

STAT 3690 Lecture 06

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Multivariate normal (MVN) distribution (J&W Sec 4.2)

- Standard normal random vector

$$- \mathbf{Z} = [Z_1, \dots, Z_p]^\top \sim MVN_p(\mathbf{0}, \mathbf{I}) \Leftrightarrow Z_1, \dots, Z_p \stackrel{\text{iid}}{\sim} N(0, 1) \Leftrightarrow$$

$$\phi_{\mathbf{Z}}(\mathbf{z}) = (2\pi)^{-p/2} \exp(-\mathbf{z}^\top \mathbf{z} / 2), \quad \mathbf{z} = [z_1, \dots, z_p]^\top \in \mathbb{R}^p$$

- (General) normal random vector

– Def: The distribution of \mathbf{X} is MVN iff there exists $q \in \mathbb{Z}^+$, $\boldsymbol{\mu} \in \mathbb{R}^q$, $\mathbf{A} \in \mathbb{R}^{q \times p}$ and $\mathbf{Z} \sim MVN_p(\mathbf{0}, \mathbf{I})$ such that $\mathbf{X} = \mathbf{AZ} + \boldsymbol{\mu}$

* Limit the discussion to non-degenerate cases, i.e., $\text{rk}(\mathbf{A}) = q$

* $\mathbf{X} \sim MVN_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, i.e.,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^q \det(\boldsymbol{\Sigma})}} \exp\{-(\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) / 2\}, \quad \mathbf{x} \in \mathbb{R}^q$$

$$\cdot \boldsymbol{\Sigma} = \text{var}(\mathbf{X}) = \mathbf{A}\mathbf{A}^\top > 0$$

Derive the pdf of MVN.

$$\mathbf{X} = \mathbf{AZ} + \boldsymbol{\mu} \text{ and } \text{rk}(\mathbf{A}) = q \Rightarrow \text{supp}(\mathbf{X}) = \mathbb{R}^q$$

If $q=p$: $\mathbf{Z} = \mathbf{A}^{-1}(\mathbf{X} - \boldsymbol{\mu})$ and $\mathbf{J} = \frac{\partial \mathbf{X}}{\partial \mathbf{Z}} = \mathbf{A}^{-1}$ (matrix calculus)

$$\therefore \phi_{\mathbf{Z}}(\mathbf{z}) = (2\pi)^{-p/2} \exp(-\frac{1}{2} \mathbf{z}^\top \mathbf{z}) \quad \mathbf{z} \in \mathbb{R}^p$$

$$\begin{aligned} \therefore f_{\mathbf{X}}(\mathbf{x}) &= (2\pi)^{-p/2} |\det(\mathbf{A}^{-1})| \exp\left\{-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (\mathbf{A}^{-1})^\top \mathbf{A}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\} \\ &= (2\pi)^{-p/2} |\det(\mathbf{A})|^{-1} \exp\left\{-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (\mathbf{A}\mathbf{A}^\top)^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\} \\ &= (2\pi)^{-p/2} |\det(\mathbf{A}\mathbf{A}^\top)|^{-1/2} \exp\left\{-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (\mathbf{A}\mathbf{A}^\top)^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\} \quad (\because \det(\mathbf{A}) = \det(\mathbf{A}^\top) = \sqrt{\det(\mathbf{A}\mathbf{A}^\top)}) \\ &\text{for all } \mathbf{x} \in \mathbb{R}^p \end{aligned}$$

If $q < p$: $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^\top$ (SVD of \mathbf{A} , $\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \mathbf{0} \end{bmatrix}$
 $\mathbf{U} \in \mathbb{R}^{p \times p}$, $\mathbf{V} \in \mathbb{R}^{p \times p}$)

Due to the results for "q=p", the pdf of $\mathbf{Y} = \mathbf{V}^\top \mathbf{Z}$ is

