

PH 712 Probability and Statistical Inference

Part IV: Point Estimation I

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Framework of statistical inference/learning

- Goal: infer/learn the distribution of RV X , say f_X , from a random sample X_1, \dots, X_n
- Assumption: $f_X \approx \hat{f}_X$ (statistical model)
 - E.g., $\hat{f}_X = \mathcal{N}(\mu, \sigma^2)$, reducing the task to estimating (μ, σ)
- Point estimation: make the “best” guess about unknown parameter(s)
 - E.g., estimate (μ, σ) by $(\hat{\mu}, \hat{\sigma})$
- Hypothesis testing
 - E.g., confirm whether $\mu = 0$ by testing $H_0 : \mu = 0$ vs. $H_1 : \mu \neq 0$
- Interval estimation: construct an interval likely to cover the unknown parameter
 - E.g., construct an interval, say (c_1, c_2) , such that $c_1 < \mu < c_2$ with a high probability

Point estimation

- θ : the unknown parameter
 - A unknown scalar (i.e., we only consider cases with one unknown parameter)
- The generation of a guess on the value of θ based on the random sample X_1, \dots, X_n
- Estimator: the generated guess, say $\hat{\theta}$
 - A statistic (why?) and hence an RV
 - E.g., sometimes, $\bar{X} = n^{-1} \sum_{i=1}^n X_i$ (sample mean) is an estimator of certain parameter θ
- Estimate: plugging the realization of the random sample, say x_1, \dots, x_n , into the estimator
 - A number (why?) and NOT randomized
 - E.g., $n^{-1} \sum_{i=1}^n x_i$ is an estimate of certain parameter θ

Maximum Likelihood (ML) Estimator (MLE)

- Θ : the set of allowed values of θ
- Likelihood function: an alias of joint pdf/pmf

$$L(\theta) = L(\theta; X_1, \dots, X_n) = f_{X_1, \dots, X_n}(X_1, \dots, X_n | \theta), \quad \theta \in \Theta$$

- f_{X_1, \dots, X_n} : the joint pdf/pmf of X_1, \dots, X_n

- Log-likelihood function: the natural logarithm of likelihood function

$$\ell(\theta) = \ln L(\theta), \quad \theta \in \Theta$$

- $\hat{\theta}_{\text{ML}}$ is the MLE for θ if $\hat{\theta}_{\text{ML}}$ is the maximizer of $L(\theta)$ (equiv. the maximizer of $\ell(\theta)$) with respect to θ constrained in Θ
 - In the math notation,
$$\hat{\theta}_{\text{ML}} = \arg \max_{\theta \in \Theta} L(\theta) = \arg \max_{\theta \in \Theta} \ell(\theta)$$
 - That is to say, $L(\hat{\theta}_{\text{ML}}) \geq L(\theta)$ and $\ell(\hat{\theta}_{\text{ML}}) \geq \ell(\theta)$, for all $\theta \in \Theta$.
- Invariance property of MLE: if $\hat{\theta}_{\text{ML}}$ is the MLE of θ , then $g(\hat{\theta}_{\text{ML}})$ is the MLE of $g(\theta)$ for any given function $g(\cdot)$.

How to locate the ML estimator (MLE) constrained in Θ ?

- If $L(\theta)$ (or equiv. $\ell(\theta)$) is monotonic with respect to $\theta \in \Theta$, then the MLE lies at one boundary point of Θ
- If $\ell(\theta)$ is non-monotonic but differentiable with respect to $\theta \in \Theta$, then
 1. Collect all the candidates including:
 - Stationary points, i.e., solutions to the equation $S(\theta) = 0$ subject to $\theta \in \Theta$
 - * Where $S(\theta) = \ell'(\theta)$ is called the score/gradient
 - Boundary points of Θ
 2. Compare the values of log-likelihood or likelihood evaluated at all the above candidates

How to locate the ML estimate constrained in Θ ?

