Optimal Piecewise-based Mechanism for Collecting Bounded Numerical Data under Local Differential Privacy

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LDP Mechanisms

- Randomization algorithm $\mathcal{M}: \mathcal{D} \to \widetilde{\mathcal{D}}$

  - quantifiable privacy for data $x \in \mathcal{D}$

$$\forall x_1, x_2 \in \mathcal{D}, \forall y \in \widetilde{\mathcal{D}} \qquad \max \boxed{\frac{\Pr[\mathcal{M}(x_1) = y]}{\Pr[\mathcal{M}(x_2) = y]}} \leq e^{\varepsilon}$$

Distinguishability of x_1 and x_2 (sensitive data)
from y (randomized data)

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Privacy	quantified by ε

$x_1 \rightarrow \mathcal{M} \rightarrow y$

Provable defense against data inference attack

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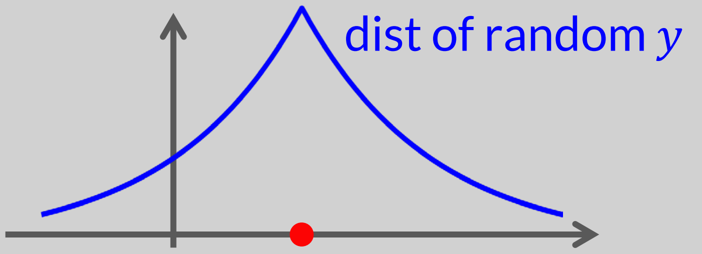
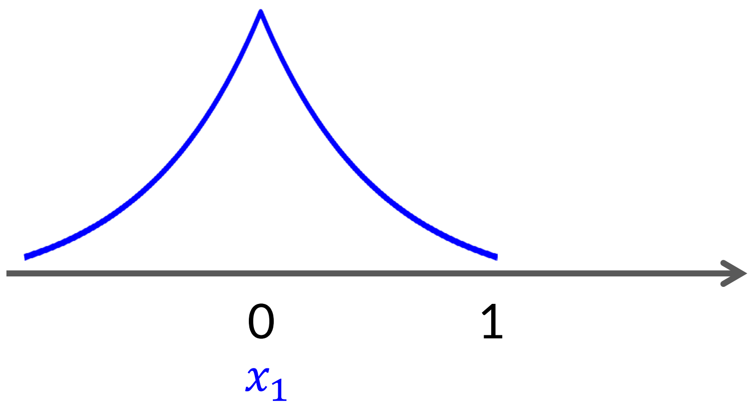
Data utility by expected errors



LDP Mechanisms for $\mathcal{D} = [0,1]$

Laplace mechanism: $[0,1] \rightarrow (-\infty, +\infty)$



LDP Mechanisms for $\mathcal{D} = [0,1]$

Symmetric (unbiased)
Useful for mean estimation
i.e. $(x_1 + x_2 + \cdots)/n$

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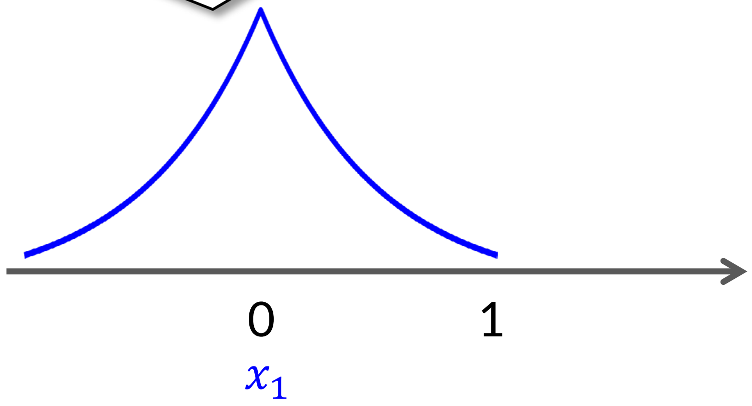

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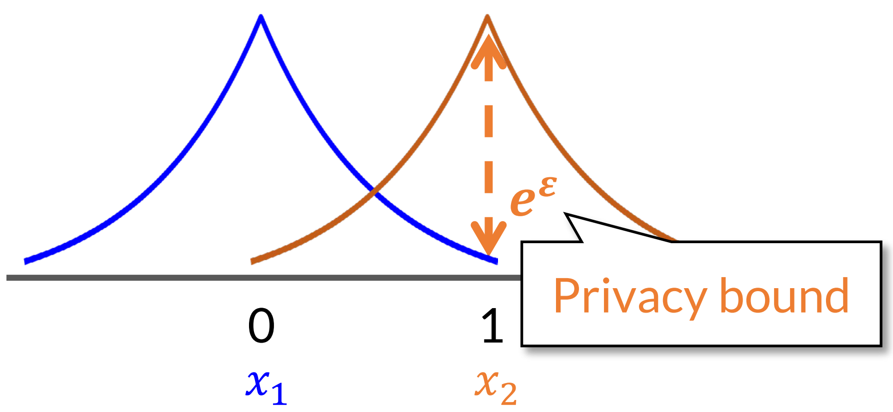

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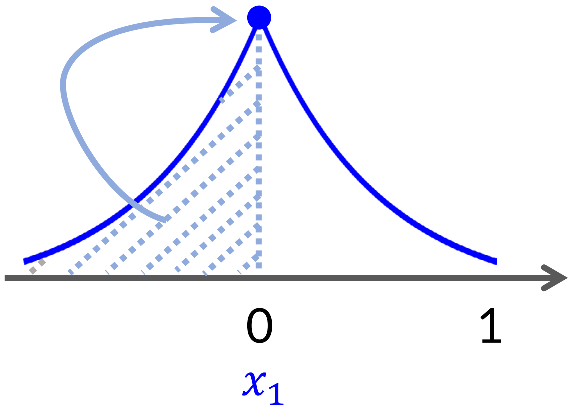

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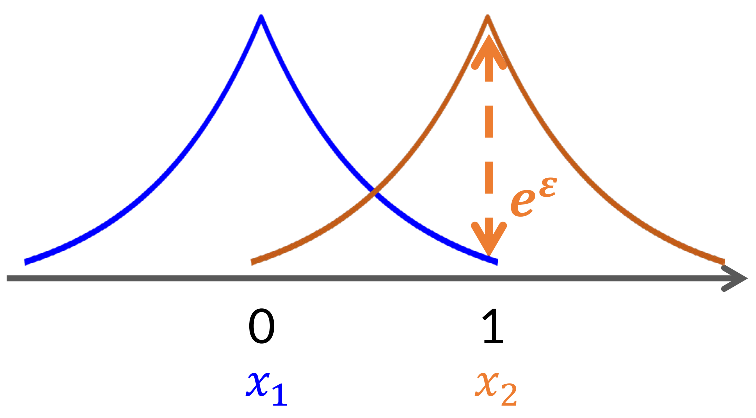

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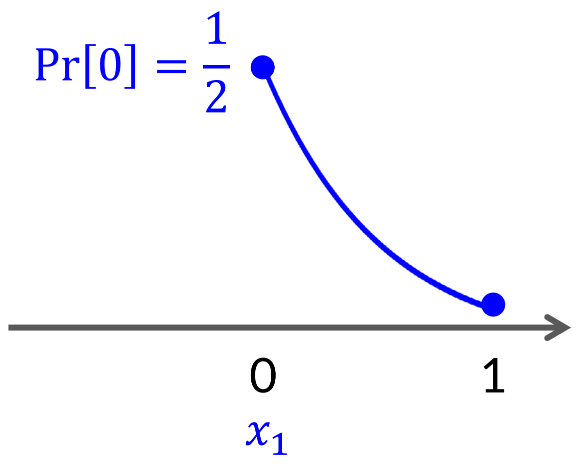

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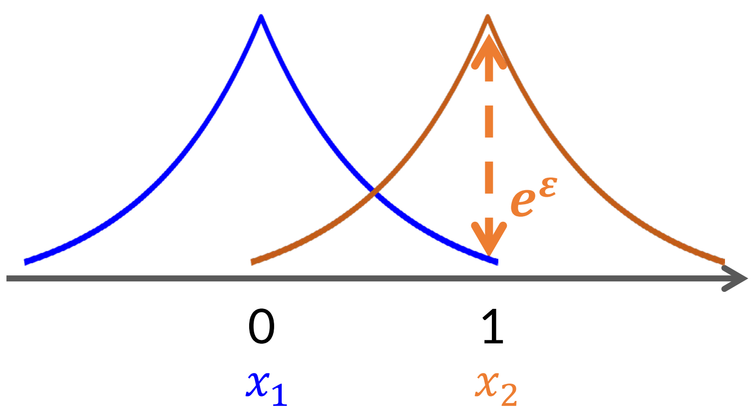

