STAT 4100 Lecture Note

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Multivariate Transformation

Multivariate distribution

- Random vector $\mathbf{X} = (X_1, \ldots, X_n)$ with realization $\boldsymbol{x} = (x_1, \ldots, x_n)$
  - cdf $F_{\mathbf{X}}(\boldsymbol{x}) = \Pr(X_1 \le x_1, \ldots, X_n \le x_n)$
- Discrete
  - Joint pmf

$$p_{\mathbf{X}}(\boldsymbol{x}) = \Pr(X_1 = x_1, \ldots, X_n = x_n)$$

  - $\mathrm{supp}(\mathbf{X}) = \mathrm{supp}(p_{\mathbf{X}}) = \{\boldsymbol{x} \in \mathbb{R}^n : p_{\mathbf{X}}(\boldsymbol{x}) > 0\}$
  - Marginal pmf of $(X_1, \ldots, X_k)$

$$p_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \sum_{(x_{k+1},\ldots,x_n)\in\mathbb{R}^{n-k}} p_{\mathbf{X}}(\boldsymbol{x})$$

- Continuous
  - Joint pdf

$$f_{\mathbf{X}}(\boldsymbol{x}) = (\partial^n / \partial x_1 \cdots \partial x_n) F_{\mathbf{X}}(\boldsymbol{x})$$

    - $\Pr(\mathbf{X} \in B) = \int_B f_{\mathbf{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$ for each Borel set $B \subset \mathbb{R}^n$
  - $\mathrm{supp}(\mathbf{X}) = \mathrm{supp}(f_{\mathbf{X}}) = \{\boldsymbol{x} \in \mathbb{R}^n : f_{\mathbf{X}}(\boldsymbol{x}) > 0\}$
  - Marginal pdf of $(X_1, \ldots, X_k)$
    - $f_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \int_{\mathbb{R}^{n-k}} f_{\mathbf{X}}(\boldsymbol{x}) \mathrm{d}x_{k+1} \cdots \mathrm{d}x_n$

Find the joint pdf of random vector $\mathbf{Y} = \boldsymbol{g}(\mathbf{X})$ by multivariate transformation (CB Sec. 4.3 & 4.6)

- Conditions
  - $\mathbf{X}$ and $\mathbf{Y}$ both of n dimensions
  - $\boldsymbol{g}(\cdot) = (g_1(\cdot), \ldots, g_n(\cdot)) : \mathrm{supp}(\mathbf{X}) \to \mathrm{supp}(\mathbf{Y})$ is one-to-one, i.e.,
    - $\boldsymbol{y} = (y_1, \ldots, y_n) = (g_1(\boldsymbol{x}), \ldots, g_n(\boldsymbol{x})) = \boldsymbol{g}(\boldsymbol{x})$
    - $\boldsymbol{x} = (x_1, \ldots, x_n) = \boldsymbol{g}^{-1}(\boldsymbol{y}) = (g_1^{-1}(\boldsymbol{y}), \ldots, g_n^{-1}(\boldsymbol{y}))$
- Jacobian matrices
  - Jacobian matrix of transformation $\boldsymbol{g}^{-1}$

$$\mathbf{J}_{\boldsymbol{g}^{-1}} = \mathbf{J}_{\boldsymbol{g}^{-1}}(\boldsymbol{y}) = \left[\frac{\partial g_i^{-1}(\boldsymbol{y})}{\partial y_j}\right]_{n\times n} = \begin{bmatrix} \frac{\partial g_1^{-1}(\boldsymbol{y})}{\partial y_1} & \cdots & \frac{\partial g_1^{-1}(\boldsymbol{y})}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n^{-1}(\boldsymbol{y})}{\partial y_1} & \cdots & \frac{\partial g_n^{-1}(\boldsymbol{y})}{\partial y_n} \end{bmatrix}$$

