STAT 3690 Lecture Note

Week Two (Jan 16, 18, & 20, 2023)

Zhiyang Zhou (zhiyang.zhou@umanitoba.ca, zhiyanggeezhou.github.io)

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Matrix basics (con'd)

Block/partitioned matrix

- A partition of matrix: Suppose $\mathbf{A}_{11}$ is of $p \times r$, $\mathbf{A}_{12}$ is of $p \times s$, $\mathbf{A}_{21}$ is of $q \times r$ and $\mathbf{A}_{22}$ is of $q \times s$. Make a new $(p+q) \times (r+s)$-matrix by organizing $\mathbf{A}_{ij}$'s in a 2 by 2 way:

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]$$

  e.g.,

$$\mathbf{A} = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 4 & 5 & 6 \end{array}\right]$$

  if

$$\mathbf{A}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{A}_{21} = \begin{bmatrix} 4 & 5 \end{bmatrix}, \quad \text{and} \quad \mathbf{A}_{22} = \begin{bmatrix} 6 \end{bmatrix}.$$

- Operations with block matrices
  - Working with partitioned matrices just like ordinary matrices
  - Matrix addition: if dimensions of $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ are quite the same, then

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}+\mathbf{B}_{11} & \mathbf{A}_{12}+\mathbf{B}_{12} \\ \mathbf{A}_{21}+\mathbf{B}_{21} & \mathbf{A}_{22}+\mathbf{B}_{22} \end{bmatrix}$$

  - Matrix multiplication: if $\mathbf{A}_{ij}\mathbf{B}_{jk}$ makes sense for each i, j, k, then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11}+\mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12}+\mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

  - Inverse: if $\mathbf{A}$, $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are all invertible, then

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11.2}^{-1} & -\mathbf{A}_{11.2}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11.2}^{-1} & \mathbf{A}_{22.1}^{-1} \end{bmatrix}$$

    * $\mathbf{A}_{11.2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$
    * $\mathbf{A}_{22.1} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$

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options(digits = 4)
set.seed(1)
## Generate an (almost surely) invertible matrix
(A = matrix(runif(9), nrow = 3, ncol = 3)) #

# Verify the inverse of partition matrix
## Method 1: following the above formula
(A11 = A[1:2, 1:2])
(A12 = matrix(A[1:2, 3], nrow = 2, ncol = 1))
(A21 = matrix(A[3, 1:2], nrow = 1, ncol = 2))
(A22 = matrix(A[3, 3], nrow = 1, ncol = 1))
(A11.2 = A11 - A12 %*% solve(A22) %*% A21)
(A22.1 = A22 - A21 %*% solve(A11) %*% A12)

(Ainv1 = rbind(
  cbind(solve(A11.2), -solve(A11.2) %*% A12 %*% solve(A22)),
  cbind(-solve(A22) %*% A21 %*% solve(A11.2), solve(A22.1))
))

## Method 2: solve()
Ainv2 = solve(A)

## Comparison
Ainv2 - Ainv1
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An example utilizing matrix basics: rephrasing the ridge estimator

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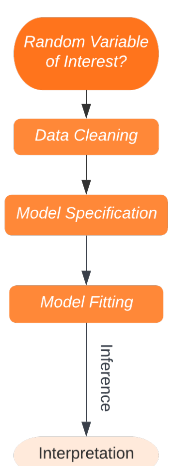
> "All models are wrong, but some are useful."
>
> — G. E. P. Box. (1976). *Journal of the American Statistical Association*, 71:791–799

Statistical modelling



What is a statistical model?

