# Real Analysis Theorems

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# 1 Limsup and Liminf

**Corollary 1.0.1.** *If* $\lim \left| \frac{s_{n+1}}{s_n} \right|$ *exists [ and equals L], then* $\lim |s_n|^{1/n}$ *exists [ and equals L].*

# 2 Uniform Continuity

**Theorem 2.1.** *Pass. ...*

# 3 Power Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$

**Theorem 3.1.** *Given any* $(a_n)$, *one of the following holds true:*

1. *The power series converges for all* $x \in \mathbb{R}$

2. *The power series converges only for* $x = 0$

3. *The power series converges for all* $x$ *in some bounded interval centered at* $0$; *the interval may be open, half-open or closed.*

**Theorem 3.2.** *Let*

$$\beta = \limsup |a_n|^{1/n} \quad and \quad R = \frac{1}{\beta}$$

*Then*

1. *The power series converges for* $|x| < R$

2. *The power series diverges for* $|x| > R$

*Also notice that* $\lim \left| \frac{a_{n+1}}{a_n} \right| = \beta$, *therefore most of the time we will use* $\lim \left| \frac{a_{n+1}}{a_n} \right|$ *as it's easier to compute thatn* $\beta$.

# 4 More on Uniform Convergence

**Theorem 4.1.** *Let $(f_n)$ be a sequence of continuous functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then*

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

**Definition 4.2.** *A sequence $(f_n)$ of functions defined on a set $S \subseteq \mathbb{R}$ is uniformly Cauchy on $S$ if*

*for each $\epsilon > 0$ there exists a number $N$ such that*
*$|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$ and all $m, n > N$*

**Theorem 4.3.** *Let $(f_n)$ be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then there exists a function $f$ on $S$ such that $f_n \to f$ uniformly on $S$.*

**Theorem 4.4.** *Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each $g_k$ is continuous on $S$ and the series converges uniformly on $S$. Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on $S$.*

**Theorem 4.5.** *If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set $S$, then the series converges uniformly on $S$.*

**Theorem 4.6.** *Let $(M_k)$ be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \le M_k$ for all $x$ in a set $S$, then $\sum g_k$ converges uniformly on $S$.*

**Theorem 4.7.** *Show that if the series $\sum g_n$ converges uniformly on a set $S$, then $\lim_{n \to \infty} \sup \{|g_n(x)| : x \in S$*
*0*

# 5 Differentiation and Integration of Power Series

**Theorem 5.1.** *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly $R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

**Lemma 5.2.** *If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R$, then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

*and*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

*also have radius of convergence $R$.*

**Theorem 5.3** (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence $R$. If the series converges at $x = R$, then $f$ is continuous at $x = R$. If the series converges at $x = -R$, then $f$ is continuous at $x = -R$.*

# 6 Basic Properties of the Derivative

**Theorem 6.1.** *Differentiability implies continuity.*

# 7 Mean Value Theorem

**Theorem 7.1.** *Let $f$ be a continuous function on $[a,b]$ that is differentiable on $(a,b)$. Then there exists [at least one] $x$ in $(a,b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

**Theorem 7.2** (Intermediate Value Theorem for Derivatives). *Let $f$ be a differentiable function on $(a,b)$. If $a < x_1 < x_2 < b$, and if $c$ lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] $x$ in $(x_1, x_2)$ such that $f'(x) = c$*

**Theorem 7.3.** *Let $f$ be a one-to-one continuous function on an open interval $I$, and let $J = f(I)$. If $f$ is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then $f^{-1}$ is differentiable at $y_0 = f(x_0)$ and*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$