PH 712 Probability and Statistical Inference

Part VIII: Point Estimation II (Aympototic Properties)

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Consistency of MM and ML estimators

- For an iid sample, under certain conditions:
  - $\hat{\theta}_{\mathrm{MM}} \approx \theta$ as $n \to \infty$
  - $\hat{\theta}_{\mathrm{ML}} \approx \theta$ as $n \to \infty$

Asymptotic efficiency of MLE (CB Thm 10.1.12 & Ex. 10.7)

- For an iid sample, under certain conditions:
  - $\sqrt{n}(\hat{\theta}_{\mathrm{ML}} - \theta) \approx \mathcal{N}(0, I_1^{-1}(\theta))$ as $n \to \infty$
    - For an iid sample, $I_1(\theta) = n^{-1} I_n(\theta)$, no longer a function of n
      - In practice, unknown $\theta \Rightarrow$ unknown $I_n(\theta)$
    - $I_n(\theta) \approx I_n(\hat{\theta}_{\mathrm{ML}}) \approx \hat{I}_n(\hat{\theta}_{\mathrm{ML}})$
      - Fisher information (evaluated at θ) $I_n(\theta) = -\mathrm{E}\{H(\theta)\}$
      - Observed Fisher information $\hat{I}_n(\hat{\theta}_{\mathrm{ML}}) = -H(\hat{\theta}_{\mathrm{ML}})$
  - Application (approximating the distribution of $\hat{\theta}_{\mathrm{ML}}$): $\hat{\theta}_{\mathrm{ML}} \approx \mathcal{N}(\theta, \hat{I}_n^{-1}(\hat{\theta}_{\mathrm{ML}}))$, i.e., approximately, $\hat{\theta}_{\mathrm{ML}}$ is normally distributed with mean θ and variance $\hat{I}_n^{-1}(\hat{\theta}_{\mathrm{ML}})$.

Delta method

- Approximating the distribution of $h(T_n)$ when T_n is normally distributed as $n \to \infty$
- (CB Thm 5.5.24, delta method) Suppose T_n is an estimator of θ. If $\sqrt{n}(T_n - \theta) \approx \mathcal{N}(0, \sigma^2)$, h is NOT a function of n, AND $h'(\theta) \neq 0$, then

$$\sqrt{n}\{h(T_n) - h(\theta)\} \approx \mathcal{N}(0, \{h'(\theta)\}^2 \sigma^2).$$

  - $\Rightarrow \mathrm{E}\{h(T_n)\} \approx h(\theta)$ AND $\mathrm{var}\{h(T_n)\} \approx \{h'(T_n)\}^2 \sigma^2 / n$ if $h'(\theta) \neq 0$.
- (CB Thm 5.5.26, second-order delta method) Suppose T_n is an estimator of θ. If $\sqrt{n}(T_n - \theta) \approx \mathcal{N}(0, \sigma^2)$, h is NOT a function of n, $h'(\theta) = 0$, AND $h''(\theta) \neq 0$, then

$$\frac{2n\{h(T_n) - h(\theta)\}}{h''(\theta)\sigma^2} \approx \chi^2(1).$$

  - $\Rightarrow \mathrm{E}\{h(T_n)\} \approx h(\theta) + h''(\theta)\sigma^2/(2n)$ AND $\mathrm{var}\{h(T_n)\} \approx \{h''(T_n)\}^2 \sigma^4 / (2n^2)$ if $h'(\theta) = 0$ but $h''(\theta) \neq 0$.

CB Example 10.1.17 & Ex. 10.9

- $X_1, \ldots, X_n \overset{\text{iid}}{\sim} p(x \mid \lambda) = \lambda^x \exp(-\lambda)/x!$, $x \in \mathbb{Z}^+ \cup \{0\}$, $\lambda > 0$. To estimate $h(\lambda) = \Pr(X_i = 0)$.

1. What is the MLE for $\Pr(X_i = 0)$, say W_n?
2. Approximate the variance of W_n.
3. Suppose $T_n = n^{-1} \sum_i \mathbf{1}_{\{0\}}(X_i)$. Approximate the variance of T_n.
4. Compute $\mathrm{ARE}(T_n, W_n)$, the ARE of T_n with respect to W_n.