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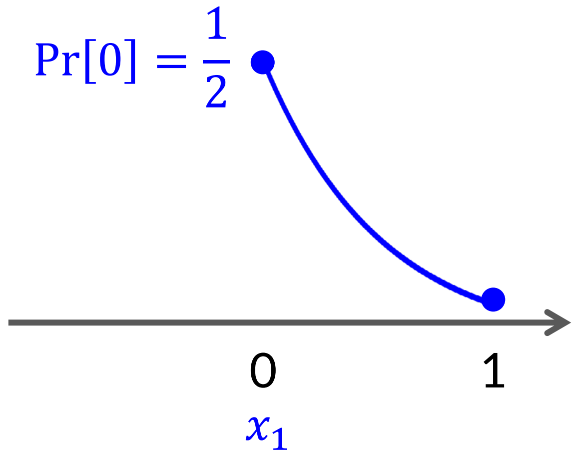
Smaller output
Useful for distribution estimation
i.e. $\text{dist}\{x_1, x_2, \cdots\}$



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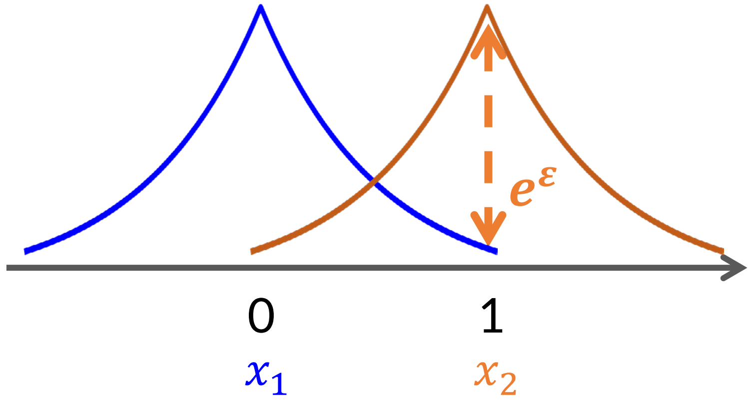

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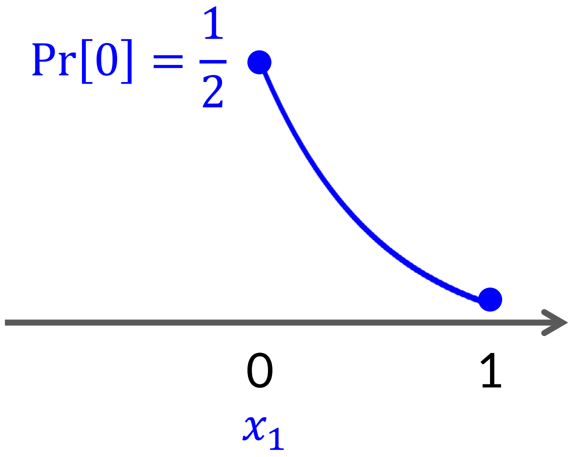

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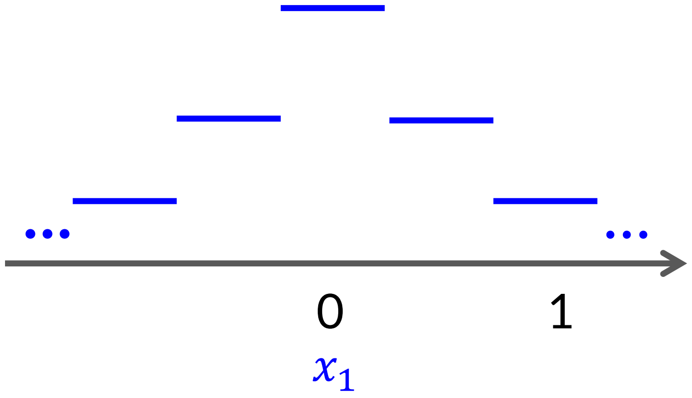
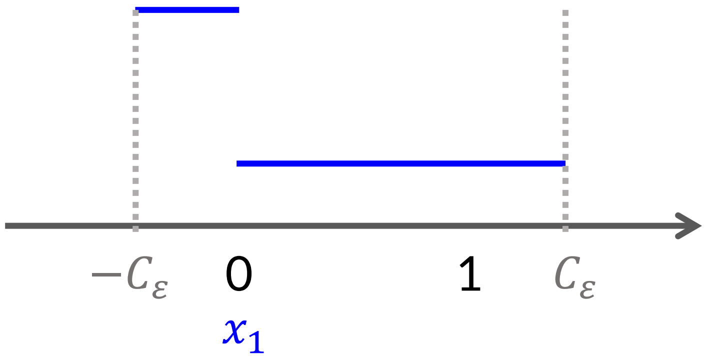


Piecewise mechanism: $[0,1] \rightarrow [-C_{\varepsilon}, C_{\varepsilon}]$



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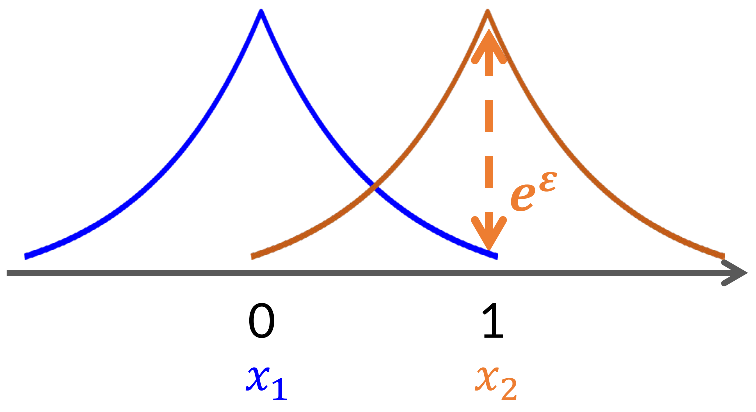

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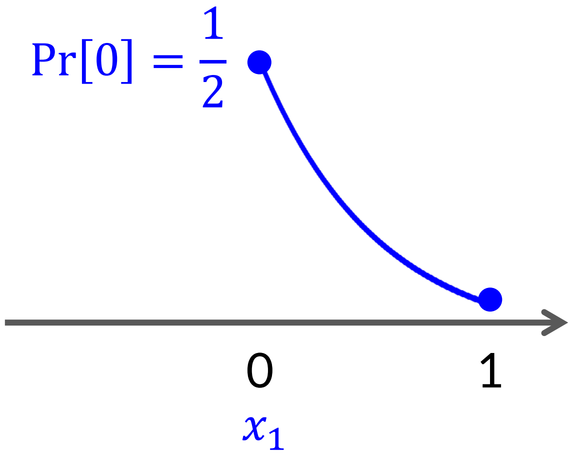

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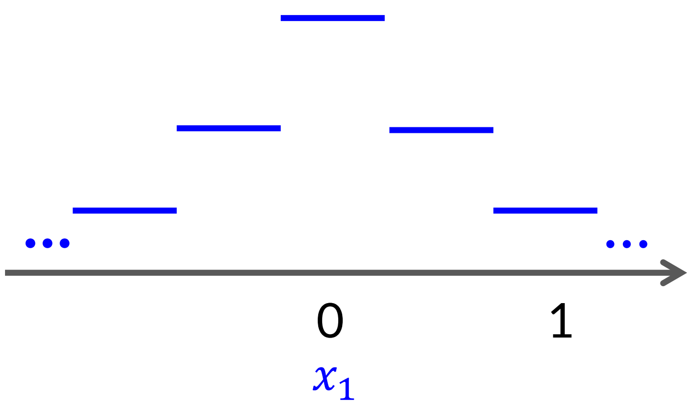

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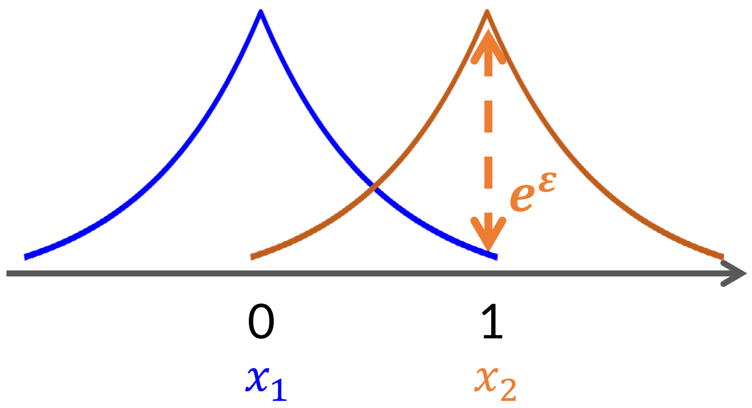

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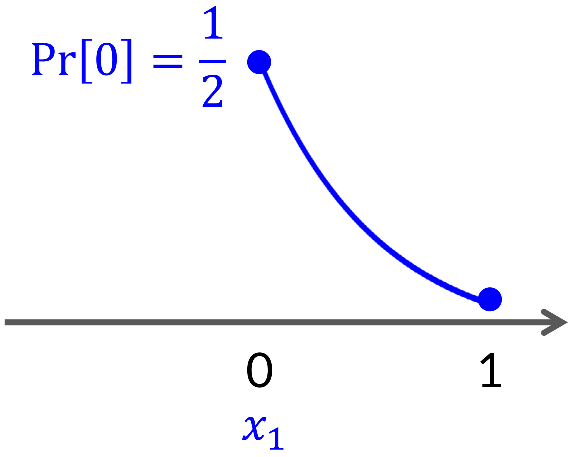

