Spectral Graph Theory

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1 Introduction

1.1 Adjacency Matrix Representation of A Graph

We represent a Graph G with a square adjacency matrix A, where entry

$$a_{ij} = \begin{cases} 1 & \text{if (i, j) is an edge in } G \\ 0 & \text{otherwise} \end{cases}$$

Evidently, the matrix is symmetric, or $a_{ij} = a_{ji}$. Therefore by spectral theorem, we have a orthonormal basis of eigenvectors, with real eigenvalues associated.

2 Counting Paths with Adjacency Matrix

Theorem L. et G be a graph on labeled vertices, let A be its adjacency matrix, and let k be a positive integer. Then $A^k_{i,j}$ is equal to the number of walks from i to j that are of length k .

Proof. The proof is fairly simple, and we will do it by induction.

When $k = 1$, we look at the original adjacency matrix, and $A_{i,j}$ indicates whether there's an edge between i, j, which is a path of length 1.

Now assume that the statement is true for k, and prove it for $k + 1$.

Let's first think about it intuitively, A^k gives the number of paths walks from i to all other points. If one such point is v, then we just need to determine if there's and edge from v to j, if so then we just add the number of walks from i to k.

Let z be any vertex of G. If there are $b_{i,z}$ walks of length k from i to z, and there are $a_{z,j}$ walks of length one (in other words, edges) from z to j, then there are $b_{i,z}a_{z,j}$ walks of length $k+1$ from i to j whose next-to-last vertex is z. Therefore, the number of all walks of length $k+1$ from i to j is:

$$c(i,j) = \sum_{z \in G} b_{i,z} a_{z,j}$$

Since $b_{i,z}$ correspond to an entry in A^k, the formula above is basically a matrix multiplication. □

At this point, it's a good habit to check our proof again. Ask ourselves this question:

We know $A^k_{i,j}$ represents some number of walks from i to j, but does it count all of them?

We'll leave it as a quick thought exercise.

2.1 Connectivity

Theorem L. et G be a simple graph on n vertices, and let A be the adjacency matrix of G. Then G is connected iff $(I+A)^{n-1}$ consists of strictly positive entries.