STAT 3690 Lecture 11

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Testing on $\mathbf{A}\boldsymbol{\mu}$ (J&W pp. 279)

- $\mathbf{A}$ is of $q \times p$ and $\mathrm{rk}(\mathbf{A}) = q$, i.e., $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top > 0$
- Model: iid $\mathbf{A}\mathbf{X}_i \sim MVN_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\Sigma\mathbf{A}^\top)$.
- LRT for $H_0 : \mathbf{A}\boldsymbol{\mu} = \boldsymbol{\nu}_0$ v.s. $H_1 : \mathbf{A}\boldsymbol{\mu} \neq \boldsymbol{\nu}_0$
  - Test statistic: $n(\mathbf{A}\bar{\mathbf{X}} - \boldsymbol{\nu}_0)^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\mathbf{X}} - \boldsymbol{\nu}_0) \sim T^2(q, n-1) = \frac{(n-1)q}{n-q}F(q, n-q)$ under H_0
  - Rejction region at level α: $R = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n : \frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0) \geq F_{1-\alpha,q,n-q}\}$
  - p-value: $p(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = 1 - F_{F(q,n-q)}\{\frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}} - \boldsymbol{\nu}_0)\}$
- Multiple comparison
  - Interested in $H_0 : \mu_1 = \cdots = \mu_p$ v.s. H_1 : Not all entries of $\boldsymbol{\mu}$ are equal.
    - μ_k: the kth entry of $\boldsymbol{\mu}$
  - Take

$$\boldsymbol{\nu}_0 = \mathbf{0}_{(p-1)\times 1}, \quad \mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}_{(p-1)\times p}.$$

  - $p = 2$ (i.e., $\mathbf{A} = [1, -1]$): A/B testing

$(1-\alpha) \times 100\%$ confidence region (CR) for $\boldsymbol{\mu}$ (J&W Sec. 5.4)

- $\Pr((1-\alpha) \times 100\%\text{CR covers } \boldsymbol{\mu}) = 1 - \alpha$
  - CR is a set made of observations and is hence random
  - $\boldsymbol{\mu}$ is fixed
  - $(1-\alpha) \times 100\%$ CR covers $\boldsymbol{\mu}$ with probability $(1-\alpha) \times 100\%$
- Dual problem of testing $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ v.s. $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ at the α level
  - Translated from rejection region. Steps:
    1. Take R as a function of $\boldsymbol{\mu}_0$;
    2. Replace $\boldsymbol{\mu}_0$ with $\boldsymbol{\mu}$;
    3. Take the complement.
  - $(1-\alpha) \times 100\%$ CR $= \{\boldsymbol{\mu} : n(\bar{\boldsymbol{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}) < \chi^2_{1-\alpha,p}\}$ if $\boldsymbol{\Sigma}$ is known
  - $(1-\alpha) \times 100\%$ CR $= \{\boldsymbol{\mu} : \frac{n(n-p)}{p(n-1)}(\bar{\boldsymbol{x}} - \boldsymbol{\mu})^\top \mathbf{S}^{-1}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}) < F_{1-\alpha,p,n-p}\}$ if $\boldsymbol{\Sigma}$ is not known

$(1-\alpha)\times 100\%$ CR for $\boldsymbol{\nu}=\mathbf{A}\boldsymbol{\mu}$

- $\mathbf{X}_1,\dots,\mathbf{X}_n \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$
  - Unknown $\boldsymbol{\Sigma}$
  - $n>p$
- $\mathbf{A}$ is of $q\times p$ and $\mathrm{rk}(\mathbf{A})=q$, i.e., $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top>0$
- Then iid $\mathbf{A}\mathbf{X}_i\sim MVN_q(\boldsymbol{\nu},\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$
- $(1-\alpha)\times 100\%$ CR for $\boldsymbol{\nu}$ is $\{\boldsymbol{\nu}:\frac{n(n-q)}{q(n-1)}(\mathbf{A}\bar{\boldsymbol{x}}-\boldsymbol{\nu})^\top(\mathbf{A}\mathbf{S}\mathbf{A}^\top)^{-1}(\mathbf{A}\bar{\boldsymbol{x}}-\boldsymbol{\nu})<F_{1-\alpha,q,n-q}\}$
- Special case: $\mathbf{A}=\boldsymbol{a}\in\mathbb{R}^p$
  - $(1-\alpha)\times 100\%$ confidence interval (CI) for scalar $\nu=\boldsymbol{a}^\top\boldsymbol{\mu}$ is

$$\{\nu:n(\boldsymbol{a}^\top\bar{\boldsymbol{x}}-\nu)^2(\boldsymbol{a}^\top\mathbf{S}\boldsymbol{a})^{-1}<F_{1-\alpha,1,n-1}\}=\left(\boldsymbol{a}^\top\bar{\boldsymbol{x}}-t_{1-\alpha/2,n-1}\sqrt{\boldsymbol{a}^\top\mathbf{S}\boldsymbol{a}/n},\boldsymbol{a}^\top\bar{\boldsymbol{x}}+t_{1-\alpha/2,n-1}\sqrt{\boldsymbol{a}^\top\mathbf{S}\boldsymbol{a}/n}\right)$$

