Optimal Piecewise-based Mechanism for Collecting Bounded Numerical Data under Local Differential Privacy

Authors: Ye Zheng, Sumita Mishra, Yidan Hu

Rochester Institute of Technology

LDP Mechanisms

- Randomized algorithm $\mathcal{M}: \mathcal{D} \rightarrow \widetilde{\mathcal{D}}$

  - provide quantifiable privacy for data $x \in \mathcal{D}$

$$\forall x_1, x_2 \in \mathcal{D}, \forall y \in \widetilde{\mathcal{D}} \qquad \max \frac{\Pr[\mathcal{M}(x_1) = y]}{\Pr[\mathcal{M}(x_2) = y]} \leq e^{\varepsilon}$$

Distinguishability of x_1 and x_2 (sensitive data)
from y (randomized data)

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Privacy

$x_1 \quad \to \quad \mathcal{M} \quad \to \quad y$

Provable defense against
data inference attack

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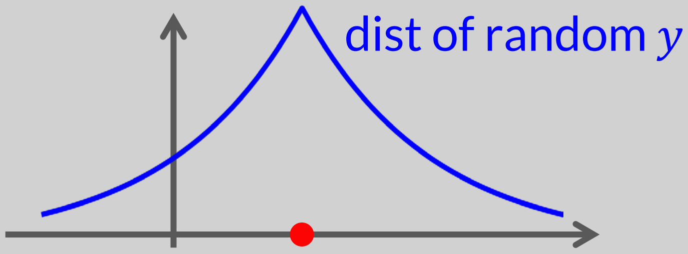
Data utility



Sensitive $x_1 = 0.5$

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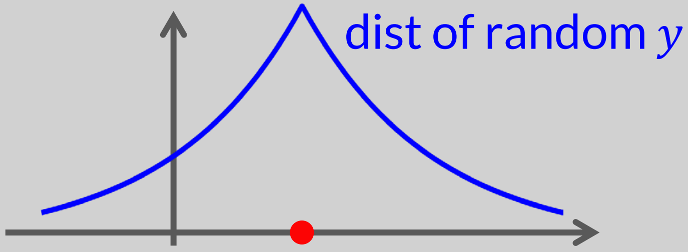
Privacy quantified by ε

$x_1 \quad \rightarrow \quad \mathcal{M} \quad \rightarrow \quad y$

Provable defense against data inference attack

Data utility by expected errors



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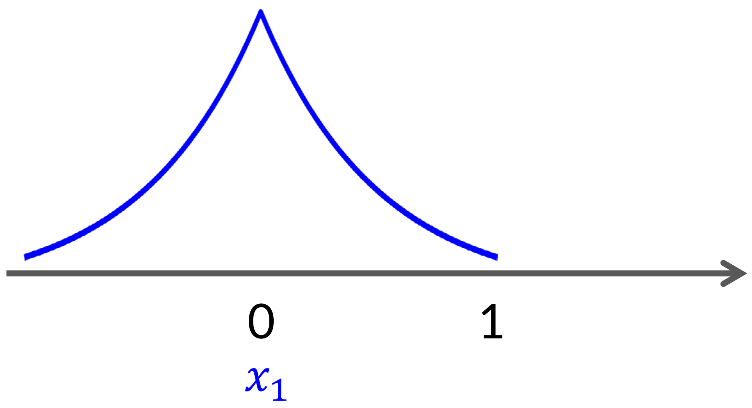
LDP Mechanisms for $\mathcal{D} = [0,1]$

Laplace mechanism: $[0,1] \to (-\infty, +\infty)$



LDP Mechanisms for $\mathcal{D} = [0,1]$

Symmetric (unbiased)
Useful for mean estimation
i.e. $(x_1 + x_2 + \cdots)/n$

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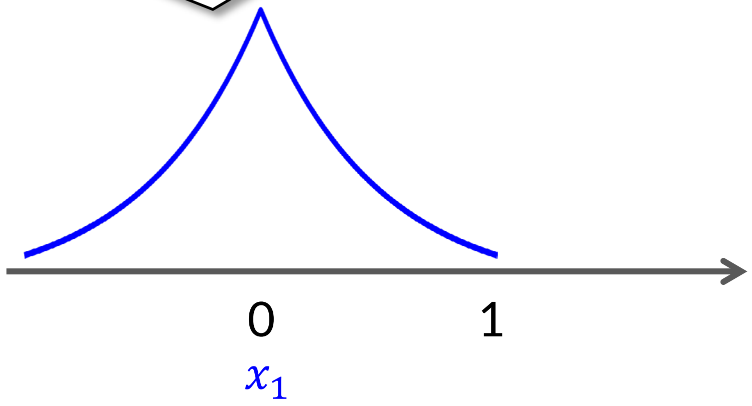

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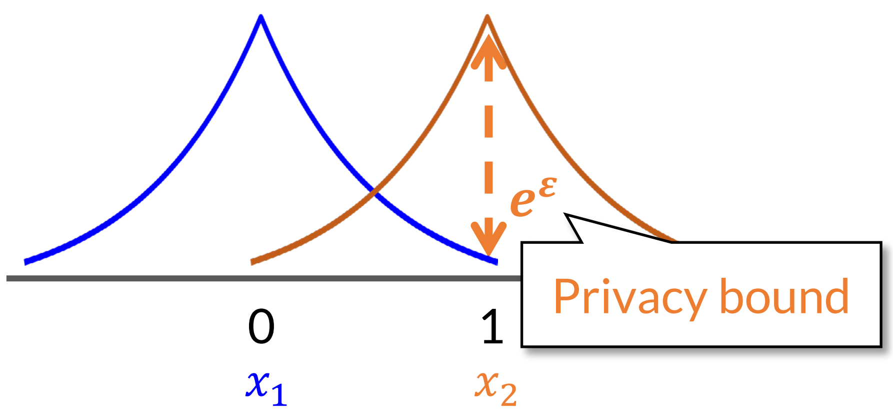

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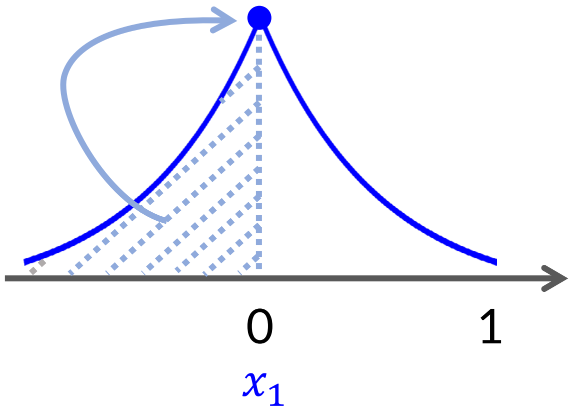

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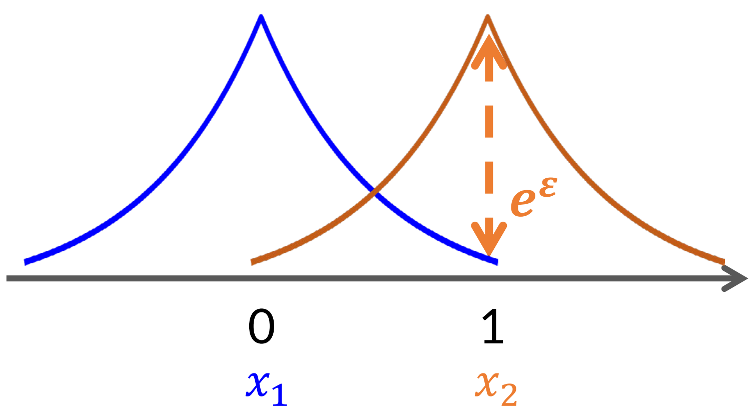

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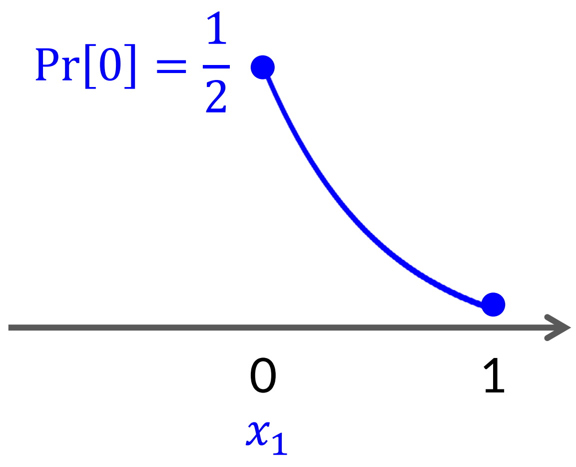

