Spectral Graph Theory

Zhiwei Zhang

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1 Introduction

1.1 Adjacency Matrix Representation of A Graph

We represent a Graph G with a square adjacency matrix A, where entry

$$a_{ij} = \begin{cases} 1 & \text{if (i, j) is an edge in } G \\ 0 & \text{otherwise} \end{cases}$$

Evidently, the matrix is symmetric, or $a_{ij} = a_{ji}$. Therefore by spectral theorem, we have a orthonormal basis of eigenvectors, with real eigenvalues associated.

2 Counting Paths with Adjacency Matrix

Theorem L. et G be a graph on labeled vertices, let A be its adjacency matrix, and let k be a positive integer. Then $A^k_{i,j}$ is equal to the number of walks from i to j that are of length k .

Proof. The proof is fairly simple, and we will do it by induction.

When $k = 1$, we look at the original adjacency matrix, and $A_{i,j}$ indicates whether there's an edge between i, j, which is a path of length 1.

Now assume that the statement is true for k, and prove it for $k + 1$.

Let's first think about it intuitively, A^k gives the number of paths walks from i to all other points. If one such point is v, then we just need to determine if there's and edge from v to j, if so then we just add the number of walks from i to k.

Let z be any vertex of G. If there are $b_{i,z}$ walks of length k from i to z, and there are $a_{z,j}$ walks of length one (in other words, edges) from z to j, then there are $b_{i,z}a_{z,j}$ walks of length $k+1$ from i to j whose next-to-last vertex is z. Therefore, the number of all walks of length $k+1$ from i to j is:

$$c(i,j) = \sum_{z \in G} b_{i,z} a_{z,j}$$

Since $b_{i,z}$ correspond to an entry in A^k, the formula above is basically a matrix multiplication. $\square$

At this point, it's a good habit to check our proof again. Ask ourselves this question:

We know $A^k_{i,j}$ represents some number of walks from i to j, but does it count all of them?

We'll leave it as a quick thought exercise.

2.1 Connectivity

Theorem L. et G be a simple graph on n vertices, and let A be the adjacency matrix of G. Then G is connected iff $(I+A)^{n-1}$ consists of strictly positive entries.

Proof. We will only give the central idea of the proof here

A path from one point to another consists at most n vertices, or $n-1$ edges. Therefore, if we cannot find a path between two points within $n-1$ edges, the graph is not connected

Using the previous theorem, we know that A_{n-1} gives the number of paths if length $n-1$ between any two points, however, this is **not enough**.

Notice that the theorem indicates it's $(I+A)^{n-1}$ instead of A^{n-1}. So what does the I do?

Graphically, it means that we assume any vertex is connected to itself. What is $(I+A)^{n-1}$ then?

For simplicity, call $(I+A) = A'$. With $A'_{i,i} = 1$, A'^k no longer counts the number of paths of length **strictly** k, but the number of paths of length $\leq k$. This is because we don't have to move to another point every time we multiply A', we can choose to stay at the same point since $A'_{i,i} = 1$.

Therefore $(I+A)^{n-1}$ counts the number of paths of length $\leq n-1$, and if there still exist an 0 entry $A'^{n-1}_{i,j}$, then it means if we cannot find a path between two points within $n-1$ edges, the graph is not connected. $\square$