STAT 3100 Lecture Note

Week Eight (Oct 25 & 27, 2022)

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Hypothesis Testing (con'd)

UMP level α test for one-sided hypotheses ($H_0 : \theta^* \leq \theta_0$ (or $\theta^* = \theta_0$) vs $H_1 : \theta^* > \theta_0$)

- Consider cases with only one unknown parameter
- Monotone likelihood ratio (MLR, CB Def 8.3.16): for each pair $\theta_2 > \theta_1$, $f(t \mid \theta_2)/f(t \mid \theta_1)$ is nondecreasing with respect to t for univariate pdfs/pmfs $\{f(t \mid \theta) : \theta \in \Theta \subset \mathbb{R}\}$
  - One-parameter exponential family with $w(\theta)$ nondecreasing w.r.t. θ bears MLR (why?)

- Karlin-Rubin (CB Thm 8.3.17): Suppose T is sufficient for θ and T follows $f_T(t \mid \theta)$ bearing MLR. A UMP level α test for $H_0 : \theta^* \leq \theta_0$ (or $\theta^* = \theta_0$) vs. $H_1 : \theta^* > \theta_0$ is

$$\phi_\lambda(\boldsymbol{x}) = \begin{cases} 1, & T(\boldsymbol{x}) > \lambda, \\ 0, & T(\boldsymbol{x}) < \lambda, \end{cases}$$

  where λ is a real number such that $\beta_\phi(\theta_0) = \mathrm{E}\{\phi_\lambda(\mathbf{X}) \mid \theta^* = \theta_0\} = \Pr\{T(\mathbf{X}) > \lambda \mid \theta^* = \theta_0\} = \alpha$.

- NOTE: in the Karlin-Rubin theorem, if the hypotheses become $H_0 : \theta^* \geq \theta_0$ (or $\theta^* = \theta_0$) vs. $H_1 : \theta^* < \theta_0$, then change the signs in the test function, i.e.,

$$\phi_\lambda(\boldsymbol{x}) = \begin{cases} 1, & T(\boldsymbol{x}) < \lambda, \\ 0, & T(\boldsymbol{x}) > \lambda, \end{cases}$$

  where λ is a real number such that $\beta_\phi(\theta_0) = \Pr\{T(\mathbf{X}) < \lambda \mid \theta^* = \theta_0\} = \alpha$.

Example Lec14.1

- iid $X_1, \ldots, X_n \sim \mathcal{N}(\mu, 1)$. Construct UMP level α test for following hypotheses.
  a. $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ with $\mu_0 < \mu_1$;
  b. $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$;
  c. $H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$;
  d. $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.

Nonexistence of UMP test for two-sided hypotheses $H_0 : \theta^* = \theta_0$ vs $H_1 : \theta^* \neq \theta_0$

- (Optional) uniformly most powerful unbiased (UMPU) level α test

Likehood ratio test (LRT, Sec 8.2.1 & 10.3.1)

- $H_0 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_0$ vs. $H_1 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_1$
- $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0 \cup \boldsymbol{\Theta}_1$
- Test statistic
$$\lambda(\boldsymbol{x}) = \frac{\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} L(\boldsymbol{\theta} \mid \boldsymbol{x})}{\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} L(\boldsymbol{\theta} \mid \boldsymbol{x})} = \frac{L(\hat{\boldsymbol{\theta}}_{0,\mathrm{ML}} \mid \boldsymbol{x})}{L(\hat{\boldsymbol{\theta}}_{\mathrm{ML}} \mid \boldsymbol{x})}$$
  - $\hat{\boldsymbol{\theta}}_{0,\mathrm{ML}}$: MLE for $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$
  - $\hat{\boldsymbol{\theta}}_{\mathrm{ML}}$: MLE for $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- Rejection region $\{\boldsymbol{x} : \lambda(\boldsymbol{x}) \le c\}$
  - c is chosen to make sure the level is α, i.e.,
$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta}) = \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \Pr\{\lambda(\mathbf{X}) \le c \mid \boldsymbol{\theta}\} \le \alpha.$$
- Asymptotially, as $n \to \infty$,
$$-2\ln\lambda(\mathbf{X}) \xrightarrow{d} \chi^2(\nu),$$
where $\nu =$ the difference of numbers of free parameters between Θ_0 and Θ_1.
- (CB Thm 10.3.3) Asymptotic LRT rejection region $\{\boldsymbol{x} : -2\ln\lambda(\boldsymbol{x}) \ge \chi^2_{\nu,1-\alpha}\} = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \le \exp(-\chi^2_{\nu,1-\alpha}/2)\}$
  - $\chi^2_{\nu,1-\alpha}$ is the $1-\alpha$ quantile of $\chi^2(\nu)$.
- (CB Ex. 8.24) For simple hypotheses, is the LRT equivalent to the UMP test?

