

PH 712 Probability and Statistical Inference

Part I: Random Variable

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Probability (HMC Sec. 1.1–1.3)

- Sample space (denoted by Ω): the set of all the possible outcomes, e.g.,
 - $\Omega = \mathbb{R}^+$ if investigating survival times of cancer patients
 - $\Omega = \{\text{yes}, \text{no}\}$ if investigating whether a treatment is effective
- Event (denoted by capital Roman letters, e.g., A): a subset of the sample space, e.g., corresponding to the previous two examples of sample spaces,
 - $A = (0, 10]$: the survival time ≤ 10
 - $B = \{\text{yes}\}$: the treatment is effective
- An event A occurs \Leftrightarrow the outcome belongs to A , e.g.,
 - The survival time is 11: A does happen
 - The treatment outcome is “yes”: B happens
- Probability (denoted by \Pr): a function quantifying the occurrence likelihood of an event
 - E.g.,
 - * $\Pr(A)$: the probability (occurrence likelihood) of event A
 - * $\Pr(A^c)$: the probability that event A does NOT occur ($A^c = \Omega \setminus A$ denoting the complement set of A)
 - * $\Pr(A \cup B)$: the probability of either A or B
 - * $\Pr(A \cap B)$: the probability of both A and B
 - Input: an event
 - Output: a real number (the occurrence probability of the input event)
 - Requirements (definition in math):
 - * $\Pr(A) \geq 0$ for any event A
 - * $\Pr(\Omega) = 1$ (i.e., the sample space as a special event always occurs)
 - * (The probability of the union of mutually exclusive countably events is the sum of the probability of each event) If $\{A_n\}_{n=1}^\infty$ is a sequence of events with $A_{n_1} \cap A_{n_2} = \emptyset$ for all $n_1 \neq n_2$, then $\Pr(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Pr(A_n)$
 - More properties (deduced from the above requirements):
 - * $\Pr(A) = 1 - \Pr(A^c)$
 - * $\Pr(\emptyset) = 0$
 - * $\Pr(A) \leq \Pr(B)$ if $A \subset B$
 - * $0 \leq \Pr(A) \leq 1$ for each A
 - * $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ for any events A and B regardless if they are disjoint or not
 - * $\Pr(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \Pr(A_n)$ for arbitrary sequence $\{A_n\}_{n=1}^\infty$

Conditional probability and independence (HMC Sec. 1.4)

- Motivating example

- A : the event that a given person recovers from a disease
- B : the event that a given person has received a certain treatment
- $\Pr(A)$: the probability that a given person recovers from the disease
- $\Pr(A | B)$: the probability that a given person recovers from the disease, given that the person has received the treatment
- If $\Pr(A | B) = \Pr(A)$, then the treatment is NOT effective for the disease
- Conditional probability of B given A (with $\Pr(A) > 0$): $\Pr(B | A) = \Pr(A \cap B) / \Pr(A)$
 - Interpretation: the occurrence probability of B , given that A has already occurred.
 - Properties:
 - * $\Pr(B | A) \geq 0$
 - * $\Pr(A | A) = 1$
 - * $\Pr(\bigcup_{n=1}^{\infty} B_n | A) = \sum_{n=1}^{\infty} \Pr(B_n | A)$ if $\{B_n\}_{n=1}^{\infty}$ are mutually exclusive
 - * (Law of total probability) $\Pr(B) = \sum_{n=1}^N \Pr(A_n) \Pr(B | A_n)$ if $\{A_n\}_{n=1}^N$ form a partition of Ω (i.e., $\{A_n\}_{n=1}^N$ are mutually exclusive and $\Omega = \bigcup_{n=1}^N A_n$)
 - * (Bayes' theorem) $\Pr(A_i | B) = \Pr(A_i) \Pr(B | A_i) / \sum_{n=1}^N \Pr(A_n) \Pr(B | A_n)$ if $\{A_n\}_{n=1}^N$ form a decomposition/partition of Ω
- Independence between two events B and A (i.e., $B \perp A$): $\Pr(B \cap A) = \Pr(A) \Pr(B)$
 - $\Leftrightarrow B \perp A^c$
 - $\Leftrightarrow \Pr(B | A) = \Pr(B)$ (if $\Pr(A) \neq 0$)
 - $\Leftrightarrow \Pr(A | B) = \Pr(A)$ (if $\Pr(B) \neq 0$)
- Mutual independence among N events A_1, \dots, A_N : for arbitrary subset of $\{A_1, \dots, A_N\}$, say $\{A_{n_1}, \dots, A_{n_K}\}$ with $2 \leq K \leq N$, $\Pr(\bigcap_{k=1}^K A_{n_k}) = \prod_{k=1}^K \Pr(A_{n_k})$

