

STAT 4100 Lecture Note

Week Eight (Oct 24, 26, & 28, 2022)

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Hypothesis Testing (con'd)

UMP level α test for one-sided hypotheses ($H_0 : \theta^* \leq \theta_0$ (or $\theta^* = \theta_0$) vs $H_1 : \theta^* > \theta_0$)

- Consider cases with only one unknown parameter
 - Monotone likelihood ratio (MLR, CB Def 8.3.16): for each pair $\theta_2 > \theta_1$, $f(t | \theta_2)/f(t | \theta_1)$ is nondecreasing with respect to t for univariate pdfs/pmfs $\{f(t | \theta) : \theta \in \Theta \subset \mathbb{R}\}$
 - One-parameter exponential family with $w(\theta)$ nondecreasing w.r.t. θ bears MLR (why?)
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- Karlin-Rubin (CB Thm 8.3.17): Suppose T is sufficient for θ and T follows $f_T(t | \theta)$ bearing MLR. A UMP level α test for $H_0 : \theta^* \leq \theta_0$ (or $\theta^* = \theta_0$) vs. $H_1 : \theta^* > \theta_0$ is

$$\phi_\lambda(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) > \lambda, \\ 0, & T(\mathbf{x}) < \lambda, \end{cases}$$

where λ is a real number such that $\beta_\phi(\theta_0) = E\{\phi_\lambda(\mathbf{X}) | \theta^* = \theta_0\} = \Pr\{T(\mathbf{X}) > \lambda | \theta^* = \theta_0\} = \alpha$.

- NOTE: in the Karlin-Rubin theorem, if the hypotheses become $H_0 : \theta^* \geq \theta_0$ (or $\theta^* = \theta_0$) vs. $H_1 : \theta^* < \theta_0$, then change the signs in the test function, i.e.,

$$\phi_\lambda(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) < \lambda, \\ 0, & T(\mathbf{x}) > \lambda, \end{cases}$$

where λ is a real number such that $\beta_\phi(\theta_0) = \Pr\{T(\mathbf{X}) < \lambda | \theta^* = \theta_0\} = \alpha$.

Example Lec14.1

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, 1)$. Construct UMP level α test for following hypotheses.
 - a. $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ with $\mu_0 < \mu_1$;
 - b. $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$;
 - c. $H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$;
 - d. $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.

Nonexistence of UMP test for two-sided hypotheses $H_0 : \theta^* = \theta_0$ vs $H_1 : \theta^* \neq \theta_0$

- (Optional) uniformly most powerful unbiased (UMPU) level α test

Likelihood ratio test (LRT, Sec 8.2.1 & 10.3.1)

- $H_0 : \theta^* \in \Theta_0$ vs. $H_1 : \theta^* \in \Theta_1$
- $\Theta = \Theta_0 \cup \Theta_1$
- Test statistic

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta | \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta | \mathbf{x})} = \frac{L(\hat{\theta}_{0,ML} | \mathbf{x})}{L(\hat{\theta}_{ML} | \mathbf{x})}$$

- $\hat{\theta}_{0,ML}$: MLE for $\theta \in \Theta_0$
- $\hat{\theta}_{ML}$: MLE for $\theta \in \Theta$

- Rejection region

$$R = \{\mathbf{x} : \lambda(\mathbf{x}) \leq c\},$$

where c is chosen to make sure the size is α , i.e.,

$$\sup_{\theta \in \Theta_0} \beta_\phi(\theta) = \sup_{\theta \in \Theta_0} \Pr\{\lambda(\mathbf{X}) \leq c | \theta\} = \alpha.$$

- Asymptotic rejection region (CB Thm 10.3.3)

$$R = \{\mathbf{x} : -2 \ln \lambda(\mathbf{x}) \geq \chi_{\nu, 1-\alpha}^2\} = \{\mathbf{x} : \lambda(\mathbf{x}) \leq \exp(-\chi_{\nu, 1-\alpha}^2/2)\},$$

where $\chi_{\nu, 1-\alpha}^2$ is the $1 - \alpha$ quantile of $\chi^2(\nu)$.

