STAT 3100 Lecture Note

Week Three (Sep 20 & 22, 2022)

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Multivariate normal (MVN) distribution (con'd)

Marginals of MVN

- Suppose p-vector $\mathbf{X} = [X_1, \ldots, X_p]^\top$ and q-vector $\mathbf{Y} = [Y_1, \ldots, Y_q]^\top$ are jointly normally distributed. Then, $\mathbf{X}$ and $\mathbf{Y}$ are independent $\Leftrightarrow$ $\mathrm{cov}(\mathbf{X}, \mathbf{Y}) = \mathbf{0}_{p\times q}$.
- If $\mathbf{X}$ is of MVN, then its all margins are still of MVN. The inverse proposition does NOT hold.
  - Cautionary example: Let $Y = XZ$, where $X \sim \mathcal{N}(0, 1)$; Z is independent of X with $\Pr(Z = 1) = \Pr(Z = -1) = .5$. X and Y both turn out to be of standard normal, but they are not jointly normal. (Why?)

```
if (!("plot3D" %in% rownames(installed.packages())))
  install.packages("plot3D")
set.seed(1)
xsize = 1e4L
X = rnorm(xsize)
Z = rbinom(n = xsize, 1, .5)
Y = (2 * Z - 1) * X
# 3d histogram of (X, Y)
plot3D::hist3D(z=table(cut(X, 100), cut(Y, 100)), border = "black")
# plot the support of joint pdf of (X, Y)
plot3D::image2D(z=table(cut(X, 100), cut(Y, 100)), border = "black")
```

Normal sampling theory (CB Sec. 5.3)

(Default) stochastic representations for χ^2-, t-, and F-r.v. (HMC Chp. 3)

- $\sum_{i=1}^n X_i^2 \sim \chi^2(n)$ if $[X_1, \ldots, X_n]^\top \sim \mathrm{MVN}(\mathbf{0}, \mathbf{I}_n)$;
- $X/\sqrt{Y/n} \sim t(n)$ if $X \sim \mathcal{N}(0, 1)$ and $Y \sim \chi^2(n)$ are independent;
- $(X/m)/(Y/n) \sim F(m, n)$ if $X \sim \chi^2(m)$ and $Y \sim \chi^2(n)$ are independent.

Important identities for iid normal samples

Let $\mathbf{X} = [X_1, \ldots, X_n]^\top \sim \mathrm{MVN}(\mu\mathbf{1}_n, \sigma^2\mathbf{I}_n)$, $\bar{X} = n^{-1}\sum_{i=1}^n X_i = n^{-1}\mathbf{1}_n^\top\mathbf{X}$, and $S^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X})^2 = (n-1)^{-1}\mathbf{X}^\top(\mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n^\top)\mathbf{X}$, where $\mathbf{1}_n = [1, \ldots, 1]^\top$, i.e., a column n-vector whose entries are all ones.

- $n^{1/2}(\bar{X}-\mu)/\sigma \sim \mathcal{N}(0,1)$
- $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$
- $\bar{X}$ and S^2 are independent of each other
- $n^{1/2}(\bar{X}-\mu)/S \sim t(n-1)$

Taylor series (CB Def 5.5.20 & Thm 5.5.21)

Taylor series about $x_0 \in \mathbb{R}$ for univariate functions

- Suppose f has derivative of order $n+1$ within an open interval of x_0, say $(x_0-\varepsilon, x_0+\varepsilon)$ with $\varepsilon > 0$. Then, for $x \in (x_0-\varepsilon, x_0+\varepsilon)$,

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(n)}(x_0)}{k!}(x-x_0)^k = f(x_0) + \sum_{k=1}^{n} \frac{f^{(n)}(x_0)}{k!}(x-x_0)^k,$$

  where $f^{(n)}(x_0) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)|_{x=x_0}$.
- Called the Maclaurin series if $x_0 = 0$

Taylor series about $\boldsymbol{x}_0 \in \mathbb{R}^p$ for multivariate functions

Under regularity conditions,

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + (\boldsymbol{x}-\boldsymbol{x}_0)^\top \nabla f(\boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^\top \mathbf{H}(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0),$$

where the gradient $\nabla f(\boldsymbol{x}_0) = [\frac{\partial}{\partial x_1} f(\boldsymbol{x}_0), \cdots, \frac{\partial}{\partial x_p} f(\boldsymbol{x}_0)]^\top$ and the Hessian $\mathbf{H}(\boldsymbol{x}_0) = [\frac{\partial^2}{\partial x_i \partial x_j} f(\boldsymbol{x}_0)]_{p\times p}$.

Application

- Approximate unknown or complex f with a polynomial
  - Δ-method
  - Asymptotic theory for maximum likelihood estimators
- Moment generating function (mgf): $M_X(t) = \mathrm{E}\{\exp(tX)\} = \sum_{n=0}^{\infty} t^n \mathrm{E}(X^n)/n!$
  - Maclaurin series of $\exp(tX)$: $\exp(tX) = \sum_{n=0}^{\infty}(tX)^n/n! \Rightarrow \mathrm{E}(X^n) = (\partial^n/\partial t^n)M_X(t)\mid_{t=0}$

Generating functions

Moment generating function (mgf, CB Sec. 2.3)

