第三章

1 (1)
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}|x|\,e^{-i\lambda x}dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^a x\,e^{-i\lambda x}dx + \int_{-a}^{0} -x\,e^{-i\lambda x}dx\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^a x\,2\cos\lambda x\,dx$$
$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1}{\lambda}x\sin\lambda x\Big|_0^a - \frac{1}{\lambda}\int_0^a \sin\lambda x\,dx\right]$$
$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1}{\lambda}a\sin a\lambda + \frac{1}{\lambda^2}(\cos\lambda a - 1)\right]$$
$$= \sqrt{\frac{2}{\pi}}\left(\frac{1}{\lambda}a\sin\lambda a + \frac{1}{\lambda^2}\cos\lambda a - \frac{1}{\lambda^2}\right]$$

(2)
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(1-\frac{|x|}{a}\right)e^{i\lambda x}dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{i\lambda}\left(e^{i\lambda a} - e^{-i\lambda a}\right) - \frac{1}{|a|}2\left(\frac{1}{\lambda}\cos\lambda a + \frac{1}{\lambda^2}\cos\lambda a - \frac{1}{\lambda^2}\right)\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{\lambda}\sin\lambda a - \frac{2}{|a|}\left(\frac{1}{\lambda}\cos\lambda a + \frac{1}{\lambda^2}\cos\lambda a - \frac{1}{\lambda^2}\right)\right]$$

(3)
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\sin\lambda_0 x\,e^{-i\lambda x}dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\lambda_0}(-\cos\lambda_0 x)e^{-i\lambda x}\Big|_{-a}^{a} - \frac{1}{\lambda_0}\int_{-a}^{a}(-\cos\lambda_0 x)e^{-i\lambda x}(-i\lambda)dx\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\lambda_0}\cos\lambda_0 a\,2\sin\lambda a - \frac{i\lambda}{\lambda_0^2}\left[\sin\lambda_0 x\cdot e^{-i\lambda x}\Big|_{-a}^{a} - \int_{-a}^{a}\sin\lambda_0 x\cdot e^{-i\lambda x}\cdot(-i\lambda)dx\right]\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\lambda_0}\cos\lambda_0 a\cdot 2\sin\lambda a - \frac{i\lambda}{\lambda_0^2}(\sin\lambda_0 a\cdot 2\cos\lambda a) + \frac{1}{\sqrt{2\pi}}\frac{\lambda^2}{\lambda_0^2}\int_{-a}^{a}\sin\lambda_0 x\,e^{-i\lambda x}dx\right]$$
$$\therefore \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\left(1+\frac{\lambda^2}{\lambda_0}\right)^{-1}\left[\frac{2i}{\lambda_0}\cos\lambda_0 a\sin\lambda a - \frac{2i\lambda}{\lambda_0^2}\sin\lambda_0 a\cos\lambda a\right]$$

(4)
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}\,e^{-i\lambda x}dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{+\infty}e^{-ax}e^{-i\lambda x}dx + \int_{-\infty}^{0}e^{ax}e^{-i\lambda x}dx\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty}e^{-ax}\,2\cos\lambda x\,dx$$
$$= \frac{2}{\sqrt{2\pi}}\left[-\frac{1}{a}\int_0^{+\infty}\cos\lambda x\,de^{-ax}\right]$$
$$= \frac{2}{\sqrt{2\pi}}\left[-\frac{1}{a}e^{-ax}\cos\lambda x\Big|_0^{a} - \frac{\lambda}{a}\int_0^{+\infty}e^{-ax}\sin\lambda x\,dx\right]$$
$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1}{a} + \frac{\lambda^2}{a^2}(-1)\int_0^{+\infty}\cos\lambda x\,e^{-ax}dx\right]$$
$$\therefore \hat{f}(\lambda) = \frac{a}{a^2+\lambda^2}\sqrt{\frac{2}{\pi}}$$

2. (1) $f(x) = x^2 U(x)$, $U = \begin{cases}1 & |x|<a\\ 0 & |x|\geq a\end{cases}$

则 $\hat{f}(\lambda) = -\dfrac{d^2\hat{U}(\lambda.t)}{d\lambda^2}$

$$\hat{U} = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{-i\lambda x}dx = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-i\lambda}\right)e^{-i\lambda x}\Big|_{-a}^{a}$$
$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{-i\lambda}\left[e^{-i\lambda a} - e^{i\lambda a}\right] = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{i\lambda}\cdot 2i\sin\lambda a$$
$$= \frac{2}{\sqrt{2\pi}}\frac{\sin\lambda a}{\lambda}$$
$$\hat{f}(\lambda) = -\frac{2}{\sqrt{2\pi}}\frac{\lambda a\cos\lambda a - \sin\lambda a}{\lambda^2}$$

(2) $f(x) = x e^{-a|x|} \quad (a>0)$

$\hat{f}(\lambda) = i\frac{d\hat{v}}{d\lambda} \qquad v = e^{-a|x|} \quad \hat{v} = \frac{a}{a^2+\lambda^2}$

$\hat{f}(\lambda) = i(-1)\frac{a\,2\lambda}{(a^2+\lambda^2)^2} = i\frac{-2a\lambda}{(a^2+\lambda^2)^2}$

(3) $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{\mu x} e^{-i\lambda x}dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{(\mu-i\lambda)x}dx$

$= \frac{1}{\sqrt{2\pi}}\frac{1}{\mu-i\lambda}[e^{a}e^{-i\lambda a} - e^{-a}e^{i\lambda a}]$

$= \frac{1}{\sqrt{2\pi}}\frac{1}{\mu-i\lambda}[e^{a}(\cos\lambda a - i\sin\lambda a) - e^{-a}(\cos\lambda a + i\sin\lambda a)]$

$= \frac{1}{\sqrt{2\pi}}\frac{1}{\mu-i\lambda}[\cos\lambda a\cdot 2\sinh a - i\sin\lambda a\cdot 2\cosh a]$

