Real Analysis Theorems

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1 Introduction

Theorem 1.1 (Complete Axiom). *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, sup$S$ exists and is a real number.*

2 Monotone Sequence and Cauchy Sequences

Theorem 2.1. *All bounded monotone sequences converge.*

Definition 2.2. *A sequence (s_n) of real numbers is called a Cauchy sequence if for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$*

Lemma 2.3. *Convergent sequences are Cauchy sequences.*

Theorem 2.4. *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

sketch proof.

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|$$

□

3 Subsequences

Theorem 3.1. *Let (s_n) be a sequence. If t is in $\mathbb{R}$, then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$*

Theorem 3.2. *If the sequence (s_n) converges, then every subsequence converges to the same limit.*

Theorem 3.3. *Every sequence (sn) has a monotonic subsequence.*

Theorem 3.4 (Bolzano-Weierstrass Theorem). *Every bounded sequence has a convergent subsequence.*

Theorem 3.5. *Let (s_n) be any sequence. There exists a monotonic subsequence whose limit is limsup s_n, and there exists a monotonic subsequence whose limit is liminf s_n.*

Theorem 3.6. *Let (s_n) be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of (s_n).*

- *S is nonempty*
- $\sup S = \limsup s_n$ *and inf* $S = \liminf s_n$
- $\lim s_n$ *exists if and only if S has exactly one element, namely* $\lim s_n$.

Theorem 3.7. *Let S denote the set of subsequential limits of a sequence (s_n). Sup- pose (t_n) is a sequence in $S\cap$ and that $t = \lim t_n$. Then t belongs to S (which means S is **closed**).*

4 Limsup and Liminf

Theorem 4.1. *If (s_n) converges to a positive real number s and (t_n) is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n$$

Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.

Theorem 4.2. *Let (s_n) be any sequence of nonzero real numbers. Then we have*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Corollary 4.2.1. *If* $\lim \left|\frac{s_{n+1}}{s_n}\right|$ *exists [and equals L], then* $\lim |s_n|^{1/n}$ *exists [and equals L].*

5 Series

Theorem 5.1 (Ratio Test)**.** *A series $\sum a_n$ of nonzero terms*

1. *Converges absolutely if limsup $|a_{n+1}/a_n| < 1$*
2. *Diverges if liminf $|a_{n+1}/a_n| > 1$*
3. *Otherwise liminf $|a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ and the test gives no information.*

Theorem 5.2 (Root Test)**.** *Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$*

1. *converges absolutely if $\alpha < 1$*
2. *diverges if $\alpha > 1$*
3. *ambiguous if $\alpha = 1$*

Theorem 5.3. *If the terms a_n are nonzero and if $\lim |a_{n+1}/a_n| = 1$, then $\alpha = \limsup_{|a_n|^{1/n}} = 1$ by Corollary 4.2.1, so neither the Ratio Test nor the Root Test gives information concerning the convergence of $\sum a_n$*

6 Alternating Series and Integral Test

Basic Idea is that to compare a series with an integral to test for divergence or convergence.

Theorem 6.1 (Alternating Series Theorem). *If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $\lim a_n = 0$, then the al- ternating series$\sum(-1)^{n+1} a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n}(-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$ for all n.*

7 Continuous Functions

Definition 7.1. *Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is continuous at x_0 in dom(f) if, for every sequence (x_n) in dom(f) converging to x_0, we have lim $_n f(x_n) = f(x_0)$. If f is continuous at each point of a set $S \subseteq$ dom (f), then f is said to be continuous on S. The function f is said to be continuous if it is continuous on* $\operatorname{dom}(f)$.

Theorem 7.2. *Let f be a real-valued function whose domain is a subset of R. Then f is continuous at x_0 in dom(f) if and only if for each $\epsilon > 0, \exists \sigma > 0$ such that, $x \in$ dom$(f) \& |x - x_0| < \sigma$ imply $|f(x) - f(x_0)| < \epsilon$.*

Theorem 7.3. *Let f be a real-valued function with $\operatorname{dom}(f) \subseteq \mathbb{R}$. If f is continuous at x_0 in $\operatorname{dom}(f)$, then $|f|$ and $kf, k \in \mathbb{R}$, are continuous at x_0.*

8 Property of Continuous Functions

Theorem 8.1. *Let f be a continuous real-valued function on a closed interval $[a, b]$. Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a, b]$; that is, there exist x_0, y_0 in $[a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.*

Theorem 8.2. *If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I : Whenever $a, b \in I, a < b$ and y lies between $f(a)$ and $f(b)$ i.e., $f(a) < y < f(b)$ or $f(b) < y < f(a)]$ there exists at least one x in (a, b) such that $f(x) = y$.*

Theorem 8.3. *If f is a continuous real-valued function on an interval I, then the set $f(I) = \{f(x) : x \in I\}$ is also an interval or a single point.*

