Real Analysis Theorems

Zhiwei Zhang

May 6, 2019

1 Introduction

Theorem 1.1 (Complete Axiom). *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, sup$S$ exists and is a real number.*

2 Monotone Sequence and Cauchy Sequences

Theorem 2.1. *All bounded monotone sequences converge.*

Definition 2.2. *A sequence (s_n) of real numbers is called a Cauchy sequence if for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$*

Lemma 2.3. *Convergent sequences are Cauchy sequences.*

Theorem 2.4. *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

sketch proof.

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|$$

□

3 Subsequences

Theorem 3.1. *Let (s_n) be a sequence. If t is in $\mathbb{R}$, then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$*

Theorem 3.2. *If the sequence (s_n) converges, then every subsequence converges to the same limit.*

Theorem 3.3. *Every sequence (sn) has a monotonic subsequence.*

Theorem 3.4 (Bolzano-Weierstrass Theorem). *Every bounded sequence has a convergent subsequence.*

Theorem 3.5. *Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is limsup s_n, and there exists a monotonic subsequence whose limit is liminf s_n.*

Theorem 3.6. *Let* (s_n) *be any sequence in* $\mathbb{R}$, *and let* S *denote the set of subsequential limits of* (s_n) *.*

- S *is nonempty*
- $\sup S = \limsup s_n$ *and inf* $S = \liminf s_n$
- $\lim s_n$ *exists if and only if* S *has exactly one element, namely* $\lim s_n$.

Theorem 3.7. *Let* S *denote the set of subsequential limits of a sequence* (s_n) *. Sup- pose* (t_n) *is a sequence in* $S \cap$ *and that* $t = \lim t_n$. *Then* t *belongs to* S *(which means* S *is* ***closed****).*

4 Limsup and Liminf

Theorem 4.1. *If* (s_n) *converges to a positive real number* s *and* (t_n) *is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ *and* $s \cdot (-\infty) = -\infty$ *for* $s > 0$.

Theorem 4.2. *Let* (s_n) *be any sequence of nonzero real numbers. Then we have*

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left| \frac{s_{n+1}}{s_n} \right|$$

Corollary 4.2.1. *If* $\lim \left| \frac{s_{n+1}}{s_n} \right|$ *exists [and equals* L], *then* $\lim |s_n|^{1/n}$ *exists [and equals* L].

5 Series

Theorem 5.1 (Ratio Test)**.** *A series* $\sum a_n$ *of nonzero terms*

1. *Converges absolutely if limsup* $|a_{n+1}/a_n| < 1$
2. *Diverges if liminf* $|a_{n+1}/a_n| > 1$
3. *Otherwise liminf* $|a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ *and the test gives no information.*

Theorem 5.2 (Root Test)**.** *Let* $\sum a_n$ *be a series and let* $\alpha = \limsup |a_n|^{1/n}$ *. The series* $\sum a_n$

1. *converges absolutely if* $\alpha < 1$
2. *diverges if* $\alpha > 1$
3. *ambiguous if* $\alpha = 1$

Theorem 5.3. *If the terms* a_n *are nonzero and if* $\lim |a_{n+1}/a_n| = 1$, *then* $\alpha = \limsup_{|a_n|^{1/n}} = 1$ *by Corollary* 4.2.1, *so neither the Ratio Test nor the Root Test gives information concerning the convergence of* $\sum a_n$

6 Alternating Series and Integral Test

Basic Idea is that to compare a series with an integral to test for divergence or convergence.

Theorem 6.1 (Alternating Series Theorem)**.** *If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $\lim a_n = 0$, then the al- ternating series$\sum(-1)^{n+1}a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ satisfy $|s - s_n| \leq a_n$ for all n.*

7 Continuous Functions

Definition 7.1. *Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is continuous at x_0 in dom(f) if, for every sequence (x_n) in dom(f) converging to x_0, we have lim $_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq$ dom (f), then f is said to be continuous on S. The function f is said to be continuous if it is continuous on* $\mathrm{dom}(f)$.

