PH 712 Probability and Statistical Inference

Part II: Mutiple Random Variables

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Recalling the cdf of a single RV

- The cdf of a RV X:
$$F_X(t) = \Pr(X \le t), \quad t \in \mathbb{R},$$
is the probability that event $X \le t$ happens.
  - $\mathbb{R}$: the set of all real numbers.
- Knowing $F_X \Leftrightarrow$ knowing the distribution of X.

Joint cdf of multiple random variables

- Extension to multiple RVs $X_1, \ldots, X_n$
- The joint cdf of n RVs $X_1, \ldots, X_n$:
$$F_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = \Pr(X_1 \le x_1, \ldots, X_n \le x_n), \quad (t_1, \ldots, t_n) \in \mathbb{R}^n,$$
is the probability that n events $X_1 \le t_1, \ldots, X_n \le t_n$ occur simultaneously.
  - $\mathbb{R}^n$: the n-dimensional Euclidean space, or roughly, the set of all real vectors of length n.
- Knowing $F_{X_1,\ldots,X_n} \Leftrightarrow$ knowing the joint distribution of $X_1, \ldots, X_n$.
- Connection to F_{X_i}, the cdf of X_i:
$$F_{X_i}(t_i) = F_{X_1,\ldots,X_n}(\infty, \ldots, \infty, t_i, \infty, \ldots, \infty), \quad t_i \in \mathbb{R}$$
  - E.g., for $n = 3$,
$$F_{X_2}(t_2) = F_{X_1,\ldots,X_3}(\infty, t_2, \infty), \quad t_2 \in \mathbb{R}$$

Joint pmf of discrete $X_1, \ldots, X_n$

- Merely existing in the case that ALL $X_1, \ldots, X_n$ are discrete RVs
- The joint pmf of n RVs $X_1, \ldots, X_n$:
$$p_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = \Pr(X_1 = t_1, \ldots, X_n = t_n), \quad (t_1, \ldots, t_n) \in \mathbb{R}^n,$$
is the probability that n events $X_1 = t_1, \ldots, X_n = t_n$ occur simultaneously.
  - $\mathrm{supp}(X_1, \ldots, X_n) = \{(t_1, \ldots, t_p) \in \mathbb{R}^n : p_{X_1,\ldots,X_n}(t_1, \ldots, t_n) > 0\}$
- Knowing $p_{X_1,\ldots,X_n} \Leftrightarrow$ knowing the joint distribution of $X_1, \ldots, X_n$.

- Connection to p_{X_i}, the pmf of X_i:

$$p_{X_i}(t_i) = \sum_{t_1=-\infty}^{\infty} \cdots \sum_{t_{i-1}=-\infty}^{\infty} \sum_{t_{i+1}=-\infty}^{\infty} \cdots \sum_{t_n=-\infty}^{\infty} p_{X_1,\dots,X_n}(t_1,\dots,t_n), \quad t_i \in \mathbb{R}$$

  - E.g., for $n = 3$,

$$p_{X_2}(t_2) = \sum_{t_1=-\infty}^{\infty} \sum_{t_3=-\infty}^{\infty} p_{X_1,X_2,X_3}(t_1,t_2,t_3), \quad t_2 \in \mathbb{R}$$

Joint pdf of continuous $X_1, \dots, X_n$

- Merely existing in the case that ALL $X_1, \dots, X_n$ are continuous RVs
- The joint pdf of n RVs $X_1, \dots, X_n$:

$$f_{X_1,\dots,X_n}(t_1,\dots,t_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} F_{X_1,\dots,X_n}(t_1,\dots,t_n), \quad (t_1,\dots,t_n) \in \mathbb{R}^n$$

  - $\text{supp}(X_1,\dots,X_n) = \{(t_1,\dots,t_n) \in \mathbb{R}^n : f_{X_1,\dots,X_n}(t_1,\dots,t_n) > 0\}$
- Knowing $f_{X_1,\dots,X_n}$ $\Leftrightarrow$ knowing the joint distribution of $X_1, \dots, X_n$.
- Connection to f_{X_i}, the pdf of X_i:

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1,\dots,X_n}(t_1,\dots,t_n) \mathrm{d}t_1 \cdots \mathrm{d}t_{i-1} \mathrm{d}t_{i+1} \cdots \mathrm{d}t_n$$

  - E.g., for $n = 3$,

$$p_{X_2}(t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(t_1,t_2,t_3) \mathrm{d}t_1 \mathrm{d}t_3, \quad t_2 \in \mathbb{R}$$

(Mutual) independence

- RVs $X_1, \dots, X_n$ are (mutually) independent $\Leftrightarrow$

$$F_{X_1,\dots,X_n}(x_1,\dots,x_n) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

  - For discrete $X_1, \dots, X_n$, joint pmf $p_{X_1,\dots,X_n}(t_1,\dots,t_n) = \prod_{i=1}^n p_{X_i}(t_i)$
  - For continuous $X_1, \dots, X_n$, joint pdf $f_{X_1,\dots,X_n}(t_1,\dots,t_n) = \prod_{i=1}^n f_{X_i}(t_i)$

Example Lec2.1

- X_1 and X_2 are independent Bernoulli RVs with pmf $p_{X_i}(0) = 1 - p_i$ and $p_{X_i}(1) = p_i$, $i = 1, 2$. Write the joint pmf of (X_1, X_2).

- Let X_1 and X_2 be independent Poisson RVs with pmf $p_{X_i}(k_i) = \frac{e^{-3} \cdot 3^{k_i}}{k_i!}$, $k_i = 0, 1, \dots$, $i = 1, 2$. Write the joint pmf of (X_1, X_2).

- X_1 and X_2 are independent uniform RVs with pdf $f_{X_i}(t_i) = \mathbf{1}_{[0,1]}(t_i)$, $i = 1, 2$. Write the joint pdf of (X_1, X_2).

- Let X_1 and X_2 be independent exponential RVs with pdf $f_{X_i}(t_i) = 2e^{-2t_i} \cdot \mathbf{1}_{(0,\infty)}(t_i)$, $i = 1, 2$. Write the joint pdf of (X_1, X_2).