STAT 4100 Lecture Note

Week Eight (Oct 24, 26, & 28, 2022)

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2022/Nov/02 18:31:54

Hypothesis Testing (con'd)

UMP level α test for one-sided hypotheses ($H_0 : \theta^* \leq \theta_0$ (or $\theta^* = \theta_0$) vs $H_1 : \theta^* > \theta_0$)

- Consider cases with only one unknown parameter
- Monotone likelihood ratio (MLR, CB Def 8.3.16): for each pair $\theta_2 > \theta_1$, $f(t \mid \theta_2)/f(t \mid \theta_1)$ is nondecreasing with respect to t for univariate pdfs/pmfs $\{f(t \mid \theta) : \theta \in \Theta \subset \mathbb{R}\}$
  - One-parameter exponential family with $w(\theta)$ nondecreasing w.r.t. θ bears MLR (why?)

- Karlin-Rubin (CB Thm 8.3.17): Suppose T is sufficient for θ and T follows $f_T(t \mid \theta)$ bearing MLR. A UMP level α test for $H_0 : \theta^* \leq \theta_0$ (or $\theta^* = \theta_0$) vs. $H_1 : \theta^* > \theta_0$ is

$$\phi_c(\boldsymbol{x}) = \begin{cases} 1, & T(\boldsymbol{x}) > c, \\ 0, & T(\boldsymbol{x}) < c, \end{cases}$$

  where c is a real number such that $\beta_\phi(\theta_0) = \mathrm{E}\{\phi_c(\mathbf{X}) \mid \theta^* = \theta_0\} = \Pr\{T(\mathbf{X}) > c \mid \theta^* = \theta_0\} = \alpha$.
  - (Optional) if $\Pr\{T(\mathbf{X}) = c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} \neq 0$, then c is taken as the largest real number satisfying that

$$\Pr\{T(\mathbf{X}) \geq c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} \geq \alpha \text{ and } \Pr\{T(\mathbf{X}) \leq c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} \geq 1 - \alpha.$$

    Meanwhile, the test function should become $\phi_{c,\gamma}$ instead of ϕ_c, where

$$\phi_{c,\gamma}(\boldsymbol{x}) = \begin{cases} 1, & T(\boldsymbol{x}) > c, \\ \gamma, & T(\boldsymbol{x}) = c, \\ 0, & T(\boldsymbol{x}) < c. \end{cases}$$

    That is, reject H_0 with probability $\gamma \in [0, 1]$ if observing $T(\boldsymbol{x}) = c$. The probability γ is chosen to make sure that the size is α, i.e.,

$$\alpha = \mathrm{E}\{\phi_{c,\gamma}(\mathbf{X}) \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} = \Pr\{T(\mathbf{X}) > c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\} + \gamma \Pr\{T(\mathbf{X}) = c \mid \boldsymbol{\theta}^* = \boldsymbol{\theta}_0\}.$$

- NOTE: in the Karlin-Rubin theorem, if the hypotheses become $H_0 : \theta^* \geq \theta_0$ (or $\theta^* = \theta_0$) vs. $H_1 : \theta^* < \theta_0$, then change the signs in the test function, i.e.,

$$\phi_c(\boldsymbol{x}) = \begin{cases} 1, & T(\boldsymbol{x}) < c, \\ 0, & T(\boldsymbol{x}) > c, \end{cases}$$

  where c is a real number such that $\beta_\phi(\theta_0) = \Pr\{T(\mathbf{X}) < c \mid \theta^* = \theta_0\} = \alpha$.

Example Lec14.1

- iid $X_1, \ldots, X_n \sim \mathcal{N}(\mu, 1)$. Construct UMP level α test for following hypotheses.
  a. $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ with $\mu_0 < \mu_1$;
  b. $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$;
  c. $H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$;
  d. $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.

Nonexistence of UMP test for two-sided hypotheses $H_0 : \theta^* = \theta_0$ vs $H_1 : \theta^* \neq \theta_0$

- (Optional) uniformly most powerful unbiased (UMPU) level α test

Likehood ratio test (LRT, Sec 8.2.1 & 10.3.1)

- $H_0 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_0$ vs. $H_1 : \boldsymbol{\theta}^* \in \boldsymbol{\Theta}_1$
- $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0 \cup \boldsymbol{\Theta}_1$
- Test statistic
$$\lambda(\boldsymbol{x}) = \frac{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} L(\boldsymbol{\theta} \mid \boldsymbol{x})}{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} L(\boldsymbol{\theta} \mid \boldsymbol{x})} = \frac{L(\hat{\boldsymbol{\theta}}_{0,\mathrm{ML}} \mid \boldsymbol{x})}{L(\hat{\boldsymbol{\theta}}_{\mathrm{ML}} \mid \boldsymbol{x})}$$
  - $\hat{\boldsymbol{\theta}}_{0,\mathrm{ML}}$: (constrained) MLE for $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$
  - $\hat{\boldsymbol{\theta}}_{\mathrm{ML}}$: MLE for $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- Rejection region
$$R = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq c\},$$
  where c is chosen to make sure the size is α, i.e.,
$$\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \beta_\phi(\boldsymbol{\theta}) = \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} \Pr\{\lambda(\mathbf{X}) \leq c \mid \boldsymbol{\theta}\} = \alpha.$$
- Asymptotic rejection region (CB Thm 10.3.3)
$$R \approx \{\boldsymbol{x} : -2 \ln \lambda(\boldsymbol{x}) \geq \chi^2_{\nu,1-\alpha}\} = \{\boldsymbol{x} : \lambda(\boldsymbol{x}) \leq \exp(-\chi^2_{\nu,1-\alpha}/2)\},$$
  where $\chi^2_{\nu,1-\alpha}$ is the $1-\alpha$ quantile of $\chi^2(\nu)$.
  - (CB Thm 10.3.1) Because, asymptotically (i.e., as $n \to \infty$), under H_0,
$$-2 \ln \lambda(\mathbf{X}) \xrightarrow{d} \chi^2(\nu),$$
    where $\nu =$ the difference of numbers of free parameters between Θ_0 and Θ.
- (CB Ex. 8.24) For simple hypotheses, is the LRT equivalent to the UMP test?

Example Lec14.3

- iid $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Test $H_0 : \mu \leq \mu_0$ vs. $H_1 : \mu > \mu_0$.
  a. σ^2 is known. Suppose test ϕ has rejection region $\{\boldsymbol{x} : \bar{x} > \mu_0 + z_{1-\alpha}\sqrt{\sigma^2/n}\}$, where $z_{1-\alpha}$ is the $(1-\alpha)$ quantile of standard normal. Show that ϕ is a UMP level α test and is equivalent to the LRT.
  b. σ^2 is unknown. Suppose test ϕ has rejection region $\{\boldsymbol{x} : \bar{x} > \mu_0 + t_{n-1,1-\alpha}\sqrt{s^2/n}\}$, where $t_{n-1,1-\alpha}$ is the $(1-\alpha)$ quantile of $t(n-1)$. Show that ϕ is of size α and is equivalent to the LRT.

Take-home exercises (NOT to be submitted; to be potentially covered in labs)

CB Ex 8.6(a–b), 8.16, 8.28(a–b), 8.33(a), 8.41