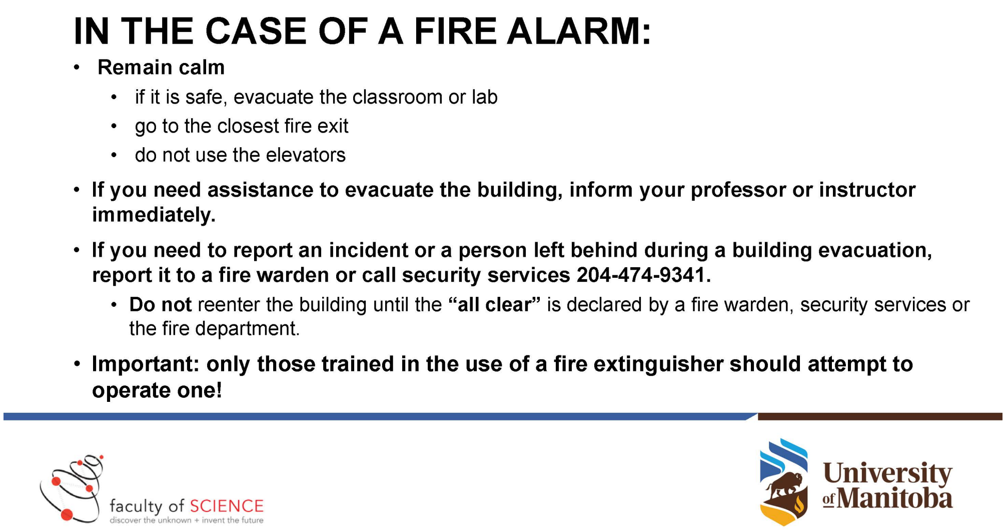
STAT 4100 Lecture Note

Week One (Sep 7 & 9, 2022)

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca, zhiyanggeezhou.github.io)

2022/Sep/08 13:50:30



Syllabus

Contact

- Zhiyang (Gee) Zhou
  - Email: zhiyang.zhou@umanitoba.ca
  - Homepage: zhiyanggeezhou.github.io
  - Office: 330 Machray Hall
- Ke (Kelly) Wang
  - Email: wangk9@myumanitoba.ca
  - Office: 355 Machray Hall

Timeline

- Lectures
  - Mon/Wed/Fri 13:30–14:20 at St. Paul's College 258
- Labs
  - Tue 16:30–18:00 at Buller 315
- Office Hours
  - (instructor) Mon 14:30–15:30 (or by appointment) at 330 Macray Hall
  - (TA) by appointment

Grading

- Assignment 20%
  - Submitted via Crowdmark
  - Attaching source codes if `R` is used in computation
  - Always including necessary interpretation
- Midterm 30%
  - In the week of Oct 17
  - Open-book, 3-hour, and online
- Final 50%
  - Open-book and in-person (?)

Materials

- References (recommended but NOT required)
  - (CB) Casella & Berger. 2002. *Statistical Inference*, 2nd Ed.
    - 2 hardcopies reserved at the Jim Peebles Science and Technology Library
  - (HMC) Hogg, Mckean & Craig. 2018. *Introduction to Mathematical Statistics*, 8th Ed.
    - Hardcopy of 6th Ed. available at the Jim Peebles Science and Technology Library
  - Salsburg. 2001. *The lady tasting tea: how statistics revolutionized science in the twentieth century.* New York: WH Freeman.
- Lecture notes
  - zhiyanggeezhou.github.io
  - UM Learn
  - **Subject to change** without prior notice
- Fall 2022 Syllabi Appendix

Outline

"All models are wrong, but some are useful."

— George Box, *Journal of American Statistical Association* 1976

- What are statistical models?
  - Distributions of random variables (r.v.s) of interest
- Statistical inference
  - To answer questions on the underlying statistical models, e.g.,
    - What is the model?
    - Is the r.v. distributed as $\mathcal{N}(0,1)$?
- Topics to be covered
  - Prerequisite
  - Estimation (finite/large sample, optimality)
  - Confidence interval (finite/large sample, interpretation)
  - Hypothesis testing (finite/large sample, optimality, interpretation)

Basics on random variables (CB/HMC Chp. 1)

