

# STAT 3100 Lecture Note

Week One (Sep 10, 2022)

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## IN THE CASE OF A FIRE ALARM:

- **Remain calm**
  - if it is safe, evacuate the classroom or lab
  - go to the closest fire exit
  - do not use the elevators
- **If you need assistance to evacuate the building, inform your professor or instructor immediately.**
- **If you need to report an incident or a person left behind during a building evacuation, report it to a fire warden or call security services 204-474-9341.**
  - **Do not** reenter the building until the “all clear” is declared by a fire warden, security services or the fire department.
- **Important: only those trained in the use of a fire extinguisher should attempt to operate one!**



## Syllabus

### Contact

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### Timeline

- Lectures
  - Tue/Thr 8:30–9:45 at EITC E2 330
- Labs
  - Wed 14:30–15:45 at EITC E2 330
- Office Hours
  - (instructor) Mon 14:30–15:30 (or by appointment) at 330 Macray Hall
  - (TA) by appointment

## Grading

- Assignment 20%
  - Submitted via Crowdmark
  - Attaching source codes if R is used in computation
  - Always including necessary interpretation
- Midterm 30%
  - In the week of Oct 17
  - Open-book, 3-hour, and online
- Final 50%
  - Open-book and in-person (?)

## Materials

- References
  - (CB) Casella & Berger. 2002. *Statistical Inference*, 2nd Ed.
    - \* 2 hardcopies reserved at Jim Peebles Science and Technology Library
  - (HMC) Hogg, Mckean & Craig. 2018. *Introduction to Mathematical Statistics*, 8th Ed.
    - \* Hardcopy of 6th Ed. available
  - Salsburg. 2001. *The lady tasting tea: how statistics revolutionized science in the twentieth century*. New York: WH Freeman.
- Lecture notes
  - zhiyanggeezhou.github.io
  - UM Learn
  - **Subject to change** without prior notice
- Fall 2022 Syllabi Appendix

## Outline

“All models are wrong, but some are useful.”

— George Box, *Journal of American Statistical Association* 1976

- What are statistical models?
  - Distributions of random variables (r.v.s) of interest
- Statistical inference
  - To answer questions on the underlying statistical models, e.g.,
    - \* What is the model?
    - \* Is the r.v. distributed as  $\mathcal{N}(0,1)$ ?
- Topics to be covered
  - Prerequisite
  - Estimation (finite/large sample, optimality)
  - Confidence interval (finite/large sample, interpretation)
  - Hypothesis testing (finite/large sample, ~~optimality~~, interpretation)

## Basics on random variables (CB/HMC Chp. 1)

### Definitions

- Definition of r.v.: a real-valued function defined on a sample space  $\Omega$ , i.e.,

$$X = X(\omega), \quad \omega \in \Omega$$

- Cumulative distribution function (cdf) of r.v.  $X$

$$F_X(x) = \Pr(X \leq x)$$

- Right continuous
  - \* Roughly speaking, a function is right-continuous if no jump occurs when the limit point is approached from the right
- Non-decreasing
- Ranging from 0 to 1, i.e.,  $F_X(-\infty) = 0$  and  $F_X(\infty) = 1$

### Example Lec1.1

- Given  $p \in (0, 1)$ , suppose

$$F(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\lfloor x \rfloor$  represents the integer part of  $x$ . Show that  $F$  is a cdf.

- Hint: Check the right-continuity of  $F$  at positive integers.

### Types of random variables

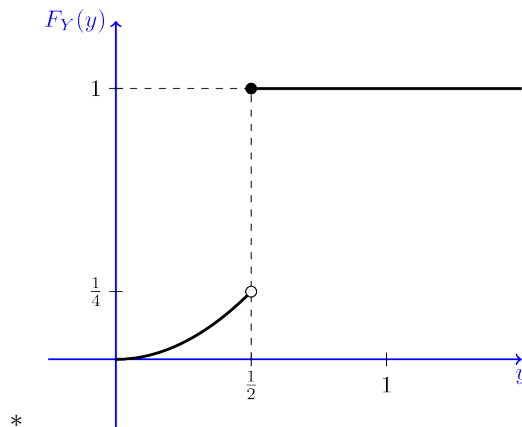
- $X$  is a discrete r.v.
  - $X$  takes countably many values
  - probability mass function (pmf):  $p_X(x) = \Pr(X = x)$
- $X$  is a continuous r.v.
  - cdf  $F_X$  is absolutely continuous, i.e.,  $\exists f_X$ , s.t.

$$F_X(x) = \int_{-\infty}^x f_X(z) dz, \quad \forall x \in \mathbb{R}.$$

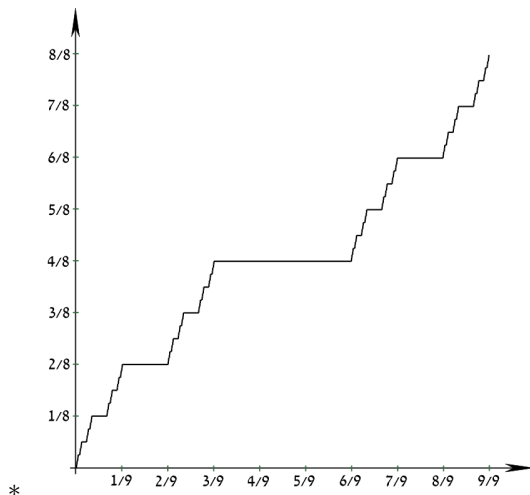
- $f_X$  is the probability density function (pdf) of  $X$ 
  - \*  $f_X(x) = (d/dx)F_X(x)$  if  $f_X$  is continuous at  $x \in \mathbb{R}$

