# STAT 3690 Lecture Note

Part VII: Principal component analysis

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# Principal component analysis (PCA)

## Population PCA

- Population PCA based upon covariance matrix $\boldsymbol{\Sigma}$
  - Random $p$-vector $\boldsymbol{X} \sim (\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - Looking for (nonrandom) $p$-vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_p \in \mathbb{R}^p$ such that

    $$\boldsymbol{w}_j^\top \boldsymbol{w}_j = 1 \text{ and } \boldsymbol{X}^\top \boldsymbol{w}_j \text{ has the maximal variance and is uncorrelated with } \boldsymbol{X}^\top \boldsymbol{w}_1, \ldots, \boldsymbol{X}^\top \boldsymbol{w}_{j-1},$$

    i.e.,

    $$\boldsymbol{w}_1 = \arg\max_{\boldsymbol{w} \in \mathbb{R}^p} \operatorname{var}(\boldsymbol{X}^\top \boldsymbol{w}) \text{ subject to } \boldsymbol{w}_1^\top \boldsymbol{w}_1 = 1$$

    and, for $j \geq 2$,

    $$\boldsymbol{w}_j = \arg\max_{\boldsymbol{w} \in \mathbb{R}^p} \operatorname{var}(\boldsymbol{X}^\top \boldsymbol{w})$$

    $$\text{subject to} \qquad \boldsymbol{w}_j^\top \boldsymbol{w}_j = 1 \text{ and } \operatorname{cov}(\boldsymbol{X}^\top \boldsymbol{w}_j, \boldsymbol{X}^\top \boldsymbol{w}_{j'}) = 0 \text{ for } j' = 1, \ldots, j-1$$
  - (PCA Theorem) Let $\lambda_1 \geq \cdots \geq \lambda_p$ be eigenvalues of $\boldsymbol{\Sigma}$. Then the above $\boldsymbol{w}_j$ is the eigenvector corresponding to $\lambda_j$.
  - Vocabulary
    - $\boldsymbol{w}_j$: the $j$th vector of loadings
    - $Z_j = (\boldsymbol{X} - \boldsymbol{\mu})^\top \boldsymbol{w}_j \sim (0, \lambda_j)$: the $j$th principal component (PC) of $\boldsymbol{X}$
  - Representation of $\boldsymbol{X}$ in terms of loadings and PCs

    $$\boldsymbol{X} = \boldsymbol{\mu} + \sum_{j=1}^p Z_j \boldsymbol{w}_j \approx \boldsymbol{\mu} + \sum_{j=1}^s Z_j \boldsymbol{w}_j$$
  - Identities
    - $\boldsymbol{w}_j^\top \boldsymbol{w}_{j'} = 1$ if $j = j'$ and 0 otherwise, i.e., $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_p\}$ is an orthogonal basis of $\mathbb{R}^p$
    - $\operatorname{cov}(Z_j, Z_{j'}) = \boldsymbol{w}_j^\top \boldsymbol{\Sigma} \boldsymbol{w}_{j'} = \lambda_j$ if $j = j'$ and 0 otherwise
    - $\sum_{j=1}^p \operatorname{var}(Z_j) = \sum_{j=1}^p \lambda_j = \operatorname{tr}(\boldsymbol{\Sigma}) = \sum_{j=1}^p \operatorname{var}(X_j)$
    - $Z_j$ contributes $\lambda_j / \sum_{j=1}^p \lambda_j \times 100\%$ of the overall variance
      - Scree plot: displaying the amount of variation in each PC
      - Stopping rule (to determine $s$)

$$s = \min\left\{k \in \mathbb{Z}^+ : \sum_{j=1}^k \lambda_j \Big/ \sum_{j=1}^p \lambda_j \geq 90\% \text{ (or another preset threshhold)}\right\}$$

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- Population PCA based upon correlation matrix
  - (Pearson) correlation matrix

$$\mathbf{R} = [\mathrm{corr}(X_i, X_j)]_{p\times p} = \mathrm{diag}\left(\frac{1}{\sqrt{\mathrm{var}(X_1)}}, \dots, \frac{1}{\sqrt{\mathrm{var}(X_p)}}\right)\boldsymbol{\Sigma}\mathrm{diag}\left(\frac{1}{\sqrt{\mathrm{var}(X_1)}}, \dots, \frac{1}{\sqrt{\mathrm{var}(X_p)}}\right)$$

  - Loadings and PCs from $\mathbf{R}$ are not identical to those obtained from $\boldsymbol{\Sigma}$
  - General advice: $\boldsymbol{\Sigma}$ is superior when entries of $\boldsymbol{X}$ are of the same units and comparable; otherwise $\mathbf{R}$ is preferred.
    - Using $\mathbf{R}$ rather than $\boldsymbol{\Sigma} \Leftrightarrow$ normalizing entries of $\boldsymbol{X}$ (i.e., $\{X_i - \mathrm{E}(X_i)\}/\sqrt{\mathrm{var}(X_i)}$) before carrying on PCA
    - Without normalizing, the component with the "smallest" units (e.g., centimeter vs. meter) could be driving most of overall variance.