Definitions

- Definition of r.v.: a real-valued function defined on a sample space Ω, i.e.,

$$X = X(\omega), \quad \omega \in \Omega$$

- Cumulative distribution function (cdf) of r.v. X

$$F_X(x) = \Pr(X \leq x)$$

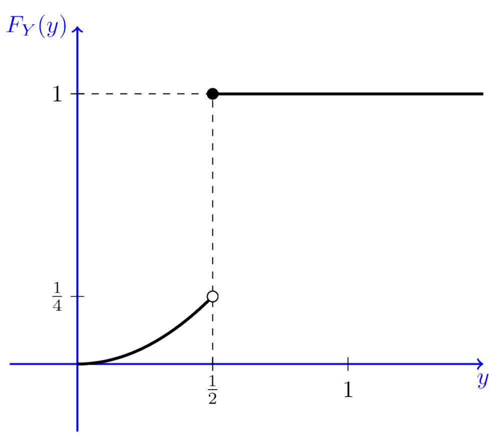
  - Right continuous
    * Roughly speaking, a function is right-continuous if no jump occurs when the limit point is approached from the right
  - Non-decreasing
  - Ranging from 0 to 1, i.e., $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

Example Lec1.1

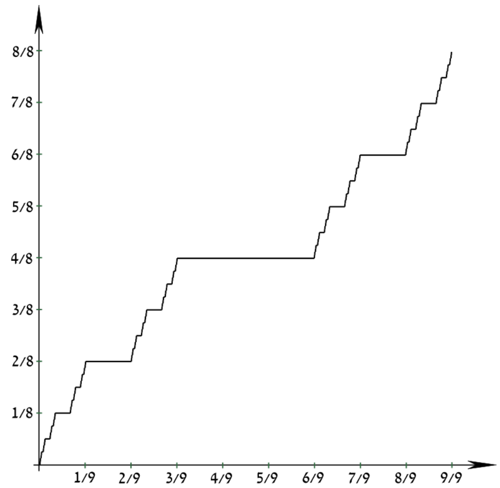
- Given $p \in (0, 1)$, suppose
$$F(x) = \begin{cases} 1 - (1-p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$
where $\lfloor x \rfloor$ represents the integer part of x. Show that F is a cdf.
- Hint: Check the right-continuity of F at positive integers.

Types of random variables

- X is a discrete r.v.
  - X takes countably many values
  - probability mass function (pmf): $p_X(x) = \Pr(X = x)$
- X is a continuous r.v.
  - cdf F_X is absolutely continuous, i.e., $\exists f_X$, s.t.
$$F_X(x) = \int_{-\infty}^{x} f_X(z) \mathrm{d}z, \quad \forall x \in \mathbb{R}.$$
  - f_X is the probability density function (pdf) of X
    * $f_X(x) = (\mathrm{d}/\mathrm{d}x) F_X(x)$ if f_X is continuous at $x \in \mathbb{R}$

- Neither discrete nor continuous
  - X is a mixed r.v., e.g.,
$$F_X(x) = \begin{cases} 1, & x \geq 1/2; \\ x^2, & 0 \leq x < 1/2; \\ 0, & \text{otherwise}. \end{cases}$$
    * 

- Neither discrete nor continuous (con'd)
  - X is following the Cantor distribution



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Univariate transformation (CB Sec. 2.1 & 2.4)

Support (CB pp. 50)

- In general, for real-valued function g
  - $\mathrm{supp}(g) = \{x \in \mathrm{domain}(g) : g(x) \neq 0\} \subset \mathrm{domain}(g)$
- For discrete r.v. X
  - pmf $p_X(\cdot)$
  - $\mathrm{supp}(X) = \mathrm{supp}(p_X) = \{x \in \mathbb{R} : p_X(x) > 0\}$
  - e.g., support of $Binom(n, p)$ is $\{0, \ldots, n\}$
- For continuous r.v. X
  - pdf $f_X(\cdot)$
  - $\mathrm{supp}(X) = \mathrm{supp}(f_X) = \{x \in \mathbb{R} : f_X(x) > 0\}$
  - e.g., support of $\mathcal{N}(0, 1)$ is $\mathbb{R}$

Indicator function

- $\mathbf{1}_A(x) = 1$ if $x \in A$ and zero otherwise

Find pmf of $Y = g(X)$ given the pmf of X

1. Figure out $\mathrm{supp}(Y) = \{y : y = g(x), x \in \mathrm{supp}(X)\}$
2. Calculate $p_Y(y) = \Pr(Y = y) = \Pr(X \in \{x \in \mathrm{supp}(X) : g(x) = y\})$

Example Lec2.1

Let X have the pmf $p_X(x) = 2^x \mathbf{1}_{\{-1,-2,\ldots\}}(x)$. Find the pmf of $Y = X^4$.