- Reachable only when the realization of X_1, \dots, X_n are available
- Theoretical way: figuring out the MLE before plugging the realization of X_1, \dots, X_n into the MLE
- Numerical way: R function `optim()`

Example Lec5.1

- Suppose X_1, \dots, X_n is an iid sample following $\mathcal{N}(\mu, \sigma^2)$, i.e., $f_{X_i}(x | \theta) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$, $x \in \mathbb{R}$, with unknown μ and known $\sigma = 1$.
 - Find the MLE of μ .
 - If the realization of the sample is $1, \dots, 10$, find the ML estimate of μ .

Ans: Theoretically, $\mu_{\text{ML}} = n^{-1} \sum_{i=1}^n X_i$. Numerically,

```
sample = 1:10
ell = function(mu){
  n = length(sample)
  sigma = 1 # known
  -n/2*log(2*pi*sigma^2) - sum((sample - mu)^2)/(2*sigma^2)
}
optim(par = 0,
      lower = -Inf, upper = Inf,
      fn=ell, method="L-BFGS-B",
      control=list(fnscale=-1))$par
```

- Suppose X_1, \dots, X_n is an iid sample following $\mathcal{N}(\mu, \sigma^2)$, i.e., $f_{X_i}(x | \theta) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$, $x \in \mathbb{R}$, with known $\mu = 5$ and unknown $\sigma > 0$.
 - Find the MLE of σ .
 - If the realization of the sample is $1, \dots, 10$, find the ML estimate of σ .

Ans: Theoretically, $\sigma_{\text{ML}} = \sqrt{n^{-1} \sum_{i=1}^n (X_i - \mu)^2}$. Numerically,

```
sample = 1:10
ell = function(sigma){
  n = length(sample)
  mu = 5 # known
  -n/2*log(2*pi*sigma^2) - sum((sample - mu)^2)/(2*sigma^2)
}
optim(par = 10,
      lower = 0.00001, upper = Inf,
      fn=ell, method="L-BFGS-B",
      control=list(fnscale=-1))$par
```

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- Suppose X_1, \dots, X_n is an iid sample following $p_{X_i}(x | \theta) = \theta^x (1 - \theta)^{1-x} \mathbf{1}_{\{0,1\}}(x)$, $\theta \in [0, 1/2]$.
 - Find the MLE of θ .
 - If the realization of the sample is $0, 1, 1, 0, 0$, find the ML estimate of θ .

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- Suppose X_1, \dots, X_n is an iid sample following an exponential distribution, i.e., $f_X(x | \beta) = \beta^{-1} \exp(-x/\beta) \mathbf{1}_{(0,\infty)}(x)$, $\beta > 0$.
 - Find the MLE of β .
 - If the realization of the sample is $1, \dots, 10$, find the ML estimate of β .

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- Suppose X_1, \dots, X_n is an iid sample following a beta distribution, i.e., $f_X(x | \theta) = \theta x^{\theta-1} \mathbf{1}_{[0,1]}(x)$, $\theta > 0$.
 - Find the MLE of θ .
 - If the realization of the sample is $0.1, \dots, 0.9$, find the ML estimate of θ .

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- The simplest linear model (or linear regression) is a collection of independent random variables Y_1, \dots, Y_n such that

$$Y_i = \beta x_i + \varepsilon_i, \quad i = 1, \dots, n,$$

where x_1, \dots, x_n are nonrandomized, and $\varepsilon_1, \dots, \varepsilon_n \stackrel{\text{iid}}{\sim} f_\varepsilon(t) = \sqrt{2\pi} \exp(-t^2/2)$ (i.e., $\mathcal{N}(0, 1)$).

- Find the MLE of β , say $\hat{\beta}_{\text{ML}}$. (Hint: create the likelihood by noting that $Y_i \sim \mathcal{N}(\beta x_i, 1)$.)
- Suppose x -values are $1, \dots, 10$. Correspondingly, observed Y -values are $2, \dots, 11$. Find the ML estimate of β .