

# PH 712 Probability and Statistical Inference

## Part I: Random Variable

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### Probability (HMC Sec. 1.1–1.3)

- Sample space (denoted by  $\Omega$ ): the set of all the possible outcomes, e.g.,
  - $\Omega = \mathbb{R}^+$  if investigating survival times of cancer patients
  - $\Omega = \{\text{yes}, \text{no}\}$  if investigating whether a treatment is effective
- Event (denoted by capital Roman letters, e.g.,  $A$ ): a subset of the sample space, e.g., corresponding to the previous sample spaces,
  - $(0, 10]$ : the survival time  $\leq 10$
  - $\{\text{yes}\}$ : the treatment is effective
- Occurrence of event: the outcome is part of the event
- Probability (denoted by  $\Pr$ ): a function quantifying the occurrence likelihood of an event
  - E.g.,
    - \*  $\Pr(A)$ : the occurrence probability of event  $A$
    - \*  $\Pr(A^c)$ : the probability that event  $A$  does NOT occur ( $A^c = \Omega \setminus A$  denoting the complement set of  $A$ )
    - \*  $\Pr(A \cup B)$ : the occurrence probability of either  $A$  or  $B$
    - \*  $\Pr(A \cap B)$ : the occurrence probability of both  $A$  and  $B$
  - Input: an event
  - Output: a real number (the occurrence probability of the input event)
  - Requirements:
    - \*  $\Pr(A) \geq 0$  for any event  $A$
    - \*  $\Pr(\Omega) = 1$  (i.e., the sample space as a special event always occurs)
    - \* (The probability of the union of mutually exclusive countably events is the sum of the probability of each event) If  $\{A_n\}_{n=1}^\infty$  is a sequence of events with  $A_{n_1} \cap A_{n_2} = \emptyset$  for all  $n_1 \neq n_2$ , then  $\Pr(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Pr(A_n)$
  - More properties (deduced from the above requirements):
    - \*  $\Pr(A) = 1 - \Pr(A^c)$
    - \*  $\Pr(\emptyset) = 0$
    - \*  $\Pr(A) \leq \Pr(B)$  if  $A \subset B$
    - \*  $0 \leq \Pr(A) \leq 1$  for each  $A$
    - \*  $\lim_{n \rightarrow \infty} \Pr(A_n) = \Pr(\lim_{n \rightarrow \infty} A_n) = \Pr(\bigcup_{n=1}^\infty A_n)$  if  $\{A_n\}_{n=1}^\infty$  is nondecreasing (i.e.,  $A_1 \subset A_2 \subset \dots$ )
    - \*  $\lim_{n \rightarrow \infty} \Pr(A_n) = \Pr(\lim_{n \rightarrow \infty} A_n) = \Pr(\bigcap_{n=1}^\infty A_n)$  if  $\{A_n\}_{n=1}^\infty$  is nonincreasing (i.e.,  $A_1 \supset A_2 \supset \dots$ )
    - \*  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$  for any events  $A$  and  $B$  regardless if they are disjoint or not

- \*  $\Pr(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \Pr(A_n)$  for arbitrary sequence  $\{A_n\}_{n=1}^{\infty}$

## Conditional probability and independence (HMC Sec. 1.4)

- Conditional probability of  $B$  given  $A$  (with  $\Pr(A) > 0$ ):  $\Pr(B | A) = \Pr(A \cap B) / \Pr(A)$ 
  - Interpretation: the occurrence probability of  $B$ , given that  $A$  has already occurred.
  - Properties:
    - \*  $\Pr(B | A) \geq 0$
    - \*  $\Pr(A | A) = 1$
    - \*  $\Pr(\bigcup_{n=1}^{\infty} B_n | A) = \sum_{n=1}^{\infty} \Pr(B_n | A)$  if  $\{B_n\}_{n=1}^{\infty}$  are mutually exclusive
    - \* (Law of total probability)  $\Pr(B) = \sum_{n=1}^N \Pr(A_n) \Pr(B | A_n)$  if  $\{A_n\}_{n=1}^N$  form a partition of  $\Omega$  (i.e.,  $\{A_n\}_{n=1}^N$  are mutually exclusive and  $\Omega = \bigcup_{n=1}^N A_n$ )
    - \* (Bayes' theorem)  $\Pr(A_i | B) = \Pr(A_i) \Pr(B | A_i) / \sum_{n=1}^N \Pr(A_n) \Pr(B | A_n)$  if  $\{A_n\}_{n=1}^N$  form a decomposition/partition of  $\Omega$
- Independence between two events  $B$  and  $A$  (i.e.,  $B \perp A$ ):  $\Pr(B \cap A) = \Pr(A) \Pr(B)$ 
  - $\Leftrightarrow B \perp A^c$
  - $\Leftrightarrow \Pr(B | A) = \Pr(B)$  (if  $\Pr(A) \neq 0$ )
- Mutual independence among  $N$  events  $A_1, \dots, A_N$ : for arbitrary subset of  $\{A_1, \dots, A_N\}$ , say  $\{A_{n_1}, \dots, A_{n_K}\}$  with  $2 \leq K \leq N$ ,  $\Pr(\bigcap_{k=1}^K A_{n_k}) = \prod_{k=1}^K \Pr(A_{n_k})$

