

PH 712 Probability and Statistical Inference

Part IX: Confidence Set/Interval

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Confidence set

- Called a confidence interval (CI) If the set is indeed an interval
- $(1 - \alpha) \times 100\%$ confidence set for θ , say $C(X_1, \dots, X_n)$: $C(X_1, \dots, X_n)$ covers θ with probability $\geq (1 - \alpha) \times 100\%$, i.e., $\Pr\{\theta \in C(X_1, \dots, X_n)\} \geq (1 - \alpha) \times 100\%$
 - $C(X_1, \dots, X_n)$ is a set defined on sample X_1, \dots, X_n and hence is randomized, while θ is fixed

Construction of a confidence set by inverting a level α two-sided test

- Implementation
 1. Construct the rejection region $R(\theta_0)$ for a level α test for hypotheses $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$.
 2. $C(x_1, \dots, x_n) = \{\theta_0 : (x_1, \dots, x_n) \in \text{supp}(X_1, \dots, X_n) \setminus R(\theta_0)\}$,
 - $\text{supp}(X_1, \dots, X_n) \setminus R(\theta_0)$: the complementary set of $R(\theta_0)$.
- If $C(X_1, \dots, X_n)$ is constructed by inverting a level α test, then
 - $C(X_1, \dots, X_n)$ does NOT cover $\theta_0 \Leftrightarrow$ reject $H_0 : \theta = \theta_0$ (vs. $H_1 : \theta \neq \theta_0$) at level α
- Special cases:
 - $(1 - \alpha) \times 100\%$ asymptotic LRT confidence set for θ : $\{\theta_0 : -2(\ell(\theta_0) - \ell(\hat{\theta}_{\text{ML}})) < \chi_{1,1-\alpha}^2\}$
 - $(1 - \alpha) \times 100\%$ Wald confidence set for θ : $\{\theta_0 : |\hat{\theta}_{\text{ML}} - \theta_0| / \sqrt{\widehat{\text{var}}(\hat{\theta}_{\text{ML}})} < \Phi_{1-\alpha/2}^{-1}\}$, i.e.,

$$\left(\hat{\theta}_{\text{ML}} - \Phi_{1-\alpha/2}^{-1} \sqrt{\widehat{\text{var}}(\hat{\theta}_{\text{ML}})}, \quad \hat{\theta}_{\text{ML}} + \Phi_{1-\alpha/2}^{-1} \sqrt{\widehat{\text{var}}(\hat{\theta}_{\text{ML}})} \right)$$

Construction of a confidence set via (nonparametric) bootstrap

- Implementation
 1. Suppose you observe x_1, \dots, x_n for an iid sample of size n .
 2. Set a seed to make your result reproducible.
 3. For b in $1 : B$, do steps a–b.
 - a. Generate a bootstrap sample $x_1^{(b)}, \dots, x_n^{(b)}$ by drawing a sample of size n with replacement from $\{x_1, \dots, x_n\}$.
 - b. Generate an estimate $\hat{\theta}^{(b)}$ from $x_1^{(b)}, \dots, x_n^{(b)}$.
 4. $(q_{\alpha/2}, q_{1-\alpha/2})$ is the $(1 - \alpha)$ bootstrap confidence interval for θ , where $q_{\alpha/2}$ and $q_{1-\alpha/2}$ are $\alpha/2$ and $(1 - \alpha/2)$ sample quantiles of $\{\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}\}$, respectively.

Example Lec9.1

- Assume $X_1, \dots, X_{10} \stackrel{\text{iid}}{\sim} f(x | p) = p^x(1 - p)^{1-x}$, $x = 0, 1$, $0 < p < 1$, and observe $0, 0, 0, 1, 1, 1, 1, 1, 1, 1$. Construct a 95% confidence set for p .

```

options(digits = 4)
alpha = .05
xs = c(rep(0,3), rep(1,7))
n = length(xs)
ell = function(p, xs){
  sum(xs*log(p) + (1-xs)*log(1-p))
}
# Asymptotic LRT CI
p_ml = optim(
  par = .5, lower = 0.000001, upper = .999999,
  fn = ell, xs = xs,
  method="L-BFGS-B",
  control=list(fnscale=-1), hessian=TRUE
)$par
ell(p_ml,xs) # log-likelihood at MLE
qchisq(1-alpha, df=1)

# Wald CI
ml = optim(
  par = .5, lower = 0.000001, upper = .999999,
  fn = ell, xs = xs,
  method="L-BFGS-B",
  control=list(fnscale=-1), hessian=TRUE
)
p_ml = ml$par
fish_info = -ml$hessian
var_p_ml = 1/fish_info
ci_wald = c(
  p_ml - qnorm(1-alpha/2)*sqrt(var_p_ml),
  p_ml + qnorm(1-alpha/2)*sqrt(var_p_ml)
); ci_wald

# Bootstrap CI
set.seed(712)
B = 1e4L
p_ml_bs = numeric(B)
for (b in 1:B) {
  xbs = sample(xs, size=n, replace=TRUE)
  p_ml_bs[b] = optim(
    par = 0.5, lower = 0.000001, upper = .999999,
    fn = ell, xs = xbs,
    method="L-BFGS-B",
    control=list(fnscale=-1))$par
}
ci_boot = quantile(p_ml_bs, probs = c(alpha/2, 1-alpha/2)); ci_boot

```