

# STAT 4100 Lecture Note

## Week Seven (Oct 17, 19, & 21, 2022)

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2022/Nov/02 17:06:12

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## Review for midterm (con'd)

To cover CB Ex. 7.66, 7.58, & 7.57

## Hypothesis Testing

### Binary classification

- Assume  $\mathbf{X} = [X_1, \dots, X_n]^\top \sim f(\mathbf{x} | \boldsymbol{\theta}^*) \in \{f(\mathbf{x} | \boldsymbol{\theta}) : \boldsymbol{\theta} \in \Theta\}$ 
  - Fixed unknown  $\boldsymbol{\theta}^*$  to be inferred
- Make a decision on  $\boldsymbol{\theta}^*$  between two hypotheses  $H_0 : \boldsymbol{\theta}^* \in \Theta_0$  and  $H_1 : \boldsymbol{\theta}^* \in \Theta_1$ 
  - $\Theta_0 \cup \Theta_1 = \Theta$
  - $\Theta_0 \cap \Theta_1 = \emptyset$
- Decision and correctness
  - True positive (TP) =  $H_0$  correctly rejected
  - False positive (FP, type I error) =  $H_0$  incorrectly rejected
  - True negative (TN) =  $H_0$  is correctly accepted
  - False negative (FN, type II error) =  $H_0$  incorrectly accepted
- E.g.,  $H_0$  : healthy vs  $H_1$  : sick
  - TP: sick people identified as sick
  - FP: healthy people identified as sick
  - TN: healthy people identified as healthy
  - FN: sick people identified as healthy

	Accept $H_0$	Reject $H_0$
$H_0$ is true	True negative (TN)	False positive (FP, type I error)
$H_0$ is false	False negative (FN, type II error)	True positive (TP)

- Misclassification rate =  $\Pr(\text{FP}) + \Pr(\text{FN})$
- False discovery rate (FDR) =  $\Pr(\text{FP}) / \{\Pr(\text{FP}) + \Pr(\text{TP})\}$

- FDR controlling for sequential/simultaneous testing
- Receiver operating characteristic curve (ROC curve): plot of TPR vs FPR
  - True positive rate (TPR, sensitivity) =  $\Pr(\text{TP})/\{\Pr(\text{TP}) + \Pr(\text{FN})\}$
  - False positive rate (FPR) =  $\Pr(\text{FP})/\{\Pr(\text{FP}) + \Pr(\text{FN})\}$
  - Area under the ROC curve (AUC)
- True negative rate (TNR, specificity) =  $\Pr(\text{TN})/\{\Pr(\text{TN}) + \Pr(\text{FP})\}$
- The optimal hypothesis testing seeking to minimize  $\Pr(\text{FN})$  subject to capped  $\Pr(\text{FP})$ , i.e.,

$$\min \Pr(\text{type II error}) \text{ subject to } \Pr(\text{type I error}) \leq \alpha$$

## Power function

- Rejection/critical region:  $R = \{\mathbf{x} : \text{data } \mathbf{x} \text{ corresponding to the rejection of } H_0\}$ 
  - Typically specified in terms of a function of the sample (called the *test statistic*); e.g., if  $R = \{\mathbf{x} : \bar{x} \geq 3\}$ , then  $\bar{X}$  is the test statistic.
- Test function  $\phi : \text{supp}(\mathbf{X}) \rightarrow \{0, 1\}$  defined as  $\phi(\mathbf{x}) = \mathbf{1}_R(\mathbf{x})$ 
  - $\phi(\mathbf{x}) = 1$  implying the rejection of  $H_0$
- Each test function  $\phi$  corresponds to a unique rejection region  $R_\phi = \{\mathbf{x} : \phi(\mathbf{x}) = 1\}$ 
  - Two tests considered to be equivalent if they correspond to the same rejection region/test function
- Power function (for  $\phi$ ):  $\beta_\phi(\boldsymbol{\theta}) = \Pr(\mathbf{X} \in R_\phi \mid \boldsymbol{\theta}) = \mathbb{E}\{\phi(\mathbf{X}) \mid \boldsymbol{\theta}\}$ 
  - $\Pr(\text{type I error}) = \beta_\phi(\boldsymbol{\theta}^*)$  if  $H_0$  is correct ( $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}_0$ )
  - $\Pr(\text{type II error}) = 1 - \beta_\phi(\boldsymbol{\theta}^*)$  if  $H_1$  is correct ( $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}_1$ )
- Prefer larger  $\beta_\phi(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_1$  and smaller  $\beta_\phi(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$  (because  $\boldsymbol{\theta}^*$  is unknown)

## Example Lec14.2

- iid  $X_1, \dots, X_n \sim N(\theta, \sigma_0^2)$  with known  $\sigma_0$ . Consider a test for  $H_0 : \theta = \theta_0$  vs  $H_1 : \theta \neq \theta_0$  with rejection region  $\{\mathbf{x} : \sqrt{n}|\bar{x} - \theta_0|/\sigma_0 > c\}$ .
  - Elaborate the power function.
  - Find sample size  $n$  and threshold  $c$  if one desires that the type I error rate is 5% and the type II error rate at  $\theta_0 + \sigma_0$  is 25%.