Theorem 8.4. *Let g be a strictly increasing function on an interval J such that $g(J)$ is an interval I. Then g is continuous on J .*

Theorem 8.5. *Let f be a one-to-one continuous function on an interval I. Then f is strictly increasing $[x_1 < x_2$ implies $f(x_1) < f(x_2)]$ or strictly decreasing $[x_1 < x_2$ implies $f(x_1) > f(x_2)]$*

9 Uniform Continuity

Definition 9.1. *Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is uniformly continuous on S if for each $\epsilon > 0, \exists \sigma > 0$ such that, $x, y \in \operatorname{dom}(f) \& |x-y| < \sigma$ imply $|f(x) - f(y)| < \epsilon$.*

Theorem 9.2. *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Theorem 9.3. *If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.*

Definition 9.4 (Extension of a fucntion). *We say a function $\tilde{f}$ is an extension of a function f if*

$$\operatorname{dom}(f) \subseteq \operatorname{dom}(\tilde{f}) \quad \textit{and} \quad f(x) = \tilde{f}(x) \textit{ for all } x \in \operatorname{dom}(f)$$

Theorem 9.5. *A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$*

Theorem 9.6. *Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I° be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I° and if f' is bounded on I° , then f is uniformly continuous on I.*

10 Power Series

Given power series $\sum_{n=0}^{\infty} a_n x^n$

Theorem 10.1. *Given any (a_n), one of the following holds true:*

1. *The power series converges for all $x \in \mathbb{R}$*
2. *The power series converges only for $x = 0$*
3. *The power series converges for all x in some bounded interval centered at 0; the intervalmay be open, half-open or closed.*

Theorem 10.2. *Let*

$$\beta = \limsup |a_n|^{1/n} \quad \textit{and} \quad R = \frac{1}{\beta}$$

Then

1. *The power series converges for $|x| < R$*
2. *The power series diverges for $|x| > R$*

Also notice that $\lim \left|\frac{a_{n+1}}{a_n}\right| = \beta$, therefore most of the time we will use $\lim \left|\frac{a_{n+1}}{a_n}\right|$ as it's easier to compute thatn β.

11 Uniform Convergence

Definition 11.1 (**Pointwise Convergence**). *Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) converges pointwise [i.e., at each point] to a function f defined on S if*

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in S$$

Definition 11.2 (**Uniform Convergence**). *Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) converges uniformly on S to a function f defined on S if for each $\epsilon > 0$ there exists a number N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in S$ and all $n > N$*

Theorem 11.3. *The uniform limit of continuous functions is continuous. More pre- cisely, let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, suppose $f_n \to f$ uniformly on S, and suppose $S = \operatorname{dom}(f)$. If each f_n is continuous at x_0 in S , then f is continuous at x_0.[So if each f_n is continuous on S, then f is continuous on S.]*

proof sketch. $\epsilon/3$ **argument.**

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

□

12 More on Uniform Convergence

Theorem 12.1. *Let (f_n) be a sequence of continuous functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Definition 12.2. *A sequence (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is uniformly Cauchy on S if*

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$|f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in S \text{ and all } m, n > N$$

Theorem 12.3. *Let (f_n) be a sequence of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then there exists a function f on S such that $f_n \to f$ uniformly on S.*

Theorem 12.4. *Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$. Suppose each g_k is continuous on S and the series converges uniformly on S . Then the series $\sum_{k=0}^{\infty} g_k$ represents a continuous function on S.*

Theorem 12.5. *If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S , then the series converges uniformly on S .*

Theorem 12.6. *Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \leq M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S.*

Theorem 12.7. *Show that if the series $\sum g_n$ converges uniformly on a set S, then $\lim_{n\to\infty} \sup \{|g_n(x)| : x \in$*
0

13 Differentiation and Integration of Power Series

Theorem 13.1. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly$R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.*

Lemma 13.2. *If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R.

Theorem 13.3 (Abel's Theorem)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence$R$. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.*

14 Basic Properties of the Derivative

Theorem 14.1. *Differentiability implies continuity.*

15 Mean Value Theorem

Theorem 15.1. *Let f be a continuous function on [a,b] that is differentiable on (a, b). Then there exists [at least one] x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Theorem 15.2 (Intermediate Value Theorem for Derivatives)**.** *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$*

Theorem 15.3. *Let f be a one-to-one continuous function on an open interval$I$, and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

Corollary 15.3.1. *Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).*

Corollary 15.3.2. *Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*

Theorem 15.4 (IVT for derivatives)**.** *Let f be a differentiable function on (a, b). If $a < x_1 < x_2 < b$, and if c lies between $f'(x_1)$ and $f'(x_2)$, there exists [at least one] x in (x_1, x_2) such that $f'(x) = c$*

Theorem 15.5 (Rolle's Theorem)**.** *Let f be a continuous function on [a,b] that is differentiable on (a,b) and satisfies $f(a) = f(b)$. There exists [at least one] x in (a, b) such that $f'(x) = 0$*