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-p/2} \exp(-\frac{1}{2} \mathbf{y}^\top \mathbf{y}) \text{ for } \mathbf{y} \in \mathbb{R}^p \quad (\because \mathbf{V}\mathbf{V}^\top = \mathbf{V}^\top\mathbf{V} = \mathbf{I})$$

i.e., $\mathbf{Y} = [\mathbf{Y}_1, \dots, \mathbf{Y}_p]^\top \sim MVN_p(\mathbf{0}, \mathbf{I}) \Leftrightarrow \mathbf{Y}_1, \dots, \mathbf{Y}_p \stackrel{\text{iid}}{\sim} N(0, 1)$

$$\Rightarrow \mathbf{X} = \mathbf{AZ} + \boldsymbol{\mu} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{Y} + \boldsymbol{\mu} = \mathbf{U}\boldsymbol{\Lambda}_1 [\mathbf{Y}_1, \dots, \mathbf{Y}_q]^\top + \boldsymbol{\mu}$$

$$\Rightarrow f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-q/2} |\det(\mathbf{A}\mathbf{A}^\top)|^{-1/2} \exp\left\{-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top (\mathbf{A}\mathbf{A}^\top)^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\} \quad (\because \mathbf{A}\mathbf{A}^\top = \mathbf{U}\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_1^\top\mathbf{U}^\top = \mathbf{U}\boldsymbol{\Lambda}_1^2\mathbf{U}^\top) \\ \text{for all } \mathbf{x} \in \mathbb{R}^p$$

• Exercise:

1. $\Sigma = \mathbf{A}\mathbf{A}^\top > 0 \Leftrightarrow \text{rk}(\mathbf{A}) = q$ (Hint: SVD of \mathbf{A});
2. $\Sigma > 0 \Rightarrow$ there exists a $p \times p$ positive definite matrix, say $\Sigma^{1/2}$, such that $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$ and $\Sigma^{-1} = \Sigma^{-1/2}\Sigma^{-1/2}$ (Hint: spectral decomposition of Σ).

$$\begin{aligned}
 1. \quad & \text{w.p. } \mathbf{A} = \mathbf{B}\mathbf{\Lambda}\mathbf{C}^\top, \text{ where } \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_q \end{bmatrix} \quad (\text{SVD of } \mathbf{A}) \\
 & \Rightarrow \mathbf{A}\mathbf{A}^\top = \mathbf{B}\mathbf{\Lambda}\mathbf{C}^\top\mathbf{C}\mathbf{\Lambda}^\top\mathbf{B}^\top \\
 & \quad = \mathbf{B}\mathbf{\Lambda}\mathbf{\Lambda}^\top\mathbf{B}^\top \\
 & \quad \text{where } \mathbf{\Lambda}\mathbf{\Lambda}^\top = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \ddots \\ 0 & 0 & \lambda_q^2 \end{bmatrix} \\
 & \therefore \mathbf{A}\mathbf{A}^\top > 0 \Leftrightarrow \lambda_1, \dots, \lambda_q \neq 0 \Leftrightarrow \text{rk}(\mathbf{A}) = q \\
 2. \quad & \Sigma = \mathbf{B}\mathbf{\Lambda}\mathbf{B}^\top, \text{ where } \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_p \end{bmatrix} \quad (\text{eigen-/spectral decomposition of } \Sigma) \\
 & \Rightarrow \mathbf{\Lambda}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_p^{-1} \end{bmatrix} \quad (\because \Sigma > 0) \\
 & \Rightarrow \Sigma^{-1} = \mathbf{B}\mathbf{\Lambda}^{-1}\mathbf{B}^\top \quad (\because (\mathbf{B}\mathbf{\Lambda}^{-1}\mathbf{B}^\top)(\mathbf{B}\mathbf{\Lambda}\mathbf{B}^\top) = \mathbf{I}) \\
 & \quad \text{let } \mathbf{\Lambda}^{1/2} = \begin{bmatrix} \lambda_1^{1/2} & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_p^{1/2} \end{bmatrix}, \quad \mathbf{\Lambda}^{-1/2} = \begin{bmatrix} \lambda_1^{-1/2} & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_p^{-1/2} \end{bmatrix} \\
 & \quad \Sigma^{1/2} = \mathbf{B}\mathbf{\Lambda}^{1/2}\mathbf{B}^\top, \quad \Sigma^{-1/2} = \mathbf{B}\mathbf{\Lambda}^{-1/2}\mathbf{B}^\top \\
 & \quad \text{then } \Sigma^{1/2}\Sigma^{1/2} = \Sigma \quad \text{and} \quad \Sigma^{-1/2}\Sigma^{-1/2} = \Sigma^{-1}
 \end{aligned}$$

