

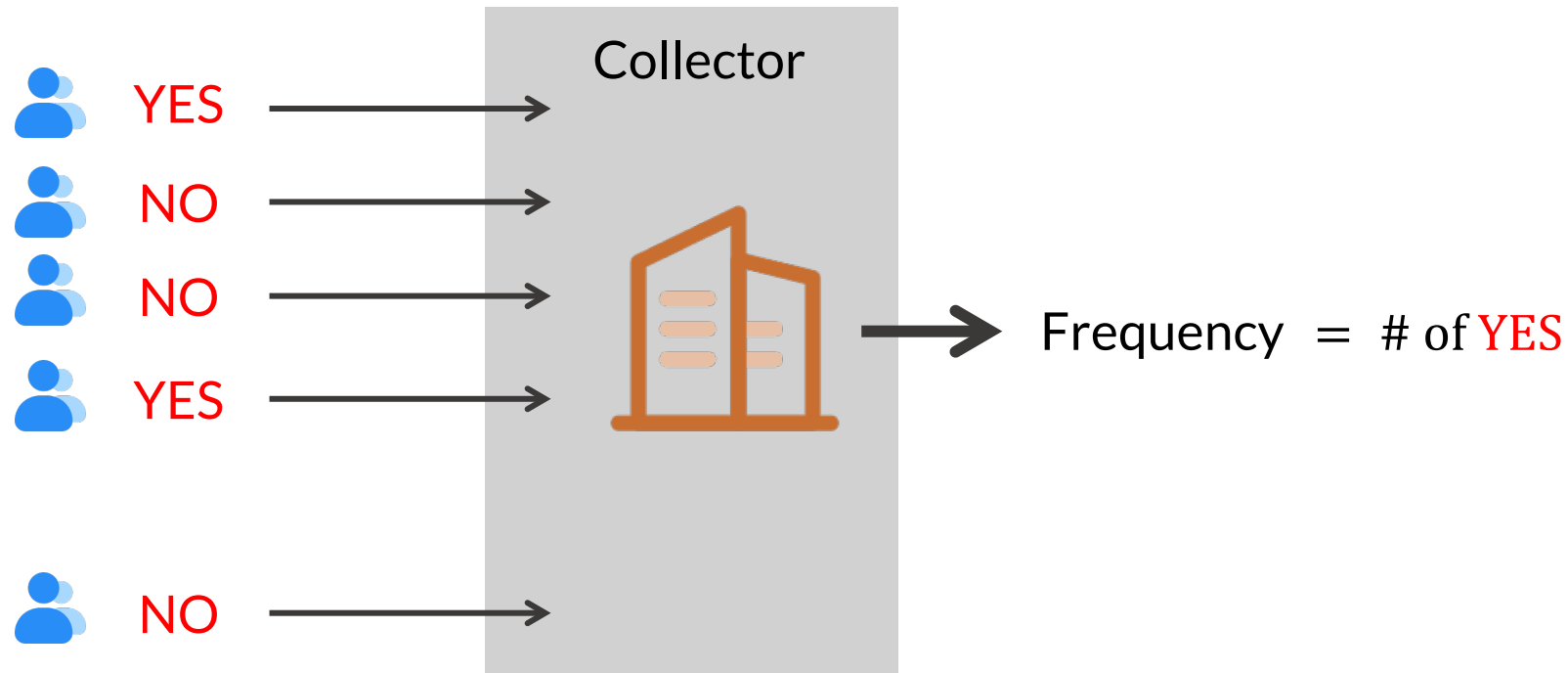
Locally Differentially Private Frequency Estimation via Joint Randomized Response

Authors: [Ye Zheng](#), Shafizur Rahman Seeam, Yidan Hu, [Rui Zhang](#), [Yanchao Zhang](#)



Frequency Estimation

- Social science: How many people engage in tax evasion?
 - ask one person if they had evaded tax
 - the person answers YES or NO



Randomized Response for Privacy

- People have **privacy concerns** on sensitive/embarrassing question
- i.e. don't want to let the collector know
- A **privacy mechanism** \mathcal{M} satisfies LDP if

For any truth x_1, x_2 ,
and randomized answer y :

$$\max \frac{\Pr[\mathcal{M}(x_1) = y]}{\Pr[\mathcal{M}(x_2) = y]} \leq e^\epsilon$$

Distinguishability of x_1 (YES) and x_2 (NO)
from y (randomized answer)



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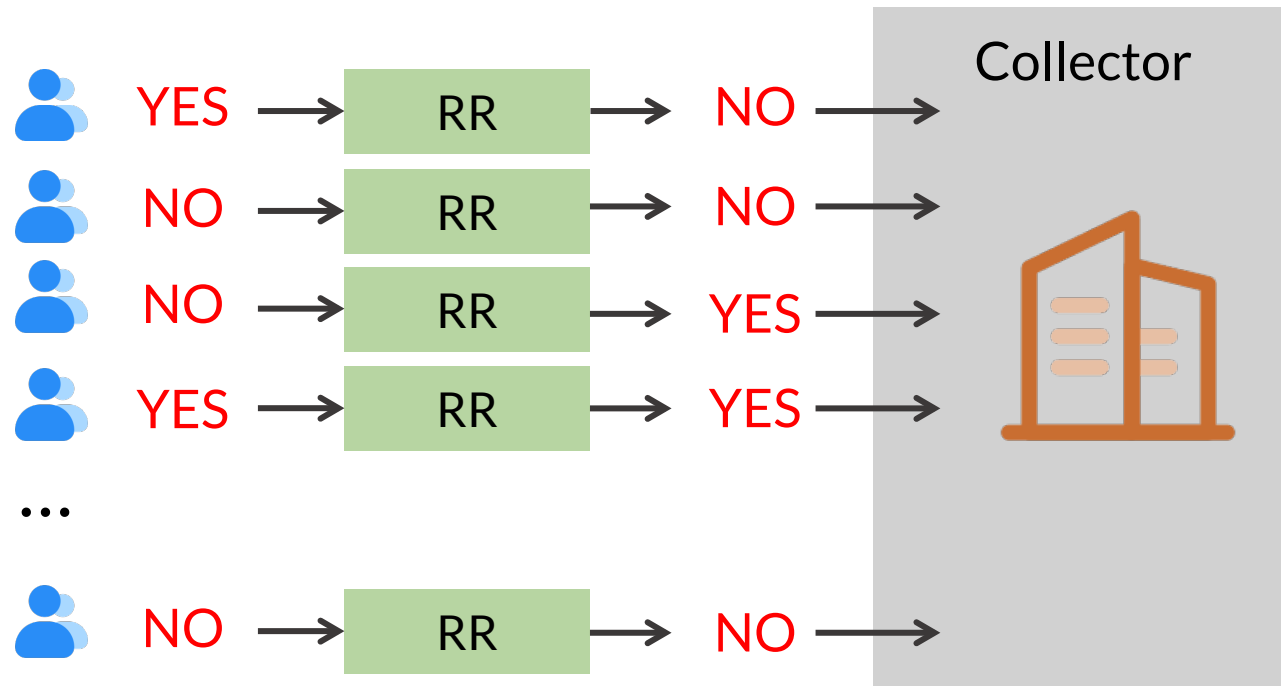
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Distinguishability of x_1 (YES) and x_2 (NO)
from y (randomized answer)

