STAT 3690 Lecture Note

Part V: Comparisons of population mean vectors

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Comparisons of population mean vectors

Comparing two population mean vectors (J&W Sec. 6.3)

- Two independent samples following two distributions with equal covariance
  - $\boldsymbol{X}_{11},\ldots,\boldsymbol{X}_{1n_1} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu}_1,\boldsymbol{\Sigma})$
  - $\boldsymbol{X}_{21},\ldots,\boldsymbol{X}_{2n_2} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu}_2,\boldsymbol{\Sigma})$
- Let $\bar{\boldsymbol{X}}_i$ and $\mathbf{S}_i$ be the sample mean and sample covariance for the ith sample, $i=1,2$.
- Hypotheses $H_0:\boldsymbol{\mu}_1=\boldsymbol{\mu}_2$ v.s. $H_1:\boldsymbol{\mu}_1\neq\boldsymbol{\mu}_2$
- Test statistic following LRT

$$T(\mathcal{X}) = (\bar{\boldsymbol{X}}_1-\bar{\boldsymbol{X}}_2)^\top\{(n_1^{-1}+n_2^{-1})\mathbf{S}_{\text{pool}}\}^{-1}(\bar{\boldsymbol{X}}_1-\bar{\boldsymbol{X}}_2) \sim \frac{(n_1+n_2-2)p}{n_1+n_2-p-1}F(p,n_1+n_2-p-1) \text{ under } H_0$$

  - $\mathbf{S}_{\text{pool}} = \frac{(n_1-1)\mathbf{S}_1+(n_2-1)\mathbf{S}_2}{n_1+n_2-2}$
- Reject H_0 at level α when

$$T(\mathcal{X}) \geq \frac{p(n_1+n_2-2)}{n_1+n_2-p-1}F_{1-\alpha,p,n_1+n_2-p-1}$$

- p-value

$$1-F_{F_{1-\alpha,p,n_1+n_2-p-1}}\left[\frac{n_1+n_2-p-1}{p(n_1+n_2-2)}T(\mathcal{X})\right]$$

```r
options(digits = 4)
install.packages(c("dslabs"))
library(dslabs)
data("gapminder")
dataset1 = gapminder[
  !is.na(gapminder$infant_mortality) &
    gapminder$continent == "Africa" &
    gapminder$year == 2012,
  c('infant_mortality', "life_expectancy")]
dataset1 = as.matrix(dataset1)
```

```r
dataset2 = gapminder[
  !is.na(gapminder$infant_mortality) &
    gapminder$continent == "Asia" &
    gapminder$year == 2012,
  c('infant_mortality', "life_expectancy")]
dataset2 = as.matrix(dataset2)

n1 <- nrow(dataset1); n2 <- nrow(dataset2); p <- ncol(dataset1)

(mu_hat1 <- colMeans(dataset1))
(mu_hat2 <- colMeans(dataset2))
(S1 <- cov(dataset1))
(S2 <- cov(dataset2))
S_pool <- ((n1 - 1)*S1 + (n2 - 1)*S2)/(n1+n2-2)

(lrt <- t(mu_hat1-mu_hat2) %*%
  solve((n1^-1 + n2^-1)*S_pool) %*%
  (mu_hat1-mu_hat2))

alpha <- .05
(cri.val <- (n1+n2-2)*p/(n1+n2-p-1)*qf(1-alpha, p, n1+n2-p-1))
lrt >= cri.val
(p.val = 1-pf((n1+n2-p-1)/(n1+n2-2)/p*lrt, p, n1+n2-p-1))
```

- Report: Testing hypotheses H_0: in 2012 Asia and Africa shared the identical mean value in both infant mortality and life expectancy v.s. H_1: otherwise, we carried on the LRT and obtained 87.65 as the value of test statistic and $[6.255, \infty)$ as the corresponding level .05 rejection region. In addition, the p-value was 4.952e-14. So, at the .05 level, there was a strong statistical evidence against H_0, i.e., we rejected H_0 and believed that in 2012 Asia and Africa didn't share the identical mean value in infant mortality and/or life expectancy.