- Univariate r.v. X
  - mgf $M_X(t) = \mathrm{E}\{\exp(tX)\}$ if $\mathrm{E}\{\exp(tX)\} < \infty$ for t in a neighborhood of 0; otherwise we say that the mgf does not exist or is undefined.
    - Continuous X: $M_X(t) = \int_{-\infty}^{\infty} \exp(tx) f_X(x)\mathrm{d}x$
      - (Two-sided) Laplace transformation of f_X
    - Discrete X: $M_X(t) = \sum_{\{x: x\in \mathrm{supp}(X)\}} \exp(tx) p_X(x)$
  - $M_{aX+b}(t) = \exp(bt) M_X(at)$

- Multivariate r.v. $\mathbf{X} = (X_1, \ldots, X_p)^\top \in \mathbb{R}^p$

- mgf $M_{\mathbf{X}}(\boldsymbol{t})$ is defined as

$$M_{\mathbf{X}}(\boldsymbol{t}) = \mathrm{E}\{\exp(\boldsymbol{t}^\top \mathbf{X})\} = \begin{cases} \int_{\mathbb{R}^p} \exp(\boldsymbol{t}^\top \mathbf{X}) f_{\mathbf{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} & \text{continuous } \mathbf{X} \\ \sum_{\{\boldsymbol{x}:\boldsymbol{x}\in\mathrm{supp}(\mathbf{X})\}} \exp(\boldsymbol{t}^\top \mathbf{X}) p_{\mathbf{X}}(\boldsymbol{x}) & \text{discrete } \mathbf{X} \end{cases}$$

  provided that $\mathrm{E}\{\exp(\boldsymbol{t}^\top \mathbf{X})\} < \infty$ for $\boldsymbol{t} = (t_1, \ldots, t_p)^\top$ in some neighborhood of $\mathbf{0}$; otherwise we say that the mgf does not exist or is undefined.
    * $X_1, \ldots, X_p$ are independent $\Rightarrow M_{\mathbf{X}}(\boldsymbol{t}) = \prod_{i=1}^p M_{X_i}(t_i)$
- $M_{\mathbf{AX}+\boldsymbol{b}}(\boldsymbol{t}) = \exp(\boldsymbol{b}^\top \boldsymbol{t}) M_{\mathbf{X}}(\mathbf{A}^\top \boldsymbol{t}) = \exp(\boldsymbol{b}^\top \boldsymbol{t}) \mathrm{E}\{\exp(\boldsymbol{t}^\top \mathbf{AX})\}$

- Application
  - Characterizing distributions: $M_{\mathbf{X}}(\boldsymbol{t})$ and $M_{\mathbf{Y}}(\boldsymbol{t})$ are both well-defined and equal for all $\boldsymbol{t}$ in a neighborhood of $\mathbf{0} \Leftrightarrow \mathbf{X} \overset{d}{=} \mathbf{Y}$
    * Proofs for laws of large numbers and central limit theorems.
  - Computing moments
    * nth raw moment $\mu'_n = \mathrm{E}X^n = \sum_{k=0}^n \binom{n}{k} \mu_k (\mu'_1)^{n-k}$
    * nth central moment $\mu_n = \mathrm{E}(X - \mathrm{E}X)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \mu'_k (\mu'_1)^{n-k}$

Example Lec6.1

- Find the mgfs of following distributions.
  - $\mathcal{N}(\mu, \sigma^2)$.
  - $\mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
  - Cauchy distribution: $f_X(x) = \{\pi(1 + x^2)\}^{-1}$, $x \in \mathbb{R}$.

Characteristic function

- For univariate X: $\phi_X(t) = \mathrm{E}\exp(itX)$ for all $t \in \mathbb{R}$
  - Fourier transform of f_X
  - Inverse: $f_X(x) = (2\pi)^{-1} \int_{\mathbb{R}} \phi_X(t) \exp(-itx) \mathrm{d}t$
  - $\mu'_n = \mathrm{E}X^n = (-i)^n \phi_X^{(n)}(0)$
- For Multivariate $\mathbf{X} = (X_1, \ldots, X_p)^\top$: $\phi_{\mathbf{X}}(\boldsymbol{t}) = \mathrm{E}\exp(i\boldsymbol{t}^\top \mathbf{X})$ for all $\boldsymbol{t} \in \mathbb{R}^p$
  - Fourier transform of $f_{\mathbf{X}}$
  - Inverse: $f_{\mathbf{X}}(\boldsymbol{x}) = (2\pi)^{-p} \int_{\mathbb{R}^p} \phi_{\mathbf{X}}(\boldsymbol{t}) \exp(-i\boldsymbol{t}^\top \boldsymbol{x}) \mathrm{d}\boldsymbol{t}$
- $\phi_{\mathbf{X}}(\boldsymbol{t}) = \phi_{\mathbf{Y}}(\boldsymbol{t})$ for all $\boldsymbol{t} \in \mathbb{R}^p \Leftrightarrow \mathbf{X} \overset{d}{=} \mathbf{Y}$

Example Lec6.2

- Find the characteristic functions of following distributions.
  - $\mathcal{N}(\mu, \sigma^2)$.
  - $\mathrm{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
  - Cauchy distribution: $f_X(x) = \{\pi(1 + x^2)\}^{-1}$, $x \in \mathbb{R}$.

Other generating functions

- Cumulant generating function
  - $K_X(t) = \ln M_X(t) = \sum_{n=0}^\infty \kappa_n t^n / n!$
  - $\kappa_n = K_X^{(n)}(0)$
- Probability-generating function
  - For discrete r.v. X taking values from $\{0, 1, \ldots\}$, $G(z) = \mathrm{E}t^X = \sum_{x=0}^\infty t^x p_X(x)$.
  - $p_X(n) = \Pr(X = n) = G^{(n)}(1)/n!$