(4) $\sin(\lambda_0 x) = \frac{1}{2i}(e^{i\lambda_0 x} - e^{-i\lambda_0 x})$

故 $f(x) = \frac{1}{2i}[e^{i\lambda_0 x}e^{-a|x|} - e^{-i\lambda_0 x}e^{-a|x|}]$

对 $\lambda_0\in R$ 易证 $\widehat{e^{i\lambda_0 x}f(x)} = \hat{f}(\lambda-\lambda_0)$

$\hat{f}(x) = \frac{1}{2i}\left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+(\lambda-\lambda_0)^2} - \sqrt{\frac{2}{\pi}}\frac{a}{a^2+(\lambda+\lambda_0)^2}\right]$

(5) $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} e^{i\lambda_0 x}e^{-i\lambda x}dx$

$= \frac{1}{\sqrt{2\pi}}\frac{1}{i(\lambda_0-\lambda)}e^{i(\lambda_0-\lambda)x}\Big|_{-L}^{L}$

$= \sqrt{\frac{2}{\pi}}\frac{\sin(\lambda_0-\lambda)L}{\lambda_0-\lambda} \quad (\lambda\neq\lambda_0)$

若 $\lambda=\lambda_0$ 则 $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L}dx = \sqrt{\frac{2}{\pi}}L$

(6) $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ax^2+ibx+c}\cdot e^{-i\lambda x}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-a(x-\frac{ib}{2a})^2+c+a(\frac{ib}{2a})^2}\cdot e^{-i\lambda x}dx$

$= \frac{1}{\sqrt{2\pi}}e^{c-\frac{b^2}{4a}}\int_{-\infty}^{+\infty} e^{-a(x-\frac{ib}{2a})^2}e^{-i\lambda x}dx$

$= \frac{1}{\sqrt{2\pi}}e^{c-\frac{b^2}{4a}}e^{-i\lambda\frac{ib}{2a}}\int_{-\infty}^{+\infty} e^{-ax^2}e^{-i\lambda x}dx$

$= e^{c-\frac{b^2}{4a}}e^{\frac{b\lambda}{2a}}\frac{1}{\sqrt{2a}}e^{-\lambda^2/(4a)}$

(7) $f(x) = f(-x) \quad (f(x))^{\vee}(\lambda) = \hat{f}(-\lambda) = (f(-x))^{\wedge}(\lambda) = \hat{f}(\lambda)$

$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}e^{-i\lambda x}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}e^{i\lambda x}dx = \left(\frac{1}{a^2+x^2}\right)^{\vee}$

$\left(\sqrt{\frac{\pi}{2}}\frac{1}{a}e^{-a|x|}\right)^{\wedge} = \frac{1}{a^2+\lambda^2}$

$\sqrt{\frac{\pi}{2}}a^{-1}e^{-a|x|} = \left(\frac{1}{a^2+\lambda^2}\right)^{\vee}$

$\hat{f}(\lambda) = a e^{-a|\lambda|}$

(8) $\hat{f}(\lambda) = i\frac{d}{d\lambda}\left(\frac{1}{a^2+x^2}\right)^{\wedge} = -ia^2 e^{-a|\lambda|}, \quad \lambda>0$

(9) $f(x) = \frac{1}{(a^2+x^2)^2} = \frac{1}{a^2}\frac{a^2+x^2-x^2}{(a^2+x^2)^2} = \frac{1}{a^2}\left[\frac{1}{a^2+x^2} - x^2\frac{1}{a^2+x^2}\right]$

$\hat{f}(\lambda) = \frac{1}{a^2}\left[a e^{-a|\lambda|} + \frac{d^2}{d\lambda^2}(a e^{-a|\lambda|})\right]$

3. (1) $(f(\lambda))^{\vee} = \frac{1}{a\sqrt{2t}}e^{-\frac{x^2}{4a^2t}}$

(2) $f(\lambda) = e^{[-a^2(\lambda-\frac{ib}{2a^2})^2 + c + a^2\cdot\frac{-b^2}{4a^4}]t}$

$$= e^{(c-\frac{b^2}{4a^2})t}\, e^{-a^2(\lambda-\frac{ib}{2a^2})^2 t}$$

平移 $(f(x-\xi))^\vee = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\lambda-\xi)\, e^{i\lambda x}\, d\lambda$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\lambda')\, e^{i(\lambda'+\xi)x}\, d\lambda'$$

$\therefore (f(\lambda))^\vee = e^{(c-\frac{b^2}{4a^2})t}\, e^{i(\frac{ib}{2a^2})x}\, (e^{-a^2\lambda^2 t})^\vee$

$$= e^{(c-\frac{b^2}{4a^2})t}\, e^{-\frac{b}{2a^2}x}\cdot\frac{1}{a\sqrt{2t}}\, e^{-x^2/(4a^2 t)}$$

(3) $(f(\lambda))^\vee = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-|\lambda|y}\, e^{i\lambda x}\, d\lambda$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_0^{+\infty} e^{-\lambda y} e^{i\lambda x}\, d\lambda + \int_{-\infty}^{0} e^{\lambda y} e^{i\lambda x}\, d\lambda\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-y+ix}(-1) + \frac{1}{y+ix}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{-y-ix+(-y)+ix}{(-y+ix)(y+ix)}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{-2y}{-y^2-x^2}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{y}{y^2+x^2}$$

4. (1) 对 $u(x,t)$ 关于 x 作 Fourier 变换 记 $\hat u(\lambda,t) = (u(x,t))^\wedge$

则有 $\frac{d\hat u(\lambda,t)}{dt} + a^2\lambda^2\hat u - ib\lambda\hat u - c\hat u = \hat f(\lambda,t)$

故 $$\begin{cases} \frac{d\hat u(\lambda,t)}{dt} + (a^2\lambda^2 - ib\lambda - c)\,\hat u = \hat f \\ \hat u|_{t=0} = \hat\varphi(\lambda) \end{cases}$$

$\Longrightarrow \hat u(\lambda,t) = e^{(-a^2\lambda^2+ib\lambda+c)t}\,\hat\varphi(\lambda) + \int_0^t e^{(-a\lambda^2+ib\lambda+c)(t-\tau)}\,\hat f(\lambda,\tau)\, d\tau$