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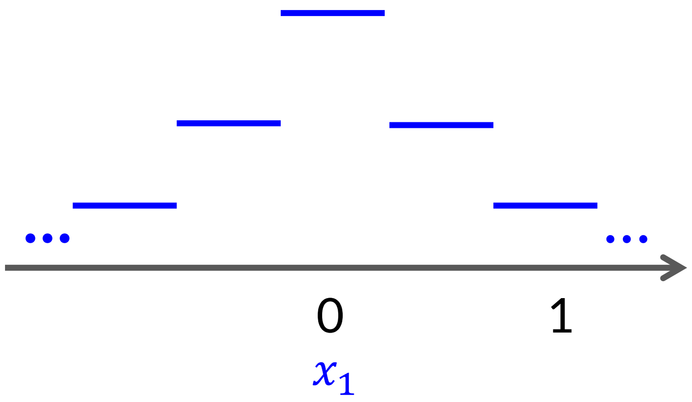

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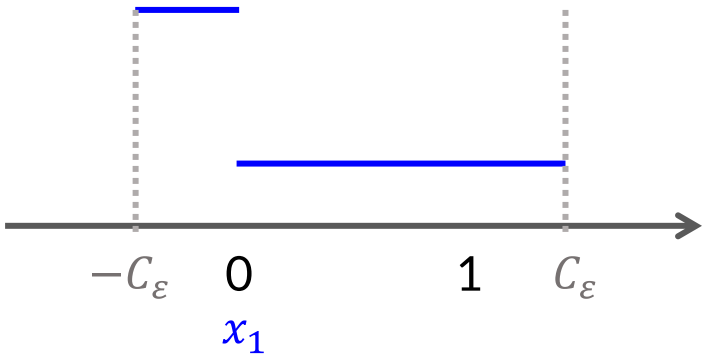

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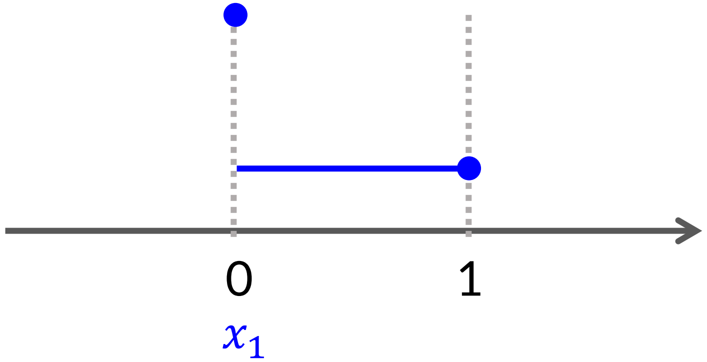

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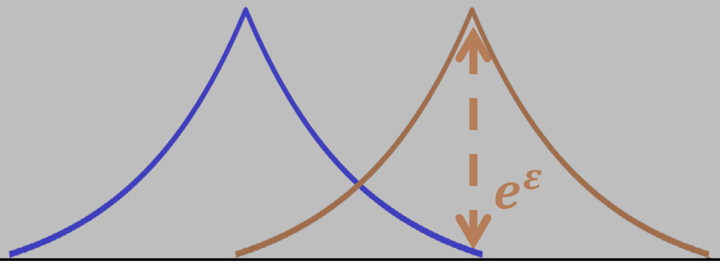

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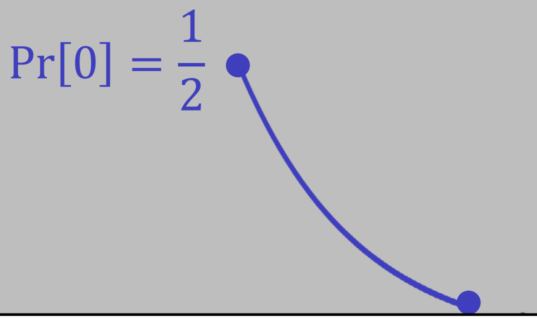

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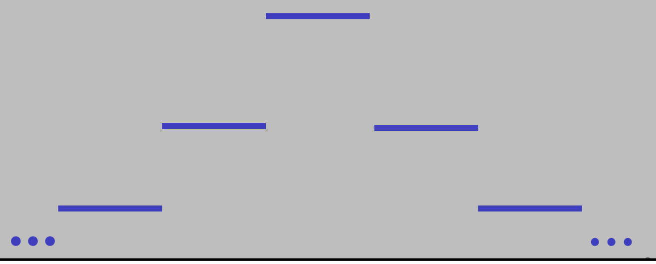
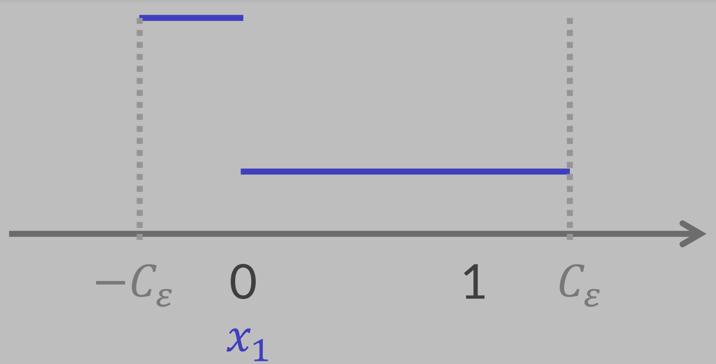
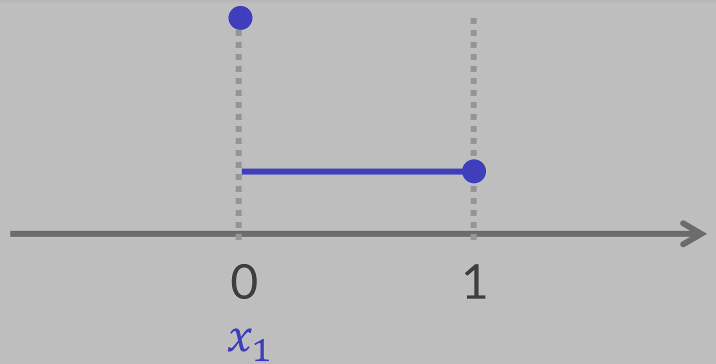


Privacy: LDP with the **same ε**

Utility: Different errors





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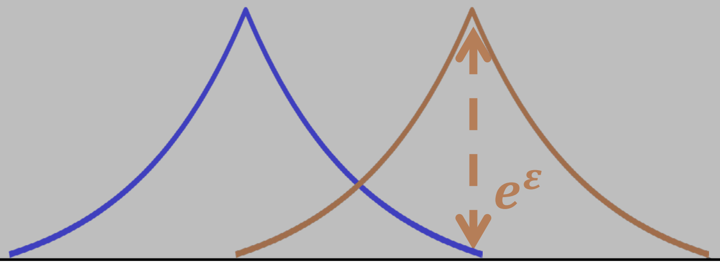

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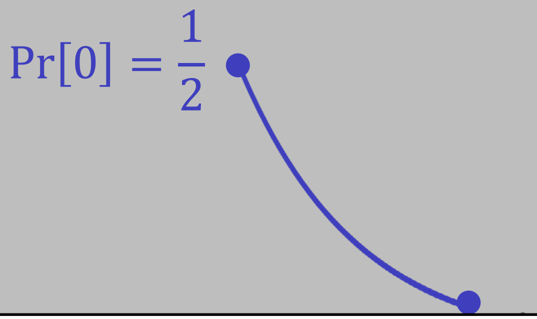

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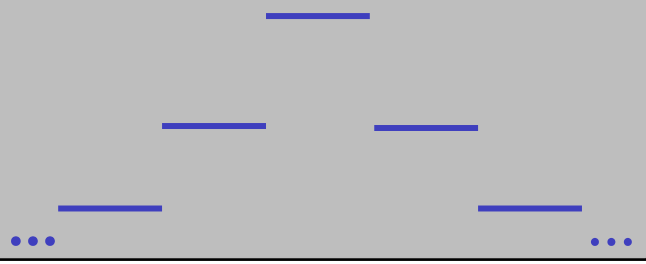

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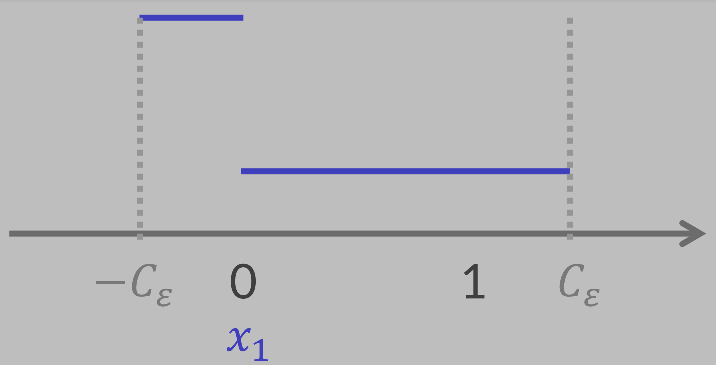
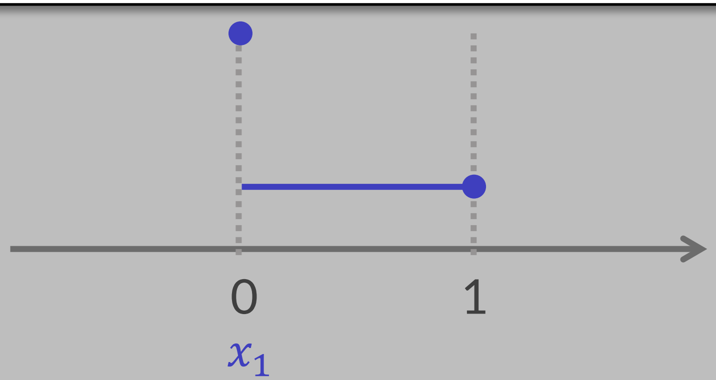
Q: What is the **optimal** LDP mechanism?





