STAT 3690 Lecture 12

zhiyanggeezhou.github.io

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca)

Feb 18, 2022

$(1-\alpha)\times 100\%$ CR for $\boldsymbol{\nu} = \mathbf{A}\boldsymbol{\mu}$

- $\mathbf{X}_1,\ldots,\mathbf{X}_n \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$
  - Unknown $\boldsymbol{\Sigma}$
  - $n > p$
- $\mathbf{A}$ is of $q\times p$ and $\mathrm{rk}(\mathbf{A}) = q$, i.e., $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top > 0$
- Then iid $\mathbf{A}\mathbf{X}_i \sim MVN_q(\boldsymbol{\nu},\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$
- $(1-\alpha)\times 100\%$ CR for $\boldsymbol{\nu}$ is $\{\boldsymbol{\nu} : \frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}}-\boldsymbol{\nu})^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}}-\boldsymbol{\nu}) < F_{1-\alpha,q,n-q}\}$
- Special case: $\mathbf{A} = \boldsymbol{a} \in \mathbb{R}^p$
  - $(1-\alpha)\times 100\%$ confidence interval (CI) for scalar $\nu = \boldsymbol{a}^\top\boldsymbol{\mu}$ is

$$\{\nu : n(\boldsymbol{a}^\top\bar{\boldsymbol{x}}-\nu)^2(\boldsymbol{a}^\top\mathbf{S}\boldsymbol{a})^{-1} < F_{1-\alpha,1,n-1}\} = \left(\boldsymbol{a}^\top\bar{\boldsymbol{x}}-t_{1-\alpha/2,n-1}\sqrt{\boldsymbol{a}^\top\mathbf{S}\boldsymbol{a}/n}, \boldsymbol{a}^\top\bar{\boldsymbol{x}}+t_{1-\alpha/2,n-1}\sqrt{\boldsymbol{a}^\top\mathbf{S}\boldsymbol{a}/n}\right)$$

  - Check the coverage probability of CI for each entry of $\boldsymbol{\mu}$

Simultaneous confidence intervals

- Interested in $(1-\alpha_k)$ CIs for scalars $\boldsymbol{a}_k^\top\boldsymbol{\mu}$, say CI_k, $k = 1,\ldots,m$, simultaneously
- Make sure $\Pr(\bigcap_k\{\boldsymbol{a}_k^\top\boldsymbol{\mu} \in \mathrm{CI}_k\}) \ge 1-\alpha$
- Bonferroni correction
  - Bonferroni inequality:

$$\Pr(\bigcap_{k=1}^m\{\boldsymbol{a}_k^\top\boldsymbol{\mu} \in \mathrm{CI}_k\}) = 1-\Pr(\bigcup_{k=1}^m\{\boldsymbol{a}_k^\top\boldsymbol{\mu} \notin \mathrm{CI}_k\}) \ge 1-\sum_{k=1}^m \Pr(\boldsymbol{a}_k^\top\boldsymbol{\mu} \notin \mathrm{CI}_k) = 1-\sum_{k=1}^m \alpha_k$$

  - Taking α_k such that $\alpha = \sum_{k=1}^m \alpha_k$, e.g., $\alpha_k = \alpha/m$, i.e.,

$$(\boldsymbol{a}_k^\top\bar{\boldsymbol{x}} - t_{1-\alpha/(2m),n-1}\sqrt{\boldsymbol{a}_k^\top\mathbf{S}\boldsymbol{a}_k/n}, \boldsymbol{a}_k^\top\bar{\boldsymbol{x}} + t_{1-\alpha/(2m),n-1}\sqrt{\boldsymbol{a}_k^\top\mathbf{S}\boldsymbol{a}_k/n})$$

  - Working for small m
- Scheffé's method
  - Let $\mathrm{CI}_{\boldsymbol{w}} = (\boldsymbol{w}^\top\bar{\boldsymbol{x}} - c\sqrt{\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w}/n}, \boldsymbol{w}^\top\bar{\boldsymbol{x}} + c\sqrt{\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w}/n})$ for all $\boldsymbol{w} \in \mathbb{R}^p$
  - Derive that $c = \sqrt{p(n-1)(n-p)^{-1}F_{1-\alpha,p,n-p}}$