### HMC Ex. 1.4.31

- A French writer, Chevalier de Méré, had asked a famous mathematician, Pascal, to explain why the following two probabilities were different (the difference had been noted from playing the game many times): (1) at least one six in four independent casts of a six-sided die; (2) at least a pair of sixes in 24 independent casts of a pair of dice. From proportions it seemed to Mr. de Méré that the two probabilities should be the same. Compute the probabilities of (1) and (2).
  - Hint:  $\Pr(\text{no six in one cast of a die}) = 5/6$ ,  $\Pr(\text{no six in one cast of a pair of dice}) = (5/6)^2$ , and  $\Pr(\text{only one six in one cast of a pair of dice}) = 2 \times (1/6) \times (5/6)$ .

## Distribution of an RV (HMC Chp. 1.5–1.7)

- RV: a function encoding the entries of  $\Omega$ 
  - Input: arbitrary entry of  $\Omega$ , say  $\omega$
  - Output:  $X(\omega) \in \mathbb{R}$
- The cumulative distribution function (cdf) of RV  $X$ , say  $F_X$ , is defined as

$$F_X(t) = \Pr(X \leq t), \quad t \in \mathbb{R}.$$

- $\{X \leq t\}$ : short for the event  $\{\omega \in \Omega : X(\omega) \leq t\}$
- $F_X$  satisfies following three properties:
  - \* (Right continuous)  $\lim_{x \rightarrow t^+} F_X(x) = F_X(t)$  (p.s.,  $\lim_{x \rightarrow t^-} F_X(x) = \Pr(X < t)$ );
  - \* (Non-decreasing)  $F_X(t_1) \leq F_X(t_2)$  for  $t_1 \leq t_2$ ;
  - \* (Ranging from 0 to 1)  $F_X(-\infty) = 0$  and  $F_X(\infty) = 1$ .
- Reversely, a function satisfying the three above properties must be a cdf for certain RV.
  - \* Indicating an one-to-one correspondence between the set of all the RVs and the set of all the cdfs
- Knowing the cdf of an RV  $\Leftrightarrow$  knowing its distribution

### Example Lec1.1

- Given  $p \in (0, 1)$ , suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lfloor x \rfloor$  represents the integer part of real  $x$ .

- Show that  $F_X$  is a cdf. (Hint: Check all the three properties of cdf, especially the right-continuity of  $F_X$  at positive integers.)
- Given  $\lambda > 0$ , suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- Show that  $F_X$  is a cdf.

## Distribution of an RV (con'd)

- Discrete RV
  - RV  $X$  merely takes countably different values
  - Probability mass function (pmf):  $p_X(t) = \Pr(X = t)$ 
    - \*  $F_X(t) = \sum_{x \leq t} p_X(x)$
    - \*  $p_X(t) = F_X(t) - \Pr(X < t) = F_X(t) - \lim_{x \rightarrow t^-} F_X(x)$
  - Knowing the pmf of a discrete RV  $\Leftrightarrow$  knowing its distribution
  - Examples:
    - \* Bernoulli: a discrete RV with two possible outcomes, typically coded as 0 (failure) and 1 (success).
      - [https://en.wikipedia.org/wiki/Bernoulli\\_distribution](https://en.wikipedia.org/wiki/Bernoulli_distribution)
    - \* Binomial (denoted by  $B(n, p)$ ): the number of successes in  $n$  independent Bernoulli trials.
      - [https://en.wikipedia.org/wiki/Binomial\\_distribution](https://en.wikipedia.org/wiki/Binomial_distribution)
      - E.g., flipping a coin 10 times and counting the number of heads.
    - \* Geometric: the number of trials until the first success in a series of independent Bernoulli trials.
      - [https://en.wikipedia.org/wiki/Geometric\\_distribution](https://en.wikipedia.org/wiki/Geometric_distribution)
      - E.g., the number of coin flips needed until the first head appears.
    - \* Poisson: the number of events that occur in a fixed interval of time or space, where events happen independently.
      - [https://en.wikipedia.org/wiki/Poisson\\_distribution](https://en.wikipedia.org/wiki/Poisson_distribution)
      - E.g., the number of emails you receive in an hour.
    - \* Uniform (the discrete version): each outcome in a finite set has an equal probability.
      - [https://en.wikipedia.org/wiki/Discrete\\_uniform\\_distribution](https://en.wikipedia.org/wiki/Discrete_uniform_distribution)
      - E.g., rolling a fair dice, where each of the six faces has an equal chance of landing.
- Continuous RV
  - RV  $X$  is continuous  $\Leftrightarrow$  its cdf  $F_X$  is absolutely continuous, i.e., there exists  $f_X$  such that

