Algebra of Uncertainty

Žiga Sajovic
ziga.sajovic@xlab.si

XLAB d.o.o.

2018

Quantifying the degrees of belief

We can quantify our beliefs about an uncertain event A by being willing to trade them for a possibility of something quantifiable.

The buy We are willing to trade certainly giving out the quantity $P\{A\}$ in exchange for a *ticket* that's redeemable for *one unit* of the quantity, in case A holds true.

The sell We are willing to trade a *ticket* that's redeemable for *one unit* of the quantity in case A holds true, in exchange for certainly receiving the quantity $P\{A\}$.

The quantity being traded may be anything that's quantifiable to us. Note that we sell and buy tickets on A at the same price $P\{A\}$.

The Dutch Book ($\mathcal{D}$)

The term *dutch book* ($\mathcal{D}$) labels a quantification of our beliefs about events A_i, by which any possible trade based on such beliefs will result in our *loss*, regardless of the truth attached to the final outcome.

With quantifiable subjective beliefs, the term is widely applicable.

Ex.: in econometrics $\mathcal{D}$ is a trade leaving one party strictly worse off.

Ex.: in courtship $\mathcal{D}$ could be a set of beliefs about the opposite sex, that leads to a sequence of interactions that always leaves you rejected and alone.

Coherence for disjoint events

Elements of discussion

- A_i is the set of disjoint events A_i
- P are the prices $P\{A_i\}$ of the ticket on A_i
- $\mathcal{C}$ is the set of rules R_i

$$\mathcal{C} = \begin{Bmatrix} R_1, \\ R_2, \\ \vdots \\ R_n \end{Bmatrix}$$

We seek a coherent set of rules $\mathcal{C}$ by which to prescribe prices $P\{A_i\}$ to events A_i in a way that prevents the possibility of being in a dutch book $\mathcal{D}$,

$$\mathcal{C} \iff \neg\mathcal{D}.$$

Constructing the argument

$$\mathcal{C} \iff \neg\mathcal{D} \sim \begin{cases} \mathcal{C} \Longleftarrow \neg\mathcal{D} \\ \mathcal{C} \Longrightarrow \neg\mathcal{D} \end{cases}$$

$\Longleftarrow$ If it is impossible for us to be in a dutch book ($\mathcal{D}$), then we must have been coherent ($\mathcal{C}$).

$\Longrightarrow$ If we are coherent ($\mathcal{C}$), then we cannot end up in a dutch book ($\mathcal{D}$).

$\mathcal{C} \impliedby \neg\mathcal{D}$

This statement is easier to frame as a story, so we being with it, to build our intuition and the set of rules itself.

$$\mathcal{C} \impliedby \neg\mathcal{D}$$

$$\iff$$

$$\neg\mathcal{C} \implies \mathcal{D}$$

By way of contrapositive, we can prove that by not being coherent ($\mathcal{C}$) we can end up in a dutch book ($\mathcal{D}$).

$\neg\mathcal{C} \implies \mathcal{D}$; non-negativity and certain (A_C) events

Any event $\implies P\{A\} \geq 0$

Imagine *You* sell me a ticket for A at price $-p < 0$. No mater what happens, your loss is at least p. Thus

$$P\{A\} < 0 \implies \mathcal{D}$$

A is certain $\implies P\{A\} = 1$

Imagine *You* sell me a ticket for A at price $p < 1$. No mater what happens, your loss is at least $1-p$. Thus

$$P\{A_C\} \neq 1 \implies \mathcal{D}$$

$\neg\mathcal{C} \implies \mathcal{D}$; finite additivity ($FA_>$)

$$\underbrace{\boldsymbol{P\{A_1 \cup A_2\} > P\{A_1\} + P\{A_2\}}}_{\neg \boldsymbol{FA}_>}$$

Imagine *You* buy from me a ticket for $A_1 \cup A_2$ at a price p and sell me tickets for both A_1 and A_2 at a price $p_1 + p_2$.

Thus

$$\neg FA_> \implies \mathcal{D}$$

- If neither event happens there is no payout and the net loss is the price of the tickets

$$-p + p_1 + p_2 < 0$$

- If either A_i happens we both pay each other 1, with net loss being the same

$$-p + p_1 + p_2 + 1 - 1 < 0$$

$\neg\mathcal{C} \implies \mathcal{D}$; finite additivity ($FA_<$)

$$\underbrace{\boldsymbol{P\{A_1 \cup A_2\} < P\{A_1\} + P\{A_2\}}}_{\neg \boldsymbol{FA_<}}$$

Imagine *You* sell me a ticket for $A_1 \cup A_2$ at price a p and buy from me tickets for both A_1 and A_2 at a price $p_1 + p_2$.

