STAT 3690 Lecture 11

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Testing on $\mathbf{A}\boldsymbol{\mu}$ (J&W pp. 279)

- $\mathbf{A}$ is of $q \times p$ and $\mathrm{rk}(\mathbf{A}) = q$, i.e., $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top > 0$
- Model: iid $\mathbf{A}\mathbf{X}_i \sim MVN_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\Sigma\mathbf{A}^\top)$.
- LRT for $H_0 : \mathbf{A}\boldsymbol{\mu} = \boldsymbol{\nu}_0$ v.s. $H_1 : \mathbf{A}\boldsymbol{\mu} \neq \boldsymbol{\nu}_0$
  - Test statistic: $n(\mathbf{A}\bar{\mathbf{X}} - \boldsymbol{\nu}_0)^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\mathbf{X}} - \boldsymbol{\nu}_0) \sim T^2(q, n-1) = \frac{(n-1)q}{n-q}F(q, n-q)$ under H_0
  - Rejction region at level α: $R = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n : \frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0) \geq F_{1-\alpha,q,n-q}\}$
  - p-value: $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1 - F_{F(q,n-q)}\{\frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)\}$
- Multiple comparison
  - Interested in $H_0 : \mu_1 = \cdots = \mu_p$ v.s. H_1 : Not all entries of $\boldsymbol{\mu}$ are equal.
    - μ_k: the kth entry of $\boldsymbol{\mu}$
  - Take

$$\boldsymbol{\nu}_0 = \mathbf{0}_{(p-1)\times 1}, \quad \mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}_{(p-1)\times p}.$$

  - $p = 2$ (i.e., $\mathbf{A} = [1, -1]$): A/B testing

```r
options(digits = 4)
install.packages(c("dslabs",'tidyverse'))
library(dslabs)
library(tidyverse)
data("gapminder")
dataset = gapminder[
  !is.na(gapminder$infant_mortality) &
    gapminder$region == 'South America' &
    gapminder$year %in% 2000:2008,
  c('country', 'year', "life_expectancy")] %>%
  spread(year, life_expectancy)
(dataset = as.matrix(dataset[, -1]))
n = nrow(dataset); p = ncol(dataset)
(mu_hat <- colMeans(dataset))

# Test H0:A %*% mu = nu_0
(nu_0 <- as.matrix(rep(0, p-1)))
```

```r
(A = cbind(rep(1, p-1), -diag(p-1)))

(test.stat <- drop(
  n * t(A %*% mu_hat - nu_0) %*%
    solve(A %*% cov(dataset) %*% t(A)) %*%
    (A %*% mu_hat - nu_0)
))
(cri.point = (n-1)*(p-1)/(n-p+1)*qf(.95, p-1, n-p+1))
test.stat >= cri.point
(p.val = 1-pf((n-p+1)/(n-1)/(p-1)*test.stat, p-1, n-p+1))
```

- Report: Testing hypotheses H_0: the average life expectancy over south american countries doesn't vary with time v.s. H_1: otherwise, we carried on the LRT and obtained 628.5 as the value of test statistic. The corresponding p-value (resp. rejection region) was 0.002858 (resp. $[132.9, \infty)$). So, at the .05 level, there was a strong statistical evidence against H_0, i.e., we believed that the average life expectancy over south american countries does vary with time.

$(1-\alpha) \times 100\%$ confidence region (CR) for $\boldsymbol{\mu}$ (J&W Sec. 5.4)

- $\Pr((1-\alpha) \times 100\%\text{CR covers } \boldsymbol{\mu}) = 1-\alpha$
  - CR is a set made of observations and is hence random
  - $\boldsymbol{\mu}$ is fixed
  - $(1-\alpha) \times 100\%$ CR covers $\boldsymbol{\mu}$ with probability $(1-\alpha) \times 100\%$
- Dual problem of testing $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ v.s. $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ at the α level
  - Translated from rejection region. Steps:
    1. Take R as a function of $\boldsymbol{\mu}_0$;
    2. Replace $\boldsymbol{\mu}_0$ with $\boldsymbol{\mu}$;
    3. Take the complement.
  - $(1-\alpha) \times 100\%$ CR $= \{\boldsymbol{\mu} : n(\bar{\boldsymbol{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}) < \chi^2_{1-\alpha,p}\}$ if $\boldsymbol{\Sigma}$ is known
  - $(1-\alpha) \times 100\%$ CR $= \{\boldsymbol{\mu} : \frac{n(n-p)}{p(n-1)}(\bar{\boldsymbol{x}} - \boldsymbol{\mu})^\top \mathbf{S}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}) < F_{1-\alpha,p,n-p}\}$ if $\boldsymbol{\Sigma}$ is not known

$(1-\alpha) \times 100\%$ CR for $\boldsymbol{\nu} = \mathbf{A}\boldsymbol{\mu}$

- $\mathbf{X}_1, \ldots, \mathbf{X}_n \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - Unknown $\boldsymbol{\Sigma}$
  - $n > p$
- $\mathbf{A}$ is of $q \times p$ and $\text{rk}(\mathbf{A}) = q$, i.e., $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top > 0$
- Then iid $\mathbf{A}\mathbf{X}_i \sim MVN_q(\boldsymbol{\nu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$
- $(1-\alpha) \times 100\%$ CR for $\boldsymbol{\nu}$ is $\{\boldsymbol{\nu} : \frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu})^\top (\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}) < F_{1-\alpha,q,n-q}\}$
- Special case: $\mathbf{A} = \boldsymbol{a} \in \mathbb{R}^p$
  - $(1-\alpha) \times 100\%$ confidence interval (CI) for scalar $\nu = \boldsymbol{a}^\top \boldsymbol{\mu}$ is

$$\{\nu : n(\boldsymbol{a}^\top \bar{\boldsymbol{x}} - \nu)^2 (\boldsymbol{a}^\top \mathbf{S}\boldsymbol{a})^{-1} < F_{1-\alpha,1,n-1}\} = \left(\boldsymbol{a}^\top \bar{\boldsymbol{x}} - t_{1-\alpha/2,n-1}\sqrt{\boldsymbol{a}^\top \mathbf{S}\boldsymbol{a}/n}, \boldsymbol{a}^\top \bar{\boldsymbol{x}} + t_{1-\alpha/2,n-1}\sqrt{\boldsymbol{a}^\top \mathbf{S}\boldsymbol{a}/n}\right)$$