- **quantifiable hardness** to distinguish x_1 (YES) and x_2 (NO) from the randomized answer y
- defense against inference from data collectors  or adversaries 

Randomized Response for Privacy

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- Randomized Response: Randomize the truth before answering the collector



Randomized Response for Privacy

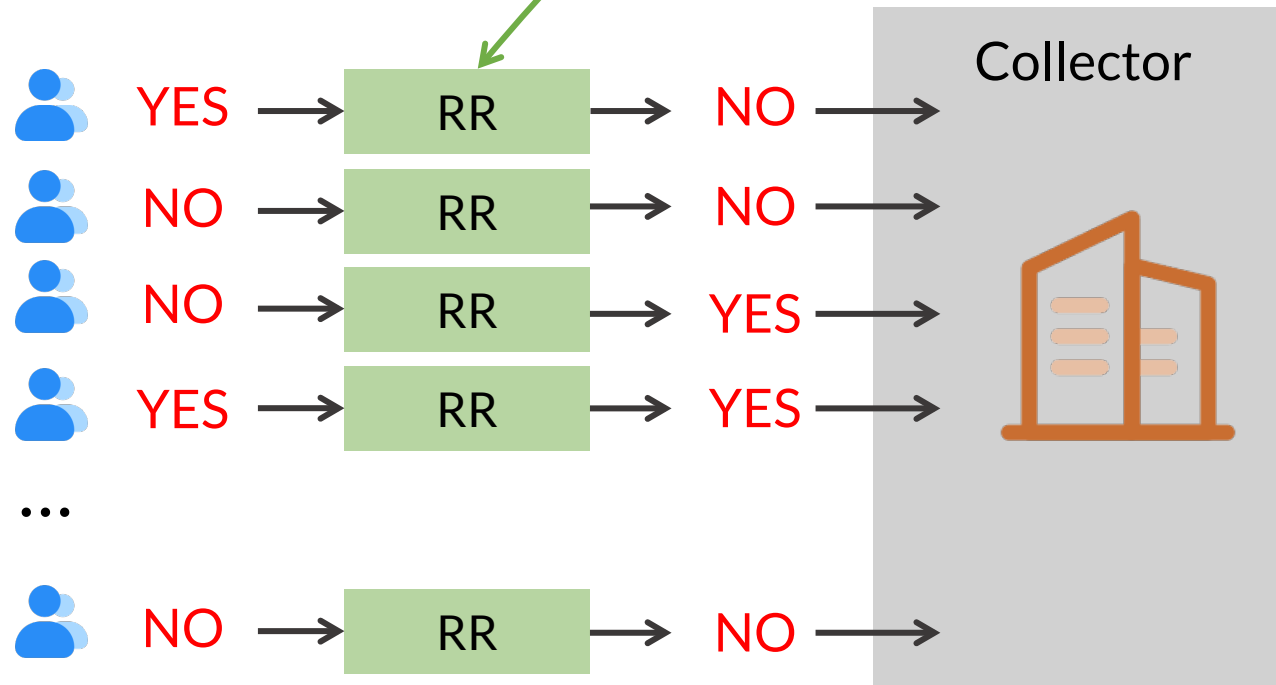
$$\max \frac{\Pr[\mathbf{RR}(x_1) = y]}{\Pr[\mathbf{RR}(x_2) = y]} \leq e^{\ln \frac{p}{1-p}}$$

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Private

RR: [Warner, 1965]
answer truth with probability p

$$\mathbf{RR}(x) = \begin{cases} x & \text{w.p. } p \\ \neg x & \text{w.p. } 1 - p \end{cases}$$



Randomized Response for Privacy

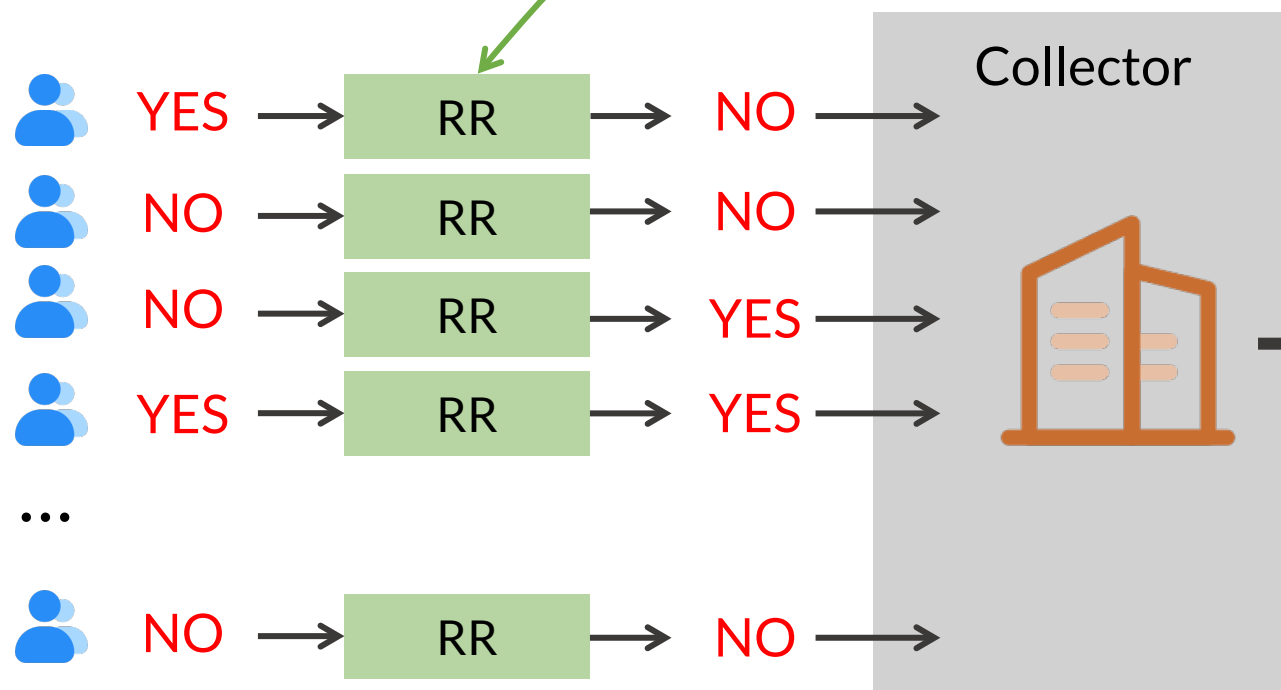
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estimated frequency

$$= \frac{\# \text{ of YES} - \# \text{ of people} \times q}{p - q}$$

Unbiased:

expectation = truth

Utility: RR's Variance

- Randomization reduces data utility

$$\text{Var}\left[\frac{\# \text{ of YES} - \# \text{ of people} \times q}{p - q}\right] = \frac{\text{Var}[\# \text{ of YES}]}{(p - q)^2} = \frac{npq}{(p - q)^2}$$

- summation of variance from all n independent randomization

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- larger $p \in (0.5, 1]$ \rightarrow lower variance \rightarrow larger privacy parameter ϵ

\uparrow data utility



\downarrow privacy

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- Q: Can we improve this privacy-utility tradeoff?

