PH 712 Probability and Statistical Inference

Part II: Mutiple Random Variables

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Recalling the cdf of a single RV

- The cdf of X:
$$F_X(t) = \Pr(X \le t), \quad t \in \mathbb{R},$$
is the probability that event $X \le t$ happens.
  - $\mathbb{R}$: the set of all real numbers.
- Knowing $F_X \Leftrightarrow$ knowing the distribution of X.

Joint cdf of multiple random variables

- Extension to multiple RVs $X_1, \ldots, X_n$
- The joint cdf of n RVs $X_1, \ldots, X_n$:
$$F_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = \Pr(\{X_1 \le t_1\} \cap \ldots \cap \{X_n \le t_n\}), \quad (t_1, \ldots, t_n) \in \mathbb{R}^n,$$
is the probability that n events $\{X_1 \le t_1\}, \ldots, \{X_n \le t_n\}$ occur simultaneously.
  - $\mathbb{R}^n$: the n-dimensional Euclidean space, or roughly, the set of all real vectors of length n.
  - When $n = 1$, $F_{X_1,\ldots,X_n}(t_1, \ldots, t_n)$ reduces to $F_{X_1}(t_1) = \Pr(X_1 \le t_1)$, the cdf of X_1.
- Knowing $F_{X_1,\ldots,X_n} \Leftrightarrow$ knowing the joint distribution of $X_1, \ldots, X_n$.
- Connection to F_{X_i}, the cdf of X_i:
$$F_{X_i}(t_i) = F_{X_1,\ldots,X_n}(\infty, \ldots, \infty, t_i, \infty, \ldots, \infty), \quad t_i \in \mathbb{R}$$
  - E.g., for $n = 3$,
$$\begin{aligned} F_{X_1}(t_1) &= F_{X_1,\ldots,X_3}(t_1, \infty, \infty), \quad t_1 \in \mathbb{R}; \\ F_{X_2}(t_2) &= F_{X_1,\ldots,X_3}(\infty, t_2, \infty), \quad t_2 \in \mathbb{R}; \\ F_{X_3}(t_3) &= F_{X_1,\ldots,X_3}(\infty, \infty, t_3), \quad t_3 \in \mathbb{R}. \end{aligned}$$
- Knowing $F_{X_1,\ldots,X_n} \Rightarrow$ knowing the joint cdf of any subcollection of $X_1, \ldots, X_n$
  - E.g., for $n = 3$,
$$\begin{aligned} F_{X_1,X_2}(t_1, t_2) &= F_{X_1,X_2,X_3}(t_1, t_2, \infty), \quad (t_1, t_2) \in \mathbb{R}^2; \\ F_{X_2,X_3}(t_2, t_3) &= F_{X_1,X_2,X_3}(\infty, t_2, t_3), \quad (t_2, t_3) \in \mathbb{R}^2; \\ F_{X_1,X_3}(t_1, t_3) &= F_{X_1,X_2,X_3}(t_1, \infty, t_3), \quad (t_1, t_3) \in \mathbb{R}^2. \end{aligned}$$

Joint pmf of discrete $X_1, \ldots, X_n$

- Merely existing in the case that ALL $X_1, \ldots, X_n$ are discrete RVs
- The joint pmf of n RVs $X_1, \ldots, X_n$:

$$p_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = \Pr(\{X_1 = t_1\} \cap \ldots \cap \{X_n = t_n\}), \quad (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

  is the probability that n events $\{X_1 = t_1\}, \ldots, \{X_n = t_n\}$ occur simultaneously.

  - $\mathrm{supp}(X_1, \ldots, X_n) = \{(t_1, \ldots, t_p) \in \mathbb{R}^n : p_{X_1,\ldots,X_n}(t_1, \ldots, t_n) > 0\}$
  - When $n = 1$, $p_{X_1,\ldots,X_n}(t_1, \ldots, t_n)$ reduces to $p_{X_1}(t_1) = \Pr(X_1 = t_1)$, the pmf of X_1.

- Knowing $p_{X_1,\ldots,X_n} \Leftrightarrow$ knowing the joint distribution of $X_1, \ldots, X_n$.
- Connection to p_{X_i}, the pmf of X_i:

$$p_{X_i}(t_i) = \sum_{t_1=-\infty}^{\infty} \cdots \sum_{t_{i-1}=-\infty}^{\infty} \sum_{t_{i+1}=-\infty}^{\infty} \cdots \sum_{t_n=-\infty}^{\infty} p_{X_1,\ldots,X_n}(t_1, \ldots, t_n), \quad t_i \in \mathbb{R}$$

  - E.g., for $n = 3$,

$$p_{X_1}(t_1) = \sum_{t_2=-\infty}^{\infty} \sum_{t_3=-\infty}^{\infty} p_{X_1,X_2,X_3}(t_1, t_2, t_3), \quad t_1 \in \mathbb{R};$$

$$p_{X_2}(t_2) = \sum_{t_1=-\infty}^{\infty} \sum_{t_3=-\infty}^{\infty} p_{X_1,X_2,X_3}(t_1, t_2, t_3), \quad t_2 \in \mathbb{R};$$

$$p_{X_3}(t_3) = \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} p_{X_1,X_2,X_3}(t_1, t_2, t_3), \quad t_3 \in \mathbb{R}.$$

