Real Analysis Theorems

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1 Limsup and Liminf

Corollary 1.0.1. *If* $\lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists [and equals* L*], then* $\lim |s_n|^{1/n}$ *exists [and equals* L*].*

2 Uniform Continuity

Theorem 2.1. *Pass. ...*

3 Power Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$

Theorem 3.1. *Given any* (a_n)*, one of the following holds true:*

1. *The power series converges for all* $x \in \mathbb{R}$
2. *The power series converges only for* $x = 0$
3. *The power series converges for all* x *in some bounded interval centered at* 0*; the intervalmay be open, half-open or closed.*

Theorem 3.2. *Let*

$$\beta = \limsup |a_n|^{1/n} \quad \text{and} \quad R = \frac{1}{\beta}$$

Then

1. *The power series converges for* $|x| < R$
2. *The power series diverges for* $|x| > R$

Also notice that $\lim \left|\frac{a_{n+1}}{a_n}\right| = \beta$*, therefore most of the time we will use* $\lim \left|\frac{a_{n+1}}{a_n}\right|$ *as it's easier to compute thatn* β*.*

4 More on Uniform Convergence

Theorem 4.1. *Let (f_n) be a sequence of continuous functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Definition 4.2. *A sequence (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is uniformly Cauchy on S if*

for each $\epsilon > 0$ there exists a number N such that
$|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$ and all $m, n > N$

Theorem 4.3. *Let (f_n) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then there exists a function f on S such that $f_n \to f$ uniformly on S.*

Theorem 4.4. *Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S . Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.*

Theorem 4.5. *If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S , then the series converges uniformly on S .*

Theorem 4.6. *Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S.*

Theorem 4.7. *Show that if the series $\sum g_n$ converges uniformly on a set S, then $\lim_{n\to\infty} \sup\{|g_n(x)| : x \in$* 0

5 Differentiation and Integration of Power Series

Theorem 5.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly$R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

Lemma 5.2. *If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Theorem 5.3 (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence$R$. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.*

6 Basic Properties of the Derivative

Theorem 6.1. *Differentiability implies continuity.*

7 Mean Value Theorem

Theorem 7.1. *Let f be a continuous function on [a,b] that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Theorem 7.2 (Intermediate Value Theorem for Derivatives)**.** *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$*

Theorem 7.3. *Let f be a one-to-one continuous function on an open intervalI, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

Corollary 7.3.1. *Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).*

Corollary 7.3.2. *Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*

Theorem 7.4 (IVT for derivatives)**.** *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$*

Theorem 7.5 (Rolle's Theorem)**.** *Let f be a continuous function on [a,b] that is differentiable on (a,b) and satisfies $f(a) = f(b)$. There exists [at least one] x in (a, b) such that $f'(x) = 0$*

8 Taylor's Theorem

Definition 8.1. *Taylor's Theorem:*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

Remainder:

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

Theorem 8.2 (Taylor's Theorem)**.** *Let f be defined on (a,b) where $a < c < b$; here we allow $a = -\infty$ or $b = \infty$. Suppose the nth derivative $f^{(n)}$ exists on (a, b). Then for each $x \neq c$ in (a, b) there is some y between c and x such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$$

Corollary 8.2.1. *Let f be defined on (a, b) where $a < c < b$. If all the derivatives $f^{(n)}$ exist on (a, b) and are bounded by a single constant C, then*

$$\lim_{n\to\infty} R_n(x) = 0 \quad \textit{for all} \quad x \in (a,b)$$

Theorem 8.3 (Taylor's Theorem (another one))**.** *Let f be defined on (a, b) where $a < c < b$, and suppose the nth derivative $f^{(n)}$ exists and is continuous on (a, b). Then for $x \in (a, b)$ we have*

$$R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$$

Corollary 8.3.1. *If f is as in Theorem 31.5, then for each x in (a, b) different from c there is some y between c and x such that*

$$R_n(x) = (x-c) \cdot \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$$

This form of R_n is known as Cauchy's form of the remainder.