Thus

$$\neg FA_< \implies \mathcal{D}$$

- If neither event happens there is no payout and the net loss is the price of the tickets

$$-p_1 - p_2 + p < 0$$

- If either A_i happens we both pay each other 1, with net loss being the same

$$-p_1 - p_2 + p + 1 - 1 < 0$$

$\neg\mathcal{C} \implies \mathcal{D}$; finite additivity ($FA$)

$$\underbrace{\boldsymbol{P\{A_1 \cup A_2\} = P\{A_1\} + P\{A_2\}}}_{\boldsymbol{FA}}$$

Combining the two arguments,

$$\neg FA_< \wedge \neg FA_< \iff \neg FA$$

we arrive at the conclusion

$$P\{A_1 \cup A_2\} \neq P\{A_1\} + P\{A_2\}$$
$$\implies \mathcal{D}$$

stating that a lack of finite additivity, leads to a dutch book.

By induction, the argument is easily extended to any finite number of disjoint events

$$P\{\cup_{i=0}^{N} A_i\} \neq \sum_{i=0}^{N} P\{A_i\} \implies \mathcal{D}$$

Thus

$$\neg FA \implies \mathcal{D}$$

$\mathcal{C} \impliedby \neg\mathcal{D}$ proven by contrapositive

$$\begin{array}{ccc} P\{A\} < 0 \implies \mathcal{D} & & P\{A\} \geq 0 \impliedby \neg\mathcal{D} \\ P\{A_C\} \neq 1 \implies \mathcal{D} & \iff & P\{A_C\} = 1 \impliedby \neg\mathcal{D} \\ \neg FA \implies \mathcal{D} & & FA \impliedby \neg\mathcal{D} \end{array}$$

Collecting the rules into a set

$$\mathcal{C} = \{P\{A\} \geq 0, P\{A_C\} = 1, FA\},$$

we state that if it is impossible to end up in a dutch book then the set of rules $\mathcal{C}$ must have been obeyed when making the trade,

$$\mathcal{C} \impliedby \neg\mathcal{D}.$$

$\mathcal{C} \implies \neg\mathcal{D}$; Describing the trades

Tickets bought for p_i

- A_i If A_i occurred your payout for α_i tickets is $\alpha_i(1 - p_i)$.
- $\neg A_i$ If $\neg A_i$ occurred your payout for α_i tickets is $\alpha_i(-p_i)$.

Tickets sold for p_i

- A_i If A_i occurred your payout for β_i tickets is $-\beta_i(1 - p_i)$.
- $\neg A_i$ If $\neg A_i$ occurred your payout for β_i tickets is $\beta_i p_i$.

After allowing trades on all events, our winnings are described by

$$W = \sum_{i=1}^{N} \lambda_i(I_{A_i} - p_i)$$

where $\lambda_i = \alpha_i - \beta_i$ (or 0 if no trades on A_i) and I_{A_i} is the indicator of A_i.

$\mathcal{C} \implies \neg\mathcal{D}$; The expected payout

Noting that the expectation of an indicator of A_i

$$E(I_{A_i}) = \sum_{k=1}^{N} p_k I_{A_i}(A_k) = p_i$$

is its probability, we see that the expected payout is $E(W) = 0$.

$$\begin{aligned} E(W) &= E(\sum_{i=1}^{N} \lambda_i (I_{A_i} - p_i)) \\ &= \sum_{i=1}^{N} E(\lambda_i (I_{A_i} - p_i)) \\ &= \sum_{i=1}^{N} \lambda_i E(I_{A_i} - p_i) \\ &= 0 \end{aligned}$$

$\mathcal{C} \implies \neg\mathcal{D}$;   A supportive theorem on expectations

Theorem

If X is a non-trivial random variable, then

$$\min X = x_{min} < E(X) < \max X = x_{max}.$$

Proof.

$$\min X = x_{min} = \sum_{i=0}^{n} p_i x_{min} < E(X) < \sum_{i=0}^{n} p_i x_{max} = x_{max} = \max X$$

□

$\mathcal{C} \implies \neg\mathcal{D}$; An immediate corollary

Corollary

If X is non-trivial, there is some positive probability $\epsilon_1 > 0$ that X exceeds its expectation $E(X)$ by a fixed amount $\eta_1 > 0$, and positive probability $\epsilon_2 > 0$ that $E(X)$ exceeds X by a fixed amount $\eta_2 > 0$.

