

STAT 4100 Lecture Note

Week Ten (Nov 14, 16, & 18, 2022)

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Important inequalities

Markov's inequality (CB Lemma 3.8.3 & HMC Thm 1.10.2)

- If $\Pr(X \geq 0) = 1$ and EX^k exists, then, for all $r, k > 0$,

$$\Pr(X \geq r) \leq EX^k / r^k.$$

Chebyshev's inequality (CB Thm 3.6.1 & Example 3.6.2 & HMC Thm 1.10.3)

- A corollary of Markov's inequality
- Let $X \sim (\mu_X, \sigma_X^2)$. Then, for each $r > 0$,

$$\Pr\{|X - \mu_X| \geq r\sigma_X\} = \Pr\{(X - \mu_X)^2 / \sigma_X^2 \geq r^2\} \leq r^{-2}.$$

Cauchy-Schwarz inequality (CB Thm 4.7.3)

- X and Y are both r.v.s. Then $|E(XY)| \leq E|XY| \leq \sqrt{EX^2}\sqrt{EY^2}$.
 - Because

$$\frac{X^2}{EX^2} + \frac{Y^2}{EY^2} \geq \frac{2|XY|}{\sqrt{EX^2}\sqrt{EY^2}}$$

Convexity and concavity

- Convex set: for any two points in the set, the whole line segment that joins them is also in the set
- Let \mathcal{D} be a convex set. Then real-valued function $f : \mathcal{D} \rightarrow \mathbb{R}$ is convex $\iff f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$ for all $x_1, x_2 \in \mathcal{D}$ and all $\lambda \in [0, 1]$.
 - If f is twice-differentiable, then f is convex (on \mathcal{D}) $\iff f''(x) \geq 0$ for each $x \in \mathcal{D}$.
- f is concave $\iff -f$ is convex.

Example Lec17.1

- Check the convexity/concavity of following functions.
 - a. $f(x) = \exp(x)$, $x \in \mathbb{R}$.
 - b. $f(x) = \ln x$, $x \in \mathbb{R}^+$.
 - c. $f(x) = x^2$, $x \in \mathbb{R}$.
 - d. $f(x) = x^{-1}$, $x \in \mathbb{R} \setminus \{0\}$.
 - e. $f(x) = x^{-2}$, $x \in \mathbb{R} \setminus \{0\}$.

Jensen's inequality (CB Thm 4.7.7 & HMC Thm 1.10.5)

- If f is convex on (a, b) and $EX \in (a, b)$, then

$$E\{f(X)\} \geq f(EX).$$

Example Lec17.2

- Let X be a positive random variable, i.e., $\Pr(X > 0) = 1$. Argue that
 - a. $E(-\ln X) \geq \ln(1/EX)$;
 - b. $EX^3 \geq (EX)^3$.

Convergence of random variables

Definitions

- Convergence in probability (CB Def 5.5.1), say $X_n \xrightarrow{P} X$: for each $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| < \varepsilon) = 1, \quad \text{or equivalently,} \quad \lim_{n \rightarrow \infty} \Pr(|X_n - X| > \varepsilon) = 0.$$

- Almost sure convergence (CB Def 5.5.6), say $X_n \xrightarrow{\text{a.s.}} X$:

$$\Pr\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1$$

or equivalently, for each $\varepsilon > 0$,

$$\Pr\left(\lim_{n \rightarrow \infty} |X_n - X| < \varepsilon\right) = 1$$

- Convergence in distribution (CB Def 5.5.10), say $X_n \xrightarrow{d} X$: for each x with $\Pr(X = x) = 0$,

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x);$$

or equivalently (and optionally), for each $t \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} E\{\exp(itX_n)\} = E\{\exp(itX)\};$$

or equivalently (and optionally), for each f continuous and bounded within $\text{supp}(X)$,

$$\lim_{n \rightarrow \infty} E\{f(X_n)\} = E\{f(X)\}.$$

- For the third equivalent statement, the boundedness of f is essential. Hence the convergence in distribution doesn't imply the convergence of moments; see CB Example 10.1.10.

Example Lec18.1 (optional)

- Assume that $X(s) = 0$ for all $s \in [0, 1]$ and

$$X_n(s) = \begin{cases} 1, & s \in [\frac{n}{2^{\lfloor \log_2 n \rfloor}} - 1, \frac{n+1}{2^{\lfloor \log_2 n \rfloor}} - 1] \\ 0, & \text{elsewhere.} \end{cases}$$

Then the convergence of X_n to X is in probability but NOT almost surely.

CB Example 5.5.11

- (Limiting distribution of the maximum of uniforms) if iid X_1, \dots, X_n follow $\text{Unif}(0, 1)$, then $n(1 - X_{(n)}) \xrightarrow{d} \text{exponential}(1)$.

Connections

- The chain of implications

$$\xrightarrow{\text{a.s.}} \Rightarrow \xrightarrow{p} \Rightarrow \xrightarrow{d}$$

- (Continuous mapping theorem, CMT) $h(\cdot)$ is continuous and $X_n \xrightarrow{\text{a.s./}p/d} X \Rightarrow h(X_n) \xrightarrow{\text{a.s./}p/d} h(X)$.
- (CB Thm 5.5.13 and Exercise 5.41) $X_n \xrightarrow{d} \text{constant } c \Rightarrow X_n \xrightarrow{p} c$.

