STAT 3100 Lecture Note

Week One (Sep 10, 2022)

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IN THE CASE OF A FIRE ALARM:

- **Remain calm**
  - if it is safe, evacuate the classroom or lab
  - go to the closest fire exit
  - do not use the elevators
- **If you need assistance to evacuate the building, inform your professor or instructor immediately.**
- **If you need to report an incident or a person left behind during a building evacuation, report it to a fire warden or call security services 204-474-9341.**
  - **Do not** reenter the building until the **"all clear"** is declared by a fire warden, security services or the fire department.
- **Important: only those trained in the use of a fire extinguisher should attempt to operate one!**

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discover the unknown + invent the future

University of Manitoba

Syllabus

Contact

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Timeline

- Lectures
  - Tue/Thr 8:30–9:45 at EITC E2 330
- Labs
  - Wed 14:30–15:45 at EITC E2 330
- Office Hours
  - (instructor) Mon 14:30–15:30 (or by appointment) at 330 Macray Hall
  - (TA) by appointment

Grading

- Assignment 20%
  - Submitted via Crowdmark
  - Attaching source codes if `R` is used in computation
  - Always including necessary interpretation
- Midterm 30%
  - In the week of Oct 17
  - Open-book, 3-hour, and online
- Final 50%
  - Open-book and in-person (?)

Materials

- References
  - (CB) Casella & Berger. 2002. *Statistical Inference*, 2nd Ed.
    * 2 hardcopies reserved at Jim Peebles Science and Technology Library
  - (HMC) Hogg, Mckean & Craig. 2018. *Introduction to Mathematical Statistics*, 8th Ed.
    * Hardcopy of 6th Ed. available
  - Salsburg. 2001. *The lady tasting tea: how statistics revolutionized science in the twentieth century.* New York: WH Freeman.
- Lecture notes
  - zhiyanggeezhou.github.io
  - UM Learn
  - **Subject to change** without prior notice
- Fall 2022 Syllabi Appendix

Outline

> "All models are wrong, but some are useful."
>
> — George Box, *Journal of American Statistical Association* 1976

- What are statistical models?
  - Distributions of random variables (r.v.s) of interest
- Statistical inference
  - To answer questions on the underlying statistical models, e.g.,
    * What is the model?
    * Is the r.v. distributed as $\mathcal{N}(0,1)$?
- Topics to be covered
  - Prerequisite
  - Estimation (finite/large sample, optimality)
  - Confidence interval (finite/large sample, interpretation)
  - Hypothesis testing (finite/large sample, ~~optimality~~, interpretation)

Basics on random variables (CB/HMC Chp. 1)

Definitions

- Definition of r.v.: a real-valued function defined on a sample space Ω, i.e.,

$$X = X(\omega), \quad \omega \in \Omega$$

- Cumulative distribution function (cdf) of r.v. X

$$F_X(x) = \Pr(X \le x)$$

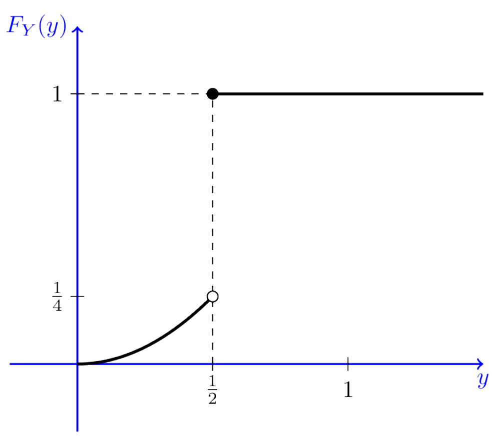
- Right continuous
    * Roughly speaking, a function is right-continuous if no jump occurs when the limit point is approached from the right
- Non-decreasing
- $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

Example Lec1.1

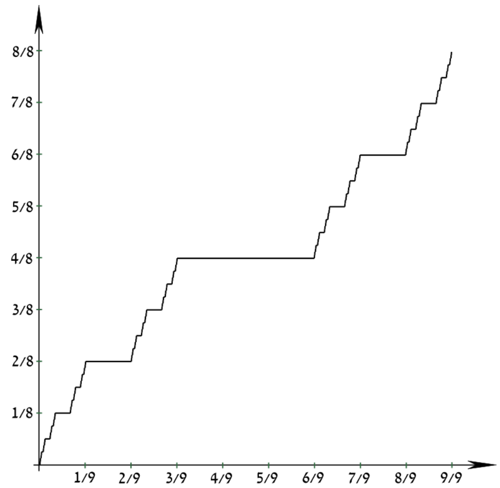
- Given $p \in (0, 1)$, suppose
$$F(x) = \begin{cases} 1 - (1-p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$
where $\lfloor x \rfloor$ represents the integer part of x. Show that F is a cdf.
- Hint: Check the right-continuity of F at positive integers.

