

STAT 4100 Lecture Note

Week Twelve (Nov 28, 30 & Dec 2, 2022)

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Approximation to the variance of $\hat{\theta}_n$

- Why?
 - Reflect the variation or dispersion of $\hat{\theta}_n$
 - Help approximate the distribution of $\hat{\theta}_n$ (and further construct the confidence region for θ) if assuming normality
- How?
 - Utilizing the asymptotic variance of $\hat{\theta}_n$
 - Resampling methods, e.g., bootstrapping

CB Example 10.1.17 & Ex. 10.9 (con'd)

- iid $X_1, \dots, X_n \sim p(x | \lambda) = \lambda^x \exp(-\lambda)/x!$, $x \in \mathbb{Z}^+$, $\lambda > 0$. Define $\theta = \Pr(X_i = 2 | \lambda) = \lambda^2 \exp(-\lambda)/2$. Approximate the variance of $\hat{\theta}_{\text{ML}} = \bar{X}_n^2 \exp(-\bar{X}_n)/2$ by delta methods.

CB Example 10.1.15

- Holding iid $X_i \sim \text{Bernoulli}(p)$, the variance of $\text{Bernoulli}(p)$ is $\tau(p) = p(1-p)$ whose MLE is $\tau(\hat{p}_{\text{ML}}) = \bar{X}_n(1 - \bar{X}_n)$. Approximate $\text{var}\{\tau(\hat{p}_{\text{ML}})\}$ by delta methods.

Bootstrapping the variance of $\hat{\theta}_n$ (CB Sec. 10.1.4)

- Nonparametric bootstrap:
 1. For j in $1 : B$, do steps 2–3.
 2. Draw the j th resample \mathbf{x}_j^* of size n from the original sample $\mathbf{x} = \{x_1, \dots, x_n\}$, with replacement, i.e., create a new iid sample \mathbf{x}_j^* from F_n (the empirical cdf of the original sample)
 3. Let $\hat{\theta}_j^* = \hat{\theta}(\mathbf{x}_j^*)$.
 4. $\text{var}(\hat{\theta}) \approx$ the sample variance of $\{\hat{\theta}_1^*, \dots, \hat{\theta}_B^*\}$.
- (Optional, see, e.g., www.stat.columbia.edu/~bodhi/Talks/Emp-Proc-Lecture-Notes.pdf) Empirical process: theoretical foundation for nonparametric bootstrap
 - (Glivenko-Cantelli) $\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow{\text{a.s.}} 0$
 - (Donsker) $\sqrt{n}(F_n - F) \xrightarrow{d} \text{BB} \circ F$, i.e., $\mathbb{E}[g\{\sqrt{n}(F_n - F)\}] \rightarrow \mathbb{E}[g(\text{BB} \circ F)]$ for all bounded, continuous and real-valued g
 - * BB is a Gaussian process (specifically, standard Brownian bridge process on $[0, 1]$), i.e.,
 - $\text{BB}(0) = \text{BB}(1) = 0$ but $\text{BB}(t) \sim \mathcal{N}(0, t(1-t))$ for $t \in (0, 1)$;

- fixing $t_1, \dots, t_p \in (0, 1)$, $[BB(t_1), \dots, BB(t_p)]^\top$ is of multivariate normal with $\text{cov}(BB(s), BB(t)) = \min(s, t) - st$;
- $BB(t)$ is continuous in t .

```
options(digits = 4)
set.seed(1)
ts = (0:1000)/1000
delta_t = 1/1000
for (i in seq_len(animation::ani.options("nmax"))) {
  dev.hold()
  W = cumsum(c(0, rnorm(n = length(ts)-1, mean = 0, sd = delta_t^.5)))
  BB = W-ts*W[length(W)]
  plot(y = BB, x = ts, xlim = c(0,1), ylim = c(-1.5,1.5), type='l', xlab = 't', ylab = 'BB(t)',
       main = paste('Sample path', i, 'of the standard Brownian Bridge on [0,1]'))
  abline(h = 0, lty = 2)
  animation::ani.pause()
}
```

- Parametric bootstrap:
 1. For j in $1 : B$, do steps 2–3.
 2. Draw the j th resample \mathbf{x}_j^* of size n from a fitted model $f(x | \hat{\theta})$.
 3. Let $\hat{\theta}_j^* = \hat{\theta}(\mathbf{x}_j^*)$.
 4. $\text{var}(\hat{\theta}) \approx$ the sample variance of $\{\hat{\theta}_1^*, \dots, \hat{\theta}_B^*\}$.

CB Example 10.1.15

- Holding iid $X_i \sim \text{Bernoulli}(p)$, the variance of $\text{Bernoulli}(p)$ is $\tau(p) = p(1 - p)$ for which the MLE is $\tau(\hat{p}_{\text{ML}}) = \bar{X}_n(1 - \bar{X}_n)$. Approximate $\text{var}\{\tau(\hat{p}_{\text{ML}})\}$ by the bootstrap.

```
options(digits = 4)
set.seed(1)
B = 1e4L
n = 30
x = rbinom(n, 1, prob = .7)
theta_ml = mean(x)
tau_theta_star_np = numeric(B)
tau_theta_star_p = numeric(B)
# Nonparametric bootstrap
for (j in 1:B){
  x_star = sample(x, size = n, replace = T)
  tau_theta_star_np[j] = mean(x_star)*(1-mean(x_star))
}
var(tau_theta_star_np)
# Parametric bootstrap
for (j in 1:B){
  x_star = rbinom(n, size = 1, prob = theta_ml)
  tau_theta_star_p[j] = mean(x_star)*(1-mean(x_star))
}
var(tau_theta_star_p)
# Estimate via the first-order delta method
theta_ml*(1-theta_ml)*(1-2*theta_ml)^2/n
# Estimate via the second-order delta method
2*theta_ml^2*(1-theta_ml)^2/n^2
```

