

# STAT 3100 Lecture Note

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## Hypothesis Testing

### Binary classification

- Assume  $\mathbf{X} = [X_1, \dots, X_n]^\top \sim f(\mathbf{x} \mid \boldsymbol{\theta}^*) \in \{f(\mathbf{x} \mid \boldsymbol{\theta}) : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ 
  - Fixed unknown  $\boldsymbol{\theta}^*$  to be inferred
- Make a decision on  $\boldsymbol{\theta}^*$  between two hypotheses  $H_0 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_0$  and  $H_1 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_1$ 
  - $\boldsymbol{\Theta}_0 \cup \boldsymbol{\Theta}_1 = \boldsymbol{\Theta}$
  - $\boldsymbol{\Theta}_0 \cap \boldsymbol{\Theta}_1 = \emptyset$
- Decision and correctness
  - True positive (TP) =  $H_0$  correctly rejected
  - False positive (FP, type I error) =  $H_0$  incorrectly rejected
  - True negative (TN) =  $H_0$  is correctly accepted
  - False negative (FN, type II error) =  $H_0$  incorrectly accepted
- E.g.,  $H_0$  : healthy vs  $H_1$  : sick
  - TP: sick people identified as sick
  - FP: healthy people identified as sick
  - TN: healthy people identified as healthy
  - FN: sick people identified as healthy

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	Accept $H_0$	Reject $H_0$
$H_0$ is true	True negative (TN)	False positive (FP, type I error)
$H_0$ is false	False negative (FN, type II error)	True positive (TP)

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- Misclassification rate =  $\Pr(\text{FP}) + \Pr(\text{FN})$
- False discovery rate (FDR) =  $\Pr(\text{FP}) / \{\Pr(\text{FP}) + \Pr(\text{TP})\}$ 
  - FDR controlling for sequential/simultaneous testing
- Receiver operating characteristic curve (ROC curve): plot of TPR vs FPR
  - True positive rate (TPR, sensitivity) =  $\Pr(\text{TP}) / \{\Pr(\text{TP}) + \Pr(\text{FN})\}$
  - False positive rate (FPR) =  $\Pr(\text{FP}) / \{\Pr(\text{FP}) + \Pr(\text{FN})\}$

- Area under the ROC curve (AUC)
- True negative rate (TNR, specificity) =  $\Pr(\text{TN}) / \{\Pr(\text{TN}) + \Pr(\text{FP})\}$
- The optimal hypothesis testing seeking to minimize  $\Pr(\text{FN})$  subject to capped  $\Pr(\text{FP})$ , i.e.,

$$\min \Pr(\text{type II error}) \text{ subject to } \Pr(\text{type I error}) \leq \alpha$$

## Power function

- Rejection/critical region:  $R = \{\mathbf{x} : \text{data } \mathbf{x} \text{ corresponding to the rejection of } H_0\}$
- Test function  $\phi : \text{supp}(\mathbf{X}) \rightarrow \{0, 1\}$  defined as  $\phi(\mathbf{x}) = \mathbf{1}_R(\mathbf{x})$ 
  - $\phi(\mathbf{x}) = 1$  implying the rejection of  $H_0$
- Each test function  $\phi$  corresponds to a unique rejection region  $R_\phi = \{\mathbf{x} : \phi(\mathbf{x}) = 1\}$
- Power function (for  $\phi$ ):  $\beta_\phi(\boldsymbol{\theta}) = \Pr(\mathbf{X} \in R_\phi \mid \boldsymbol{\theta}) = \mathbb{E}\{\phi(\mathbf{X}) \mid \boldsymbol{\theta}\}$ 
  - $\Pr(\text{type I error}) = \beta_\phi(\boldsymbol{\theta}^*)$  if  $H_0$  is correct ( $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}_0$ )
  - $\Pr(\text{type II error}) = 1 - \beta_\phi(\boldsymbol{\theta}^*)$  if  $H_1$  is correct ( $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}_1$ )
- Prefer larger  $\beta_\phi(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_1$  and smaller  $\beta_\phi(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$  (because  $\boldsymbol{\theta}^*$  is unknown)

## Example Lec14.2

- iid  $X_1, \dots, X_n \sim N(\theta, \sigma_0^2)$  with known  $\sigma_0$ . Consider a test for  $H_0 : \theta = \theta_0$  vs  $H_1 : \theta \neq \theta_0$  with rejection region  $\{\mathbf{x} : \sqrt{n}|\bar{\mathbf{x}} - \theta_0|/\sigma_0 > c\}$ .
  - Elaborate the power function.
  - Find sample size  $n$  and threshold  $c$  if one desires that the type I error rate is 5% and the maximal type II error rate is 25% and attained at  $\theta = \theta_0 + \sigma_0$ .

