# Real Analysis Theorems

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## 1 Introduction

**Theorem 1.1** (Complete Axiom). *Every nonempty subset $S$ of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, sup$S$ exists and is a real number.*

## 2 Monotone Sequence and Cauchy Sequences

**Theorem 2.1.** *All bounded monotone sequences converge.*

**Definition 2.2.** *A sequence $(s_n)$ of real numbers is called a Cauchy sequence if for each $\epsilon > 0$ there exists a number $N$ such that $m, n > N$ implies $|s_n - s_m| < \epsilon$*

**Lemma 2.3.** *Convergent sequences are Cauchy sequences.*

**Theorem 2.4.** *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

*sketch proof.*

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|$$

□

## 3 Subsequences

**Theorem 3.1.** *Let $(s_n)$ be a sequence. If $t$ is in $\mathbb{R}$, then there is a subsequence of $(s_n)$ converging to $t$ if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$*

**Theorem 3.2.** *If the sequence $(s_n)$ converges, then every subsequence converges to the same limit.*

**Theorem 3.3.** *Every sequence $(sn)$ has a monotonic subsequence.*

**Theorem 3.4** (Bolzano-Weierstrass Theorem). *Every bounded sequence has a convergent subsequence.*

**Theorem 3.5.** *Let $(s_n)$ be any sequence. There exists a monotonic subsequence whose limit is limsup $s_n$, and there exists a monotonic subsequence whose limit is liminf $s_n$.*

**Theorem 3.6.** *Let $(s_n)$ be any sequence in $\mathbb{R}$, and let $S$ denote the set of subsequential limits of $(s_n)$.*

- *$S$ is nonempty*
- $\sup S = \limsup s_n$ *and inf* $S = \liminf s_n$
- $\lim s_n$ *exists if and only if $S$ has exactly one element, namely* $\lim s_n$.

**Theorem 3.7.** *Let $S$ denote the set of subsequential limits of a sequence $(s_n)$. Sup- pose $(t_n)$ is a sequence in $S\cap$ and that $t = \lim t_n$. Then $t$ belongs to $S$ (which means $S$ is* ***closed****).*

# 4 Limsup and Liminf

**Theorem 4.1.** *If $(s_n)$ converges to a positive real number $s$ and $(t_n)$ is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n$$

*Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.*

**Theorem 4.2.** *Let $(s_n)$ be any sequence of nonzero real numbers. Then we have*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

**Corollary 4.2.1.** *If* $\lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists [ and equals $L$], then* $\lim |s_n|^{1/n}$ *exists [ and equals $L$].*

# 5 Series

**Theorem 5.1** (Ratio Test)**.** *A series $\sum a_n$ of nonzero terms*

1. *Converges absolutely if limsup $|a_{n+1}/a_n| < 1$*
2. *Diverges if liminf $|a_{n+1}/a_n| > 1$*
3. *Otherwise liminf $|a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ and the test gives no information.*

**Theorem 5.2** (Root Test)**.** *Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$*

1. *converges absolutely if $\alpha < 1$*
2. *diverges if $\alpha > 1$*
3. *ambiguous if $\alpha = 1$*

**Theorem 5.3.** *If the terms $a_n$ are nonzero and if $\lim |a_{n+1}/a_n| = 1$, then $\alpha = \limsup_{|a_n|^{1/n}} = 1$ by Corollary 4.2.1, so neither the Ratio Test nor the Root Test gives information concerning the convergence of $\sum a_n$*

# 6 Alternating Series and Integral Test

**Basic Idea is that to compare a series with an integral to test for divergence or convergence.**

**Theorem 6.1** (Alternating Series Theorem). *If* $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ *and* $\lim a_n = 0$, *then the al- ternating series* $\sum(-1)^{n+1} a_n$ *converges. Moreover, the partial sums* $s_n = \sum_{k=1}^{n}(-1)^{k+1} a_k$ *satisfy* $|s - s_n| \leq a_n$ *for all* $n$.

