

PH 712 Probability and Statistical Inference

Part I: Random Variable

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2024/09/23 10:59:42

Probability (HMC Sec. 1.1–1.3)

- Sample space (denoted by Ω): the set of all the possible outcomes, e.g.,
 - $\Omega = \mathbb{R}^+$ if investigating survival times of cancer patients
 - $\Omega = \{\text{yes, no}\}$ if investigating whether a treatment is effective
- Event (denoted by capital Roman letters, e.g., A): a subset of the sample space, e.g., corresponding to the previous sample spaces,
 - $(0, 10]$: the survival time ≤ 10
 - $\{\text{yes}\}$: the treatment is effective
- Occurrence of event: the outcome is part of the event
- Probability (denoted by \Pr): a function quantifying the occurrence likelihood of an event
 - E.g.,
 - * $\Pr(A)$: the occurrence probability of event A
 - * $\Pr(A^c)$: the probability that event A does NOT occur ($A^c = \Omega \setminus A$ denoting the complement set of A)
 - * $\Pr(A \cup B)$: the occurrence probability of either A or B
 - * $\Pr(A \cap B)$: the occurrence probability of both A and B
 - Input: an event
 - Output: a real number (the occurrence probability of the input event)
 - Requirements:
 - * $\Pr(A) \geq 0$ for any event A
 - * $\Pr(\Omega) = 1$ (i.e., the sample space as a special event always occurs)
 - * (The probability of the union of mutually exclusive countably events is the sum of the probability of each event) If $\{A_n\}_{n=1}^\infty$ is a sequence of events with $A_{n_1} \cap A_{n_2} = \emptyset$ for all $n_1 \neq n_2$, then $\Pr(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Pr(A_n)$
 - More properties (deduced from the above requirements):
 - * $\Pr(A) = 1 - \Pr(A^c)$
 - * $\Pr(\emptyset) = 0$
 - * $\Pr(A) \leq \Pr(B)$ if $A \subset B$
 - * $0 \leq \Pr(A) \leq 1$ for each A
 - * $\lim_{n \rightarrow \infty} \Pr(A_n) = \Pr(\lim_{n \rightarrow \infty} A_n) = \Pr(\bigcup_{n=1}^\infty A_n)$ if $\{A_n\}_{n=1}^\infty$ is nondecreasing (i.e., $A_1 \subset A_2 \subset \dots$)
 - * $\lim_{n \rightarrow \infty} \Pr(A_n) = \Pr(\lim_{n \rightarrow \infty} A_n) = \Pr(\bigcap_{n=1}^\infty A_n)$ if $\{A_n\}_{n=1}^\infty$ is nonincreasing (i.e., $A_1 \supset A_2 \supset \dots$)
 - * $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ for any events A and B regardless if they are disjoint or not

- * $\Pr(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \Pr(A_n)$ for arbitrary sequence $\{A_n\}_{n=1}^{\infty}$

Conditional probability and independence (HMC Sec. 1.4)

- Conditional probability of B given A (with $\Pr(A) > 0$): $\Pr(B | A) = \Pr(A \cap B) / \Pr(A)$
 - Interpretation: the occurrence probability of B , given that A has already occurred.
 - Properties:
 - * $\Pr(B | A) \geq 0$
 - * $\Pr(A | A) = 1$
 - * $\Pr(\bigcup_{n=1}^{\infty} B_n | A) = \sum_{n=1}^{\infty} \Pr(B_n | A)$ if $\{B_n\}_{n=1}^{\infty}$ are mutually exclusive
 - * (Law of total probability) $\Pr(B) = \sum_{n=1}^N \Pr(A_n) \Pr(B | A_n)$ if $\{A_n\}_{n=1}^N$ form a partition of Ω (i.e., $\{A_n\}_{n=1}^N$ are mutually exclusive and $\Omega = \bigcup_{n=1}^N A_n$)
 - * (Bayes' theorem) $\Pr(A_i | B) = \Pr(A_i) \Pr(B | A_i) / \sum_{n=1}^N \Pr(A_n) \Pr(B | A_n)$ if $\{A_n\}_{n=1}^N$ form a decomposition/partition of Ω
- Independence between two events B and A (i.e., $B \perp A$): $\Pr(B \cap A) = \Pr(A) \Pr(B)$
 - $\Leftrightarrow B \perp A^c$
 - $\Leftrightarrow \Pr(B | A) = \Pr(B)$ (if $\Pr(A) \neq 0$)
- Mutual independence among N events A_1, \dots, A_N : for arbitrary subset of $\{A_1, \dots, A_N\}$, say $\{A_{n_1}, \dots, A_{n_K}\}$ with $2 \leq K \leq N$, $\Pr(\bigcap_{k=1}^K A_{n_k}) = \prod_{k=1}^K \Pr(A_{n_k})$

