STAT 3690 Lecture 02

zhiyanggeezhou.github.io

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca)

Jan 26th, 2022

Eigendecomposition

- $\mathbf{A}$ is a real square $n \times n$ matrix
- Characteristic equation of $\mathbf{A}$: $\det(\lambda \mathbf{I}_n - \mathbf{A}) = 0$, with identity matrix $\mathbf{I}$
- Eigenvalues of $\mathbf{A}$, say $\lambda_1 \geq \cdots \geq \lambda_n$: n roots of characteristic equation are
- (Right) eigenvector $\boldsymbol{v}_i$: $\mathbf{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$
- Eigendecomposition: $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$
  - $\mathbf{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ are both $n \times n$ matrices
- Implementation in *R*: eigen()

Spectral decomposition

- $\mathbf{A}$ is a real symmetric square $n \times n$ matrix
- Then $\mathbf{V}$ is orthogonal, i.e., $\mathbf{V}^\top\mathbf{V} = \mathbf{V}\mathbf{V}^\top = \mathbf{I}$ and $\mathbf{V}^\top = \mathbf{V}^{-1}$
- Spectral decomposition : $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^\top$

Singular value decomposition (SVD)

- Consider a general real $n \times p$ matrix $\mathbf{B}$
- But, obviously, $\mathbf{B}^\top\mathbf{B}$ and $\mathbf{B}\mathbf{B}^\top$ are both symmetric and square
  - They have identical non-zero eigenvalues
  - They are even positive semi-definite, i.e., their eigenvalues are non-nagative
- Then $\mathbf{B}\mathbf{B}^\top = \mathbf{U}_{n\times n}\Gamma_{n\times n}\mathbf{U}_{n\times n}^\top$ and $\mathbf{B}^\top\mathbf{B} = \mathbf{W}_{p\times p}\boldsymbol{\Delta}_{p\times p}\mathbf{W}_{p\times p}^\top$
  - $\mathbf{U}$ and $\mathbf{W}$ are both orthogonal
- SVD:
$$\mathbf{B} = \mathbf{U}_{n\times n}\mathbf{S}_{n\times p}\mathbf{W}_{p\times p}^\top = s_{11}\boldsymbol{u}_1\boldsymbol{w}_1^\top + \cdots + s_{rr}\boldsymbol{u}_r\boldsymbol{w}_r^\top$$
  - Singular values s_{ii} is the ith diagonal entry of $\mathbf{S}_{n\times p}$
  - $s_{11} \geq \cdots \geq s_{rr} > 0$ are square roots of non-zero eigenvalues of $\mathbf{B}^\top\mathbf{B}$ and $\mathbf{B}\mathbf{B}^\top$
  - $\boldsymbol{u}_i$ (resp. $\boldsymbol{w}_i$) is the ith column of $\mathbf{U}_{n\times n}$ (resp. $\mathbf{W}_{p\times p}$)
  - r is the rank of diagonal $\mathbf{S}_{n\times p}$
- Thin/compact SVD
  - Implementation in *R*: svd()

- Exercise: Is it feasible to apply eigen() only in conducting the thin SVD for a matrix with non-negative singular values (λ_i's)?

```
options(digits = 4) # control the number of significant digits
set.seed(1)
A = matrix(runif(12), nrow = 2, ncol = 6)
svdResult = svd(A)
eigenResult = eigen(tcrossprod(A))
# respective set of eigenvalues from each method
svdResult$d; eigenResult$values^.5
# respective eigenvectors from each method
svdResult$u; eigenResult$vectors
# respective eigenvectors from each method
svdResult$v; t(diag(eigenResult$values^-.5) %*% t(eigenResult$vectors) %*% A)
```