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options(digits = 4)
(Sigma = matrix(c(10, 2, 2, 5), ncol = 2, nrow = 2))
(spectral = eigen(Sigma))
(SigmaRoot = spectral$eigenvectors %*% diag(spectral$values^.5) %*% t(spectral$eigenvectors))
(SigmaRootInv = spectral$eigenvectors %*% diag(spectral$values^-.5) %*% t(spectral$eigenvectors))
# Check properties of root of Sigma
(SigmaRoot %*% SigmaRoot - Sigma)
(solve(SigmaRoot) - SigmaRootInv)
(SigmaRootInv %*% SigmaRootInv - solve(Sigma))
# SVD <=> spectral decomposition if Sigma is positive (semi-)definite
svd(Sigma)
eigen(Sigma)

```

• Useful properties of MVN

- $\mathbf{X} \sim \text{MVN}_p(\boldsymbol{\mu}, \Sigma) \Leftrightarrow \mathbf{Z} = \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim \text{MVN}_p(\mathbf{0}, \mathbf{I})$. So, we have a stochastic representation of arbitrary $\mathbf{X} \sim \text{MVN}_p(\boldsymbol{\mu}, \Sigma)$: $\mathbf{X} = \Sigma^{1/2}\mathbf{Z} + \boldsymbol{\mu}$, where $\mathbf{Z} \sim \text{MVN}_p(\mathbf{0}, \mathbf{I})$.
- $\mathbf{X} \sim \text{MVN}$ iff, for all $a \in \mathbb{R}^p$, $a^\top \mathbf{X}$ has a (univariate) normal distribution.
- If $\mathbf{X} \sim \text{MVN}_p(\boldsymbol{\mu}, \Sigma)$, then $\mathbf{A}\mathbf{X} + \mathbf{b} \sim \text{MVN}_q(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\Sigma\mathbf{A}^\top)$ for $\mathbf{A} \in \mathbb{R}^{q \times p}$ and $\text{rk}(\mathbf{A}) = q$.

• Exercise: Generate six iid samples following bivariate normal $\text{MVN}_2(\boldsymbol{\mu}, \Sigma)$ with

$$\boldsymbol{\mu} = [3, 6]^\top, \quad \Sigma = \begin{bmatrix} 10 & 2 \\ 2 & 5 \end{bmatrix}.$$

```

options(digits = 4)
set.seed(1)
Mu = matrix(c(3, 6), ncol = 1, nrow = 2)
Sigma = matrix(c(10, 2, 2, 5), ncol = 2, nrow = 2)
n = 1000
# Method 1: via rnorm()
spectral = eigen(Sigma)
SigmaRoot = spectral$vectors %*% diag(spectral$values^.5) %*% t(spectral$vectors)
A1 = matrix(0, nrow = n, ncol = length(Mu))
for (i in 1:n) {
  A1[i, ] = t(SigmaRoot %*% matrix(rnorm(2), nrow = 2, ncol = 1) + Mu)
}
# Method 2: via MASS::mvrnorm()
A2 = MASS::mvrnorm(n = 6, Mu, Sigma)

```