STAT 4100 Lecture Note

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Evaluating estimators

Mean squared error (MSE)

- Univariate: $\mathrm{E}(\hat{\theta}-\theta_0)^2 = \{\mathrm{E}(\hat{\theta})-\theta_0\}^2 + \mathrm{var}(\hat{\theta})$
- Multivariate: $\mathrm{E}\{(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0)^\top(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0)\} = \{\mathrm{E}(\hat{\boldsymbol{\theta}})-\boldsymbol{\theta}_0\}^\top\{\mathrm{E}(\hat{\boldsymbol{\theta}})-\boldsymbol{\theta}_0\} + \mathrm{cov}(\hat{\boldsymbol{\theta}})$
- Best unbiased estimator (i.e., (uniform) minimun variance unbiased estimator, abbr. UMVUE/MVUE): if $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\mathbf{X})$ satisfies that
  - $\hat{\boldsymbol{\theta}}$ is unbiased for $\boldsymbol{\theta}$, i.e., $\mathrm{E}(\hat{\boldsymbol{\theta}}) = \boldsymbol{\theta}$
  - $\mathrm{var}(\hat{\boldsymbol{\theta}}) - \mathrm{var}(\hat{\boldsymbol{\theta}}^*) \le 0$ for all unbiased $\hat{\boldsymbol{\theta}}^*$
- UMVUE is unique (CB Thm 7.3.19)

Cramer-Rao lower bound (CB Thm 7.3.9 & Lemma 7.3.11)

- Only consider the univariate case, i.e., one-dimensional unknown parameter θ
  - Score function $S(\theta;\mathbf{X})$ and Hessian $H(\theta;\mathbf{X})$ both scalar
- Cramer-Rao lower bound: $\mathrm{var}(\hat{\theta}) \ge \{(\mathrm{d}/\mathrm{d}\theta)\mathrm{E}(\hat{\theta})\}^2/I(\theta)$ for $\hat{\boldsymbol{\theta}}$ satisfying regularity conditions
  - Fisher information: $I(\theta) = \mathrm{var}(S(\theta;\mathbf{X})) = \mathrm{E}[\{S(\theta;\mathbf{X})\}^2] = -\mathrm{E}\{H(\theta;\mathbf{X})\}$
- (CB Coro 7.3.15) $\hat{\theta}$ attains the lower bound $\Leftrightarrow$ there is $a(\theta)$ such that $S(\theta;\mathbf{X}) = a(\theta)\{\hat{\theta} - \mathrm{E}(\hat{\theta})\}$
- The unbiased $\hat{\theta}$ attaining the lower bound is UMVUE

Example Lec8.1

- Find the lower bound for unbiased estimators for σ^2 in the following cases.
  a. iid $X_1,\ldots,X_n \sim \mathcal{N}(\mu_0,\sigma^2)$ with known μ_0 and unknown σ^2.
  b. iid $X_1,\ldots,X_n \sim \mathcal{N}(\mu,\sigma^2)$ with unknown (μ,σ^2).

Sufficiency (CB Sec 6.2.1)

- A statistic $\mathbf{T} = \mathbf{T}(\mathbf{X})$ is sufficient for $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_p)$ $\Leftrightarrow$ the distribution of $\mathbf{X}$ conditioning on $\mathbf{T}$ and $\boldsymbol{\theta}$, say $f_{\mathbf{X}|\mathbf{T},\boldsymbol{\theta}}(\boldsymbol{x} \mid \boldsymbol{t},\boldsymbol{\theta})$, is free of $\boldsymbol{\theta}$.

- Fisher-Neyman factorization theorem (CB Thm 6.2.6; HMC Thm 7.2.1): $\mathbf{T}$ is sufficient for $\boldsymbol{\theta} \Leftrightarrow$ the likelihood function can be factored into two parts, one of them not depending on $\boldsymbol{\theta}$, i.e.,

$$L(\boldsymbol{\theta}; \boldsymbol{x}) = f_{\mathbf{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) = h(\boldsymbol{x}) g(\mathbf{T}(\boldsymbol{x}), \boldsymbol{\theta}), \text{for all } \boldsymbol{x} \text{ and } \boldsymbol{\theta}$$

- (HMC Thm 7.3.2) If $\mathbf{T}$ is sufficient for θ and $\hat{\theta}$ is the unique MLE of θ, then $\hat{\theta}$ must be a function of $\mathbf{T}$.

- Nonuniqueness of sufficient statistics
  - Trivial examples
    - $\mathbf{X}$ is always sufficient.
    - $(X_{(1)}, \ldots, X_{(n)})$ is always sufficient if X_i's are iid, with $X_{(1)} \leq \cdots \leq X_{(n)}$.
  - $\mathbf{T}$ is sufficient and $g(\cdot)$ is a one-to-one mapping $\Rightarrow g(\mathbf{T})$ is also sufficient.
- Minimal sufficiency: a sufficient statistc that is a function of all the other sufficient statistics.
  - (CB Thm 6.2.13) How to find a minimal sufficient sufficient statistic:
    1. Find the sufficient and necessary condition for $L(\boldsymbol{\theta}; \boldsymbol{x})/L(\boldsymbol{\theta}; \boldsymbol{y})$ to be free of $\boldsymbol{\theta}$;
    2. If the above condition is of the form $\mathbf{T}(\boldsymbol{x}) = \mathbf{T}(\boldsymbol{y})$, then $\mathbf{T}(\mathbf{X})$ is a minimal sufficient statistic for $\boldsymbol{\theta}$.