## Uniformly most powerful (UMP) level $\alpha$ test (CB Sec 8.3.2)

- $\phi$  is of level  $\alpha$  iff  $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta}) \leq \alpha$ 
  - $\phi$  is of size  $\alpha$  iff  $\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta}) = \alpha$
- Let  $\phi$  is a level  $\alpha$  test for  $H_0 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_0$  vs  $H_1 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_1$ . If  $\beta_\phi(\boldsymbol{\theta}) \geq \beta_{\phi'}(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_1$  and all  $\phi'$  of level  $\alpha$ , then  $\phi$  is a UMP level  $\alpha$  test.
- If  $\phi$  is a UMP level  $\alpha$  test, then  $\beta_\phi(\boldsymbol{\theta}) \geq \alpha \geq \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta})$  for all  $\boldsymbol{\theta} \in \boldsymbol{\Theta}_1$  (unbiasedness for testing, CB Def 8.3.9)

## UMP level $\alpha$ test for simple hypotheses ( $H_0 : \boldsymbol{\theta}^* = \boldsymbol{\theta}_0$ vs $H_1 : \boldsymbol{\theta}^* = \boldsymbol{\theta}_1$ )

- To maximize  $\beta_\phi(\boldsymbol{\theta}_1)$  with respect to  $\phi$  subject to  $\beta_\phi(\boldsymbol{\theta}_0) \leq \alpha$
- Neymann-Pearson (NP) Lemma (CB Thm 8.3.12):  $\phi$  is the UMP test of level  $\alpha$  for simple hypotheses  $\iff \exists c > 0$  such that  $\beta_\phi(\boldsymbol{\theta}_0) = \mathbb{E}\{\phi_c(\mathbf{X}) \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} = \alpha$ , where

$$\phi_c(\mathbf{x}) = \begin{cases} 1, & f(\mathbf{x} \mid \boldsymbol{\theta}_1) > cf(\mathbf{x} \mid \boldsymbol{\theta}_0), \\ 0, & f(\mathbf{x} \mid \boldsymbol{\theta}_1) < cf(\mathbf{x} \mid \boldsymbol{\theta}_0). \end{cases}$$

- (Optional) if  $\Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) = c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} \neq 0$ , then  $c$  is taken as the largest real number satisfying that

$$\Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) \geq c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} \geq \alpha \text{ and } \Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) \leq c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} \geq 1 - \alpha.$$

Meanwhile, the test function should become  $\phi_{c,\gamma}$  instead of  $\phi_c$ , where

$$\phi_{c,\gamma}(\mathbf{x}) = \begin{cases} 1, & f(\mathbf{x} \mid \boldsymbol{\theta}_1)/f(\mathbf{x} \mid \boldsymbol{\theta}_0) > c, \\ \gamma, & f(\mathbf{x} \mid \boldsymbol{\theta}_1)/f(\mathbf{x} \mid \boldsymbol{\theta}_0) = c, \\ 0, & f(\mathbf{x} \mid \boldsymbol{\theta}_1)/f(\mathbf{x} \mid \boldsymbol{\theta}_0) < c, \end{cases}$$

That is, reject  $H_0$  with probability  $\gamma \in [0, 1]$  if observing  $f(\mathbf{x} \mid \boldsymbol{\theta}_1)/f(\mathbf{x} \mid \boldsymbol{\theta}_0) = c$ . The probability  $\gamma$  is chosen to make sure that the size is  $\alpha$ , i.e.,

$$\alpha = E\{\phi_{c,\gamma}(\mathbf{X}) \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} = \Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) > c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} + \gamma \Pr\{f(\mathbf{X} \mid \boldsymbol{\theta}_1)/f(\mathbf{X} \mid \boldsymbol{\theta}_0) = c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\}.$$

- According to NP Lemma, for simple hypotheses, UMP test at level  $\alpha \iff$  UMP test at size  $\alpha$ .
- UMP test and sufficiency (CB Coro 8.3.13): sufficient statistics can be taken as test statistics for UMP test.