Find cdf of $Y = g(X)$ given the distribution of X

- Calculate $F_Y(y) = \Pr\{g(X) \leq y\} = \Pr[X \in g^{-1}\{(-\infty, y]\}]$
  - $g^{-1}\{(-\infty, y]\} = \{x : g(x) \leq y\}$

Example Lec2.2

Let X have the uniform pdf $f_X(x) = \pi^{-1} \mathbf{1}_{(-\pi/2,\pi/2)}(x)$. Find the cdf of $Y = \tan X$.

Find pdf of $Y = g(X)$ given the distribution of X

1. Figure out $\text{supp}(Y) = \{y : y = g(x), x \in \text{supp}(X)\}$
2. (Generically) If the cdf F_Y is known OR pdf f_X is easy to be integrated, then

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} \int_{\{x : g(x) \le y\}} f_X(x) \mathrm{d}x$$

   - The integration of f_X is often avoidable by employing the Leibniz Rule (CB Thm. 2.4.1):

$$\frac{\mathrm{d}}{\mathrm{d}y} \int_{a(y)}^{b(y)} f(x) \mathrm{d}x = f\{b(y)\} \frac{\mathrm{d}}{\mathrm{d}y} b(y) - f\{a(y)\} \frac{\mathrm{d}}{\mathrm{d}y} a(y)$$

   with $a(y)$ and $b(y)$ both differentiable with respect to y.

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2. (Alternatively) According to CB Ex. 2.7(b), i.e., an extension of CB Thm. 2.1.5 & 2.1.8 and HMC Thm 1.7.1.

$$f_Y(y) = \sum_{k=1}^{k} f_X\{g_k^{-1}(y)\} \left| J_{g_k^{-1}} \right| \mathbf{1}_{B_k}(y)$$

   - g_k is strictly monotonic on A_k and $g(x) = g_k(x)$ for all $x \in A_k$
     - $\{A_1, \ldots, A_K\}$ is a partition of $\text{supp}(X)$, i.e., $A_k \cap A_{k'} = \emptyset$ if $k \neq k'$ and $\bigcap_{k=1}^{K} A_k = \text{supp}(X)$
   - g_k^{-1} is continuously differentiable on $B_k = \{g_k(x) : x \in A_k\}$
   - Jacobian of transformation g_k^{-1}

$$J_{g_k^{-1}} = \frac{\mathrm{d}}{\mathrm{d}y} g_k^{-1}(y)$$

Example Lec2.2'

Let X have the uniform pdf $f_X(x) = \pi^{-1} \mathbf{1}_{(-\pi/2, \pi/2)}(x)$. Find the pdf of $Y = \tan X$.

Example Lec2.3

$X \sim \text{Weibull}(\text{shape} = \alpha, \text{scale} = \beta)$, viz. $f_X(x) = (\alpha/\beta)(x/\beta)^{\alpha-1} \exp\{-(x/\beta)^\alpha\} \mathbf{1}_{(0,\infty)}(x)$. Find the pdf of $Y = \ln(X)$.

Example Lec2.4

Let X have the pdf $f_X(x) = 2^{-1} \mathbf{1}_{(0,2)}(x)$. Find the pdf of $Y = X^2$.

Example Lec2.5

Let $f_X(x) = 3^{-1} \mathbf{1}_{(-1,2)}(x)$. Find the pdf of $Y = X^2$.

Distribution of $Y = F_X(X)$ (Probability integral transformation, CB Thm. 2.1.10)

- If
  - $X \sim F_X$ (not necessarily continuous)
  - $Y = F_X(X)$
- Then $Y \sim \text{unif}(\text{image}(F_X))$
  - Specifically $Y \sim \text{unif}([0, 1])$ if X is continuous
- Application: inverse transform sampling
  - Goal: generate independent and identically distributed (iid) random samples following F_X
  - Implementation
    1. Sample iid $U_1, \ldots, U_n \sim \text{unif}(\text{image}(F_X))$

2. Then iid $F_X^{-1}(U_1), \ldots, F_X^{-1}(U_n) \sim F_X$
   * $F_X^{-1}(y) = \inf\{x \in \mathrm{supp}(X) : F_X(x) \geq y\}$

- Pros & cons
  * (Theoretically) applicable to arbitrary F_X
  * The closed form of F_X^{-1} NOT always reachable