# 7 Continuous Functions

**Definition 7.1.** *Let* $f$ *be a real-valued function whose domain is a subset of* $\mathbb{R}$ *. The function* $f$ *is continuous at* $x_0$ *in dom(f) if, for every sequence* $(x_n)$ *in dom(f) converging to* $x_0$*, we have lim* ${}_n f\left(x_n\right)=f\left(x_0\right)$*. If* $f$ *is continuous at each point of a set* $S \subseteq$ *dom* $(f)$*, then* $f$ *is said to be continuous on* $S$*. The function* $f$ *is said to be continuous if it is continuous on* $\operatorname{dom}(f)$.

**Theorem 7.2.** *Let* $f$ *be a real-valued function whose domain is a subset of* $R$*. Then* $f$ *is continuous at* $x_0$ *in dom(f) if and only if for each* $\epsilon > 0, \exists \sigma > 0$ *such that,* $x \in dom(f) \& |x - x_0| < \sigma$ *imply* $|f(x) - f(x_0)| < \epsilon$.

**Theorem 7.3.** *Let* $f$ *be a real-valued function with* $\operatorname{dom}(f) \subseteq \mathbb{R}$*. If* $f$ *is continuous at* $x_0$ *in* $\operatorname{dom}(f)$*, then* $|f|$ *and* $kf, k \in \mathbb{R}$*, are continuous at* $x_0$.

# 8 Property of Continuous Functions

**Theorem 8.1.** *Let* $f$ *be a continuous real-valued function on a closed interval* $[a, b]$ *. Then* $f$ *is a bounded function. Moreover,* $f$ *assumes its maximum and minimum values on* $[a, b]$*; that is, there exist* $x_0, y_0$ *in* $[a, b]$ *such that* $f(x_0) \leq f(x) \leq f(y_0)$ *for all* $x \in [a, b]$.

**Theorem 8.2.** *If* $f$ *is a continuous real-valued function on an interval* $I$*, then* $f$ *has the intermediate value property on* $I$ *: Whenever* $a, b \in I, a < b$ *and* $y$ *lies between* $f(a)$ *and* $f(b)$ *i.e.,* $f(a) < y < f(b)$ *or* $f(b) < y < f(a)]$ *there exists at least one* $x$ *in* $(a, b)$ *such that* $f(x) = y$.

**Theorem 8.3.** *If* $f$ *is a continuous real-valued function on an interval* $I$*, then the set* $f(I) = \{f(x) : x \in I\}$ *is also an interval or a single point.*

**Theorem 8.4.** *Let* $g$ *be a strictly increasing function on an interval* $J$ *such that* $g(J)$ *is an interval I. Then* $g$ *is continuous on* $J$ *.*

**Theorem 8.5.** *Let* $f$ *be a one-to-one continuous function on an interval I. Then* $f$ *is strictly increasing* $[x_1 < x_2$ *implies* $f\left(x_1\right) < f\left(x_2\right)]$ *or strictly decreasing* $[x_1 < x_2$ *implies* $f\left(x_1\right) > f\left(x_2\right)]$

# 9 Uniform Continuity

**Definition 9.1.** *Let $f$ be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then $f$ is uniformly continuous on $S$ if for each $\epsilon > 0, \exists \sigma > 0$ such that, $x, y \in \operatorname{dom}(f) \& |x - y| < \sigma$ imply $|f(x) - f(y)| < \epsilon$.*

**Theorem 9.2.** *If $f$ is continuous on a closed interval $[a, b]$, then $f$ is uniformly continuous on $[a, b]$.*

**Theorem 9.3.** *If $f$ is uniformly continuous on a set $S$ and $(s_n)$ is a Cauchy sequence in $S$, then $(f(s_n))$ is a Cauchy sequence.*

**Definition 9.4** (Extension of a fucntion)**.** *We say a function $\tilde{f}$ is an extension of a function $f$ if*

$$\operatorname{dom}(f) \subseteq \operatorname{dom}(\tilde{f}) \quad \text{and} \quad f(x) = \tilde{f}(x) \text{ for all } x \in \operatorname{dom}(f)$$

