

STAT 3690 Lecture Note

Week One (Jan 9, 11, & 13, 2023)

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2023/Jan/08 19:45:36

IN THE CASE OF A FIRE ALARM:

- **Remain calm**
 - if it is safe, evacuate the classroom or lab
 - go to the closest fire exit
 - do not use the elevators
- **If you need assistance to evacuate the building, inform your professor or instructor immediately.**
- **If you need to report an incident or a person left behind during a building evacuation, report it to a fire warden or call security services 204-474-9341.**
 - **Do not** reenter the building until the “all clear” is declared by a fire warden, security services or the fire department.
- **Important: only those trained in the use of a fire extinguisher should attempt to operate one!**



Syllabus

Contact

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Timeline

- Lectures
 - Mon/Wed/Fri 9:30–10:20 am
- Office Hour
 - Wed 10:30–11:20 am
- Assessments
 - Assignments (4/5 times)
 - Midterm
 - Final project

Grading

- Assignments (30%)
 - Scanned/photographed and submitted to Crowdmark
 - Attaching both outputs and source codes (if applicable)
 - Including necessary interpretation
 - Organized in a clear and readable way
 - Accepting NO late submission
- Midterm (35%)
 - Open-book
 - In-person on Mar 10 6–8 pm OR take-home (webcam-invigilated) NOT later than Mar. 20
- Final project (35%)
 - Individual report analyzing recently collected datasets
 - See the Project Guideline posted at UM Learn

Materials

- Reading list (recommended but not required)
 - [J&W] R. A. Johnson & D. W. Wichern. (2007). *Applied Multivariate Statistical Analysis*, 5/6th Ed. London: Pearson Education.
 - * 2HR print reserve in the Sciences and Technology Library
 - [R&C] A. C. Rencher & W. F. Christensen. (2012). *Methods of Multivariate Analysis*, 3rd Ed. Hoboken: Wiley.
 - * Digital copy accessible via the library
 - D. Salsburg (2001). *The Lady Tasting Tea: How Statistics Revolutionized Science in the Twentieth Century*. New York: WH Freeman.
- Lecture notes and beyond
 - zhiyanggeezhou.github.io
 - UM Learn

Outline

- Topics to be covered
 - Matrix manipulation
 - Basics of statistical modeling
 - Multivariate normal distribution
 - Inference on a mean vector
 - Comparisons of several multivariate means
 - Multivariate linear regression
 - Principal component analysis
 - Factor analysis
 - Canonical correlation analysis
 - and so forth

R basics

- Installation
 - download and install BASE *R* from <https://cran.r-project.org>
 - download and install *Rstudio* from <https://www.rstudio.com>
 - download and install packages via *Rstudio*
- Working directory
 - When you ask *R* to open a certain file, it will look in the working directory for this file.
 - When you tell *R* to save a data file or figure, it will save it in the working directory.
- Packages

- installation: `install.packages()`
 - loading: `library()`
 - Help manual: `help()`, `?`, google, stackoverflow, etc.
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- *R* is free but not cheap
 - Open-source
 - Citing packages
 - NO quality control
 - Requiring statistical sophistication
 - Time-consuming to become a master
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- References for *R*
 - M. L. Rizzo (2019) Statistical Computing with R, 2nd Ed. (forthcoming)
 - O. Jones, R. Maillardet, A. Robinson (2014) Introduction to Scientific Programming and Simulation Using R, 2nd Ed.
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 - Courses online
 - <https://www.pluralsight.com/search?q=R>
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 - Data types: let `str()` or `class()` tell you
 - numbers (integer, real, or complex)
 - characters (“abc”)
 - logical (TRUE or FALSE)
 - date & time
 - factor (commonly encountered in this course)
 - NA (different from Inf, “ ’”, 0, NaN etc.)
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- Data structures: let `str()` or `class()` tell you
 - vector: an ordered collection of the same data type
 - matrix: two-dimensional collection of the same data type
 - array: more than two dimensional collection of the same data type
 - data frame: collection of vectors of same length but of arbitrary data types
 - list: collection of arbitrary objects
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- Data input and output
 - create
 - * vector: `c()`, `seq()`, `rep()`
 - * matrix: `matrix()`, `cbind()`, `rbind()`
 - * data frame
 - output: `write.table()`, `write.csv()`, `write.xlsx()`
 - import: `read.table()`, `read.csv()`, `read.xlsx()`
 - * header: whether or not assume variable names in first row
 - * stringsAsFactors: whether or not convert character string to factors
 - `scan()`: a more general way to input data
 - `save.image()` and `load()`: save and reload workspace
 - `source()`: run R script
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- Parenthesis in *R*
 - parenthesis `()` to enclose inputs for functions

- square brackets [], [[]] for indexing
- braces {} to enclose forloop or statements such as if or ifelse

