STAT 3690 Homework 1

zhiyanggeezhou.github.io

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca)

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Answers must be submitted electronically via Crowdmark. Please enclose your R source code (if applicable) as well.

1. The function $\mathrm{cov}(\cdot,\cdot)$ is bilinear, i.e., for random vectors $\mathbf{W}$, $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ and fixed matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, one has $\mathrm{cov}(\mathbf{AW}+\mathbf{BX},\mathbf{Y}) = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{WY}} + \mathbf{B}\boldsymbol{\Sigma}_{\mathbf{XY}}$ and $\mathrm{cov}(\mathbf{W},\mathbf{CY}+\mathbf{DZ}) = \boldsymbol{\Sigma}_{\mathbf{WY}}\mathbf{C}^\top + \boldsymbol{\Sigma}_{\mathbf{WZ}}\mathbf{D}^\top$, where $\mathbf{AW}+\mathbf{BX}$ and $\mathbf{CY}+\mathbf{DZ}$ both make sense.
   a. Prove this bilinearity.
   b. Rephrase $\mathrm{cov}(\mathbf{AW}+\mathbf{BX},\mathbf{CY}+\mathbf{DZ})$ in terms of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\boldsymbol{\Sigma}_{\mathbf{WY}}$, $\boldsymbol{\Sigma}_{\mathbf{WZ}}$, $\boldsymbol{\Sigma}_{\mathbf{XY}}$ and $\boldsymbol{\Sigma}_{\mathbf{XZ}}$.
2. Let $\mathbf{A}$ be a square matrix with eigendecomposition $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^{-1}$. Given a real number c ($\neq$ any eigenvalue of $\mathbf{A}$), express the eigendecomposition of $(\mathbf{A}-c\mathbf{I})^{-1}$ in terms of $\mathbf{U}$, Λ, $\mathbf{I}$ and c.
3. Let W be a discrete random variable such that $\Pr(W=1) = \Pr(W=-1) = 1/2$. Define $Y = WX$ with $X \sim N(0,1)$ and $X \perp\!\!\!\perp W$. Prove the following identities.
   a. $Y \sim N(0,1)$.
   b. X and Y are uncorrelated with each other.
   c. X is not independent of Y.
4. Let $\mathbf{X} = [X_1, X_2, X_3]^\top \sim MVN_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with
$$\boldsymbol{\mu} = [6,1,4]^\top, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$
   a. Find the conditional distribution of X_2 given $X_1 = 2$ and $X_3 = 1$.
   b. Find the distribution of random 2-vector $\mathbf{Y} = [3X_1 - 2X_2 + X_3, X_2 - X_3]^\top$.
   c. Find $w_1, w_2 \in \mathbb{R}$ such that $W = w_1X_1 + w_2X_2 + X_3$ is independent of $\mathbf{Y}$. (Hint: don't forget to verify the normality of random 3-vector $[W, \mathbf{Y}^\top]^\top$ after figuring out values of w_1 and w_2.)