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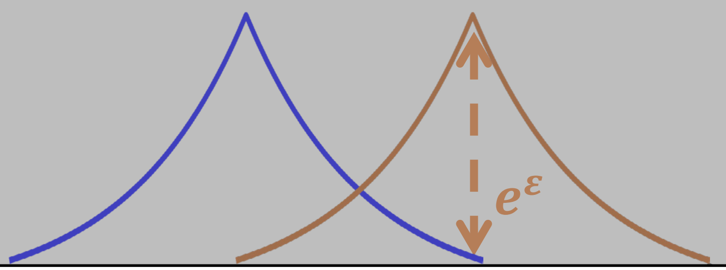

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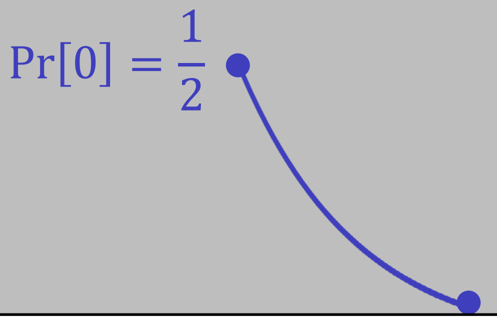

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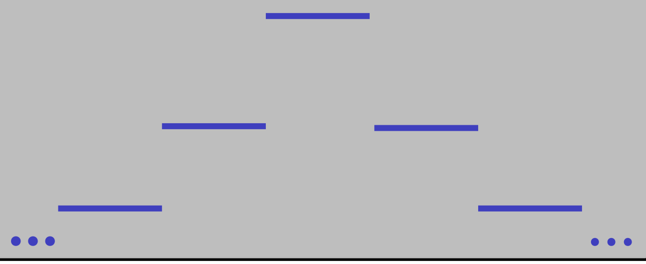

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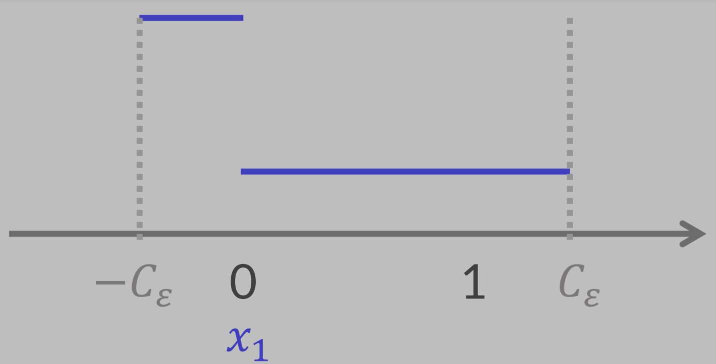
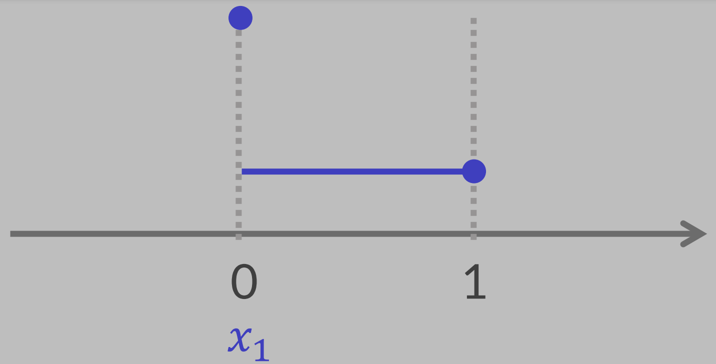
Q: What is the **optimal piecewise-based** mechanism?





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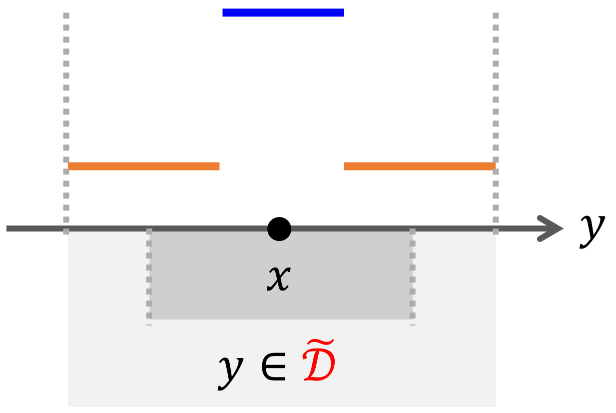
3-Piecewise Mechanism

- 3-piecewise distributions on bounded numerical domain $\mathcal{D} \to \widetilde{\mathcal{D}}$

  - given input x, sample output y from a distribution



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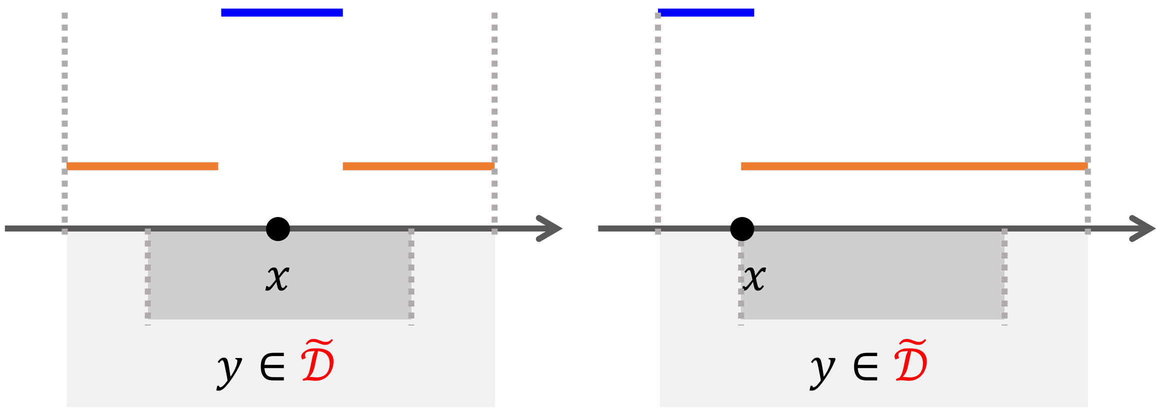

Sampling probability depends on ε

Sampling pieces depends on x and ε

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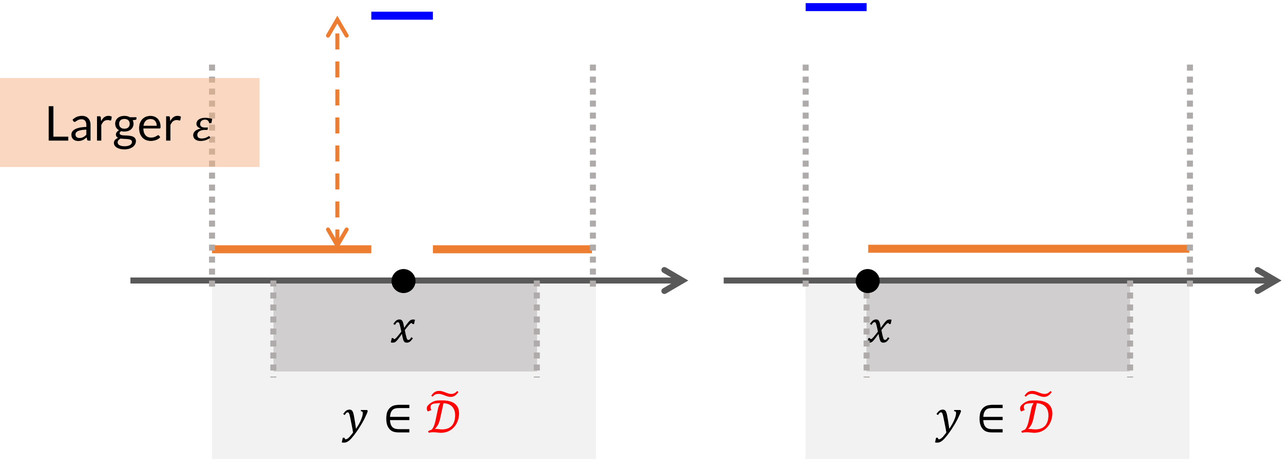

