

STAT 4100 Lecture Note

Week Five (Oct 3, 5, & 7, 2022)

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Evaluating estimators

Mean squared error (MSE)

- Univariate: $E(\hat{\theta} - \theta)^2 = \{E(\hat{\theta}) - \theta\}^2 + \text{var}(\hat{\theta})$
- Multivariate: $E\{(\hat{\theta} - \theta)^\top (\hat{\theta} - \theta)\} = \{E(\hat{\theta}) - \theta\}^\top \{E(\hat{\theta}) - \theta\} + \text{cov}(\hat{\theta})$
- Best unbiased estimator (i.e., (uniform) minimum variance unbiased estimator, abbr. UMVUE/MVUE): if $\hat{\theta} = \hat{\theta}(\mathbf{X})$ satisfies that
 - $\hat{\theta}$ is unbiased for θ , i.e., $E(\hat{\theta}) = \theta$
 - $\text{var}(\hat{\theta}) - \text{var}(\hat{\theta}^*) \leq 0$ for all unbiased $\hat{\theta}^*$
- UMVUE is unique (CB Thm 7.3.19)

Cramer-Rao lower bound (CB Thm 7.3.9 & Lemma 7.3.11)

- Only consider the univariate case, i.e., one-dimensional unknown parameter θ
 - Score function $S(\theta; \mathbf{X})$ and Hessian $H(\theta; \mathbf{X})$ both scalar
- Cramer-Rao lower bound: $\text{var}(\hat{\theta}) \geq \{(d/d\theta)E(\hat{\theta})\}^2 / I(\theta)$ for $\hat{\theta}$ satisfying regularity conditions
 - Fisher information: $I(\theta) = \text{var}(S(\theta; \mathbf{X})) = E[\{S(\theta; \mathbf{X})\}^2] = -E\{H(\theta; \mathbf{X})\}$
 - Proof: Applying the Cauchy-Schwarz inequality (CB Thm 4.7.3)
- (CB Coro 7.3.15) $\hat{\theta}$ attains the lower bound \Leftrightarrow there is $a(\theta)$ such that $S(\theta; \mathbf{X}) = a(\theta)\{\hat{\theta} - E(\hat{\theta})\}$
- The unbiased $\hat{\theta}$ attaining the lower bound is UMVUE

Example Lec8.1

- Find the lower bound for unbiased estimators for σ^2 in the following cases.
 - a. iid $X_1, \dots, X_n \sim \mathcal{N}(\mu_0, \sigma^2)$ with known μ_0 and unknown σ^2 .

Sufficiency (CB Sec 6.2.1)

- A statistic $\mathbf{T} = \mathbf{T}(\mathbf{X})$ is sufficient for $\theta = (\theta_1, \dots, \theta_p) \Leftrightarrow$ the distribution of \mathbf{X} conditioning on \mathbf{T} and θ , say $f_{\mathbf{X}|\mathbf{T},\theta}(\mathbf{x} | \mathbf{t}, \theta)$, is free of θ .

- Fisher-Neyman factorization theorem (CB Thm 6.2.6; HMC Thm 7.2.1): \mathbf{T} is sufficient for $\theta \Leftrightarrow$ the likelihood function can be factored into two parts, one of them not depending on θ , i.e.,

$$L(\theta; \mathbf{x}) = f_{\mathbf{X}}(\mathbf{x} | \theta) = h(\mathbf{x})g(\mathbf{T}(\mathbf{x}), \theta), \text{ for all } \mathbf{x} \text{ and } \theta$$

- (HMC Thm 7.3.2) If \mathbf{T} is sufficient for θ and $\hat{\theta}$ is the unique MLE of θ , then $\hat{\theta}$ must be a function of \mathbf{T} .

- Nonuniqueness of sufficient statistics
 - Trivial examples
 - * \mathbf{X} is always sufficient.
 - * $(X_{(1)}, \dots, X_{(n)})$ is always sufficient if X_i 's are iid, with $X_{(1)} \leq \dots \leq X_{(n)}$.
 - \mathbf{T} is sufficient and $g(\cdot)$ is a one-to-one mapping $\Rightarrow g(\mathbf{T})$ is also sufficient.
- Minimal sufficiency: a sufficient statistic that is a function of all the other sufficient statistics.
 - (CB Thm 6.2.13) How to find a minimal sufficient statistic:
 1. Find the sufficient and necessary condition for $L(\theta; \mathbf{x})/L(\theta; \mathbf{y})$ to be free of θ ;
 2. If the above condition is of the form $\mathbf{T}(\mathbf{x}) = \mathbf{T}(\mathbf{y})$, then $\mathbf{T}(\mathbf{X})$ is a minimal sufficient statistic for θ .

Example Lec8.2

- Find the minimal sufficient statistics in the following scenarios.
 - a. iid $X_1, \dots, X_n \sim \text{Unif}\{1, \dots, \theta\}$ with unknown positive integer θ .
 - b. iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ with unknown μ and σ^2 .