$\therefore u(x,t) = e^{(c-\frac{b^2}{4a^2})t}\,\frac{1}{a\sqrt{2t}}\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/(4a^2t)}\cdot e^{-\frac{b}{2a^2}(x-\xi)}\,\varphi(\xi)\, d\xi$

$$+ \int_0^t \frac{1}{a\sqrt{t-\tau}}\, e^{(c-\frac{b^2}{4a^2})(t-\tau)} \int_{-\infty}^{+\infty} e^{-(x-\xi)^2/4a^2(t-\tau)}\cdot e^{-\frac{b}{2a^2}(x-\xi)}\cdot f(\xi,\tau)\, d\xi\, d\tau$$

(2) $(u(x,y))^\wedge = \hat u(\lambda,y)$ 对方程关于 x 作 Fourier 变换, 得

$$\begin{cases} (i\lambda)^2\,\hat u(\lambda,y) + \frac{\partial^2\hat u}{\partial y^2} = 0 \quad\Rightarrow\quad \frac{d^2\hat u}{dy^2} - \lambda^2\hat u(\lambda,y) = 0 \\ \hat u|_{y=0} = \hat\varphi(\lambda) \end{cases}$$

$\Longrightarrow \hat u(\lambda,y) = c\,e^{-|\lambda|y}$ 由初始条件 $\Longrightarrow \hat u(\lambda,y) = \hat\varphi(\lambda)\,e^{-|\lambda|y}$

$u(x,y) = (e^{-|\lambda|y})^\vee * \varphi$

$$= \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{y}{y^2+(x-\xi)^2}\,\varphi(\xi)\, d\xi$$

5.(1) $\forall \varphi \in \mathcal{D}(R)$ 有

$$\langle \varphi(x)\delta(x), \psi(x)\rangle = \langle \delta(x), \varphi(x)\psi(x)\rangle$$

$$= \varphi(0)\psi(0) = \varphi(0)\langle \delta, \psi\rangle = \langle \varphi(0)\delta, \psi\rangle$$

$$\therefore \varphi(x)\delta(x) = \varphi(0)$$

(2) $\forall \psi \in \mathcal{D}(R)$ 有

$$\langle \varphi(x)\delta'(x), \psi\rangle = \langle \delta', \varphi(x)\psi\rangle = -\langle \delta, (\varphi\psi)'\rangle$$

$$= -\langle \delta, \varphi\psi' + \varphi'\psi\rangle = -(\varphi(0)\psi'(0) + \varphi'(0)\psi(0))$$

$$= \varphi(0)\langle \delta', \psi\rangle - \varphi'(0)\langle \delta, \psi\rangle$$

$$= \langle \varphi(0)\delta' - \varphi'(0)\delta, \psi\rangle$$

$$\therefore \varphi(x)\delta'(x) = \varphi(0)\delta' - \varphi'(0)\delta$$

(3) $\langle x\delta^{(m)}(x), \varphi\rangle = \langle \delta^{(m)}(x), x\varphi\rangle = (-1)^m\langle \delta, (x\varphi)^{(m)}\rangle$

$$= (-1)^m\langle \delta, x\cdot\varphi^{(m)}(x) + m\varphi^{(m-1)}(x)\rangle$$

$$= (-1)^m m\cdot\varphi^{(m-1)}(0)$$

$$= (-1)^m m\langle \delta, \varphi^{(m-1)}\rangle$$

$$= -m\langle \delta^{(m-1)}, \varphi\rangle$$

$$\therefore x\delta^{(m)}(x) = -m\delta^{(m-1)}$$

(4) $\forall \varphi \in \mathcal{D}(R)$, 有

$$\langle x^m\delta^{(m)}(x), \varphi\rangle = \langle \delta^{(m)}, x^m\varphi(x)\rangle$$

$$= \langle \delta, (x^m\varphi(x))^{(m)}\rangle$$

$$= (-1)^{(m)}\langle \delta, C_m^0 x^m\varphi^{(m)} + \cdots + C_m^m m!\varphi(x)\rangle$$

$$= (-1)^{(m)} m!\varphi(0) = (-1)^m m!\langle \delta, \varphi\rangle$$

$$\therefore x^m\delta^{(m)}(x) = (-1)^m m!\delta(x)$$

(5) $\forall \varphi \in \mathcal{D}(R)$, 有

$$\langle (H(x)\rho(x))', \varphi\rangle = -\langle H(x)\rho(x), \varphi'\rangle$$

$$= -\int_0^{+\infty}\rho(x)\varphi'(x)dx$$

$$= \rho(0)\varphi(0) + \int_{-\infty}^{+\infty}H(x)\rho'(x)\varphi(x)dx$$

$$= \rho(0)\langle \delta, \varphi\rangle + \langle H\rho', \varphi\rangle$$

$$= \langle \rho(0)\delta + H\rho', \varphi\rangle$$

$$\therefore (H(x)\rho(x))' = \delta(x)\rho(0) + H(x)\rho'(x)$$

6.(1) $\forall \varphi \in \mathcal{D}(R)$, 若 $m \geq 2$

$$\langle |x|^{(m)}, \varphi\rangle = (-1)^m\langle |x|, \varphi^{(m)}\rangle$$

$$= (-1)^m\int_{-\infty}^{+\infty}|x|\varphi^{(m)}dx$$

$$= (-1)^m\left[\int_0^{+\infty}x\varphi^{(m)}(x)dx + \int_{-\infty}^0(-x)\varphi^{(m)}(x)dx\right]$$

$$= (-1)^m[2\varphi^{(m-2)}(0)]$$

$$= 2\cdot(-1)^m\langle \delta, \varphi^{(m-2)}\rangle$$

$$= 2\langle \delta^{(m-2)}, \varphi\rangle$$

$$\therefore (|x|)^{(m)} = 2\delta^{(m-2)}$$

若 $m = 1$, 则

$$\langle |x|', \varphi\rangle = -\langle |x|, \varphi'\rangle = -\left(\int_0^{+\infty}x\varphi'(x)dx + \int_{-\infty}^0 -x\varphi'(x)dx\right)$$

$$= -\left[-\int_0^{+\infty} \varphi\, dx + \int_{-\infty}^0 \varphi\, dx\right]$$
$$= \int_{-\infty}^{+\infty} H(x)\varphi(x)\,dx - \int_{-\infty}^{+\infty} H(-x)\varphi(x)\,dx$$
$$= \langle H(x) - H(-x), \varphi \rangle$$
$$\therefore (|x|)' = H(x) - H(-x)$$