Theorem 8.4 (Binomial Series Theorem)**.** *If $\alpha \in \mathbb{R}$ and $|x| < 1$, then*

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

Theorem 8.5 (Newton's Method)**.** *Newton's method for finding an approximate solution to $f(x) = 0$ is to begin with a reasonable initial guess x_0 and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \textit{for} \quad n \geq 1$$

Theorem 8.6 (Secant Method)**.** *A similar approach to approximating solutions of $f(x) = 0$ is to start with two reasonable guesses x_0 and x_1 and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} \quad \textit{for} \quad n \geq 2$$

9 The Riemann Integral

Definition 9.1.

$$M(f, S) = \sup\{f(x) : x \in S\} \quad \textit{and} \quad m(f, S) = \inf\{f(x) : x \in S\}$$

A partition of $[a,b]$ *is any finite ordered subset* P *having the form*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

The upper Darboux sum $U(f,P)$ *of* f *with respect to* P *is the sum*

$$U(f,P) = \sum_{k=1}^{n} M\left(f, [t_{k-1}, t_k]\right) \cdot (t_k - t_{k-1})$$

and the lower Darboux sum $L(f,P)$ *is*

$$L(f,P) = \sum_{k=1}^{n} m\left(f, [t_{k-1}, t_k]\right) \cdot (t_k - t_{k-1})$$

Lemma 9.2. *Let* f *be a bounded function on* $[a,b]$*. If* P *and* Q *are partitions of* $[a,b]$ *and* $P \subseteq Q$*, then*

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

.

Lemma 9.3. *If* f *is a bounded function on* $[a,b]$*, and if* P *and* Q *are partitions of* $[a,b]$*, then* $L(f,P) \le U(f,Q)$

Theorem 9.4. *A bounded function* f *on* $[a,b]$ *is integrable if and only if for each* $\epsilon > 0$ *there exists a partition* P *of* $[a,b]$ *such that*

$$U(f,P) - L(f,P) < \epsilon$$

Definition 9.5. *The mesh of a partition* P *is the maximum length of the subintervals comprising* P*. Thus if*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

then

$$\mathrm{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}$$

Theorem 9.6. *A bounded function* f *on* $[a,b]$ *is integrable if and only if for each* $\epsilon > 0$ *there exists a*$\delta > 0$ *such that*

$$\mathrm{mesh}(P) < \delta \quad \textit{implies} \quad U(f,P) - L(f,P) < \epsilon$$

for all partitions P *of* $[a,b]$*.*

Definition 9.7. *The function* f *is Riemann integrable on* $[a,b]$ *if there exists a number* r *with the following property. For each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|S - r| < \epsilon$$

for every Riemann sum S *of* f *associated with a partition* P *having* $\mathrm{mesh}(P) < \delta$*.*

Theorem 9.8. *A bounded function f on $[a,b]$ is Riemann integrable if and only if it is [Darboux] integrable, in which case the values of the integrals agree.*

Corollary 9.8.1. *Let f be a bounded Riemann integrable function on [a,b]. Suppose (S_n) is a sequence of Riemann sums, with corresponding partitions P_n, satisfying lim $_n$ $\operatorname{mesh}(P_n) = 0$. Then the sequence (S_n) converges to $\int_a^b f$.*

Theorem 9.9. *Every monotonic function f on $[a,b]$ is integrable.*

Theorem 9.10. *Every continuous function f on $[a,b]$ is integrable.*

Theorem 9.11. *If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

.

Theorem 9.12. *If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a,b]$, then f is integrable on $[a,b]$.*

Theorem 9.13 (IVT for integrals)**.** *If f is a continuous function on $[a,b]$, then for at least one x in (a,b) we have*

$$f(x) = \frac{1}{b-a}\int_a^b f$$

Theorem 9.14 (Dominated Convergence Theorem)**.** *Suppose (f_n) is a sequence of integrable functions on $[a,b]$ and $f_n \to f$ pointwise where f is an integrable function on $[a,b]$. If there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all n and all x in $[a,b]$, then*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$