HMC Ex. 1.4.31

- A French writer, Chevalier de Méré, had asked a famous mathematician, Pascal, to explain why the following two probabilities were different (the difference had been noted from playing the game many times): (1) at least one six in four independent casts of a six-sided die; (2) at least a pair of sixes in 24 independent casts of a pair of dice. From proportions it seemed to Mr. de Méré that the two probabilities should be the same. Compute the probabilities of (1) and (2).
 - Hint: $\Pr(\text{no six in one cast of a die}) = 5/6$, $\Pr(\text{no six in one cast of a pair of dice}) = (5/6)^2$, and $\Pr(\text{only one six in one cast of a pair of dice}) = 2 \times (1/6) \times (5/6)$.

RV and events

- RV: an encoder (function) mapping entries of sample space to real numbers,
 - Input: an element of sample space
 - Output: a real number
- Example of RVs: Severity of a patient's cold symptoms
 - Sample space $\Omega = \{\text{no reaction, mild, moderate, severe}\}$
 - RV X : $X(\text{no reaction}) = 0$, $X(\text{mild}) = 1$, $X(\text{moderate}) = 2$, $X(\text{severe}) = 3$
- Using values of an RV to define events
 - For the above example, $\{X \leq .7\} = \{\text{no reaction}\}$, $\{X \leq 2.3\} = \{\text{no reaction, mild, moderate}\}$
 - What is $\{1.1 \leq X < 2\}$? How about $\{1.1 \leq X < 2.1\}$?

Distribution of an RV (HMC Chp. 1.5–1.7)

- The cumulative distribution function (cdf) of RV X , say F_X , is defined as

$$F_X(t) = \Pr(X \leq t), \quad t \in \mathbb{R}.$$

- F_X satisfies following three properties:
 - * (Right continuous) $\lim_{x \rightarrow t^+} F_X(x) = F_X(t)$ (p.s., $\lim_{x \rightarrow t^-} F_X(x) = \Pr(X < t)$);
 - * (Non-decreasing) $F_X(t_1) \leq F_X(t_2)$ for $t_1 \leq t_2$;

- * (Ranging from 0 to 1) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.
- Reversely, a function satisfying the three above properties must be a cdf for certain RV.
 - * Indicating an one-to-one correspondence between the set of all the RVs and the set of all the cdfs
- Knowing the cdf of an RV \Leftrightarrow knowing its distribution

Example Lec1.1

- Given $p \in (0, 1)$, suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of real x .

- Plot the curve of F_X .

```
p = .3
F_X = function(x) {
  return((1 - (1 - p)^floor(x))*ifelse(x >= 1, 1, 0))
}
curve(F_X, from = -10, to = 10, n = 1000, col = "blue", lwd = 2,
      xlab = "x", ylab = expression(F[X](x)), main = "Cumulative Distribution Function")
```

- Given $\lambda > 0$, suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- Plot the curve of F_X .

```
lambda = 2
F_X = function(x) {
  return((1 - exp(- x/lambda))*ifelse(x > 0, 1, 0))
}
curve(F_X, from = -10, to = 10, n = 1000, col = "blue", lwd = 2,
      xlab = "x", ylab = expression(F[X](x)), main = "Cumulative Distribution Function")
```

Distribution of an RV (con'd)