Example Lec8.2

- Find the minimal sufficient statistics in the following scenarios.
  a. iid $X_1, \ldots, X_n \sim \text{Unif}\{1, \ldots, \theta\}$ with unknown positive integer θ.
  b. iid $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ with unknown μ and σ^2.

Rao-Blackwellization (CB Thm 7.3.17)

- Only consider one-dimensional cases
- Improve the variance of statistic W, an estimator of θ: take use of $\mathrm{E}(W \mid T)$ (a function of T only) instead with sufficient T
- T sufficient for $\theta \Rightarrow \mathrm{E}(W \mid T, \theta) = \mathrm{E}(W \mid T) \Rightarrow \mathrm{var}\{\mathrm{E}(W \mid T, \theta) \mid \theta\} = \mathrm{var}\{\mathrm{E}(W \mid T) \mid \theta\} \leq \mathrm{var}(W \mid \theta)$ for all $\theta \in \Theta$
  - No impact on the bias
  - Not working if W is already a function of T
    - (HMC Thm 7.3.2) MLE is usually a function of (non-trivial) sufficient statitics

Example Lec9.1

- Improve statistic W in terms of variance.
  a. $W = X_1$, where iid $X_1, X_2 \sim \mathcal{N}(\theta, 1)$ with unknown θ.
  b. $W = 2X_1 - X_2$, where iid $X_1, X_2 \sim f(x \mid \theta) = \theta^{-1} \exp(-x\theta^{-1}) \mathbf{1}_{\mathbb{R}^+ \times \mathbb{R}^+}(x, \theta)$ with unknown θ.

Completeness (CB Def 6.2.21)

- Only consider one-dimensional cases
- T is a complete statistic if we have the following identity: for any (measurable) function g,

$$\mathrm{E}(g(T) \mid \theta) = 0 \text{ for all } \theta \in \Theta \Rightarrow \Pr(g(T) = 0 \mid \theta) = 1 \text{ for all } \theta \in \Theta.$$

  - Geometrical interpretation: $\mathrm{span}\{f_{T|\theta}(t \mid \theta) : \theta \in \Theta\} = \{g(\cdot) : \text{(measurable) } g \text{ is defined on } \mathrm{supp}(T)\}$
  - Bounded completeness: restricted to bounded g only
- (CB Thm 6.2.28) Minimal sufficient statistics exist $\Rightarrow$ complete sufficient statistics are minimally sufficient

- (HMC Thm 7.5.2) iid $X_1, \dots, X_n \sim f(x \mid \boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp\left\{\sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x)\right\}$, i.e., following the exponential family, $\Rightarrow (\sum_{i=1}^n t_1(X_i), \dots, \sum_{i=1}^n t_k(X_i))$ is both sufficient and complete

Example Lec9.2

- Find the complete statistic for iid $X_1, \dots, X_n \sim f(x \mid \theta) = (x!)^{-1}\theta^x e^{-\theta}\mathbf{1}_{\mathbb{R}^+ \times \{0,1,\dots\}}(\theta, x)$.

Lehmann-Scheffe (CB Thm 7.3.23 & 7.5.1; HMC Thm 7.4.1)

- The unbiased estimator only depending on complete sufficient statistics is the UMVUE.
- Application to the construction of UMVUE
  1. Find the minimal sufficient T.
  2. Check the completeness of T.
  3. Find unbiased $g(T)$, e.g.,
     - $\mathrm{E}(W \mid T)$ with certain unbiased W
     - debiased MLE (if it is a funtion of T).

Example Lec9.3

- iid $X_1, \dots, X_n \sim \mathrm{Unif}\{1, \dots, \theta\}$, integer $\theta \geq 2$.
  a. Find the complete statistic for θ.
  b. Prove that $[X_{(n)}^{n+1} - (X_{(n)} - 1)^{n+1}]/[X_{(n)}^n - (X_{(n)} - 1)^n]$ is the UMVUE for θ.

Verifying the independence

Ancillary Statistics

- Statistics whose distribution does not depend on unknown θ.

Example Lec10.1

- Verify the following statistics are ancillary for θ.
  a. Range $X_{(n)} - X_{(1)}$ with $X_1, \dots, X_n \sim \mathrm{Unif}(\theta, \theta + 1)$.
  b. X_1/X_2 with $X_1, X_2 \sim \mathcal{N}(0, \theta^2)$.

Basu's theorem (CB Thm 6.2.4)

- T is complete and sufficient, while S is ancillary. Then T and S are independent of each other.
  - The completeness of T can be relaxed to be bounded completeness.

Example Lec10.2

- Let iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Deduce the independence of $\bar{X}$ and S^2 by applying Basu's theorem for the case with unknown μ and known σ^2.

Summary on how to verify the independence of X and Y

- Joint cdf: $F_{X,Y}(x, y) = F_X(x)F_Y(y)$
- Joint pdf or pmf: $f_{X,Y}(x, y) = f_X(x)f_Y(y)$
- conditional pdf or pmf: $f_{X|Y}(x \mid y) = f_X(x)$
- mgf: $\mathrm{E}(e^{t_1X+t_2Y}) = \mathrm{E}(e^{t_1X})\mathrm{E}(e^{t_2Y})$

- cf: $\mathrm{E}(e^{it_1X+it_2Y}) = \mathrm{E}(e^{it_1X})\mathrm{E}(e^{it_2Y})$
- Basu's theorem
  - Sometimes it is even more complex to find complete statisitcs than to obtain the joint pdf
- Zero covariance matrix for normal cases