- The (joint) distribution of the random variable(s) of interest
  - E.g., reformulate linear regression and logit regression models in terms of distributions

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Recall the characterization of univariate distributions

- A random variable (RV), say X, is a real-valued function (defined on a sample space).
- The cumulative distribution function (cdf) of X, say $F_X(x) = \Pr(X \le x)$, $x \in \mathbb{R}$, if (right continuous) $\lim_{t\to x_0^+} F_X(x) = F_X(x_0)$, (non-decreasing) $F_X(x_0) \le F_X(x_1)$ for $x_0 < x_1$, and (ranging from 0 to 1) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.
  - Reversely, any function satisfying the three properties must be a cdf for certain RV.
- Discrete RV
  - RV X takes countable different values
  - Probability mass function (pmf): $p_X(x) = \Pr(X = x)$
- Continuous RV
  - RV X is continuous iff its cdf F_X is (absolutely) continuous, i.e., there exists f_X, s.t.

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\mathrm{d}u, \quad \forall x \in \mathbb{R}.$$

  - Probability density function (pdf): $f_X(x) = F'_X(x)$.
- Moment-generating function (mgf) $M_X(t) = \mathrm{E}\{\exp(tX)\}$ if $\mathrm{E}\{\exp(tX)\} < \infty$ for t in a neighbourhood of 0
  - If the mgf exists, then $\mathrm{E}(X^k) = M_X^{(k)}(t) \mid_{t=0}$.

Support of RV

- Support of X, say $\mathrm{supp}(X)$, is $\{x \in \mathbb{R} : p_X(x) \text{ (or } f_X(x)) > 0\}$
  - e.g., support of $Binom(n, p)$ is $\{0, \dots, n\}$; support of $\mathcal{N}(0, 1)$ is $\mathbb{R}$.

Indicator function

- Given a set A, the indicator function of A is

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

  - Hence, e.g., if $X \sim Binom(n, p)$, then $p_X(x) = \binom{x}{n} p^x (1-p)^{1-x}$, $x \in \{0, \dots, n\}$, $p \in (0, 1)$, or equivalently, $p_X(x) = \binom{x}{n} p^x (1-p)^{1-x} \mathbf{1}_{\{0,\dots,n\}}(x)\mathbf{1}_{(0,1)}(p)$

Characterization of joint/multivariate distributions

- Random (column) vector/vector-valued RV
  - $\boldsymbol{X} = [X_1, \dots, X_p]^\top$
- Joint cdf: $F_{\boldsymbol{X}}(x_1, \dots, x_p) = \Pr(X_1 \le x_1, \dots, X_p \le x_p)$
- Joint distribution of continuous RVs
  - Joint pdf:
$$f_{\boldsymbol{X}}(x_1, \dots, x_p) = \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{\boldsymbol{X}}(x_1, \dots, x_p)$$
  - E.g., multivariate normal (MVN) distribution
- Joint distribution of discrete RVs
  - Joint pmf:
$$p_{\boldsymbol{X}}(x_1, \dots, x_p) = \Pr(X_1 = x_1, \dots, X_p = x_p)$$
  - E.g., categorical distribution & multinomial distribution

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- Exercise: Suppose that we independently observe an experiment that has m possible outcomes $O_1, \dots, O_m$ for n times; e.g., sample n balls with replacement from a pool of balls of m colors. Let $p_1, \dots, p_m$ denote probabilities of $O_1, \dots, O_m$ in each experiment respectively. Let X_i denote the number of times that outcome O_i occurs in the n repetitions.
  - What is the distribution of X_i?
  - What is the joint pmf of $\boldsymbol{X} = [X_1, \dots, X_m]^\top$?

- Moment-generating function (mgf) $M_{\boldsymbol{X}}(\boldsymbol{t}) = \mathrm{E}\{\exp(\boldsymbol{t}^\top \boldsymbol{X})\}$ if there exists $\delta > 0$ s.t. $\mathrm{E}\{\exp(\boldsymbol{t}^\top \boldsymbol{X})\} < \infty$ for all $\boldsymbol{t} \in \{\boldsymbol{t} : \boldsymbol{t}^\top \boldsymbol{t} < \delta\}$
  - If the mgf of $\boldsymbol{X}$ exists and X_i are independent of each other, then $M_{\boldsymbol{X}}(\boldsymbol{t}) = \prod_{i=1}^p M_{X_i}(t_i)$.