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- Q: Can we improve this privacy-utility tradeoff?
 - yes, by correlated (joint) randomization

This Paper: Joint RR (JRR)

- JRR: Better data utility by joint randomization

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- JRR: Better data utility by joint randomization
- **Example:** 2-person ($x_1 = \text{YES}$ and $x_2 = \text{YES}$) with $p = 0.8$ ($P[T = 1] = 0.8$)

RR: Joint distribution

	$T_1 = 1$	$T_1 = 0$	Truthfulness of x_1
$T_2 = 1$	0.64 ($= p^2$)	0.16 ($= pq$)	
$T_2 = 0$	0.16 ($= pq$)	0.04 ($= q^2$)	

Truthfulness
of x_2

This Paper: Joint RR (JRR)

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Independent T_1 and T_2 ($P[T_1 \cap T_2] = P[T_1] \cdot P[T_2]$)

Joint probability = Π of marginal probabilities

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	$T_1 = 1$	$T_1 = 0$
$T_2 = 1$	0.6 ($= p^2 + \rho pq$)	0.2 ($= pq - \rho pq$)
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Frequency Estimation via Joint Randomized Response

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$P[T_1 = 1] = 0.8$

$P[T_1 = 0 \cap T_2 = 0] = 0 \neq P[T_1 = 0] \cdot P[T_2 = 0] = 0.04$

NOT independent T_1 and T_2

Joint probability \neq Π of marginal probabilities

Frequency Estimation

Utility: JRR's Variance

- Same estimator as RR

$$\underset{\substack{\uparrow \\ \text{Expectation}}}{E[\# \text{ of YES}]} = \sum_{i=1}^{\# \text{ 👤 }} P[y_i = \text{YES}] = n_{\text{YES}} \cdot p + (\# \text{ 👤 } - n_{\text{YES}}) \cdot q$$

$\underset{\substack{\uparrow \\ \text{Ground truth}}}{n_{\text{YES}}}$ \nearrow

Utility: JRR's Variance

- Same estimator as RR

$$E[\# \text{ of YES}] = \sum_{i=1}^{\# \text{ people}} P[y_i = \text{YES}] = n_{\text{YES}} \cdot p + (\# \text{ people} - n_{\text{YES}}) \cdot q$$

→ Unbiased estimator $\hat{n}_{\text{YES}} = \frac{\# \text{ of YES} - 2q}{p - q}$

Identical to RR

Utility: JRR's Variance

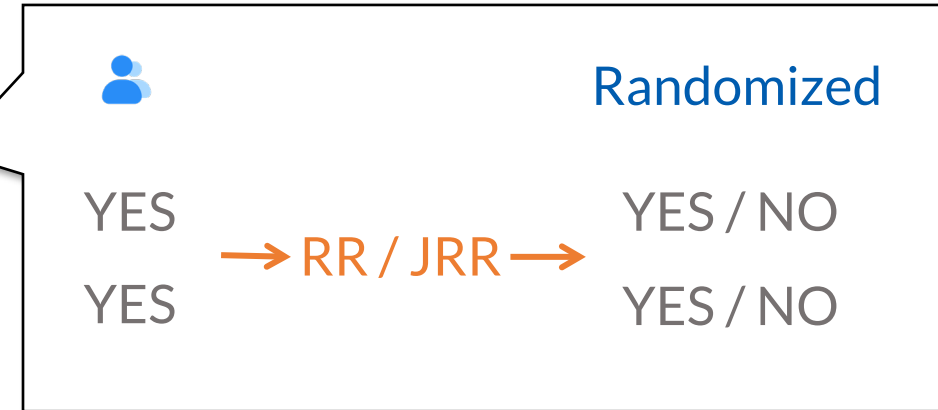
- Variance: ($\# \text{ people} = 2, p = 0.8$)

$$\text{Var}[\hat{n}_{\text{YES}}] = \frac{\text{Var}[\# \text{ of YES}]}{(0.8 - 0.2)^2}$$

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- Distribution table:

RR

# of YES	0	1	2
Probability	0.04	0.16 + 0.16	0.64

$$\text{Var}[\# \text{ of YES}] = E[(X - \mu)^2] = \mathbf{0.32}$$

$$= \sum_{X=0,1,2} (X - 1.6)^2 \cdot \text{Pr}[X] \approx \mathbf{0.1 + 0.12 + 0.1}$$

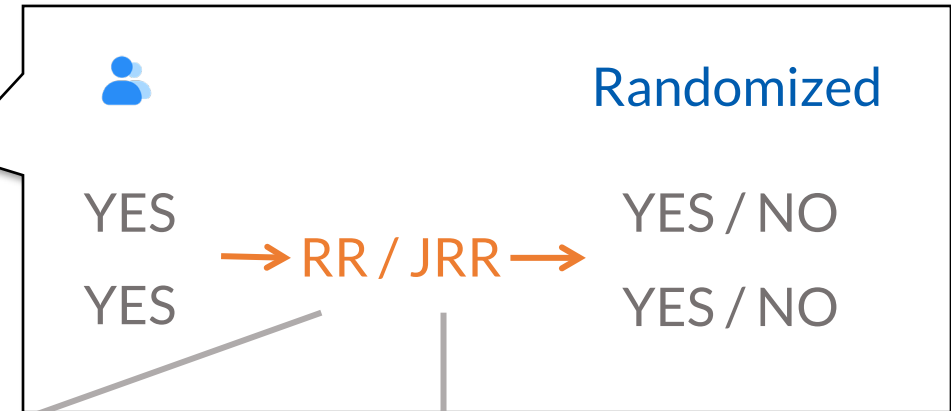
JRR

# of YES	0	1	2
Probability	0	0.2 + 0.2	0.6

$$\text{Var}[\# \text{ of YES}] = E[(X - \mu)^2] = \mathbf{0.24}$$

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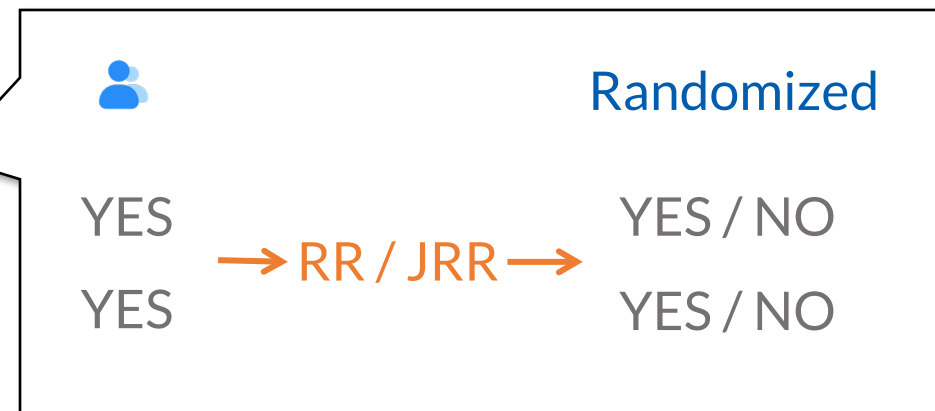
Frequency Estimation via Joint Randomization