HMC Ex. 1.4.31

- A French writer, Chevalier de Méré, had asked a famous mathematician, Pascal, to explain why the following two probabilities were different (the difference had been noted from playing the game many times): (1) at least one six in four independent casts of a six-sided die; (2) at least a pair of sixes in 24 independent casts of a pair of dice. From proportions it seemed to Mr. de Méré that the two probabilities should be the same. Compute the probabilities of (1) and (2).
 - Hint: $\Pr(\text{no six in one cast of a die}) = 5/6$, $\Pr(\text{no six in one cast of a pair of dice}) = (5/6)^2$, and $\Pr(\text{only one six in one cast of a pair of dice}) = 2 \times (1/6) \times (5/6)$.

Distribution of an RV (HMC Chp. 1.5–1.7)

- RV: a function encoding the entries of Ω
 - Input: arbitrary entry of Ω , say ω
 - Output: $X(\omega) \in \mathbb{R}$
- The cumulative distribution function (cdf) of RV X , say F_X , is defined as

$$F_X(t) = \Pr(X \leq t), \quad t \in \mathbb{R}.$$

- $\{X \leq t\}$: short for the event $\{\omega \in \Omega : X(\omega) \leq t\}$
- F_X satisfies following three properties:
 - * (Right continuous) $\lim_{x \rightarrow t^+} F_X(x) = F_X(t)$ (p.s., $\lim_{x \rightarrow t^-} F_X(x) = \Pr(X < t)$);
 - * (Non-decreasing) $F_X(t_1) \leq F_X(t_2)$ for $t_1 \leq t_2$;
 - * (Ranging from 0 to 1) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.
- Reversely, a function satisfying the three above properties must be a cdf for certain RV.
 - * Indicating an one-to-one correspondence between the set of all the RVs and the set of all the cdfs
- Knowing the cdf of an RV \Leftrightarrow knowing its distribution

Example Lec1.1

- Given $p \in (0, 1)$, suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of real x .

- Show that F_X is a cdf. (Hint: Check all the three properties of cdf, especially the right-continuity of F_X at positive integers.)
- Given $\lambda > 0$, suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- Show that F_X is a cdf.

Distribution of an RV (con'd)

- Discrete RV
 - RV X merely takes countably different values
 - Probability mass function (pmf): $p_X(t) = \Pr(X = t)$
 - * $F_X(t) = \sum_{x \leq t} p_X(x)$
 - * $p_X(t) = F_X(t) - \Pr(X < t) = F_X(t) - \lim_{x \rightarrow t^-} F_X(x)$
 - Knowing the pmf of a discrete RV \Leftrightarrow knowing its distribution
 - Examples:
 - * Bernoulli: a discrete RV with two possible outcomes, typically coded as 0 (failure) and 1 (success).
 - https://en.wikipedia.org/wiki/Bernoulli_distribution
 - * Binomial (denoted by $B(n, p)$): the number of successes in n independent Bernoulli trials.
 - https://en.wikipedia.org/wiki/Binomial_distribution
 - E.g., flipping a coin 10 times and counting the number of heads.
 - * Geometric: the number of trials until the first success in a series of independent Bernoulli trials.
 - https://en.wikipedia.org/wiki/Geometric_distribution
 - E.g., the number of coin flips needed until the first head appears.
 - * Poisson: the number of events that occur in a fixed interval of time or space, where events happen independently.
 - https://en.wikipedia.org/wiki/Poisson_distribution
 - E.g., the number of emails you receive in an hour.
 - * Uniform (the discrete version): each outcome in a finite set has an equal probability.
 - https://en.wikipedia.org/wiki/Discrete_uniform_distribution
 - E.g., rolling a fair dice, where each of the six faces has an equal chance of landing.
- Continuous RV
 - RV X is continuous \Leftrightarrow its cdf F_X is absolutely continuous, i.e., there exists f_X such that