- Discrete RV
 - RV X merely takes countably different values
 - Probability mass function (pmf): $p_X(t) = \Pr(X = t)$
 - * $F_X(t) = \sum_{x \leq t} p_X(x)$
 - * $p_X(t) = F_X(t) - \Pr(X < t)$
 - Knowing the pmf of a discrete RV \Leftrightarrow knowing its distribution
 - Examples:
 - * Uniform (the discrete version): each outcome in a finite set has an equal probability.
 - E.g., the outcome of rolling a fair dice is following the uniform distribution.
 - https://en.wikipedia.org/wiki/Discrete_uniform_distribution
 - * Bernoulli: a discrete RV with two possible outcomes, typically coded as 0 (failure) and 1 (success).
 - https://en.wikipedia.org/wiki/Bernoulli_distribution
 - * Binomial (denoted by $B(n, p)$): the number of successes in n independent Bernoulli trials.
 - E.g., after flipping a coin 10 times, the number of heads is following the binomial distribution.
 - https://en.wikipedia.org/wiki/Binomial_distribution
 - * Geometric: the number of trials until the first success in a series of independent Bernoulli trials.

- E.g., the number of coin flips needed until the first head appears is following the geometric distribution.
 - https://en.wikipedia.org/wiki/Geometric_distribution
- * Poisson: the number of events that occur in a fixed interval of time or space, where events happen independently.
 - E.g., the number of emails you receive in an hour.
 - https://en.wikipedia.org/wiki/Poisson_distribution
- Continuous RV
 - RV X is continuous \Leftrightarrow there exists f_X such that

$$F_X(t) = \int_{-\infty}^t f_X(x)dx, \quad \forall t \in \mathbb{R}.$$
 - * Probability density function (pdf): $f_X(t) = dF_X(t)/dt$ (nonnegative for all t).
 - $\int_{-\infty}^{\infty} f_X(x)dx = F_X(\infty) = 1$
 - * $\Pr(X = x_0) = 0$ for all $x_0 \in \mathbb{R}$
 - Because $\Pr(X = x_0) = \Pr(X \leq x_0) - \Pr(X < x_0) = F_X(x_0) - \lim_{x \rightarrow x_0^-} F_X(x) = 0$ (The proof is not required.)
 - Knowing the pdf of a continuous RV \Leftrightarrow knowing its distribution
 - Examples:
 - * Uniform (the continuous version): all outcomes in a continuous range are equally likely.
 - [https://en.wikipedia.org/wiki/Uniform_distribution_\(continuous\)](https://en.wikipedia.org/wiki/Uniform_distribution_(continuous))
 - * Normal/Gaussian (denoted by $\mathcal{N}(\mu, \sigma^2)$): the most important and widely used distributions, where data is symmetrically distributed around the mean.
 - https://en.wikipedia.org/wiki/Normal_distribution
 - * Exponential: often used to describe waiting times.
 - https://en.wikipedia.org/wiki/Exponential_distribution

Example Lec1.2

- Given $p \in (0, 1)$, suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of x .

- What is the pmf/pdf of X ?
- Given $\lambda > 0$, suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- What is the pmf/pdf of X ?

Support of RV (HMC pp. 46)

- For discrete RV X with pmf p_X
 - $\text{supp}(X) = \{x \in \mathbb{R} : p_X(x) > 0\}$
 - E.g., support of $B(n, p)$ is $\{0, \dots, n\}$
 - $\sum_{x \in \text{supp}(X)} p_X(x) = 1$
- For continuous RV X with pdf f_X
 - $\text{supp}(X) = \{x \in \mathbb{R} : f_X(x) > 0\}$
 - E.g., support of $\mathcal{N}(0, 1)$ is \mathbb{R}
 - $\int_{\text{supp}(X)} f_X(x)dx = 1$

Example Lec1.3

- Given $p \in (0, 1)$, suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of x .

- What is the support of X ?
- Given $\lambda > 0$, suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- What is the support of X ?

Indicator function

Given a set A , the indicator function of A is

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Example Lec1.4

- Given $p \in (0, 1)$, suppose

$$F_X(x) = \begin{cases} 1 - (1 - p)^{\lfloor x \rfloor}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ represents the integer part of x .