Simultaneous confidence intervals

- Construct $(1-\alpha_k)$ CI for scalars $\boldsymbol{a}_k^\top\boldsymbol{\mu}$, say CI_k, $k=1,\dots,m$, simultaneously
- Make sure $\Pr(\bigcap_k\{\boldsymbol{a}_k^\top\boldsymbol{\mu}\in\mathrm{CI}_k\})\geq 1-\alpha$
- Bonferroni correction
  - Bonferroni inequality:

$$\Pr(\bigcap_{k=1}^m\{\boldsymbol{a}_k^\top\boldsymbol{\mu}\in\mathrm{CI}_k\})=1-\Pr(\bigcup_{k=1}^m\{\boldsymbol{a}_k^\top\boldsymbol{\mu}\notin\mathrm{CI}_k\})\geq 1-\sum_{k=1}^m\Pr(\boldsymbol{a}_k^\top\boldsymbol{\mu}\notin\mathrm{CI}_k)=1-\sum_{k=1}^m\alpha_k$$

  - Taking α_k such that $\alpha=\sum_{k=1}^m\alpha_k$, e.g., $\alpha_k=\alpha/m$, i.e.,

$$(\boldsymbol{a}_k^\top\bar{\boldsymbol{x}}-t_{1-\alpha/(2m),n-1}\sqrt{\boldsymbol{a}_k^\top\mathbf{S}\boldsymbol{a}_k/n},\boldsymbol{a}_k^\top\bar{\boldsymbol{x}}+t_{1-\alpha/(2m),n-1}\sqrt{\boldsymbol{a}_k^\top\mathbf{S}\boldsymbol{a}_k/n})$$

  - Working for small m
- Scheffé's method
  - Let $\mathrm{CI}_{\boldsymbol{w}}=(\boldsymbol{w}^\top\bar{\boldsymbol{x}}-c\sqrt{\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w}/n},\boldsymbol{w}^\top\bar{\boldsymbol{x}}+c\sqrt{\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w}/n})$ for all $\boldsymbol{w}\in\mathbb{R}^p$
  - Derive that $c=\sqrt{p(n-1)(n-p)^{-1}F_{1-\alpha,p,n-p}}$
    - By Cauchy–Schwarz: $\{\boldsymbol{w}^\top(\bar{\boldsymbol{x}}-\boldsymbol{\mu})\}^2=[(\mathbf{S}^{1/2}\boldsymbol{w})^\top\{\mathbf{S}^{-1/2}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})\}]^2\leq\{(\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w})^\top/n\}\{n(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^\top\mathbf{S}^{-1}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})\}\Rightarrow$

$$\begin{aligned}\Pr(\bigcap_{k=1}^m\{\boldsymbol{a}_k^\top\boldsymbol{\mu}\in\mathrm{CI}_k\})&\geq\Pr(\bigcap_{\boldsymbol{w}\in\mathbb{R}^p}\{\boldsymbol{w}^\top\boldsymbol{\mu}\in\mathrm{CI}_{\boldsymbol{w}}\})=1-\Pr(\bigcup_{\boldsymbol{w}\in\mathbb{R}^p}\{\boldsymbol{w}^\top\boldsymbol{\mu}\notin\mathrm{CI}_{\boldsymbol{w}}\})\\&=1-\Pr(\bigcup_{\boldsymbol{w}\in\mathbb{R}^p}[\{\boldsymbol{w}^\top(\bar{\mathbf{X}}-\boldsymbol{\mu})\}^2/\{(\boldsymbol{w}^\top\mathbf{S}\boldsymbol{w})^\top/n\}>c^2])\\&\geq 1-\Pr(\{n(\bar{\mathbf{X}}-\boldsymbol{\mu})^\top\mathbf{S}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu})>c^2\})\end{aligned}$$

    - $\Pr(\{n(\bar{\mathbf{X}}-\boldsymbol{\mu})^\top\mathbf{S}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu})>c^2\})=\alpha\Rightarrow c=\sqrt{p(n-1)(n-p)^{-1}F_{1-\alpha,p,n-p}}$
  - Working for large even infinite m

Comparing two multivariate means (J&W Sec. 6.3)

- Two independent samples of (potentially) different sizes from two distributions with equal covariance
  - $\mathbf{X}_{11}, \ldots, \mathbf{X}_{1n_1} \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$
  - $\mathbf{X}_{21}, \ldots, \mathbf{X}_{2n_2} \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$
- Let $\bar{\mathbf{X}}_i$ and $\mathbf{S}_i$ be the sample mean and sample covariance for the ith sample
- Hypotheses $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ v.s. $H_1 : \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$
- Test statistic following LRT

$$(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)^\top \{(n_1^{-1} + n_2^{-1})\mathbf{S}_{\text{pool}}\}^{-1}(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2) \sim \frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1} F(p, n_1 + n_2 - p - 1)$$

- Rejection region at level α

$$\left\{ x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2} : (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)^\top \{(n_1^{-1} + n_2^{-1})\mathbf{S}_{\text{pool}}\}^{-1}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2) \geq \frac{p(n_1 + n_2 - 2)}{n_1 + n_2 - p - 1} F_{1-\alpha, p, n_1+n_2-p-1} \right\}$$

  - $\mathbf{S}_{\text{pool}} = \frac{(n_1-1)\mathbf{S}_1 + (n_2-1)\mathbf{S}_2}{n_1+n_2-2}$
- p-value

$$1 - F_{F_{1-\alpha, p, n_1+n_2-p-1}} \left[\frac{n_1 + n_2 - p - 1}{p(n_1 + n_2 - 2)} (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)^\top \{(n_1^{-1} + n_2^{-1})\mathbf{S}_{\text{pool}}\}^{-1}(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2) \right]$$