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- Elementary arithmetic operators
 - +, -, *, /, ^
 - log, exp, sin, cos, tan, sqrt
 - FALSE and TRUE becoming 0 and 1, respectively
 - sum(), mean(), median(), min(), max(), var(), sd(), summary()
 - Matrix calculation
 - element-wise multiplication: A * B
 - matrix multiplication: A %*% B
 - singular value decomposition: eigen(A)
 - Loops: for() and while()

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- Probabilities
 - normal distribution: dnorm(), pnorm(), qnorm(), rnorm()
 - uniform distribution: dunif(), punif(), qunif(), runif()
 - multivariate normal distribution: dmnorm(), rmvnorm()

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- Basic plots
 - strip chart, histogram, box plot, scatter plot
 - Package ggplot2 (RECOMMENDED)
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Matrix basics

Matrix decomposition

- Eigen-decomposition (for square matrix $\mathbf{A}_{n \times n}$): $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$
 - $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n)$
 - * $\lambda_1 \geq \dots \geq \lambda_n$ are the eigenvalues of \mathbf{A} , i.e., n roots of characteristic equation $\det(\lambda \mathbf{I}_n - \mathbf{A}) = 0$
 - $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_n]_{n \times n}$
 - * $\mathbf{v}_1, \dots, \mathbf{v}_n$ are (right) eigenvectors of \mathbf{A} , i.e., $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$
 - Implementation in R: eigen()
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- Spectral decomposition (for symmetric \mathbf{A}): $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$
 - \mathbf{V} is orthogonal, i.e., $\mathbf{V}^T = \mathbf{V}^{-1}$
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- Singular value decomposition (SVD) for $n \times p$ matrix \mathbf{B} : $\mathbf{B} = \mathbf{U}\mathbf{S}\mathbf{W}^T$
 - $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_n]_{n \times n}$ with \mathbf{u}_i the i th eigenvector of $\mathbf{B}\mathbf{B}^T$
 - * \mathbf{U} is orthogonal
 - $\mathbf{W} = [\mathbf{w}_1, \dots, \mathbf{w}_p]_{p \times p}$ with \mathbf{w}_i the i th eigenvector of $\mathbf{B}^T\mathbf{B}$
 - * \mathbf{W} is orthogonal

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$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0}_{n \times (p-n)} \end{bmatrix}_{n \times p} \text{ if } n \leq p \text{ AND } \begin{bmatrix} \mathbf{S}_1 & - & - \\ \mathbf{0}_{(n-p) \times p} & - & - \end{bmatrix}_{n \times p} \text{ if } n > p$$

- * $\mathbf{S}_1 = \text{diag}(s_1, \dots, s_n)$ if $n \leq p$ and $\text{diag}(s_1, \dots, s_p)$ if $n > p$
- * $s_1 \geq \dots \geq s_n$ are square roots of eigenvalues of $\mathbf{B}\mathbf{B}^\top$
- * $s_1 \geq \dots \geq s_p$ are square roots of eigenvalues of $\mathbf{B}^\top\mathbf{B}$
- Thin/compact SVD for $n \times p$ matrix \mathbf{B} :

$$\mathbf{B} = [\mathbf{u}_1, \dots, \mathbf{u}_r] \text{diag}(s_1, \dots, s_r) [\mathbf{w}_1, \dots, \mathbf{w}_r]^\top = s_1 \mathbf{u}_1 \mathbf{w}_1^\top + \dots + s_r \mathbf{u}_r \mathbf{w}_r^\top$$

- * $r = \text{rank}(\mathbf{B}) \leq \min\{n, p\}$
- * $s_1 \geq \dots \geq s_r > 0$ are square roots of non-zero eigenvalues of $\mathbf{B}^\top\mathbf{B}$ or $\mathbf{B}\mathbf{B}^\top$
- * Implementation via R: `svd()`

- The connection of decompositions

$$\begin{array}{ccc} \text{Spectral decomposition} & \Rightarrow & \text{Eigen-decomposition} \\ \text{(for symmetric matrices)} & & \text{(for square matrices)} \\ \Downarrow & & \\ \text{(Thin) SVD} & & \\ \text{(for any matrices)} & & \end{array}$$

- Exercise: Is it feasible to apply R function `eigen()` only if one would like to implement the thin SVD for a square matrix with non-negative singular values (say s_i 's)?