(2) $\langle (H(x)\sin x)', \varphi \rangle = -\langle H(x)\sin x, \varphi' \rangle$
$$= -\int_0^{+\infty} \sin x \cdot \varphi'(x)\,dx$$
$$= -\sin x\, \varphi(x)\Big|_0^{+\infty} + \int_0^{+\infty} \cos x\, \varphi(x)\,dx$$
$$= \int_{-\infty}^{+\infty} H(x)\cos x\, \varphi(x)\,dx$$
$$\therefore (H(x)\sin x)' = H(x)\cos x$$

(3) $\langle (H(x)e^{ax})'', \varphi \rangle = \langle H(x)e^{ax}, \varphi'' \rangle$
$$= \int_0^{+\infty} e^{ax}\varphi''(x)\,dx$$
$$= e^{ax}\cdot\varphi'(x)\Big|_0^{+\infty} - a\int_0^{+\infty} e^{ax}\varphi'(x)\,dx$$
$$= \varphi'(0) - a\left(e^{ax}\varphi\Big|_0^{+\infty} - a\int_0^{+\infty} e^{ax}\varphi(x)\,dx\right)$$
$$= \varphi'(0) + a\varphi(0) + a^2\int_{-\infty}^{+\infty} H(x)e^{ax}\varphi(x)\,dx$$
$$= -\langle \delta', \varphi \rangle + a\langle \delta, \varphi \rangle + a^2\langle H(x)e^{ax}, \varphi \rangle$$
$$\therefore (H(x)e^{ax})'' = -\delta' + a\delta + a^2 H(x)e^{ax}$$

7. (1) $\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_0^{+\infty} \sin x\, \varphi'(x)\,dx$
$$= -\sin x \cdot \varphi\Big|_0^{+\infty} + \int_0^{+\infty} \cos x \cdot \varphi(x)\,dx$$
$$= \langle H(x)\cos x, \varphi \rangle$$
$$\therefore f' = H(x)\cos x$$

(2) $\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_0^{+\infty} \cos x\, \varphi'(x)\,dx$
$$= -\varphi'(0) - \int_{-\infty}^{+\infty} H(x)\sin x\, \varphi(x)\,dx$$
$$= \langle \delta', \varphi \rangle - \langle H(x)\sin x, \varphi \rangle$$
$$\therefore f' = \delta' - H(x)\sin x$$

(3) $\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_{-1}^1 x^2 \varphi'(x)\,dx$
$$= -x^2\varphi\Big|_{-1}^1 + \int_{-1}^1 2x\cdot\varphi(x)\,dx$$
$$= -\varphi(1) + \varphi(-1) + \int_{-1}^1 2x\,\varphi(x)\,dx$$
$$= -\langle \delta(x-1), \varphi \rangle + \langle \delta(x+1), \varphi \rangle + \int_{-1}^1 2x\,\varphi(x)\,dx$$

记 $g(x) = \begin{cases} 2x, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

则 $f' = -\delta(x-1) + \delta(x+1) + g(x)$

8. (1) 令 $z=\frac{x}{2a\sqrt{t}}$

$$\frac{\partial\Phi}{\partial t}-a^2\frac{\partial^2\Phi}{\partial x^2}=\Phi'(z)\frac{1}{2a}\left(-\frac{x}{2\sqrt{t}\cdot t}\right)-a^2\left(\Phi'(z)\frac{1}{2a\sqrt{t}}\right)_x$$

$$a^2\left(\Phi'(z)\frac{1}{2a\sqrt{t}}\right)_x=a^2\frac{1}{2a\sqrt{t}}\left(\Phi'(z)\cdot\left(-2\frac{x}{2a\sqrt{t}}\right)\frac{1}{2a\sqrt{t}}\right)$$

$$=\Phi'(z)\cdot\frac{1}{2t}\cdot\frac{-x}{2a\sqrt{t}}$$

$$\therefore\ \frac{\partial\Phi}{\partial t}-a^2\frac{\partial^2\Phi}{\partial x^2}=0$$

令 $v=u-U_0$ 则

$$\begin{cases} v_t-a^2v_{xx}=0\\ v(x,0)=u(x,0)-U_0=-U_0\\ v|_{x=0}=0\end{cases}$$

令 $v=c\cdot\Phi\left(\frac{x}{2a\sqrt{t}}\right)$ 则 v满足方程 且 $v|_{x=0}=0$

由 $v|_{t=0}=-U_0$ 得

$$c\frac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-\xi^2}d\xi=c=-U_0$$

$$\therefore\ u(x,t)=-U_0\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-\xi^2}d\xi+U_0$$

(2) 偶延拓 $\bar{u}(x,0)=\begin{cases}U_0 & 0\le x\le 1\\ 0 & x>1\\ U_0 & -1\le x<0\\ 0 & x<-1\end{cases}$

则 $\begin{cases}\bar{u}_t-a^2\bar{u}_{xx}=0\\ \frac{\partial\bar{u}}{\partial x}\Big|_{x=0}=0\\ \bar{u}|_{t=0}=\text{given}\end{cases}$

令 $\bar{u}=C_1\left[\Phi\left(\frac{x}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x}{2a\sqrt{t}}\right)\right]+C_2\left[\Phi\left(\frac{x-1}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x-1}{2a\sqrt{t}}\right)\right]$