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## Sample PCA

- $\boldsymbol{X} = [\boldsymbol{X}_{1\cdot}, \dots, \boldsymbol{X}_{n\cdot}]^\top_{n\times p}$
  - i.e., $\boldsymbol{X}_{i\cdot}^\top$ is the $i$th row of $\boldsymbol{X}$
  - Assuming $\boldsymbol{X}_{i\cdot} \overset{\text{iid}}{\sim} (\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- Estimate the loadings $\boldsymbol{w}_j$ through the eigenvectors of sample covariance matrix $\mathbf{S}$ or sample correlation matrix $\hat{\mathbf{R}}$
- Score matrix of the first $s$ PCs

$$\boldsymbol{Z} = [Z_{ij}]_{n\times s} = \tilde{\boldsymbol{X}}\widehat{\mathbf{W}}$$

  - $\tilde{\boldsymbol{X}} = [\boldsymbol{X}_{1\cdot} - \bar{\boldsymbol{X}}_{\text{row}}, \dots, \boldsymbol{X}_{n\cdot} - \bar{\boldsymbol{X}}_{\text{row}}]^\top_{n\times p}$: row-centered $\boldsymbol{X}$ (i.e. the sample mean has been subtracted from each row of $\boldsymbol{X}$)
    - $\bar{\boldsymbol{X}}_{\text{row}} = n^{-1}\sum_{i=1}^n \boldsymbol{X}_{i\cdot}$
  - $\widehat{\mathbf{W}} = [\hat{\boldsymbol{w}}_1, \dots, \hat{\boldsymbol{w}}_s]_{p\times s}$: $\hat{\boldsymbol{w}}_j$ is the estimate of $\boldsymbol{w}_j$
  - $Z_{ij} = (\boldsymbol{X}_{i\cdot} - \bar{\boldsymbol{X}}_{\text{row}})^\top \hat{\boldsymbol{w}}_j$: the $j$th PC score for the $i$th observation

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## Geometric interpretation of (sample) PCA

- The definition of PCA as a linear combination that maximises variance is due to H. Hotelling (1933, Journal of Educational Psychology, 24, 417–441).
- PCA was introduced earlier by K. Pearson (1901, Philosophical Magazine, Series 6, 2(11), 559–572) to minimize the overall error in reconstructing data points

$$(\bar{\boldsymbol{X}}_{\text{row}}, \widehat{\mathbf{W}}, \boldsymbol{Z}_{i\cdot}) = \arg\min_{\boldsymbol{\theta}, \mathbf{A}, \mathbf{B}_i} \sum_{i=1}^n (\boldsymbol{X}_{i\cdot} - \boldsymbol{\theta} - \mathbf{A}\mathbf{B}_i)^\top (\boldsymbol{X}_{i\cdot} - \boldsymbol{\theta} - \mathbf{A}\mathbf{B}_i)$$

  - $\boldsymbol{Z}_{i\cdot} = [Z_{i1}, \dots, Z_{is}]$: the $i$th row of score matrix $\boldsymbol{Z}$

## Application of (sample) PCA

- Image compression: `mnist` is a list with two components: `train` and `test`. Each of these is a list with two components: images and labels.
  - The `images` component is a matrix with each row for one image consisting of $28 \times 28 = 784$ entries (pixels). Their value are integers between 0 and 255 representing grey scale.

- The `labels` components is a vector representing the digit shown in the image.

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- PC regression (PCR): regression on PC scores
  1. Perform PCA on the observed data matrix of explanatory variables, usually centered
  2. Regress the outcome vector(s) on the selected PCs as covariates using linear regression to get a vector of estimated regression coefficients
  3. Transform this coefficient vector back to the scale of the actual covariates
- Note that the prediction of PCR is identical to that of linear regression, when all the PCs are included.

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- Example of PCR: dataset `Prostate` comes from a study that examined the correlation between the level of prostate-specific antigen and a number of clinical measures in men who were about to receive a radical prostatectomy; see Stamey et al, 1989, Journal of Urology 141(5), 1076–1083.
  - `lcavol`: log(cancer volume)
  - `lweight`: log(prostate weight)
  - `age`: patient age
  - `lbph`: log(benign prostatic hyperplasia amount)
  - `svi`: seminal vesicle invasion
  - `lcp`: log(capsular penetration)
  - `gleason`: Gleason score
  - `pgg45`: percentage Gleason scores 4 or 5
  - `lpsa`: log(prostate specific antigen)

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## Summary of PCA

- Procedure
  1. Create PCs which are weighted sums of (centered) explanatory variables, with eigenvectors of (sample) correlation/covariance matrix taken as weights.
  2. Take PCs as surrogates of (centered) explanatory variables for various techniques
- Pros and cons
  - Doable without strong distribution assumption
  - Uninterpretable PCs
  - Not involving response; abandoned PCs possibly related to response