

STAT 3690 Lecture Note

Week Three (Jan 23, 25, & 27, 2023)

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Statistical modelling (con'd)

Transformation of random vectors

- Derive the pdf of continuous $\mathbf{Y} = \mathbf{g}(\mathbf{X})$ from the pdf of continuous \mathbf{X}
- Prerequisite
 - $\mathbf{X} = [X_1, \dots, X_p]^\top$ and $\mathbf{Y} = [Y_1, \dots, Y_p]^\top$
 - $\mathbf{g} = (g_1, \dots, g_p): \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a continuous one-to-one map with inverse $\mathbf{g}^{-1} = (h_1, \dots, h_p)$, i.e., $Y_i = g_i(\mathbf{X})$ and $X_i = h_i(\mathbf{Y})$
- Elaborate $\text{supp}(\mathbf{Y}) = \{[y_1, \dots, y_p]^\top : [h_1(y_1, \dots, y_p), \dots, h_p(y_1, \dots, y_p)]^\top \in \text{supp}(\mathbf{X})\}$
- Jacobian matrix of \mathbf{g}^{-1} is $\mathbf{J}_{\mathbf{g}^{-1}} = [\partial x_i / \partial y_j]_{p \times p} = [\partial h_i(y_1, \dots, y_p) / \partial y_j]_{p \times p}$
 - Also, $|\det(\mathbf{J}_{\mathbf{g}^{-1}})| = |\det([\partial y_i / \partial x_j]_{p \times p})|^{-1} = |\det([\partial g_i(x_1, \dots, x_p) / \partial x_j]_{p \times p})|^{-1}$
- Then

$$f_{\mathbf{Y}}(y_1, \dots, y_p) = f_{\mathbf{X}}(h_1(y_1, \dots, y_p), \dots, h_p(y_1, \dots, y_p)) |\det(\mathbf{J}_{\mathbf{g}^{-1}})| \mathbf{1}_{\text{supp}(\mathbf{Y})}(y_1, \dots, y_p)$$

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- Exercise: Let $\mathbf{X} = [X_1, X_2]^\top$ follow the standard bivariate normal, i.e., its pdf is

$$f_{\mathbf{X}}(x_1, x_2) = (2\pi)^{-1} \exp\{-(x_1^2 + x_2^2)/2\} \mathbf{1}_{\mathbb{R}^2}(x_1, x_2).$$

Find out the joint pdf of $\mathbf{Y} = [Y_1, Y_2]^\top$, where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $0 \leq Y_2 < 2\pi$ is the angle from the positive x -axis to the ray from the origin to the point (X_1, X_2) , that is, Y is X in the polar coordinate.

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- Exercise: Given positive α, β and θ , $\mathbf{X} = [X_1, X_2]^\top$ follow

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} x_1^{\alpha-1} x_2^{\beta-1} \exp\left(-\frac{x_1 + x_2}{\theta}\right) \mathbf{1}_{\mathbb{R}^+ \times \mathbb{R}^+}(x_1, x_2).$$

Find out the joint pdf of $\mathbf{Y} = [Y_1, Y_2]^\top$, where $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$.

Expectation of random matrix

- $E(\mathbf{X}) = [E(X_{ij})]_{n \times p}$, where
 - Random $n \times p$ matrix $\mathbf{X} = [X_{ij}]_{n \times p}$
 - (Linearity) $E(\mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{Y}) = \mathbf{A}E(\mathbf{X}) + \mathbf{B}E(\mathbf{Y})$, where
 - Fixed $\mathbf{A} \in \mathbb{R}^{\ell \times n}$ and $\mathbf{B} \in \mathbb{R}^{\ell \times m}$
 - Random matrices $\mathbf{X} = [X_{ij}]_{n \times p}$ and $\mathbf{Y} = [Y_{ij}]_{m \times p}$
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Covariance matrix