满足方程和边界条件,再由初始条件知

$x\in[0,1]\Rightarrow$

$$C_1\frac{2}{\sqrt{\pi}}\left(\int_0^{+\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)+C_2\frac{2}{\sqrt{\pi}}\left(\int_0^{-\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)=U_0$$

$x\in(1,+\infty)\Rightarrow$

$$C_1\frac{2}{\sqrt{\pi}}\left(\int_0^{+\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)+C_2\frac{2}{\sqrt{\pi}}\left(\int_0^{+\infty}e^{-\xi^2}d\xi+\int_0^{-\infty}e^{-\xi^2}d\xi\right)=0$$

$\Longrightarrow\ C_1\cdot 0+C_2(-2)=U_0 \quad\Rightarrow\ C_2=-\frac{U_0}{2}$

$\quad C_1\cdot 0+C_2\cdot 0=0$

解为 $u=-\frac{U_0}{2}\left[\Phi\left(\frac{x-1}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x-1}{2a\sqrt{t}}\right)\right]+C\left[\Phi\left(\frac{x}{2a\sqrt{t}}\right)+\Phi\left(\frac{-x}{2a\sqrt{t}}\right)\right]$

(解不唯一)

(3)

(4)

9. (1) 令 $u = X(x)T(t)$ 代入方程

$$X(x)T'(t) - X''(x)T(t) - X(x)T(t) = 0$$

$$\therefore \frac{X''(x)}{X(x)} = -\lambda = \frac{T'(t) - T(t)}{T(t)}$$

由边界条件 $\Rightarrow X_n(x) = \sin\frac{n\pi}{l}x \qquad \lambda_n = \left(\frac{n\pi}{l}\right)^2$

代入 $T'(t) + (\lambda - 1)T(t) = 0 \Rightarrow T_n(t) = C_n e^{-a^2\left[\left(\frac{n\pi}{l}\right)^2 - 1\right]t}$

$$u = \sum_{n=1}^{\infty} C_n e^{-a^2\left[\left(\frac{n\pi}{l}\right)^2 - 1\right]t} \sin\frac{n\pi}{l}x$$

由初始条件得 $\sum C_n \sin\frac{n\pi}{l}x = 1 \qquad C_n = 1$

$$\therefore u = \sum_{n=1}^{\infty} e^{-\left[\left(\frac{n\pi}{l}\right)^2 - 1\right]} \sin\frac{n\pi}{l}x$$

(2) $u = \sum_{n=1}^{\infty} C_n e^{-a^2\left(\frac{n\pi}{\pi}\right)^2 t} \cdot \cos\frac{n\pi}{\pi}x$

$$= \sum_{n=0}^{\infty} C_n e^{-a^2n^2t} \cos nx$$

由初始条件 得 $C_n = \dfrac{\int_0^{\pi} \sin x \cos nx\, dx}{\int_0^{\pi} \cos^2 nx\, dx} = \begin{cases} 0 & n \geq 1 \\ \frac{2}{\pi} & n = 0 \end{cases}$

(3) $u(x,t) = \sum_{n=0}^{\infty} C_n e^{-a^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)^2 t} \cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x$

由 $\sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x = x^2(1-x)$

$$C_n = \frac{\int_0^l x^2(1-x)\cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x\, dx}{\int_0^l \cos\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x\, dx}$$

(4) 令 $V = u - Atx/l$. 则有

$$\begin{cases} V_t - a^2 V_{xx} = -Ax/l \\ V|_{x=0} = 0 \\ V|_{x=l} = u|_{x=0} - At = 0 \\ V|_{t=0} = u|_{t=0} = 0 \end{cases}$$

$$V(x,t) = \int_0^t \sum_{n=1}^{\infty} \left(f_n(\tau) e^{-a^2\left(\frac{n\pi}{l}\right)^2(t-\tau)} d\tau \sin\frac{n\pi}{l}x\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{l}\int_0^l (-Ax/l)\sin\frac{n\pi}{l}x\, dx \int_0^t e^{-a^2\left(\frac{n\pi}{l}\right)^2(t-\tau)} d\tau \cdot \sin\frac{n\pi}{l}x$$

(5) 令 $V = u - x$. 则有

$$\begin{cases} V_t - a^2 V_{xx} = x(l - x) = f(x) \\ V|_{t=0} = \sin\frac{n\pi}{l}x - x = \varphi(x) \\ V|_{x=0} = 0 \\ V_x|_{x=l} = u_x|_{x=l} - 1 = 0 \end{cases}$$

$$V(x,t) = \sum_{n=1}^{\infty} \varphi_n \cdot e^{-a^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)^2 t} \sin\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x + \sum_{n=1}^{\infty} \int_0^t f_n e^{-a^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)^2 (t-\tau)} d\tau \sin\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x$$

其中 $\varphi_n = \int_0^l \varphi(x)\sin\frac{n\pi + \frac{\pi}{2}}{l}x\, dx \Big/ \int_0^l \sin^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x\, dx$

$$f_n = \int_0^l f(x)\sin\frac{n\pi + \frac{\pi}{2}}{l}x\, dx \Big/ \int_0^l \sin^2\left(\frac{n\pi + \frac{\pi}{2}}{l}\right)x\, dx$$

(6) 令 $V = u - q x^2/2l$ 则有

$$\begin{cases} V_t - a^2 V_{xx} = a^2 q/l = f(x) \\ V|_{t=0} = -q x^2/2l = \varphi(x) \\ V_x|_{x=0} = 0 \\ V_x|_{x=l} = 0 \end{cases}$$

$$V(x,t) = \sum_{n=0}^{\infty} \varphi_n \cdot e^{-(\frac{n\pi}{l})^2 t} \cos\frac{n\pi}{l}x + \sum_{n=0}^{\infty} \int_0^t f_n e^{-(\frac{n\pi}{l})^2(t-\tau)} d\tau \cos\frac{n\pi}{l}x$$

$$\sum \varphi_n \cos\beta_n x = -u_0$$

$$\therefore \varphi_n = \int_0^l -u_0 \cos\beta_n x \, dx \Big/ \int_0^l \cos^2\beta_n x \, dx$$