Utility: JRR's Variance

- Variance: (# 👤 = 2, $p = 0.8$)

$$\text{Var}[\hat{n}_{\text{YES}}] = \frac{\text{Var}[\# \text{ of YES}]}{(0.8 - 0.2)^2}$$



- Distribution table:

	RR		
# of YES	0	1	2
Probability	0.04	0.16 + 0.16	0.64

	JRR (near to μ)		
# of YES	0	1	2
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Better utility

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Frequency Estimation via Joint Randomized Response

JRR's General Form

- Correlated randomization with 2 persons x_{2i-1} and x_{2i}

JRR: Joint distribution

	$T_{2i-1} = 1$	$T_{2i-1} = 0$
$T_{2i} = 1$	$p^2 + \rho pq$	$(1 - \rho)pq$
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$\rho \in [-1,1]$:
correlation coefficient



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Utility Theorem. The variance of JRR's estimator \hat{n}_v is

$$\text{Var}[\hat{n}_v] = \frac{pq}{(p - q)^2} \cdot \left(n + \frac{\rho((2n_{\text{YES}} - n)^2 - n)}{n - 1} \right).$$

Privacy: NOT as Simple as RR

- If any person can be an adversary

JRR: Joint distribution

	$T_1 = 1$	$T_1 = 0$
$T_2 = 1$	0.6	0.2
$T_2 = 0$	0.2	0

T_1 : I am an adversary 🤖

$\Pr[T_2 = 1 | T_1 = 0] = 1$ 🤖

When I report untruthfully ($T_1 = 0$),
My partner will report truthfully ($T_2 = 1$)

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- Correlation results in privacy leakage

JRR – Privacy Model in This Paper

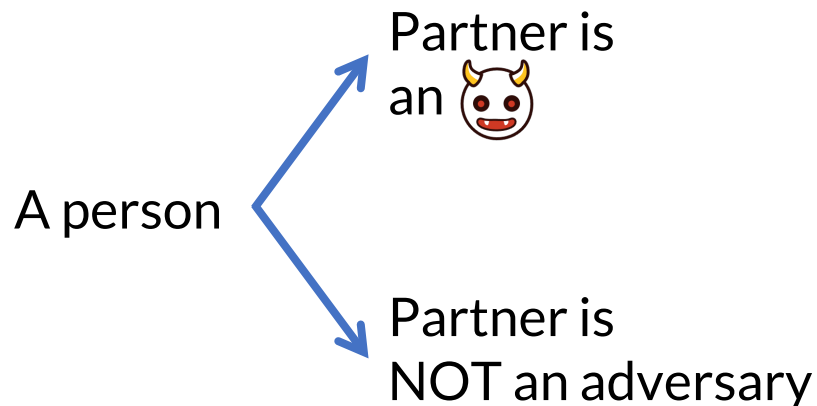
- Form random 2-person groups for correlated randomization

JRR – Privacy Model in This Paper

- Form random 2-person groups for correlated randomization
- **Threat model:**
 - any person can be an adversary 
 - if a group contains an adversary, the adversary knows **who is their partner** (after random grouping)

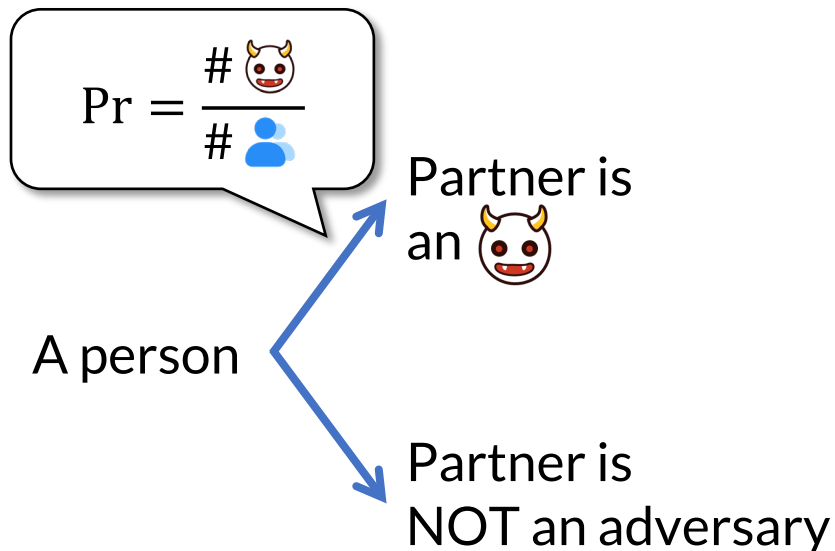
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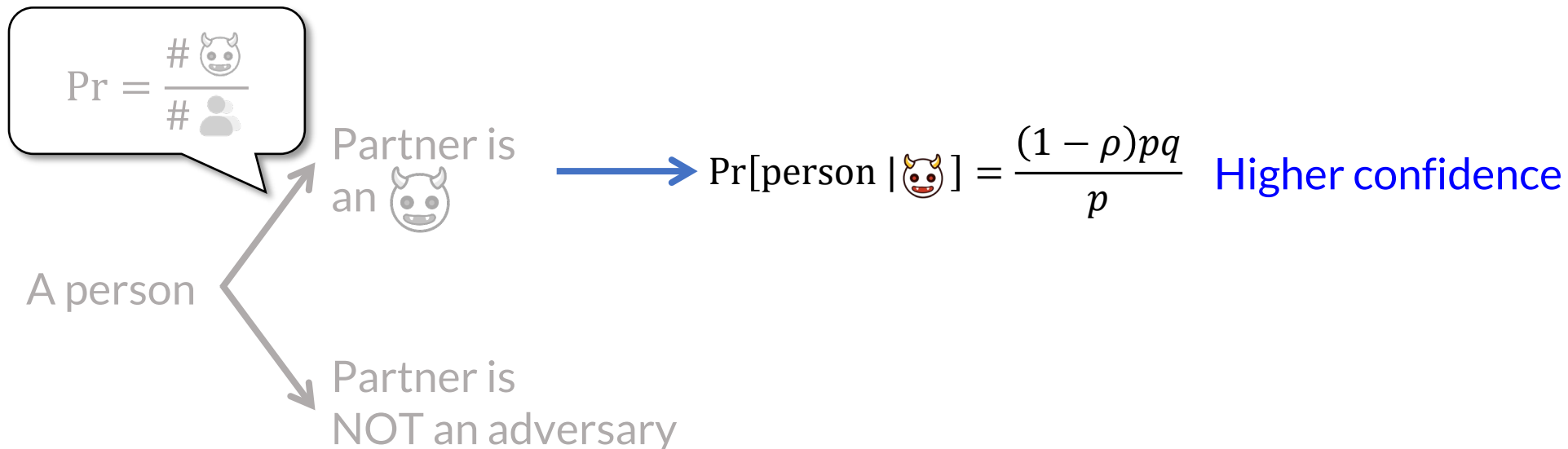
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- Form random 2-person groups for correlated randomization
- Threat model:**
 - if a group contains an adversary, the adversary knows who is their partner (after random grouping)
 - the adversary cannot control randomness, but can **infer their partner's**