- Random p -vector $\mathbf{X} = [X_1, \dots, X_p]^\top$ and random q -vector $\mathbf{Y} = [Y_1, \dots, Y_q]^\top$
 - Covariance matrix (defined via expectation) $\Sigma_{\mathbf{XY}} = \text{cov}(\mathbf{X}, \mathbf{Y}) = E[\{\mathbf{X} - E(\mathbf{X})\}\{\mathbf{Y} - E(\mathbf{Y})\}^\top]$
 - Also, $\Sigma_{\mathbf{XY}} = E(\mathbf{XY}^\top) - E(\mathbf{X})E(\mathbf{Y}^\top)$
 - The (i, j) -entry of $\Sigma_{\mathbf{XY}}$ is $\text{cov}(X_i, Y_j)$
 - $\Sigma_{\mathbf{AX} + \mathbf{a}, \mathbf{BY} + \mathbf{b}} = \mathbf{A}\Sigma_{\mathbf{XY}}\mathbf{B}^\top$ for fixed $\mathbf{A} \in \mathbb{R}^{m \times p}$, $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{B} \in \mathbb{R}^{\ell \times q}$ and $\mathbf{b} \in \mathbb{R}^\ell$
 - $\Sigma_{\mathbf{X}} \geq 0$, where $\Sigma_{\mathbf{X}} = \text{cov}(\mathbf{X})$ is short for $\Sigma_{\mathbf{XX}} = \text{cov}(\mathbf{X}, \mathbf{X})$
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- Exercise: Verify the following properties of covariance matrix
 1. $\Sigma_{\mathbf{AX} + \mathbf{a}, \mathbf{BY} + \mathbf{b}} = \mathbf{A}\Sigma_{\mathbf{XY}}\mathbf{B}^\top$
 2. $\Sigma_{\mathbf{X}} \geq 0$

Sample covariance matrix

- Samples $\mathbf{X}_k = [X_{k1}, \dots, X_{kp}]^\top$ and $\mathbf{Y}_k = [Y_{k1}, \dots, Y_{kq}]^\top$, $k = 1, \dots, n$
- $(\mathbf{X}_k, \mathbf{Y}_k) \stackrel{\text{iid}}{\sim} (\mathbf{X}, \mathbf{Y})$, where $\mathbf{X} = [X_1, \dots, X_p]^\top$ and $\mathbf{Y} = [Y_1, \dots, Y_q]^\top$
- Sample means
 - $\bar{\mathbf{X}} = n^{-1} \sum_{k=1}^n \mathbf{X}_k = [\bar{X}_{\cdot 1}, \dots, \bar{X}_{\cdot p}]^\top$
 - $\bar{\mathbf{Y}} = n^{-1} \sum_{k=1}^n \mathbf{Y}_k = [\bar{Y}_{\cdot 1}, \dots, \bar{Y}_{\cdot q}]^\top$
- Sample covariance matrix:

$$\mathbf{S}_{\mathbf{XY}} = \frac{1}{n-1} \sum_{k=1}^n \{(\mathbf{X}_k - \bar{\mathbf{X}})(\mathbf{Y}_k - \bar{\mathbf{Y}})^\top\}$$

- The (i, j) -entry of $\mathbf{S}_{\mathbf{XY}}$ is $(n-1)^{-1} \sum_{k=1}^n (X_{ki} - \bar{X}_{\cdot i})(Y_{kj} - \bar{Y}_{\cdot j})$, i.e., the sample covariance between X_i and Y_j
 - Unbiasedness: $E(\mathbf{S}_{\mathbf{XY}}) = \Sigma_{\mathbf{XY}}$
 - $\mathbf{S}_{\mathbf{AX} + \mathbf{a}, \mathbf{BY} + \mathbf{b}} = \mathbf{A}\mathbf{S}_{\mathbf{XY}}\mathbf{B}^\top$ for $\mathbf{A} \in \mathbb{R}^{m \times p}$, $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{B} \in \mathbb{R}^{\ell \times q}$ and $\mathbf{b} \in \mathbb{R}^\ell$
 - $\mathbf{S}_{\mathbf{X}} \geq 0$
 - Implementation in R: `cov()` (or `var()` if $\mathbf{X} = \mathbf{Y}$)
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- Exercise: Verify the following properties of sample covariance matrix
 1. $E(\mathbf{S}_{\mathbf{XY}}) = \Sigma_{\mathbf{XY}}$
 2. $\mathbf{S}_{\mathbf{AX} + \mathbf{a}, \mathbf{BY} + \mathbf{b}} = \mathbf{A}\mathbf{S}_{\mathbf{XY}}\mathbf{B}^\top$
 3. $\mathbf{S}_{\mathbf{X}} \geq 0$

Computing mean vectors and covariance matrices via R