PH 712 Probability and Statistical Inference

Part VI: Evaluating Estimators I

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Bias

- Bias of $\hat{\theta}$: $\mathrm{Bias}_\theta(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta$
- Unbiased estimator: $\hat{\theta}$ satisfying $\mathrm{E}(\hat{\theta}) = \theta$

Mean squared error (MSE)

- $\mathrm{MSE}_\theta(\hat{\theta}) = \mathrm{E}(\hat{\theta} - \theta)^2$
  - The lower the better
  - $= \mathrm{Bias}^2_\theta(\hat{\theta}) + \mathrm{var}(\hat{\theta})$
- For unbiased estimators, minimizing the MSE $\Leftrightarrow$ minimizing the variance

Cramér-Rao lower bound (CRLB, CB Thm 7.3.9 & Lemma 7.3.11)

- Recall the score $S(\theta; x_1, \ldots, x_n) = \ell'(\theta; x_1, \ldots, x_n)$
- Hessian: $H(\theta; x_1, \ldots, x_n) = \ell''(\theta; x_1, \ldots, x_n)$
- CRLB $= I^{-1}(\theta) \left\{ \frac{\mathrm{d}}{\mathrm{d}\theta} \mathrm{E}(\hat{\theta}) \right\}^2$
  - Reducing to $I^{-1}(\theta)$ if $\mathrm{E}(\hat{\theta}) = \theta$ (i.e., unbiased $\hat{\theta}$)
  - Fisher information:

$$I(\theta) = I(\theta; X_1, \ldots, X_n) = \mathrm{var}\{S(\theta; X_1, \ldots, X_n)\} = \mathrm{E}[\{S(\theta; X_1, \ldots, X_n)\}^2] = -\mathrm{E}\{H(\theta; X_1, \ldots, X_n)\}$$

- Under regularity conditions, $\mathrm{var}(\hat{\theta}) \geq$ CRLB.

Example Lec6.1

- Find the CRLB for all the UNBIASED estimators in the following cases.
  a. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with UNKNOWN μ and GIVEN σ^2.
  b. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with GIVEN μ and UNKNOWN σ^2.