JRR – Formal Privacy & Utility

Privacy Theorem. Assume a set of data contributors \mathcal{T}_m whose reporting truthfulness is known to the adversary. For any data contributor i , the JRR mechanism satisfies:

$$\frac{\Pr[\text{JRR}(x_i) | \mathcal{T}_m]}{\Pr[\text{JRR}(x'_i) | \mathcal{T}_m]} \leq e^\varepsilon, \text{ where } \varepsilon = \ln \frac{mp_{\max} + (n - m - 1)p}{mp_{\min} + (n - m - 1)q}.$$

Privacy affected by

m	# adversaries 🐉
n	# of persons 👤
ρ	Correlation coefficient

$p_{\max} = \max\{(1 - \rho)p, p + \rho q\}$:
confidence of adversaries
inferring a specific value

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$$\text{Var}[\hat{n}_v] = \frac{pq}{(p - q)^2} \cdot \left(n + \frac{\rho((2n_{\text{YES}} - n)^2 - n)}{n - 1} \right).$$

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privacy constraint

$$\varepsilon = \ln \frac{mp_{\max} + (n - m - 1)p}{mp_{\min} + (n - m - 1)q}.$$

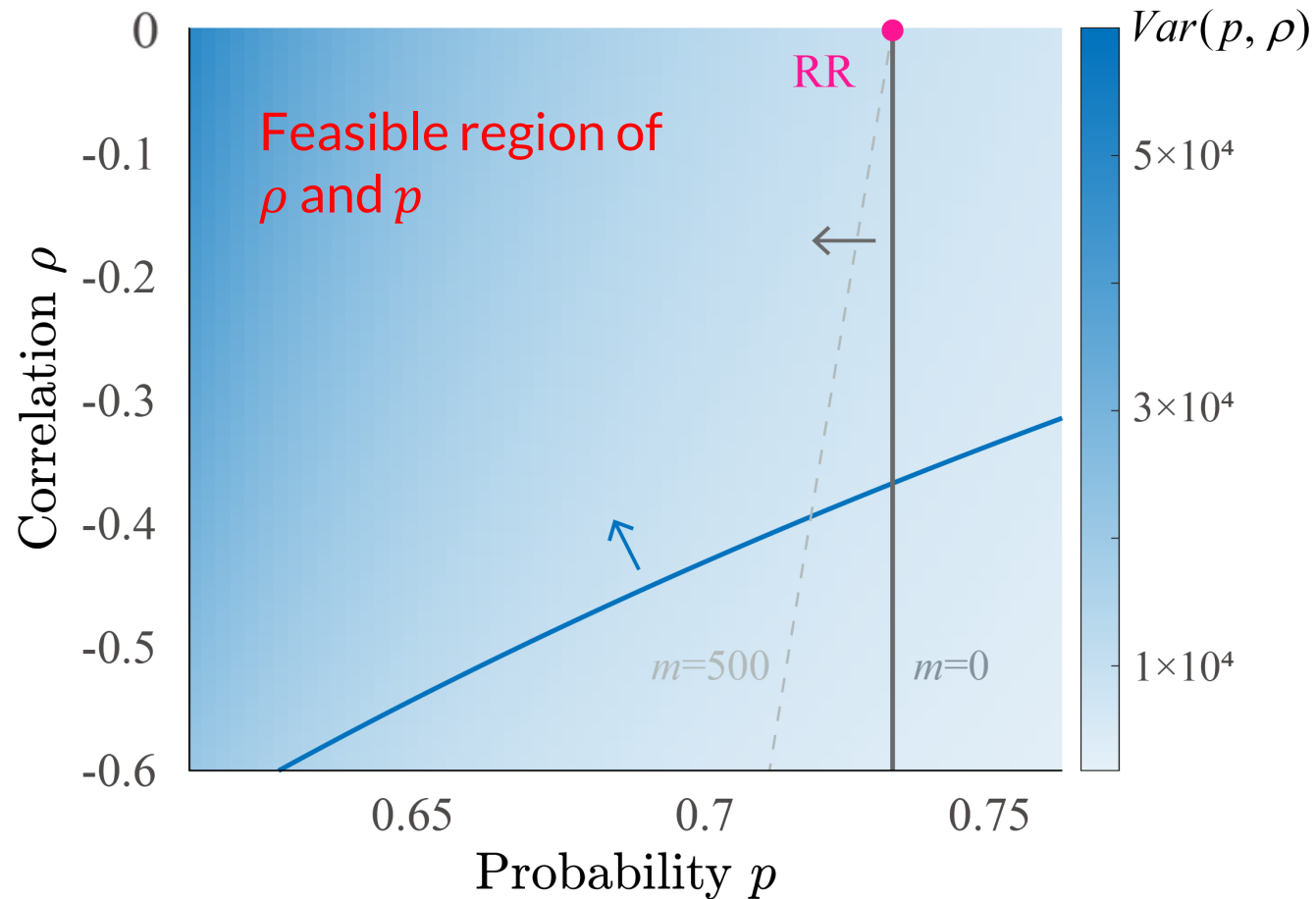
Utility Theorem. The variance of JRR's estimator \hat{n}_v is

minimize

$$\text{Var}[\hat{n}_v] = \frac{pq}{(p - q)^2} \cdot \left(n + \frac{\rho((2n_{\text{YES}} - n)^2 - n)}{n - 1} \right).$$

