

# STAT 4100 Lecture Note

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Zhiyang Zhou (zhiyang.zhou@umanitoba.ca, zhiyanggeezhou.github.io)

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## Evaluating estimators (con'd)

### Completeness (CB Def 6.2.21)

- Only consider one-dimensional cases
- $T$  is a complete statistic if we have the following identity: for any (measurable) function  $g$ ,

$$E(g(T) \mid \theta) = 0 \text{ for all } \theta \in \Theta \Rightarrow \Pr(g(T) = 0 \mid \theta) = 1 \text{ for all } \theta \in \Theta.$$

- Geometrical interpretation:  $\text{span}\{f_{T|\theta}(t \mid \theta) : \theta \in \Theta\} = \{g(\cdot) : (\text{measurable}) \text{ } g \text{ is defined on } \text{supp}(T)\}$
- Bounded completeness: restricted to bounded  $g$  only
- (CB Thm 6.2.28) Minimal sufficient statistics exist  $\Rightarrow$  complete sufficient statistics are minimally sufficient
- (HMC Thm 7.5.2) iid  $X_1, \dots, X_n \sim f(x \mid \theta) = h(x)c(\theta) \exp \left\{ \sum_{i=1}^n w_i(\theta) t_i(x) \right\}$ , i.e., following the exponential family,  $\Rightarrow (\sum_{i=1}^n t_1(X_i), \dots, \sum_{i=1}^n t_k(X_i))$  is complete sufficient

### Example Lec9.2

- Find the complete statistic for the following scenarios:
  - a. iid  $X_1, \dots, X_n \sim f(x \mid \theta) = (x!)^{-1} \theta^x e^{-\theta} \mathbf{1}_{\mathbb{R}^+ \times \{0,1,\dots\}}(\theta, x)$ ;
  - b. iid  $X_1, \dots, X_n \sim \text{Unif}\{1, \dots, \theta\}$ , integer  $\theta \geq 2$ .

### Lehmann-Scheffe (CB Thm 7.3.23 & 7.5.1; HMC Thm 7.4.1)

- The unbiased estimator only depending on complete sufficient statistics is the UMVUE.
- Application to the construction of UMVUE
  1. Find the minimal sufficient  $T$ .
  2. Check the completeness of  $T$ .
  3. Find unbiased  $g(T)$ , e.g.,
    - $E(W \mid T)$  with certain unbiased  $W$
    - debiased MLE (if it is a function of  $T$ ).

### Example Lec9.3

- Suppose that iid  $X_1, \dots, X_n$  are following  $\text{Unif}\{1, \dots, \theta\}$ , integer  $\theta \geq 2$ . Prove that  $[X_{(n)}^{n+1} - (X_{(n)} - 1)^{n+1}] / [X_{(n)}^n - (X_{(n)} - 1)^n]$  is the UMVUE for  $\theta$ .

# Verifying the independence

## Ancillary Statistics

- Statistics whose distribution does not depend on unknown  $\theta$ .

### Example Lec10.1

- Verify the following statistics are ancillary for  $\theta$ .
  - Range  $X_{(n)} - X_{(1)}$  with  $X_1, \dots, X_n \sim \text{Unif}(\theta, \theta + 1)$ .
  - $X_1/X_2$  with  $X_1, X_2 \sim \mathcal{N}(0, \theta^2)$ .

### Basu's theorem (CB Thm 6.2.4)

- $T$  is complete and sufficient, while  $S$  is ancillary. Then  $T$  and  $S$  are independent of each other.
  - The completeness of  $T$  can be relaxed to be bounded completeness.

### Example Lec10.2

- Let iid  $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ . Deduce the independence of  $\bar{X}$  and  $S^2$  by applying Basu's theorem for the case with unknown  $\mu$  and known  $\sigma^2$ .

