STAT 3690 Lecture 10

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Testing on $\boldsymbol{\mu}$ (J&W Sec. 5.2 & 5.3)

- $\mathbf{X}_1, \ldots, \mathbf{X}_n \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu}, \Sigma)$ $n > p$
- Hypotheses: $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ v.s. $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$
- Recall the univariate case ($p = 1$)
  - The model reduces to $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$
  - Hypotheses reduces to $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$
  - $\bar{X}$ and s^2 are sample mean and sample variance, respectively
  - Known σ^2
    * Name of approach: Z-test (also LRT)
    * Test statistic: $\sqrt{n}(\bar{X} - \mu_0)/\sigma \sim N(0, 1)$ under H_0
      · OR $n(\bar{X} - \mu_0)^2/\sigma^2 \sim \chi(1)$ under H_0
    * Rejction region at level α: $R = \{x_1, \ldots, x_n : \sqrt{n}|\bar{x} - \mu_0|/\sigma \geq \Phi^{-1}_{1-\alpha/2}\} = \{x_1, \ldots, x_n : n(\bar{x} - \mu_0)^2/\sigma^2 \geq \chi^2_{1-\alpha,1}\}$
      · $\Phi^{-1}_{1-\alpha/2}$: the $(1 - \alpha/2)$-quantile of $N(0, 1)$
      · $\chi^2_{1-\alpha,1}$: the $(1 - \alpha)$-quantile of $\chi^2(1)$
  - Unknown σ^2
    * Name of approach: t-test (also LRT)
    * Test statistic: $\sqrt{n}(\bar{X} - \mu_0)/s \sim t(n - 1)$ under H_0
      · OR $n(\bar{X} - \mu_0)^2/s^2 \sim F(1, n - 1)$ under H_0
    * Rejction region at level α: $R = \{x_1, \ldots, x_n : \sqrt{n}|\bar{x} - \mu_0|/s \geq t_{1-\alpha/2,n-1}\} = \{x_1, \ldots, x_n : n(\bar{x} - \mu_0)^2/s^2 \geq F_{1-\alpha,1,n-1}\}$
      · $t_{1-\alpha/2,n-1}$: the $(1 - \alpha/2)$-quantile of $t(n - 1)$
      · $F_{1-\alpha,1,n-1}$: the $(1 - \alpha)$-quantile of $F(1, n - 1)$

- Multivariate case (with known $\boldsymbol{\Sigma}$)
  - Name of approach: LRT
  - Test statistic: $n(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0) \sim \chi^2(p)$ under H_0
  - Rejction region at level α: $R = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n : n(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0) \geq \chi^2_{1-\alpha,p}\}$
  - p-value: $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1 - F_{\chi^2(p)}\{n(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0)^\top \boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0)\}$
    * $F_{\chi^2(p)}$: the cdf of $\chi^2(p)$

- Multivariate case (with unknown $\boldsymbol{\Sigma}$)
  - Name of approach: LRT
  - Test statistic: $n(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0) \sim T^2(p, n - 1) = \frac{(n-1)p}{n-p} F(p, n - p)$ under H_0

- Rejction region at level α: $R = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n : \frac{n(n-p)}{p(n-1)}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0) \geq F_{1-\alpha,p,n-p}\}$
- p-value: $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1 - F_{F(p,n-p)}\{\frac{n(n-p)}{p(n-1)}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0)\}$
  * $F_{F(p,n-p)}$: the cdf of $F(p, n-p)$

Testing on $\mathbf{A}\boldsymbol{\mu}$

- $\mathbf{A}$ is of $q \times p$ and $\text{rk}(\mathbf{A}) = q$, i.e., $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top > 0$
- Model: iid $\mathbf{A}\mathbf{X}_i \sim MVN_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\Sigma\mathbf{A}^\top)$.
- LRT for $H_0 : \mathbf{A}\boldsymbol{\mu} = \boldsymbol{\nu}_0$ v.s. $H_1 : \mathbf{A}\boldsymbol{\mu} \neq \boldsymbol{\nu}_0$
  - Test statistic: $n(\mathbf{A}\bar{\mathbf{X}} - \boldsymbol{\nu}_0)^\top (\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\mathbf{X}} - \boldsymbol{\nu}_0) \sim T^2(q, n-1) = \frac{(n-1)q}{n-q}F(q, n-q)$ under H_0
  - Rejction region at level α: $R = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n : \frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)^\top (\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0) \geq F_{1-\alpha,q,n-q}\}$
  - p-value: $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1 - F_{F(q,n-q)}\{\frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)^\top (\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)\}$
- Multiple comparison
  - Interested in $H_0 : \mu_1 = \cdots = \mu_p$ v.s. H_1 : Not all entries of $\boldsymbol{\mu}$ are equal.
    * μ_k: the kth entry of $\boldsymbol{\mu}$
  - Take
$$\boldsymbol{\nu}_0 = \mathbf{0}_{(p-1)\times 1}, \quad \mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}_{(p-1)\times p}.$$
  - $p = 2$: A/B testing