Proof.

Denote by p_1 the probability of x_{min} and p_2 the probability of x_{max}. Than $\eta_1 = x_{max} - E(X) > 0$ and $\eta_2 = E(X) - x_{min} > 0$, with $\epsilon_1 = p_1$ and $\epsilon_2 = p_2$. □

$\mathcal{C} \implies \neg\mathcal{D}$; Implications of $E(W)$

Implications of the $E(W) = 0$

- Either W is trivial and there is no uncertainty, no gambles and no loss, or
- there is a positive probability ϵ, that you will gain at least the amount η, as by Corollary, i.e. no dutch book.

$$\boldsymbol{E(W) = 0 \implies \neg\mathcal{D}}$$

The calculation of the expected payout was dependent on the rules of probability which coincide with the rules of coherence $\mathcal{C}$. Thus

$$\mathcal{C} \implies \neg\mathcal{D},$$

proving the statement.

$\mathcal{C} \iff \neg\mathcal{D}$; Theorem of Coherence

Theorem (Coherence) Your prices $P\{A_i\}$ at which you are willing to buy and sell tickets cannot lead you into a dutch book if and only if they are coherent,

$$\mathcal{C} \iff \neg\mathcal{D}.$$

$$\mathcal{C} = \begin{Bmatrix} P\{A_i\} \geq 0, \\ P\{A_C\} = 1, \\ FA \end{Bmatrix}$$

Coherence for joint and conditional events ($\mathcal{C}^*$)

With coherent views on disjoint events, we seek to quantify our conditional beliefs through joint and conditional trades.

The conditional trade on $A_1|A_2$

Given the outcome

A_2 $\sim$ the ticket is:

- A_1 $\sim$ redeemable for a unit of the quantity
- $\neg A_1$ $\sim$ worth nothing, or

$\neg A_2$ $\sim$ the trade is annulled.

The trade on disjoint A_i

The prices $P\{A_i\}$ are assumed to be coherent ($\mathcal{C}$).

The joint trade on $A_1 A_2$

The trades are priced as if $A_1 A_2$ was a single event $\tilde{A}$.

But we need additional rules $\mathcal{C}^* = \mathcal{C} \cup \{R_i\}$ to avoid being in a dutch book.

$\mathcal{C}^* \impliedby \neg\mathcal{D}$; The space of possible outcomes of a trade

$$W_1 = \lambda_1(1 - P\{A_1 A2\}) + \lambda_2(1 - P\{A_2\}) + \lambda_3(1 - P\{A_1|A_2\})$$

$$W_2 = -\lambda_1 P\{A_1 A2\} + \lambda_2(1 - P\{A_2\}) - \lambda_3 P\{A_1|A_2\}$$

$$W_3 = -\lambda_1 P\{A_1 A2\} - \lambda_2 P\{A_2\}$$

The possible payouts are:

W_1 if A_1 and A_2 happen,

W_2 if $\neg A_1$ and A_2 happen,

W_3 if A_2 does not happen.

It is clear that we need such prices P, that it is impossible to uniquely determine such trades λ_j, that the payout W_i is a loss in any outcome.

$$\exists_{\lambda_j}\forall_i(W_i < 0) \iff \mathcal{D}$$

$\mathcal{C}^* \Longleftarrow \neg\mathcal{D}$; On solutions of linear equations

Three planes can intersect in

0. a unique point,
1. a line,
2. a plane,
3. or do not intersect.

$$M \cdot \vec{\lambda} = \vec{W} \begin{cases} a\lambda_1 + b\lambda_2 + c\lambda_3 = & W_1 \\ d\lambda_1 + e\lambda_2 + f\lambda_3 = & W_2 \\ g\lambda_1 + h\lambda_2 + i\lambda_3 = & W_3 \end{cases}$$