### How to verify the independence of $X$ and $Y$

- Joint cdf:  $F_{X,Y}(x, y) = F_X(x)F_Y(y)$
- Joint pdf or pmf:  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$
- conditional pdf or pmf:  $f_{X|Y}(x | y) = f_X(x)$
- mgf:  $E(e^{t_1 X + t_2 Y}) = E(e^{t_1 X})E(e^{t_2 Y})$
- cf:  $E(e^{it_1 X + it_2 Y}) = E(e^{it_1 X})E(e^{it_2 Y})$
- Basu's theorem
  - Sometimes it is even more complex to find complete statistics than to obtain the joint pdf
- Zero covariance matrix for normal cases

## Review for midterm

### Find the distribution of $Y = g(X)$ given the distribution of $X$

- First figure out  $\text{support}(Y)$
- Univariate transformation
  - For discrete  $Y$ : find the pmf of  $Y$  by definition
  - For continuous  $Y$ : by CB Ex. 2.7(b),

$$f_Y(y) = \sum_{k=1}^K f_X\{g_k^{-1}(y)\} \left| J_{g_k^{-1}}(y) \right| \mathbf{1}_{B_k}(y)$$

- \* Partition  $\text{supp}(X)$  into  $K$  intervals  $A_1, \dots, A_K$  such that
  - $\bigcup_{k=1}^K A_k = \text{supp}(X)$  and  $A_k \cap A_{k'} = \emptyset$  if  $k \neq k'$
  - $g$  is strictly monotonic and continuously differentiable on  $A_k$
- \*  $g_k = g_k(x) = g(x)$ ,  $x \in A_k$

- \* Jacobian of transformation  $g_k^{-1}$

$$J_{g_k^{-1}} = \frac{d}{dy} g_k^{-1}(y)$$

- \*  $B_k = \{g(x) : x \in A_k\}$

- Bivariate transformation

- Consider

- \*  $\mathbf{X}$  and  $\mathbf{Y}$  both two-dimensional

- \*  $\mathbf{g}(\cdot) = (g_1(\cdot), g_2(\cdot)) : \text{supp}(\mathbf{X}) \rightarrow \text{supp}(\mathbf{Y})$  is one-to-one, i.e.,

$$\cdot \mathbf{y} = (y_1, y_2) = (g_1(x_1, x_2), g_2(x_1, x_2)) = \mathbf{g}(x_1, x_2)$$

$$\cdot \mathbf{x} = (x_1, x_2) = \mathbf{g}^{-1}(y_1, y_2) = (h_1(y_1, y_2), h_2(y_1, y_2))$$

- If  $\mathbf{g}^{-1}$  is continuously differentiable,

$$f_{\mathbf{Y}}(y_1, y_2) = f_{\mathbf{X}}\{\mathbf{g}^{-1}(y_1, y_2)\} |\det\{\mathbf{J}_{\mathbf{g}^{-1}}(y_1, y_2)\}| \mathbf{1}_{\text{supp}(\mathbf{Y})}(y_1, y_2)$$

- \* Jacobian matrix of transformation  $\mathbf{g}^{-1}$

$$\mathbf{J}_{\mathbf{g}^{-1}}(y_1, y_2) = \left[ \frac{\partial h_i(y_1, y_2)}{\partial y_j} \right]_{2 \times 2} = \begin{bmatrix} \frac{\partial h_1(y_1, y_2)}{\partial y_1} & \frac{\partial h_1(y_1, y_2)}{\partial y_2} \\ \frac{\partial h_2(y_1, y_2)}{\partial y_1} & \frac{\partial h_2(y_1, y_2)}{\partial y_2} \end{bmatrix} = \mathbf{J}_{\mathbf{g}}^{-1}\{\mathbf{g}^{-1}(y_1, y_2)\}$$

$$\cdot \text{Hence } \det\{\mathbf{J}_{\mathbf{g}^{-1}}(y_1, y_2)\} = 1 / \det[\mathbf{J}_{\mathbf{g}}\{\mathbf{g}^{-1}(y_1, y_2)\}]$$

