Spectral Graph Theory

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1 Introduction

1.1 Adjacency Matrix Representation of A Graph

We represent a Graph G with a square adjacency matrix A, where entry

$$a_{ij} = \begin{cases} 1 & \text{if (i, j) is an edge in } G \\ 0 & \text{otherwise} \end{cases}$$

Evidently, the matrix is symmetric, or $a_{ij} = a_{ji}$. Therefore by spectral theorem, we have a orthonormal basis of eigenvectors, with real eigenvalues associated.

2 Counting Paths with Adjacency Matrix

Theorem L. et G be a graph on labeled vertices, let A be its adjacency matrix, and let k be a positive integer. Then $A^k_{i,j}$ is equal to the number of walks from i to j that are of length k .

Proof. The proof is fairly simple, and we will do it by induction.

When $k = 1$, we look at the original adjacency matrix, and $A_{i,j}$ indicates whether there's an edge between i, j, which is a path of length 1.

Now assume that the statement is true for k, and prove it for $k + 1$.

Let's first think about it intuitively, A^k gives the number of paths walks from i to all other points. If one such point is v, then we just need to determine if there's and edge from v to j, if so then we just add the number of walks from i to k.

Let z be any vertex of G. If there are $b_{i,z}$ walks of length k from i to z, and there are $a_{z,j}$ walks of length one (in other words, edges) from z to j, then there are $b_{i,z}a_{z,j}$ walks of length $k+1$ from i to j whose next-to-last vertex is z. Therefore, the number of all walks of length $k+1$ from i to j is:

$$c(i,j) = \sum_{z \in G} b_{i,z} a_{z,j}$$

Since $b_{i,z}$ correspond to an entry in A^k, the formula above is basically a matrix multiplication. $\square$

At this point, it's a good habit to check our proof again. Ask ourselves this question:

We know $A^k_{i,j}$ represents some number of walks from i to j, but does it count all of them?

We'll leave it as a quick thought exercise.

2.1 Connectivity

Theorem L. et G be a simple graph on n vertices, and let A be the adjacency matrix of G. Then G is connected iff $(I+A)^{n-1}$ consists of strictly positive entries.

Proof. We will only give the central idea of the proof here

A path from one point to another consists at most n vertices, or $n-1$ edges. Therefore, if we cannot find a path between two points within $n-1$ edges, the graph is not connected

Using the previous theorem, we know that A_{n-1} gives the number of paths if length $n-1$ between any two points, however, this is **not enough**.

Notice that the theorem indicates it's $(I+A)^{n-1}$ instead of A^{n-1}. So what does the I do?

Graphically, it means that we assume any vertex is connected to itself. What is $(I+A)^{n-1}$ then?

For simplicity, call $(I+A) = A'$. With $A'_{i,i} = 1$, A'^k no longer counts the number of paths of length **strictly** k, but the number of paths of length $\leq k$. This is because we don't have to move to another point every time we multiply A', we can choose to stay at the same point since $A'_{i,i} = 1$.

Therefore $(I+A)^{n-1}$ counts the number of paths of length $\leq n-1$, and if there still exist an 0 entry $A'^{n-1}_{i,j}$, then it means if we cannot find a path between two points within $n-1$ edges, the graph is not connected. $\square$

Some Remarks: Bellman-Ford algorithm also uses the property that a path between two vertices in a connected graph has edge count at most $n-1$ to check for negative cycles.

3 Matrix Tree Theorem (Many Versions)

It turns out that we can use matrix to count the number of spanning trees with matrices. There are many matrix tree theorems, and we here will just talk about a few of them.

3.1 Incidency Matrix

We first define the incidency matrix A for a graph G

The incidency matrix of $G(V, E)$ is a $n \times m$ matrix, where $n = |V|, m = |E|$. We label the edges $e_1, \cdots, e_m$ and vertices $v_1, \cdots, v_n$. Then

$$A_{i,k} = \begin{cases} 1 \text{ if } i \text{ is the head of the edge of } e_k \\ -1 \text{ if } i \text{ is the tail of the edge of } e_k \\ 0 \text{ otherwise} \end{cases}$$