10. (1) 令 $V = u - \left(\frac{1+hl}{2h+h^2 l} - \frac{hx}{2h+h^2 l}\right)hu_1 + \left(\frac{1}{2h+h^2 l} + \frac{h}{2h+h^2 l}x\right)hu_2$

$= u - w$

则 $\begin{cases} V_t - a^2 V_{xx} = 0 \\ V|_{t=0} = \varphi(x) - w \\ \left[-\frac{\partial V}{\partial x} + hV\right]\Big|_{x=0} = 0 \\ \left(\frac{V}{\partial x} + hV\right)\Big|_{x=l} = 0 \end{cases}$

特征值 $\lambda_n = \beta_n^2$ β_n 满足 $\tan\beta l = \frac{2h\beta}{\beta^2 - h^2}$

记正根为 $0 < \beta_1 < \beta_2 < \cdots$ $\lim_{n\to\infty}\beta_n = +\infty$ 则

$$V(x,t) = \sum_{n=1}^{\infty}(\varphi_n - W_n)e^{-a^2\beta_n^2 t}\left(\cos\beta_n x + \frac{h}{\beta_n}\sin\beta_n x\right)$$

易证 $\lim_{t\to+\infty} V(x,t) = 0$

故 $\lim_{t\to+\infty} u = \lim_{t\to+\infty}(V+W) = W = \frac{xh}{2+hl}(u_2 - u_1) + \frac{u_1 + u_2 + hlu_1}{2+hl}$

(2) 令 $V = e^{ht}(u - u_0)$ 则

$$\begin{cases} V_t - a^2 V_{xx} = he^{ht}(u-u_0) + e^{ht}u_t - a^2 u_{xx}e^{ht} = 0 \\ V|_{t=0} = \varphi(x) - u_0 \\ V_x|_{x=0,l} = e^{ht}u|_{x=0,l} = 0 \end{cases}$$

$$\therefore V(x,t)=\sum_{n=0}^{\infty}(\varphi_n-V_{0n})e^{-(\frac{an\pi}{l})^2 t}\cos\frac{n\pi}{l}x$$

$$=(\varphi_0-V_{00})+\sum_{n=1}^{\infty}(\varphi_n-V_{0n})e^{-(\frac{an\pi}{l})^2 t}\cos\frac{n\pi}{l}x$$

$$\longrightarrow \varphi_0-V_{00}\quad (t\to+\infty)$$

$$\lim_{t\to+\infty}u=\lim_{t\to+\infty}(e^{-ht}V+V_0)=V_0$$

11. 记 $Q_T=\{(x,t)\mid 0<x<l=l_1+l_2,\ 0<t\le T\}$

$$u_t-a_1^2u_{xx}=0\qquad 0<x<l_1,\ t>0$$

$$u_t-a_2^2u_{xx}=0\qquad 0<x<l_1+l_2,\ t>0$$

$$u|_{t=0}=\varphi(x)\qquad 0<x<l_1+l_2$$

$$u|_{x=0,l_1+l_2}=0$$

$$\begin{cases}u(l_1-0,t)=u(l_1+0,t)\\ k_1u_x(l_1-0,t)=k_2u_x(l_1+0,t)\end{cases}$$

设 $u(l_1-0,t)=u(l+0,t)=g(t)$ (待定)

求解 $\begin{cases}u_t-a_1^2u_{xx}=0 & 0<x<l_1\\ u|_{t=0}=\varphi(x)\\ u|_{x=0}=0\\ u|_{x=l}=g\end{cases}$

令 $v=u-\frac{x}{l}g$ 则

$$v_t-a^2v_{xx}=-\frac{x}{l}g'(t)+0=-\frac{x}{l}g'(t)=f(x,t)$$

$$v|_{t=0}=-\frac{x}{l}g(0)=G$$

$$v|_{x=0}=0$$

$$v|_{x=l}=0$$

$$V(x,t)=\sum_{n=1}^{\infty}G_n e^{-(\frac{an\pi}{l_1})^2 t}\sin\frac{n\pi}{l_1}x+\sum_{n=1}^{\infty}\int_0^t f_n(\tau)e^{-(\frac{an\pi}{l_1})^2(t-\tau)}d\tau\sin\frac{n\pi}{l_1}x$$

$$\therefore u=v+\frac{x}{l}g\qquad \text{in } 0<x<l_1$$

考虑 $\begin{cases}u_t-a^2u_{xx}=0 & \text{in } l_1<x<l_1+l_2\\ u|_{t=0}=\varphi\\ u|_{x=l_1}=g\\ u|_{x=l_1+l_2}=0\end{cases}$

令 $h(x,t)=u(x,t)-\frac{l_2+l_1-x}{l_2}g(t)$ 则

$$\begin{cases}h_t-a^2h_{xx}=-\frac{l_2+l_1-x}{l_2}g'(t)=\tilde f(x,t) & l_1<x<l_1+l_2\\ h|_{t=0}=\varphi(x)-\frac{l_2+l_1-x}{l_2}g(0)=\Phi(x)\\ h|_{x=l_1}=0\\ h|_{x=l_1+l_2}=0\end{cases}$$

故 $$h(x,t)=\sum C_n e^{-(\frac{an\pi}{l_2})^2 t}\sin\frac{n\pi}{l_2}(x-l_1)+\sum\int_0^t\tilde f_n(\tau)e^{-(\frac{an\pi}{l_2})^2(t-\tau)}d\tau\sin\frac{n\pi}{l_2}(x-l_1)$$

$$=\sum\Phi_n e^{-(\frac{an\pi}{l_2})^2 t}\sin\frac{n\pi}{l_2}(x-l_1)+\sum\int_0^t\tilde f_n(\tau)e^{-(\frac{an\pi}{l_2})^2(t-\tau)}d\tau\sin\frac{n\pi}{l_2}(x-l_1)$$