- (CB Thm 10.3.1) Because, asymptotically (i.e., as $n \rightarrow \infty$), under H_0 ,

$$-2 \ln \lambda(\mathbf{X}) \xrightarrow{d} \chi^2(\nu),$$

where ν = the difference of numbers of free parameters between Θ_0 and Θ .

- (CB Ex. 8.24) For simple hypotheses, is the LRT equivalent to the UMP test?

Example Lec14.3

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Test $H_0 : \mu \leq \mu_0$ vs. $H_1 : \mu > \mu_0$.
 - σ^2 is known. Suppose test ϕ has rejection region $\{\mathbf{x} : \bar{x} > \mu_0 + z_{1-\alpha} \sqrt{\sigma^2/n}\}$, where $z_{1-\alpha}$ is the $(1 - \alpha)$ quantile of standard normal. Show that ϕ is a UMP level α test and is equivalent to the LRT.
 - σ^2 is unknown. Suppose test ϕ has rejection region $\{\mathbf{x} : \bar{x} > \mu_0 + t_{n-1, 1-\alpha} \sqrt{s^2/n}\}$, where $t_{n-1, 1-\alpha}$ is the $(1 - \alpha)$ quantile of $t(n - 1)$. Show that ϕ is of size α and is equivalent to the LRT.

p -value (CB Sec 8.3.4)

- The p -value $p(\mathbf{X})$ is valid (to be taken as a test statistic) iff $\sup_{\theta \in \Theta_0} \Pr\{p(\mathbf{X}) \leq \alpha | \theta\} \leq \alpha$ for each $\alpha \in [0, 1]$.
 - i.e., it is possible to define “level” and “size” if we take $\{\mathbf{x} : p(\mathbf{x}) \leq \alpha\}$ as the rejection region
 - $p(\mathbf{X})$ is valid $\Rightarrow p(\mathbf{X})$ is a test statistic with rejection region $\{\mathbf{x} : p(\mathbf{x}) \leq \alpha\}$.
- A special case of valid $p(\mathbf{X})$
 - (CB Thm 8.3.27) if H_0 is rejected when $T(\mathbf{x})$ is too large, then $p(\mathbf{x}) = \sup_{\theta \in \Theta_0} \Pr\{T(\mathbf{X}) \geq T(\mathbf{x}) | \theta\}$.

Example Lec16.1

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Consider $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.
 - Verify that the size α LRT rejects H_0 when $|\bar{x} - \mu_0| > t_{n-1, 1-\alpha/2}(s/\sqrt{n})$.
 - Find the expression of p -value for LRT.

Confidence set (CB Sec 9.2.1 & 9.3.1)

- Confidence set of θ^* : $C(\mathbf{X})$
- Coverage probability of confidence set $C(\mathbf{X})$: $\Pr\{\theta^* \in C(\mathbf{X})\}$
- $1 - \alpha$ confidence set: $C(\mathbf{X})$ with $\inf_{\theta \in \Theta} \Pr\{\theta \in C(\mathbf{X})\} = 1 - \alpha$
- (CB Thm 9.2.2) construct the confidence set by inverting the acceptance region
 1. For each $\theta_0 \in \Theta$, find the rejection region, say $R(\theta_0)$, of a level α test of $H_0 : \theta^* = \theta_0$ vs. $H_1 : \theta^* \neq \theta_0$
 2. $C(\mathbf{x}) = \{\theta_0 : \mathbf{x} \in \text{supp}(\mathbf{X})/R(\theta_0)\}$

Example Lec16.2

- iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, 1)$. For each of the following cases, write down the rejection region of the level α LRT and then invert it to obtain the $1 - \alpha$ confidence interval.
 - a. $H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1$ with $\mu_0 < \mu_1$;
 - b. $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$;
 - c. $H_0 : \mu \geq \mu_0$ vs $H_1 : \mu < \mu_0$;
 - d. $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.

Take-home exercises (NOT to be submitted; to be potentially covered in labs)

CB Ex 8.2, 8.6(a–b), 8.16, 8.28, 8.33, 8.41, 9.33(a)