

# STAT 3100 Lecture Note

Week Eight (Oct 25 & 27, 2022)

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## Hypothesis Testing (con'd)

**UMP level  $\alpha$  test for one-sided hypotheses ( $H_0 : \theta^* \leq \theta_0$  (or  $\theta^* = \theta_0$ ) vs  $H_1 : \theta^* > \theta_0$ )**

- Consider cases with only one unknown parameter
- Monotone likelihood ratio (MLR, CB Def 8.3.16): for each pair  $\theta_2 > \theta_1$ ,  $f(t | \theta_2)/f(t | \theta_1)$  is nondecreasing with respect to  $t$  for univariate pdfs/pmfs  $\{f(t | \theta) : \theta \in \Theta \subset \mathbb{R}\}$ 
  - One-parameter exponential family with  $w(\theta)$  nondecreasing w.r.t.  $\theta$  bears MLR (why?)

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- Karlin-Rubin (CB Thm 8.3.17): Suppose  $T$  is sufficient for  $\theta$  and  $T$  follows  $f_T(t | \theta)$  bearing MLR. A UMP level  $\alpha$  test for  $H_0 : \theta^* \leq \theta_0$  (or  $\theta^* = \theta_0$ ) vs.  $H_1 : \theta^* > \theta_0$  is

$$\phi_\lambda(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) > \lambda, \\ 0, & T(\mathbf{x}) < \lambda, \end{cases}$$

where  $\lambda$  is a real number such that  $\beta_\phi(\theta_0) = E\{\phi_\lambda(\mathbf{X}) | \theta^* = \theta_0\} = \Pr\{T(\mathbf{X}) > \lambda | \theta^* = \theta_0\} = \alpha$ .

- NOTE: in the Karlin-Rubin theorem, if the hypotheses become  $H_0 : \theta^* \geq \theta_0$  (or  $\theta^* = \theta_0$ ) vs.  $H_1 : \theta^* < \theta_0$ , then change the signs in the test function, i.e.,

$$\phi_\lambda(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) < \lambda, \\ 0, & T(\mathbf{x}) > \lambda, \end{cases}$$

where  $\lambda$  is a real number such that  $\beta_\phi(\theta_0) = \Pr\{T(\mathbf{X}) < \lambda | \theta^* = \theta_0\} = \alpha$ .

### Example Lec14.1

- iid  $X_1, \dots, X_n \sim \mathcal{N}(\mu, 1)$ . Construct UMP level  $\alpha$  test for following hypotheses.
  - a.  $H_0 : \mu = \mu_0$  vs  $H_1 : \mu = \mu_1$  with  $\mu_0 < \mu_1$ ;
  - b.  $H_0 : \mu = \mu_0$  vs  $H_1 : \mu > \mu_0$ ;
  - c.  $H_0 : \mu \geq \mu_0$  vs  $H_1 : \mu < \mu_0$ ;
  - d.  $H_0 : \mu = \mu_0$  vs  $H_1 : \mu \neq \mu_0$ .

**Nonexistence of UMP test for two-sided hypotheses  $H_0 : \theta^* = \theta_0$  vs  $H_1 : \theta^* \neq \theta_0$**

- (Optional) uniformly most powerful unbiased (UMPU) level  $\alpha$  test

## Likelihood ratio test (LRT, Sec 8.2.1 & 10.3.1)

- $H_0 : \theta^* \in \Theta_0$  vs.  $H_1 : \theta^* \in \Theta_1$
- $\Theta = \Theta_0 \cup \Theta_1$
- Test statistic

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta | \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta | \mathbf{x})} = \frac{L(\hat{\theta}_{0,ML} | \mathbf{x})}{L(\hat{\theta}_{ML} | \mathbf{x})}$$

- $\hat{\theta}_{0,ML}$ : (constrained) MLE for  $\theta \in \Theta_0$
- $\hat{\theta}_{ML}$ : MLE for  $\theta \in \Theta$

- Rejection region

$$R = \{\mathbf{x} : \lambda(\mathbf{x}) \leq c\},$$

where  $c$  is chosen to make sure the size is  $\alpha$ , i.e.,

$$\sup_{\theta \in \Theta_0} \beta_\phi(\theta) = \sup_{\theta \in \Theta_0} \Pr\{\lambda(\mathbf{X}) \leq c | \theta\} = \alpha.$$

- Asymptotic rejection region (CB Thm 10.3.3)

$$R = \{\mathbf{x} : -2 \ln \lambda(\mathbf{x}) \geq \chi_{\nu, 1-\alpha}^2\} = \{\mathbf{x} : \lambda(\mathbf{x}) \leq \exp(-\chi_{\nu, 1-\alpha}^2/2)\},$$

where  $\chi_{\nu, 1-\alpha}^2$  is the  $1 - \alpha$  quantile of  $\chi^2(\nu)$ .

- (CB Thm 10.3.1) Because, asymptotically (i.e., as  $n \rightarrow \infty$ ), under  $H_0$ ,

$$-2 \ln \lambda(\mathbf{X}) \xrightarrow{d} \chi^2(\nu),$$

where  $\nu$  = the difference of numbers of free parameters between  $\Theta_0$  and  $\Theta$ .

- (CB Ex. 8.24) For simple hypotheses, is the LRT equivalent to the UMP test?

## Example Lec14.3

- iid  $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ . Test  $H_0 : \mu \leq \mu_0$  vs.  $H_1 : \mu > \mu_0$ .
  - a.  $\sigma^2$  is known. Suppose test  $\phi$  has rejection region  $\{\mathbf{x} : \bar{x} > \mu_0 + z_{1-\alpha} \sqrt{\sigma^2/n}\}$ , where  $z_{1-\alpha}$  is the  $(1 - \alpha)$  quantile of standard normal. Show that  $\phi$  is a UMP level  $\alpha$  test and is equivalent to the LRT.
  - b.  $\sigma^2$  is unknown. Suppose test  $\phi$  has rejection region  $\{\mathbf{x} : \bar{x} > \mu_0 + t_{n-1, 1-\alpha} \sqrt{s^2/n}\}$ , where  $t_{n-1, 1-\alpha}$  is the  $(1 - \alpha)$  quantile of  $t(n - 1)$ . Show that  $\phi$  is of size  $\alpha$  and is equivalent to the LRT.

## Take-home exercises (NOT to be submitted; to be potentially covered in labs)

CB Ex 8.2, 8.6(a–b), 8.16, 8.28, 8.33, 8.41