$$F_X(t) = \int_{-\infty}^t f_X(x) dx, \quad \forall t \in \mathbb{R}.$$

- * Probability density function (pdf): $f_X(t) = dF_X(t)/dt = \lim_{\delta \rightarrow 0^+} \Pr(t < X \leq t + \delta)/\delta (\geq 0)$.
 - $\int_{-\infty}^{\infty} f_X(x) dx = \lim_{t \rightarrow \infty} \int_{-\infty}^t f_X(x) dx = \lim_{t \rightarrow \infty} F_X(t) = 1$
 - * $\Pr(X = x_0) = 0$ for all $x_0 \in \mathbb{R}$
 - Because $\Pr(X = x_0) = \Pr(X \leq x_0) - \Pr(X < x_0) = F_X(x_0) - \lim_{x \rightarrow x_0^-} F_X(x) = 0$
- Knowing the pdf of a continuous RV \Leftrightarrow knowing its distribution
- Examples:
 - * Uniform (the continuous version): all outcomes in a continuous range are equally likely.
 - [https://en.wikipedia.org/wiki/Uniform_distribution_\(continuous\)](https://en.wikipedia.org/wiki/Uniform_distribution_(continuous))
 - * Normal/Gaussian (denoted by $\mathcal{N}(\mu, \sigma^2)$): the most important and widely used distributions, where data is symmetrically distributed around the mean.
 - https://en.wikipedia.org/wiki/Normal_distribution
 - * Exponential: the time between events in a Poisson process, often used to describe waiting times.
 - https://en.wikipedia.org/wiki/Exponential_distribution

Example Lec1.2

- Given $\lambda > 0$, suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- What is the type of X , discrete or continuous?
- Given $p \in (0, 1)$, suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of x .

- What is the type of X , discrete or continuous?

Support of RV (CB pp. 50 & HMC pp. 46)

- For discrete RV X with pmf p_X
 - $\text{supp}(X) = \{x \in \mathbb{R} : p_X(x) > 0\}$
 - E.g., support of $B(n, p)$ is $\{0, \dots, n\}$
 - $\sum_{x \in \text{supp}(X)} p_X(x) = 1$
- For continuous RV X with pdf f_X
 - $\text{supp}(X) = \{x \in \mathbb{R} : f_X(x) > 0\}$
 - E.g., support of $\mathcal{N}(0, 1)$ is \mathbb{R}
 - $\int_{\text{supp}(X)} f_X(x) dx = 1$

Example Lec1.3

- Revisit F_X defined in Example Lec1.1, i.e.,

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of real x .

- What is the support of X ?

Indicator function

Given a set A , the indicator function of A is

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Example Lec1.4

- Revisit F_X defined in Example Lec1.1, i.e.,

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of x .

- Please reformulate F_X with the indicator function of $A = \{x : x \geq 1\}$.

Indicating the support when writing pmf and pdf

- Bernoulli: a discrete RV with two possible outcomes, typically coded as 0 (failure) and 1 (success).
 - https://en.wikipedia.org/wiki/Bernoulli_distribution
- Binomial (denoted by $B(n, p)$): the number of successes in n independent Bernoulli trials.