- Polar coordinate system

## Bivariate normal (BVN) distribution

- Random 2-vector  $\mathbf{X} = [X_1, X_2]^\top \sim \text{BVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Leftrightarrow \mathbf{X} = \boldsymbol{\Sigma}^{1/2} \mathbf{Z} + \boldsymbol{\mu}$  with  $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim \text{BVN}(0, \mathbf{I}_2) \Rightarrow$

$$\mathbf{E}(\mathbf{X}) = [\mathbf{E}(X_1), \mathbf{E}(X_2)]^\top = \boldsymbol{\mu} \quad \text{and} \quad \text{cov}(\mathbf{X}) = [\text{cov}(X_i, X_j)]_{2 \times 2} = \boldsymbol{\Sigma}$$

- Random 2-vector  $\mathbf{X} \sim \text{BVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Rightarrow \mathbf{B}\mathbf{X} + \mathbf{b} \sim \text{BVN}(\mathbf{B}\boldsymbol{\mu} + \mathbf{b}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^\top)$
- If  $[X_1, X_2]^\top$  is of BVN, then the marginal distributions of  $X_1$  and  $X_2$  are both normal. The inverse proposition does NOT hold.
- Normal sampling theory
  - $\sum_{i=1}^n X_i^2 \sim \chi^2(n)$  if iid  $X_1, \dots, X_n \sim \mathcal{N}(0, 1)$
  - $X/\sqrt{Y/n} \sim t(n)$  if  $X \sim \mathcal{N}(0, 1)$  and  $Y \sim \chi^2(n)$  are independent
  - $(X/m)/(Y/n) \sim F(m, n)$  if  $X \sim \chi^2(m)$  and  $Y \sim \chi^2(n)$  are independent
  - $n^{1/2}(\bar{X} - \mu)/\sigma \sim \mathcal{N}(0, 1)$  if iid  $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$
  - $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$  if iid  $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$
  - $\bar{X}$  and  $S^2$  are independent of each other if iid  $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$
  - $n^{1/2}(\bar{X} - \mu)/S \sim t(n-1)$  if iid  $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$

## Generating functions

- Univariate mgf  $M_X(t) = \mathbf{E}\{\exp(tX)\}$  if  $\mathbf{E}\{\exp(tX)\} < \infty$  for all  $t$  inside a neighbourhood of 0
  - Characterizing distributions: identical mgfs implying identical distributions
  - $M_Y(t) = \exp(bt) \prod_i M_{X_i}(a_i t)$  if  $Y = b + \sum_i a_i X_i$ , where  $b$  and  $a_i$  are constants,  $X_1, \dots, X_p$  are independent, and each  $M_{X_i}(\cdot)$  exists

## Exponential family

- If the pdf or pmf of  $X$  is of the following form

$$f(x | \boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp \left\{ \sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x) \right\}$$

- $\mathcal{N}(\mu, \sigma^2)$  (CB Example 3.4.4):
  - $h(x) = \mathbf{1}_{\mathbb{R}}(x)$
  - $c(\mu, \sigma) = (2\pi\sigma^2)^{-1/2} \exp\{-\mu^2/(2\sigma^2)\} \mathbf{1}_{\mathbb{R} \times \mathbb{R}^+}(\mu, \sigma)$
  - $w_1(\mu, \sigma) = \sigma^{-2} \mathbf{1}_{\mathbb{R}^+}(\sigma)$
  - $w_2(\mu, \sigma) = \mu \sigma^{-2} \mathbf{1}_{\mathbb{R}^+}(\sigma)$
  - $t_1(x) = -x^2/2$
  - $t_2(x) = x$
- $\text{Binom}(n, p)$  (CB Example 3.4.1):
  - $h(x) = \binom{n}{x} \mathbf{1}_{\{0, \dots, n\}}(x)$  (What happens if  $n$  is also an unknown parameter?)
  - $c(p) = (1-p)^n \mathbf{1}_{(0,1)}(p)$
  - $w_1(p) = \ln\{p/(1-p)\} \mathbf{1}_{(0,1)}(p)$
  - $t_1(x) = x$
- Other special cases of exponential family: gamma, beta, Poisson, negative binomial
- $(\sum_{i=1}^n t_1(X_i), \dots, \sum_{i=1}^n t_k(X_i))$  is sufficient complete