LDP Mechanisms for $\mathcal{D} = [0,1]$

Smaller output
Useful for distribution estimation
i.e. $\{x_1, x_2, \cdots\}$

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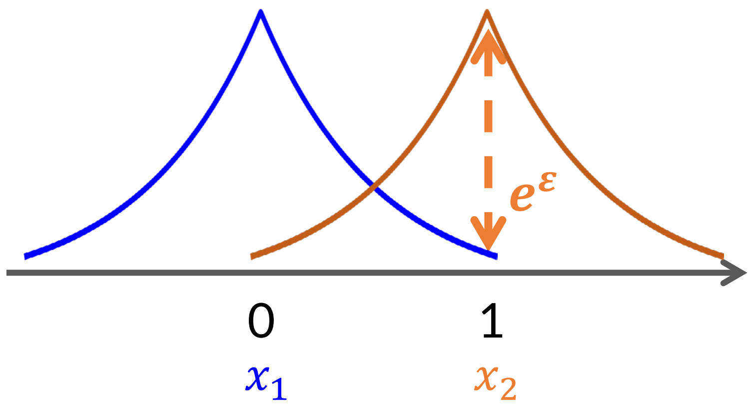

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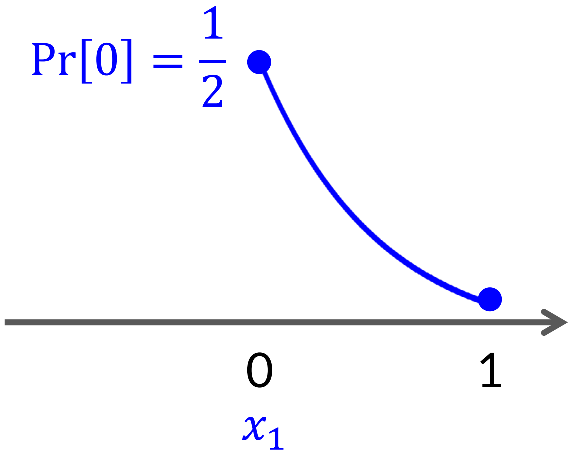

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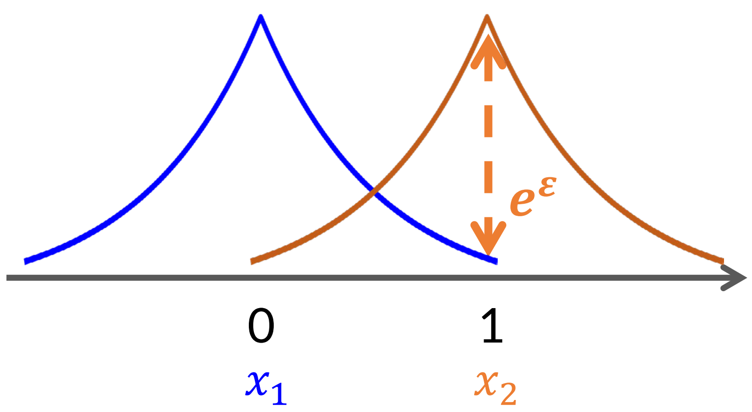

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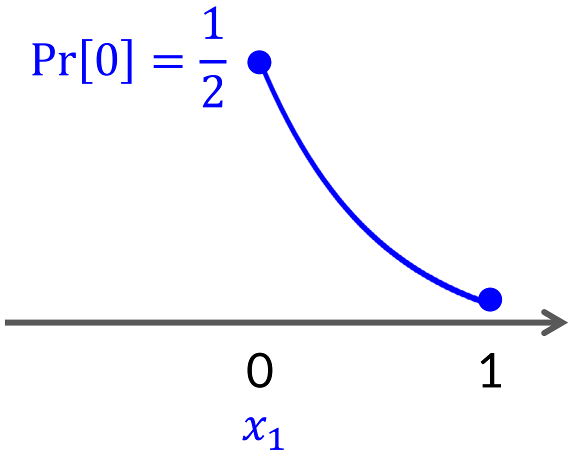

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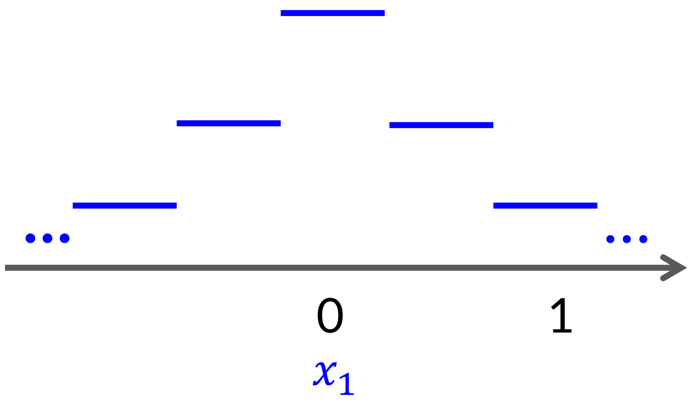
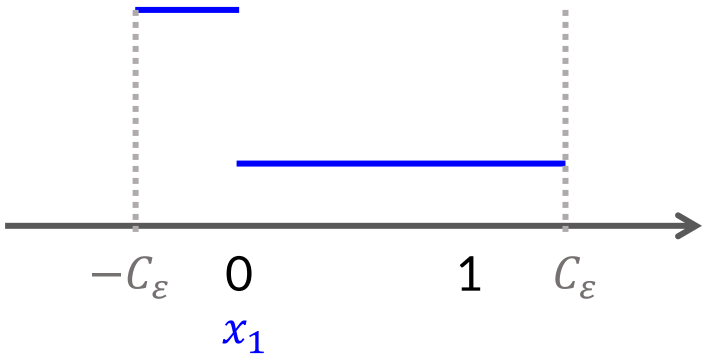


Piecewise mechanism: $[0,1] \to [-C_{\varepsilon}, C_{\varepsilon}]$



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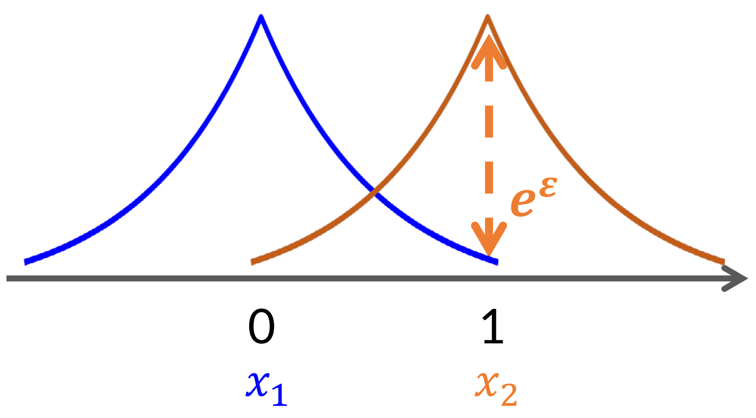

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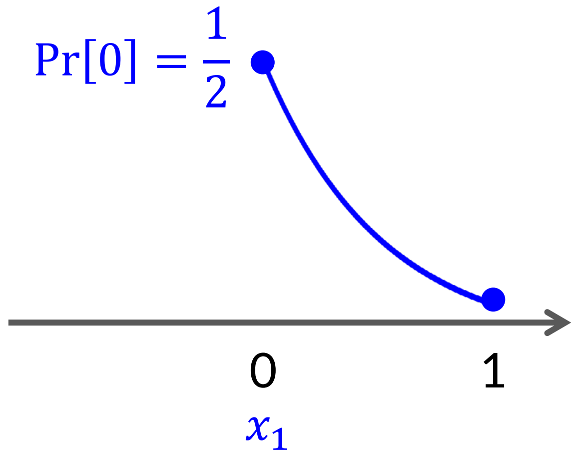

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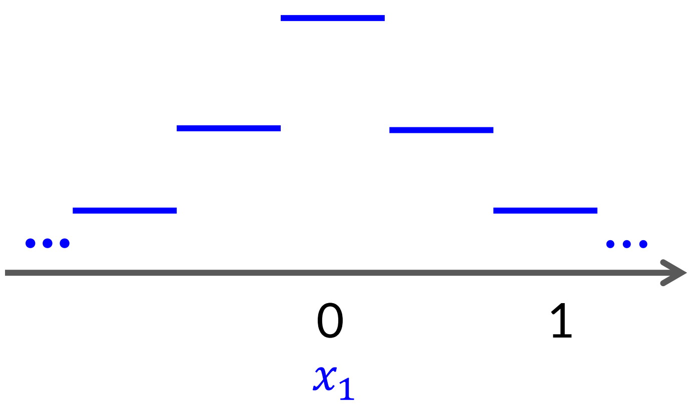

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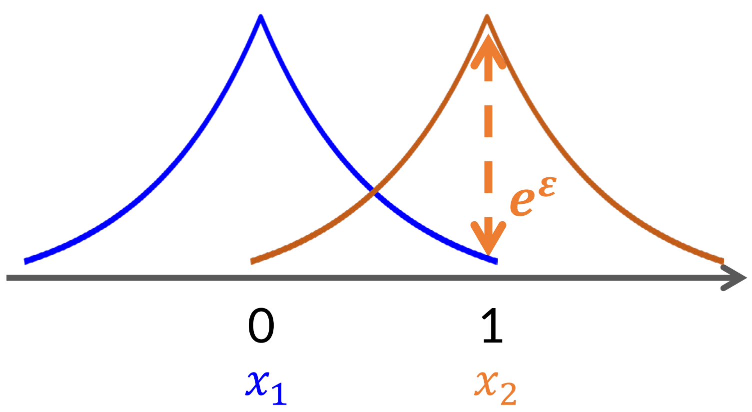

