

STAT 3100 Lecture Note

Week Eleven (Nov 22 & 24, 2022)

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Asymptotic properties of MLE (con'd)

Consistency (or consistence, CB Sec 10.1.1)

- $T_n = T_n(X_1, \dots, X_n)$ is consistent for θ iff $T_n \xrightarrow{p} \theta$ as $n \rightarrow \infty$
 - A sufficient condition for consistency: $E(T_n | \theta) \rightarrow \theta$ and $\text{var}(T_n | \theta) \rightarrow 0$ as $n \rightarrow \infty$

CB Example 5.5.3

- Suppose that iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Prove that
 - $S_n^2 = (n-1)^{-1} \sum_i (X_i - \bar{X}_n)^2$ is consistent for σ^2 ;
 - $\widehat{\sigma}_{\text{ML}}^2 = n^{-1} \sum_i (X_i - \bar{X}_n)^2$ is consistent for σ^2 too.

Consistency of MLE (univariate case, CB Thm 10.1.6)

- $\hat{\theta}_{\text{ML}} \xrightarrow{p} \theta_0$, provided that $\hat{\theta}_{\text{ML}}$ is the MLE for θ_0 , under (sufficient) regularity conditions (CB Sec 10.6.2):
 - iid $X_1, \dots, X_n \sim f(x | \theta_0)$;
 - for $\theta_1, \theta_2 \in \Theta$, if $\theta_1 \neq \theta_2$, then $f_X(x | \theta_1) \neq f_X(x | \theta_2)$;
 - $f_X(x | \theta)$ has a common support for each $\theta \in \Theta$ and is differentiable with respect to θ ;
 - * Violated by, e.g., $\text{Unif}(0, \theta)$;
 - θ_0 is an interior point of parameter space Θ .

Example of inconsistent MLE

There are independent $X_{i1}, X_{i2} \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, \dots, n$. Then $\widehat{\sigma}_{\text{ML}}^2$ is NOT consistent for σ^2 .

Examples of consistent MLE with the regularity conditions violated

- iid $X_1, \dots, X_n \sim \text{Ber}(1)$
- iid $X_1, \dots, X_n \sim \text{Unif}(0, \theta)$

Efficiency

- (HMC Def 6.2.2) For an estimator, say T_n , unbiased for $\tau(\theta)$, the (finite-sample) *efficiency* of T_n is the ratio of the CRLB to $\text{var}(T_n)$, i.e., $[\{\tau'(\theta)\}^2 / I_n(\theta)] / \text{var}(T_n | \theta)$.
 - The higher efficiency the better;
 - the efficiency = 1 \iff an efficient estimator.
- (CB Def 10.1.9) If $k_n\{T_n - \tau(\theta)\} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, then σ^2 is the *asymptotic variance* of T_n .

- (CB Def 10.1.11) T_n is *asymptotically efficient* for $\tau(\theta) \iff \sqrt{n}\{T_n - \tau(\theta)\} \xrightarrow{d} \mathcal{N}(0, \{\tau'(\theta)\}^2/I_1(\theta))$, where

$$I_1(\theta) = -\mathbb{E} \left\{ \frac{\partial^2}{\partial \theta^2} \ln f(X_i | \theta) \mid \theta \right\} = -\mathbb{E}\{H(\theta; X_i) \mid \theta\}$$

is the Fisher information of one single observation.

– i.e., the asymptotic variance of T_n is $\{\tau'(\theta)\}^2/I_1(\theta)$, attaining the CRLB

- (CB Def 10.1.16) Denote by T_n and W_n two estimators for $\tau(\theta)$. Suppose that $\sqrt{n}\{T_n - \tau(\theta)\} \xrightarrow{d} \mathcal{N}(0, \sigma_T^2)$ and $\sqrt{n}\{W_n - \tau(\theta)\} \xrightarrow{d} \mathcal{N}(0, \sigma_W^2)$. The *asymptotic relative efficiency* (ARE) of T_n with respect to W_n is defined as

$$\text{ARE}(T_n, W_n) = \sigma_W^2 / \sigma_T^2.$$

- T_n is asymptotically more efficient than $W_n \iff \text{ARE}(T_n, W_n) > 1$
- T_n is asymptotically efficient $\iff \{\{\tau'(\theta)\}^2/I_1(\theta)\}/\sigma_T^2 = 1$

CB Example 10.1.17 & Ex. 10.9

- iid $X_1, \dots, X_n \sim p(x | \lambda) = \lambda^x \exp(-\lambda)/x!$, $x \in \mathbb{Z}^+$, $\lambda > 0$. To estimate $\Pr(X_i = 0) = \exp(-\lambda)$.
 - Consider $T_n = n^{-1} \sum_i \mathbf{1}_{\{0\}}(X_i)$ and MLE $W_n = \exp(-\bar{X}_n)$. Compute $\text{ARE}(T_n, W_n)$, the ARE of T_n with respect to W_n .
 - Find the UMVUE for $\Pr(X_i = 0)$, say U_n , and then calculate $\text{ARE}(U_n, W_n)$.

