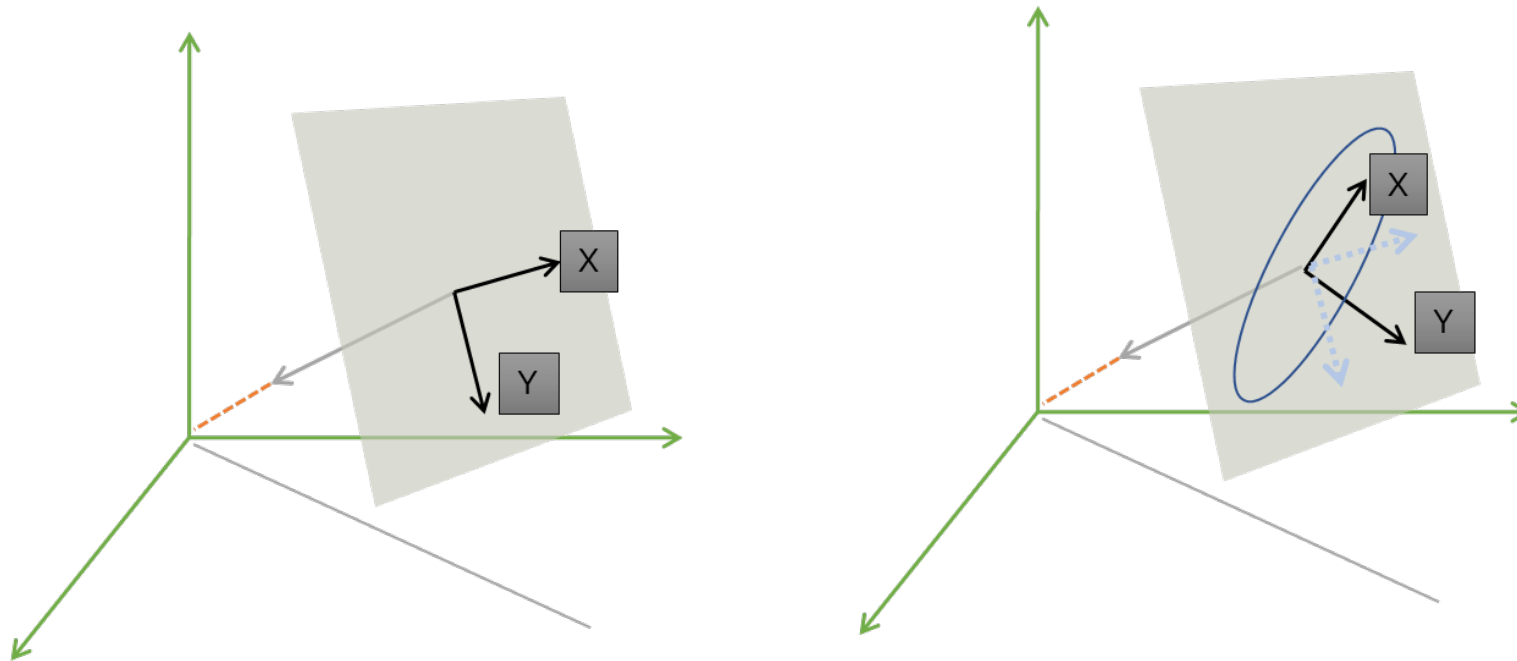
- Matlab can calculate direct products of vectors:

$$\underbrace{\begin{matrix} a = [a_1, & a_2, & a_3] \\ b = [b_1, & b_2, & b_3] \end{matrix}}_{\substack{2 \text{ row vectors in} \\ \text{Matlab}}} \xrightarrow{\text{Matlab}} \underbrace{a' * b}_{\text{Matlab}} \rightarrow \hat{a} \otimes \hat{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1 \quad b_2 \quad b_3) = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

- Strain signal: “Contraction of wave and detector tensors”

$$s(t) = \sum_{i,j=1}^3 W_{ij} D_{ij} = W^{ij} D_{ij} = \vec{W} : \vec{D} = h_+(t) \overbrace{\vec{W} : \vec{e}_+}^{F_+(\hat{n})} + h_\times(t) \overbrace{\vec{W} : \vec{e}_\times}^{F_\times(\hat{n})}$$

