PH 712 Probability and Statistical Inference

Part VI: Evaluating Estimators I

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Bias

- Bias of $\hat{\theta}$: $\mathrm{Bias}(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta$
- Unbiased: if $\mathrm{E}(\hat{\theta}) = \theta$

Mean squared error (MSE)

- $\mathrm{MSE}(\hat{\theta}) = \mathrm{E}(\hat{\theta} - \theta)^2 = \mathrm{Bias}^2(\hat{\theta}) + \mathrm{var}(\hat{\theta})$
  - The lower the better
- For unbiased estimators, minimizing the MSE $\Leftrightarrow$ minimizing the variance

Cramér-Rao lower bound (CRLB, CB Thm 7.3.9 & Lemma 7.3.11)

- Recall the score $S(\theta) = \ell'(\theta)$
- $\mathrm{CRLB} = I_n^{-1}(\theta)\left\{\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{E}(\hat{\theta})\right\}^2$
  - Reducing to $I_n^{-1}(\theta)$ if $\mathrm{E}(\hat{\theta}) = \theta$ (i.e., unbiased $\hat{\theta}$)
  - Where $I_n(\theta) = \mathrm{var}\{S(\theta)\} = \mathrm{E}[\{S(\theta)\}^2] = -\mathrm{E}\{H(\theta)\}$ is called the Fisher information
    - Where $H(\theta) = S'(\theta) = \ell''(\theta)$ is called the Hessian
    - The most convenient way to calculate $I_n(\theta)$: $I_n(\theta) = -\mathrm{E}\{H(\theta)\}$
- Under regularity conditions, $\mathrm{var}(\hat{\theta}) \geq \mathrm{CRLB}$.

Example Lec6.1

- Find the CRLB for all the UNBIASED estimators in the following cases.
  a. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with UNKNOWN μ and GIVEN σ^2.
  b. $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with GIVEN μ and UNKNOWN σ^2.

Efficiency (HMC Def 6.2.2)

- For an estimator, say T_n, unbiased for $g(\theta)$, i.e., $\mathrm{E}(T_n) = g(\theta)$, the *efficiency* of T_n is the ratio of the CRLB to $\mathrm{var}(T_n)$, i.e., $\mathrm{CRLB}/\mathrm{var}(T_n)$, regularly up to 1.
  - The higher efficiency the better;
  - T_n is an efficient estimator for $g(\theta) \iff \mathrm{E}(T_n) = g(\theta)$ and its efficient = 1.