Comparing multiple population mean vectors (one-way multivariate analysis of variance (One-way MANOVA), J&W Sec. 6.4)

- Generalization of two-sample problem
  - Model: m independent samples, where
    - $\boldsymbol{X}_{11}, \ldots, \boldsymbol{X}_{1n_1} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$
    - $\vdots$
    - $\boldsymbol{X}_{m1}, \ldots, \boldsymbol{X}_{mn_m} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu}_m, \boldsymbol{\Sigma})$
  - Hypotheses $H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_m$ v.s. H_1 : otherwise
- Alternatively
  - Model: m independent samples, where
    - $\boldsymbol{X}_{11}, \ldots, \boldsymbol{X}_{1n_1} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu} + \boldsymbol{\tau}_1, \boldsymbol{\Sigma})$
    - $\vdots$
    - $\boldsymbol{X}_{m1}, \ldots, \boldsymbol{X}_{mn_m} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu} + \boldsymbol{\tau}_m, \boldsymbol{\Sigma})$
      - Identifiability: $\sum_i \boldsymbol{\tau}_i = 0$ otherwise there are infinitely many models that lead to the same data-generating mechanism.
  - Hypotheses $H_0 : \boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_m = 0$ v.s. H_1 : otherwise
- Alternatively
  - Model: $\boldsymbol{X}_{ij} = \boldsymbol{\mu} + \boldsymbol{\tau}_i + \mathbf{E}_{ij}$ with $\mathbf{E}_{ij} \overset{\text{iid}}{\sim} \text{MVN}_p(\mathbf{0}, \boldsymbol{\Sigma})$
    - Identifiability: $\sum_i \boldsymbol{\tau}_i = 0$
  - Hypotheses $H_0 : \boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_m = 0$ v.s. H_1 : otherwise

- Sample means and sample covariances
  - Sample mean for the ith sample $\bar{\boldsymbol{X}}_{i\cdot} = n_i^{-1} \sum_j \boldsymbol{X}_{ij}$
  - Sample covariance for the ith sample $\mathbf{S}_i = (n_i - 1)^{-1} \sum_j (\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})^\top$
  - Grand mean $\bar{\boldsymbol{X}}_{\cdot\cdot} = \sum_i n_i \bar{\boldsymbol{X}}_{i\cdot} / \sum_i n_i = \sum_{ij} \boldsymbol{X}_{ij} / \sum_i n_i$
  - Decomposition of total (corrected) sum of squares and cross products matrix (SSP):

$$\mathbf{SSP}_{\mathrm{t}} = \mathbf{SSP}_{\mathrm{w}} + \mathbf{SSP}_{\mathrm{b}}$$

    - Total (corrected) SSP: $\mathbf{SSP}_{\mathrm{t}} = \sum_{ij} (\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{\cdot\cdot})(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{\cdot\cdot})^\top = \mathbf{SSP}_{\mathrm{w}} + \mathbf{SSP}_{\mathrm{b}}$
    - Within-group SSP: $\mathbf{SSP}_{\mathrm{w}} = \sum_i (n_i - 1)\mathbf{S}_i = \sum_{ij} (\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})^\top$
    - Between-group SSP: $\mathbf{SSP}_{\mathrm{b}} = \sum_i n_i (\bar{\boldsymbol{X}}_{i\cdot} - \bar{\boldsymbol{X}}_{\cdot\cdot})(\bar{\boldsymbol{X}}_{i\cdot} - \bar{\boldsymbol{X}}_{\cdot\cdot})^\top$

- ML estimator of $(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m, \boldsymbol{\Sigma})$
  - Unconstrained
    - $\hat{\boldsymbol{\mu}}_i = \bar{\boldsymbol{X}}_{i\cdot} = n_i^{-1} \sum_j \boldsymbol{X}_{ij}$
    - $\widehat{\boldsymbol{\Sigma}} = (\sum_i n_i)^{-1} \mathbf{SSP}_{\mathrm{w}}$
  - Under H_0
    - $\hat{\boldsymbol{\mu}}_i = \bar{\boldsymbol{X}}_{\cdot\cdot}$ for each i
    - $\widehat{\boldsymbol{\Sigma}} = (\sum_i n_i)^{-1} \mathbf{SSP}_{\mathrm{t}}$
- Likelihood ratio

$$\lambda = \left\{ \frac{\det(\mathbf{SSP}_{\mathrm{w}})}{\det(\mathbf{SSP}_{\mathrm{t}})} \right\}^{\sum_i n_i / 2}$$

  monotonic increasing with respect to the Wilk's lambda test statistic

$$\Lambda = \frac{\det(\mathbf{SSP}_{\mathrm{w}})}{\det(\mathbf{SSP}_{\mathrm{t}})} = \frac{\det(\mathbf{SSP}_{\mathrm{w}})}{\det(\mathbf{SSP}_{\mathrm{w}} + \mathbf{SSP}_{\mathrm{b}})}$$