    - Jacobian matrix of transformation $\boldsymbol{g}$

$$\mathbf{J}_{\boldsymbol{g}} = \mathbf{J}_{\boldsymbol{g}}(\boldsymbol{x}) = \left[\frac{\partial g_i(\boldsymbol{x})}{\partial x_j}\right]_{n\times n} = \begin{bmatrix} \frac{\partial g_1(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial g_1(\boldsymbol{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial g_n(\boldsymbol{x})}{\partial x_n} \end{bmatrix}$$

    - $\mathbf{J}_{\boldsymbol{g}^{-1}}(\boldsymbol{y}) = \{\mathbf{J}_{\boldsymbol{g}}(\boldsymbol{g}^{-1}(\boldsymbol{y}))\}^{-1}$
        - Alternative way to reach $\mathbf{J}_{\boldsymbol{g}^{-1}}(\boldsymbol{y})$
- Then

$$f_{\mathbf{Y}}(\boldsymbol{y}) = f_{\mathbf{X}}\{g^{-1}(\boldsymbol{y})\}|\det\{\mathbf{J}_{\boldsymbol{g}^{-1}}(\boldsymbol{y})\}|\mathbf{1}_{\mathrm{supp}(\mathbf{Y})}(\boldsymbol{y}).$$

    - Never miss $\mathbf{1}_{\mathrm{supp}(\mathbf{Y})}(\boldsymbol{y})$
- If $\boldsymbol{g}$ is NOT one-to-one, one may figure out the cdf of $\mathbf{Y}$ and then differentiate it.

Example Lec3.1

X_1 and X_2 are iid from $\mathcal{N}(0,1)$. Find the joint pdf of $Y_1 = (X_1+X_2)/\sqrt{2}$ and $Y_2 = (X_1 - X_2)/\sqrt{2}$ and show their independence.

Note: the sample mean and standard deviation are respectively $\bar{X} = (X_1+X_2)/2 = Y_1/\sqrt{2}$ and $S = \sqrt{(X_1-\bar{X})^2 + (X_2-\bar{X})^2} = |Y_2|$.

Find the marginal pdf

1. Figure out the joint pdf first
2. Taking the Integral

Example Lec3.2

X_1 and X_2 are iid from $\mathcal{N}(0,1)$. Find the pdf of $U = \sqrt{X_1^2 + X_2^2}$.

Basics on square matrices

Eigen-decomposition

- $\mathbf{A}$ is a real $n \times n$ matrix
- Eigenvalues of $\mathbf{A}$, say $\lambda_1 \geq \cdots \geq \lambda_n$: n roots of characteristic equation $\det(\lambda\mathbf{I}_n - \mathbf{A}) = 0$
- The ith (Right) eigenvector $\boldsymbol{v}_i$: $\mathbf{A}\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$
- Eigen-decomposition: $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$
    - $\mathbf{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ both $n \times n$ matrices
    - Specifically $\mathbf{V}^{-1} = \mathbf{V}^\top$ for symmetric $\mathbf{A}$
- Numerical implementation in R: eigen()
- Connection to determinant and trace
    - Determinant
        - $\det\mathbf{A} = \prod_{i=1}^n \lambda_i$
        - $\det(\mathbf{A}^\top) = \det\mathbf{A}$
        - $\det(\mathbf{A}^{-1}) = (\det\mathbf{A})^{-1}$
        - $\det(c\mathbf{A}) = c^n\det\mathbf{A}$ for $n\times n$ matrix $\mathbf{A}$ and scalar c
        - $\det(\mathbf{AB}) = (\det\mathbf{A})(\det\mathbf{B})$ for squared $\mathbf{A}$ and $\mathbf{B}$
    - Trace

    * $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$
    * $\operatorname{tr}(c\mathbf{A}) = c\operatorname{tr}(\mathbf{A})$ for scalar c
    * $\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$ for squared $\mathbf{A}$ and $\mathbf{B}$
    * $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$