## Uniformly most powerful (UMP) level $\alpha$ test (CB Sec 8.3.2)

- $\phi$  is of level  $\alpha$  iff  $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta}) \leq \alpha$ 
  - $\phi$  is of size  $\alpha$  iff  $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta}) = \alpha$
- Let  $\phi$  is a level  $\alpha$  test for  $H_0 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_0$  vs  $H_1 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_1$ . If  $\beta_\phi(\boldsymbol{\theta}) \geq \beta_{\phi'}(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_1$  and all  $\phi'$  of level  $\alpha$ , then  $\phi$  is a UMP level  $\alpha$  test.
- If  $\phi$  is a UMP level  $\alpha$  test, then  $\beta_\phi(\boldsymbol{\theta}) \geq \alpha \geq \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_1$  (unbiasedness for testing, CB Def 8.3.9)

## UMP level $\alpha$ test for simple hypotheses ( $H_0 : \boldsymbol{\theta}^* = \boldsymbol{\theta}_0$ vs $H_1 : \boldsymbol{\theta}^* = \boldsymbol{\theta}_1$ )

- To maximize  $\beta_\phi(\boldsymbol{\theta}_1)$  with respect to  $\phi$  subject to  $\beta_\phi(\boldsymbol{\theta}_0) \leq \alpha$
- Neymann-Pearson (NP) Lemma (CB Thm 8.3.12):  $\phi$  is the UMP test of level  $\alpha$  for simple hypotheses  $\iff \exists k > 0$  such that  $\beta_\phi(\boldsymbol{\theta}_0) = \mathbb{E}\{\phi(\mathbf{X}) \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0\} = \alpha$ , where

$$\phi(\mathbf{x}) = \begin{cases} 1, & f(\mathbf{x} \mid \boldsymbol{\theta}_1) > k f(\mathbf{x} \mid \boldsymbol{\theta}_0), \\ 0, & f(\mathbf{x} \mid \boldsymbol{\theta}_1) < k f(\mathbf{x} \mid \boldsymbol{\theta}_0). \end{cases}$$

In practice (especially for discrete distributions),  $k$  is the largest real number such that

$$\Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) \geq k \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0\} \geq \alpha$$

and

$$\Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) \leq k \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0\} \geq 1 - \alpha.$$

- What shall we do if  $\Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1) = kf(\mathbf{X} \mid \boldsymbol{\theta}_0) \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0\} \neq 0$ ? Take a randomized test! Specifically, if  $f_{\boldsymbol{\theta}_1}(\mathbf{X})/f_{\boldsymbol{\theta}_0}(\mathbf{X}) = k$ , let  $\phi(\mathbf{x}) = \gamma \in [0, 1]$  such that

$$\Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) > k \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0\} + \gamma \Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) = k \mid \boldsymbol{\theta} = \boldsymbol{\theta}_0\} = \alpha.$$

That is, we reject  $H_0$  with probability  $\gamma$ .

- For simple hypotheses, UMP test at level  $\alpha \iff$  UMP test at size  $\alpha$ .

- UMP test and sufficiency (CB Coro 8.3.13): sufficient statistics can be taken as test statistics for UMP  $\phi$ .

### UMP level $\alpha$ test for one-sided hypotheses ( $H_0 : \theta^* \leq \theta_0$ (or $\theta^* = \theta_0$ ) vs $H_1 : \theta^* > \theta_0$ )

- Consider cases with only one unknown parameter
- Monotone likelihood ratio (MLR, CB Def 8.3.16): fixing  $\theta_1 > \theta_2$ ,  $g(t \mid \theta_1)/g(t \mid \theta_2)$  is monotonic with respect to  $t$  for  $\{g(t \mid \theta) : \theta \in \Theta \subset \mathbb{R}\}$ 
  - E.g., one-parameter exponential family bears MLR
- Karlin-Rubin (CB Thm 8.3.17): Suppose  $T$  is sufficient for  $\theta$  and  $T \sim g(t \mid \theta)$  bearing MLR. A UMP level  $\alpha$  test for  $H_0 : \theta^* \leq \theta_0$  (or  $\theta^* = \theta_0$ ) vs  $H_1 : \theta^* > \theta_0$  is

$$\phi(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) > t_\alpha, \\ 0, & T(\mathbf{x}) < t_\alpha, \end{cases}$$

where  $t_\alpha$  is a real number such that  $\beta_\phi(\theta_0) = \mathbb{E}\{\phi(\mathbf{X}) \mid \theta = \theta_0\} = \Pr(T > t_\alpha \mid \theta = \theta_0) = \alpha$ .

### Example Lec14.1

- iid  $X_1, \dots, X_n \sim N(\mu, 1)$ . Construct UMP level  $\alpha$  test for following hypotheses.
  - $H_0 : \mu = \mu_0$  vs  $H_1 : \mu = \mu_1$  with  $\mu_0 < \mu_1$ ;
  - $H_0 : \mu = \mu_0$  vs  $H_1 : \mu > \mu_0$ ;
  - $H_0 : \mu \geq \mu_0$  vs  $H_1 : \mu < \mu_0$ ;
  - $H_0 : \mu = \mu_0$  vs  $H_1 : \mu \neq \mu_0$ .