# PH 712 Probability and Statistical Inference

Part VIII: Point Estimation II (Aympototic Properties)

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## Consistency of MM and ML estimators

- For an iid sample, under certain conditions:
  - $\hat{\theta}_{\mathrm{MM}} \approx \theta$ as $n \to \infty$
  - $\hat{\theta}_{\mathrm{ML}} \approx \theta$ as $n \to \infty$

## Asymptotic efficiency of ML estimator (CB Thm 10.1.12 & Ex. 10.7)

- For an iid sample, under certain conditions:
  - $\sqrt{n}(\hat{\theta}_{\mathrm{ML}} - \theta) \approx \mathcal{N}(0, I_1^{-1}(\theta))$ as $n \to \infty$
    - For an iid sample, $I_1(\theta) = n^{-1} I_n(\theta)$, no longer a function of $n$

## Approximating the distribution of $\hat{\theta}_{\mathrm{ML}}$:

- In practice, unknown $\theta \Rightarrow$ unknown $I_n(\theta)$
- $I_n(\theta) \approx I_n(\hat{\theta}_{\mathrm{ML}}) \approx \hat{I}_n(\hat{\theta}_{\mathrm{ML}})$
  - Fisher information evaluated at $\hat{\theta}_{\mathrm{ML}}$: $I_n(\hat{\theta}_{\mathrm{ML}}) = \mathrm{E}\{-\ell'(\theta)\} \mid_{\theta=\hat{\theta}_{\mathrm{ML}}}$
  - Observed Fisher information (i.e., the minus Hessian evaluated at $\hat{\theta}_{\mathrm{ML}}$): $\hat{I}_n(\hat{\theta}_{\mathrm{ML}}) = -\ell''(\hat{\theta}_{\mathrm{ML}})$
- Approximately, $\hat{\theta}_{\mathrm{ML}}$ is normally distributed with mean $\theta$ and variance $I_n^{-1}(\theta)$, $\hat{I}_n^{-1}(\hat{\theta}_{\mathrm{ML}})$ OR $\hat{I}_n^{-1}(\hat{\theta}_{\mathrm{ML}})$, depending on 1) whether $\theta$ is allowed in the result AND 2) how convenient it is to take the expectation of $\ell''(\theta)$.

## Delta method

- Approximating the distribution of $h(T_n)$ when $T_n$ is normally distributed as $n \to \infty$
- (CB Thm 5.5.24, delta method) Suppose $T_n$ is an estimator of $\theta$. If $\sqrt{n}(T_n - \theta) \approx \mathcal{N}(0, \sigma^2)$, $h$ is NOT a function of $n$, AND $h'(\theta) \neq 0$, then
$$\sqrt{n}\{h(T_n) - h(\theta)\} \approx \mathcal{N}(0, \{h'(\theta)\}^2\sigma^2).$$
  - $\Rightarrow \mathrm{E}\{h(T_n)\} \approx h(\theta)$ AND $\mathrm{var}\{h(T_n)\} \approx \{h'(\theta)\}^2\sigma^2/n$ if $h'(\theta) \neq 0$.
- (CB Thm 5.5.26, second-order delta method) Suppose $T_n$ is an estimator of $\theta$. If $\sqrt{n}(T_n - \theta) \approx \mathcal{N}(0, \sigma^2)$, $h$ is NOT a function of $n$, $h'(\theta) = 0$, AND $h''(\theta) \neq 0$, then
$$\frac{2n\{h(T_n) - h(\theta)\}}{h''(\theta)\sigma^2} \approx \chi^2(1).$$
  - $\Rightarrow \mathrm{E}\{h(T_n)\} \approx h(\theta) + h''(\theta)\sigma^2/(2n)$ AND $\mathrm{var}\{h(T_n)\} \approx \{h''(\theta)\}^2\sigma^4/(2n^2)$ if $h'(\theta) = 0$ and $h''(\theta) \neq 0$.

## CB Example 10.1.17 & Ex. 10.9

- $X_1, \ldots, X_n \overset{\text{iid}}{\sim} p(x \mid \lambda) = \lambda^x \exp(-\lambda)/x!$, $x \in \mathbb{Z}^+ \cup \{0\}$, $\lambda > 0$. To estimate $h(\lambda) = \Pr(X_i = 0)$.
  1. What is the MLE for $\Pr(X_i = 0)$, say $W_n$?
  2. Approximate the variance of $W_n$.
  3. Suppose $T_n = n^{-1} \sum_i \mathbf{1}_{\{0\}}(X_i)$. Approximate the variance of $T_n$.
  4. Compute $\text{ARE}(T_n, W_n)$, the ARE of $T_n$ with respect to $W_n$.