Spectral Graph Theory – Electric Flow

Zhiwei Zhang

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1 Introduction

In this notes, we are going to discuss the interestingproperties when turning a graph into a network of resistors.

1.1 Electrical Laws

First recall some E&M Laws:

$$\begin{aligned}
I &= \frac{U}{R} && \text{Ohm's law} \\
E &= I^2 R && \text{Energy formula} \\
|I_{v,in}| &= |I_{v,out}| && \text{conservation of flow}
\end{aligned}$$

Note that the last law only holds for nodes that are not source or sink.

1.2 Matrices

Also recall the laplacian of a graph:

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Weighted laplacian of a graph:

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \sim j \\ w_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $w_i = \sum_{j \sim i} w_{ij}$ is the sum of the weights of edges incident on vertex i.

Pseudoinverse of laplacian: Let $0 = \lambda_1 < \lambda_2 \leqslant \ldots \leq \lambda_n$ be the eigenvalues of L_G with associated eigenvectors $u_1, u_2, \ldots, u_n$. Then

$$L_G = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

. The pseudo-inverse of L_G is

$$L_G^+ = \sum_{i=2}^{n} \frac{1}{\lambda_i} u_i n_i^T$$

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1.3 Formation on Graph

We will write a matrix formation of the problem.

- Let $G(V, E)$ be an undirected graph with $|V| = n, |E| = m$.
- Let $v \in \mathbb{R}^n$ be the vector representing the potentials of vertices.
- Edges represent the resistors, and $\forall e(u, v) \in E$. Edge e has resistance r_e.
- Let $f \in \mathbb{R}^m$ representing the flow of all edges, where $f(a, b)$ represents the flow from a to b with. Since $f(a, b)$ is directed, we have $f(a, b) = -f(b, a)$.
- Let weight $w_e = \frac{1}{r_e}$, or the "conductance" of e.

2 Matrix Formation

We also define $f_{ext}(a) = \sum_{b=(a,b)\in E} f(a, b)$, and $f_ex t(a)$ basically denotes the external current on a, which is **positive number if a is souce, negative number with equal magnitude if a is source, and zero otherwise**. So f_{ext} is a very sparse vector.

Ohm's law directly states that $f(a, b) = \frac{v(a)-v(b)}{r_{a,b}} = w_{a,b}(v(a) - v(b))$, therefore

$$\sum_{b:(a,b)\in E} f(a, b) = \sum_{b:(a,b)\in E} w_{a,b}(v(a) - v(b)) = d(a)v(a) - \sum_{b:(a,b)\in E} w_{a,b} v(b)$$

where $d(a) = \sum_{b:(a,b)\in E} w_{a,b}$, the weighted degree of a.

Notice that $d(a), w_{a,b}$ are entries of the weighted laplacian L_G, and through simple verification, we can show that the equation above the equivalent to $L_G v = f_{ext}$

3 Computing Voltages

Since we know that $Nul(L_G) = \vec{1}$, it's trivial to see that for any vector x, $L_G x$ is perpendicular to $\vec{1}$, which implies there's a solution to $L_G v = f_{ext}$ iff f_{ext} is perpendicular to 1. This is also simply true since the two non-zero entries of f_{ext} have the same magnitude with opposite sign.

Therefore $v = L_G^+ f_{ext}$ is the only solution with $v \perp \vec{1}$, and the whole set of solution is $\{v + c\vec{1} | c \in \mathbb{R}\}$

This also makes sense in the physical way, as if we increase the potential of all nodes, the physical flow or energy will not change as electrical potentials are only significant when taking differences.

4 Computing Currents

First we reintroduce the incidence matrix B of dimension $m \times n$, and the row corresponding to the edge $e = (a, b)$ where $a < b$ is $(x_a - x_b)^T$, where x_a is the characteristic with the only non-zero entry at $a - th$ entry of value 1.

Let w be the mxm diagonal matrix where $W_{e,e} = w_e$ is the weight of edge e.

Notice that Bv gives the potential difference on each edge, and therefore $f = WBv$.

It's also true that $L_G = \sum_{e:(a,b)} w_e (x_a - x_b)(x_a - x_b)^T = B^T W B$.

Therefore $f_{ext} = L_G v = B^T W B v = B^T f$

5 Random Walk and Effective Resistance