

PH 712 Probability and Statistical Inference

Part II: Mutiple Random Variables

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Recalling the cdf of a single RV

- The cdf of X :

$$F_X(t) = \Pr(X \leq t), \quad t \in \mathbb{R},$$

is the probability that event $X \leq t$ happens.

– \mathbb{R} : the set of all real numbers.

- Knowing $F_X \Leftrightarrow$ knowing the distribution of X .

Joint cdf of multiple random variables

- Extension to multiple RVs X_1, \dots, X_n
- The joint cdf of n RVs X_1, \dots, X_n :

$$F_{X_1, \dots, X_n}(t_1, \dots, t_n) = \Pr(\{X_1 \leq t_1\} \cap \dots \cap \{X_n \leq t_n\}), \quad (t_1, \dots, t_n) \in \mathbb{R}^n,$$

is the probability that n events $\{X_1 \leq t_1\}, \dots, \{X_n \leq t_n\}$ occur simultaneously.

– \mathbb{R}^n : the n -dimensional Euclidean space, or roughly, the set of all real vectors of length n .

– When $n = 1$, $F_{X_1, \dots, X_n}(t_1, \dots, t_n)$ reduces to $F_{X_1}(t_1) = \Pr(X_1 \leq t_1)$, the cdf of X_1 .

- Knowing $F_{X_1, \dots, X_n} \Leftrightarrow$ knowing the joint distribution of X_1, \dots, X_n .
- Connection to F_{X_i} , the cdf of X_i :

$$F_{X_i}(t_i) = F_{X_1, \dots, X_n}(\infty, \dots, \infty, t_i, \infty, \dots, \infty), \quad t_i \in \mathbb{R}$$

– E.g., for $n = 3$,

$$F_{X_1}(t_1) = F_{X_1, \dots, X_3}(t_1, \infty, \infty), \quad t_1 \in \mathbb{R};$$

$$F_{X_2}(t_2) = F_{X_1, \dots, X_3}(\infty, t_2, \infty), \quad t_2 \in \mathbb{R};$$

$$F_{X_3}(t_3) = F_{X_1, \dots, X_3}(\infty, \infty, t_3), \quad t_3 \in \mathbb{R}.$$

- Knowing $F_{X_1, \dots, X_n} \Rightarrow$ knowing the joint cdf of any subcollection of X_1, \dots, X_n
 - E.g., for $n = 3$,

$$F_{X_1, X_2}(t_1, t_2) = F_{X_1, X_2, X_3}(t_1, t_2, \infty), \quad (t_1, t_2) \in \mathbb{R}^2;$$

$$F_{X_2, X_3}(t_2, t_3) = F_{X_1, X_2, X_3}(\infty, t_2, t_3), \quad (t_2, t_3) \in \mathbb{R}^2;$$

$$F_{X_1, X_3}(t_1, t_3) = F_{X_1, X_2, X_3}(t_1, \infty, t_3), \quad (t_1, t_3) \in \mathbb{R}^2.$$

Joint pmf of discrete X_1, \dots, X_n

- Merely existing in the case that ALL X_1, \dots, X_n are discrete RVs
- The joint pmf of n RVs X_1, \dots, X_n :

$$p_{X_1, \dots, X_n}(t_1, \dots, t_n) = \Pr(\{X_1 = t_1\} \cap \dots \cap \{X_n = t_n\}), \quad (t_1, \dots, t_n) \in \mathbb{R}^n,$$

is the probability that n events $\{X_1 = t_1\}, \dots, \{X_n = t_n\}$ occur simultaneously.

- $\text{supp}(X_1, \dots, X_n) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : p_{X_1, \dots, X_n}(t_1, \dots, t_n) > 0\}$
- When $n = 1$, $p_{X_1, \dots, X_n}(t_1, \dots, t_n)$ reduces to $p_{X_1}(t_1) = \Pr(X_1 = t_1)$, the pmf of X_1 .

- Knowing $p_{X_1, \dots, X_n} \Leftrightarrow$ knowing the joint distribution of X_1, \dots, X_n .
- Connection to p_{X_i} , the pmf of X_i :

$$p_{X_i}(t_i) = \sum_{t_1=-\infty}^{\infty} \cdots \sum_{t_{i-1}=-\infty}^{\infty} \sum_{t_{i+1}=-\infty}^{\infty} \cdots \sum_{t_n=-\infty}^{\infty} p_{X_1, \dots, X_n}(t_1, \dots, t_n), \quad t_i \in \mathbb{R}$$

- E.g., for $n = 3$,

$$p_{X_1}(t_1) = \sum_{t_2=-\infty}^{\infty} \sum_{t_3=-\infty}^{\infty} p_{X_1, X_2, X_3}(t_1, t_2, t_3), \quad t_1 \in \mathbb{R};$$

$$p_{X_2}(t_2) = \sum_{t_1=-\infty}^{\infty} \sum_{t_3=-\infty}^{\infty} p_{X_1, X_2, X_3}(t_1, t_2, t_3), \quad t_2 \in \mathbb{R};$$

$$p_{X_3}(t_3) = \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} p_{X_1, X_2, X_3}(t_1, t_2, t_3), \quad t_3 \in \mathbb{R}.$$