Rao-Blackwellization (CB Thm 7.3.17)

- Only consider one-dimensional cases
- Improve the variance of statistic W , an estimator of θ : take use of $E(W | T)$ (a function of T only) instead with sufficient T
- T sufficient for $\theta \Rightarrow E(W | T, \theta) = E(W | T) \Rightarrow \text{var}\{E(W | T, \theta) | \theta\} = \text{var}\{E(W | T) | \theta\} \leq \text{var}(W | \theta)$ for all $\theta \in \Theta$
 - No impact on the bias
 - Not working if W is already a function of T
 - * (HMC Thm 7.3.2) MLE is usually a function of (non-trivial) sufficient statistics

Example Lec9.1

- Improve statistic W in terms of variance.
 - a. $W = X_1$, where iid $X_1, X_2 \sim \mathcal{N}(\theta, 1)$ with unknown θ .
 - b. $W = 2X_1 - X_2$, where iid $X_1, X_2 \sim f(x | \theta) = \theta^{-1} \exp(-x\theta^{-1}) \mathbf{1}_{\mathbb{R}^+ \times \mathbb{R}^+}(x, \theta)$ with unknown θ .

Completeness (CB Def 6.2.21)

- Only consider one-dimensional cases
- T is a complete statistic if we have the following identity: for any (measurable) function g ,

$$E(g(T) | \theta) = 0 \text{ for all } \theta \in \Theta \Rightarrow \Pr(g(T) = 0 | \theta) = 1 \text{ for all } \theta \in \Theta.$$

- Geometrical interpretation: $\text{span}\{f_{T|\theta}(t | \theta) : \theta \in \Theta\} = \{g(\cdot) : (\text{measurable}) g \text{ is defined on } \text{supp}(T)\}$
- Bounded completeness: restricted to bounded g only
- (CB Thm 6.2.28) Minimal sufficient statistics exist \Rightarrow complete sufficient statistics are minimally sufficient

- (HMC Thm 7.5.2) iid $X_1, \dots, X_n \sim f(x | \theta) = h(x)c(\theta) \exp \left\{ \sum_{i=1}^k w_i(\theta) t_i(x) \right\}$, i.e., following the exponential family, $\Rightarrow (\sum_{i=1}^n t_1(X_i), \dots, \sum_{i=1}^n t_k(X_i))$ is both sufficient and complete

Example Lec9.2

- Find the complete statistic for the following scenarios:
 - a. iid $X_1, \dots, X_n \sim f(x | \theta) = (x!)^{-1} \theta^x e^{-\theta} \mathbf{1}_{\mathbb{R}^+ \times \{0,1,\dots\}}(\theta, x)$;
 - b. iid $X_1, \dots, X_n \sim \text{Unif}\{1, \dots, \theta\}$, integer $\theta \geq 2$.

Lehmann-Scheffe (CB Thm 7.3.23 & 7.5.1; HMC Thm 7.4.1)

- The unbiased estimator only depending on complete sufficient statistics is the UMVUE.
- Application to the construction of UMVUE
 1. Find the minimal sufficient T .
 2. Check the completeness of T .
 3. Find unbiased $g(T)$, e.g.,
 - $E(W | T)$ with certain unbiased W
 - debiased MLE (if it is a function of T).

Example Lec9.3

- Suppose that iid X_1, \dots, X_n are following $\text{Unif}\{1, \dots, \theta\}$, integer $\theta \geq 2$. Prove that $[X_{(n)}^{n+1} - (X_{(n)} - 1)^{n+1}] / [X_{(n)}^n - (X_{(n)} - 1)^n]$ is the UMVUE for θ .

Verifying the independence

Ancillary Statistics

- Statistics whose distribution does not depend on unknown θ .

Example Lec10.1

- Verify the following statistics are ancillary for θ .
 - a. Range $X_{(n)} - X_{(1)}$ with $X_1, \dots, X_n \sim \text{Unif}(\theta, \theta + 1)$.
 - b. X_1/X_2 with $X_1, X_2 \sim \mathcal{N}(0, \theta^2)$.

Basu's theorem (CB Thm 6.2.4)

- T is complete and sufficient, while S is ancillary. Then T and S are independent of each other.
 - The completeness of T can be relaxed to be bounded completeness.

Example Lec10.2

- Let iid $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Deduce the independence of \bar{X} and S^2 by applying Basu's theorem for the case with unknown μ and known σ^2 .

Summary on how to verify the independence of X and Y

- Joint cdf: $F_{X,Y}(x, y) = F_X(x)F_Y(y)$
- Joint pdf or pmf: $f_{X,Y}(x, y) = f_X(x)f_Y(y)$
- conditional pdf or pmf: $f_{X|Y}(x | y) = f_X(x)$
- mgf: $E(e^{t_1 X + t_2 Y}) = E(e^{t_1 X})E(e^{t_2 Y})$

- cf: $E(e^{it_1 X + it_2 Y}) = E(e^{it_1 X})E(e^{it_2 Y})$
- Basu's theorem
 - Sometimes it is even more complex to find complete statisitcs than to obtain the joint pdf
- Zero covariance matrix for normal cases