PH 712 Probability and Statistical Inference

Part II: Transformation Between RVs

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Find the pmf of $Y = g(X)$, given the pmf of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. Calculate $p_Y(y) = \Pr(Y = y) = \Pr(X \in \{x \in \text{supp}(X) : y = g(x)\})$

Example Lec2.1

Let X have the pmf $p_X(x) = 2^x \mathbf{1}_{\{-1,-2,\ldots\}}(x)$. Find the pmf of $Y = X^2$.

Find the cdf of $Y = g(X)$, given the distribution of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. Calculate $F_Y(y) = \Pr\{g(X) \le y\} = \Pr(X \in \{x : g(x) \le y\})$

Example Lec2.2

Suppose $X \sim U([-\pi/2, \pi/2])$, i.e., its pdf is $f_X(x) = \pi^{-1}\mathbf{1}_{[-\pi/2,\pi/2]}(x)$. Find the cdf of $Y = X^2$.

$Y = F_X(X) \sim U([0, 1])$ (CB Thm. 2.1.10)

- If $X \sim F_X$ and $Y = F_X(X)$
- Then $Y \sim U(\text{supp}(Y))$
  - Specifically $Y \sim U([0, 1])$ if X is continuous
- Application: inverse transform sampling
  - Goal: generate an independent and identically distributed (iid) sample following F_X
  - Implementation
    1. Draw $U_1, \ldots, U_n \overset{\text{iid}}{\sim} U(\text{supp}(Y))$ with $Y = F_X(X)$
    2. Then $F_X^{-1}(U_1), \ldots, F_X^{-1}(U_n) \overset{\text{iid}}{\sim} F_X$
       * $F_X^{-1}(y) = \inf\{x : F_X(x) \ge y\}$
  - Pros & cons
    * (Theoretically) applicable to arbitrary F_X
    * The closed form of F_X^{-1} NOT always available

Find the pdf of $Y = g(X)$, given the pdf of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. $$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\int_{\{x: g(x) \le y\}} f_X(x)\mathrm{d}x$$

- The integration of f_X at the right-hand side is often avoidable by employing the Leibniz Rule (CB Thm. 2.4.1):
$$\frac{\mathrm{d}}{\mathrm{d}y}\int_{a(y)}^{b(y)} f(x)\mathrm{d}x = f\{b(y)\}\frac{\mathrm{d}}{\mathrm{d}y}b(y) - f\{a(y)\}\frac{\mathrm{d}}{\mathrm{d}y}a(y)$$
with $a(y)$ and $b(y)$ both differentiable with respect to y.

Example Lec2.2'

Let X have the uniform pdf $f_X(x) = \pi^{-1}\mathbf{1}_{[-\pi/2,\pi/2]}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.3

$X \sim \text{Weibull}(\text{shape} = \alpha, \text{scale} = \beta)$, i.e., $f_X(x) = (\alpha/\beta)(x/\beta)^{\alpha-1}\exp\{-(x/\beta)^\alpha\}\mathbf{1}_{(0,\infty)}(x)$. Find the pdf of $Y = \ln(X)$.

Example Lec2.4

Let X have the pdf $f_X(x) = 2^{-1}\mathbf{1}_{(0,2)}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.5

Suppose $f_X(x) = 3^{-1}\mathbf{1}_{(-1,2)}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.6

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$, i.e., $f_X(x) = \sigma\sqrt{2\pi}\exp\{-(x-\mu)^2/(2\sigma^2)\}$. Find the pdf of $Y = aX + b$ with $a \neq 0$.