

PH 712 Probability and Statistical Inference

Part VII: Point Estimation II

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Consistency of MLE

- For an iid sample, under certain conditions:
 - $\hat{\theta}_{\text{ML}} \approx \theta$ as $n \rightarrow \infty$

Example of inconsistent MLE

There are independent $X_{i1}, X_{i2} \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, \dots, n$. Then $\widehat{\sigma^2}_{\text{ML}}$ is NOT consistent for σ^2 .

Asymptotic efficiency of MLE

- For an iid sample, under certain conditions (more than those for consistency):
 - $\sqrt{n}(\hat{\theta}_{\text{ML}} - \theta) \approx \mathcal{N}(0, I_1^{-1}(\theta))$ as $n \rightarrow \infty$
 - * For an iid sample, $I_1(\theta) = n^{-1}I_n(\theta)$, no longer a function of n

Implications of consistency and asymptotic efficiency of MLE

- As $n \rightarrow \infty$,
 - $E(\hat{\theta}_{\text{ML}}) \approx \theta$ (asymptotically unbiased)
 - $\text{var}(\hat{\theta}_{\text{ML}}) \approx I_n^{-1}(\theta)$ (asymptotically attains CRLB)
 - $\text{MSE}(\hat{\theta}_{\text{ML}}) \approx I_n^{-1}(\theta)$
- Promoting the use of MLE in practice
- Approximating the distribution of MLE
 - As $n \rightarrow \infty$, $\hat{\theta}_{\text{ML}} - \theta \approx \mathcal{N}(0, I_n^{-1}(\theta))$ or equiv. $\hat{\theta}_{\text{ML}} \approx \mathcal{N}(\theta, I_n^{-1}(\theta))$
- Unknown Fisher information $I_n(\theta)$? Recall that $I_n(\theta)$ may be approximated by the observed Fisher information $\hat{I}_n(\hat{\theta}_{\text{ML}})$ (as mentioned in Part V when introducing CRLB)
 - Observed Fisher information (i.e., the minus Hessian evaluated at $\hat{\theta}_{\text{ML}}$): $\hat{I}_n(\hat{\theta}_{\text{ML}}) = -\ell''(\hat{\theta}_{\text{ML}})$

Example Lec7.1

- Suppose X_1, \dots, X_n is an iid sample following $p_{X_i}(x | \theta) = \theta^x(1 - \theta)^{1-x}\mathbf{1}_{\{0,1\}}(x)$, $\theta \in [0, 1/2]$.
 - Observing the sample 0, 1, 1, 1, 0, give the approximate distribution of MLE $\hat{\theta}_{\text{ML}}$.
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- Suppose X_1, \dots, X_n is an iid sample following an exponential distribution, i.e., $f_X(x | \beta) = \beta^{-1} \exp(-x/\beta)\mathbf{1}_{(0,\infty)}(x)$, $\beta > 0$.

- Observing the sample $1, \dots, 10$, give the approximate distribution of MLE $\hat{\beta}_{\text{ML}}$.

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- $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} p(x | \lambda) = \lambda^x \exp(-\lambda)/x!, x \in \mathbb{Z}^+ \cup \{0\}, \lambda > 0$.
 - Observing the sample $1, \dots, 10$, give the approximate distribution of MLE $\hat{\lambda}_{\text{ML}}$.

Delta method

- Used to approximate the distribution of $h(T_n)$ when
 - T_n is asymptotically normally distributed
 - $h(\cdot)$ doesn't rely on n
- (Delta method) Suppose T_n is an estimator of θ . If $\sqrt{n}(T_n - \theta) \approx \mathcal{N}(0, \sigma^2)$, h is NOT a function of n , AND $h'(\theta) \neq 0$, then as $n \rightarrow \infty$,
 - $E\{h(T_n)\} \approx h(\theta)$
 - $\text{var}\{h(T_n)\} \approx \{h'(\theta)\}^2 \sigma^2 / n$
 - $\sqrt{n}\{h(T_n) - h(\theta)\} \approx \mathcal{N}(0, \{h'(\theta)\}^2 \sigma^2)$

Example Lec7.2

- $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} p(x | \lambda) = \lambda^x \exp(-\lambda)/x!, x \in \mathbb{Z}^+ \cup \{0\}, \lambda > 0$. Suppose the sample is $1, \dots, 10$.
 1. What is the ML estimate of $\theta = \Pr(X_i = 0)$?
 2. Give the approximate distribution of MLE of θ .