Square root of positive (semi-)definite matrix

- \mathbf{A} is positive semi-definite (say $\mathbf{A} \geq 0$) iff \mathbf{A} is symmetric and its eigenvalues are all non-negative
 - Equiv., $\mathbf{u}^\top \mathbf{A} \mathbf{u} \geq 0$ for any $n \times 1$ real matrix \mathbf{u} (say $\mathbf{u} \in \mathbb{R}^{n \times 1}$ OR $\mathbf{u} \in \mathbb{R}^n$)
- \mathbf{A} is positive definite (say $\mathbf{A} > 0$) iff \mathbf{A} is symmetric and its eigenvalues are all positive
 - Equiv., $\mathbf{u}^\top \mathbf{A} \mathbf{u} > 0$ for all $\mathbf{u} \in \mathbb{R}^n$
- If $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$ is the spectral decomposition of positive semi-definite \mathbf{A} , then $\mathbf{A}^{1/2} = \mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^\top$ satisfies that
 - $\mathbf{\Lambda}^{1/2} = \text{diag}(\lambda_1^{1/2}, \dots, \lambda_n^{1/2})$
 - $\mathbf{A}^{1/2} \mathbf{A}^{1/2} = \mathbf{A}$
- If $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$ is the spectral decomposition of positive definite \mathbf{A} , then $\mathbf{A}^{-1/2} = \mathbf{V}\mathbf{\Lambda}^{-1/2}\mathbf{V}^\top$ satisfies that
 - $\mathbf{\Lambda}^{-1/2} = \text{diag}(\lambda_1^{-1/2}, \dots, \lambda_n^{-1/2})$
 - $\mathbf{A}^{-1/2} \mathbf{A}^{-1/2} = \mathbf{A}^{-1}$
 - $\mathbf{A}^{1/2} \mathbf{A}^{-1/2} = \mathbf{I}_n$

Determinant and trace

- Applicable only to square matrices
- Properties for determinant
 - $\det(\mathbf{A}) = \prod_i \lambda_i$
 - $\det(\mathbf{A}^\top) = \det(\mathbf{A})$
 - $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$
 - $\det(c \cdot \mathbf{A}) = c^n \det(\mathbf{A})$ for $n \times n$ matrix \mathbf{A} and scalar c
 - $\det(\mathbf{A}\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$ if \mathbf{A} and \mathbf{B} are square matrices of the identical dimension

- Properties for trace
 - $\text{tr}(\mathbf{A}) = \sum_i \lambda_i$
 - $\text{tr}(c \cdot \mathbf{A}) = c \cdot \text{tr}(\mathbf{A})$ for scalar c
 - $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$ if \mathbf{A} and \mathbf{B} are square matrices of the identical dimension
 - $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$ for $m \times n$ matrix \mathbf{A} and $n \times m$ matrix \mathbf{B}
- Remark: $\det(\mathbf{A})$ and $\text{tr}(\mathbf{A})$ can be taken as measures of the size of \mathbf{A} when \mathbf{A} is positive definite (i.e., its eigenvalues are all positive).

- Exercise: Prove that
 1. $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$ for $m \times n$ \mathbf{A} and $n \times m$ \mathbf{B} .
 2. (The trace trick) $\text{tr}(\mathbf{A}_1 \cdots \mathbf{A}_k) = \text{tr}(\mathbf{A}_{k'+1} \cdots \mathbf{A}_k \mathbf{A}_1 \cdots \mathbf{A}_{k'})$ for $1 < k' < k$.
 3. $\text{tr}(\mathbf{A}) = \sum_i \lambda_i$.
 4. $\det(\mathbf{A}) = \prod_i \lambda_i$. Hint: Jordan matrix decomposition, i.e., there exists a Jordan normal (or canonical) form \mathbf{J} and invertible \mathbf{U} such that $\mathbf{A} = \mathbf{U}\mathbf{J}\mathbf{U}^{-1}$ for any square \mathbf{A} .

Block/partitioned matrix

- A partition of matrix: Suppose \mathbf{A}_{11} is of $p \times r$, \mathbf{A}_{12} is of $p \times s$, \mathbf{A}_{21} is of $q \times r$ and \mathbf{A}_{22} is of $q \times s$. Make a new $(p+q) \times (r+s)$ -matrix by organizing \mathbf{A}_{ij} 's in a 2 by 2 way:

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array} \right]$$

e.g.,

$$\mathbf{A} = \left[\begin{array}{cc|cc} 1 & 0 & 2 & \\ 0 & 1 & 3 & \\ \hline 4 & 5 & 6 & \end{array} \right]$$

if

$$\mathbf{A}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{A}_{21} = \begin{bmatrix} 4 & 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_{22} = \begin{bmatrix} 6 \end{bmatrix}.$$

- Operations with block matrices
 - Working with partitioned matrices just like ordinary matrices
 - Matrix addition: if dimensions of \mathbf{A}_{ij} and \mathbf{B}_{ij} are quite the same, then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} + \mathbf{B}_{11} & \mathbf{A}_{12} + \mathbf{B}_{12} \\ \mathbf{A}_{21} + \mathbf{B}_{21} & \mathbf{A}_{22} + \mathbf{B}_{22} \end{bmatrix}$$

- Matrix multiplication: if $\mathbf{A}_{ij}\mathbf{B}_{jk}$ makes sense for each i, j, k , then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

- Inverse: if \mathbf{A} , \mathbf{A}_{11} and \mathbf{A}_{22} are all invertible, then

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11.2}^{-1} & -\mathbf{A}_{11.2}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11.2}^{-1} & \mathbf{A}_{22.1}^{-1} \end{bmatrix}$$

- * $\mathbf{A}_{11.2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$
- * $\mathbf{A}_{22.1} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$