Types of random variables

- X is a discrete r.v.
    - X takes countably many values
    - probability mass function (pmf): $p_X(x) = \Pr(X = x)$
- X is a continuous r.v.
    - cdf F_X is absolutely continuous, i.e., $\exists f_X$, s.t.
$$F_X(x) = \int_{-\infty}^{x} f_X(z)\mathrm{d}z, \quad \forall x \in \mathbb{R}.$$
    - f_X is the probability density function (pdf) of X
        * $f_X(x) = (\mathrm{d}/\mathrm{d}x)F_X(x)$ if f_X is continuous at $x \in \mathbb{R}$

- Neither discrete nor continuous
    - X is a mixed r.v., e.g.,
$$F_X(x) = \begin{cases} 1, & x \geq 1/2; \\ x^2, & 0 \leq x < 1/2; \\ 0, & \text{otherwise.} \end{cases}$$
        * 

- Neither discrete nor continuous (con'd)
    - X is following the Cantor distribution



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Univariate transformation (CB Sec. 2.1 & 2.4)

Support (CB pp. 50 & HMC pp. 46)

- In general, for real-valued function g
  - $\text{supp}(g) = \{x \in \text{domain}(g) : g(x) \neq 0\} \subset \text{domain}(g)$
- For discrete r.v. X
  - pmf $p_X(\cdot)$
  - $\text{supp}(X) = \text{supp}(p_X) = \{x \in \mathbb{R} : p_X(x) > 0\}$
  - e.g., Support of $Binom(n, p)$ is $\{0, \ldots, n\}$
- For continuous r.v. X
  - pdf $f_X(\cdot)$
  - $\text{supp}(X) = \text{supp}(f_X) = \{x \in \mathbb{R} : f_X(x) > 0\}$
  - e.g., Support of $N(0, 1)$ is $\mathbb{R}$

Indicator function

Given a set A,

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Find pmf of $Y = g(X)$ given the pmf of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. Calculate $p_Y(y) = \Pr(Y = y) = \Pr(X \in \{x \in \text{supp}(X) : g(x) = y\})$

Example Lec2.1

Let X have the pmf $p_X(x) = 2^x \mathbf{1}_{\{-1,-2,\ldots\}}(x)$. Find the pmf of $Y = X^4$.

Find cdf of $Y = g(X)$ given the distribution of X

- Calculate $F_Y(y) = \Pr\{g(X) \leq y\} = \Pr[X \in \{x : g(x) \leq y\}]$

Example Lec2.2

Let X have the uniform pdf $f_X(x) = \pi^{-1} \mathbf{1}_{(-\pi/2, \pi/2)}(x)$. Find the cdf of $Y = \tan X$.

Find pdf of $Y = g(X)$ given the distribution of X

1. Figure out $\mathrm{supp}(Y) = \{y : y = g(x), x \in \mathrm{supp}(X)\}$
2. (Generically) If the cdf F_Y is known OR pdf f_X is easy to be integrated, then

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} \int_{\{x : g(x) \le y\}} f_X(x) \mathrm{d}x$$

- The integration of f_X is often avoidable by employing the Leibniz Rule (CB Thm. 2.4.1):

$$\frac{\mathrm{d}}{\mathrm{d}y} \int_{a(y)}^{b(y)} f(x) \mathrm{d}x = f\{b(y)\} \frac{\mathrm{d}}{\mathrm{d}y} b(y) - f\{a(y)\} \frac{\mathrm{d}}{\mathrm{d}y} a(y)$$

with $a(y)$ and $b(y)$ both differentiable with respect to y.

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2. (Alternatively) According to CB Ex. 2.7(b), i.e., an extension of CB Thm. 2.1.5 & 2.1.8 and HMC Thm 1.7.1.

$$f_Y(y) = \sum_{k=1}^{k} f_X\{g_k^{-1}(y)\} \left| J_{g_k^{-1}} \right| \mathbf{1}_{B_k}(y)$$

- Partition $\mathrm{supp}(X)$ into K intervals $A_1, \ldots, A_K$ such that $\bigcup_{k=1}^{K} A_k = \mathrm{supp}(X)$ and $A_k \cap A_{k'} = \emptyset$ if $k \ne k'$
- g_k is strictly monotonic on A_k and $g(x) = g_k(x)$ for all $x \in A_k$
- g_k^{-1} is continuously differentiable on $B_k = \{g_k(x) : x \in A_k\}$
- Jacobian of transformation g_k^{-1}

$$J_{g_k^{-1}} = \frac{\mathrm{d}}{\mathrm{d}y} g_k^{-1}(y)$$

Example Lec2.2'

Let X have the uniform pdf $f_X(x) = \pi^{-1} \mathbf{1}_{(-\pi/2, \pi/2)}(x)$. Find the pdf of $Y = \tan X$.

Example Lec2.3

$X \sim \mathrm{Weibull}(\mathrm{shape} = \alpha, \mathrm{scale} = \beta)$, viz. $f_X(x) = (\alpha/\beta)(x/\beta)^{\alpha-1} \exp\{-(x/\beta)^\alpha\} \mathbf{1}_{(0,\infty)}(x)$. Find the pdf of $Y = \ln(X)$.

Example Lec2.4

Let X have the pdf $f_X(x) = 2^{-1} \mathbf{1}_{(0,2)}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.5

Let $f_X(x) = 3^{-1} \mathbf{1}_{(-1,2)}(x)$. Find the pdf of $Y = X^2$.

cdf of $Y = F_X(X)$ (probability integral transformation, CB Thm. 2.1.10)

- If
  - $X \sim F_X$ (not necessarily continuous)
  - $Y = F_X(X)$
- Then $Y \sim \mathrm{unif}(\mathrm{image}(F_X))$
  - Specifically $Y \sim \mathrm{unif}([0, 1])$ if X is continuous
- Application: inverse transform sampling
  - Goal: generate independent and identically distributed (iid) random samples following F_X
  - Implementation

1. Sample iid $U_1, \ldots, U_n \sim \text{unif}(\text{image}(F_X))$
2. Then iid $F_X^{-1}(U_1), \ldots, F_X^{-1}(U_n) \sim F_X$
    * $F_X^{-1}(y) = \inf\{x : F_X(x) \geq y\}$

- Pros & cons
    * (Theoretically) applicable to arbitrary F_X
    * The closed form of F_X^{-1} NOT always reachable