Real Analysis Theorems

Zhiwei Zhang

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1 Limsup and Liminf

Corollary 1.0.1. *If* $\lim\left|\frac{s_{n+1}}{s_n}\right|$ *exists [and equals* L*], then* $\lim|s_n|^{1/n}$ *exists [and equals* L*].*

2 Power Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$

Theorem 2.1. *Given any* (a_n)*, one of the following holds true:*

1. *The power series converges for all* $x \in \mathbb{R}$
2. *The power series converges only for* $x = 0$
3. *The power series converges for all* x *in some bounded interval centered at* 0*; the intervalmay be open, half-open or closed.*

Theorem 2.2. *Let*

$$\beta = \limsup |a_n|^{1/n} \qquad \textit{and} \qquad R = \frac{1}{\beta}$$

Then

1. *The power series converges for* $|x| < R$
2. *The power series diverges for* $|x| > R$

Also notice that $\lim\left|\frac{a_{n+1}}{a_n}\right| = \beta$*, therefore most of the time we will use* $\lim\left|\frac{a_{n+1}}{a_n}\right|$ *as it's easier to compute thatn* β*.*

3 More on Uniform Convergence

Theorem 3.1. *Let* (f_n) *be a sequence of continuous functions on* $[a, b]$*, and suppose* $f_n \to f$ *uniformly on* $[a, b]$*. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Definition 3.2. *A sequence (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is uniformly Cauchy on S if*

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$|f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in S \text{ and all } m, n > N$$

Theorem 3.3. *Let (f_n) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then there exists a function f on S such that $f_n \to f$ uniformly on S.*

Theorem 3.4. *Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S . Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.*

Theorem 3.5. *If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S , then the series converges uniformly on S .*

Theorem 3.6. *Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S.*

Theorem 3.7. *Show that if the series $\sum g_n$ converges uniformly on a set S, then $\lim_{n\to\infty} \sup \{|g_n(x)| : x \in S$*
0

4 Differentiation and Integration of Power Series

Theorem 4.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly$R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

Lemma 4.2. *If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Theorem 4.3 (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence$R$. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.*