- Neither discrete nor continuous
  - $X$  is a mixed r.v., e.g.,

$$F_X(x) = \begin{cases} 1, & x \geq 1/2; \\ x^2, & 0 \leq x < 1/2; \\ 0, & \text{otherwise.} \end{cases}$$



- Neither discrete nor continuous (con'd)
  - $X$  is following the Cantor distribution



## Univariate transformation (CB Sec. 2.1 & 2.4)

### Support (CB pp. 50)

- In general, for real-valued function  $g$ 
  - $\text{supp}(g) = \{x \in \text{domain}(g) : g(x) \neq 0\} \subset \text{domain}(g)$
- For discrete r.v.  $X$ 
  - pmf  $p_X(\cdot)$
  - $\text{supp}(X) = \text{supp}(p_X) = \{x \in \mathbb{R} : p_X(x) > 0\}$
  - e.g., Support of  $\text{Binom}(n, p)$  is  $\{0, \dots, n\}$
- For continuous r.v.  $X$ 
  - pdf  $f_X(\cdot)$
  - $\text{supp}(X) = \text{supp}(f_X) = \{x \in \mathbb{R} : f_X(x) > 0\}$
  - e.g., Support of  $N(0, 1)$  is  $\mathbb{R}$

### Indicator function

- $\mathbf{1}_A(x) = 1$  if  $x \in A$  and zero otherwise

### Find pmf of $Y = g(X)$ given the pmf of $X$

1. Figure out  $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. Calculate  $p_Y(y) = \Pr(Y = y) = \Pr(X \in \{x \in \text{supp}(X) : g(x) = y\})$

### Example Lec2.1

Let  $X$  have the pmf  $p_X(x) = 2^x \mathbf{1}_{\{-1, -2, \dots\}}(x)$ . Find the pmf of  $Y = X^4$ .

### Find cdf of $Y = g(X)$ given the distribution of $X$

- Calculate  $F_Y(y) = \Pr\{g(X) \leq y\} = \Pr[X \in \{x : g(x) \leq y\}]$

### Example Lec2.2

Let  $X$  have the uniform pdf  $f_X(x) = \pi^{-1} \mathbf{1}_{(-\pi/2, \pi/2)}(x)$ . Find the cdf of  $Y = \tan X$ .

### Find pdf of $Y = g(X)$ given the distribution of $X$

1. Figure out  $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. (Generically) If the cdf  $F_Y$  is known OR pdf  $f_X$  is easy to be integrated, then

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{\{x: g(x) \leq y\}} f_X(x) dx$$

- The integration of  $f_X$  is often avoidable by employing the Leibniz Rule (CB Thm. 2.4.1):

$$\frac{d}{dy} \int_{a(y)}^{b(y)} f(x) dx = f\{b(y)\} \frac{d}{dy} b(y) - f\{a(y)\} \frac{d}{dy} a(y)$$

with  $a(y)$  and  $b(y)$  both differentiable with respect to  $y$ .

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2. (Alternatively) According to CB Ex. 2.7(b), i.e., an extension of CB Thm. 2.1.5 & 2.1.8 and HMC Thm 1.7.1.

$$f_Y(y) = \sum_{k=1}^k f_X\{g_k^{-1}(y)\} \left| J_{g_k^{-1}} \right| \mathbf{1}_{B_k}(y)$$

- $g_k$  is strictly monotonic on  $A_k$  and  $g(x) = g_k(x)$  for all  $x \in A_k$ 
  - $\{A_1, \dots, A_K\}$  is a partition of  $\text{supp}(X)$ , i.e.,  $A_k \cap A_{k'} = \emptyset$  if  $k \neq k'$  and  $\bigcap_{k=1}^K A_k = \text{supp}(X)$
- $g_k^{-1}$  is continuously differentiable on  $B_k = \{g_k(x) : x \in A_k\}$
- Jacobian of transformation  $g_k^{-1}$

$$J_{g_k^{-1}} = \frac{d}{dy} g_k^{-1}(y)$$

### Example Lec2.2'

Let  $X$  have the uniform pdf  $f_X(x) = \pi^{-1} \mathbf{1}_{(-\pi/2, \pi/2)}(x)$ . Find the pdf of  $Y = \tan X$ .

### Example Lec2.3

$X \sim \text{Weibull}(\text{shape} = \alpha, \text{scale} = \beta)$ , viz.  $f_X(x) = (\alpha/\beta)(x/\beta)^{\alpha-1} \exp\{-(x/\beta)^\alpha\} \mathbf{1}_{(0, \infty)}(x)$ . Find the pdf of  $Y = \ln(X)$ .

### Example Lec2.4

Let  $X$  have the pdf  $f_X(x) = 2^{-1} \mathbf{1}_{(0, 2)}(x)$ . Find the pdf of  $Y = X^2$ .

### Example Lec2.5

Let  $f_X(x) = 3^{-1} \mathbf{1}_{(-1, 2)}(x)$ . Find the pdf of  $Y = X^2$ .