Asymptotic efficiency of MLE (CB Thm 10.1.12 & Ex. 10.7)

- $\sqrt{n}\{\tau(\hat{\theta}_{\text{ML}}) - \tau(\theta_0)\} \xrightarrow{d} \mathcal{N}(0, \{\tau'(\theta_0)\}^2/I_1(\theta_0))$, provided that $\hat{\theta}_{\text{ML}}$ is the MLE for θ_0 , τ is differentiable and we have the previous four regularity conditions (for the consistency of MLE) plus the following two more (CB Sec 10.6.2):
 - For each $x \in \text{supp}(X)$, $f(x | \theta)$ is three times continuously differentiable with respect to θ ; and $\int f(x | \theta) dx$ can be differentiated three times under the integral sign;
 - for each $\theta \in \Theta$, there exists $c(\theta) > 0$ and $M(x, \theta)$ such that $|\frac{\partial^3}{\partial \theta^3} \ln f_X(x | \theta)| \leq M(x, \theta)$ for all $x \in \text{supp}(X)$ and $\theta \in (\theta - c(\theta), \theta + c(\theta))$.
- In practice,
 - $nI_1(\theta_0) = I_n(\theta_0) \approx I_n(\hat{\theta}_{\text{ML}}) \approx \hat{I}_n(\hat{\theta}_{\text{ML}})$
 - * (Expected) Fisher information (number) $I_n(\theta_0) = -\mathbb{E}\{H(\theta_0; \mathbf{X})\}$
 - * Observed Fisher information (number) $\hat{I}_n(\hat{\theta}_{\text{ML}}) = -\frac{\partial^2}{\partial \theta^2} \ln L(\theta; \mathbf{x})|_{\theta=\hat{\theta}_{\text{ML}}} = -H(\hat{\theta}_{\text{ML}}; \mathbf{x})$
 - Hence $\text{var}(\tau(\hat{\theta}_{\text{ML}})) \approx \{\tau'(\theta_0)\}^2/I_n(\theta_0) \approx \{\tau'(\hat{\theta}_{\text{ML}})\}^2/I_n(\hat{\theta}_{\text{ML}}) \approx \{\tau'(\hat{\theta}_{\text{ML}})\}^2/\hat{I}_n(\hat{\theta}_{\text{ML}})$

Approximation to variances

Delta method

- (CB Thm 5.5.24, delta method) If $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ and $\tau'(\theta) \neq 0$, then

$$\sqrt{n}\{\tau(\hat{\theta}_n) - \tau(\theta)\} \xrightarrow{d} \mathcal{N}(0, \{\tau'(\theta)\}^2 \sigma^2).$$

– Hence $\text{var}\{\tau(\hat{\theta}_n)\} \approx \{\tau'(\theta)\}^2 \sigma^2 / n$ if $\tau'(\theta) \neq 0$

- (CB Thm 5.5.26, second-order delta method) If $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, $\tau'(\theta) = 0$, and $\tau''(\theta) \neq 0$, then

$$n\{\tau(\hat{\theta}_n) - \tau(\theta)\} \xrightarrow{d} \frac{\tau''(\theta)\sigma^2}{2} \chi^2(1).$$

– Hence $\text{var}\{\tau(\hat{\theta}_n)\} \approx \{\tau''(\theta)\}^2 \sigma^4 / (2n^2)$ if $\tau'(\theta) = 0$ but $\tau''(\theta) \neq 0$

CB Example 10.1.15

- Holding iid $X_i \sim \text{Bernoulli}(p)$, the variance of $\text{Bernoulli}(p)$ is $\tau(p) = p(1-p)$ whose MLE is $\tau(\hat{p}_{\text{mle}}) = \bar{X}_n(1 - \bar{X}_n)$. Approximate $\text{var}\{\tau(\hat{p}_{\text{mle}})\}$ by the delta method.

Bootstrapping the variance of $\hat{\theta} = \hat{\theta}(\mathbf{X})$ (CB Sec. 10.1.4)

- Nonparametric bootstrap:
 1. For j in $1 : B$, do steps 2–3.
 2. Draw the j th resample \mathbf{x}_j^* of size n from the original sample $\mathbf{x} = \{x_1, \dots, x_n\}$, with replacement.
 3. Let $\hat{\theta}_j^* = \hat{\theta}(\mathbf{x}_j^*)$.
 4. $\text{var}\{\hat{\theta}(\mathbf{X})\} \approx$ the sample variance of $\{\hat{\theta}_1^*, \dots, \hat{\theta}_B^*\}$.
- Parametric bootstrap:
 1. For j in $1 : B$, do steps 2–3.
 2. Draw the j th resample \mathbf{x}_j^* of size n from a fitted model $f(x | \hat{\theta})$.
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Take-home exercises (NOT to be submitted; to be potentially covered in labs)

- CB Ex. 10.3, 10.17(a–c) $\sqrt{n}(T_n - \tau(\theta)) \xrightarrow{d} N(0, \sigma_T^2)$ and $\sqrt{n}(W_n - \tau(\theta)) \xrightarrow{d} N(0, \sigma_W^2)$. The asymptotic relative efficiency (ARE) of T_n with respect to W_n is defined as

$$\text{ARE}(T_n, W_n) = \sigma_W^2 / \sigma_T^2.$$

$$- \text{ARE}(T_n, W_n) > 1 \iff T_n \text{ is asymptotically more efficient than } W_n$$

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 - a. Consider $T_n = n^{-1} \sum_i \mathbf{1}_{\{0\}}(X_i)$ and MLE $W_n = \exp(-\bar{X}_n)$. Compute $\text{ARE}(T_n, W_n)$, the ARE of T_n with respect to W_n .
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