**Theorem 9.5.** *A real-valued function $f$ on $(a, b)$ is uniformly continuous on $(a, b)$ if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$*

**Theorem 9.6.** *Let $f$ be a continuous function on an interval $I$ [$I$ may be bounded or unbounded]. Let $I^\circ$ be the interval obtained by removing from $I$ any endpoints that happen to be in I. If $f$ is differentiable on $I^\circ$ and if $f'$ is bounded on $I^\circ$ , then $f$ is uniformly continuous on $I$.*

# 10 Power Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$

**Theorem 10.1.** *Given any $(a_n)$, one of the following holds true:*

1. *The power series converges for all $x \in \mathbb{R}$*
2. *The power series converges only for $x = 0$*
3. *The power series converges for all $x$ in some bounded interval centered at $0$; the intervalmay be open, half-open or closed.*

**Theorem 10.2.** *Let*

$$\beta = \limsup |a_n|^{1/n} \quad \text{and} \quad R = \frac{1}{\beta}$$

*Then*

1. *The power series converges for $|x| < R$*
2. *The power series diverges for $|x| > R$*

*Also notice that $\lim \left|\frac{a_{n+1}}{a_n}\right| = \beta$, therefore most of the time we will use $\lim \left|\frac{a_{n+1}}{a_n}\right|$ as it's easier to compute thatn $\beta$.*

# 11 Uniform Convergence

**Definition 11.1** (**Pointwise Convergence**). *Let $(f_n)$ be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence $(f_n)$ converges pointwise [i.e., at each point] to a function $f$ defined on $S$ if*

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in S$$

**Definition 11.2** (**Uniform Convergence**). *Let $(f_n)$ be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence $(f_n)$ converges uniformly on $S$ to a function $f$ defined on $S$ if for each $\epsilon > 0$ there exists a number $N$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$ and all $n > N$*

**Theorem 11.3.** *The uniform limit of continuous functions is continuous. More pre- cisely, let $(f_n)$ be a sequence of functions on a set $S \subseteq \mathbb{R}$, suppose $f_n \to f$ uniformly on $S$, and suppose $S = \operatorname{dom}(f)$. If each $f_n$ is continuous at $x_0$ in $S$ , then $f$ is continuous at $x_0$.[So if each $f_n$ is continuous on $S$, then $f$ is continuous on $S$.]*

*proof sketch.* $\epsilon/3$ **argument.**

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

□

# 12 More on Uniform Convergence

**Theorem 12.1.** *Let $(f_n)$ be a sequence of continuous functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

**Definition 12.2.** *A sequence $(f_n)$ of functions defined on a set $S \subseteq \mathbb{R}$ is uniformly Cauchy on $S$ if*

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$|f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in S \text{ and all } m, n > N$$

**Theorem 12.3.** *Let $(f_n)$ be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then there exists a function $f$ on $S$ such that $f_n \to f$ uniformly on $S$.*

**Theorem 12.4.** *Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each $g_k$ is continuous on $S$ and the series converges uniformly on $S$ . Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on $S$.*

**Theorem 12.5.** *If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set $S$ , then the series converges uniformly on $S$ .*

**Theorem 12.6.** *Let $(M_k)$ be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all $x$ in a set $S$, then $\sum g_k$ converges uniformly on $S$.*

**Theorem 12.7.** *Show that if the series $\sum g_n$ converges uniformly on a set $S$, then $\lim_{n\to\infty} \sup\{|g_n(x)| : x \in$
0*

# 13 Differentiation and Integration of Power Series

**Theorem 13.1.** *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly$R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

**Lemma 13.2.** *If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R$, then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

*and*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

*also have radius of convergence $R$.*

**Theorem 13.3** (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence$R$. If the series converges at $x = R$, then $f$ is continuous at $x = R$. If the series converges at $x = -R$, then $f$ is continuous at $x = -R$.*