Square root of matrices

- $\mathbf{A}^{1/2} = \mathbf{V}\Lambda^{1/2}\mathbf{V}^\top$ if for semi-positive definite $\mathbf{A}$
  - Semi-positive/non-negative definite: symmetric $\mathbf{A}$ with eigenvalues all non-negative
  - $\Lambda^{1/2} = \operatorname{diag}(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2})$
  - $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$
- $\mathbf{A}^{-1/2} = \mathbf{V}\Lambda^{-1/2}\mathbf{V}^\top$ for positive definite $\mathbf{A}$
  - Positive definite: symmetric $\mathbf{A}$ with eigenvalues all positive
  - $\Lambda^{-1/2} = \operatorname{diag}(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2})$
  - $\mathbf{A}^{-1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1}$ and $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{I}$

Multivariate normal (MVN) distribution

$\mathrm{MVN}(\mathbf{0}, \mathbf{I}_p)$

- Random p-vector $\mathbf{Z} = (Z_1, \ldots, Z_p)^\top \sim \mathrm{MVN}(\mathbf{0}, \mathbf{I}_p) \Leftrightarrow$ iid $Z_1 \ldots, Z_p \sim \mathcal{N}(0,1)$.
- pdf of $\mathrm{MVN}(0, \mathbf{I}_p)$:

$$
\begin{aligned}
f_{\mathbf{Z}}(\boldsymbol{z}) &= \prod_{i=1}^p (2\pi)^{-1/2} \exp(-z_i^2/2) \\
&= (2\pi)^{-p/2} \exp(-\boldsymbol{z}^\top \boldsymbol{z}/2), \quad \boldsymbol{z} \in \mathbb{R}^p
\end{aligned}
$$

$\mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

- pdf of $\mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\Sigma} > 0$:

$$
f_{\mathbf{X}}(\boldsymbol{x}) = (2\pi)^{-p/2} (\det \boldsymbol{\Sigma})^{-1/2} \exp\{-(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})/2\}, \quad \boldsymbol{x} \in \mathbb{R}^p
$$

- $\mathbf{X} \sim \mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Rightarrow \mathbf{AX} + \boldsymbol{a} \sim \mathrm{MVN}(\mathbf{A}\boldsymbol{\mu} + \boldsymbol{a}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$ for arbitrary $\mathbf{A} \in \mathbb{R}^{q\times p}$ and $\boldsymbol{a} \in \mathbb{R}^q$
- $\mathbf{X} \sim \mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Leftrightarrow \mathbf{X} = \boldsymbol{\Sigma}^{1/2}\mathbf{Z} + \boldsymbol{\mu}$ with $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim \mathrm{MVN}(0, \mathbf{I}_p)$

Marginals of MVN

- Suppose p-vector $\mathbf{X} = (X_1, \ldots, X_p)^\top$ and q-vector $\mathbf{Y} = (Y_1, \ldots, Y_q)^\top$ are jointly normally distributed. Then, $\mathbf{X}$ and $\mathbf{Y}$ are independent $\Leftrightarrow \operatorname{cov}(\mathbf{X}, \mathbf{Y}) = \mathbf{0}_{p\times q}$.
- If $\mathbf{X}$ is of MVN, then its all margins are still of MVN. The inverse proposition does NOT hold.
  - Cautionary example: Let $Y = XZ$, where $X \sim \mathcal{N}(0,1)$; Z is independent of X with $\Pr(Z=1) = \Pr(Z=-1) = .5$. X and Y both turn out to be of standard normal, but they are not jointly normal.

Normal sampling theory (CB Sec. 5.3)

Stochastic representations for χ^2-, t-, and F-r.v. (HMC Chp. 3)

- If iid $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$,then
  - $\sum_{i=1}^n X_i^2 \sim \chi^2(n)$ if iid $X_1, \ldots, X_n \sim \mathcal{N}(0,1)$;
  - $X/\sqrt{Y/n} \sim t(n)$ if $X \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2(n)$ are independent;

- $(X/m)/(Y/n) \sim F(m,n)$ if $X \sim \chi^2(m)$ and $Y \sim \chi^2(n)$ are independent.

Important identities for normal samples

- $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ and $S^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ are independent
- $n^{1/2}(\bar{X} - \mu)/\sigma \sim \mathcal{N}(0,1)$
- $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$
- $n^{1/2}(\bar{X} - \mu)/S \sim t(n-1)$