

STAT 3690 Homework 1

zhiyanggeezhou.github.io

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca)

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Answers must be submitted electronically via Crowdmark. Please enclose your R source code (if applicable) as well.

1. The function $\text{cov}(\cdot, \cdot)$ is bilinear, i.e., for random vectors \mathbf{W} , \mathbf{X} , \mathbf{Y} and \mathbf{Z} and fixed matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} , one has $\text{cov}(\mathbf{AW} + \mathbf{BX}, \mathbf{Y}) = \mathbf{A}\Sigma_{\mathbf{WY}} + \mathbf{B}\Sigma_{\mathbf{XY}}$ and $\text{cov}(\mathbf{W}, \mathbf{CY} + \mathbf{DZ}) = \Sigma_{\mathbf{WY}}\mathbf{C}^\top + \Sigma_{\mathbf{WZ}}\mathbf{D}^\top$, where $\mathbf{AW} + \mathbf{BX}$ and $\mathbf{CY} + \mathbf{DZ}$ both make sense.
 - a. Prove this bilinearity.
 - b. Rephrase $\text{cov}(\mathbf{AW} + \mathbf{BX}, \mathbf{CY} + \mathbf{DZ})$ in terms of matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} , $\Sigma_{\mathbf{WY}}$, $\Sigma_{\mathbf{WZ}}$, $\Sigma_{\mathbf{XY}}$ and $\Sigma_{\mathbf{XZ}}$.
2. Let \mathbf{A} be a square matrix with eigendecomposition $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$. Given a real number c (\neq any eigenvalue of \mathbf{A}), express the eigendecomposition of $(\mathbf{A} - c\mathbf{I})^{-1}$ in terms of \mathbf{U} , $\mathbf{\Lambda}$, \mathbf{I} and c . (Hint: $\mathbf{I} = \mathbf{U}\mathbf{U}^{-1}$)
3. Let W be a discrete random variable such that $\Pr(W = 1) = \Pr(W = -1) = 1/2$. Define $Y = WX$ with $X \sim N(0, 1)$ and $X \perp\!\!\!\perp W$. Prove the following identities.
 - a. $Y \sim N(0, 1)$.
 - b. X and Y are uncorrelated with each other.
 - c. X is not independent of Y .
4. Let $\mathbf{X} = [X_1, X_2, X_3]^\top \sim MVN_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with

$$\boldsymbol{\mu} = [6, 1, 4]^\top, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

- a. Find the conditional distribution of X_2 given $X_1 = 2$ and $X_3 = 1$.
- b. Find the distribution of random 2-vector $\mathbf{Y} = [3X_1 - 2X_2 + X_3, X_2 - X_3]^\top$.
- c. Find $w_1, w_2 \in \mathbb{R}$ such that $W = w_1X_1 + w_2X_2 + X_3$ is independent of \mathbf{Y} . (Hint: don't forget to verify the normality of random 3-vector $[W, \mathbf{Y}^\top]^\top$ after figuring out values of w_1 and w_2 .)