Theorem 7.2. *Let f be a real-valued function whose domain is a subset of R. Then f is continuous at x_0 in dom(f) if and only if for each $\epsilon > 0, \exists \sigma > 0$ such that, $x \in$ dom$(f)\&|x - x_0| < \sigma$ imply $|f(x) - f(x_0)| < \epsilon$.*

Theorem 7.3. *Let f be a real-valued function with $\mathrm{dom}(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in $\mathrm{dom}(f)$, then $|f|$ and $kf, k \in \mathbb{R}$, are continuous at x_0.*

8 Uniform Continuity

Theorem 8.1. *Pass. ...*

9 Power Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$

Theorem 9.1. *Given any (a_n), one of the following holds true:*

1. *The power series converges for all $x \in \mathbb{R}$*
2. *The power series converges only for $x = 0$*
3. *The power series converges for all x in some bounded interval centered at 0; the intervalmay be open, half-open or closed.*

Theorem 9.2. *Let*

$$\beta = \limsup |a_n|^{1/n} \qquad \textit{and} \qquad R = \frac{1}{\beta}$$

Then

1. *The power series converges for $|x| < R$*

2. The power series diverges for $|x| > R$

Also notice that $\lim \left|\frac{a_{n+1}}{a_n}\right| = \beta$, *therefore most of the time we will use* $\lim \left|\frac{a_{n+1}}{a_n}\right|$ *as it's easier to compute thatn* β.

10 More on Uniform Convergence

Theorem 10.1. *Let* (f_n) *be a sequence of continuous functions on* $[a, b]$, *and suppose* $f_n \to f$ *uniformly on* $[a, b]$. *Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Definition 10.2. *A sequence* (f_n) *of functions defined on a set* $S \subseteq \mathbb{R}$ *is uniformly Cauchy on* S *if*

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$|f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in S \text{ and all } m, n > N$$

Theorem 10.3. *Let* (f_n) *be a sequence of functions defined and uniformly Cauchy on a set* $S \subseteq \mathbb{R}$. *Then there exists a function* f *on* S *such that* $f_n \to f$ *uniformly on* S.

Theorem 10.4. *Consider a series* $\sum_{k=0}^{\infty} g_k$ *of functions on a set* $S \subseteq \mathbb{R}$. *Suppose each* g_k *is continuous on* S *and the series converges uniformly on* S . *Then the series* $\sum_{k=0}^{\infty} g_k$ *represents a continuous function on* S.

Theorem 10.5. *If a series* $\sum_{k=0}^{\infty} g_k$ *of functions satisfies the Cauchy criterion uniformly on a set* S , *then the series converges uniformly on* S .

Theorem 10.6. *Let* (M_k) *be a sequence of nonnegative real numbers where* $\sum M_k < \infty$. *If* $|g_k(x)| \leq M_k$ *for all* x *in a set* S, *then* $\sum g_k$ *converges uniformly on* S.

Theorem 10.7. *Show that if the series* $\sum g_n$ *converges uniformly on a set* S, *then* $\lim_{n\to\infty} \sup \{|g_n(x)| : x \in$
0

11 Differentiation and Integration of Power Series

Theorem 11.1. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with radius of convergence* $R > 0$ [*possibly*$R = +\infty$]. *If* $0 < R_1 < R$, *then the power series converges uniformly on* $[-R_1, R_1]$ *to a continuous function.*

Lemma 11.2. *If the power series* $\sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* R, *then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Theorem 11.3 (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergenceR. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.*

12 Basic Properties of the Derivative

Theorem 12.1. *Differentiability implies continuity.*

13 Mean Value Theorem

Theorem 13.1. *Let f be a continuous function on [a,b] that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Theorem 13.2 (Intermediate Value Theorem for Derivatives)**.** *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$*

Theorem 13.3. *Let f be a one-to-one continuous function on an open intervalI, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

Corollary 13.3.1. *Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).*

Corollary 13.3.2. *Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*

Theorem 13.4 (IVT for derivatives)**.** *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$*

Theorem 13.5 (Rolle's Theorem)**.** *Let f be a continuous function on [a,b] that is differentiable on (a,b) and satisfies $f(a) = f(b)$. There exists [at least one] x in (a, b) such that $f'(x) = 0$*

14 Taylor's Theorem

Definition 14.1. *Taylor's Theorem:*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

Remainder:

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Theorem 14.2 (Taylor's Theorem)**.** *Let f be defined on (a,b) where $a < c < b$; here we allow $a = -\infty$ or $b = \infty$. Suppose the nth derivative $f^{(n)}$ exists on (a, b). Then for each $x \neq c$ in (a, b) there is some y between c and x such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$$

Corollary 14.2.1. *Let f be defined on (a, b) where $a < c < b$. If all the derivatives $f^{(n)}$ exist on (a, b) and are bounded by a single constant C, then*

$$\lim_{n\to\infty} R_n(x) = 0 \quad \textit{for all} \quad x \in (a, b)$$

Theorem 14.3 (Taylor's Theorem (another one))**.** *Let f be defined on (a, b) where $a < c < b$, and suppose the nth derivative $f^{(n)}$ exists and is continuous on (a, b). Then for $x \in (a, b)$ we have*

$$R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$$

Corollary 14.3.1. *If f is as in Theorem 31.5, then for each x in (a, b) different from c there is some y between c and x such that*

$$R_n(x) = (x-c) \cdot \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$$

This form of R_n is known as Cauchy's form of the remainder.