- https://en.wikipedia.org/wiki/Binomial_distribution
- $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \cdot \mathbf{1}_{\{0,1,\dots,n\}}(k)$
 * OR $\binom{n}{k} p^k (1-p)^{n-k}$, $k \in \{0,1,\dots,n\}$
- Geometric: the number of trials until the first success in a series of independent Bernoulli trials.
 - https://en.wikipedia.org/wiki/Geometric_distribution
 - $p_X(k) = (1-p)^{k-1} p \cdot \mathbf{1}_{\mathbb{Z}^+}(k)$
 * OR $(1-p)^{k-1} p$, $k \in \mathbb{Z}^+$
- Poisson: the number of events that occur in a fixed interval of time or space, where events happen independently.
 - https://en.wikipedia.org/wiki/Poisson_distribution
 - $p_X(k) = \lambda^k \exp(-\lambda)/k! \cdot \mathbf{1}_{\{0,1,2,\dots\}}(k)$
 * OR $\lambda^k \exp(-\lambda)/k!$, $k \in \{0,1,2,\dots\}$
- Uniform (the discrete version; denoted by $U([a,b])$ with integers $a < b$): each outcome in a finite set has an equal probability.
 - https://en.wikipedia.org/wiki/Discrete_uniform_distribution
 - $p_X(k) = 1/(b-a+1) \cdot \mathbf{1}_{\{a,a+1,\dots,b-1,b\}}(k)$
 * OR $1/(b-a+1)$, $k \in \{a,a+1,\dots,b-1,b\}$
- Uniform (the continuous version): all outcomes in a continuous range are equally likely.
 - [https://en.wikipedia.org/wiki/Uniform_distribution_\(continuous\)](https://en.wikipedia.org/wiki/Uniform_distribution_(continuous))
- Normal/Gaussian (denoted by $\mathcal{N}(\mu, \sigma^2)$): the most important and widely used distributions, where data is symmetrically distributed around the mean.
 - https://en.wikipedia.org/wiki/Normal_distribution
- Exponential: the time between events in a Poisson process, often used to describe waiting times.
 - https://en.wikipedia.org/wiki/Exponential_distribution
 - $f_X(x) = \lambda \exp(-\lambda x) \cdot \mathbf{1}_{[0,\infty)}(x)$
 * OR $\lambda \exp(-\lambda x)$, $x \geq 0$

Expectations (HMC Sec. 1.8–1.9)

- Given RV X and function g , the expectation of $g(X)$ is

$$\mathbb{E}\{g(X)\} = \begin{cases} \int_{x \in \text{supp}(X)} g(x) f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} g(x) p_X(x) & \text{for discrete } X \end{cases}$$

- Weighted average of values of $g(X)$
- $\mathbb{E}\{a_1 g_1(X) + a_2 g_2(X)\} = a_1 \mathbb{E}\{g_1(X)\} + a_2 \mathbb{E}\{g_2(X)\}$ for constants a_1 and a_2
- Examples
 - Taking $g(X) = X$

$$\mathbb{E}(X) = \begin{cases} \int_{x \in \text{supp}(X)} x f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} x p_X(x) & \text{for discrete } X \end{cases}$$

- * The mean of X (a.k.a. the 1st raw moment/moment about 0 of X)
- * $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$ for constants a and b
- Taking $g(X) = X^k$ with positive integer k :

$$\mathbb{E}(X^k) = \begin{cases} \int_{x \in \text{supp}(X)} x^k f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} x^k p_X(x) & \text{for discrete } X \end{cases}$$

- * The k th raw moment/moment about 0 of X
- Taking $g(X) = \{X - \mathbb{E}(X)\}^2$:

$$\text{Var}(X) = \mathbb{E}[\{X - \mathbb{E}(X)\}^2] = \begin{cases} \int_{x \in \text{supp}(X)} \{x - \mathbb{E}(X)\}^2 f_X(x) dx & \text{for continuous } X \\ \sum_{x \in \text{supp}(X)} \{x - \mathbb{E}(X)\}^2 p_X(x) & \text{for discrete } X \end{cases}$$