- $X_n \xrightarrow{\text{a.s./}p} X$ and $Y_n \xrightarrow{\text{a.s./}p} Y \Rightarrow$
 - $aX_n + bY_n \xrightarrow{\text{a.s./}p} aX + bY$
 - $X_n Y_n \xrightarrow{\text{a.s./}p} XY$
- (Slutsky's theorem, CB Thm 5.5.17) $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} \text{constant } c \Rightarrow$
 - $aX_n + bY_n \xrightarrow{d} aX + bc$
 - * The requirement that $Y_n \xrightarrow{d} \text{constant } c$ is important. Otherwise, consider the following counterexample: assuming $X_n, X, Y \sim \text{Unif}(-1, 1)$ and $Y_n = -X_n$, one may find that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ but $X_n + Y_n$ does not converge to $X + Y$ in distribution.
 - $X_n Y_n \xrightarrow{d} cX$

CB Exercise 5.43

- $\sqrt{n}(X_n - c) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \Rightarrow X_n \xrightarrow{p} c$.

Laws of large numbers (LLN, CB Thm 5.5.2 & 5.5.9)

- If X_1, X_2, \dots are iid with finite mean μ , then
 - (Weak law of large numbers, WLLN) $\bar{X}_n \xrightarrow{p} \mu$;
 - * Proof using Chebyshev's inequality (if assuming finite variance as well)
 - (Strong law of large numbers, SLLN) $\bar{X}_n \xrightarrow{\text{a.s.}} \mu$.

Central limit theorem (CLT)

- (CB Thm 5.5.15) if X_1, \dots, X_n are iid with finite mean μ and finite variance σ^2 , then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} \mathcal{N}(0, 1).$$

- A generic approximation to the distribution of \bar{X}_n
- (CB Example 5.5.18) assuming conditions for CLT and that $T_n \xrightarrow{d} \sigma > 0$,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{T_n} \xrightarrow{d} \mathcal{N}(0, 1).$$

- by the CMT, CLT and Slutsky's theorem

CB Exercise 5.35 (optional)

- Derive the Stirling's formula: $n! \approx n^{n+1/2} \exp(-n) \sqrt{2\pi}$.

Asymptotic properties of MLE

Consistency (or consistence, CB Sec 10.1.1)

- $T_n = T_n(X_1, \dots, X_n)$ is consistent for θ iff $T_n \xrightarrow{P} \theta$ as $n \rightarrow \infty$
 - A sufficient condition for consistency: $E(T_n | \theta) \rightarrow \theta$ and $\text{var}(T_n | \theta) \rightarrow 0$ as $n \rightarrow \infty$

CB Example 5.5.3

- Suppose that iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Prove that
 - $S_n^2 = (n-1)^{-1} \sum_i (X_i - \bar{X}_n)^2$ is consistent for σ^2 ;
 - $\widehat{\sigma}_{\text{ML}}^2 = n^{-1} \sum_i (X_i - \bar{X}_n)^2$ is consistent for σ^2 too.

Consistency of MLE (univariate case, CB Thm 10.1.6)

- $\hat{\theta}_{\text{ML}} \xrightarrow{P} \theta$ and $h(\hat{\theta}_{\text{ML}}) \xrightarrow{P} h(\theta)$ with continuous function h under regularity conditions (CB Sec 10.6.2):
 - X_1, \dots, X_n are iid;
 - If $\theta_1 \neq \theta_2$, then $f_X(x | \theta_1) \neq f_X(x | \theta_2)$;
 - $f_X(x | \theta)$ has a common support for each $\theta \in \Theta$ and is differentiable with respect to θ ;
 - * Violated by, e.g., $\text{Unif}(0, \theta)$;
 - true θ is an interior point of parameter space Θ .

Asymptotic distribution of MLE (CB Thm 10.1.2)

- $\sqrt{n}(\hat{\theta}_{\text{ML}} - \theta) \xrightarrow{d} \mathcal{N}(0, 1/I(\theta))$, where

$$I(\theta) = -E \left\{ \frac{\partial^2}{\partial \theta^2} \ln L(\theta; X_i) \mid \theta \right\},$$

under the previous four regularity conditions plus two more (CB Sec 10.6.2):

- For each $x \in \text{supp}(X)$, $f(x | \theta)$ is three times continuously differentiable with respect to θ ; and $\int f(x | \theta) dx$ can be differentiated three times under the integral sign;
- for each $\theta_0 \in \Theta$, there exists $c > 0$ and $M(x)$ (with finite mean) such that $|\frac{\partial^3}{\partial \theta^3} \ln f_X(x | \theta)| \leq M(x)$ for all $x \in \text{supp}(X)$ and $\theta \in (\theta_0 - c, \theta_0 + c)$.
- In practise,
 - $I(\theta) = n^{-1} I_n(\theta) \approx n^{-1} I_n(\hat{\theta}_{\text{ML}}) \approx n^{-1} \hat{I}_n(\hat{\theta}_{\text{ML}})$
 - * Expected (Fisher) information (number) $I_n(\theta) = -E \left\{ \frac{\partial^2}{\partial \theta^2} \ln L(\theta; \mathbf{X}) \mid \theta \right\}$
 - * Observed (Fisher) information (number) $\hat{I}_n(\hat{\theta}_{\text{ML}}) = -\frac{\partial^2}{\partial \theta^2} \ln L(\theta; \mathbf{X}) \Big|_{\theta=\hat{\theta}_{\text{ML}}}$
 - Hence $\text{var}(\hat{\theta}_{\text{ML}}) \approx 1/I_n(\theta) \approx 1/I_n(\hat{\theta}_{\text{ML}}) \approx 1/\hat{I}_n(\hat{\theta}_{\text{ML}})$

Take-home exercises (NOT to be submitted; to be potentially covered in labs)