- Contraction of matrices A and B in Matlab $\rightarrow \text{sum}(A(:) .* B(:))$
- Compare the antenna patterns obtained using tensor contractions and analytical forms
- Demo codes: GWSC / GW/ detframefpfc.m and testdetframefpfc.m



Exercise: Wave frame conventions

Extend the previous exercise to include rotation due to polarization angle into the polarization tensors (Hint: the new wave frame X, Y vector components will be linear combinations of the old X, Y components)

Strain signal from a non-evolving binary

- Use the sinusoidal signal generation function to generate

$$h_+(t) = A \sin(2\pi f_0 t)$$

$$h_\times(t) = B \sin(2\pi f_0 t + \phi_0)$$

- Pick your own values of A, B, f_0, ϕ_0 (Respect Nyquist theorem!)
- Plot the strain signal for different values of θ, ϕ , and ψ

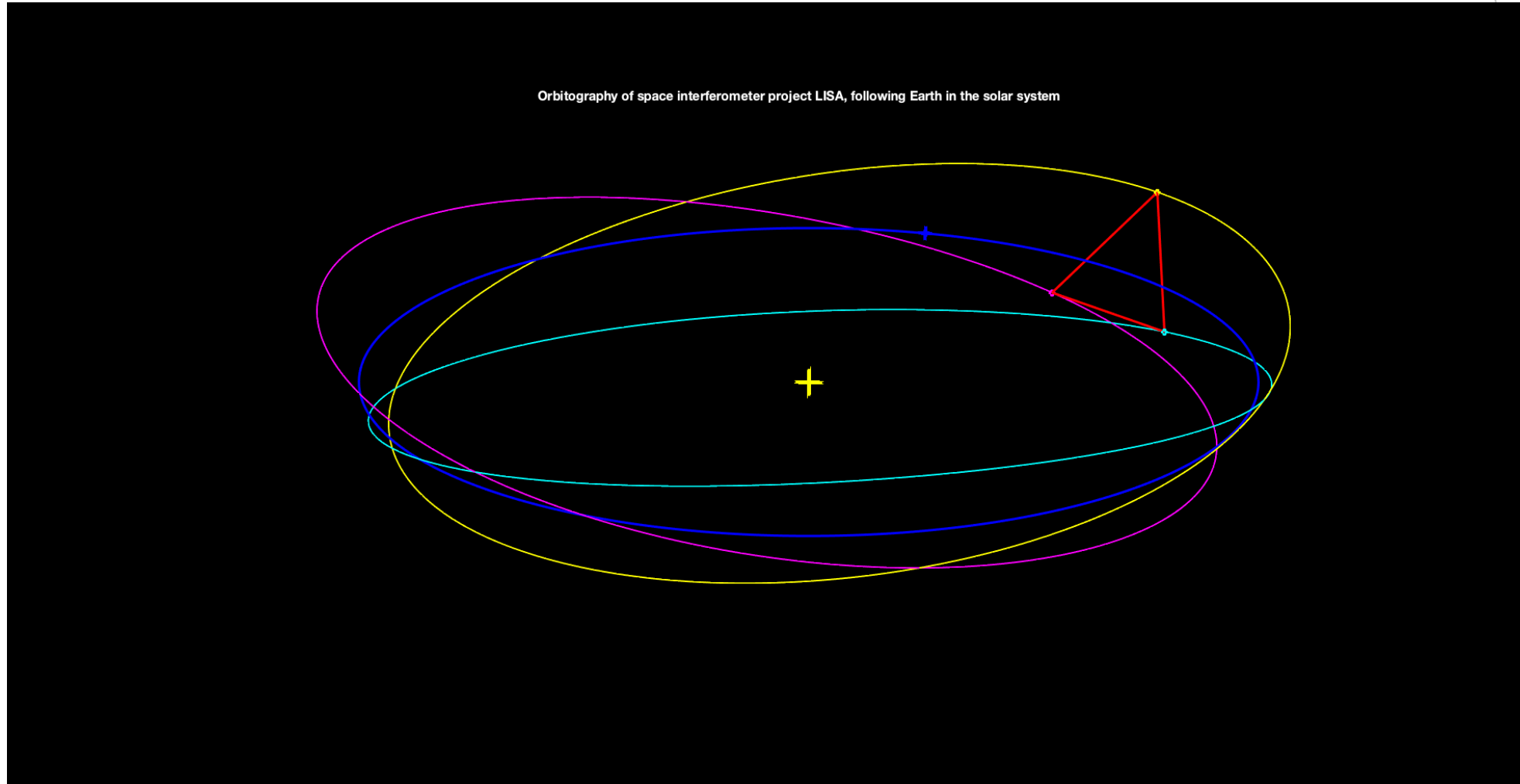
General strain signal for a static interferometer