* By Cauchy–Schwarz: $\{\boldsymbol{w}^\top(\bar{\boldsymbol{x}}-\boldsymbol{\mu})\}^2 = [(\mathbf{S}^{1/2}\boldsymbol{w})^\top\{\mathbf{S}^{-1/2}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})\}]^2 \le \{(\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w})^\top/n\}\{n(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^\top\mathbf{S}^{-1}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})\} \Rightarrow$

$$
\begin{aligned}
\Pr(\bigcap_{k=1}^{m}\{\boldsymbol{a}_k^\top\boldsymbol{\mu}\in \mathrm{CI}_k\}) \ge \Pr(\bigcap_{\boldsymbol{w}\in\mathbb{R}^p}\{\boldsymbol{w}^\top\boldsymbol{\mu}\in\mathrm{CI}_{\boldsymbol{w}}\}) &= 1-\Pr(\bigcup_{\boldsymbol{w}\in\mathbb{R}^p}\{\boldsymbol{w}^\top\boldsymbol{\mu}\notin\mathrm{CI}_{\boldsymbol{w}}\}) \\
&= 1-\Pr(\bigcup_{\boldsymbol{w}\in\mathbb{R}^p}[\{\boldsymbol{w}^\top(\bar{\mathbf{X}}-\boldsymbol{\mu})\}^2/\{(\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w})^\top/n\} > c^2]) \\
&\ge 1-\Pr(\{n(\bar{\mathbf{X}}-\boldsymbol{\mu})^\top\mathbf{S}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu}) > c^2\})
\end{aligned}
$$

* $\Pr(\{n(\bar{\mathbf{X}}-\boldsymbol{\mu})^\top\mathbf{S}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu}) > c^2\}) = \alpha \Rightarrow c = \sqrt{p(n-1)(n-p)^{-1}F_{1-\alpha,p,n-p}}$

- Working for large even infinite m

Comparing two multivariate means (J&W Sec. 6.3)

- Two independent samples of (potentially) different sizes from two distributions with equal covariance
  - $\mathbf{X}_{11},\ldots,\mathbf{X}_{1n_1} \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu}_1,\boldsymbol{\Sigma})$
  - $\mathbf{X}_{21},\ldots,\mathbf{X}_{2n_2} \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu}_2,\boldsymbol{\Sigma})$
- Let $\bar{\mathbf{X}}_i$ and $\mathbf{S}_i$ be the sample mean and sample covariance for the ith sample
- Hypotheses $H_0:\boldsymbol{\mu}_1=\boldsymbol{\mu}_2$ v.s. $H_1:\boldsymbol{\mu}_1\ne\boldsymbol{\mu}_2$
- Test statistic following LRT

$$(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2)^\top\{(n_1^{-1}+n_2^{-1})\mathbf{S}_{\text{pool}}\}^{-1}(\bar{\mathbf{X}}_1-\bar{\mathbf{X}}_2) \sim \frac{(n_1+n_2-2)p}{n_1+n_2-p-1}F(p,n_1+n_2-p-1)$$

- Rejection region at level α

$$\left\{x_{11},\ldots,x_{1n_1},x_{21},\ldots,x_{2n_2} : (\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)^\top\{(n_1^{-1}+n_2^{-1})\mathbf{S}_{\text{pool}}\}^{-1}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2) \ge \frac{p(n_1+n_2-2)}{n_1+n_2-p-1}F_{1-\alpha,p,n_1+n_2-p-1}\right\}$$

  - $\mathbf{S}_{\text{pool}} = \frac{(n_1-1)\mathbf{S}_1+(n_2-1)\mathbf{S}_2}{n_1+n_2-2}$
- p-value

$$1-F_{F_{1-\alpha,p,n_1+n_2-p-1}}\left[\frac{n_1+n_2-p-1}{p(n_1+n_2-2)}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)^\top\{(n_1^{-1}+n_2^{-1})\mathbf{S}_{\text{pool}}\}^{-1}(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)\right]$$