  - $\Lambda \sim$ Wilk's lambda distribution $\Lambda(\boldsymbol{\Sigma}, \sum_i n_i - m, m - 1)$ under H_0
    - Since $\mathbf{SSP}_{\mathrm{w}} \sim W_p(\boldsymbol{\Sigma}, \sum_i n_i - m)$ and $\mathbf{SSP}_{\mathrm{b}} \sim W_p(\boldsymbol{\Sigma}, m - 1)$ under H_0
    - Bartlett's approximation (when $\sum_i n_i - m$ is large)

$$\{(p + m)/2 - \sum_i n_i + 1\} \ln \Lambda \approx \chi^2(p(m - 1))$$

    - Rao's approximation (default for `R` functions `manova` and `car::Manova`)
- Reject H_0 when Λ is small (equiv. $\mathbf{SSP}_{\mathrm{b}}$ is large); specifically, H_0 is rejected when

$$\{(p + m)/2 - \sum_i n_i + 1\} \ln \Lambda \geq \chi^2_{1-\alpha, p(m-1)} \quad \text{OR} \quad \Lambda \leq \exp\left\{ \frac{\chi^2_{1-\alpha, p(m-1)}}{(p + m)/2 - \sum_i n_i + 1} \right\}$$

- p-value

$$1 - F_{\chi^2(p(m-1))} \left[\{(p + m)/2 - \sum_i n_i + 1\} \ln \Lambda \right]$$

  - $F_{\chi^2(p(m-1))}$: the cdf of $\chi^2(p(m - 1))$

- Exercise 5.1: investigating the factors in manufacturing plastic film (see W. J. Krzanowski (1988) *Principles of Multivariate Analysis*. A User's Perspective. Oxford UP, pp. 381.)

- Three response variables (tear, gloss and opacity) describing measured characteristics of the resultant film
- A total of 20 runs
- One factor `RATE` (rate of extrusion, 2-level, low or high) in the production test

```
options(digits = 4)
install.packages('car')
tear <- c(
  6.5, 6.2, 5.8, 6.5, 6.5, 6.9, 7.2, 6.9, 6.1, 6.3,
  6.7, 6.6, 7.2, 7.1, 6.8, 7.1, 7.0, 7.2, 7.5, 7.6
)
gloss <- c(
  9.5, 9.9, 9.6, 9.6, 9.2, 9.1, 10.0, 9.9, 9.5, 9.4,
  9.1, 9.3, 8.3, 8.4, 8.5, 9.2, 8.8, 9.7, 10.1, 9.2
)
opacity <- c(
  4.4, 6.4, 3.0, 4.1, 0.8, 5.7, 2.0, 3.9, 1.9, 5.7,
  2.8, 4.1, 3.8, 1.6, 3.4, 8.4, 5.2, 6.9, 2.7, 1.9
)
(X <- cbind(tear, gloss, opacity))
(rate <- factor(gl(2,10,length=nrow(X)), labels=c("Low", "High")))

# Bartlett's approximation to Wilks lambda distribution
X_low <- X[rate == 'Low',]
X_high <- X[rate == 'High',]
n <- nrow(X); p <- ncol(X); m <- length(levels(rate))
(SSPcor = (n-1)*cov(X))
(SSPw <- (nrow(X_low) - 1)*cov(X_low) + (nrow(X_high) - 1)*cov(X_high))
(Lambda <- det(SSPw)/det(SSPcor))
(cri.point = exp(qchisq(0.95, p*(m-1))/((p+m)/2-n+1)))
Lambda <= cri.point
(p.val = 1-pchisq(((p+m)/2-n+1)*log(Lambda), p*(m-1)))

# Rao's approximation to Wilks lambda distribution
summary(manova(X ~ rate), test = 'Wilks')
summary(car::Manova(lm(X ~ rate)), test.statistic='Wilks')
```