- ▶ Write a function:
 - ▶ Inputs:
 - ▶ h_+ (vector) and h_\times (vector): time series of the polarizations (don't have to be sinusoidal)
 - ▶ θ, ϕ
 - ▶ Output:
 - ▶ Strain signal (for perpendicular arm interferometer)

Antenna patterns for LISA

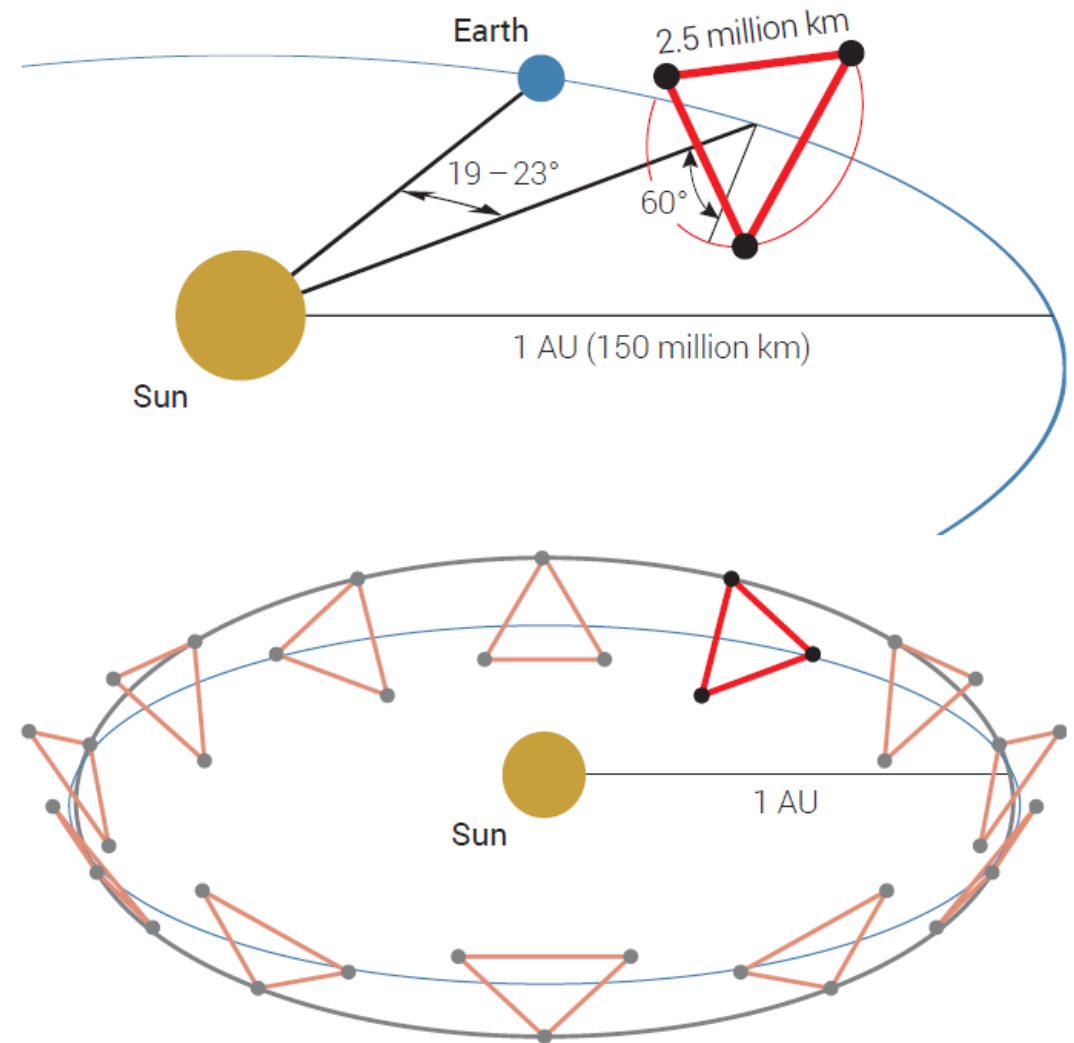
Toy LISA

- ▶ **Toy LISA:** Rigid equilateral triangle formation of three satellites
- ▶ Actual LISA cannot be rigid because the satellites must follow Keplerian orbits
- ▶ Toy LISA is good for practicing data analysis because it allows fast generation of signals and templates