- Please reformulate F_X with the indicator function of $A = \{x : x \geq 1\}$.
- Given $\lambda > 0$, suppose

$$F_X(x) = \begin{cases} 1 - \exp(-x/\lambda), & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

- Please reformulate F_X with the indicator function of $A = \{x : x > 0\}$.

Indicating the support when writing pmf and pdf

- Bernoulli: https://en.wikipedia.org/wiki/Bernoulli_distribution
- Binomial (denoted by $B(n, p)$): https://en.wikipedia.org/wiki/Binomial_distribution
 - $p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k} \cdot \mathbf{1}_{\{0, 1, \dots, n\}}(k)$
 - * OR $\binom{n}{k} p^k (1 - p)^{n-k}$, $k \in \{0, 1, \dots, n\}$
- Geometric: https://en.wikipedia.org/wiki/Geometric_distribution
 - $p_X(k) = (1 - p)^{k-1} p \cdot \mathbf{1}_{\mathbb{Z}^+}(k)$
 - * OR $(1 - p)^{k-1} p$, $k \in \mathbb{Z}^+$
- Poisson: https://en.wikipedia.org/wiki/Poisson_distribution
 - $p_X(k) = \lambda^k \exp(-\lambda) / k! \cdot \mathbf{1}_{\{0, 1, 2, \dots\}}(k)$
 - * OR $\lambda^k \exp(-\lambda) / k!$, $k \in \{0, 1, 2, \dots\}$
- Uniform (the discrete version; denoted by $U([a, b])$ with integers $a < b$): https://en.wikipedia.org/wiki/Discrete_uniform_distribution
 - $p_X(k) = 1/(b - a + 1) \cdot \mathbf{1}_{\{a, a+1, \dots, b-1, b\}}(k)$
 - * OR $1/(b - a + 1)$, $k \in \{a, a + 1, \dots, b - 1, b\}$
- Uniform (the continuous version): [https://en.wikipedia.org/wiki/Uniform_distribution_\(continuous\)](https://en.wikipedia.org/wiki/Uniform_distribution_(continuous))

- Exponential: https://en.wikipedia.org/wiki/Exponential_distribution
 - $f_X(x) = \lambda \exp(-\lambda x) \cdot \mathbf{1}_{[0,\infty)}(x)$
 - * OR $\lambda \exp(-\lambda x), x \geq 0$
- Normal/Gaussian (denoted by $\mathcal{N}(\mu, \sigma^2)$): https://en.wikipedia.org/wiki/Normal_distribution
 - Don't have to specify the support for normal RVs because it is \mathbb{R}
 - $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
 - * OR $\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
 - Specifically, if $\mu = 0$ and $\sigma = 1$, then it is called the standard normal (denoted by $\mathcal{N}(0, 1)$):
 - * $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$
 - OR $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

Expectation (HMC Sec. 1.8–1.9)

- Definition: given RV X and function g , the expectation of $g(X)$ is

$$\mathbb{E}\{g(X)\} = \begin{cases} \sum_{x \in \text{supp}(X)} g(x)p_X(x) & \text{for discrete } X \\ \int_{x \in \text{supp}(X)} g(x)f_X(x)dx & \text{for continuous } X \end{cases}$$

- $\mathbb{E}\{g(X)\}$ is a average of values of $g(X)$ weighted by the distribution of X
- $\mathbb{E}\{g(X)\}$ is a fixed real number
- Special cases with different $g(\cdot)$
 - If $g(X) = X$, then $\mathbb{E}\{g(X)\}$ becomes the expectation/mean of X (a.k.a. the 1st raw moment/moment about 0 of X):

$$\mathbb{E}(X) = \begin{cases} \sum_{x \in \text{supp}(X)} xp_X(x) & \text{for discrete } X \\ \int_{x \in \text{supp}(X)} xf_X(x)dx & \text{for continuous } X \end{cases}$$