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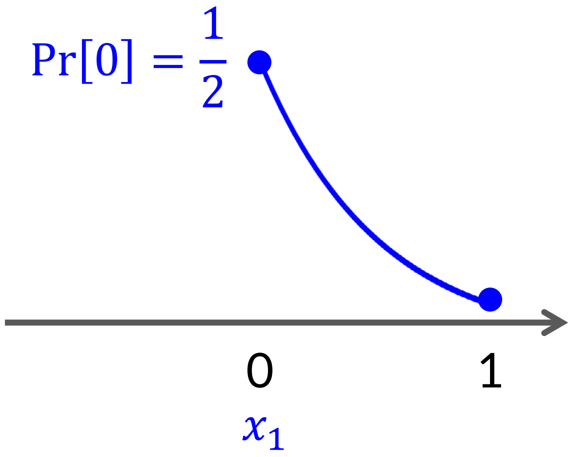

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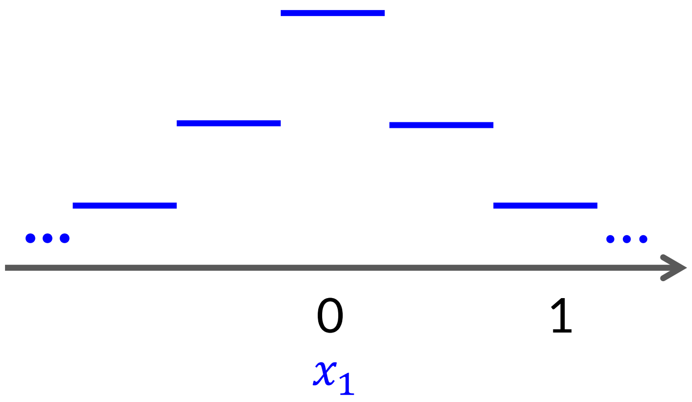

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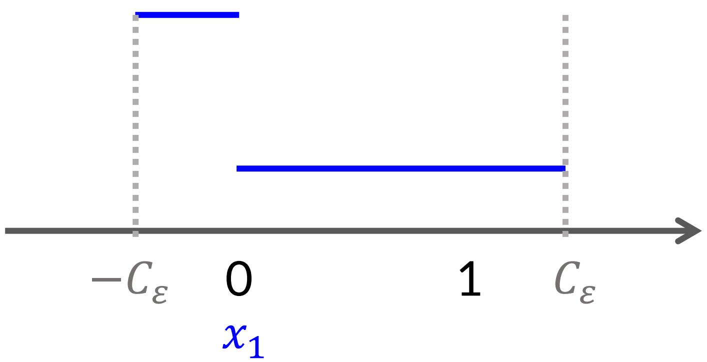

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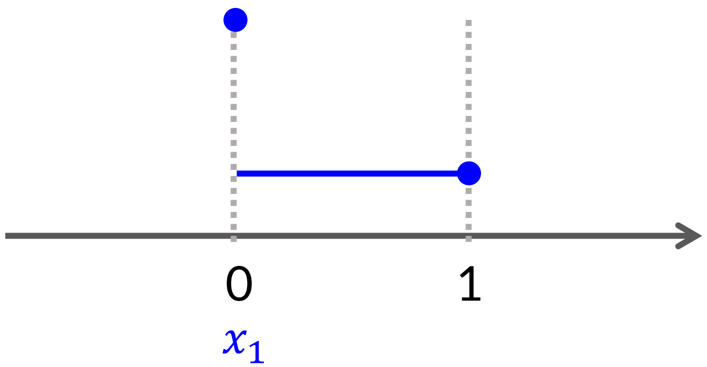

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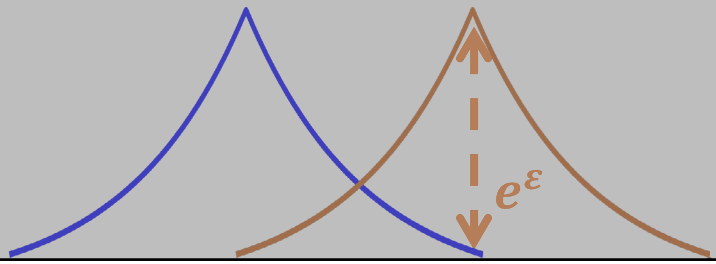

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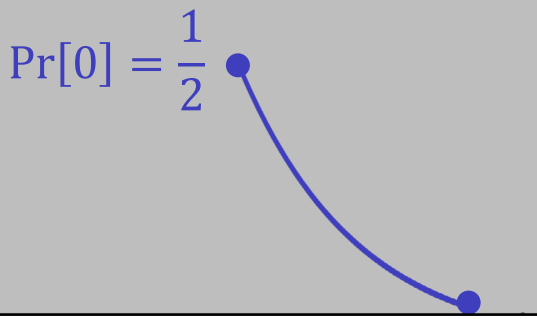

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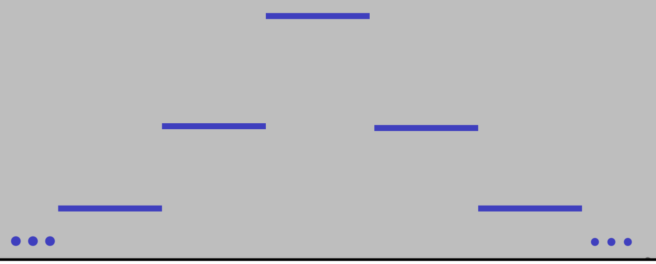
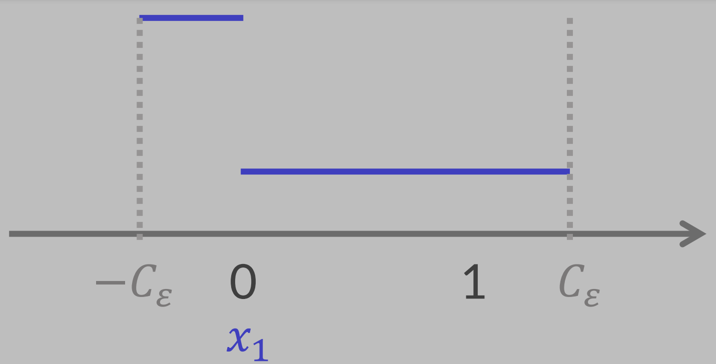
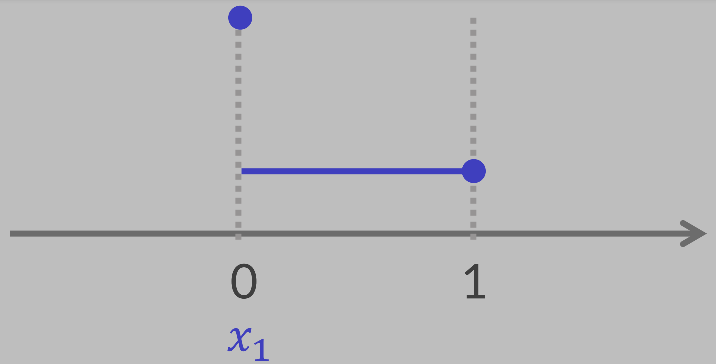


Privacy: LDP with the **same ε**

Utility: **Different errors**





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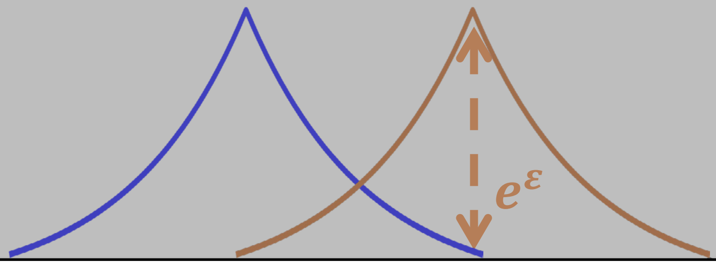

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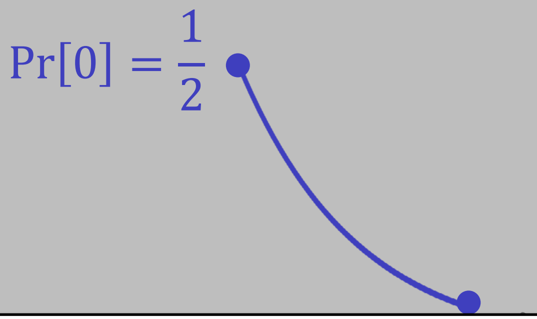

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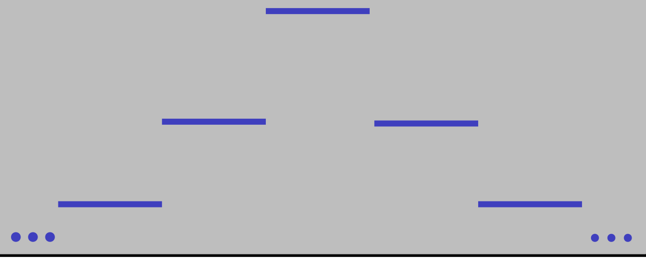