$\therefore\ u = h(x,t) + \frac{l_1+l_2-x}{l_2} g(t)$, $l_1 < x < l_1+l_2$

由连接条件得

$V|_{x=l_1} + g\cdot\frac{x}{l} = h|_{x=l_1} + \frac{l_2+l_1-x}{l_2} g$ 自然成立

另外 $k_1\Big[\sum_{n=1}^{\infty} \int_0^t f_n(\tau) e^{-(\frac{an\pi}{l_1})^2(t-\tau)} d\tau\, \frac{n\pi}{l_1}(-1)^n$

$+ \sum_{n=1}^{\infty} C_n e^{-(\frac{an\pi}{l_1})^2 t}\cdot\frac{n\pi}{l_1}(-1)^n + \frac{g}{l_1}\Big]$

$= k_2\Big[\sum_{n=1}^{\infty} \Phi_n e^{-(\frac{an\pi}{l_2})^2 t}\frac{n\pi}{l_2} + \sum_{n=1}^{\infty}\int_0^t f_n(\tau) e^{-(\frac{an\pi}{l_2})^2(t-\tau)}d\tau$

$\cdot\frac{n\pi}{l_2} - \frac{g(t)}{l_2}\Big]$

找 $g(t)$ 满足的方程和定解条件 ？？

13. 证明：令 $V=u_t$ 则有

$$\begin{cases} V_t - V_{xx} = f_t(x,t) \\ V|_{x=0,l} = 0 \\ V|_{t=0} = u_t|_{t=0} = (f+u_{xx})|_{t=0} = f(x,0)+\varphi''(x) \end{cases}$$

由最大模估计，知

$\max_{\bar Q}|V(x,t)| \le C[\|f\|_{C(\bar Q)} + \|f(x,0)\|_{C[0,l]} + \|\varphi''\|_{C[0,l]}]$

即 $\max_{\bar Q}|u_t(x,t)| \le C[\|f\|_{C^1(\bar Q)} + \|\varphi''\|_{C[0,l]}]$

14 (1) 因为 $\varphi\in C^1[0,l]$ $\forall x\in[0,l]$

有 $\varphi(x)=\varphi(x)-\varphi(0)=\varphi'(\xi)x \le \|\varphi'\|_{C[0,l]}\cdot x$

记 $V(x,t)=\|\varphi'\|_{C[0,l]}\cdot x$

$u_t-u_{xx}=0 \le V_t-V_{xx}=0$

$u|_{t=0}=\varphi(x)\le V(x,0)=V(x)$

$u|_{x=0}=0\le V|_{x=0}$

$u|_{x=l}=0\le V|_{x=l}$

由比较原理 $\Rightarrow$ $u\le V$ in $\bar Q$

$\frac{u(x,t)-u(0,t)}{x} \le \frac{V(x,t)-V(0,t)}{x}$, $x>0$

令 $x\to0^+$ 得 $\frac{\partial u(0,t)}{\partial x} \le \lim_{x\to0^+}\|\varphi'\|_{C[0,l]} = \|\varphi'\|_{C[0,l]}$

同法 由 $\varphi(x)\ge -\|\varphi'\|_C\cdot x$

$\frac{u(x,t)-u(0,t)}{x} \ge \frac{V(x,t)-\|\varphi'\|_C}{x} = -\|\varphi'\|_{C[0,l]}$

令 $x\to0^+$，得 $\frac{\partial u(0,t)}{\partial x}\ge -\|\varphi'\|_{C[0,l]}$

$\Rightarrow \|\frac{\partial u}{\partial x}(0,t)\| \le \|\varphi'\|_{C[0,l]} \le \|\varphi\|_{C^1[0,l]}$

再由 $\varphi(x)-\varphi(l)=\varphi'(\xi)(l-x)$ $\Rightarrow$

$-\|\varphi'\|_C(l-x)\le\varphi(x)\le\|\varphi'\|_C\cdot(l-x)$

类似证明 $-\|\varphi'\|_C(l-x)\le u(x,t)\le\|\varphi'\|_C(l-x)$

$\frac{-\|\varphi'\|_C\cdot(l-x)}{l-x} \le \frac{u(x,t)-u(l,t)}{l-x} \le \frac{\|\varphi'\|_C\cdot(l-x)-0}{l-x}$

$-\|\varphi'\|_C\le -\frac{\partial u(l,t)}{\partial x}\le\|\varphi'\|_C$

$|\frac{\partial u}{\partial x}(l,t)|\le\|\varphi'\|_C\le\|\varphi\|_{C^1[0,l]}$

(2) 令 $v=u_x$，则

$$\begin{cases} v_t - v_{xx} = 0 \\ v|_{t=0} = \varphi'(x) \\ v|_{x=0} = \frac{\partial u}{\partial x}(0,t) \\ v|_{x=l} = \frac{\partial u}{\partial x}(l,t) \end{cases}$$

由最大模估计 得

$$\max_{\bar{Q}} |v| \leq C \max\left\{ \|\varphi'\|_{C[0,l]},\ \left\|\frac{\partial u(0,t)}{\partial x}\right\|_{C[0,T]},\ \left\|\frac{\partial u(l,T)}{\partial x}\right\|_{C[0,T]} \right\}$$

$$\leq \tilde{C}$$

$\tilde{C}$ 和 $\|\varphi\|_{C^1}$，C 有关

15. 令 $v=\frac{\partial u}{\partial x}$，则有

$$\begin{cases} Lv = f_x(x,t) \\ v|_{t=0} = \varphi'(x) \\ v|_{x=0} = u_x|_{x=0} = u(0,t) + g_1(t) \\ v|_{x=l} = g_2(t) - \beta u(l,t) \end{cases}$$