- * Variance of X (a.k.a. the 2nd central moment of X)
- * Measuring how spread out the data are if they are independently generated following F_X
- * $\text{Var}(X) = E(X^2) - \{E(X)\}^2$
- * $\text{Var}(aX + b) = a^2 \text{Var}(X)$
- * $\text{sd}(X) = \sqrt{\text{Var}(X)}$: the standard deviation of X

Example Lec1.5

- Find the mean and variance of $X \sim \mathcal{N}(0, 1)$, i.e., $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$$E(X) = \int_{\mathbb{R}} x f_X(x) dx \stackrel{x \exp(-x^2/2) \text{ is odd}}{=} \int_{\mathbb{R}} \frac{x}{\sqrt{2\pi}} \exp(-x^2/2) dx = 0$$

$$\text{Var}(X) \stackrel{\text{even } x^2 \exp(-x^2/2)}{=} 2 \int_0^\infty \frac{x^2 \exp(-x^2/2)}{\sqrt{2\pi}} dx \stackrel{u=x^2/2}{=} 2 \int_0^\infty \frac{2u \exp(-u)}{\sqrt{2\pi}} d\sqrt{2u} = \frac{2\Gamma(3/2)}{\sqrt{\pi}} = 1$$

- Find the mean and variance of $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, i.e., $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ (p.s. $X \sim \mathcal{N}(\mu, \sigma^2) \Leftrightarrow Z = (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$)
- Find the mean and variance of Cauchy distribution, i.e., $f_X(x) = \{\pi(1+x^2)\}^{-1}$, $x \in \mathbb{R}$

$$\int_1^\infty \frac{x^2}{\pi(1+x^2)} dx \geq \int_1^\infty \frac{x}{\pi(1+x^2)} dx = \infty$$

Distribution of an RV (con'd)

- Moment generating function (mgf, HMC Sec. 1.9/CB Sec. 2.3)
 - $M_X(t) = E\{\exp(tX)\}$
 - * Continuous X : $M_X(t) = \int_{x \in \text{supp}(X)} \exp(tx) f_X(x) dx$
 - * Discrete X : $M_X(t) = \sum_{x \in \text{supp}(X)} \exp(tx) p_X(x)$
 - The mgf of X is $M_X(t)$, $t \in A$, $\Leftrightarrow M_X(t)$ is finite for t in a neighborhood of 0, say A ; otherwise the mgf does NOT exist or is NOT well defined.
 - $M_{aX+b}(t) = \exp(bt) M_X(at)$
 - Knowing the mgf (if any) of an RV \Leftrightarrow knowing its distribution
 - If mgf $M(t)$ is well-defined, then the k th raw moment is the k th-order derivative of $M(t)$ evaluated at 0, i.e., $E(X^k) = M^{(k)}(0)$

Example Lec1.6

- Find the mgf of $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, i.e., $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$$E(e^{tX}) = \int_{-\infty}^\infty e^{tx} f_X(x) dx = \frac{\int_{-\infty}^\infty \exp\left(tx - \frac{(x-\mu)^2}{2\sigma^2}\right) dx}{\sqrt{2\pi\sigma^2}} = \frac{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty \exp\left(-\frac{(x - (\mu + \sigma^2 t))^2}{2\sigma^2}\right) dx$$

- Find the mgf of Cauchy distribution, i.e., $f_X(x) = \{\pi(1+x^2)\}^{-1}$, $x \in \mathbb{R}$

$$E\{\exp(tX)\} = \int_{-\infty}^\infty \frac{\exp(tx)}{\pi(1+x^2)} dx$$

- $\frac{1}{1+x^2}$ decreases to 0 polynomially as $x \rightarrow \infty$ or $x \rightarrow -\infty$.
- If $t > 0$, then $\exp(tx)$ grows exponentially as $x \rightarrow \infty$; if $t < 0$, then $\exp(tx)$ grows exponentially as $x \rightarrow -\infty$.
- Therefore, $\frac{\exp(tx)}{1+x^2} \rightarrow \infty$ as $x \rightarrow \infty$ when $t > 0$, and as $x \rightarrow -\infty$ when $t < 0$. The integral $E\{\exp(tx)\}$ does not converge for any nonzero t .