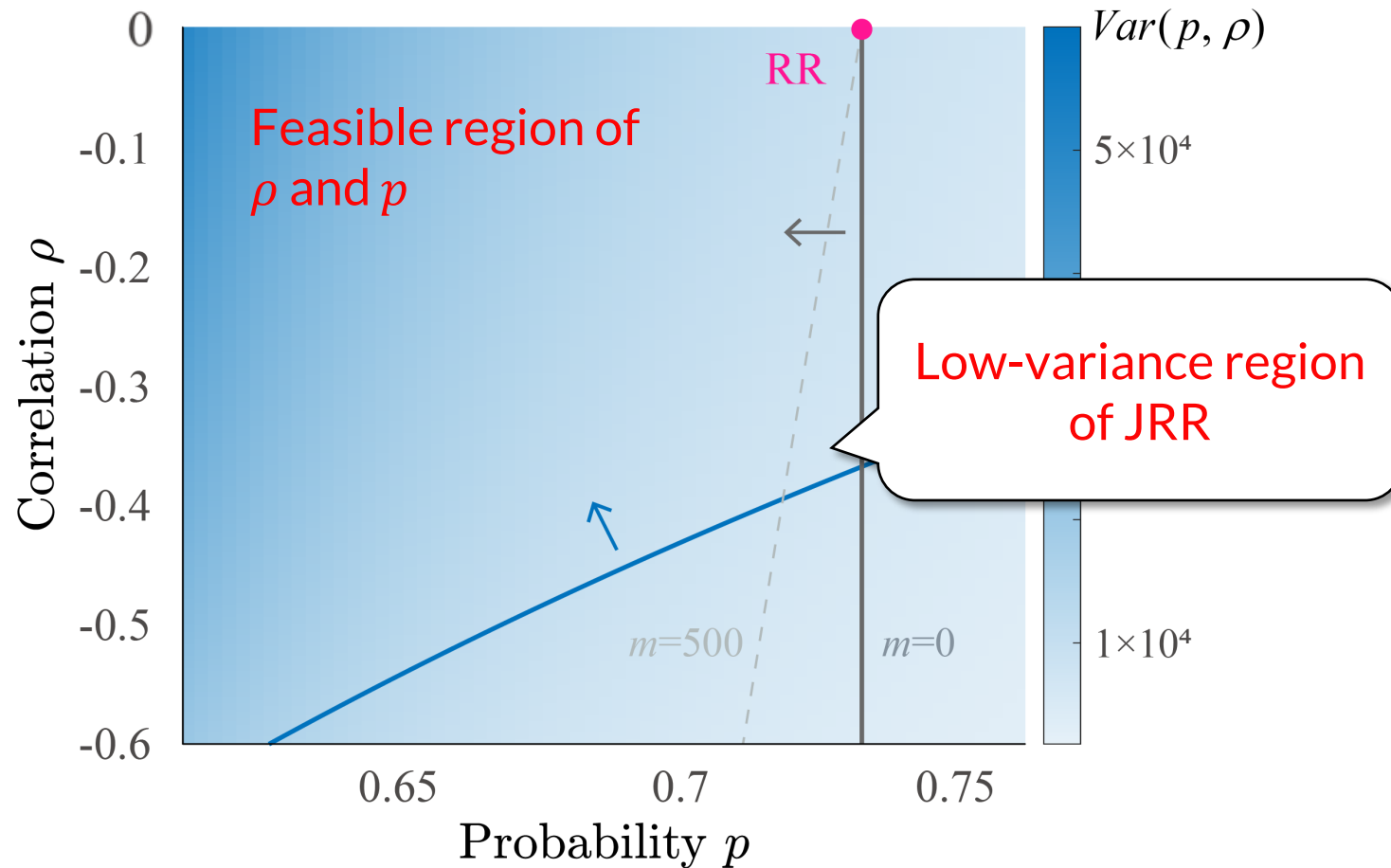
JRR – Variance Heatmap

- Effect of ρ and p (when $\varepsilon = 1, n = 10^4, n_{\text{Yes}} = 200$, and $m = 0$ & 500)



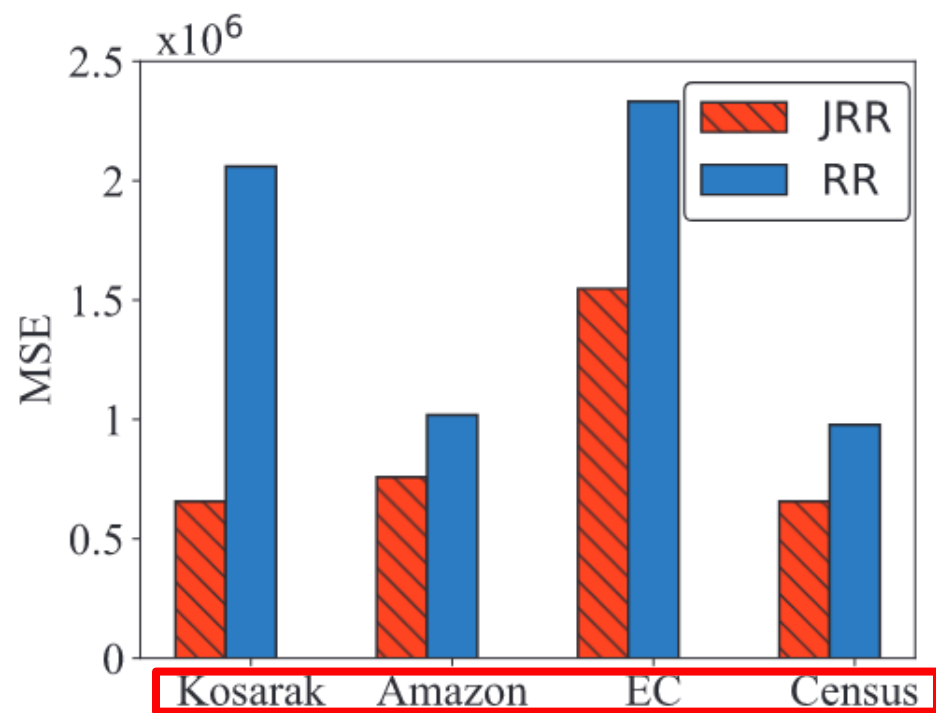
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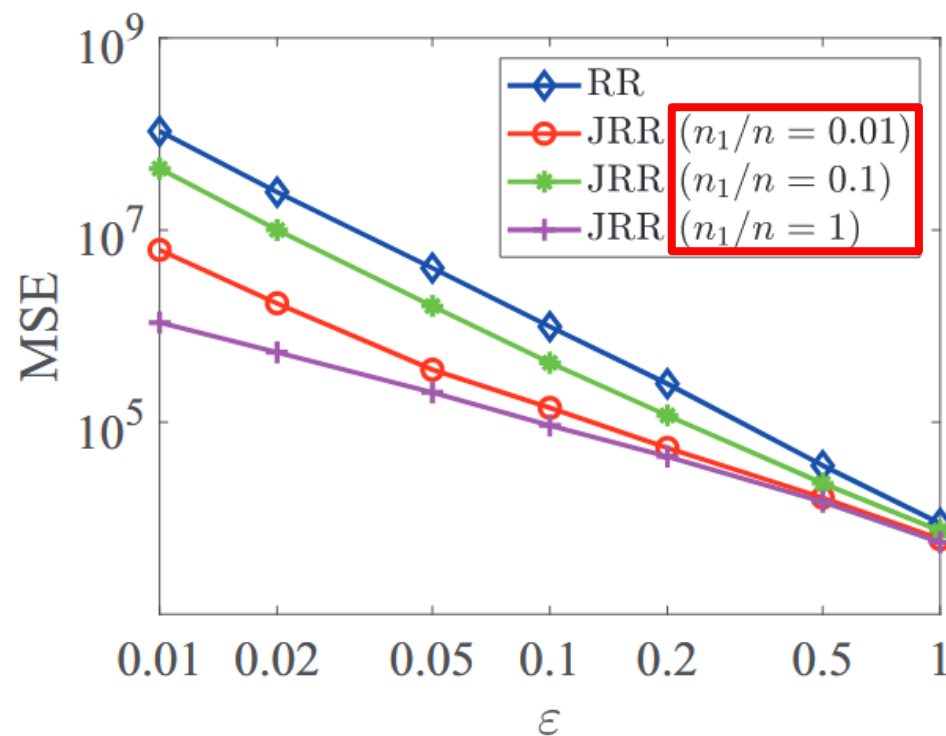