## Point estimation

- Method of moments (MOM)
  - Equate raw moments to their empirical counterparts (Why is it reasonable?)
  - Pros and cons
- Maximum likelihood (ML)
  - $\hat{\boldsymbol{\theta}}_{\text{ML}}$  is a statistic such that

$$\hat{\boldsymbol{\theta}}_{\text{ML}} = \arg \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} L(\boldsymbol{\theta}; \mathbf{x}) = \arg \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \ell(\boldsymbol{\theta}; \mathbf{x})$$

- Invariance property:  $\widehat{g(\boldsymbol{\theta})}_{\text{ML}} = g(\hat{\boldsymbol{\theta}}_{\text{ML}})$
- Maximizing  $L(\boldsymbol{\theta}; \mathbf{x})$  or  $\ell(\boldsymbol{\theta}; \mathbf{x})$  with respect to  $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ 
  - \* For discrete  $\boldsymbol{\Theta}$ : compare  $L(\boldsymbol{\theta}; \mathbf{x})$  or  $\ell(\boldsymbol{\theta}; \mathbf{x})$  over all the possible values of  $\boldsymbol{\theta}$
  - \* For continuous  $\boldsymbol{\Theta}$ : is a continuous set Solving simultaneous equations  $\mathbf{S}(\boldsymbol{\theta}) = \mathbf{0}$  for  $\boldsymbol{\theta}$ 
    - If  $\mathbf{S}(\boldsymbol{\theta}) = \mathbf{0}$  does not exist: utilize the monotonicity
    - If  $\mathbf{S}(\boldsymbol{\theta}) = \mathbf{0}$  exists: compare  $L(\boldsymbol{\theta}; \mathbf{x})$  or  $\ell(\boldsymbol{\theta}; \mathbf{x})$  over all the stationary points and boundary points

## Evaluating estimators

- Mean squared error (MSE):  $E(\hat{\theta} - \theta)^2 = \{E(\hat{\theta}) - \theta\}^2 + \text{var}(\hat{\theta})$ 
  - UMVUE/MVUE/Best unbiased estimator: minimize  $\text{var}(\hat{\theta})$  subject to  $E(\hat{\theta}) = \theta$
- Cramer-Rao lower bound (one-dimensional case):  $\text{var}(\hat{\theta}) \geq \{(d/d\theta)E(\hat{\theta})\}^2 / I(\theta)$ 
  - Fisher information:  $I(\theta) = \text{var}\{S(\theta; \mathbf{X})\} = E[\{S(\theta; \mathbf{X})\}^2] = -E\{H(\theta; \mathbf{X})\}$ 
    - \*  $I(\theta) = n \text{var}\{S(\theta; X_1)\} = nE[\{S(\theta; X_1)\}^2] = -nE\{H(\theta; X_1)\}$  for iid sample  $\mathbf{X} = [X_1, \dots, X_n]$
  - For unbiased estimators
    - \*  $\text{var}(\hat{\theta}) \geq 1/I(\theta)$
    - \* The unbiased estimator attaining the lower bound is UMVUE
- Alternative ways to find UMVUE

- Rao-Blackwellization with sufficient complete statistics
- Debiased MLE if the MLE is a function only based on sufficient complete statistics

**Take-home exercises (NOT to be submitted; to be potentially covered in labs)**

CB Ex. 7.46, 7.48, 7.57, 7.58, 7.66