$$F_X(t) = \int_{-\infty}^t f_X(x) dx, \quad \forall t \in \mathbb{R}.$$

- \* Probability density function (pdf):  $f_X(t) = dF_X(t)/dt$  (nonnegative for all  $t$ ).
  - $\int_{-\infty}^{\infty} f_X(x) dx = \lim_{t \rightarrow \infty} \int_{-\infty}^t f_X(x) dx = \lim_{t \rightarrow \infty} F_X(t) = 1$
- \*  $\Pr(X = x_0) = 0$  for all  $x_0 \in \mathbb{R}$ 
  - Because  $\Pr(X = x_0) = \Pr(X \leq x_0) - \Pr(X < x_0) = F_X(x_0) - \lim_{x \rightarrow x_0^-} F_X(x) = 0$
- Knowing the pdf of a continuous RV  $\Leftrightarrow$  knowing its distribution
- Examples:
  - \* Uniform (the continuous version): all outcomes in a continuous range are equally likely.
    - [https://en.wikipedia.org/wiki/Uniform\\_distribution\\_\(continuous\)](https://en.wikipedia.org/wiki/Uniform_distribution_(continuous))
  - \* Normal/Gaussian (denoted by  $\mathcal{N}(\mu, \sigma^2)$ ): the most important and widely used distributions, where data is symmetrically distributed around the mean.
    - [https://en.wikipedia.org/wiki/Normal\\_distribution](https://en.wikipedia.org/wiki/Normal_distribution)
  - \* Exponential: often used to describe waiting times.
    - [https://en.wikipedia.org/wiki/Exponential\\_distribution](https://en.wikipedia.org/wiki/Exponential_distribution)

### Example Lec1.2

- Given  $\lambda > 0$ , suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- What is the type of  $X$ , discrete or continuous?
- Given  $p \in (0, 1)$ , suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lfloor x \rfloor$  represents the integer part of  $x$ .

- What is the type of  $X$ , discrete or continuous?

### Support of RV (CB pp. 50 & HMC pp. 46)

- For discrete RV  $X$  with pmf  $p_X$ 
  - $\text{supp}(X) = \{x \in \mathbb{R} : p_X(x) > 0\}$
  - E.g., support of  $B(n, p)$  is  $\{0, \dots, n\}$
  - $\sum_{x \in \text{supp}(X)} p_X(x) = 1$
- For continuous RV  $X$  with pdf  $f_X$ 
  - $\text{supp}(X) = \{x \in \mathbb{R} : f_X(x) > 0\}$
  - E.g., support of  $\mathcal{N}(0, 1)$  is  $\mathbb{R}$
  - $\int_{\text{supp}(X)} f_X(x) dx = 1$

### Example Lec1.3

- Revisit  $F_X$  defined in Example Lec1.1, i.e.,

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lfloor x \rfloor$  represents the integer part of real  $x$ .

- What is the support of  $X$ ?

### Indicator function

Given a set  $A$ , the indicator function of  $A$  is

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

### Example Lec1.4

- Revisit  $F_X$  defined in Example Lec1.1, i.e.,

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lfloor x \rfloor$  represents the integer part of  $x$ .

- Please reformulate  $F_X$  with the indicator function of  $A = \{x : x \geq 1\}$ .

