STAT 4100 Lecture Note

Week Nine (Oct 31 & Nov 2 & 4, 2022)

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca, zhiyanggeezhou.github.io)

2022/Nov/02 15:56:44

Hypothesis Testing (con'd)

Example Lec14.3

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Test $H_0 : \mu \le \mu_0$ vs. $H_1 : \mu > \mu_0$.
  a. σ^2 is known. Suppose test ϕ has rejection region $\{\boldsymbol{x} : \bar{x} > \mu_0 + z_{1-\alpha}\sqrt{\sigma^2/n}\}$, where $z_{1-\alpha}$ is the $(1-\alpha)$ quantile of standard normal. Show that ϕ is a UMP level α test and is equivalent to the LRT.
  b. σ^2 is unknown. Suppose test ϕ has rejection region $\{\boldsymbol{x} : \bar{x} > \mu_0 + t_{n-1,1-\alpha}\sqrt{s^2/n}\}$, where $t_{n-1,1-\alpha}$ is the $(1-\alpha)$ quantile of $t(n-1)$. Show that ϕ is of size α and is equivalent to the LRT.

CB Ex 8.2

- For a given city in a given year, assume that the number of automobile accidents follows a Poisson distribution. In past years the average number of accidents per year was 15, and this year it was 10. Is it justified to claim that the accident rate has dropped?
- Demo report: Testing hypotheses H_0 : ___ vs. H_1 : ___ , we carried on the ___ test and obtained ___ as the value of test statistic. The corresponding rejection region is ___ . So, at the ___ level, there was/wasn't a strong statistical evidence against H_0, i.e., we believed that ___ .

p-value (CB Sec 8.3.4)

- The p-value $p(\mathbf{X})$ is valid (to be taken as a test statistic) iff $\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \Pr\{p(\mathbf{X}) \le \alpha \mid \boldsymbol{\theta}\} \le \alpha$ for each $\alpha \in [0, 1]$.
  - i.e., it is possible to define "level" and "size" if we take $\{\boldsymbol{x} : p(\boldsymbol{x}) \le \alpha\}$ as the rejection region
  - $p(\mathbf{X})$ is valid $\Rightarrow$ $p(\mathbf{X})$ is a test statistic with rejection region $\{\boldsymbol{x} : p(\boldsymbol{x}) \le \alpha\}$.
- A special case of valid $p(\mathbf{X})$
  - (CB Thm 8.3.27) if H_0 is rejected when test statistic $T(\boldsymbol{x})$ is larger than a constant, then $p(\boldsymbol{x}) = \sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0} \Pr\{T(\mathbf{X}) \ge T(\boldsymbol{x}) \mid \boldsymbol{\theta}\}$.

Example Lec16.1

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Consider $H_0 : \mu = \mu_0$ vs $H_1 : \mu \ne \mu_0$.
  a. Verify that the size α LRT rejects H_0 when $|\bar{x} - \mu_0| > t_{n-1,1-\alpha/2}(s/\sqrt{n})$.
  b. Find the the expression of p-value for LRT.

Confidence set (CB Sec 9.2.1 & 9.3.1)

- Confidence set of $\boldsymbol{\theta}^*$ ($\in \boldsymbol{\Theta}$), say $C(\mathbf{X})$

- $C(\mathbf{X})$ is randomized, while $\boldsymbol{\theta}^*$ is fixed

- Coverage probability of $C(\mathbf{X})$: the probability for $C(\mathbf{X})$ to cover the true value $\boldsymbol{\theta}^*$, i.e., $\Pr\{\boldsymbol{\theta}^* \in C(\mathbf{X})\}$
- $1-\alpha$ confidence set: $C(\mathbf{X})$ with confidence coefficient $1-\alpha$
  - Confidence coefficient: $\inf_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \Pr\{\boldsymbol{\theta} \in C(\mathbf{X}) \mid \boldsymbol{\theta}\}$
- (CB Thm 9.2.2) construct the confidence set by inverting the acceptance region
  1. For each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, find the rejection region, say $R(\boldsymbol{\theta})$, of a level α test of $H_0 : \boldsymbol{\theta}^* = \boldsymbol{\theta}$ vs. $H_1 : \boldsymbol{\theta}^* \neq \boldsymbol{\theta}$
  2. $C(\boldsymbol{x}) = \{\boldsymbol{\theta} : \boldsymbol{x} \in \mathrm{supp}(\mathbf{X})/R(\boldsymbol{\theta})\}$
- $1-\alpha$ confidence set $C(\mathbf{X})$ does not cover $\boldsymbol{\theta}_0 \iff$ reject $H_0 : \boldsymbol{\theta}^* = \boldsymbol{\theta}_0$ (vs. $H_1 : \boldsymbol{\theta}^* \neq \boldsymbol{\theta}_0$) at level α