- Knowing $p_{X_1, \dots, X_n} \Rightarrow$ knowing the joint pmf of any subcollection of X_1, \dots, X_n
- E.g., for $n = 3$,

$$p_{X_1, X_2}(t_1, t_2) = \sum_{t_3=-\infty}^{\infty} p_{X_1, X_2, X_3}(t_1, t_2, t_3), \quad (t_1, t_2) \in \mathbb{R}^2;$$

$$p_{X_2, X_3}(t_2, t_3) = \sum_{t_1=-\infty}^{\infty} p_{X_1, X_2, X_3}(t_1, t_2, t_3), \quad (t_2, t_3) \in \mathbb{R}^2;$$

$$p_{X_1, X_3}(t_1, t_3) = \sum_{t_2=-\infty}^{\infty} p_{X_1, X_2, X_3}(t_1, t_2, t_3), \quad (t_1, t_3) \in \mathbb{R}^2.$$

Joint pdf of continuous X_1, \dots, X_n

- Merely existing in the case that ALL X_1, \dots, X_n are continuous RVs
- The joint pdf of n RVs X_1, \dots, X_n :

$$f_{X_1, \dots, X_n}(t_1, \dots, t_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} F_{X_1, \dots, X_n}(t_1, \dots, t_n), \quad (t_1, \dots, t_n) \in \mathbb{R}^n$$

- $\text{supp}(X_1, \dots, X_n) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : f_{X_1, \dots, X_n}(t_1, \dots, t_n) > 0\}$
- When $n = 1$, $f_{X_1, \dots, X_n}(t_1, \dots, t_n)$ reduces to $f_{X_1}(t_1) = \frac{d}{dt_1} F_{X_1}(t_1)$, the pdf of X_1 .

- Knowing $f_{X_1, \dots, X_n} \Leftrightarrow$ knowing the joint distribution of X_1, \dots, X_n .
- Connection to f_{X_i} , the pdf of X_i :

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1, \dots, X_n}(t_1, \dots, t_n) dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_n,$$

- E.g., for $n = 3$,

$$p_{X_1}(t_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(t_1, t_2, t_3) dt_2 dt_3, \quad t_1 \in \mathbb{R};$$

$$p_{X_2}(t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(t_1, t_2, t_3) dt_1 dt_3, \quad t_2 \in \mathbb{R};$$

$$p_{X_3}(t_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(t_1, t_2, t_3) dt_1 dt_2, \quad t_3 \in \mathbb{R}.$$

- Knowing $f_{X_1, \dots, X_n} \Rightarrow$ knowing the joint pdf of any subcollection of X_1, \dots, X_n

- E.g., for $n = 3$,

$$f_{X_1, X_2}(t_1, t_2) = \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(t_1, t_2, t_3) dt_3, \quad (t_1, t_2) \in \mathbb{R}^2;$$

$$f_{X_2, X_3}(t_2, t_3) = \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(t_1, t_2, t_3) dt_1, \quad (t_2, t_3) \in \mathbb{R}^2;$$

$$f_{X_1, X_3}(t_1, t_3) = \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(t_1, t_2, t_3) dt_2, \quad (t_1, t_3) \in \mathbb{R}^2.$$

(Mutual) independence

- RVs X_1, \dots, X_n are (mutually) independent \Leftrightarrow

$$F_{X_1, \dots, X_n}(t_1, \dots, t_n) = \prod_{i=1}^n F_{X_i}(t_i)$$

- For discrete X_1, \dots, X_n , joint pmf $p_{X_1, \dots, X_n}(t_1, \dots, t_n) = \prod_{i=1}^n p_{X_i}(t_i)$
- For continuous X_1, \dots, X_n , joint pdf $f_{X_1, \dots, X_n}(t_1, \dots, t_n) = \prod_{i=1}^n f_{X_i}(t_i)$

Example Lec2.1

- X_1 and X_2 are independent Bernoulli RVs with pmf $p_{X_i}(0) = 1 - p_i$ and $p_{X_i}(1) = p_i$, $i = 1, 2$. Write the joint pmf of X_1, X_2 .

- Let X_1 and X_2 be independent Poisson RVs with pmf $p_{X_i}(k_i) = \frac{e^{-3} \cdot 3^{k_i}}{k_i!}$, $k_i = 0, 1, \dots$, $i = 1, 2$. Write the joint pmf of X_1, X_2 .

- X_1 and X_2 are independent uniform RVs with pdf $f_{X_i}(t_i) = \mathbf{1}_{[0,1]}(t_i)$, $i = 1, 2$. Write the joint pdf of X_1, X_2 .

- Let X_1 and X_2 be independent exponential RVs with pdf $f_{X_i}(t_i) = 2e^{-2t_i} \cdot \mathbf{1}_{(0,\infty)}(t_i)$, $i = 1, 2$. Write the joint pdf of X_1, X_2 .