Experiments

- Comparison with RR under the same privacy level - JRR: $\varepsilon(n, m, \rho, p)$, RR: $\varepsilon(p)$



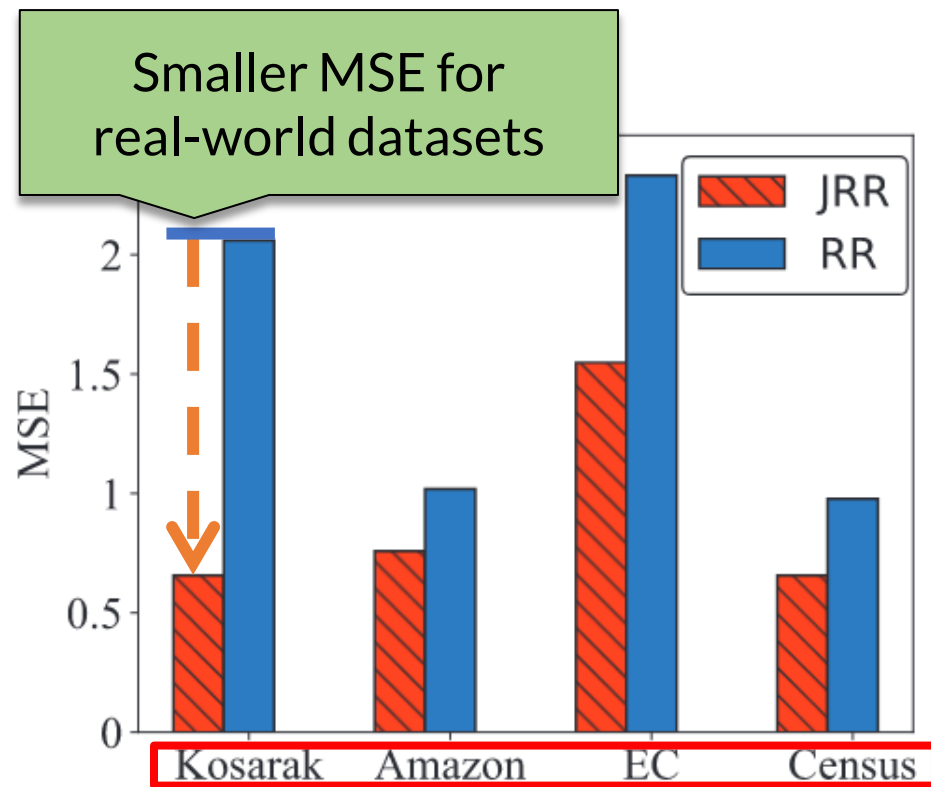
Real-world datasets ($\varepsilon = 0.1$)



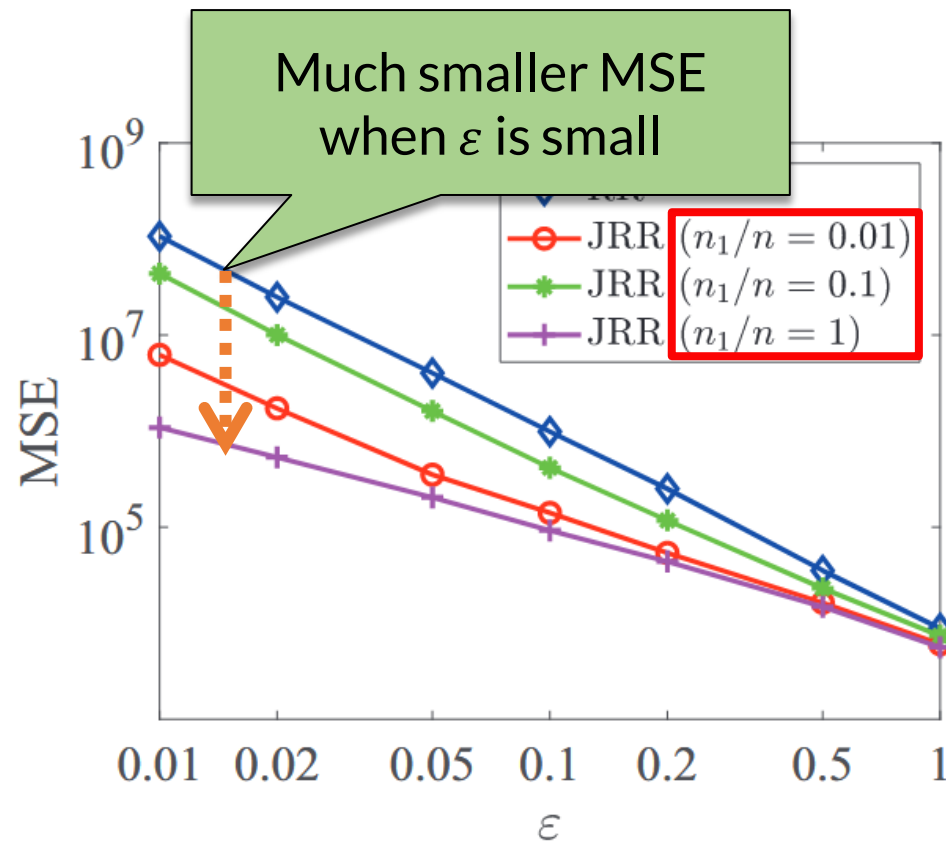
Synthetic datasets ($n = 10^4$)