### Indicating the support when writing pmf and pdf

- Bernoulli: [https://en.wikipedia.org/wiki/Bernoulli\\_distribution](https://en.wikipedia.org/wiki/Bernoulli_distribution)
- Binomial (denoted by  $B(n, p)$ ): [https://en.wikipedia.org/wiki/Binomial\\_distribution](https://en.wikipedia.org/wiki/Binomial_distribution)

- $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \cdot \mathbf{1}_{\{0,1,\dots,n\}}(k)$
  - \* OR  $\binom{n}{k} p^k (1-p)^{n-k}, k \in \{0,1,\dots,n\}$
- Geometric: [https://en.wikipedia.org/wiki/Geometric\\_distribution](https://en.wikipedia.org/wiki/Geometric_distribution)
  - $p_X(k) = (1-p)^{k-1} p \cdot \mathbf{1}_{\mathbb{Z}^+}(k)$
  - \* OR  $(1-p)^{k-1} p, k \in \mathbb{Z}^+$
- Poisson: [https://en.wikipedia.org/wiki/Poisson\\_distribution](https://en.wikipedia.org/wiki/Poisson_distribution)
  - $p_X(k) = \lambda^k \exp(-\lambda)/k! \cdot \mathbf{1}_{\{0,1,2,\dots\}}(k)$
  - \* OR  $\lambda^k \exp(-\lambda)/k!, k \in \{0,1,2,\dots\}$
- Uniform (the discrete version; denoted by  $U([a,b])$  with integers  $a < b$ ): [https://en.wikipedia.org/wiki/Discrete\\_uniform\\_distribution](https://en.wikipedia.org/wiki/Discrete_uniform_distribution)
  - $p_X(k) = 1/(b-a+1) \cdot \mathbf{1}_{\{a,a+1,\dots,b-1,b\}}(k)$
  - \* OR  $1/(b-a+1), k \in \{a,a+1,\dots,b-1,b\}$
- Uniform (the continuous version): [https://en.wikipedia.org/wiki/Uniform\\_distribution\\_\(continuous\)](https://en.wikipedia.org/wiki/Uniform_distribution_(continuous))
- Normal/Gaussian (denoted by  $\mathcal{N}(\mu, \sigma^2)$ ): [https://en.wikipedia.org/wiki/Normal\\_distribution](https://en.wikipedia.org/wiki/Normal_distribution)
- Exponential: [https://en.wikipedia.org/wiki/Exponential\\_distribution](https://en.wikipedia.org/wiki/Exponential_distribution)
  - $f_X(x) = \lambda \exp(-\lambda x) \cdot \mathbf{1}_{[0,\infty)}(x)$
  - \* OR  $\lambda \exp(-\lambda x), x \geq 0$

## Expectations (HMC Sec. 1.8–1.9)

- Given RV  $X$  and function  $g$ , the expectation of  $g(X)$  is

$$\mathbb{E}\{g(X)\} = \begin{cases} \int_{x \in \text{supp}(X)} g(x) f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} g(x) p_X(x) & \text{for discrete } X \end{cases}$$

- Weighted average of values of  $g(X)$
  - $\mathbb{E}\{a_1 g_1(X) + a_2 g_2(X)\} = a_1 \mathbb{E}\{g_1(X)\} + a_2 \mathbb{E}\{g_2(X)\}$  for constants  $a_1$  and  $a_2$
- Examples
  - Taking  $g(X) = X$

$$\mathbb{E}(X) = \begin{cases} \int_{x \in \text{supp}(X)} x f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} x p_X(x) & \text{for discrete } X \end{cases}$$

- \* The mean of  $X$  (a.k.a. the 1st raw moment/moment about 0 of  $X$ )
  - \*  $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$  for constants  $a$  and  $b$
- Taking  $g(X) = X^k$  with positive integer  $k$ :

$$\mathbb{E}(X^k) = \begin{cases} \int_{x \in \text{supp}(X)} x^k f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} x^k p_X(x) & \text{for discrete } X \end{cases}$$

- \* The  $k$ th raw moment/moment about 0 of  $X$
- Taking  $g(X) = \{X - \mathbb{E}(X)\}^2$ :

$$\text{Var}(X) = \mathbb{E}[\{X - \mathbb{E}(X)\}^2] = \begin{cases} \int_{x \in \text{supp}(X)} \{x - \mathbb{E}(X)\}^2 f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} \{x - \mathbb{E}(X)\}^2 p_X(x) & \text{for discrete } X \end{cases}$$

- \* Variance of  $X$  (a.k.a. the 2nd central moment of  $X$ )
  - \* Measuring how spread out the data are if they are independently generated following  $F_X$
  - \*  $\text{Var}(X) = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2$
  - \*  $\text{Var}(aX + b) = a^2 \text{Var}(X)$
  - \*  $\text{sd}(X) = \sqrt{\text{Var}(X)}$ : the standard deviation of  $X$