16 Taylor's Theorem

Definition 16.1. *Taylor's Theorem:*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Remainder:

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Theorem 16.2 (Taylor's Theorem)**.** *Let f be defined on (a,b) where $a < c < b$; here we allow $a = -\infty$ or $b = \infty$. Suppose the nth derivative $f^{(n)}$ exists on (a, b). Then for each $x \neq c$ in (a, b) there is some y between c and x such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n$$

Corollary 16.2.1. *Let f be defined on (a, b) where $a < c < b$. If all the derivatives $f^{(n)}$ exist on (a, b) and are bounded by a single constant C, then*

$$\lim_{n\to\infty} R_n(x) = 0 \quad \textit{for all} \quad x \in (a, b)$$

Theorem 16.3 (Taylor's Theorem (another one))**.** *Let f be defined on (a, b) where $a < c < b$, and suppose the nth derivative $f^{(n)}$ exists and is continuous on (a, b). Then for $x \in (a, b)$ we have*

$$R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$$

Corollary 16.3.1. *If f is as in Theorem 31.5, then for each x in (a, b) different from c there is some y between c and x such that*

$$R_n(x) = (x-c) \cdot \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$$

This form of R_n is known as Cauchy's form of the remainder.

Theorem 16.4 (Binomial Series Theorem)**.** *If $\alpha \in \mathbb{R}$ and $|x| < 1$, then*

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k$$

Theorem 16.5 (Newton's Method)**.** *Newton's method for finding an approximate solution to $f(x) = 0$ is to begin with a reasonable initial guess x_0 and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{for} \quad n \geq 1$$

Theorem 16.6 (Secant Method)**.** *A similar approach to approximating solutions of $f(x) = 0$ is to start with two reasonable guesses x_0 and x_1 and then compute*

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-2} - x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} \quad \text{for} \quad n \geq 2$$

17 The Riemann Integral

Definition 17.1.

$$M(f, S) = \sup\{f(x) : x \in S\} \quad \text{and} \quad m(f, S) = \inf\{f(x) : x \in S\}$$

A partition of $[a, b]$ is any finite ordered subset P having the form

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

The upper Darboux sum $U(f, P)$ of f with respect to P is the sum

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

and the lower Darboux sum $L(f, P)$ is

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

Lemma 17.2. *Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and $P \subseteq Q$, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

.

Lemma 17.3. *If f is a bounded function on $[a, b]$, and if P and Q are partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$*

Theorem 17.4. *A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \epsilon$$

Definition 17.5. *The mesh of a partition P is the maximum length of the subintervals comprising P. Thus if*

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

then

$$\text{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}$$

Theorem 17.6. *A bounded function f on $[a,b]$ is integrable if and only if for each $\epsilon > 0$ there exists a$\delta > 0$ such that*

$$\mathrm{mesh}(P) < \delta \quad \textit{implies} \quad U(f,P) - L(f,P) < \epsilon$$

for all partitions P of $[a,b]$.

Definition 17.7. *The function f is Riemann integrable on $[a,b]$ if there exists a number r with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|S - r| < \epsilon$$

for every Riemann sum S of f associated with a partition P having $\mathrm{mesh}(P) < \delta$.

Theorem 17.8. *A bounded function f on $[a,b]$ is Riemann integrable if and only if it is [Darboux] integrable, in which case the values of the integrals agree.*

Corollary 17.8.1. *Let f be a bounded Riemann integrable function on [a,b]. Suppose (S_n) is a sequence of Riemann sums, with corresponding partitions P_n, satisfying lim $_n \,\mathrm{mesh}\,(P_n) = 0$. Then the sequence (S_n) converges to $\int_a^b f$.*

Theorem 17.9. *Every monotonic function f on $[a,b]$ is integrable.*

Theorem 17.10. *Every continuous function f on $[a,b]$ is integrable.*

Theorem 17.11. *If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

.

Theorem 17.12. *If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a,b]$, then f is integrable on $[a,b]$.*

Theorem 17.13 (IVT for integrals)**.** *If f is a continuous function on $[a,b]$, then for at least one x in (a,b) we have*

$$f(x) = \frac{1}{b-a}\int_a^b f$$

Theorem 17.14 (Dominated Convergence Theorem)**.** *Suppose (f_n) is a sequence of integrable functions on $[a,b]$ and $f_n \to f$ pointwise where f is an integrable function on $[a,b]$. If there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all n and all x in $[a,b]$, then*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

Corollary 17.14.1 (Monotone Convergence Theorem)**.** *Suppose (f_n) is a sequence of integrable functions on $[a,b]$ such that $f_1(x) \leq f_2(x) \leq \cdots$ for all x in $[a,b]$. Suppose also that $f_n \to f$ pointwise where f is an integrable function on $[a,b]$. Then*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

18 Fundamental Theorem of Calculus