

# PH 712 Probability and Statistical Inference

## Part V: Point Estimation I

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### Recall types of the stochastic model

- Stochastic model: the distribution of RVs of interest
  - Parametric model
  - Non-parametric model
  - Semi-parametric model

### Parametric model

- A set of pdfs/pmf indexed by  $p$ -dimensional unknown  $\theta$  (constrained in  $\Theta$ ) with small or moderate dimension  $p$ , i.e.,  $\{f(\cdot | \theta) : \theta \in \Theta \subset \mathbb{R}^p\}$  with
  - $f(\cdot | \theta)$ : either a pdf or a pmf
  - $\Theta$ : the set of allowed values of  $\theta$
- True parameters, say  $\theta_0$ , believed to be fixed (frequentist statistics)
  - Rather than randomizing  $\theta_0$  (Bayesian statistics)
- Estimator: a statistic (i.e., a function of the sample); a guess about  $\theta_0$
- Estimate: plugging the realization into the estimator
- $p = 1$  hereafter, i.e., considering only one unknown parameter

### Maximum Likelihood Estimator (mle, CB Sec 7.2.2)

- Likelihood:
  - a real-valued function of unknown  $\theta$

$$L(\theta) = L(\theta; x_1, \dots, x_n) = f_{X_1, \dots, X_n}(x_1, \dots, x_n | \theta), \quad \theta \in \Theta$$

- $f_{X_1, \dots, X_n}$ : the joint pdf or pmf of  $X_1, \dots, X_n$
- Log-likelihood: the natural logarithm of likelihood

$$\ell(\theta) = \ell(\theta; x_1, \dots, x_n) = \ln L(\theta; x_1, \dots, x_n), \quad \theta \in \Theta$$

- If  $\hat{\theta}(x_1, \dots, x_n)$  is the maximizer of  $L(\theta)$  (equiv. the maximizer of  $\ell(\theta)$ ) with respect to  $\theta$  constrained in  $\Theta$ , i.e.,

$$\hat{\theta}(x_1, \dots, x_n) = \arg \max_{\theta \in \Theta} L(\theta) = \arg \max_{\theta \in \Theta} \ell(\theta),$$

then the statistic  $\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)$  is the mle for  $\theta$ .

- Invariance (CB Thm 7.2.10): if  $\hat{\theta}$  is the mle of  $\theta$ , then  $g(\hat{\theta})$  is the mle of  $g(\theta)$  for any given function  $g(\cdot)$ .

## How to locate the mle constrained in $\Theta$ ?

- If  $\ell(\theta)$  is monotonic with respect to  $\theta \in \Theta$ , then the mle lies at one boundary point of  $\Theta$
- If  $\ell(\theta)$  is non-monotonic but differentiable with respect to  $\theta \in \Theta$ , then
  1. Collect all the candidates:
    - Stationary points, i.e., solutions to the equation  $S(\theta) = 0$  subject to  $\theta \in \Theta$ 
      - \*  $S(\theta) = S(\theta; x_1, \dots, x_n) = \ell'(\theta; x_1, \dots, x_n)$ : the score
    - Boundary points of  $\Theta$
  2. Compare the values of log-likelihood or likelihood evaluated at all the above candidates

## Example Lec5.1

- Suppose  $X_1, \dots, X_n$  are iid as the following distributions. Find the mles for corresponding parameters.
  - a.  $\mathcal{N}(\mu, \sigma^2)$ , with unknown  $\mu \in \mathbb{R}$  and known  $\sigma \in \mathbb{R}^+$ .
  - b.  $\mathcal{N}(\mu, \sigma^2)$ , with known  $\mu \in \mathbb{R}$  and unknown  $\sigma \in \mathbb{R}^+$ .
  - c.  $Bern(\theta)$ :  $p(x | \theta) = \theta^x (1 - \theta)^{1-x} \mathbf{1}_{\{0,1\}}(x)$ ,  $\theta \in [0, 1/2]$ .
  - d. Exponential:  $f(x | \beta) = \beta^{-1} \exp(-x/\beta) \mathbf{1}_{(0,\infty)}(x)$ ,  $\beta \in \mathbb{R}^+$ .