Theorem 14.4 (Binomial Series Theorem)**.** *If $\alpha \in \mathbb{R}$ and $|x| < 1$, then*

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

Theorem 14.5 (Newton's Method)**.** *Newton's method for finding an approximate solution to $f(x) = 0$ is to begin with a reasonable initial guess x_0 and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \textit{for} \quad n \geq 1$$

Theorem 14.6 (Secant Method)**.** *A similar approach to approximating solutions of $f(x) = 0$ is to start with two reasonable guesses x_0 and x_1 and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} \quad \textit{for} \quad n \geq 2$$

15 The Riemann Integral

Definition 15.1.

$$M(f,S) = \sup\{f(x) : x \in S\} \quad \textit{and} \quad m(f,S) = \inf\{f(x) : x \in S\}$$

A partition of $[a,b]$ *is any finite ordered subset* P *having the form*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

The upper Darboux sum $U(f,P)$ *of* f *with respect to* P *is the sum*

$$U(f,P) = \sum_{k=1}^{n} M\left(f, [t_{k-1}, t_k]\right) \cdot (t_k - t_{k-1})$$

and the lower Darboux sum $L(f,P)$ *is*

$$L(f,P) = \sum_{k=1}^{n} m\left(f, [t_{k-1}, t_k]\right) \cdot (t_k - t_{k-1})$$

Lemma 15.2. *Let* f *be a bounded function on* $[a,b]$*. If* P *and* Q *are partitions of* $[a,b]$ *and* $P \subseteq Q$*, then*

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

.

Lemma 15.3. *If* f *is a bounded function on* $[a,b]$*, and if* P *and* Q *are partitions of* $[a,b]$*, then* $L(f,P) \leq U(f,Q)$

Theorem 15.4. *A bounded function* f *on* $[a,b]$ *is integrable if and only if for each* $\epsilon > 0$ *there exists a partition* P *of* $[a,b]$ *such that*

$$U(f,P) - L(f,P) < \epsilon$$

Definition 15.5. *The mesh of a partition* P *is the maximum length of the subintervals comprising* P*. Thus if*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

then

$$\mathrm{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}$$

Theorem 15.6. *A bounded function* f *on* $[a,b]$ *is integrable if and only if for each* $\epsilon > 0$ *there exists a*$\delta > 0$ *such that*

$$\mathrm{mesh}(P) < \delta \quad \textit{implies} \quad U(f,P) - L(f,P) < \epsilon$$

for all partitions P *of* $[a,b]$*.*

Definition 15.7. *The function f is Riemann integrable on $[a, b]$ if there exists a number r with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|S - r| < \epsilon$$

for every Riemann sum S of f associated with a partition P having $\mathrm{mesh}(P) < \delta$.

Theorem 15.8. *A bounded function f on $[a, b]$ is Riemann integrable if and only if it is [Darboux] integrable, in which case the values of the integrals agree.*

Corollary 15.8.1. *Let f be a bounded Riemann integrable function on [a,b]. Suppose (S_n) is a sequence of Riemann sums, with corresponding partitions P_n, satisfying lim $_n \mathrm{mesh}\,(P_n) = 0$. Then the sequence (S_n) converges to $\int_a^b f$.*

Theorem 15.9. *Every monotonic function f on $[a, b]$ is integrable.*

Theorem 15.10. *Every continuous function f on $[a, b]$ is integrable.*

Theorem 15.11. *If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

.

Theorem 15.12. *If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a, b]$, then f is integrable on $[a, b]$.*

Theorem 15.13 (IVT for integrals)**.** *If f is a continuous function on $[a, b]$, then for at least one x in (a, b) we have*

$$f(x) = \frac{1}{b-a} \int_a^b f$$

Theorem 15.14 (Dominated Convergence Theorem)**.** *Suppose (f_n) is a sequence of integrable functions on $[a, b]$ and $f_n \to f$ pointwise where f is an integrable function on $[a, b]$. If there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all n and all x in $[a, b]$, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

Corollary 15.14.1 (Monotone Convergence Theorem)**.** *Suppose (f_n) is a sequence of integrable functions on $[a, b]$ such that $f_1(x) \leq f_2(x) \leq \cdots$ for all x in $[a, b]$. Suppose also that $f_n \to f$ pointwise where f is an integrable function on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

16 Fundamental Theorem of Calculus