Example Lec16.2

- iid $X_1, \ldots, X_n \sim \mathcal{N}(\mu, 1)$. For each of the following cases, write down the rejection region of the level α LRT and then invert it to obtain the $1-\alpha$ confidence interval.
  a. $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ with $\mu_0 < \mu_1$;
  b. $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$;
  c. $H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$;
  d. $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.

Important inequalities

Markov's inequality (CB Lemma 3.8.3 & HMC Thm 1.10.2)

- If $\Pr(X \geq 0) = 1$ and $\mathrm{E}X^k$ exists, then, for all $r, k > 0$,

$$\Pr(X \geq r) \leq \mathrm{E}X^k / r^k.$$

Chebychev's inequality (CB Thm 3.6.1 & Example 3.6.2 & HMC Thm 1.10.3)

- A corollary of Markov's inequality
- Let $X \sim (\mu_X, \sigma_X^2)$. Then, for each $r > 0$,

$$\Pr\{|X - \mu_X| \geq r\sigma_X\} = \Pr\{(X-\mu_X)^2/\sigma_X^2 \geq r^2\} \leq r^{-2}.$$

Cauchy-Schwarz inequality (CB Thm 4.7.3)

- X and Y are both r.v.s. Then $|\mathrm{E}(XY)| \leq \mathrm{E}|XY| \leq \sqrt{\mathrm{E}X^2}\sqrt{\mathrm{E}Y^2}$.
  - Because

$$\frac{X^2}{\mathrm{E}X^2} + \frac{Y^2}{\mathrm{E}Y^2} \geq \frac{2|XY|}{\sqrt{\mathrm{E}X^2}\sqrt{\mathrm{E}Y^2}}$$

Jensen's inequality (CB Thm 4.7.7 & HMC Thm 1.10.5)

- $g(x)$ is convex on $\mathcal{X} \iff g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$ for all $x, y \in \mathcal{X}$ and all $\lambda \in [0,1]$.
  - Let univariate $g(x)$ be twice-differentiable. Then $g(x)$ is convex in $\mathcal{X} \iff (\mathrm{d}^2/\mathrm{d}x^2)g(x) \geq 0$ for each $x \in \mathcal{X}$.
  - $g(x)$ is concave $\iff -g(x)$ is convex.
- Jensen's inequality: if $g(x)$ is convex on (a, b) and $\mathrm{E}X \in (a,b)$, then $\mathrm{E}\{g(X)\} \geq g(\mathrm{E}X)$.

Example Lec17.1

- Check the convexity of following functions.
  a. $g(x) = \exp(x)$, $x \in \mathbb{R}$.
  b. $g(x) = \ln x$, $x \in \mathbb{R}^+$.
  c. $g(x) = x^2$, $x \in \mathbb{R}$.
  d. $g(x) = x^{-1}$, $x \in \mathbb{R} \setminus \{0\}$.
  e. $g(x) = x^{-2}$, $x \in \mathbb{R} \setminus \{0\}$.

Example Lec17.2

- Let X be a positive random variable, i.e., $\Pr(X > 0) = 1$. Argue that
  a. $\mathrm{E}(-\ln X) \geq \ln(1/\mathrm{E}X)$;
  b. $\mathrm{E}X^3 \geq (\mathrm{E}X)^3$.

Take-home exercises (NOT to be submitted; to be potentially covered in labs)

CB Ex 9.33(a); HMC Ex 4.2.17, 4.2.21, 4.6.8