Marginalization

- $\boldsymbol{X} = [X_1, \dots, X_m]^\top$,
- $\boldsymbol{Y} = [X_1, \dots, X_q]^\top$, $p > q$, as part of $\boldsymbol{X}$
- Marginal cdf of $\boldsymbol{Y}$
$$F_{\boldsymbol{Y}}(x_1, \dots, x_q) = \lim_{x_{q+1}, \dots, x_m \to \infty} F_{\boldsymbol{X}}(x_1, \dots, x_m)$$
- Marginal pdf of $\boldsymbol{Y}$ (when $X_1, \dots, X_m$ are all continous)
$$f_{\boldsymbol{Y}}(x_1, \dots, x_q) = \int_{\mathbb{R}^{m-q}} f_{\boldsymbol{X}}(x_1, \dots, x_m) \mathrm{d}x_{q+1} \cdots x_m$$
- Marginal pmf of $\boldsymbol{Y}$ (when $X_1, \dots, X_m$ are all discrete)
$$p_{\boldsymbol{Y}}(x_1, \dots, x_q) = \sum_{x_{q+1}, \dots, x_m} p_{\boldsymbol{X}}(x_1, \dots, x_m)$$

Conditioning

- $\boldsymbol{X} = [X_1, \dots, X_m]^\top$ and $\boldsymbol{Y} = [Y_1, \dots, Y_q]^\top$
- Conditional pdf of $\boldsymbol{Y}$ given $\boldsymbol{X}$

$$f_{\boldsymbol{Y}|\boldsymbol{X}}(y_1, \dots, y_q \mid x_1, \dots, x_m) = \frac{f_{\boldsymbol{X},\boldsymbol{Y}}(x_1, \dots, x_m, y_1, \dots, y_q)}{f_{\boldsymbol{X}}(x_1, \dots, x_m)}$$

- Conditional pmf of $\boldsymbol{Y}$ given $\boldsymbol{X}$

$$p_{\boldsymbol{Y}|\boldsymbol{X}}(y_1, \dots, y_q \mid x_1, \dots, x_m) = \frac{p_{\boldsymbol{X},\boldsymbol{Y}}(x_1, \dots, x_m, y_1, \dots, y_q)}{p_{\boldsymbol{X}}(x_1, \dots, x_m)}$$

Transformation of random vectors

- Derive the pdf of continuous $\boldsymbol{Y} = \boldsymbol{g}(\boldsymbol{X})$ from the pdf of continuous $\boldsymbol{X}$
- Prerequisite
  - $\boldsymbol{X} = [X_1, \dots, X_p]^\top$ and $\boldsymbol{Y} = [Y_1, \dots, Y_p]^\top$
  - $\boldsymbol{g} = (g_1, \dots, g_p)$: $\mathbb{R}^p \to \mathbb{R}^p$ is a continuous one-to-one map with inverse $\boldsymbol{g}^{-1} = (h_1, \dots, h_p)$, i.e., $Y_i = g_i(\boldsymbol{X})$ and $X_i = h_i(\boldsymbol{Y})$
- Elaborate $\operatorname{supp}(\boldsymbol{Y}) = \{[y_1, \dots, y_p]^\top : [h_1(y_1, \dots, y_p), \dots, h_p(y_1, \dots, y_p)]^\top \in \operatorname{supp}(\boldsymbol{X})\}$
- Jacobian matrix of $\boldsymbol{g}^{-1}$ is $\mathbf{J}_{\boldsymbol{g}^{-1}} = [\partial x_i/\partial y_j]_{p\times p} = [\partial h_i(y_1, \dots, y_p)/\partial y_j]_{p\times p}$
  - Also, $|\det(\mathbf{J}_{\boldsymbol{g}^{-1}})| = |\det([\partial y_i/\partial x_j]_{p\times p})|^{-1} = |\det([\partial g_i(x_1, \dots, x_p)/\partial x_j]_{p\times p})|^{-1}$
- Then

$$f_{\boldsymbol{Y}}(y_1, \dots, y_p) = f_{\boldsymbol{X}}(h_1(y_1, \dots, y_p), \dots, h_p(y_1, \dots, y_p))|\det(\mathbf{J}_{\boldsymbol{g}^{-1}})|\mathbf{1}_{\operatorname{supp}(\boldsymbol{Y})}(y_1, \dots, y_p)$$