# 14 Basic Properties of the Derivative

**Theorem 14.1.** *Differentiability implies continuity.*

# 15 Mean Value Theorem

**Theorem 15.1.** *Let $f$ be a continuous function on [a,b] that is differentiable on $(a, b)$. Then there exists [at least one] $x$ in $(a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

**Theorem 15.2** (Intermediate Value Theorem for Derivatives)**.** *Let $f$ be a differentiable function on $(a, b)$. If $a < x_1 < x_2 < b$, and if $c$ lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] $x$ in $(x_1, x_2)$ such that $f'(x) = c$*

**Theorem 15.3.** *Let $f$ be a one-to-one continuous function on an open interval$I$, and let $J = f(I)$. If $f$ is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then $f^{-1}$ is differentiable at $y_0 = f(x_0)$ and*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

**Corollary 15.3.1.** *Let $f$ be a differentiable function on $(a, b)$ such that $f'(x) = 0$ for all $x \in (a, b)$. Then $f$ is a constant function on $(a, b)$.*

**Corollary 15.3.2.** *Let $f$ and $g$ be differentiable functions on $(a, b)$ such that $f' = g'$ on $(a, b)$. Then there exists a constant $c$ such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*

**Theorem 15.4** (IVT for derivatives). *Let $f$ be a differentiable function on $(a, b)$. If $a < x_1 < x_2 < b$, and if $c$ lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] $x$ in $(x_1, x_2)$ such that $f'(x) = c$*

**Theorem 15.5** (Rolle's Theorem). *Let $f$ be a continuous function on [a,b] that is differentiable on (a,b) and satisfies $f(a) = f(b)$. There exists [at least one] $x$ in $(a, b)$ such that $f'(x) = 0$*

# 16 Taylor's Theorem

**Definition 16.1.** *Taylor's Theorem:*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

*Remainder:*

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

**Theorem 16.2** (Taylor's Theorem). *Let $f$ be defined on (a,b) where $a < c < b$; here we allow $a = -\infty$ or $b = \infty$. Suppose the nth derivative $f^{(n)}$ exists on $(a, b)$. Then for each $x \neq c$ in $(a, b)$ there is some $y$ between $c$ and $x$ such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$$

**Corollary 16.2.1.** *Let $f$ be defined on $(a, b)$ where $a < c < b$. If all the derivatives $f^{(n)}$ exist on $(a, b)$ and are bounded by a single constant $C$, then*

$$\lim_{n \to \infty} R_n(x) = 0 \quad \textit{for all} \quad x \in (a, b)$$

**Theorem 16.3** (Taylor's Theorem (another one)). *Let $f$ be defined on $(a, b)$ where $a < c < b$, and suppose the nth derivative $f^{(n)}$ exists and is continuous on $(a, b)$. Then for $x \in (a, b)$ we have*

$$R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$$

**Corollary 16.3.1.** *If $f$ is as in Theorem 31.5, then for each $x$ in $(a, b)$ different from $c$ there is some $y$ between $c$ and $x$ such that*

$$R_n(x) = (x-c) \cdot \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$$

*This form of $R_n$ is known as Cauchy's form of the remainder.*

**Theorem 16.4** (Binomial Series Theorem). *If $\alpha \in \mathbb{R}$ and $|x| < 1$, then*

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

**Theorem 16.5** (Newton's Method)**.** *Newton's method for finding an approximate solution to $f(x) = 0$ is to begin with a reasonable initial guess $x_0$ and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{for} \quad n \geq 1$$

**Theorem 16.6** (Secant Method)**.** *A similar approach to approximating solutions of $f(x) = 0$ is to start with two reasonable guesses $x_0$ and $x_1$ and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} \quad \text{for} \quad n \geq 2$$

# 17 The Riemann Integral

**Definition 17.1.**

$$M(f, S) = \sup\{f(x) : x \in S\} \quad \text{and} \quad m(f, S) = \inf\{f(x) : x \in S\}$$

*A partition of $[a, b]$ is any finite ordered subset $P$ having the form*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

*The upper Darboux sum $U(f, P)$ of $f$ with respect to $P$ is the sum*

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

*and the lower Darboux sum $L(f, P)$ is*

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

**Lemma 17.2.** *Let $f$ be a bounded function on $[a, b]$. If $P$ and $Q$ are partitions of $[a, b]$ and $P \subseteq Q$, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

.