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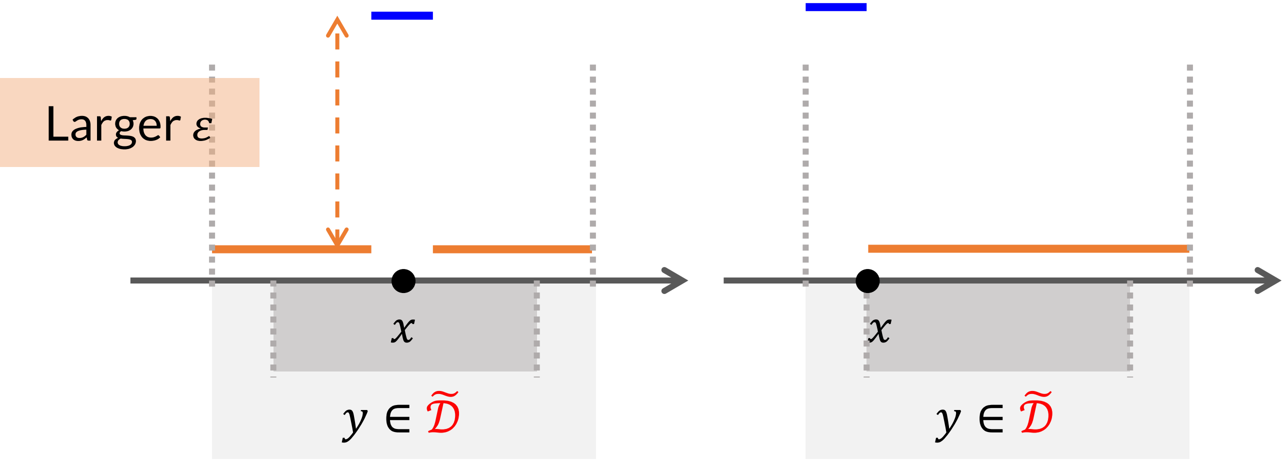

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  - different errors, but **without optimality**

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NOT enough to study optimality of piecewise-based mechanism

- only 3 pieces, two probabilities

- 3-piecewise distributions on boun

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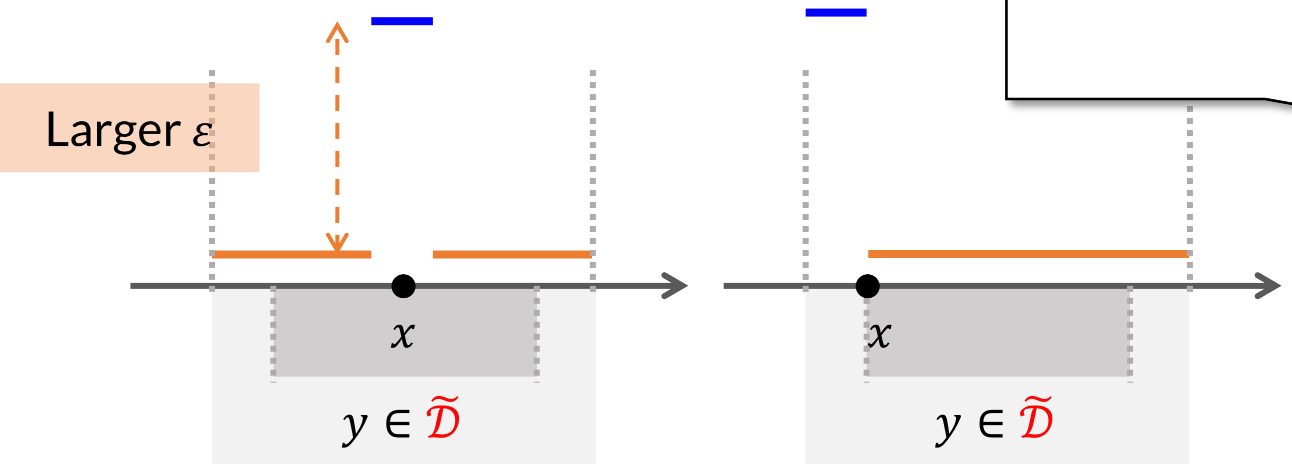

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- piecewise distribution can have more pieces and probabilities

- 3-piecewise distributions on boun

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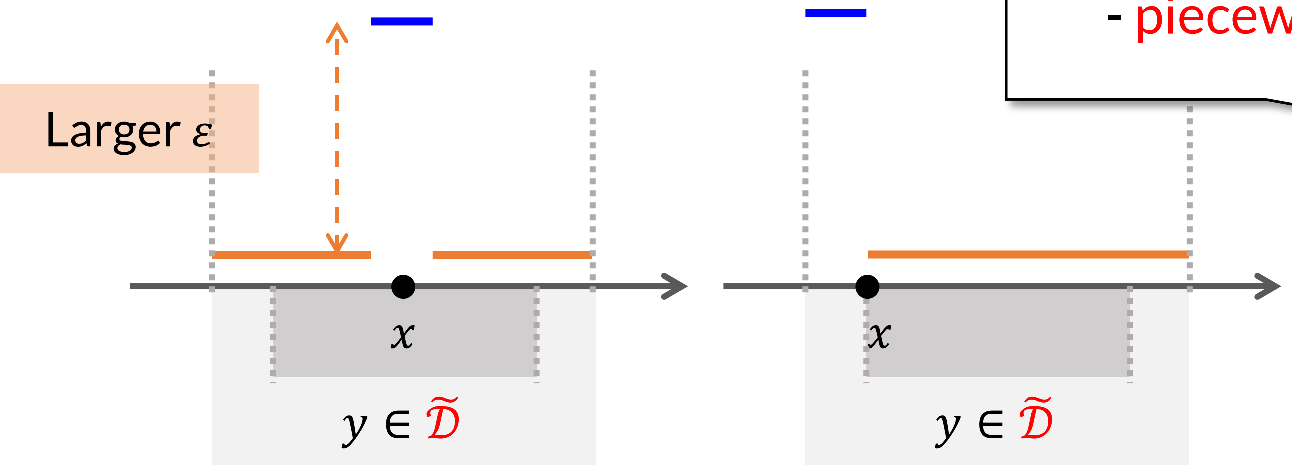

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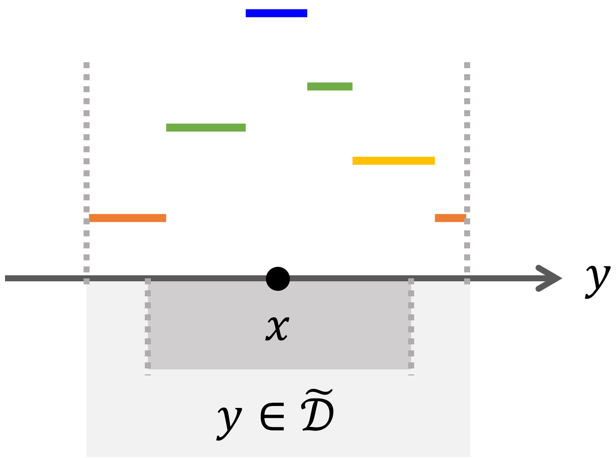
Generalized Piecewise-based Mechanism

- **Most generalized version:** m-piecewise distributions



$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_{1,\varepsilon} & \text{if } y \in [l_{1,x,\varepsilon}, r_{1,x,\varepsilon}] \\ p_{2,\varepsilon} & \text{if } y \in [l_{2,x,\varepsilon}, r_{2,x,\varepsilon}] \\ & \dots \\ p_{m,\varepsilon} & \text{if } y \in [l_{m,x,\varepsilon}, r_{m,x,\varepsilon}] \end{cases}$$

$$\max \frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon}$$ (LDP constraint)

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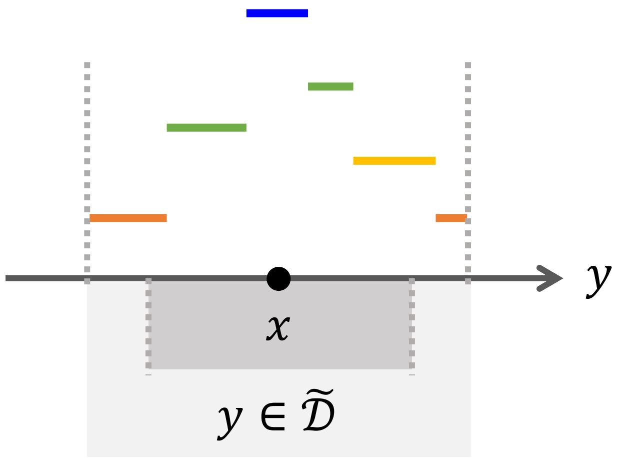

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$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon}$ (LDP constraint)

