

Spectral Graph Theory

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1 Introduction

1.1 Adjacency Matrix Representation of A Graph

We represent a Graph G with a square adjacency matrix A , where entry

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is an edge in } G \\ 0 & \text{otherwise} \end{cases}$$

Evidently, the matrix is symmetric, or $a_{ij} = a_{ji}$. Therefore by spectral theorem, we have a orthonormal basis of eigenvectors, with real eigenvalues associated.

2 Counting Paths with Adjacency Matrix

Theorem L. Let G be a graph on labeled vertices, let A be its adjacency matrix, and let k be a positive integer. Then $A_{i,j}^k$ is equal to the number of walks from i to j that are of length k .

Proof. The proof is fairly simple, and we will do it by induction.

When $k = 1$, we look at the original adjacency matrix, and $A_{i,j}$ indicates whether there's an edge between i, j , which is a path of length 1.

Now assume that the statement is true for k , and prove it for $k + 1$.

Let's first think about it intuitively, A^k gives the number of paths walks from i to all other points. If one such point is v , then we just need to determine if there's an edge from v to j , if so then we just add the number of walks from i to v .

Let z be any vertex of G . If there are $b_{i,z}$ walks of length k from i to z , and there are $a_{z,j}$ walks of length one (in other words, edges) from z to j , then there are $b_{i,z}a_{z,j}$ walks of

length $k + 1$ from i to j whose next-to-last vertex is z . Therefore, the number of all walks of length $k + 1$ from i to j is:

$$c(i, j) = \sum_{z \in G} b_{i,z} a_{z,j}$$

Since $b_{i,z}$ correspond to an entry in A^k , the formula above is basically a matrix multiplication. \square