- Report: Testing hypotheses H_0: no `RATE` effect on film characteristics v.s. H_1: otherwise, we carried on the Wilk's lambda test and obtained 0.4136 as the value of test statistic and $(-\infty, 0.6227]$ as the corresponding level .05 rejection region. In addition, the p-value was 0.002227. So, at the .05 level, there was statistical evidence against H_0, i.e., we rejected H_0 and believed that there was an effect from `RATE` on film characteristics.

Testing for equality of covariance matrices (J&W Sec. 6.6)

- Model: m independent samples, where
  - $\boldsymbol{X}_{11}, \ldots, \boldsymbol{X}_{1n_1} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$
  - $\vdots$
  - $\boldsymbol{X}_{m1}, \ldots, \boldsymbol{X}_{mn_m} \overset{\text{iid}}{\sim} \text{MVN}_p(\boldsymbol{\mu}_m, \boldsymbol{\Sigma}_m)$
- Hypotheses $H_0 : \boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_m$ v.s. H_1 : otherwise
- MLE of $(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m, \boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_m)$

- Under H_0
  * $\hat{\boldsymbol{\mu}}_i = \bar{\boldsymbol{X}}_{i\cdot} = n_i^{-1} \sum_j \boldsymbol{X}_{ij}$
  * $\widehat{\boldsymbol{\Sigma}}_i = (\sum_i n_i)^{-1} \mathbf{SSP}_{\mathrm{w}} = (\sum_i n_i)^{-1} \sum_{ij} (\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})^\top$ for all i
- No restriction on $\boldsymbol{\Sigma}_i$
  * $\hat{\boldsymbol{\mu}}_i = \bar{\boldsymbol{X}}_{i\cdot} = n_i^{-1} \sum_j \boldsymbol{X}_{ij}$
  * $\widehat{\boldsymbol{\Sigma}}_i = n_i^{-1}(n_i - 1)\mathbf{S}_i = n_i^{-1} \sum_j (\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_{i\cdot})^\top$

- Likelihood ratio
$$\lambda = \prod_i \left[\frac{\det\{n_i^{-1}(n_i - 1)\mathbf{S}_i\}}{\det\{(\sum_i n_i)^{-1}(\sum_i n_i - m)\mathbf{S}_{\mathrm{pool}}\}} \right]^{n_i/2}$$
  - $\mathbf{S}_{\mathrm{pool}} = (\sum_i n_i - m)^{-1} \mathbf{SSP}_{\mathrm{w}}$

- Box's M test statistic (a modification of LRT)
$$M = -2 \ln \prod_i \left(\frac{\det \mathbf{S}_i}{\det \mathbf{S}_{\mathrm{pool}}} \right)^{(n_i - 1)/2}$$
  - Under H_0
$$(1 - u)M \approx \chi^2(p(p+1)(m-1)/2)$$
    * $u = \{\sum_i (n_i - 1)^{-1} - (\sum_i n_i - m)^{-1}\}\{6(p+1)(m-1)\}^{-1}(2p^2 + 3p - 1)$
  - Reject H_0 at level α when
$$(1 - u)M \geq \chi^2_{1-\alpha, p(p+1)(m-1)/2}$$
  - p-value
$$1 - F_{\chi^2_{1-\alpha, p(p+1)(m-1)/2}}\{(1 - u)M\}$$

- Exercise 5.2: factors in producing plastic film (continued)
  - Check the equality of covariance matrices for `RATE="Low"` and `RATE="High"`

```r
install.packages('heplots')
options(digits = 4)
tear <- c(
  6.5, 6.2, 5.8, 6.5, 6.5, 6.9, 7.2, 6.9, 6.1, 6.3,
  6.7, 6.6, 7.2, 7.1, 6.8, 7.1, 7.0, 7.2, 7.5, 7.6
)
gloss <- c(
  9.5, 9.9, 9.6, 9.6, 9.2, 9.1, 10.0, 9.9, 9.5, 9.4,
  9.1, 9.3, 8.3, 8.4, 8.5, 9.2, 8.8, 9.7, 10.1, 9.2
)
opacity <- c(
  4.4, 6.4, 3.0, 4.1, 0.8, 5.7, 2.0, 3.9, 1.9, 5.7,
  2.8, 4.1, 3.8, 1.6, 3.4, 8.4, 5.2, 6.9, 2.7, 1.9
)
(X <- cbind(tear, gloss, opacity))
(rate <- factor(gl(2,10,length=nrow(X)), labels=c("Low", "High")))

result = heplots::boxM(lm(X~rate))
result$p.value
```

- Report: Testing hypotheses H_0: the covariance matrix does not vary with the level of `RATE` v.s. H_1: otherwise, we carried on the Box's M test and obtained .6743 as the p-value. So, at the .05 level, there was no strong statistical evidence against H_0, i.e., we did not reject H_0 and believed that the covariance matrix does not vary with the level of `RATE`.

Two-way MANOVA (J&W Sec. 6.7)

- Model: $\boldsymbol{X}_{ijk} = \boldsymbol{\mu}+\boldsymbol{\tau}_i+\boldsymbol{\beta}_j+\boldsymbol{\gamma}_{ij}+\mathbf{E}_{ijk}$ with $\mathbf{E}_{ijk} \overset{\text{iid}}{\sim} \text{MVN}_p(\mathbf{0}, \boldsymbol{\Sigma})$, $i = 1, \ldots, m$, $j = 1, \ldots, b$, $k = 1, \ldots, n$
  - $\boldsymbol{\tau}_i$: the main effect of factor 1 at level i
  - $\boldsymbol{\beta}_j$: the main effect of factor 2 at level j
  - $\boldsymbol{\gamma}_{ij}$: the interaction effect of factors 1 and 2 when their levels are i and j, respectively
  - Constraints for identifiability: $\sum_i \boldsymbol{\tau}_i = \sum_j \boldsymbol{\beta}_j = \sum_i \boldsymbol{\gamma}_{ij} = \sum_j \boldsymbol{\gamma}_{ij} = \mathbf{0}$

- Decomposition of total (corrected) SSP

$$\mathbf{SSP}_{\text{t}} = \mathbf{SSP}_{\text{m1}} + \mathbf{SSP}_{\text{m2}} + \mathbf{SSP}_{\text{2fi}} + \mathbf{SSP}_{\text{r}}$$

  - Total (corrected) SSP

$$\mathbf{SSP}_{\text{t}} = \sum_{i=1}^{m}\sum_{j=1}^{b}\sum_{k=1}^{n}(\boldsymbol{X}_{ijk} - \bar{\boldsymbol{X}}_{\cdots})(\boldsymbol{X}_{ijk} - \bar{\boldsymbol{X}}_{\cdots})^{\top}$$

    * $\bar{\boldsymbol{X}}_{\cdots} = (mbn)^{-1}\sum_{i,j,k}\boldsymbol{X}_{ijk}$
  - SSP for main effect of factor 1

$$\mathbf{SSP}_{\text{m1}} = \sum_{i=1}^{m} bn(\bar{\boldsymbol{X}}_{i\cdot\cdot} - \bar{\boldsymbol{X}}_{\cdots})(\bar{\boldsymbol{X}}_{i\cdot\cdot} - \bar{\boldsymbol{X}}_{\cdots})^{\top}$$

    * $\bar{\boldsymbol{X}}_{i\cdot\cdot} = (bn)^{-1}\sum_{j,k}\boldsymbol{X}_{ijk}$
  - SSP for main effect of factor 2

$$\mathbf{SSP}_{\text{m2}} = \sum_{j=1}^{b} mn(\bar{\boldsymbol{X}}_{\cdot j\cdot} - \bar{\boldsymbol{X}}_{\cdots})(\bar{\boldsymbol{X}}_{\cdot j\cdot} - \bar{\boldsymbol{X}}_{\cdots})^{\top}$$

    * $\bar{\boldsymbol{X}}_{\cdot j\cdot} = (mn)^{-1}\sum_{i,k}\boldsymbol{X}_{ijk}$
  - SSP for 2-factor-interaction (2fi)

$$\mathbf{SSP}_{\text{2fi}} = \sum_{i=1}^{m}\sum_{j=1}^{b} n(\bar{\boldsymbol{X}}_{ij\cdot} - \bar{\boldsymbol{X}}_{i\cdot\cdot} - \bar{\boldsymbol{X}}_{\cdot j\cdot} + \bar{\boldsymbol{X}}_{\cdots})(\bar{\boldsymbol{X}}_{ij\cdot} - \bar{\boldsymbol{X}}_{i\cdot\cdot} - \bar{\boldsymbol{X}}_{\cdot j\cdot} + \bar{\boldsymbol{X}}_{\cdots})^{\top}$$

    * $\bar{\boldsymbol{X}}_{ij\cdot} = n^{-1}\sum_{k}\boldsymbol{X}_{ijk}$
  - SSP for residual

$$\mathbf{SSP}_{\text{r}} = \sum_{i=1}^{m}\sum_{j=1}^{b}\sum_{k=1}^{n}(\boldsymbol{X}_{ijk} - \bar{\boldsymbol{X}}_{ij\cdot})(\boldsymbol{X}_{ijk} - \bar{\boldsymbol{X}}_{ij\cdot})^{\top}$$

- Testing interaction
  - Hypotheses $H_0 : \boldsymbol{\gamma}_{11} = \cdots = \boldsymbol{\gamma}_{mb} = \mathbf{0}$ v.s. H_1 : otherwise
  - Wilk's lambda test statistic

$$\Lambda = \frac{\det \mathbf{SSP}_{\text{r}}}{\det(\mathbf{SSP}_{\text{r}} + \mathbf{SSP}_{\text{2fi}})}$$

    * Under H_0, by Bartlett's approximation

$$[\{p+1-(m-1)(b-1)\}/2 - mb(n-1)]\ln\Lambda \approx \chi^2((m-1)(b-1))$$

- Reject H_0 at level α when

$$[\{p+1-(m-1)(b-1)\}/2 - mb(n-1)] \ln \Lambda \geq \chi^2_{1-\alpha,(m-1)(b-1)}$$

- p-value

$$1 - F_{\chi^2((m-1)(b-1))}([\{p+1-(m-1)(b-1)\}/2 - mb(n-1)] \ln \Lambda)$$

- Testing main effects
  - Testing factor 1 main effects
    - Hypotheses $H_0 : \boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_m = \mathbf{0}$ v.s. H_1 : otherwise
    - Wilk's lambda test statistic

$$\Lambda = \frac{\det \mathbf{SSP}_{\mathrm{r}}}{\det(\mathbf{SSP}_{\mathrm{r}} + \mathbf{SSP}_{\mathrm{m1}})}$$

      - Under H_0, by Bartlett's approximation

$$[\{p+1-(m-1)\}/2 - mb(n-1)] \ln \Lambda \approx \chi^2(m-1)$$

    - Reject H_0 at level α when

$$[\{p+1-(m-1)\}/2 - mb(n-1)] \ln \Lambda \geq \chi^2_{1-\alpha,m-1}$$

    - p-value

$$1 - F_{\chi^2(m-1)}([\{p+1-(m-1)\}/2 - mb(n-1)] \ln \Lambda)$$

  - Testing factor 2 main effects
    - Hypotheses $H_0 : \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_b = \mathbf{0}$ v.s. H_1 : otherwise
    - Wilk's lambda test statistic

$$\Lambda = \frac{\det \mathbf{SSP}_{\mathrm{r}}}{\det(\mathbf{SSP}_{\mathrm{r}} + \mathbf{SSP}_{\mathrm{2fi}})}$$

      - Under H_0, by Bartlett's approximation

$$[\{p+1-(b-1)\}/2 - mb(n-1)] \ln \Lambda \approx \chi^2(b-1)$$

    - Reject H_0 at level α when

$$[\{p+1-(b-1)\}/2 - mb(n-1)] \ln \Lambda \geq \chi^2_{1-\alpha,b-1}$$

    - p-value

$$1 - F_{\chi^2(b-1)}([\{p+1-(b-1)\}/2 - mb(n-1)] \ln \Lambda)$$

- Exercise 5.3: factors in producing plastic film (continued)
  - One more factor `ADDITIVE` (amount of an additive, 2-level, low or high) in the production test