- Error (data utility): $\mathcal{L}(y, x)$

$\mathcal{L}(y, x) := |y - x|^p$

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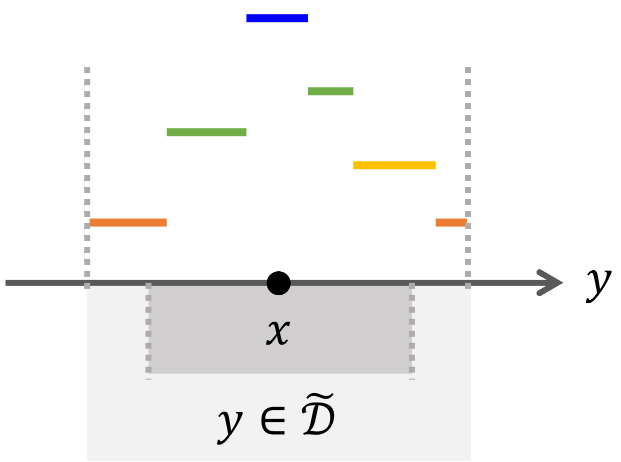

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- Expected error:

$$\int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

Optimal Piecewise-based Mechanism

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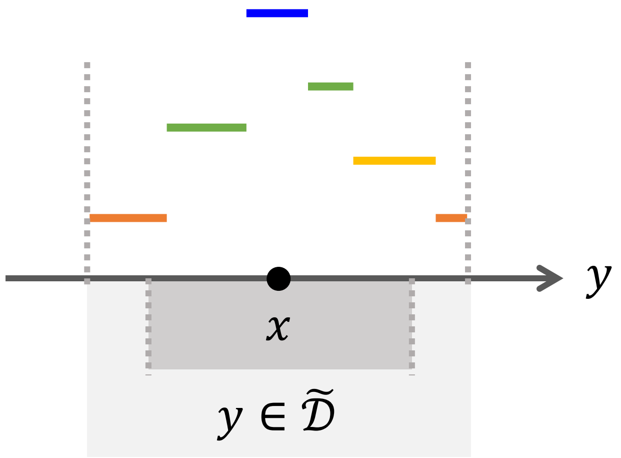


$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_{1,\varepsilon} & \text{if } y \in [l_{1,x,\varepsilon}, r_{1,x,\varepsilon}] \\ p_{2,\varepsilon} & \text{if } y \in [l_{2,x,\varepsilon}, r_{2,x,\varepsilon}] \\ & \dots \\ p_{m,\varepsilon} & \text{if } y \in [l_{m,x,\varepsilon}, r_{m,x,\varepsilon}] \end{cases}$$

$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \leq e^{\varepsilon}$ (LDP constraint)

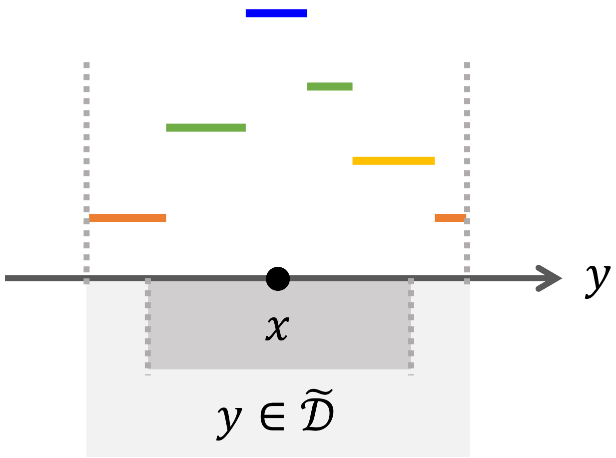
- Expected error:

$$\min_{\mathcal{M}:\, p_i, l_i, r_i} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y$$

Find $\mathcal{M}$ to minimize the error at x

Optimal Piecewise-based Mechanism

- **Most generalized version:** m-piecewise distributions



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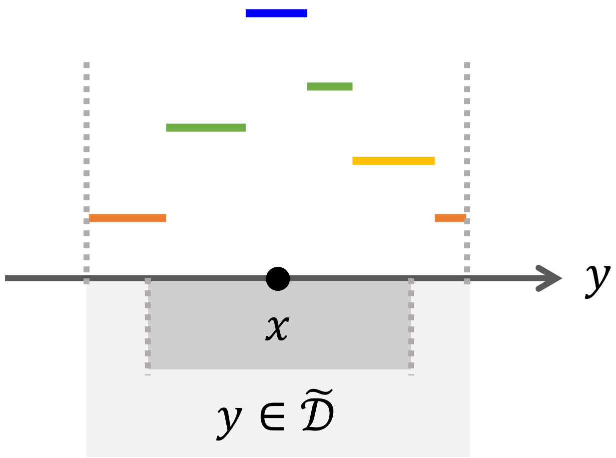

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Solved $\mathcal{M}$ is
the optimal piecewise-based mechanism

Mathematically $\equiv$ to find the **optimal piecewise distribution under the LDP constraint**

Challenges & Proofs

- Challenges

1. min-max problem & multiple variables
2. optimal results only for a specific m

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1. variables p_i, l_i, r_i

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Inner max
has a closed form

Worst-case from
x = endpoints of $\mathcal{D}$

Reduced to:
minimization problem

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Worst-case from $x =$ endpoints of $\mathcal{D}$

Reduced to:
minimization problem

2

The optimal #piece is m

If optimal m-piecewise = $(m+1)$-piecewise

Reduced to:
Check $m = 1,2, \ldots$ until optimal

"Solvable"

NOT Manually "Solvable" When $m \geq 4$

- **Too many variables & non-linear**

$$\max_{x \in \{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x)\, \mathrm{d}y$$

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Monte Carlo testing with 10^4 random samples

Hypothesis. For any domain $\mathcal{D} \rightarrow \mathcal{D}$, under error metrics $\mathcal{L}(y,x) := |y-x|$ and $\mathcal{L}(y,x) := (y-x)^2$, the optimal piecewise-based mechanism falls into 3-piecewise mechanism.

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closed-form $\mathcal{M}$ can be manually solved
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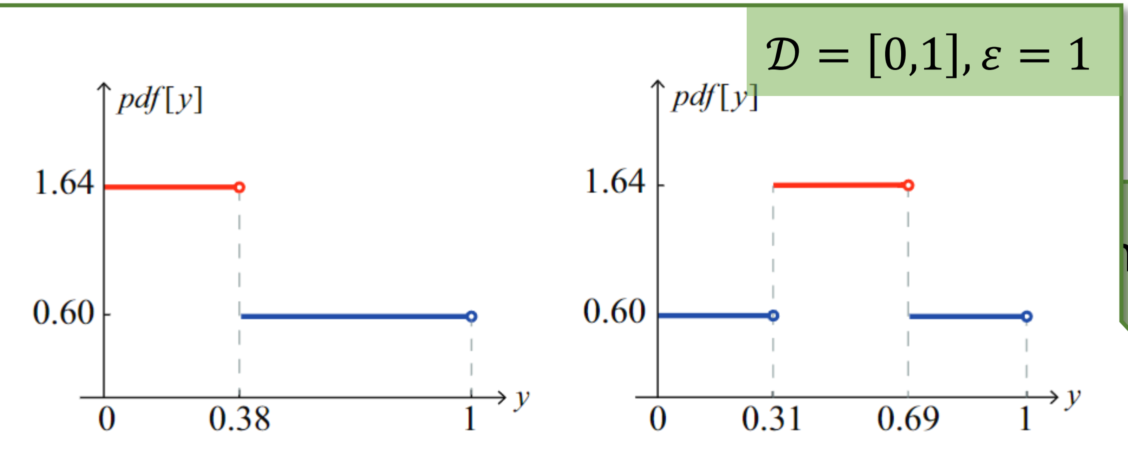

(a) $x = 0$. (b) $x = 0.5$.