## Example Lec1.5

- Find the mean and variance of  $X \sim \mathcal{N}(0, 1)$ , i.e.,  $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$$E(X) = \int_{\mathbb{R}} x f_X(x) dx \stackrel{x \exp(-x^2/2) \text{ is odd}}{=} \int_{\mathbb{R}} \frac{x}{\sqrt{2\pi}} \exp(-x^2/2) dx = 0$$

$$\text{Var}(X) \stackrel{\text{even } x^2 \exp(-x^2/2)}{=} 2 \int_0^\infty \frac{x^2 \exp(-x^2/2)}{\sqrt{2\pi}} dx \stackrel{u=x^2/2}{=} 2 \int_0^\infty \frac{2u \exp(-u)}{\sqrt{2\pi}} d\sqrt{2u} = \frac{2\Gamma(3/2)}{\sqrt{\pi}} = 1$$

- Find the mean and variance of  $X \sim \mathcal{N}(\mu, \sigma^2)$  with  $\mu \in \mathbb{R}$  and  $\sigma \in \mathbb{R}^+$ , i.e.,  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$  (p.s.  $X \sim \mathcal{N}(\mu, \sigma^2) \Leftrightarrow Z = (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$ )
- Find the mean and variance of Cauchy distribution, i.e.,  $f_X(x) = \{\pi(1 + x^2)\}^{-1}$ ,  $x \in \mathbb{R}$

$$\int_1^\infty \frac{x^2}{\pi(1+x^2)} dx \geq \int_1^\infty \frac{x}{\pi(1+x^2)} dx = \infty$$

## Distribution of an RV (con'd)

- Moment generating function (mgf, HMC Sec. 1.9/CB Sec. 2.3)
  - $M_X(t) = E\{\exp(tX)\}$ 
    - Continuous  $X$ :  $M_X(t) = \int_{x \in \text{supp}(X)} \exp(tx) f_X(x) dx$
    - Discrete  $X$ :  $M_X(t) = \sum_{x \in \text{supp}(X)} \exp(tx) p_X(x)$
  - The mgf of  $X$  is  $M_X(t)$ ,  $t \in A$ ,  $\Leftrightarrow M_X(t)$  is finite for  $t$  in a neighborhood of 0, say  $A$ ; otherwise the mgf does NOT exist or is NOT well defined.
    - A neighborhood of 0:  $(-\epsilon_1, \epsilon_2)$  for certain  $\epsilon_1, \epsilon_2 > 0$ , e.g., an open interval including both positive and negative numbers
  - $M_{aX+b}(t) = \exp(bt) M_X(at)$
  - Knowing the mgf (if any) of an RV  $\Leftrightarrow$  knowing its distribution
  - If mgf  $M(t)$  is well-defined, then the  $k$ th raw moment is the  $k$ th-order derivative of  $M(t)$  evaluated at 0, i.e.,  $E(X^k) = M^{(k)}(0)$

## Example Lec1.6

- Find the mgf of  $X \sim \mathcal{N}(\mu, \sigma^2)$  with  $\mu \in \mathbb{R}$  and  $\sigma \in \mathbb{R}^+$ , i.e.,  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$$E(e^{tX}) = \int_{-\infty}^\infty e^{tx} f_X(x) dx = \frac{\int_{-\infty}^\infty \exp\left(tx - \frac{(x-\mu)^2}{2\sigma^2}\right) dx}{\sqrt{2\pi\sigma^2}} = \frac{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty \exp\left(-\frac{(x - (\mu + \sigma^2 t))^2}{2\sigma^2}\right) dx$$

- Find the mgf of Cauchy distribution, i.e.,  $f_X(x) = \{\pi(1 + x^2)\}^{-1}$ ,  $x \in \mathbb{R}$

$$E\{\exp(tX)\} = \int_{-\infty}^\infty \frac{\exp(tx)}{\pi(1+x^2)} dx$$

- $\frac{1}{1+x^2}$  decreases to 0 polynomially as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ .
- If  $t > 0$ , then  $\exp(tx)$  grows exponentially as  $x \rightarrow \infty$ ; if  $t < 0$ , then  $\exp(tx)$  grows exponentially as  $x \rightarrow -\infty$ .
- Therefore,  $\frac{\exp(tx)}{1+x^2} \rightarrow \infty$  as  $x \rightarrow \infty$  when  $t > 0$ , and as  $x \rightarrow -\infty$  when  $t < 0$ . The integral  $E\{\exp(tx)\}$  does not converge for any nonzero  $t$ .