PH 712 Probability and Statistical Inference

Part IX: Confidence Set/Interval

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Confidence set

- Called a confidence interval (CI) If the set is indeed an interval
- $(1-\alpha)\times 100\%$ confidence set for θ, say $C(X_1,\ldots,X_n)$: $C(X_1,\ldots,X_n)$ covers θ with probability $\geq (1-\alpha)\times 100\%$, i.e., $\Pr\{\theta\in C(X_1,\ldots,X_n)\}\geq(1-\alpha)\times 100\%$
  - $C(X_1,\ldots,X_n)$ is a set defined on sample $X_1,\ldots,X_n$ and hence is randomized, while θ is fixed

Construction of a confidence set by inverting a level α two-sidede test

- Implementation
  1. Construct the rejection region $R(\theta_0)$ for a level α test for hypotheses $H_0:\theta=\theta_0$ vs. $H_1:\theta\neq\theta_0$.
  2. $C(x_1,\ldots,x_n)=\{\theta_0:(x_1,\ldots,x_n)\in\mathrm{supp}(X_1,\ldots,X_n)/R(\theta_0)\}$,
     - $\mathrm{supp}(X_1,\ldots,X_n)/R(\theta_0)$: the complementary set of $R(\theta_0)$.
- If $C(X_1,\ldots,X_n)$ is constructed by inverting a level α a level α test, then
  - $C(X_1,\ldots,X_n)$ does NOT cover θ_0 $\Leftrightarrow$ reject $H_0:\theta=\theta_0$ (vs. $H_1:\theta\neq\theta_0$) at level α
- Special cases:
  - $(1-\alpha)\times 100\%$ asymptotic LRT confidence set for θ: $\{\theta_0:-2(\ell(\theta_0)-\ell(\hat\theta_{\mathrm{ML}}))<\chi^2_{1,1-\alpha}\}$
  - $(1-\alpha)\times 100\%$ Wald confidence set for θ: $\{\theta_0:|\hat\theta_{\mathrm{ML}}-\theta_0|/\sqrt{\widehat{\mathrm{var}}(\hat\theta_{\mathrm{ML}})}<\Phi^{-1}_{1-\alpha/2}\}$, i.e.,

$$\left(\hat\theta_{\mathrm{ML}}-\Phi^{-1}_{1-\alpha/2}\sqrt{\widehat{\mathrm{var}}(\hat\theta_{\mathrm{ML}})},\quad \hat\theta_{\mathrm{ML}}+\Phi^{-1}_{1-\alpha/2}\sqrt{\widehat{\mathrm{var}}(\hat\theta_{\mathrm{ML}})}\right)$$

Construction of a confidence set via (nonparametric) bootstrap

- Implementation
  1. Suppose you observe $x_1,\ldots,x_n$ for an iid sample of size n.
  2. Set a seed to make your result reproducible.
  3. For b in $1:B$, do steps a–b.
     a. Generate a bootstrap sample $x_1^{(b)},\ldots,x_n^{(b)}$ by drawing a sample of size n with replacement from $\{x_1,\ldots,x_n\}$.
     b. Generate a estimate $\hat\theta^{(b)}$ from $x_1^{(b)},\ldots,x_n^{(b)}$.
  4. $(q_{\alpha/2},q_{1-\alpha/2})$ is the $(1-\alpha)$ bootstrap confidence interval for θ, where $q_{\alpha/2}$ and $q_{1-\alpha/2}$ are $\alpha/2$ and $(1-\alpha/2)$ sample quantiles of $\{\hat\theta^{(1)},\ldots,\hat\theta^{(B)}\}$, respectively.

Example Lec9.1

- Assume $X_1,\ldots,X_{10}\overset{\text{iid}}{\sim}f(x\mid p)=p^x(1-p)^{1-x}$, $x=0,1$, $0<p<1$, and observe $0,0,0,1,1,1,1,1,1$. Construct a 95% confidence set for p.

```r
options(digits = 4)
alpha = .05
xs = c(rep(0,3), rep(1,7))
n = length(xs)
ell = function(p, xs){
  sum(xs*log(p) + (1-xs)*log(1-p))
}
# Asymptotic LRT CI
ml = optim(
  par = 0, lower = 0.000001, upper = .999999,
  fn = ell, xs = xs,
  method="L-BFGS-B",
  control=list(fnscale=-1), hessian=TRUE
)
ml$value # log-likelihood at MLE
qchisq(1-alpha, df=1)
# Wald CI
fish_info = -ml$hessian
var_p_hat = 1/fish_info
ci_wald = c(
  p_hat - qnorm(1-alpha/2)*sqrt(var_p_hat),
  p_hat + qnorm(1-alpha/2)*sqrt(var_p_hat)
); ci_wald
# Bootstrap CI
set.seed(712)
B = 1e4L
p_hat_bs = numeric(B)
for (b in 1:B) {
  xbs = sample(xs, size=n, replace=TRUE)
  p_hat_bs[b] = optim(
    par = 0, lower = 0.000001, upper = .999999,
    fn = ell, xs = xbs,
    method="L-BFGS-B",
    control=list(fnscale=-1))$par
}
ci_boot = quantile(p_hat_bs, probs = c(alpha/2, 1-alpha/2)); ci_boot
```