PH 712 Probability and Statistical Inference

Recap for Final

Zhiyang Zhou (zhou67@uwm.edu, zhiyanggeezhou.github.io)

2024/Dec/11 09:57:42

Hypothesis testing (Part IX)

- Hypotheses
  - $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_1$.
    * $\Theta = \Theta_0 \cup \Theta_1$
    * $\emptyset = \Theta_0 \cap \Theta_1$
    * Θ_0 must be a closed set, i.e., include "=" in H_0
- Rejection region R: reject H_0 if $\{x_1, \ldots, x_n\} \in R$
- Power function: $\beta(\theta) = \Pr(\{X_1, \ldots, X_n\} \in R \mid \theta)$, $\theta \in \Theta$
  - Pr(type I error) $= \beta_\phi(\theta)$ if H_0 is correct (i.e., $\theta \in \Theta_0$)
  - Pr(type II error) $= 1 - \beta_\phi(\theta)$ if H_1 is correct (i.e., $\theta \in \Theta_1$)
- Level α: $\sup_{\theta\in\Theta_0} \beta(\theta) \le \alpha$
  - Size α: $\sup_{\theta\in\Theta_0} \beta(\theta) = \alpha$
- Uniformly most powerful (UMP) level α test $\iff$ its power function $\ge$ power functions of any level α tests at each $\theta \in \Theta_1$
- Likelihood ratio test (LRT, equivalent to the UMP test when the UMP test exists)
  - Test statistic
$$\lambda(X_1, \ldots, X_n) = \frac{L(\hat\theta_{\mathrm{ML},0})}{L(\hat\theta_{\mathrm{ML}})}$$
    * $\hat\theta_{\mathrm{ML},0}$: MLE with constraint $\theta \in \Theta$
    * $\hat\theta_{\mathrm{ML},0}$: MLE with constraint $\theta \in \Theta_0$
  - Level α LRT rejection region: $\{\{x_1, \ldots, x_n\} : \lambda(x_1, \ldots, x_n) \le c_\alpha\}$ where c_α is the solution to
$$\sup_{\theta\in\Theta_0} \beta(\theta) = \sup_{\theta\in\Theta_0} \Pr\{\lambda(X_1, \ldots, X_n) \le c_\alpha \mid \theta\} = \alpha$$
  - Special cases: Z-test, t-test and F-test
- Asymptotic LRT
  - More feasible than LRT
  - Test statistic
$$-2\ln\lambda(X_1, \ldots, X_n) = -2\{\ell(\hat\theta_{\mathrm{ML},0}) - \ell(\hat\theta_{\mathrm{ML}})\}$$
  - Level α asymptotic LRT rejection region: $\{\{x_1, \ldots, x_n\} : -2\ln\lambda(x_1, \ldots, x_n) \ge \chi^2_{\nu,1-\alpha}\}$
    * ν: the difference of numbers of free parameters between Θ_0 and Θ

- Wald test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$
  - Test statistic $(\hat{\theta}_{\mathrm{ML}} - \theta_0)/\sqrt{\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})}$
    * Obtain $\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})$ via the Fisher information and/or delta methods
  - Level α Wald rejection region: $\{\{x_1, \ldots, x_n\} : |\hat{\theta}_{\mathrm{ML}} - \theta_0|/\sqrt{\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})} \geq \Phi^{-1}_{1-\alpha/2}\}$
- Score test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$
  - Test statistic: $\ell'(\theta_0)/\sqrt{I_n(\theta_0)}$
  - Level α score rejection region: $\{\{x_1, \ldots, x_n\} : |\ell'(\theta_0)|/\sqrt{I_n(\theta_0)} \geq \Phi^{-1}_{1-\alpha/2}\}$
- p-value
  - A test statistic facilitating the report of testing result because its critical point is by default α, i.e., the rejection region in terms of $p(x_1, \ldots, x_n)$ is always

$$\{\{x_1, \ldots, x_n\} : p(x_1, \ldots, x_n) \leq \alpha\}$$

    * The null distribution of $p(X_1, \ldots, X_n)$ is $U(0, 1)$? Wrong!
  - Not always well-defined
  - Special cases
    * Asymptotic LRT: $p(x_1, \ldots, x_n) = 1 - F_{\chi^2(\nu)}(-2\lambda(x_1, \ldots, x_n))$
      · $F_{\chi^2(\nu)}(\cdot)$: the cdf of $\chi^2(\nu)$
    * Wald test: $p(x_1, \ldots, x_n) = 2\Phi\left(-|\hat{\theta}_{\mathrm{ML}} - \theta_0|/\sqrt{\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})}\right)$
    * Score test: $p(x_1, \ldots, x_n) = 2\Phi\left(-|\ell'(\theta_0)|/\sqrt{I_n(\theta_0)}\right)$

$(1 - \alpha) \times 100\%$ confidence set of θ (Part X)

- A set covering true θ with probability AT LEAST $(1 - \alpha) \times 100\%$
- Constructed by inverting a level α rejection region for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$
- The narrower the better
- Special cases:
  - $(1 - \alpha) \times 100\%$ asymptotic LRT confidence set for θ: $\{\theta : -2(\ell(\theta) - \ell(\hat{\theta}_{\mathrm{ML}})) < \chi^2_{1,1-\alpha}\}$
  - $(1 - \alpha) \times 100\%$ Wald confidence set for θ: $\{\theta : |\hat{\theta}_{\mathrm{ML}} - \theta|/\sqrt{\widehat{\mathrm{var}}(\hat{\theta}_{\mathrm{ML}})} < \Phi^{-1}_{1-\alpha/2}\}$
    * Why is it preferable to use $\hat{\theta}_{\mathrm{ML}}$ rather than other estimators? Hint: Bridge the width of Wald confidence set to the efficiency (Part VI) and asymptotic efficiency (Part VIII).
  - $(1 - \alpha) \times 100\%$ score confidence set for θ: $\{\theta : |\ell'(\theta)|/\sqrt{I_n(\theta)} < \Phi^{-1}_{1-\alpha/2}\}$

Techniques involved in inference

- MLE (Part V)
  - Its consistency and asymptotic efficiency (Part VIII)
- Approximating the variance of $h(\hat{\theta}_{\mathrm{ML}})$ (Part VIII)
  - Fisher information
  - Delta methods