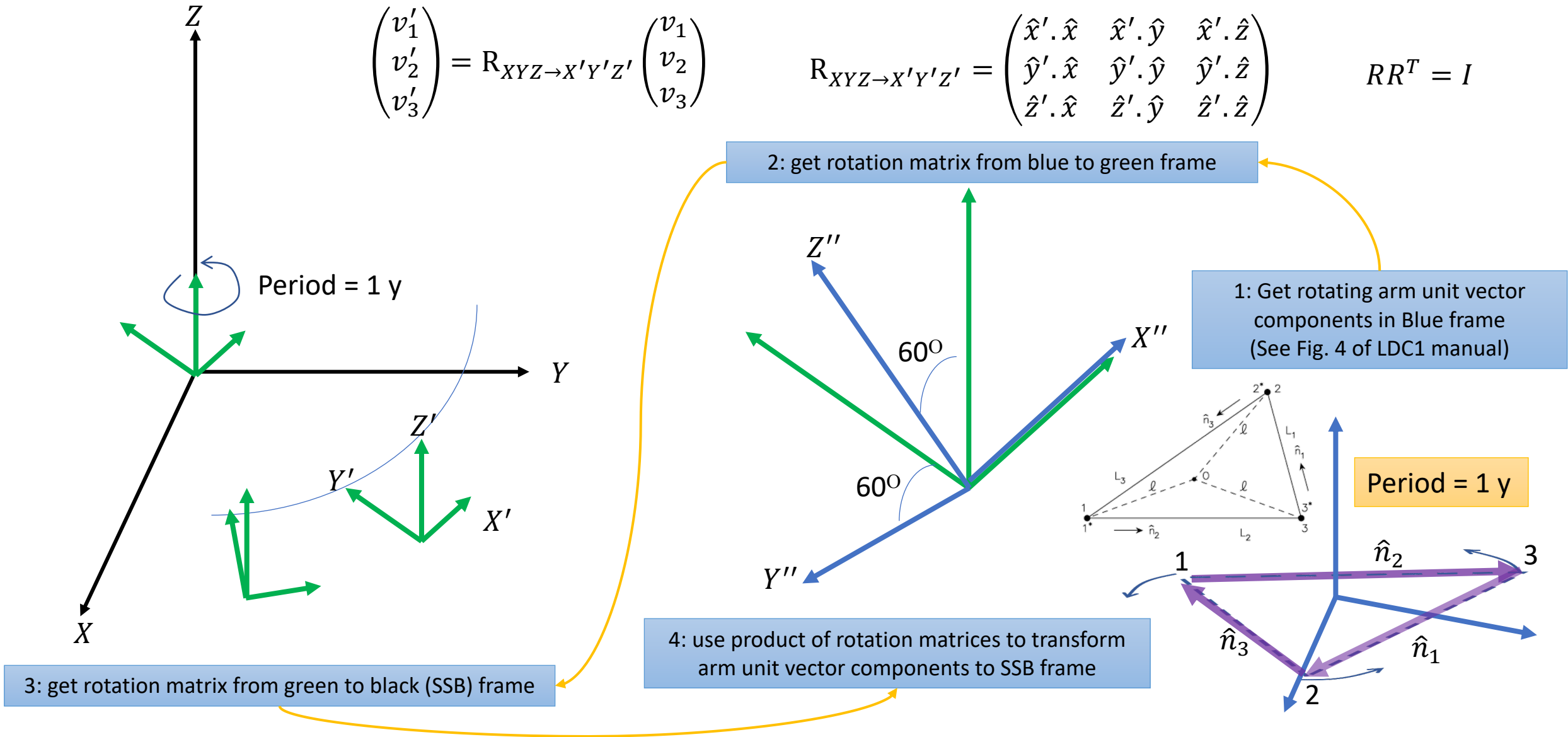


Reference frames and rotations

- ▶ The common reference frame to use here is the **Solar System Barycentric** (SSB) frame
- ▶ The polarization tensors will be computed in the same way as for the detector frame (see previous exercises) but now the frame is the SSB
- ▶ We need to find the components of LISA arm unit vectors in the SSB frame and then obtain the detector tensor from the arm vectors
- ▶ Finally, contract the detector and polarization tensors to get antenna pattern functions