Monte Carlo testing
with 10^4 random samples

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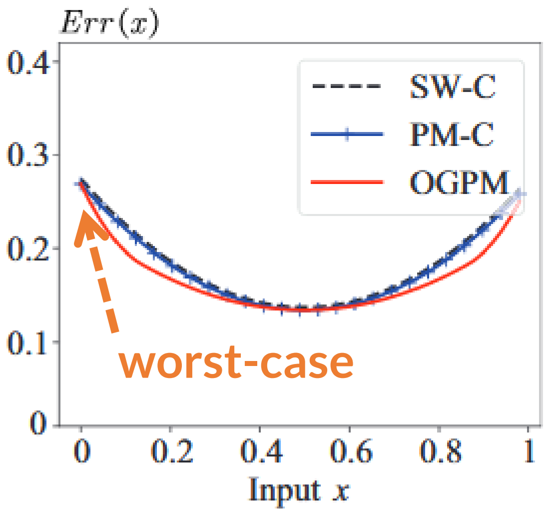
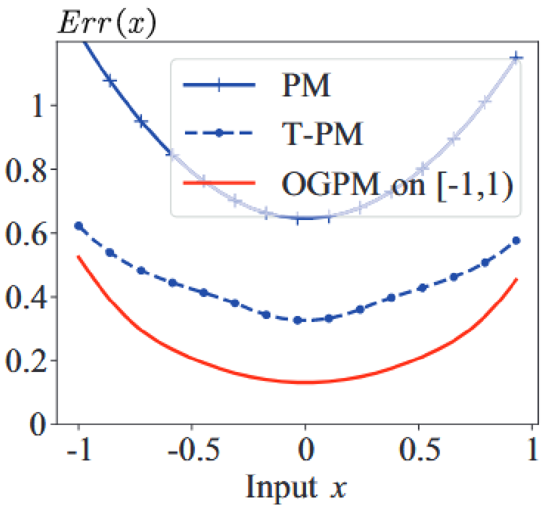
Comparison of Expected Errors

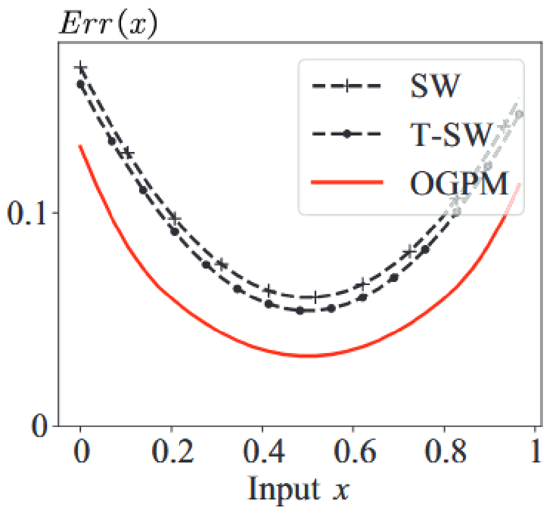
Whole-domain error (i.e. each point in $\mathcal{D}$) ($\varepsilon = 2$)

Compressed PM, SW



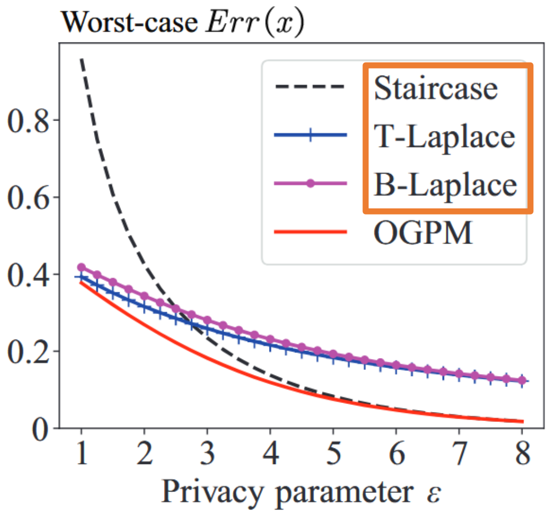
Original and **truncated** PM, SW





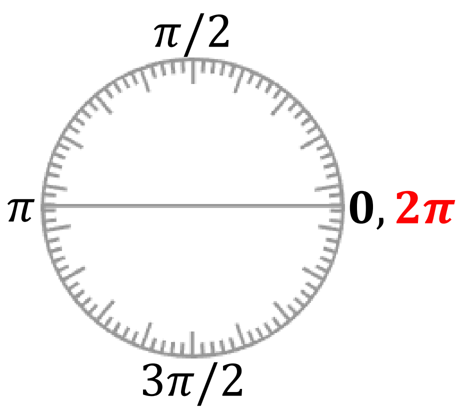
Worst-case error

Non-piecewise-based



Circular Domain

- Different meaning of distance, e.g. $\text{distance}(0, 2\pi) = 0$



$\mathcal{L} \to \mathcal{L}_{\text{mod}}$

$$\mathcal{L}_{\text{mod}}(y, x) \coloneqq \min(\mathcal{L}(y, x), \mathcal{L}(y, 2\pi - x))$$

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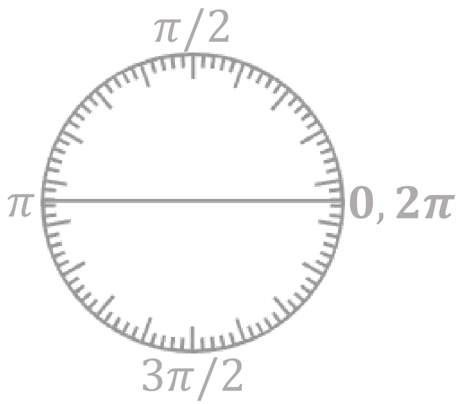

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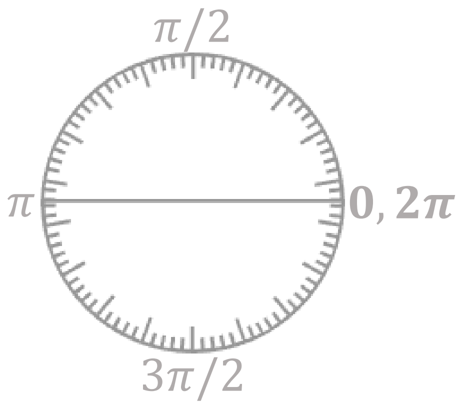

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- **Linking** to problems in the classical domain

min-max under $\mathcal{L}_{\text{mod}}$ = min under $\mathcal{L}$ at π

p_i

min under $\mathcal{L}_{\text{mod}}$ at x = min under $\mathcal{L}$ at π + Transform

$l_{i,x}^{\text{mod}}, r_{i,x}^{\text{mod}}$

Circular Domain

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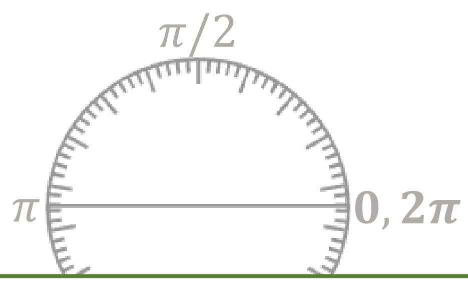

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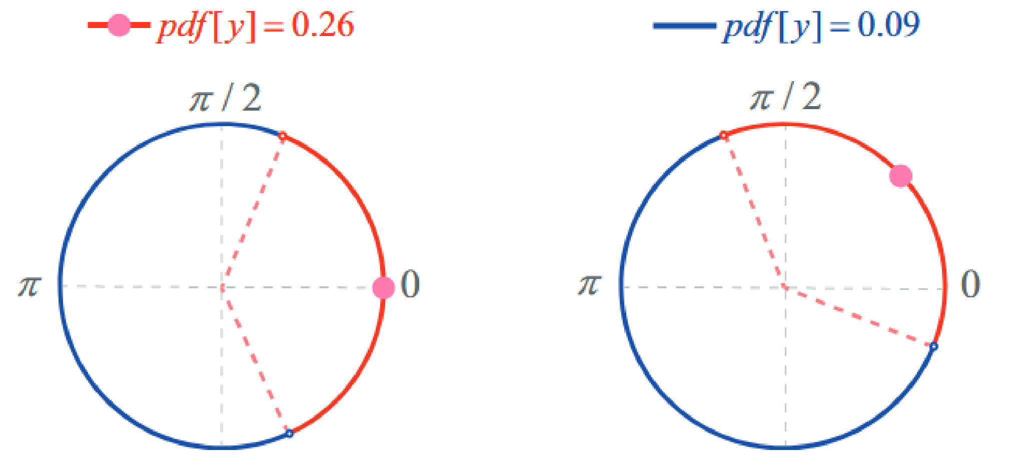


(a) $x = 0$. (b) $x = \pi/4$.

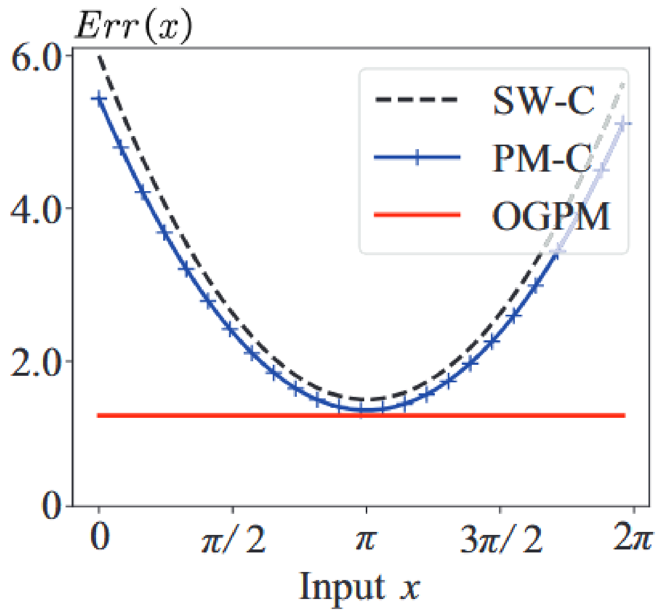
min under $\mathcal{L}_{\text{mod}}$ at x	=	min under $\mathcal{L}$ at π + Transform

$l_{i,x}^{\text{mod}}, r_{i,x}^{\text{mod}}$

Comparison of Expected Errors

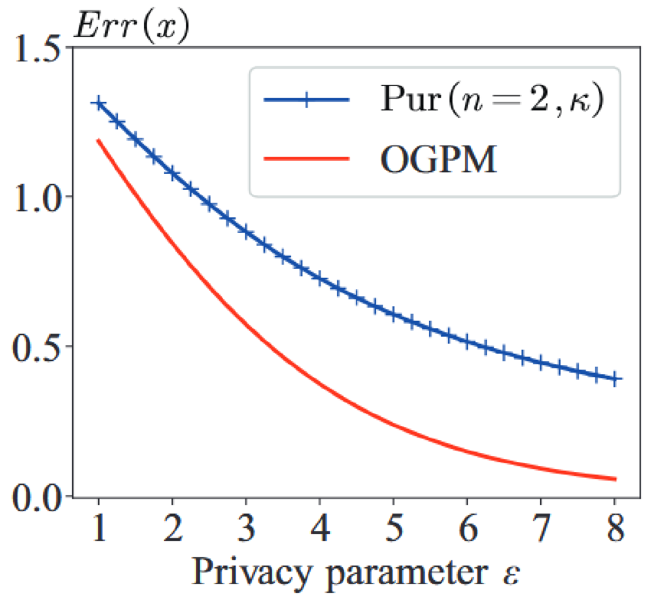
Whole-domain error ($\varepsilon = 2$)

PM-C, SW-C on the **flatten** domain



Worst-case error

Purkayastha mechanism [CCS'21]*



* Differential Privacy for Directional Data

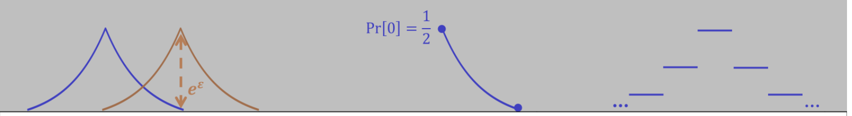
Optimal Piecewise-based Mechanism for Collecting Bounded Numerical Data under LDP

LDP Mechanisms for $\mathcal{D} = [0,1]$

Laplace mechanism: $[0,1] \to (-\infty, +\infty)$

Laplace + truncation: $[0,1] \to [0,1]$

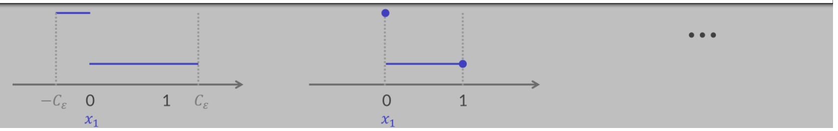
Staircase: $[0,1] \to (-\infty, +\infty)$



Privacy: LDP with the **same** ε

Utility: Different errors

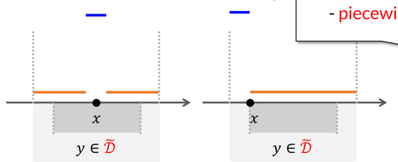
Q: What is the **optimal piecewise-based** mechanism?



3-Piecewise Mechanism

- 3-piecewise distributions on boun…
  - given input x, samples output y f…

NOT enough to study optimality of piecewise-based mechanism
- only 3 pieces, two probabilities
- piecewise distribution can has more pieces and probabilities

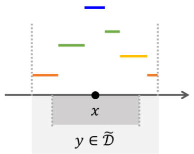


$$pdf[\mathcal{M}(x) = y] = \begin{cases} p_\varepsilon & \text{if } y \in [l_{x,\varepsilon}, r_{x,\varepsilon}] \\ \frac{p_\varepsilon}{\exp(\varepsilon)} & \text{if } y \in \tilde{\mathcal{D}} \setminus [l_{x,\varepsilon}, r_{x,\varepsilon}] \end{cases}$$

- Instantiations: PM [2019], SW [2020], PTT [2024] (design different $p_\varepsilon, l_{x,\varepsilon}, r_{x,\varepsilon}$)
  - different errors, but without optimality

Optimal Piecewise-based Mechanism

- Most generalized version: m-piecewise distributions



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$\frac{p_{i,\varepsilon}}{p_{j,\varepsilon}} \le e^{\varepsilon}$ (LDP constraint)

Solved $\mathcal{M}$ is the optimal piecewise-based mechanism

Mathematically: To find the optimal piecewise distribution under the LDP constraint

$$\min_{\mathcal{M}: p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\tilde{\mathcal{D}}} \mathcal{L}(y,x) \cdot pdf[\mathcal{M}(x) = y] dy$$

Find $\mathcal{M}$ to minimize worst-case error

Challenges & Proofs

- Challenges
  1. min-max problem & multiple variables
  2. optimal results only for a specific m

$$\min_{\mathcal{M}: p_i, l_i, r_i} \max_{x \in \mathcal{D}} \int_{\tilde{\mathcal{D}}} \mathcal{L}(y,x) \cdot pdf[\mathcal{M}(x) = y] dy$$

2. $i \in [m]$ 1. variables p_i, l_i, r_i

1 Inner max has a closed form — Worst-case from x = endpoints of $\mathcal{D}$ — Reduced to: minimization problem

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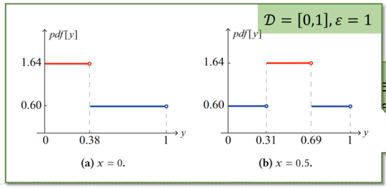

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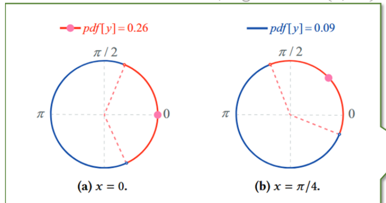

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min under $\mathcal{L}_{\text{mod}}$ at x = min under $\mathcal{L}$ at π + Transform — $l_{i,x}^{\text{mod}}, r_{i,x}^{\text{mod}}$

Thank you!

& Q A

Optimality of LDP Mechanisms

- Optimal error (utility) under privacy level ε

  - many mechanisms are optimal in **order-of-magnitude**, e.g. $\Omega(\frac{1}{\sqrt{n}})$ for the counting query*

  - the staircase mechanism is optimal for **domain** $[0,1] \rightarrow (-\infty, +\infty)^{\dagger}$

  - the geometric mechanism is universally optimal if any **post-processing** is allowed, e.g. truncation††

* The Complexity of Differential Privacy, *book section of "Tutorials on the Foundations of Cryptography", 2017*

† The Staircase Mechanism in Differential Privacy, *journal version of ISIT'14*

†† Universally Utility-maximizing Privacy Mechanisms, *STOC'09*

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- **Specify the utility model** (conditions for optimality)

1 Error metric

$Err(\text{truth}, \text{rand})$

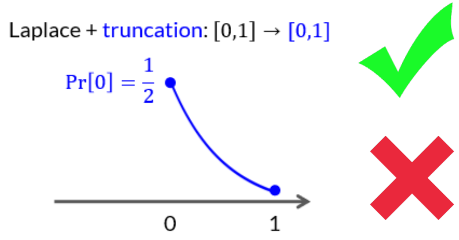
Err or $\boldsymbol{\Omega}(Err)$

2 Data domain & type of mechanisms

Discrete / cont. $\mathcal{D} \rightarrow \widetilde{\mathcal{D}}$

Laplace-shape / piecewise

3 Post-processing



Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x) = y] \mathrm{d}y = \max_{x \in \mathcal{D}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y, x) \, \mathrm{d}y \qquad (m\text{-piecewise distribution})$$

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convex function w.r.t x

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$$= \max_{x\in\{a,b\}} \sum_{i=1}^{m} p_i \int_{l_i}^{r_i} \mathcal{L}(y,x)\,\mathrm{d}y \qquad \text{(maximum principle)}$$

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x \in \mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y, x) \cdot pdf[\mathcal{M}(x)] = \sum^{m} \ldots c^{r_i} \ldots$$

After merging redundant pieces

- Optimal #piece is m **if** optimal m-piecewise = (m+1)-piecewise

Error from an arbitrary piece

(≥ 0 variable)

$$\textbf{if:} \quad \min_{e_1, e_2, e_3} e_1 + e_2 + e_3 = \min_{e_1, e_2, e_3, e_4} e_1 + e_2 + e_3 + e_4$$

i.e. the error can't be lowered by arbitrary ≥ 0 variable

Proof Intuitions

- Worst-case error is achieved at endpoints

$$\max_{x\in\mathcal{D}} \int_{\widetilde{\mathcal{D}}} \mathcal{L}(y,x)\cdot pdf[\mathcal{M}(x) =$$

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(≥ 0 variable)

if: $$\min_{e_1,e_2,e_3} e_1+e_2+e_3 = \min_{e_1,e_2,e_3,e_4} e_1+e_2+e_3+e_4$$

i.e. the error can't be lowered by arbitrary ≥ 0 variable

then: $$= \min_{e_1,e_2,e_3,e_4,e_5} e_1+e_2+e_3+e_4+e_5$$

otherwise, $e_4 \leftarrow e_4 + e_5$ can further lower the error