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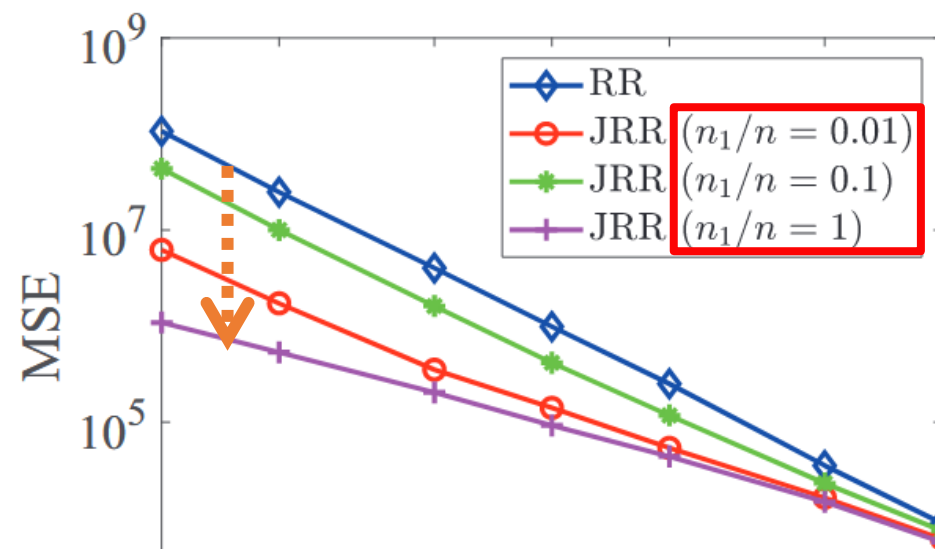
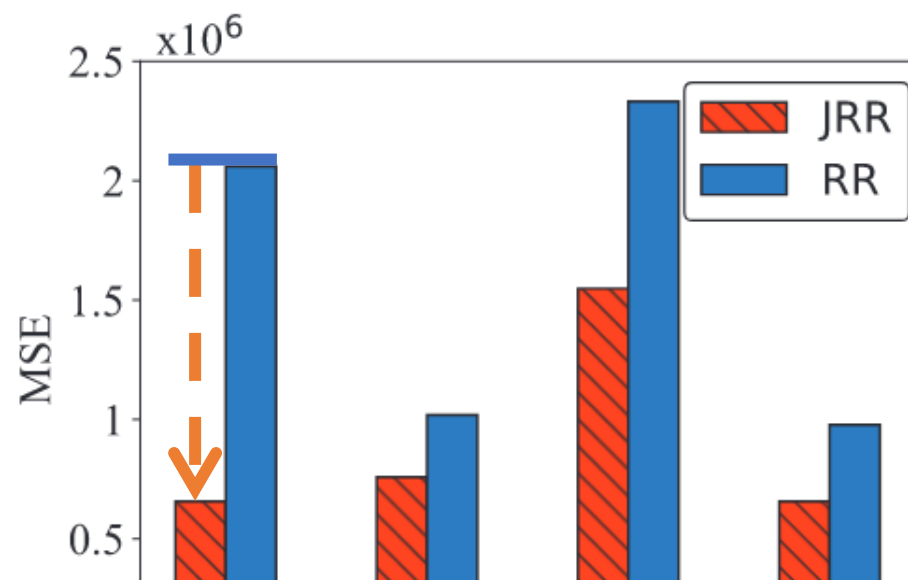
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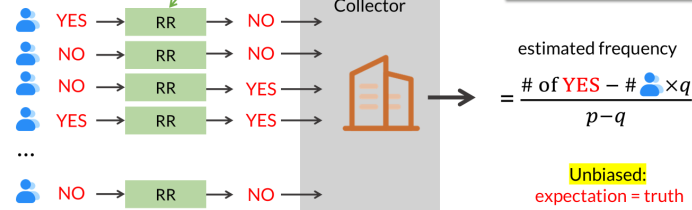


- Correlated randomization can improve the data utility of frequency estimation
- JRR: Privacy & utility model for correlated randomization**

Locally Differentially Private Frequency Estimation via Joint Randomized Response

Randomized Response for Privacy

- People have **privacy concerns** on sensitive/embarrassing questions - i.e. don't want to let the collector know
- Randomized Response: Randomize the truth before answering



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Locally Differentially Private Frequency Estimation via Joint Randomized Response

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$\rho \in [-1, 1]$: correlated coefficient

- RR is a special case of JRR with $\rho = 0$ (no correlation)

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Locally Differentially Private Frequency Estimation via Joint Randomized Response

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This Paper: Joint RR (JRR)

- JRR: Better data utility by joint randomization
- Example: 2-person ($x_1 = \text{YES}$ and $x_2 = \text{YES}$) with $p = 0.8$ ($P[T = 1] = 0.8$)

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Joint probability = Π of marginal probabilities

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$P[T_1 = 0 \cap T_2 = 0] = 0 \neq P[T_1 = 0] \cdot P[T_2 = 0] = 0.04$

NOT independent T_1 and T_2

Joint probability $\neq \Pi$ of marginal probabilities

JRR - Privacy Model in This Paper

- Randomly groups into form 2-person groups for correlated randomization
- Threat model:**
 - if a group contains an adversary, the adversary knows who is their partner (after random grouping)
 - the adversary cannot control randomness, but can **infer their partner's**



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Locally Differentially Private Frequency Estimation via Joint Randomized Response

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Utility: JRR's Variance

- Variance: ($\# = 2, p = 0.8$)

$$\text{Var}[\hat{n}_{\text{YES}}] = \frac{\text{Var}[\# \text{ of YES}]}{(0.8 - 0.2)^2}$$

- Distribution table:

# of YES	0	1	2
Probability	0.04	0.16 + 0.16	0.64

$$\text{Var}[\# \text{ of YES}] = E[(X - \mu)^2] = 0.32$$

$$= \sum_{x=0,1,2} (x - 1.6)^2 \cdot \Pr[X] \approx 0.1 + 0.12 + 0.1$$

Better utility

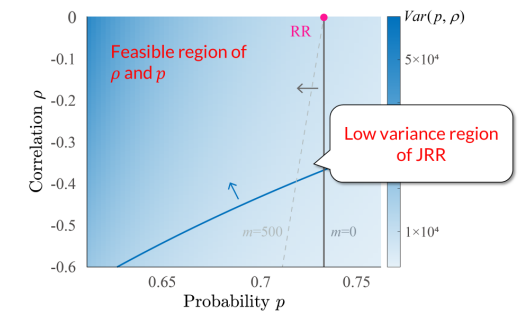
# of YES	0	1	2
Probability	0	0.2 + 0.2	0.6

$$\text{Var}[\# \text{ of YES}] = E[(X - \mu)^2] = 0.24$$

$$= \sum_{x=0,1,2} (x - 1.6)^2 \cdot \Pr[X] = 0 + 0.14 + 0.1$$

JRR - Variance Heatmap

- Effect of ρ and p (when $\epsilon = 1, n = 10^4, n_{\text{YES}} = 200$, and $m = 0$ & 500)



Ye Zheng

Locally Differentially Private Frequency Estimation via Joint Randomized Response

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Thank you!



Privacy Model

- No need of random grouping:
 - when **one** person hold **multiple** items

