Johnson-Lindenstrauss, Compress Sensing

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1 Johnson-Lindenstrauss (JL)

1.1 Formal Theorem

For a set of n points $x_1, \dots, x_n \in \mathbb{R}^d$, then for a random $k = \frac{c \log n}{\varepsilon^2}$ dimensional subspace, we project these points to this subspace.

JL states that with probability $1 - \frac{1}{n^{c-2}}$, we have:

$$(1-\varepsilon)\sqrt{\frac{k}{d}}\,|x_i - x_j| \leq |y_i - y_j| \leq (1+\varepsilon)\sqrt{\frac{k}{d}}\,|x_i - x_j|$$

Furthermore, y_i is the projection of x_i on to the subspace.

1.2 Intuitive Interpretation

$|x_i - x_j|$ is the original distance between two points, and similar for $|y_i - y_j|$. Therefore the inequality is stating that after projecting to the subspace of dimension k, with high probability that the pariwise distances are scaled by $\sqrt{\frac{k}{d}}$ within factor of $1 \pm \varepsilon$

2 Projection Method

We select a set of k orthogonal unit vectors

$$v_1, \cdots, v_k$$

as the basis of the subspace, and project each vector on to them.

For a vector $x \in x_1, \dots, x_n$, we look at its coordinate vector $y_1, \dots, x_k$ in the projection onto the subspace, where $y_i = x \cdot v_i$

Now, consider a rotation U that transform v_i to e_i (elementary vector where the i-th entry is 1 and all others are 0). Let $z = Ux$, therefore we have y_i be the i-th coordinate of z (we basically transform the coordinate vector to the standard basis).

3 "Expected Length"

Notice that z is still a unit vector, therefore:

$$\sum_{i \in [d]} z_i^2 = 1.$$

After the projection, we only take the first k coordinates:

$$E\left[\sum_{i \in [k]} z_i^2\right] = \frac{k}{d}.\text{ Linearity of Expectation.}$$

(Terefore the length is roughly expected to be $\sqrt{\frac{k}{d}}$, but it's not yet rigorously proven yet).