- Exercise: Let $\boldsymbol{X} = [X_1, X_2]^\top$ follow the standard bivariate normal, i.e., its pdf is

$$f_{\boldsymbol{X}}(x_1, x_2) = (2\pi)^{-1} \exp\{-(x_1^2 + x_2^2)/2\}\mathbf{1}_{\mathbb{R}^2}(x_1, x_2).$$

  Find out the joint pdf of $\boldsymbol{Y} = [Y_1, Y_2]^\top$, where $Y_1 = \sqrt{X_1^2 + X_2^2}$ and $0 \le Y_2 < 2\pi$ is the angle from the positive x-axis to the ray from the origin to the point (X_1, X_2), that is, Y is X in the polar coordinate.

Expectation of random matrix

- $\mathrm{E}(\boldsymbol{X}) = [\mathrm{E}(X_{ij})]_{n\times p}$, where
  - Random $n \times p$ matrix $\boldsymbol{X} = [X_{ij}]_{n\times p}$
- (Linearity) $\mathrm{E}(\mathbf{A}\boldsymbol{X} + \mathbf{B}\boldsymbol{Y}) = \mathbf{A}\mathrm{E}(\boldsymbol{X}) + \mathbf{B}\mathrm{E}(\boldsymbol{Y})$, where
  - Fixed $\mathbf{A} \in \mathbb{R}^{\ell\times n}$ and $\mathbf{B} \in \mathbb{R}^{\ell\times m}$
  - Random matrices $\boldsymbol{X} = [X_{ij}]_{n\times p}$ and $\boldsymbol{Y} = [Y_{ij}]_{m\times p}$

Covariance matrix

- Random p-vector $\boldsymbol{X} = [X_1, \dots, X_p]^\top$ and random q-vector $\boldsymbol{Y} = [Y_1, \dots, Y_q]^\top$
- Covariance matrix (defined via expectation) $\boldsymbol{\Sigma}_{\boldsymbol{X}\boldsymbol{Y}} = \operatorname{cov}(\boldsymbol{X}, \boldsymbol{Y}) = \mathrm{E}[\{\boldsymbol{X} - \mathrm{E}(\boldsymbol{X})\}\{\boldsymbol{Y} - \mathrm{E}(\boldsymbol{Y})\}^\top]$

- Also, $\boldsymbol{\Sigma}_{\boldsymbol{XY}} = \mathrm{E}(\boldsymbol{X}\boldsymbol{Y}^\top) - \mathrm{E}(\boldsymbol{X})\mathrm{E}(\boldsymbol{Y}^\top)$
- The (i,j)-entry of $\boldsymbol{\Sigma}_{\boldsymbol{XY}}$ is $\mathrm{cov}(X_i, Y_j)$

- $\boldsymbol{\Sigma}_{\mathbf{A}\boldsymbol{X}+\boldsymbol{a},\mathbf{B}\boldsymbol{Y}+\boldsymbol{b}} = \mathbf{A}\Sigma_{\boldsymbol{XY}}\mathbf{B}^\top$ for fixed $\mathbf{A} \in \mathbb{R}^{m\times p}$, $\boldsymbol{a} \in \mathbb{R}^m$, $\mathbf{B} \in \mathbb{R}^{\ell\times q}$ and $\boldsymbol{b} \in \mathbb{R}^\ell$
- $\boldsymbol{\Sigma}_{\boldsymbol{X}} \geq 0$, where $\boldsymbol{\Sigma}_{\boldsymbol{X}} = \mathrm{cov}(\boldsymbol{X})$ is short for $\boldsymbol{\Sigma}_{\boldsymbol{XX}} = \mathrm{cov}(\boldsymbol{X}, \boldsymbol{X})$