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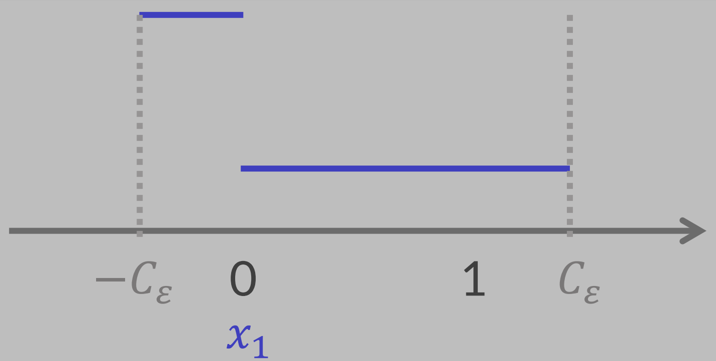
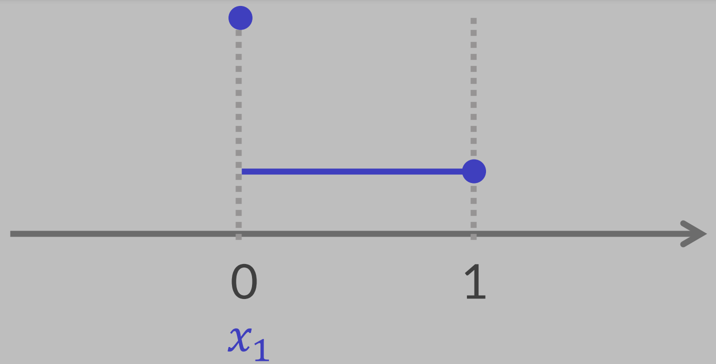
Q: What is the **optimal** LDP mechanism?





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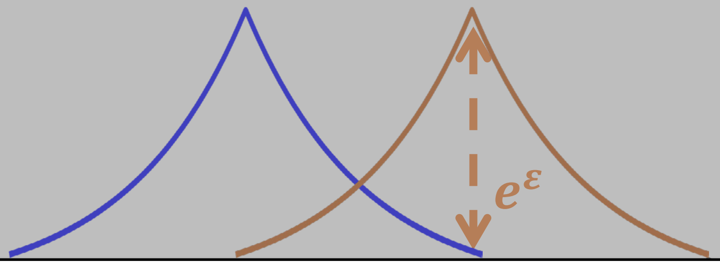

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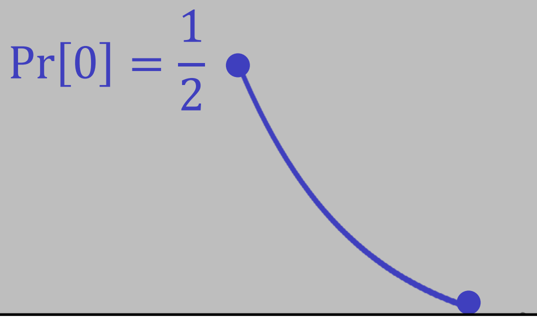

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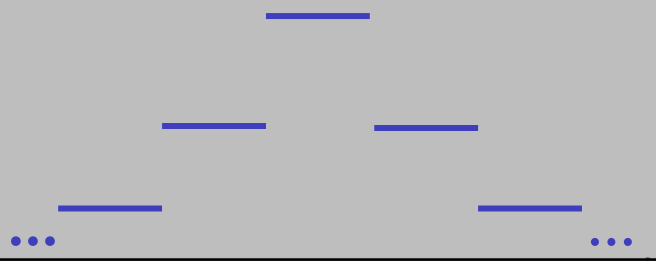

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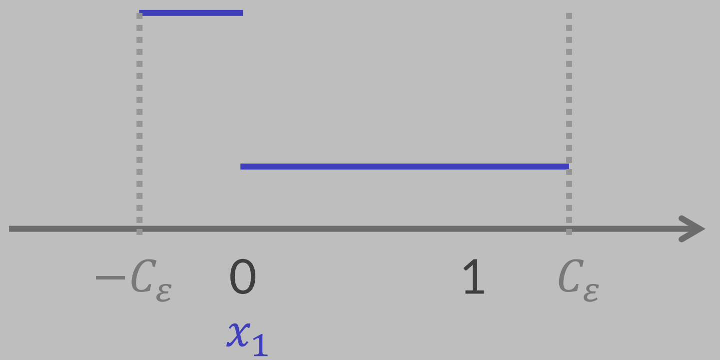
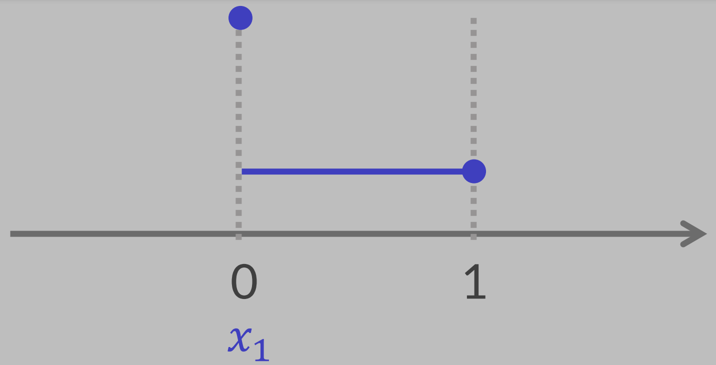
Q: What is the **optimal piecewise-based** mechanism?





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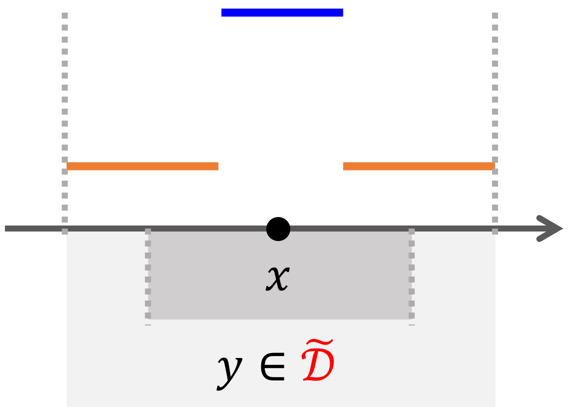
3-Piecewise Mechanism

- 3-piecewise distributions on bounded numerical domain $\mathcal{D} \rightarrow \widetilde{\mathcal{D}}$

  - given input x, samples output y from a distribution



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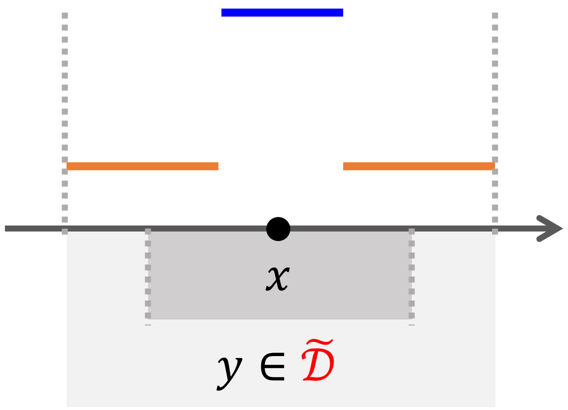

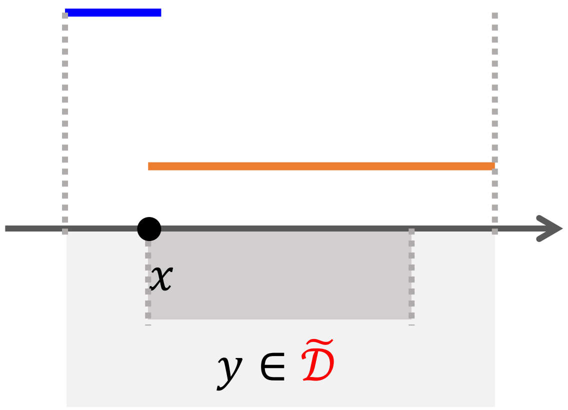

Sampling probability depends on ε

Sampling pieces depends on x and ε

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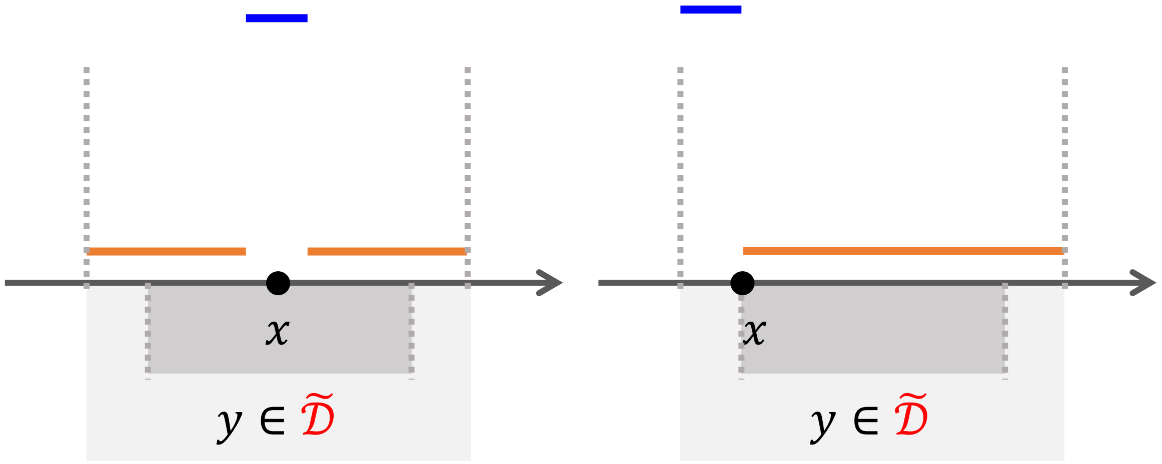

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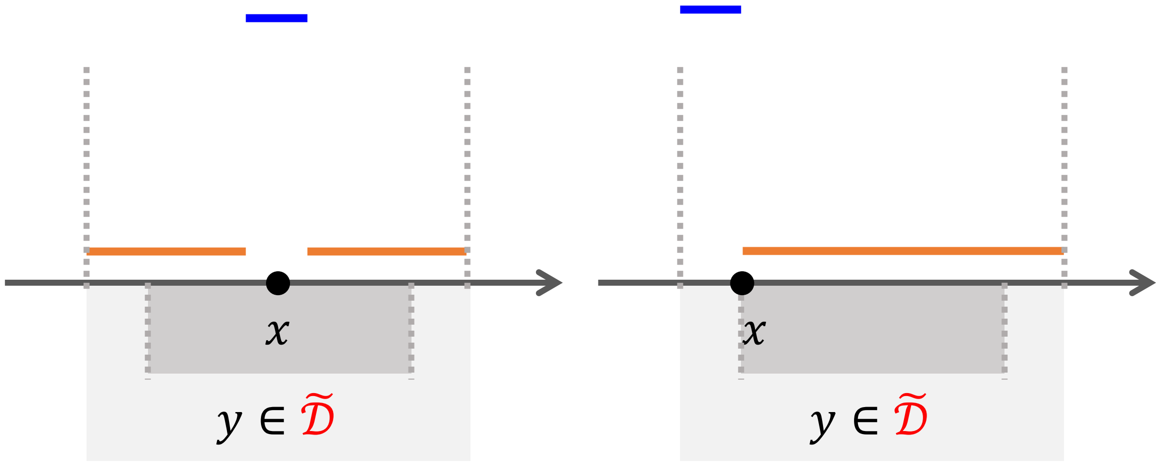

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  - different errors, but **without optimality**

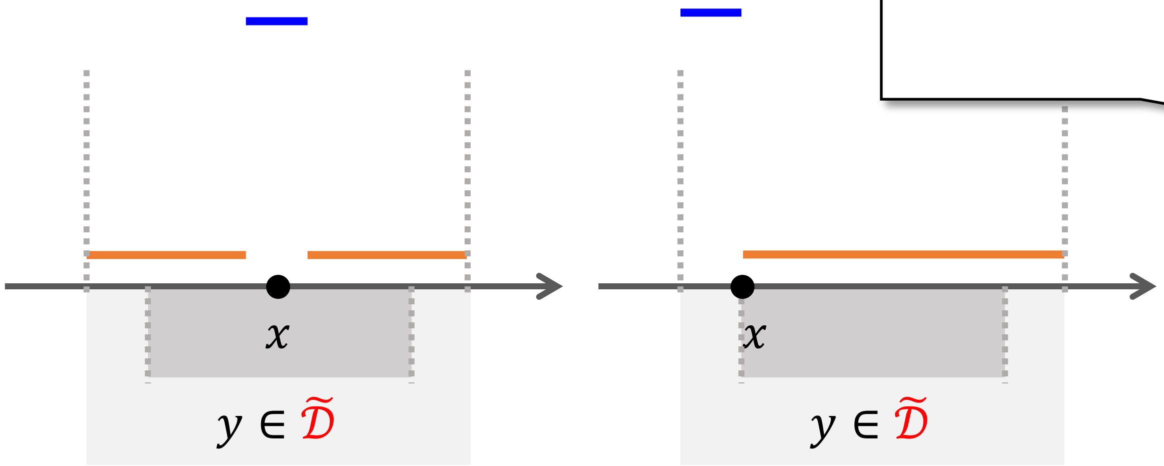
3-Piecewise Mechanism

NOT enough to study optimality of piecewise-based mechanism

- only 3 pieces, two probabilities

- 3-piecewise distributions on boun

  - given input x, samples output y fr



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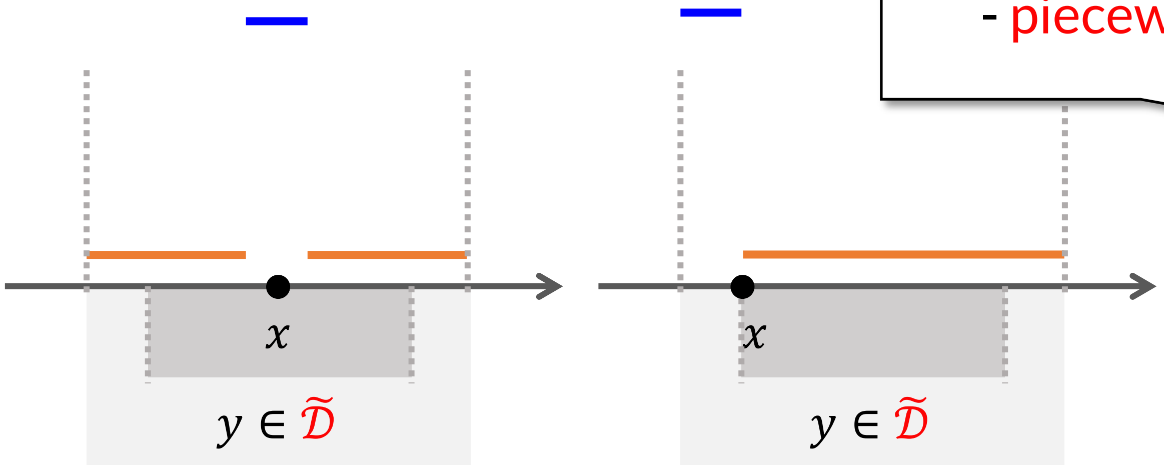

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NOT enough to study optimality of piecewise-based mechanism

- only 3 pieces, two probabilities

- piecewise distribution can has more pieces and probabilities

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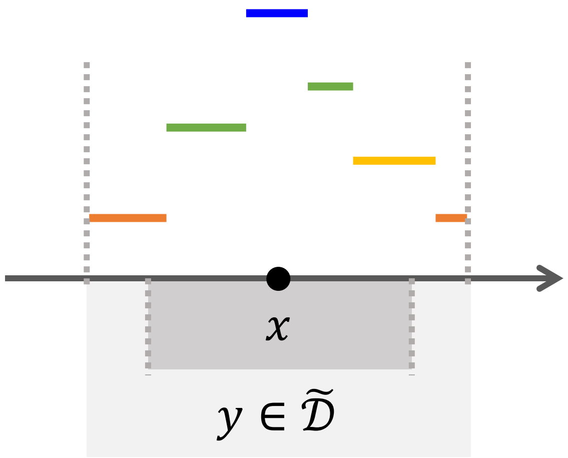
  - different errors, but **without optimality**

Generalized Piecewise-based Mechanism

- **Most generalized version:** m-piecewise distributions



$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_{1,\varepsilon} & \text{if } y \in [l_{1,x,\varepsilon}, r_{1,x,\varepsilon}] \\ p_{2,\varepsilon} & \text{if } y \in [l_{2,x,\varepsilon}, r_{2,x,\varepsilon}] \\ & \dots \\ p_{m,\varepsilon} & \text{if } y \in [l_{m,x,\varepsilon}, r_{m,x,\varepsilon}] \end{cases}$$

$$\max \frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon}$$ (LDP constraint)

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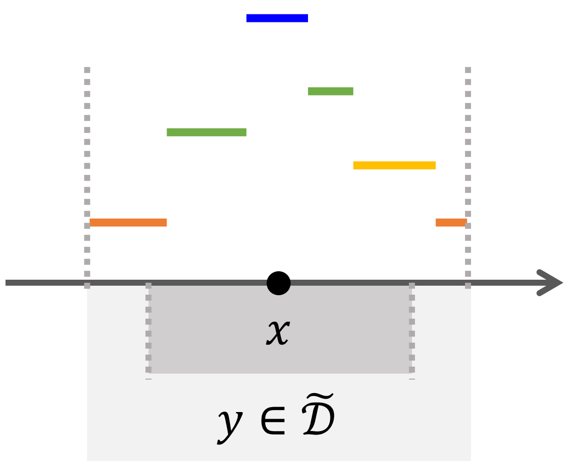

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$$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon}$$ (LDP constraint)

- Error (data utility): $\mathcal{L}(y, x)$

$\mathcal{L}(y, x) \coloneqq |y - x|^p$

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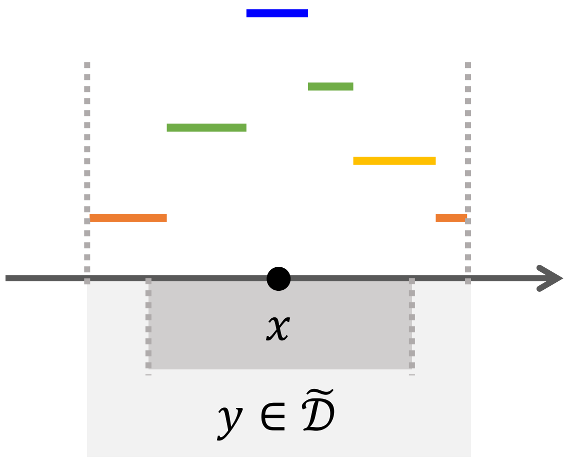


$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_{1,\varepsilon} & \text{if } y \in [l_{1,x,\varepsilon}, r_{1,x,\varepsilon}] \\ p_{2,\varepsilon} & \text{if } y \in [l_{2,x,\varepsilon}, r_{2,x,\varepsilon}] \\ & \dots \\ p_{m,\varepsilon} & \text{if } y \in [l_{m,x,\varepsilon}, r_{m,x,\varepsilon}] \end{cases}$$

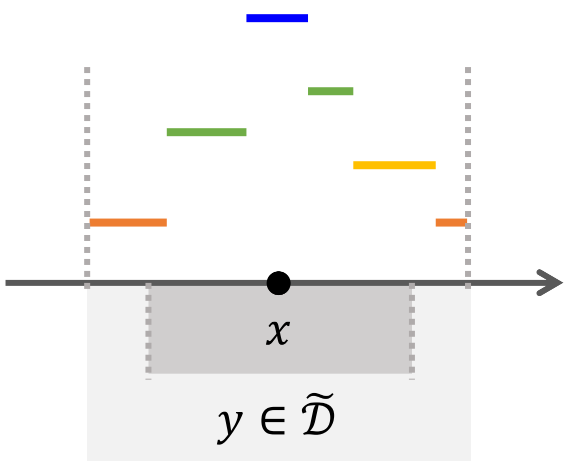
$$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon} \text{ (LDP constraint)}$$

- Expected error:

$$\int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

Generalized Piecewise-based Mechanism

- Most generalized version: m-piecewise distributions



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$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon}$ (LDP constraint)

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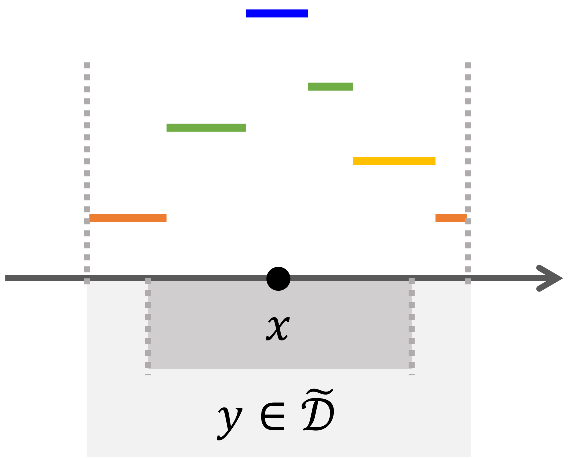
Find $\mathcal{M}$ to minimize error at x

Optimal Piecewise-based Mechanism

- Most generalized version: m-piecewise distributions



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Optimal Piecewise-based Mechanism

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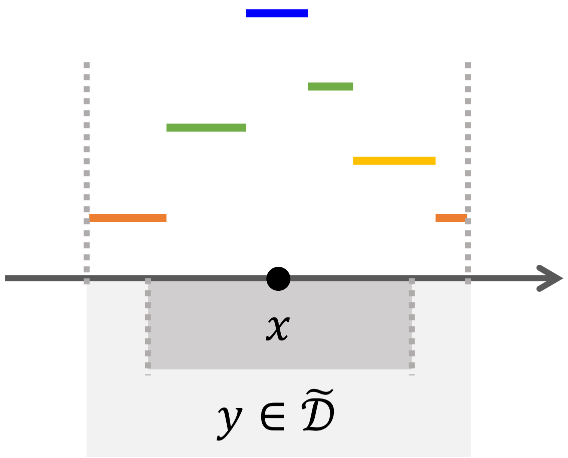

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Find $\mathcal{M}$ to minimize worst-case error

Solved $\mathcal{M}$ is
the optimal piecewise-based mechanism

Mathematically: To find the **optimal piecewise distribution under the LDP constraint**

Challenges & Proofs

- Challenges
  1. min-max problem & multiple variables
  2. optimal results only for a specific m

$$\min_{\mathcal{M}:\, p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

2. $i \in [m]$

1. variables p_i, l_i, r_i

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1

Inner max
has a closed form

Worst-case from
$x =$ endpoints of $\mathcal{D}$

Reduced to:
minimization problem

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Worst-case from $x =$ endpoints of $\mathcal{D}$

Reduced to: minimization problem

2 The optimal #piece is m

If optimal m-piecewise = $(m+1)$-piecewise

Reduced to: Check $m = 1,2,\ldots$ until optimal

Solvable

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y = \max_{x \in \mathcal{D}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x) \, \mathrm{d}y$$

(m-piecewise distribution)

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convex function w.r.t x

Proof Intuitions

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$$= \max_{x\in\{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y,x)\,\mathrm{d}y \qquad (\text{maximum principle})$$

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x)] = \sum^{m} \ldots c^{r_i}$$

After merging redundant pieces

- Optimal #piece is m **if** optimal m-piecewise = (m+1)-piecewise

Error from an arbitrary piece
(≥ 0 variable)

if: $\min_{e_1, e_2, e_3} e_1 + e_2 + e_3 = \min_{e_1, e_2, e_3, e_4} e_1 + e_2 + e_3 + e_4$

i.e. the error can't be lowered by arbitrary ≥ 0 variable

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$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x)] = \sum^{m} \ldots c^{r_i}$$

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i.e. the error can't be lowered by arbitrary ≥ 0 variable

then: $= \min_{e_1, e_2, e_3, e_4, e_5} e_1 + e_2 + e_3 + e_4 + e_5$

otherwise, $e_4 \leftarrow e_4 + e_5$ can further lower the error

NOT Manually "Solvable" When $m \geq 4$

- Too many variables & non-linear

$$\max_{x \in \{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x)\, \mathrm{d}y$$

NOT Manually "Solvable" When $m \geq 4$

- Too many variables & non-linear
- Efficiently solved by off-the-shelf solvers, e.g. Gurobi

  - limitation: needs given ε

  - limitation: cannot provide closed-form $\mathcal{M}$: p_i, l_i, r_i

  - can be used to analyze optimality

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Hypothesis. For any domain $\mathcal{D} \to \mathcal{D}$, under error metrics $\mathcal{L}(y, x) \coloneqq |y - x|$ and $\mathcal{L}(y, x) \coloneqq (y - x)^2$, the optimal piecewise-based mechanism falls into 3-piecewise mechanism.

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closed-form $\mathcal{M}$ can be solved
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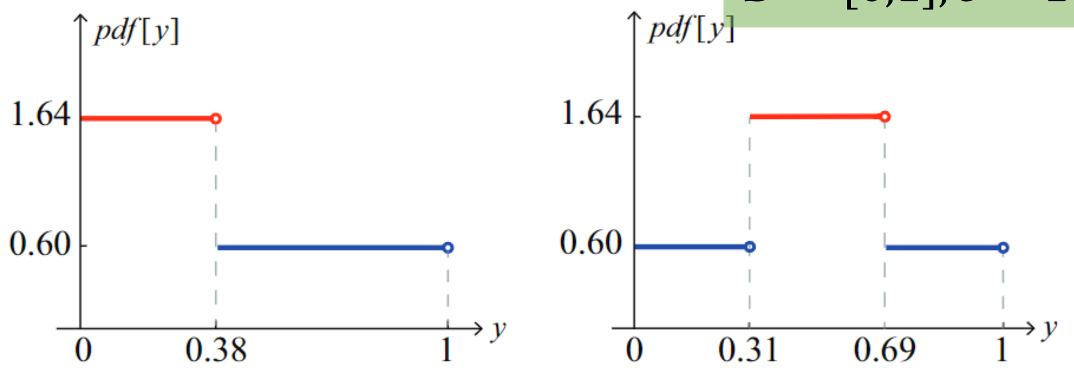
- Efficiently solved by off-the-shelf solvers, e.g. Gurobi

  \- limitation: needs given ε

  \- limitation: cannot provide closed-form $\mathcal{M}$: p_i, l_i, r_i

$\mathcal{D} = [0,1], \varepsilon = 1$



(a) $x = 0$. (b) $x = 0.5$.

er error metrics $\mathcal{L}(y,x) := |y-x|$ and $\mathcal{L}(y,x) :=$
echanism falls into 3-piecewise mechanism.

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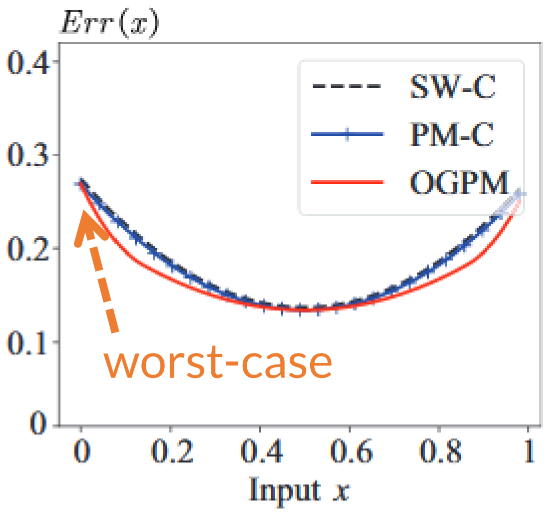
(different from existing instantiations)

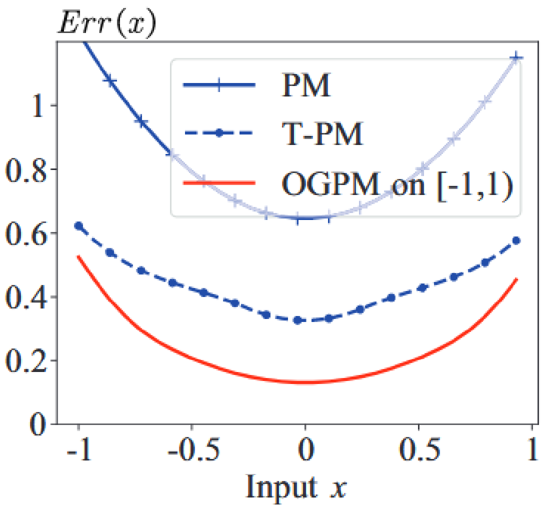
Error Comparisons

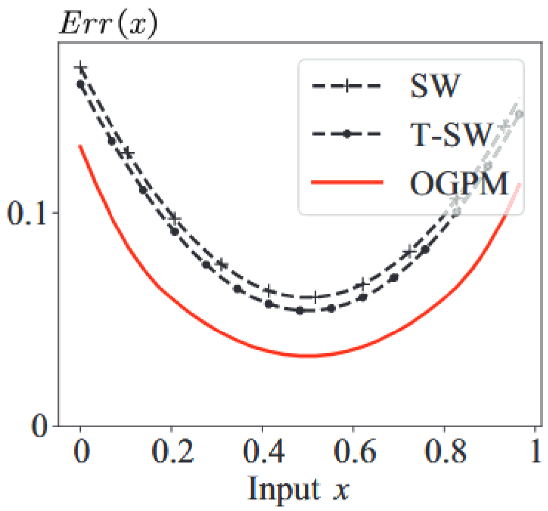
Whole-domain error ($\varepsilon = 2$)

Compressed PM, SW



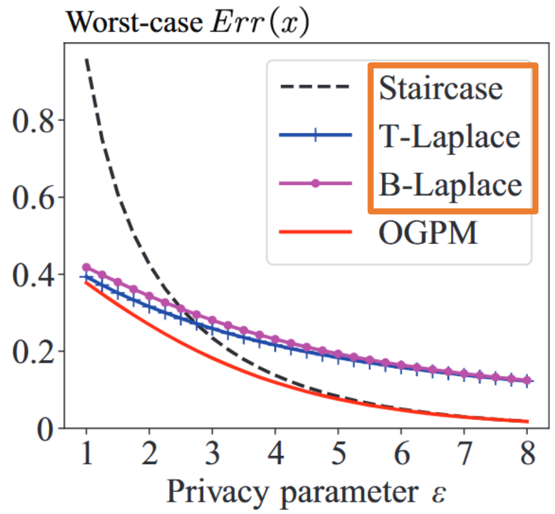
Original and **truncated** PM, SW





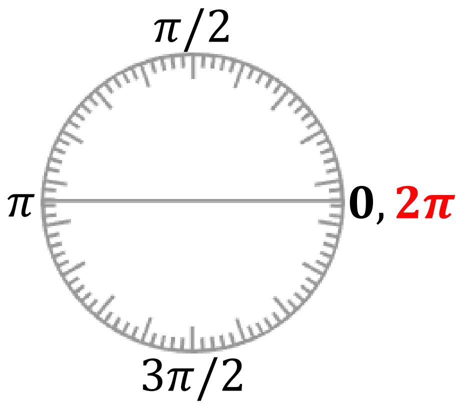
Worst-case error

Non-piecewise-based



Circular Domain

- Different distance, e.g. distance $(0, 2\pi) = 0$

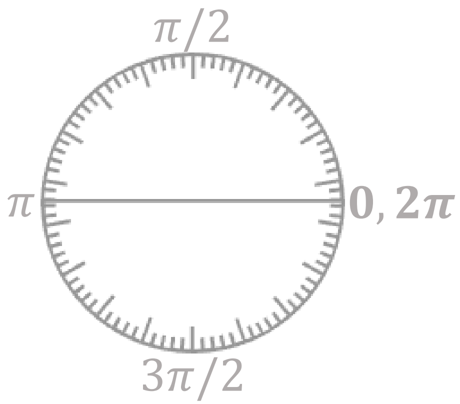


$\mathcal{L} \to \mathcal{L}_{\text{mod}}$

$$\mathcal{L}_{\text{mod}}(y, x) \coloneqq \min(\mathcal{L}(y, x), \mathcal{L}(y, 2\pi - x))$$

Circular Domain

- Different distance, e.g. distance $(0, 2\pi) = 0$



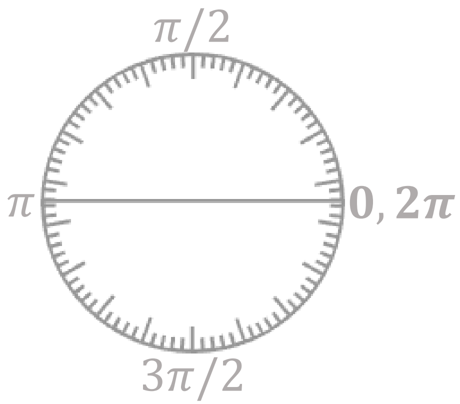
$\mathcal{L} \rightarrow \mathcal{L}_{\mathrm{mod}}$

$\mathcal{L}_{\mathrm{mod}}(y, x) \coloneqq \min(\mathcal{L}(y, x), \mathcal{L}(y, 2\pi - x))$

$$\min_{\mathcal{M}:\, p_i, l_i, r_i} \max_{x \in [0, 2\pi]} \int_{\widetilde{\mathcal{D}}} \mathcal{L}_{\mathrm{mod}}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

Circular Domain

- Different distance, e.g. distance $(0, 2\pi) = 0$



$\mathcal{L} \rightarrow \mathcal{L}_{\mathrm{mod}}$

$\mathcal{L}_{\mathrm{mod}}(y, x) \coloneqq \min(\mathcal{L}(y, x), \mathcal{L}(y, 2\pi - x))$

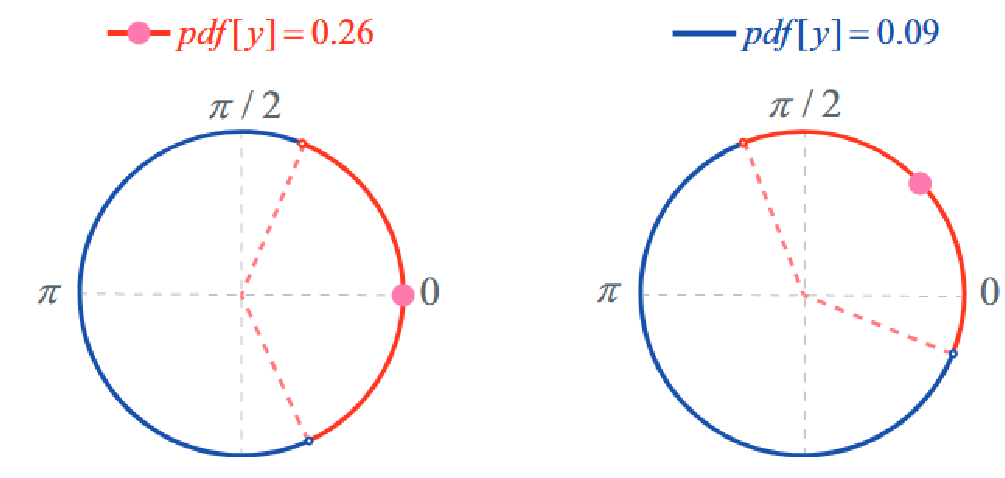
$$\Rightarrow \min_{\mathcal{M}:\, p_i, l_i, r_i} \max_{x \in [0, 2\pi]} \int_{\widetilde{\mathcal{D}}} \mathcal{L}_{\mathrm{mod}}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

- Link to problems in the classical domain

min-max under $\mathcal{L}_{\mathrm{mod}}$	=	min under $\mathcal{L}$ at π		min under $\mathcal{L}_{\mathrm{mod}}$ at x	=	min under $\mathcal{L}$ at π + Transform
	p_i				$l_{i,x}^{\mathrm{mod}}, r_{i,x}^{\mathrm{mod}}$	

Circular Domain

- Different distance, e.g. distance $(0, 2\pi) = 0$



(a) $x = 0$. **(b)** $x = \pi/4$.

$\mathcal{L} \rightarrow \mathcal{L}_{\text{mod}}$

$\mathcal{L}_{\text{mod}}(y, x) \coloneqq \min(\mathcal{L}(y, x), \mathcal{L}(y, 2\pi - x))$

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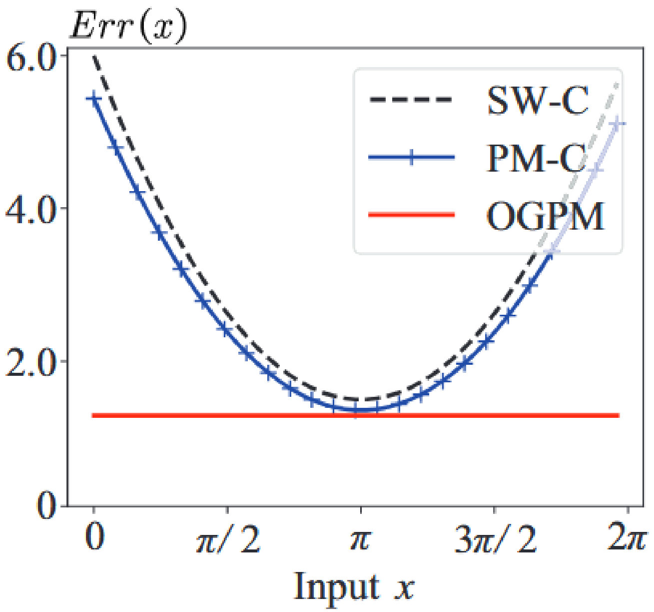
p_i

$l_{i,x}^{\text{mod}}, r_{i,x}^{\text{mod}}$

Error Comparison

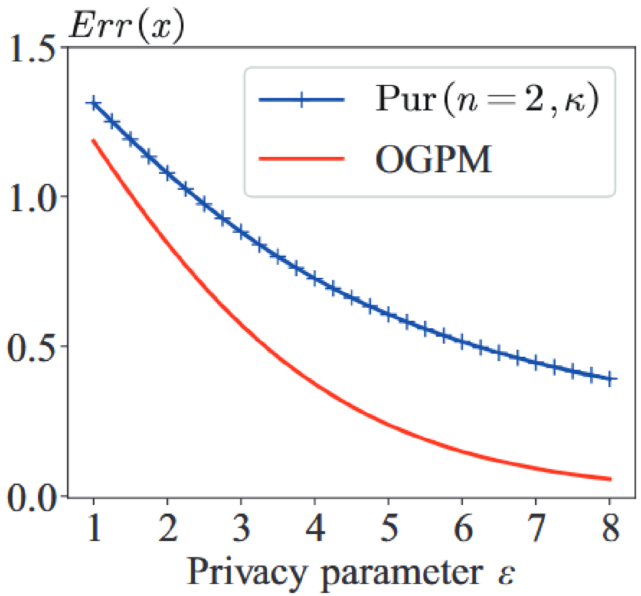
Whole-domain error ($\varepsilon = 2$)

Flatten PM, SW



Worst-case error

Purkayastha mechanism*



* Differential Privacy for Directional Data, [CCS'21]

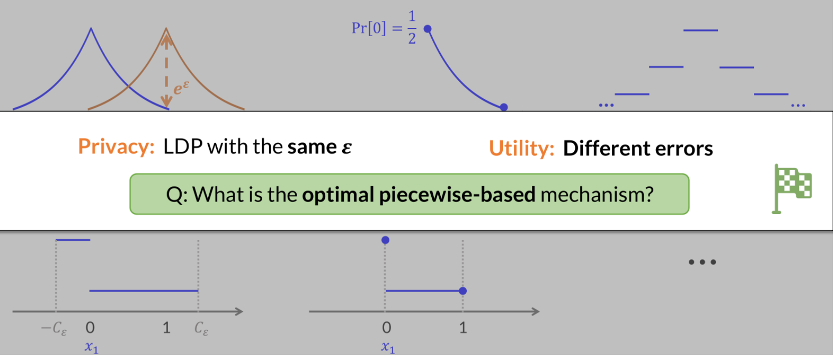
Optimal Piecewise-based Mechanism for Collecting Bounded Numerical Data under LDP

LDP Mechanisms for $\mathcal{D} = [0,1]$

Laplace mechanism: $[0,1] \to (-\infty, +\infty)$

Laplace + truncation: $[0,1] \to [0,1]$

Staircase: $[0,1] \to (-\infty, +\infty)$



Privacy: LDP with the same ε **Utility:** Different errors

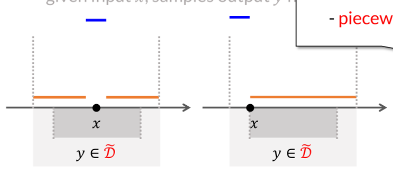
Q: What is the **optimal piecewise-based** mechanism?

3-Piecewise Mechanism

- 3-piecewise distributions on boun
  - given input x, samples output y f

NOT enough to study optimality of piecewise-based mechanism

- only 3 pieces, two probabilities
- piecewise distribution can has more pieces and probabilities

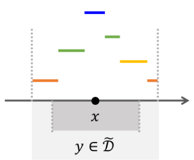


$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_\varepsilon & \text{if } y \in [l_{x,\varepsilon}, r_{x,\varepsilon}] \\ \frac{p_\varepsilon}{\exp(\varepsilon)} & \text{if } y \in \tilde{\mathcal{D}} \setminus [l_{x,\varepsilon}, r_{x,\varepsilon}] \end{cases}$$

- Instantiations: PM [2019], SW [2020], PTT [2024] (design different $p_\varepsilon, l_{x,\varepsilon}, r_{x,\varepsilon}$)
  - different errors, but without optimality

Optimal Piecewise-based Mechanism

- Most generalized version: m-piecewise distributions



$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_{1,\varepsilon} & \text{if } y \in [l_{1,x,\varepsilon}, r_{1,x,\varepsilon}] \\ p_{2,\varepsilon} & \text{if } y \in [l_{2,x,\varepsilon}, r_{2,x,\varepsilon}] \\ & \dots \\ p_{m,\varepsilon} & \text{if } y \in [l_{m,x,\varepsilon}, r_{m,x,\varepsilon}] \end{cases}$$

$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon}$ (LDP constraint)

Solved $\mathcal{M}$ is the optimal piecewise-based mechanism

Mathematically: To find the optimal piecewise distribution under the LDP constraint

$$\min_{\mathcal{M}: p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\tilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] dy$$

Find $\mathcal{M}$ to minimize worst-case error

Challenges & Proofs

- Challenges
  1. min-max problem & multiple variables
  2. optimal results only for a specific m

$$\min_{\mathcal{M}: p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\tilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] dy$$

2. $i \in [m]$ 1. variables p_i, l_i, r_i

1. Inner max has a closed form. Worst-case from x = endpoints of $\mathcal{D}$. Reduced to: minimization problem

2. The optimal #piece is m. If optimal m-piecewise = $(m+1)$-piecewise. Reduced to: Check $m = 1, 2, \dots$ until optimal

Solvable

NOT Manually "Solvable" When $m \geq 4$

- Too many variables & non-linear
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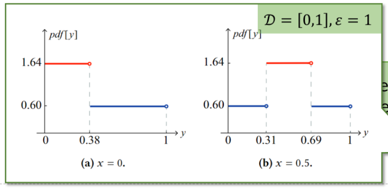
  - limitation: needs given ε
  - limitation: cannot provide closed-form $\mathcal{M}: p_i, l_i, r_i$

$$\max_{x \in \{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x)\, dy$$



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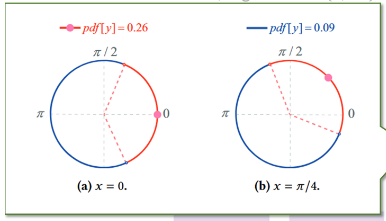

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p_i			$l_{i,x}^{\text{mod}}, r_{i,x}^{\text{mod}}$		

Thank you!

Q & A

Optimality of LDP Mechanisms

- Optimal error (utility) under privacy level ε

  - many mechanisms are optimal in **order-of-magnitude**, e.g. $\Omega(\frac{1}{\sqrt{n}})$ for counting query*

  - the staircase mechanism is optimal for **domain** $[0,1] \rightarrow (-\infty, +\infty)^{\dagger}$

  - the geometric mechanism is universally optimal if any **post-processing** is allowed, e.g. truncation††

* The Complexity of Differential Privacy, 2017

† The Staircase Mechanism in Differential Privacy, 2016

†† Universally Utility-maximizing Privacy Mechanisms, 2009

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  - the geometric mechanism is universally optimal if any **post-processing** is allowed, e.g. truncation††

- **Specify the utility model** (conditions for optimality)

1 Error metric

$Err(\text{truth}, \text{rand})$

Err or $\boldsymbol{\Omega}(Err)$

2 Data domain & type of mechanisms

Discrete / cont. $\mathcal{D} \rightarrow \widetilde{\mathcal{D}}$

Laplace-shape / piecewise

3 Post-processing