STAT 3690 Lecture 07

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Useful properties of MVN

- $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Leftrightarrow \mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim MVN_p(\mathbf{0}, \mathbf{I})$. So, we have a stochastic representation of arbitrary $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: $\mathbf{X} = \boldsymbol{\Sigma}^{1/2}\mathbf{Z} + \boldsymbol{\mu}$, where $\mathbf{Z} \sim MVN_p(\mathbf{0}, \mathbf{I})$.
- $\mathbf{X} \sim MVN$ iff, for all $a \in \mathbb{R}^p$, $a^\top \mathbf{X}$ has a (univariate) normal distribution.
- If $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\mathbf{AX} + \boldsymbol{b} \sim MVN_q(\mathbf{A}\boldsymbol{\mu} + \boldsymbol{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$ for $\mathbf{A} \in \mathbb{R}^{q \times p}$ and $\mathrm{rk}(\mathbf{A}) = q$.

- Exercise: Generate six iid samples following bivariate normal $MVN_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with

$$\boldsymbol{\mu} = [3, 6]^\top, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 10 & 2 \\ 2 & 5 \end{bmatrix}.$$

- Exercise:
  1. Prove that $(\mathbf{X} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim \chi^2(p)$ if $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
  2. Suppose $X_1 \sim N(0, 1)$. In the following two cases, verify that $X_2 \sim N(0, 1)$ as well. Does $\mathbf{X} = [X_1, X_2]^\top$ follow an MVN in both cases?
     a. $X_2 = -X_1$;
     b. $X_2 = (2Y - 1)X_1$, where $Y \sim Ber(p)$ and $\mathbf{Y} \perp\!\!\!\perp \mathbf{X}$.
        - P.S.: $\mathbf{Y} \perp\!\!\!\perp \mathbf{X} \Leftrightarrow f_{\mathbf{Z}}(\boldsymbol{z}) = f_{\mathbf{X}}(\boldsymbol{x}) f_{\mathbf{Y}}(\boldsymbol{y})$, where $\mathbf{Z} = [\mathbf{X}^\top, \mathbf{Y}^\top]^\top$

Joint, marginal and conditional MVN

- If $\mathbf{X} \sim MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

  with $\boldsymbol{\Sigma}_{11} > 0$ and $\boldsymbol{\Sigma}_{22} > 0$, then
  - $\mathbf{X}_i \sim MVN_{p_i}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_{ii})$, i.e., marginals of MVN are MVN.
  - $\mathbf{X}_i \mid \mathbf{X}_j = \boldsymbol{x}_j \sim MVN_{p_i}(\boldsymbol{\mu}_{i|j}, \boldsymbol{\Sigma}_{i|j})$, i.e., conditionals of MVN are MVN.
  - $\boldsymbol{\mu}_{i|j} = \boldsymbol{\mu}_i + \boldsymbol{\Sigma}_{ij}\boldsymbol{\Sigma}_{jj}^{-1}(\boldsymbol{x}_j - \boldsymbol{\mu}_j)$
  - $\boldsymbol{\Sigma}_{i|j} = \boldsymbol{\Sigma}_{ii} - \boldsymbol{\Sigma}_{ij}\boldsymbol{\Sigma}_{jj}^{-1}\boldsymbol{\Sigma}_{ji}$
  - $\mathbf{X}_1 \perp\!\!\!\perp \mathbf{X}_2 \Leftrightarrow \boldsymbol{\Sigma}_{12} = \mathbf{0}$

- Exercise: The argument $\mathbf{X}_1 \perp\!\!\!\perp \mathbf{X}_2 \Leftrightarrow \boldsymbol{\Sigma}_{12} = 0$ is based on the assumption that $\mathbf{X} = [\mathbf{X}_1^\top, \mathbf{X}_2^\top]^\top$ is of MVN. That is, if $\mathbf{X}_1$ and $\mathbf{X}_2$ are both MVN BUT they are not jointly normal, a zero $\boldsymbol{\Sigma}_{12}$ doesn't suffice for the independence between $\mathbf{X}_1$ and $\mathbf{X}_2$. A counter-example will be part of Assignment 1.

Checking normality (J&W Sec 4.6)

- Checking the univariate marginal distributions
  - Normal Q-Q plot
    - qqnorm(); car::qqPlot()
  - Normality test
    - shapiro.test()
- Checking the quadratic form
  - χ^2 Q-Q plot
    - $D_i^2 = (\mathbf{X}_i - \bar{\mathbf{X}})^\top \mathbf{S}^{-1}(\mathbf{X}_i - \bar{\mathbf{X}}) \approx \chi^2(p)$ if $\mathbf{X}_i \overset{\text{iid}}{\sim} MVN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
    - qqplot(); car::qqPlot()

Detecting outliers (J&W Sec 4.7)

- Scatter plot of standardized values
- Check the points farthest from the origin in χ^2 Q-Q plot

Improving normality (J&W Sec 4.8)

- Box-cox transformation: for $x > 0$,

$$x^*(\lambda) = \begin{cases} (x^\lambda - 1)/\lambda & \lambda \neq 0 \\ \ln(x) & \lambda = 0 \end{cases}$$

  - If $x \leq 0$, change it to be positive first.
- Exploratory data analysis (EDA)
  - J. Tukey (1977). Exploratory Data Analysis. Addison-Wesley. ISBN 978-0-201-07616-5.

R package "MVN"