Large-sample hypothesis testing

Recall the LRT

- $H_0 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_0$ v.s. $H_1 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_1$, where $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0 \cup \boldsymbol{\Theta}_1$
- LRT statistic

$$\lambda(\mathbf{x}) = \frac{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} L(\boldsymbol{\theta}; \mathbf{x})}{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} L(\boldsymbol{\theta}; \mathbf{x})} = \frac{L(\hat{\boldsymbol{\theta}}_{0, \text{ML}}; \mathbf{x})}{L(\hat{\boldsymbol{\theta}}_{\text{ML}}; \mathbf{x})}$$

- $\hat{\boldsymbol{\theta}}_{0, \text{ML}}$: constrained MLE for $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$
- $\hat{\boldsymbol{\theta}}_{\text{ML}}$: unconstrained MLE for $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- $\{\mathbf{x} : \lambda(\mathbf{x}) \leq c_\alpha\}$: rejection region of level α LRT
 - c_α is such defined that $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \Pr(\lambda(\mathbf{X}) \leq c_\alpha \mid \boldsymbol{\theta}) = \alpha$

Asymptotic distribution of LRT statistic (CB Thm 10.3.1 & 10.3.3)

- Under H_0 , as $n \rightarrow \infty$,

$$-2 \ln \lambda(\mathbf{X}) \xrightarrow{d} \chi^2(\nu),$$

where ν = difference of numbers of free parameters in $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}$.

- (CB Thm 10.3.3) $\{\mathbf{x} : -2 \ln \lambda(\mathbf{x}) \geq \chi_{\nu, 1-\alpha}^2\}$: asymptotic rejection region of level α LRT
 - $\chi_{\nu, 1-\alpha}^2$ is the $1 - \alpha$ quantile of $\chi^2(\nu)$.

CB Example 10.3.4

- iid $X_1, \dots, X_n \sim \Pr(X_i = j) = p_j, j = 1, \dots, 5$. Specify the $1 - \alpha$ LRT rejection region for $H_0 : p_1 = p_2 = p_3$ and $p_4 = p_5$ vs. H_1 : Otherwise.

Wald test (CB pp. 493)

- $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$
 - Wald statistic: $(\hat{\theta}_n - \theta_0) / \sqrt{\text{var}(\hat{\theta}_n)}$ (if $(\hat{\theta}_n - \theta_0) / \sqrt{\text{var}(\hat{\theta}_n)} \xrightarrow{d} \mathcal{N}(0, 1)$ under H_0 as $n \rightarrow \infty$)
 - * Asymptotically equivalent to LRT for this two sided test if $\hat{\theta}_n = \hat{\theta}_{\text{ML}}$
 - * Substitute $\widehat{\text{var}}(\hat{\theta}_n)$ for $\text{var}(\hat{\theta}_n)$ if $\text{var}(\hat{\theta}_n)$ is well approximated by $\widehat{\text{var}}(\hat{\theta}_n)$
 - Level α rejection region: $\{\mathbf{x} : |\hat{\theta}_n - \theta_0| / \sqrt{\text{var}(\hat{\theta}_n)} \geq \Phi_{1-\alpha/2}^{-1}\}$

Score test (CB pp. 494)

- $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$
 - Score statistic: $S(\theta_0; \mathbf{X}) / \sqrt{I_n(\theta_0)}$ ($\xrightarrow{d} \mathcal{N}(0, 1)$ under H_0 as $n \rightarrow \infty$)
 - Level α rejection region: $\{\mathbf{x} : |S(\theta_0; \mathbf{x})| / \sqrt{I_n(\theta_0)} \geq \Phi_{1-\alpha/2}^{-1}\}$.
- If $\boldsymbol{\Theta}_0$ contains more than one points, then substitute $\hat{\theta}_{0, \text{ML}}$ for θ_0 . So the score test at most involves the constrained MLE.

CB Examples 10.3.5 & 10.3.6

- iid $X_1, \dots, X_n \sim \text{Bernoulli}(p), p \in (0, 1)$. Derive LRT, Wald and score tests for $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

Asymptotic confidence regions

- Constructed by reverting rejection regions
- Examples
 - $1 - \alpha$ LRT confidence region for θ : $\{\theta : -2 \ln\{L(\theta; \mathbf{x})/L(\hat{\theta}_{\text{ML}}; \mathbf{x})\} < \chi^2_{1,1-\alpha}\}$
 - $1 - \alpha$ Wald confidence region for θ : $\{\theta : |\hat{\theta}_n - \theta|/\sqrt{\text{var}(\hat{\theta}_n)} < \Phi_{1-\alpha/2}^{-1}\}$
 - $1 - \alpha$ score confidence region for θ : $\{\theta : |S(\theta; \mathbf{x})|/\sqrt{I_n(\theta)} < \Phi_{1-\alpha/2}^{-1}\}$

CB Examples 10.4.2, 10.4.3 & 10.4.5

- iid $X_1, \dots, X_n \sim \text{Bernoulli}(p)$, construct $1 - \alpha$ confidence intervals for p .

Take-home exercises (NOT to be submitted; to be potentially covered in labs)

- CB Ex. 10.17(a-c), 10.36, 10.38
- HMC Ex. 6.3.16–6.3.18