Example Lec14.3

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Test $H_0 : \mu \le \mu_0$ vs. $H_1 : \mu > \mu_0$.
  a. σ^2 is known. Suppose test ϕ has rejection region $\{\boldsymbol{x} : \bar{x} > \mu_0 + z_{1-\alpha}\sqrt{\sigma^2/n}\}$, where $z_{1-\alpha}$ is the $(1-\alpha)$ quantile of standard normal. Show that ϕ is a UMP level α test and is equivalent to the LRT.
  b. σ^2 is unknown. Suppose test ϕ has rejection region $\{\boldsymbol{x} : \bar{x} > \mu_0 + t_{n-1,1-\alpha}\sqrt{s^2/n}\}$, where $t_{n-1,1-\alpha}$ is the $(1-\alpha)$ quantile of $t(n-1)$. Show that ϕ is of size α and is equivalent to the LRT.

p-value (CB Sec 8.3.4)

- $p(\mathbf{X})$ is valid iff $\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \Pr\{p(\mathbf{X}) \le \alpha \mid \boldsymbol{\theta}\} \le \alpha$ for each $\alpha \in [0,1]$.
  - i.e., it is possible to define "level" and "size" if we take $\{\boldsymbol{x} : p(\boldsymbol{x}) \le \alpha\}$ as the rejection region
  - $p(\mathbf{X})$ is valid $\Rightarrow$ $p(\mathbf{X})$ is a test statistic with rejection region $\{\boldsymbol{x} : p(\boldsymbol{x}) \le \alpha\}$.
- (CB Thm 8.3.27) H_0 is rejected when $T(\boldsymbol{x})$ is too large $\Rightarrow$ $p(\boldsymbol{x}) = \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \Pr\{T(\mathbf{X}) \ge T(\boldsymbol{x}) \mid \boldsymbol{\theta}\}$.
- (CB Ex 8.51, another interpretation of p-values) suppose that, for each $\alpha \in [0,1]$, $R_\alpha = \{\boldsymbol{x} : T(\boldsymbol{x}) \ge c_\alpha\}$ is the rejection region of a size α test $\Rightarrow$ $p(\boldsymbol{x}) = \inf\{\alpha \in [0,1] : \boldsymbol{x} \in R_\alpha\}$.

Example Lec16.1

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Consider $H_0 : \mu = \mu_0$ vs $H_1 : \mu \ne \mu_0$.
  a. Verify that the size α LRT rejects H_0 when $|\bar{x} - \mu_0| > t_{n-1,1-\alpha/2}(s/\sqrt{n})$.
  b. Find the the expression of p-value for LRT.

Confidence set (CB Sec 9.2.1 & 9.3.1)

- Confidence set of $\boldsymbol{\theta}^*$: $C(\mathbf{X})$
- Coverage probability of confidence set $C(\mathbf{X})$: $\Pr\{\boldsymbol{\theta}^* \in C(\mathbf{X})\}$
- $1-\alpha$ confidence set: $C(\mathbf{X})$ with $\inf_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \Pr\{\boldsymbol{\theta} \in C(\mathbf{X})\} = 1-\alpha$
- (CB Thm 9.2.2) construct the confidence set by inverting the acceptance region
    1. For each $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$, find the rejection region, say $R(\boldsymbol{\theta}_0)$, of a level α test of $H_0 : \boldsymbol{\theta}^* = \boldsymbol{\theta}_0$ vs. $H_1 : \boldsymbol{\theta}^* \neq \boldsymbol{\theta}_0$
    2. $C(\boldsymbol{x}) = \{\boldsymbol{\theta}_0 : \boldsymbol{x} \in \mathrm{supp}(\mathbf{X})/R(\boldsymbol{\theta}_0)\}$

Example Lec16.2

- iid $X_1, \ldots, X_n \sim \mathcal{N}(\mu, 1)$. For each of the following cases, write down the rejection region of the level α LRT and then invert it to obtain the $1-\alpha$ confidence interval.
    a. $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ with $\mu_0 < \mu_1$;
    b. $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$;
    c. $H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$;
    d. $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.

Take-home exercises (NOT to be submitted; to be potentially covered in labs)

CB Ex 8.2, 8.6(a–b), 8.16, 8.28, 8.33, 8.41, 9.33(a)