Obtain the arm components in the SSB frame



Antenna patterns

- Use the expressions in Sec II B of the paper arXiv:1207.4956v1 to obtain the detector tensors for the two **Michelson TDI combinations**

the $\vec{G}\vec{W}$ propagation direction. The two detector tensors are defined as $D_I \equiv \frac{1}{2}(\hat{n}_1 \otimes \hat{n}_1 - \hat{n}_2 \otimes \hat{n}_2)$, $D_{II} \equiv \frac{1}{2\sqrt{3}}(\hat{n}_1 \otimes \hat{n}_1 + \hat{n}_2 \otimes \hat{n}_2 - 2\hat{n}_3 \otimes \hat{n}_3)$, where $\hat{n}_1, \hat{n}_2, \hat{n}_3$ denote the unit vectors along each arm of LISA. Here we assume

- Obtain the $F_{+,\times}(t; \hat{n})$ for each TDI combination by contracting the respective detector tensor above with each polarization tensor
- Write a code:
 - Inputs: Source direction, vector of sample times
 - Outputs: F_+ , F_\times time series for a Michelson TDI combination

Toy LISA response: Partial

- ▶ Write a Matlab script to do the following
- ▶ Generate h_+ , h_\times that are sinusoidal
 - ▶ The script should have user-specified sky location and polarization angle for the GW source
- ▶ Generate any one of the Michelson TDI response (no doppler shift included) using the code from the previous exercise to generate the $F_{+,\times}$ time series
$$s(t) = F_+(t; \theta, \phi)h_+(t) + F_\times(t; \theta, \phi)h_\times(t)$$
- ▶ Take FFT of the detector response and compare to the FFT of h_+

Toy LISA response: Full

- ▶ LISA detector response including doppler shift
 - ▶ $h_{+,\times}(t) \rightarrow h_{+,\times}(t - \frac{\hat{n} \cdot \bar{x}_d}{c})$
 - ▶ \hat{n} : Wave propagation direction
 - ▶ $\bar{x}_d(t)$: LISA centroid
- ▶ Write a code to calculate the components of the position vector, \bar{x}_d , of the LISA centroid (simple circular orbit) in the SSB frame
- ▶ Plot any one of the LISA Michelson responses for a monochromatic source in the SSB frame
 - ▶ In SSB frame: $h_+(t) = A \sin(\omega_0 t)$; $h_\times(t) = \left(\frac{A}{2}\right) \cos \omega_0 t$
 - ▶ Generate doppler modulated sinusoids h_+ , h_\times :
$$h_+(t) = A \sin\left(\omega_0 \left(t - \frac{\hat{n} \cdot \bar{x}_d}{c}\right)\right)$$
$$h_\times(t) = B \cos\left(\omega_0 \left(t - \frac{\hat{n} \cdot \bar{x}_d}{c}\right)\right)$$
 - ▶ One parameter is missing here: polarization angle (but we will ignore it)
- ▶ Compare FFT of the response to that of $h_{+,\times}(t)$ in the SSB frame

Effect of sky location

- ▶ Take the same SSB frame h_+, h_\times but at a different sky location and verify that the LISA response is different
- ▶ This allows LISA to have directional sensitivity to long-lived sources.

Advanced

- ▶ Find out about the Tianqin detector configuration, which is a geocentric one
 - ▶ J. Luo et al., “TianQin: a space-borne gravitational wave detector,” *Class. Quant. Grav.*, vol. 33, no. 3, p. 035010, 2016.
- ▶ Write code using tensors to compute the response of Tianqin in the long wavelength approximation
- ▶ (Taiji configuration is essentially the same as LISA and, hence, does not lead to a very different result)