# PH 712 Probability and Statistical Inference

Part V: Point Estimation I

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## Recall types of the stochastic model

- Stochastic model: the distribution of RVs of interest
  - Parametric model
  - Non-parametric model
  - Semi-parametric model

## Parametric model

- A set of pdfs/pmfs indexed by $p$-dimensional unknown $\boldsymbol{\theta}$ (constrained in $\boldsymbol{\Theta}$) with small or moderate dimension $p$, i.e., $\{f(\cdot \mid \boldsymbol{\theta}) : \boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbb{R}^p\}$ with
  - $f(\cdot \mid \boldsymbol{\theta})$: either a pdf or a pmf
  - $\boldsymbol{\Theta}$: the set of allowed values of $\boldsymbol{\theta}$
- True parameters, say $\boldsymbol{\theta}_0$, believed to be fixed (frequentist statistics)
  - Rather than randomizing $\boldsymbol{\theta}_0$ (Bayesian statistics)
- Estimator: a statistic (i.e., a function of the sample); a guess about $\boldsymbol{\theta}_0$
- Estimate: plugging the realization into the estimator
- $p = 1$ hereafter, i.e., considering only one unknown parameter

## Method of moments (MM, CB Sec 7.2.1)

- Procedure
  1. Equate RAW moments ($\mathrm{E}(X_i^k)$) to their empirical counterparts ($n^{-1}\sum_{i=1}^n X_i^k$).
  2. Solve the resulting simultaneous equations for $\theta$.
- Pros and cons
  - Easy implementation
  - Start point for more complex methods
  - No constraint
  - Not uniquely defined

## Example Lec5.1

- Suppose $X_1, \ldots, X_n$ is an iid sample following distributions as below. Find the MM estimator in each scenario.
  a. $\mathcal{N}(\mu, \sigma^2)$, with unknown $\mu \in \mathbb{R}$ and known $\sigma > 0$.
  b. $\mathcal{N}(\mu, \sigma^2)$, with known $\mu \in \mathbb{R}$ and unknown $\sigma > 0$.
  c. $Bern(\theta)$: $p_X(x \mid \theta) = \theta^x(1-\theta)^{1-x}\mathbf{1}_{\{0,1\}}(x)$, $\theta \in [0, 1/2]$.

d. Exponential: $f_X(x \mid \beta) = \beta^{-1} \exp(-x/\beta)\mathbf{1}_{(0,\infty)}(x)$, $\beta > 0$.
e. $f_X(x \mid \theta) = \theta x^{\theta-1}\mathbf{1}_{[0,1]}(x)$, $\theta > 0$.

## Maximum Likelihood (ML) Estimator (MLE, CB Sec 7.2.2)

- Likelihood:
  - a real-valued function of unknown $\theta$

$$L(\theta) = L(\theta; X_1, \ldots, X_n) = f_{X_1,\ldots,X_n}(X_1, \ldots, X_n \mid \theta), \quad \theta \in \Theta$$

  - $f_{X_1,\ldots,X_n}$: the joint pdf/pmf of $X_1, \ldots, X_n$
- Log-likelihood: the natural logarithm of likelihood

$$\ell(\theta) = \ln L(\theta), \quad \theta \in \Theta$$

- If $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$ is the maximizer of $L(\theta)$ (equiv. the maximizer of $\ell(\theta)$) with respect to $\theta$ constrained in $\Theta$, i.e.,

$$\hat{\theta}(X_1, \ldots, X_n) = \arg\max_{\theta\in\Theta} L(\theta) = \arg\max_{\theta\in\Theta} \ell(\theta),$$

  then $\hat{\theta}$ is the MLE for $\theta$.
- Invariance (CB Thm 7.2.10): if $\hat{\theta}$ is the MLE of $\theta$, then $g(\hat{\theta})$ is the MLE of $g(\theta)$ for any given function $g(\cdot)$.

## How to locate the MLE constrained in $\Theta$?

- If $\ell(\theta)$ is monotonic with respect to $\theta \in \Theta$, then the MLE lies at one boundary point of $\Theta$
- If $\ell(\theta)$ is non-monotonic but differentiable with respect to $\theta \in \Theta$, then
  1. Collect all the candidates including:
     - Stationary points, i.e., solutions to the equation $S(\theta) = 0$ subject to $\theta \in \Theta$
       - Where $S(\theta) = \ell'(\theta)$ is called the score
     - Boundary points of $\Theta$
  2. Compare the values of log-likelihood or likelihood evaluated at all the above candidates

## Example Lec5.1'

- Suppose $X_1, \ldots, X_n$ is an iid sample following distributions as below. Find the MLE in each scenario.
  a. $\mathcal{N}(\mu, \sigma^2)$, with unknown $\mu \in \mathbb{R}$ and known $\sigma > 0$.
  b. $\mathcal{N}(\mu, \sigma^2)$, with known $\mu \in \mathbb{R}$ and unknown $\sigma > 0$.
  c. $Bern(\theta)$: $p_X(x \mid \theta) = \theta^x(1-\theta)^{1-x}\mathbf{1}_{\{0,1\}}(x)$, $\theta \in [0, 1/2]$.
  d. Exponential: $f_X(x \mid \beta) = \beta^{-1} \exp(-x/\beta)\mathbf{1}_{(0,\infty)}(x)$, $\beta > 0$.
  e. $f_X(x \mid \theta) = \theta x^{\theta-1}\mathbf{1}_{[0,1]}(x)$, $\theta > 0$.