STAT 3690 Lecture 06

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Feb 4nd, 2022

Multivariate normal (MVN) distribution

- Standard normal random vector
  - $\mathbf{Z} = [Z_1, \ldots, Z_p]^\top \sim MVN_p(\mathbf{0}, \mathbf{I}) \Leftrightarrow Z_1, \ldots, Z_p \overset{\text{iid}}{\sim} N(0,1) \Leftrightarrow$

$$\phi_{\mathbf{Z}}(\boldsymbol{z}) = (2\pi)^{-p/2} \exp(-\boldsymbol{z}^\top \boldsymbol{z}/2), \quad \boldsymbol{z} = [z_1, \ldots, z_p]^\top \in \mathbb{R}^p$$

- (General) normal random vector
  - Def: The distribution of $\mathbf{X}$ is MVN iff there exists $q \in \mathbb{Z}^+$, $\boldsymbol{\mu} \in \mathbb{R}^q$, $\mathbf{A} \in \mathbb{R}^{q \times p}$ and $\mathbf{Z} \sim MVN_p(\mathbf{0}, \mathbf{I})$ such that $\mathbf{X} = \mathbf{A}\mathbf{Z} + \boldsymbol{\mu}$
    - Limit the discussion to non-degenerate cases, i.e., $\mathrm{rk}(\mathbf{A}) = q$
    - $\mathbf{X} \sim MVN_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, i.e.,

$$f_{\mathbf{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^q \det(\boldsymbol{\Sigma})}} \exp\{-(\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})/2\}, \quad \boldsymbol{x} \in \mathbb{R}^q$$

      - $\boldsymbol{\Sigma} = \mathrm{var}(\mathbf{X}) = \mathbf{A}\mathbf{A}^\top > 0$

- Exercise:
  1. $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^\top > 0 \Leftrightarrow \mathrm{rk}(\mathbf{A}) = q$ (Hint: SVD of $\mathbf{A}$);
  2. $\boldsymbol{\Sigma} > 0 \Rightarrow$ there exists a $p \times p$ positive definite matrix, say $\boldsymbol{\Sigma}^{1/2}$, such that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}$ and $\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{-1/2}$ (Hint: spectral decomposition of $\boldsymbol{\Sigma}$).

1. $\underset{q\times p}{A} = \underset{q\times q}{B}\,\underset{q\times p}{\Lambda}\,\underset{p\times p}{C}$, where $\Lambda = \left[\begin{array}{ccc|c} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_q & \end{array}\right]$ (SVD of A)

$\Rightarrow AA^T = B\Lambda C^T C \Lambda^T B^T$

$\quad = B\Lambda\Lambda^T B^T$,

where $\Lambda\Lambda^T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_q \end{bmatrix}$

$\therefore AA^T > 0 \Leftrightarrow \lambda_1, \cdots, \lambda_q > 0 \Leftrightarrow rk(A) = q$

2. $\Sigma = B\Lambda B^T$, where $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_q \end{bmatrix}$ (eigen-/spectral decomposition of Σ)

$\Rightarrow \Lambda^{-1} = \begin{bmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_q^{-1} \end{bmatrix}$ $(\because \Sigma > 0)$

$\Rightarrow \Sigma^{-1} = B\Lambda^{-1}B^T$ $(\because (B\Lambda^{-1}B^T)(B\Lambda B^T) = I)$

Let $\Lambda^{1/2} = \begin{bmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_q^{-1} \end{bmatrix}$, $\Lambda^{-\frac{1}{2}} = \begin{bmatrix} \lambda_1^{-\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & \lambda_q^{-\frac{1}{2}} \end{bmatrix}$

$\Sigma^{1/2} = B\Lambda^{1/2}B^T$, $\Sigma^{-\frac{1}{2}} = B\Lambda^{-\frac{1}{2}}B^T$

then $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$ and $\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}} = \Sigma^{-1}$

* Useful properties of MVN
  - $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Leftrightarrow \mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim MVN_p(\mathbf{0}, \mathbf{I})$. So, we have a stochastic representation of arbitrary $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: $\mathbf{X} = \boldsymbol{\Sigma}^{1/2}\mathbf{Z} + \boldsymbol{\mu}$, where $\mathbf{Z} \sim MVN_p(\mathbf{0}, \mathbf{I})$.
  - $\mathbf{X} \sim MVN$ iff, for all $a \in \mathbb{R}^p$, $a^\top \mathbf{X}$ has a (univariate) normal distribution.
  - If $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\mathbf{A}\mathbf{X} + \boldsymbol{b} \sim MVN_q(\mathbf{A}\boldsymbol{\mu} + \boldsymbol{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$ for $\mathbf{A} \in \mathbb{R}^{q \times p}$ and $\text{rk}(\mathbf{A}) = q$.

* Exercise: Generate six iid samples following bivariate normal $MVN_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with

$$\boldsymbol{\mu} = [3, 6]^\top, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 10 & 2 \\ 2 & 5 \end{bmatrix}.$$

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* Exercise:
  1. Prove that $(\mathbf{X} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim \chi^2(p)$ if $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
  2. Suppose $X_1 \sim N(0, 1)$ and $\mathbf{X} = [X_1, X_2]^\top$. Does $\mathbf{X}$ follow an MVN in the following two cases?
     a. $X_2 = -X_1$;
     b. $X_2 = (2Y - 1)X_1$, where $Y \sim Ber(p)$ is independent of $\mathbf{X}$.

Joint, marginal and conditional MVN

* If $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

  with $\boldsymbol{\Sigma}_{11} > 0$ and $\boldsymbol{\Sigma}_{22} > 0$, then
  - $\mathbf{X}_i \sim MVN_{p_i}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_{ii})$, i.e., marginals of MVN are MVN.
  - $\mathbf{X}_i \mid \mathbf{X}_j = \boldsymbol{x}_j \sim MVN_{p_i}(\boldsymbol{\mu}_{i|j}, \boldsymbol{\Sigma}_{i|j})$, i.e., conditionals of MVN ar MVN.
  - $\boldsymbol{\mu}_{i|j} = \boldsymbol{\mu}_i + \boldsymbol{\Sigma}_{ij}\boldsymbol{\Sigma}_{jj}^{-1}(\boldsymbol{x}_j - \boldsymbol{\mu}_j)$
  - $\boldsymbol{\Sigma}_{i|j} = \boldsymbol{\Sigma}_{ii} - \boldsymbol{\Sigma}_{ij}\boldsymbol{\Sigma}_{jj}^{-1}\boldsymbol{\Sigma}_{ji}$
  - $\mathbf{X}_i \perp \mathbf{X}_j \Leftrightarrow \boldsymbol{\Sigma}_{ij} = 0$