Spectral Graph Theory – Electric Flow

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1 Introduction

In this notes, we are going to discuss the interestingproperties when turning a graph into a network of resistors.

1.1 Electrical Laws

First recall some E&M Laws:

$$
\begin{aligned}
I &= \frac{U}{R} && \text{Ohm's law}\\
E &= I^2R && \text{Energy formula}\\
|I_{v,in}| &= |I_{v,out}| && \text{conservation of flow}
\end{aligned}
$$

Note that the last law only holds for nodes that are not source or sink.

1.2 Matrices

Also recall the laplacian of a graph:

$$
L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}
$$

Weighted laplacian of a graph:

$$
L_{ij} = \begin{cases} -w_{ij} & \text{if } i \sim j \\ w_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
$$

where $w_i = \sum_{j \sim i} w_{ij}$ is the sum of the weights of edges incident on vertex i.

Pseudoinverse of laplacian: Let $0 = \lambda_1 < \lambda_2 \leqslant \ldots \leq \lambda_n$ be the eigenvalues of L_G with associated eigenvectors $u_1, u_2, \ldots, u_n$. Then

$$L_G = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

. The pseudo-inverse of L_G is

$$L_G^+ = \sum_{i=2}^{n} \frac{1}{\lambda_i} u_i n_i^T$$

.

1.3 Formation on Graph

We will write a matrix formation of the problem.

- Let $G(V, E)$ be an undirected graph with $|V| = n, |E| = m$.
- Let $v \in \mathbb{R}^n$ be the vector representing the potentials of vertices.
- Edges represent the resistors, and $\forall e(u, v) \in E$. Edge e has resistance r_e.
- Let $f \in \mathbb{R}^m$ representing the flow of all edges, where $f(a, b)$ represents the flow from a to b with. Since $f(a, b)$ is directed, we have $f(a, b) = -f(b, a)$.
- Let weight $w_e = \frac{1}{r_e}$, or the "conductance" of e.

2 Matrix Formation

We also define $f_{ext}(a) = \sum_{b=(a,b)\in E} f(a, b)$, and $f_{ex}t(a)$ basically denotes the external current on a, which is **positive number if a is souce, negative number with equal magnitude if a is source, and zero otherwise**. So f_{ext} is a very sparse vector.

Ohm's law directly states that $f(a, b) = \frac{v(a)-v(b)}{r_{a,b}} = w_{a,b}(v(a) - v(b))$, therefore

$$\sum_{b:(a,b)\in E} f(a, b) = \sum_{b:(a,b)\in E} w_{a,b}(v(a) - v(b)) = d(a)v(a) - \sum_{b:(a,b)\in E} w_{a,b}v(b)$$

where $d(a) = \sum_{b:(a,b)\in E} w_{a,b}$, the weighted degree of a.

Notice that $d(a), w_{a,b}$ are entries of the weighted laplacian L_G, and through simple verification, we can show that the equation above the equivalent to $L_G v = f_{ext}$