- Knowing $p_{X_1,\ldots,X_n} \Rightarrow$ knowing the joint pmf of any subcollection of $X_1, \ldots, X_n$
  - E.g., for $n = 3$,

$$p_{X_1,X_2}(t_1, t_2) = \sum_{t_3=-\infty}^{\infty} p_{X_1,X_2,X_3}(t_1, t_2, t_3), \quad (t_1, t_2) \in \mathbb{R}^2;$$

$$p_{X_2,X_3}(t_2, t_3) = \sum_{t_1=-\infty}^{\infty} p_{X_1,X_2,X_3}(t_1, t_2, t_3), \quad (t_2, t_3) \in \mathbb{R}^2;$$

$$p_{X_1,X_3}(t_1, t_3) = \sum_{t_2=-\infty}^{\infty} p_{X_1,X_2,X_3}(t_1, t_2, t_3), \quad (t_1, t_3) \in \mathbb{R}^2.$$

Joint pdf of continuous $X_1, \ldots, X_n$

- Merely existing in the case that ALL $X_1, \ldots, X_n$ are continuous RVs
- The joint pdf of n RVs $X_1, \ldots, X_n$:

$$f_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} F_{X_1,\ldots,X_n}(t_1, \ldots, t_n), \quad (t_1, \ldots, t_n) \in \mathbb{R}^n$$

  - $\mathrm{supp}(X_1, \ldots, X_n) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : f_{X_1,\ldots,X_n}(t_1, \ldots, t_n) > 0\}$
  - When $n = 1$, $f_{X_1,\ldots,X_n}(t_1, \ldots, t_n)$ reduces to $f_{X_1}(t_1) = \frac{\mathrm{d}}{\mathrm{d}t_1} F_{X_1}(t_1)$, the pdf of X_1.

- Knowing $f_{X_1,\ldots,X_n} \Leftrightarrow$ knowing the joint distribution of $X_1, \ldots, X_n$.
- Connection to f_{X_i}, the pdf of X_i:

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1,\ldots,X_n}(t_1, \ldots, t_n) \mathrm{d}t_1 \cdots \mathrm{d}t_{i-1} \mathrm{d}t_{i+1} \cdots \mathrm{d}t_n,$$

- E.g., for $n = 3$,

$$f_{X_1}(t_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(t_1,t_2,t_3)\mathrm{d}t_2\mathrm{d}t_3, \quad t_1 \in \mathbb{R};$$
$$f_{X_2}(t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(t_1,t_2,t_3)\mathrm{d}t_1\mathrm{d}t_3, \quad t_2 \in \mathbb{R};$$
$$f_{X_3}(t_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(t_1,t_2,t_3)\mathrm{d}t_1\mathrm{d}t_2, \quad t_3 \in \mathbb{R}.$$

- Knowing $f_{X_1,\ldots,X_n} \Rightarrow$ knowing the joint pdf of any subcollection of $X_1,\ldots,X_n$
  - E.g., for $n = 3$,

$$f_{X_1,X_2}(t_1,t_2) = \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(t_1,t_2,t_3)\mathrm{d}t_3, \quad (t_1,t_2) \in \mathbb{R}^2;$$
$$f_{X_2,X_3}(t_2,t_3) = \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(t_1,t_2,t_3)\mathrm{d}t_1, \quad (t_2,t_3) \in \mathbb{R}^2;$$
$$f_{X_1,X_3}(t_1,t_3) = \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(t_1,t_2,t_3)\mathrm{d}t_2, \quad (t_1,t_3) \in \mathbb{R}^2.$$

(Mutual) independence

- RVs $X_1,\ldots,X_n$ are (mutually) independent $\Leftrightarrow$

$$F_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = \prod_{i=1}^{n} F_{X_i}(t_i)$$

  - For discrete $X_1,\ldots,X_n$, joint pmf $p_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = \prod_{i=1}^{n} p_{X_i}(t_i)$
  - For continuous $X_1,\ldots,X_n$, joint pdf $f_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = \prod_{i=1}^{n} f_{X_i}(t_i)$

Example Lec2.1

- X_1 and X_2 are independent Bernoulli RVs with pmf $p_{X_i}(0) = 1 - p_i$ and $p_{X_i}(1) = p_i$, $i = 1,2$. Write the joint pmf of X_1, X_2.

Ans: Since X_1 and X_2 are independent, $\mathrm{supp}(X_1,X_2) = \{0,1\} \times \{0,1\}$.

$$p_{X_1,X_2}(t_1,t_2) = p_{X_1}(t_1)p_{X_2}(t_2) = \begin{cases} (1-p_1)(1-p_2), & (t_1,t_2) = (0,0), \\ p_1(1-p_2), & (t_1,t_2) = (1,0), \\ (1-p_1)p_2, & (t_1,t_2) = (0,1), \\ p_1p_2, & (t_1,t_2) = (1,1), \\ 0, & \text{otherwise.} \end{cases}$$

- Let X_1 and X_2 be independent Poisson RVs with pmf $p_{X_i}(k_i) = \frac{e^{-3} \cdot 3^{k_i}}{k_i!}$, $k_i = 0,1,\ldots$, $i = 1,2$. Write the joint pmf of X_1, X_2.

Ans: Since X_1 and X_2 are independent, $\mathrm{supp}(X_1,X_2) = \{0,1,\ldots\} \times \{0,1,\ldots\}$. It follows that for any $(k_1,k_2) \in \mathrm{supp}(X_1,X_2)$,

$$p_{X_1,X_2}(k_1,k_2) = p_{X_1}(k_1)p_{X_2}(k_2) = \frac{e^{-6} \cdot 3^{k_1+k_2}}{k_1!k_2!}.$$

- X_1 and X_2 are independent uniform RVs with pdf $f_{X_i}(t_i) = \mathbf{1}_{[0,1]}(t_i)$, $i = 1,2$. Write the joint pdf of X_1, X_2.

- Let X_1 and X_2 be independent exponential RVs with pdf $f_{X_i}(t_i) = 2e^{-2t_i} \cdot \mathbf{1}_{(0,\infty)}(t_i)$, $i = 1,2$. Write the joint pdf of X_1, X_2.