- If $g(X) = X^k$ with positive integer k , then $\mathbb{E}\{g(X)\}$ becomes the k th raw moment/moment about 0 of X :

$$\mathbb{E}(X^k) = \begin{cases} \sum_{x \in \text{supp}(X)} x^k p_X(x) & \text{for discrete } X \\ \int_{x \in \text{supp}(X)} x^k f_X(x)dx & \text{for continuous } X \end{cases}$$

- If $g(X) = \{X - \mathbb{E}(X)\}^2$, then $\mathbb{E}\{g(X)\}$ becomes the variance of X (a.k.a. the 2nd central moment of X):

$$\text{Var}(X) = \mathbb{E}[\{X - \mathbb{E}(X)\}^2] = \begin{cases} \sum_{x \in \text{supp}(X)} \{x - \mathbb{E}(X)\}^2 p_X(x) & \text{for discrete } X \\ \int_{x \in \text{supp}(X)} \{x - \mathbb{E}(X)\}^2 f_X(x)dx & \text{for continuous } X \end{cases}$$

- * Measuring how spread out the data are if they are independently generated following F_X
- * $\text{sd}(X) = \sqrt{\text{Var}(X)}$: the standard deviation of X
- If $g(X) = \mathbf{1}_A(X)$, then $\mathbb{E}\{g(X)\}$ becomes the probability that X belongs to event A :

$$\mathbb{E}\{\mathbf{1}_A(X)\} = \Pr(X \in A)$$

- If $g(X) = c$ for certain constant c , then $\mathbb{E}\{g(X)\}$ remains c :

$$\mathbb{E}(c) = c.$$

- Linearity of expectation: $\mathbb{E}\{a_1 g_1(X) + a_2 g_2(X)\} = a_1 \mathbb{E}\{g_1(X)\} + a_2 \mathbb{E}\{g_2(X)\}$ for constants a_1 and a_2 , implying that
 - $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$ for constants a and b
 - $\text{Var}(X) = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2$
 - $\text{Var}(aX + b) = a^2 \text{Var}(X)$

Example Lec1.5

- Find the mean and variance of the following RV

x	0	2.4	3
$p_X(x)$	0.2	0.5	0.3

Answer:

$$\begin{aligned} E(X) &= 0 \times 0.2 + 2.4 \times 0.5 + 3 \times 0.3 = 2.55; \\ E(X^2) &= 0^2 \times 0.2 + 2.4^2 \times 0.5 + 3^2 \times 0.3 = 7.05; \\ \text{Var}(X) &= E(X^2) - \{E(X)\}^2 = 7.05 - 2.55^2 = 0.4275. \end{aligned}$$

- RV X follows a uniform distribution over the interval $[0, 2]$, i.e., $f_X(x) = .5 \times \mathbf{1}_{[0,2]}(x)$. Find the mean and variance of X .

Answer:

$E(X) = \int_0^2 x \times .5 dx = 1$ is given below:

```
integrand = function(x){
    .5*x
}
integrate(integrand, lower = 0, upper = 2)$value
```

$E(X^2) = \int_0^2 x^2 \times .5 dx = 4/3$ is given below:

```
integrand = function(x){
    .5*x^2
}
integrate(integrand, lower = 0, upper = 2)$value
```

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 = 4/3 - 1^2 = 1/3.$$

- A continuous RV X has an exponential distribution with pdf $f_X(x) = 2 \exp(-2x) \cdot \mathbf{1}_{[0,\infty)}(x)$. Find the mean and variance of X .

Answer:

- Find the mean and variance of $X \sim \mathcal{N}(0, 1)$, i.e., $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$.

Answer:

- Find the mean and variance of $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, i.e., $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.

Identity on normal RVs:

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ for $a > 0$.
 - $aX + b$ is an affine transformation of X .
 - After an affine transformation, a normal RV remains normal.
 - Specifically, $X \sim \mathcal{N}(\mu, \sigma^2) \Leftrightarrow (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$

Answer:

-
- Find the mean and variance of Cauchy distribution, i.e., $f_X(x) = \{\pi(1+x^2)\}^{-1}$, $x \in \mathbb{R}$.

Answer: