

STAT 4100 Lecture Note

Week One (Sep 7 & 9, 2022)

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Fire Safety Orientation

IN THE CASE OF A FIRE ALARM:

- **Remain calm**
 - if it is safe, evacuate the classroom or lab
 - go to the closest fire exit
 - do not use the elevators
- **If you need assistance to evacuate the building, inform your professor or instructor immediately.**
- **If you need to report an incident or a person left behind during a building evacuation, report it to a fire warden or call security services 204-474-9341.**
 - **Do not** reenter the building until the “all clear” is declared by a fire warden, security services or the fire department.
- **Important: only those trained in the use of a fire extinguisher should attempt to operate one!**



Contact

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Timeline

- Lectures
 - Mon/Wed/Fri 13:30–14:20 at St. Paul’s College 258
- Labs
 - Tue 16:30–18:00 at Buller 315
- Office Hours
 - (instructor) Mon 14:30–15:30 (or by appointment) at 330 Macray Hall
 - (TA) TBD

Grading

- Assignment 20%
 - Submitted via Crowdmark
 - Attaching source codes if R is used in computation
 - Always including necessary interpretation
- Midterm 30%
 - In the week of Oct 17
 - Open-book, 3-hour, and online
- Final 50%
 - Open-book and in-person (?)

Materials

- References
 - (CB) Casella & Berger. 2002. *Statistical Inference*, 2nd Ed.
 - * 2 hardcopies reserved at Jim Peebles Science and Technology Library
 - (HMC) Hogg, McKean & Craig. 2018. *Introduction to Mathematical Statistics*, 8th Ed.
 - * Hardcopy of 6th Ed. available
- Lecture notes
 - zhiyanggeezhou.github.io
 - UM Learn
 - Subject to change from time to time without prior notice

Outline

- Topics to be covered
 - Prerequisite
 - Estimation (finite/large sample, optimality)
 - Confidence interval (finite/large sample, interpretation)
 - Hypothesis testing (finite/large sample, optimality, interpretation)

“All models are wrong, but some are useful.”

— George Box, *Journal of American Statistical Association* 1976

Statistical modelling & inference

- What are statistical models?
 - Distributions of random variables (r.v.s) of interest
- Statistical inference
 - To answer questions on the underlying statistical models, e.g.,
 - * What the model is?
 - * Is the r.v. distributed as $N(0, 1)$?

Definitions on random variables (CB/HMC Chp. 1)

- Definition of r.v.: a real-valued function defined on a sample space Ω , i.e.,

$$X = X(\omega), \quad \omega \in \Omega$$

- Cumulative distribution function (cdf) of r.v. X

$$F_X(x) = \Pr(X \leq x)$$

- Right continuous
- Non-decreasing
- Ranging from 0 to 1, i.e., $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

Exercise Lec1.1

- Given $p \in (0, 1)$, suppose

$$F(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of x . Show that F is a cdf.

- Hint: Check the right-continuity of F at positive integers.

Definitions on random variables (con'd)

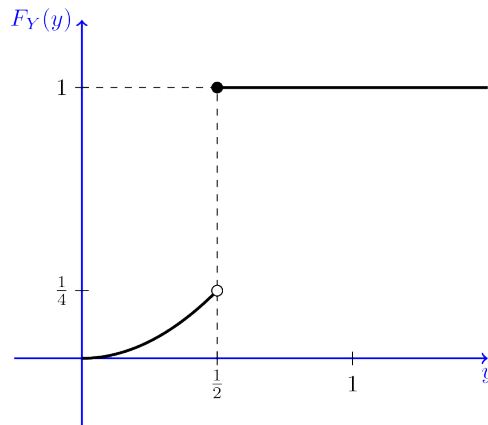
- X is a discrete r.v.
 - X takes countably many values
 - probability mass function (pmf): $p_X(x) = \Pr(X = x)$
- X is a continuous r.v.
 - cdf F_X is absolutely continuous, i.e., $\exists f_X$, s.t.

$$F_X(x) = \int_{-\infty}^x f_X(z) dz, \quad \forall x \in \mathbb{R}.$$

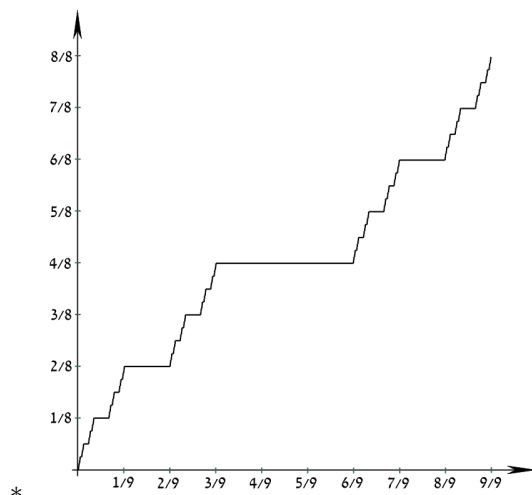
- f_X is the probability density function (pdf) of X
 - * $f_X(x) = (d/dx)F_X(x)$ if f_X is continuous at $x \in \mathbb{R}$

- Neither discrete nor continuous
 - X is a mixed r.v., e.g.,

$$F_X(x) = \begin{cases} 1, & x \geq 1/2; \\ x^2, & 0 \leq x < 1/2; \\ 0, & \text{otherwise.} \end{cases}$$



- *
 - X is following the Cantor distribution



Univariate transformation (CB Sec. 2.1 & 2.4)

Support (CB pp. 50)

- In general, for real-valued function g
 - $\text{supp}(g) = \{x \in \text{domain}(g) : g(x) \neq 0\} \subset \text{domain}(g)$
- For discrete r.v. X
 - pmf $p_X(\cdot)$
 - $\text{supp}(X) = \text{supp}(p_X) = \{x \in \mathbb{R} : p_X(x) > 0\}$
 - e.g., Support of $\text{Binom}(n, p)$ is $\{0, \dots, n\}$
- For continuous r.v. X
 - pdf $f_X(\cdot)$
 - $\text{supp}(X) = \text{supp}(f_X) = \{x \in \mathbb{R} : f_X(x) > 0\}$
 - e.g., Support of $N(0, 1)$ is \mathbb{R}

Indicator function

- $I_A(x) = 1$ if $x \in A$ and zero otherwise

Find pmf of $Y = g(X)$ given the pmf of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. Calculate $p_Y(y) = \Pr(Y = y) = \Pr(X \in \{x \in \text{supp}(X) : g(x) = y\})$

Example Lec2.1

Let X have the pmf $p_X(x) = 2^x I_{\{-1, -2, \dots\}}(x)$. Find the pmf of $Y = X^4$.

Find cdf of $Y = g(X)$ given the distribution of X

- Calculate $F_Y(y) = \Pr\{g(X) \leq y\} = \Pr[X \in g^{-1}\{(-\infty, y]\}]$
 - $g^{-1}\{(-\infty, y]\} = \{x : g(x) \leq y\}$

Example Lec2.2

Let X have the uniform pdf $f_X(x) = \pi^{-1} I_{(-\pi/2, \pi/2)}(x)$. Find the cdf of $Y = \tan X$.

Find pdf of $Y = g(X)$ given the distribution of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. (Generically) If the cdf F_Y is known OR pdf f_X is easy to be integrated, then

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{\{x: g(x) \leq y\}} f_X(x) dx$$

- The integration of f_X is often avoidable by employing the Leibniz Rule (CB Thm. 2.4.1):

$$\frac{d}{dy} \int_{a(y)}^{b(y)} f(x) dx = f\{b(y)\} \frac{d}{dy} b(y) - f\{a(y)\} \frac{d}{dy} a(y)$$

with $a(y)$ and $b(y)$ both differentiable with respect to y .

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2. (Alternatively) According to CB Ex. 2.7(b), i.e., an extension of CB Thm. 2.1.5 & 2.1.8 and HMC Thm 1.7.1.

$$f_Y(y) = \sum_{k=1}^k f_X\{g_k^{-1}(y)\} \left| \frac{d}{dy} g_k^{-1}(y) \right| I_{B_k}(y)$$

- g_k is strictly monotonic on A_k and $g(x) = g_k(x)$ for all $x \in A_k$
 - $\{A_1, \dots, A_K\}$ is a partition of $\text{supp}(X)$, i.e., $A_k \cap A_{k'} = \emptyset$ if $k \neq k'$ and $\bigcap_{k=1}^K A_k = \text{supp}(X)$
- g_k^{-1} is continuously differentiable on $B_k = \{g_k(x) : x \in A_k\}$
- Jacobian (determinant) of g_k^{-1} : $(d/dy)g_k^{-1}(y)$

Example Lec2.2'

Let X have the uniform pdf $f_X(x) = \pi^{-1} I_{(-\pi/2, \pi/2)}(x)$. Find the pdf of $Y = \tan X$.

Example Lec2.3

$X \sim \text{Weibull}(\text{shape} = \alpha, \text{scale} = \beta)$, viz. $f_X(x) = (\alpha/\beta)(x/\beta)^{\alpha-1} \exp\{-(x/\beta)^\alpha\} I_{(0, \infty)}(x)$. Find the pdf of $Y = \ln(X)$.

Example Lec2.4

Let X have the pdf $f_X(x) = 2^{-1} I_{(0, 2)}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.5

Let $f_X(x) = 3^{-1} I_{(-1, 2)}(x)$. Find the pdf of $Y = X^2$.