**Lemma 17.3.** *If $f$ is a bounded function on $[a, b]$, and if $P$ and $Q$ are partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$*

**Theorem 17.4.** *A bounded function $f$ on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition $P$ of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \epsilon$$

**Definition 17.5.** *The mesh of a partition $P$ is the maximum length of the subintervals comprising $P$. Thus if*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

*then*

$$\operatorname{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}$$

**Theorem 17.6.** *A bounded function $f$ on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a$\delta > 0$ such that*

$$\text{mesh}(P) < \delta \quad \textit{implies} \quad U(f, P) - L(f, P) < \epsilon$$

*for all partitions $P$ of $[a, b]$.*

**Definition 17.7.** *The function $f$ is Riemann integrable on $[a, b]$ if there exists a number $r$ with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|S - r| < \epsilon$$

*for every Riemann sum $S$ of $f$ associated with a partition $P$ having* $\text{mesh}(P) < \delta$.

**Theorem 17.8.** *A bounded function $f$ on $[a, b]$ is Riemann integrable if and only if it is [ Darboux] integrable, in which case the values of the integrals agree.*

**Corollary 17.8.1.** *Let $f$ be a bounded Riemann integrable function on [a,b]. Suppose $(S_n)$ is a sequence of Riemann sums, with corresponding partitions $P_n$, satisfying lim $_n$* $\text{mesh}\,(P_n) =$ 0. *Then the sequence $(S_n)$ converges to $\int_a^b f$.*

**Theorem 17.9.** *Every monotonic function $f$ on $[a, b]$ is integrable.*

**Theorem 17.10.** *Every continuous function $f$ on $[a, b]$ is integrable.*

**Theorem 17.11.** *If $f$ is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

.

**Theorem 17.12.** *If $f$ is a piecewise continuous function or a bounded piecewise monotonic function on $[a, b]$, then $f$ is integrable on $[a, b]$.*

**Theorem 17.13** (IVT for integrals)**.** *If $f$ is a continuous function on $[a, b]$, then for at least one $x$ in $(a, b)$ we have*

$$f(x) = \frac{1}{b - a} \int_a^b f$$

**Theorem 17.14** (Dominated Convergence Theorem)**.** *Suppose $(f_n)$ is a sequence of integrable functions on $[a, b]$ and $f_n \to f$ pointwise where $f$ is an integrable function on $[a, b]$ . If there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n$ and all $x$ in $[a, b]$, then*

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n \to \infty} f_n(x)dx$$

**Corollary 17.14.1** (Monotone Convergence Theorem)**.** *Suppose $(f_n)$ is a sequence of integrable functions on $[a, b]$ such that $f_1(x) \leq f_2(x) \leq \cdots$ for all $x$ in $[a, b]$. Suppose also that $f_n \to f$ pointwise where $f$ is an integrable function on $[a, b]$. Then*

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n \to \infty} f_n(x)dx$$

# 18 Fundamental Theorem of Calculus

**Theorem 18.1.** *If $g$ is a continuous function on [a,b] that is differentiable on $(a, b)$ and if $g'$ is integrable on $[a, b]$, then*

$$\int_a^b g' = g(b) - g(a).$$

**Theorem 18.2.** *Let $f$ be an integrable function on $[a, b]$. For $x$ in $[a, b]$, let*

$$F(x) = \int_a^x f(t)dt$$

*Then $F$ is continuous on $[a, b]$. If $f$ is continuous at $x_0$ in $(a, b)$, then $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$.*