The 0. scenario is the only one that would allow *Them* to uniquely determine the needed trades λ_i to put *Us* in a dutch book. Thus

$$\det(M) \neq 0 \implies \mathcal{D}.$$

$\mathcal{C}^* \impliedby \neg\mathcal{D}$; The impossibility of a solution

$$\begin{vmatrix} 1-P\{A_1A_2\} & 1-P\{A_2\} & 1-P\{A_1|A_2\} \\ -P\{A_1A2\} & 1-P\{A_2\} & -P\{A1|A_2\} \\ -P\{A_1\} & -P\{A_2\} & 0 \end{vmatrix} = P\{A_1A_2\} - P\{A_1|A_2\}\cdot P\{A_2\}$$

If the determinant is not zero, it is always possible to determine such trades λ_j, that the payout W_i is a loss in any outcome.

$$\det \neq 0 \implies \mathcal{D}$$

$$\underbrace{\boldsymbol{P\{A_1A_2\} = P\{A_1|A_2\}\cdot P\{A_2\}}}_{\mathcal{B}}$$

Thus

$$\det = 0 \impliedby \neg\mathcal{D}$$

$$\iff$$

$$\mathcal{B} \impliedby \neg\mathcal{D}$$

$\mathcal{C}^* \implies \neg\mathcal{D}$; Describing the trades

The derived model in $\mathcal{C}$ is easily extended to cover the trade on A_1A_2 ; a trade on a simple event with indicator an $I_{A_1A_2}$.

The conditional trade on $A_1|A_2$ costing $P\{A_1|A_2\}$ and paying 1 if A_1, but only if A_2 is described by,

$$I_{A_2}(I_{A_1} - P\{A_1|A_2\})$$

After allowing trades on all events, the complete payout is described by

$$W = \underbrace{\sum_{i=1}^{2} \lambda_i(I_{A_i} - P\{A_i\})}_{W'} + \underbrace{\lambda_3(I_{A_1A_2} - P\{A_1A_2\})}_{W_{1,2}} + \underbrace{\lambda_4 I_{A_2}(I_{A_1} - P\{A_1|A_2\})}_{W_{1|2}}$$

$\mathcal{C}^* \implies \neg\mathcal{D}$; Implications of $E(W)$

By applying the same mechanics as in the proof of $\mathcal{C}$, we see that both $E(W')$ and $E(W_{1,2})$ are zero.

$$\begin{aligned}
E(W_{1|2}) &= E\Big[\lambda_4 I_{A_2}(I_{A_1} - P\{A_1|A_2\})\Big] \\
&= \lambda_4 E(I_{A_2} I_{A_1}) - \lambda_4 E(I_{A_2} P\{A_1|A_2\}) \\
&= \lambda_4 \Big(E(I_{A_2} I_{A_1}) - P\{A_1|A_2\} E(I_{A_2})\Big) \\
&= \lambda_4 \Big(P\{A_1 A_2\} - P\{A_1|A_2\} P\{A_2\}\Big)
\end{aligned}$$

If we require $\overbrace{P\{A_1 A_2\} = P\{A_1|A_2\} \cdot P\{A_2\}}^{\mathcal{B}}$, we have $E(W) = 0$ and can again apply the Corollary; i.e. no dutch book. Thus

$$\mathcal{B} \implies \neg\mathcal{D}.$$

$\mathcal{C}^* \iff \neg\mathcal{D}$; Theorem of Conditional Coherence

Theorem (Coherence*) Your prices $P\{A_i\}$ and $P\{A_i|A_j\}$ at which you are willing to buy and sell tickets cannot lead you into a dutch book if and only if they are coherent,

$$\mathcal{C}^* \iff \neg\mathcal{D}.$$

$$\mathcal{C}^* = \begin{Bmatrix} P\{A_i\} \geq 0, \\ P\{A_C\} = 1, \\ FA, \\ \mathcal{B} \end{Bmatrix}$$

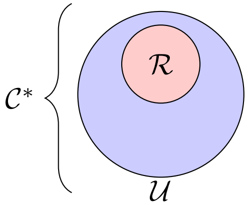
Conclusions: Uncertain and Random events

There was no mention of randomness during our construction. Coherence ($\mathcal{C}^*$) allows us to treat all uncertain ($\mathcal{U}$) events about which we hold subjective beliefs, while the frequentist interpretation of probability can only treat random events ($\mathcal{R}$).

Note that the claim

random $\subset$ uncertain

is justified by de Finetti's theorem.



$\mathcal{R}$ are random events

$\mathcal{U}$ are uncertain events