STAT 3690 Lecture 14

zhiyanggeezhou.github.io

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca)

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2-way MANOVA (J&W Sec. 6.7)

- Model: $\mathbf{X}_{ijk} = \boldsymbol{\mu} + \boldsymbol{\tau}_i + \boldsymbol{\beta}_j + \boldsymbol{\gamma}_{ij} + \mathbf{E}_{ijk}$ with $\mathbf{E}_{ij} \overset{\text{iid}}{\sim} MVN_p(\mathbf{0}, \boldsymbol{\Sigma})$, $i = 1, \ldots, m$, $j = 1, \ldots, b$, $k = 1, \ldots, n$
    - $\boldsymbol{\tau}_i$: the main effect of factor 1 at level i
    - $\boldsymbol{\beta}_j$: the main effect of factor 2 at level j
    - $\boldsymbol{\gamma}_{ij}$: the interaction of factors 1 and 2 whose levels are i and j, respectively
    - Identifiability: $\sum_i \boldsymbol{\tau}_i = \sum_j \boldsymbol{\beta}_j = \sum_i \boldsymbol{\gamma}_{ij} = \sum_j \boldsymbol{\gamma}_{ij} = \mathbf{0}$

- Sum of squares and cross products matrix (SSP)
    - Total corrected SSP
    $$\mathbf{SSP}_{\text{cor}} = \sum_{i=1}^{m} \sum_{j=1}^{b} \sum_{k=1}^{n} (\mathbf{X}_{ijk} - \bar{\mathbf{X}})(\mathbf{X}_{ijk} - \bar{\mathbf{X}})^\top$$
        * $\bar{\mathbf{X}} = (mbn)^{-1} \sum_{i,j,k} \mathbf{X}_{ijk}$
    - SSP for factor 1
    $$\mathbf{SSP}_{\text{f1}} = \sum_{i=1}^{m} bn(\bar{\mathbf{X}}_{i\cdot} - \bar{\mathbf{X}})(\bar{\mathbf{X}}_{i\cdot} - \bar{\mathbf{X}})^\top$$
        * $\bar{\mathbf{X}}_{i\cdot} = (bn)^{-1} \sum_{j,k} \mathbf{X}_{ijk}$
    - SSP for factor 2
    $$\mathbf{SSP}_{\text{f2}} = \sum_{j=1}^{b} mn(\bar{\mathbf{X}}_{\cdot j} - \bar{\mathbf{X}})(\bar{\mathbf{X}}_{\cdot j} - \bar{\mathbf{X}})^\top$$
        * $\bar{\mathbf{X}}_{\cdot j} = (mn)^{-1} \sum_{i,k} \mathbf{X}_{ijk}$
    - SSP for interaction
    $$\mathbf{SSP}_{\text{int}} = \sum_{i=1}^{m} \sum_{j=1}^{b} n(\bar{\mathbf{X}}_{ij} - \bar{\mathbf{X}}_{i\cdot} - \bar{\mathbf{X}}_{\cdot j} + \bar{\mathbf{X}})(\bar{\mathbf{X}}_{ij} - \bar{\mathbf{X}}_{i\cdot} - \bar{\mathbf{X}}_{\cdot j} + \bar{\mathbf{X}})^\top$$
        * $\bar{\mathbf{X}}_{ij} = n^{-1} \sum_k \mathbf{X}_{ijk}$
    - SSP for residual
    $$\mathbf{SSP}_{\text{res}} = \sum_{i=1}^{m} \sum_{j=1}^{b} \sum_{k=1}^{n} (\mathbf{X}_{ijk} - \bar{\mathbf{X}}_{ij})(\mathbf{X}_{ijk} - \bar{\mathbf{X}}_{ij})^\top$$
    - $\mathbf{SSP}_{\text{cor}} = \mathbf{SSP}_{\text{f1}} + \mathbf{SSP}_{\text{f2}} + \mathbf{SSP}_{\text{int}} + \mathbf{SSP}_{\text{res}}$

- Testing interaction
    - Hypotheses $H_0 : \boldsymbol{\gamma}_{11} = \cdots = \boldsymbol{\gamma}_{mb} = \mathbf{0}$ v.s. $H_1 :$ otherwise

– Wilk's lambda test statistic
$$\Lambda = \frac{\det \mathbf{SSP}_{\mathrm{res}}}{\det(\mathbf{SSP}_{\mathrm{res}} + \mathbf{SSP}_{\mathrm{int}})}$$

* Under H_0, by Bartlett's approximation

$$[\{p + 1 - (m-1)(b-1)\}/2 - mb(n-1)]\ln \Lambda \approx \chi^2((m-1)(b-1))$$

– Rejection H_0 at level α when

$$[\{p + 1 - (m-1)(b-1)\}/2 - mb(n-1)]\ln \Lambda \geq \chi^2_{1-\alpha,(m-1)(b-1)}$$

– p-value

$$1 - F_{\chi^2((m-1)(b-1))}([\{p + 1 - (m-1)(b-1)\}/2 - mb(n-1)]\ln \Lambda)$$

- Testing main effects
  - Testing factor 1 main effects
    * Hypotheses $H_0 : \boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_m = \mathbf{0}$ v.s. H_1 : otherwise
    * Wilk's lambda test statistic
    $$\Lambda = \frac{\det \mathbf{SSP}_{\mathrm{res}}}{\det(\mathbf{SSP}_{\mathrm{res}} + \mathbf{SSP}_{\mathrm{f1}})}$$

    · Under H_0, by Bartlett's approximation

    $$[\{p + 1 - (m-1)\}/2 - mb(n-1)]\ln \Lambda \approx \chi^2(m-1)$$

    * Rejection H_0 at level α when

    $$[\{p + 1 - (m-1)\}/2 - mb(n-1)]\ln \Lambda \geq \chi^2_{1-\alpha,m-1}$$

    * p-value

    $$1 - F_{\chi^2(m-1)}([\{p + 1 - (m-1)\}/2 - mb(n-1)]\ln \Lambda)$$

  - Testing factor 2 main effects
    * Hypotheses $H_0 : \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_b = \mathbf{0}$ v.s. H_1 : otherwise
    * Wilk's lambda test statistic
    $$\Lambda = \frac{\det \mathbf{SSP}_{\mathrm{res}}}{\det(\mathbf{SSP}_{\mathrm{res}} + \mathbf{SSP}_{\mathrm{f2}})}$$

    · Under H_0, by Bartlett's approximation

    $$[\{p + 1 - (b-1)\}/2 - mb(n-1)]\ln \Lambda \approx \chi^2(b-1)$$

    * Rejection H_0 at level α when

    $$[\{p + 1 - (b-1)\}/2 - mb(n-1)]\ln \Lambda \geq \chi^2_{1-\alpha,b-1}$$

    * p-value

    $$1 - F_{\chi^2(b-1)}([\{p + 1 - (b-1)\}/2 - mb(n-1)]\ln \Lambda)$$

- Exercise: factors in producing plastic film (continued)
  - One more factor `ADDITIVE` (amount of an additive, 2-level, low or high) in the production test