由最大模估计，得

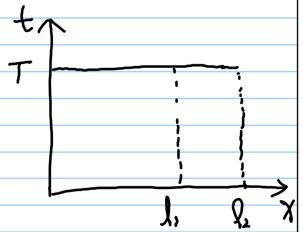
$$\max_{\bar{Q}} |v(x,t)| \leq C\Big(\|f_x\|_{C(\bar{Q})} + \max\big\{\|\varphi'\|_{C[0,l]},$$
$$\|g_1\|_{C[0,T]} + \alpha\|u(0,t)\|_{C[0,T]},\ \|g_2\|_{C[0,T]} + \beta\|u(l,t)\|_{C[0,T]}\big\}\Big)$$

$$\leq C\big[\|f_x\|_{C(\bar{Q})} + \|\varphi'\|_{C[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]} +$$
$$\alpha(\|f\|_{C(\bar{Q})} + \|\varphi\|_{C[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]}) +$$
$$\beta(\|f\|_{C(\bar{Q})} + \|\varphi\|_{C[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]})\big]$$

$$= C(\alpha,\beta)\big[\|f\|_{C'(\bar{Q})} + \|\varphi\|_{C'[0,l]} + \|g_1\|_{C[0,T]} + \|g_2\|_{C[0,T]}\big)$$

16. 因为

$$\begin{cases} Lu_{l_2} = \frac{\partial u_{l_2}}{\partial t} - \frac{\partial^2 u_{l_2}}{\partial x^2} = 0 \quad \text{in } Q^{l_2} \\ u_{l_2}|_{t=0} = 0 \\ u_{l_2}(0,t) = g_1(t) \\ u_{l_2}(l_2,t) = 0 \end{cases}$$



$g_1(t) \geq 0 \Longrightarrow u_{l_2}(x,t) \geq 0 \quad \text{in } \bar{Q}^{l_2}$

$\Longrightarrow u_{l_2}(l_1,t) \geq 0$

在 Q^{l_1} 上． $Lu_{l_1} = Lu_{l_2} = 0$

$u_{l_1}|_{t=0} = u_{l_2}|_{t=0} = 0$

$u_{l_1}|_{x=0} = u_{l_2}|_{x=0} = g_1(t)$

$u_{l_1}|_{x=l_1} = 0 \leq u_{l_2}|_{x=l_1}$

$\Longrightarrow u_{l_1}(x,t) \leq u_{l_2}(x,t) \quad \forall (x,t) \in Q^{l_1}$

17. (1) 反证：设 $u \geq 0$ 不成立，即存在 $(x_0,t_0) \in Q$，

s.t. $u(x_0,t_0) < 0$ 故 $\min_{\bar{Q}} u < 0$

由极值原理：$\min_{\bar{Q}} u = \min_{\Gamma} u$ 设 $(x^*,t^*) \in \Gamma$

s.t. $u(x^*,t^*) = \min_{\bar{Q}} u < 0$

由条件 $(x^*,t^*) \in \Gamma \cap \{x=0\}$ 由极值必要条件，得

$\frac{\partial u}{\partial x}\Big|_{(x^*,t^*)} \geq 0$

$h(u_0 - u|_{(x^*,t^*)}) > 0$

$\Longrightarrow \left[\frac{\partial u}{\partial x}+h(u_0-u)\right]\Big|_{(x^*,t^*)}>0$ 与条件矛盾

故 $u\geq 0$ in $\overline{Q}$

令 $v=u_0-u$ 则有

$$v_t-v_{xx}=0$$
$$v|_{t=0}=u_0$$
$$\left[\frac{\partial v}{\partial x}+h(v_0-v)\right]\Big|_{x=0}=h(u_0-u_0)=0$$
$$v|_{x=l}=u_0$$

同前证明 有 $v\geq 0$ in $\overline{Q}$ $\Rightarrow u_0\geq u$ in $\overline{Q}$

(2) 设 $h_1<h_2$ u_{h_1}, u_{h_2} 是相应的解，则有 $v=u_{h_1}-u_{h_2}$

满足

$$v_t-v_{xx}=0 \quad \text{in } Q$$
$$v|_{t=0}=0$$
$$v|_{t=l}=0$$
$$\frac{\partial v}{\partial x}+(h_2-h_1)(u_0-u_{h_2})-h_1 v$$
$$=\frac{\partial u_{h_2}}{\partial x}-\frac{\partial u_{h_1}}{\partial x}+(h_2-h_1)(u_0-u_{h_2})-h_1(u_{h_2}-u_{h_1})$$
$$=\frac{\partial u_{h_2}}{\partial x}-h_2u_{h_2}+h_2u_0-\left[\frac{\partial u_{h_1}}{\partial x}-h_1u_{h_1}+h_1u_0\right]=0$$

故 $\frac{\partial v}{\partial x}+h_1\left[\frac{h_2-h_1}{h_1}(u_0-u_{h_2})-v\right]=0$

$h_1>0$ $\frac{h_2-h_1}{h_1}>0$ $u_0-u_{h_2}>0$ 故 $v\geq 0$

即 $u_{h_2}\geq u_{h_1}$ u_h 关于 h 单调递增

18. 设 u 在 $\overline{Q}$ 上存在非负最大值，即 $\max\limits_{\overline{Q}} u\geq 0$. 设 $(x_0,t_0)\in\overline{Q}$

s.t $u(x_0,t_0)=\max\limits_{\overline{Q}} u\geq 0$

分两种情况

i) $Lu<0$ 则证明 $(x_0,t_0)\in\Gamma$

反证 若 $(x_0,t_0)\overline{\in}\Gamma$，则 $(x_0,t_0)\in Q$ 则有

$$\frac{\partial u}{\partial x}\Big|_{(x_0,t_0)}=0$$
$$\frac{\partial^2 u}{\partial x^2}\Big|_{(x_0,t_0)}\leq 0$$
$$\frac{\partial u}{\partial t}\Big|_{(x_0,t_0)}\geq 0$$

故 $Lu|_{(x_0,t_0)}=\left(\frac{\partial u}{\partial t}-a^2u_{xx}+c(x,t)u\right)\Big|_{(x_0,t_0)}\geq 0$ 与 $Lu<0$ 矛盾

因此 $(x_0,t_0)\in\Gamma$，即 $\max\limits_{\overline{Q}} u=\max\limits_{\Gamma} u$

ii) 若 $Lu\leq 0$. 作辅助函数 $v(x,t)=u+\varepsilon e^{-Mt}$, $\varepsilon>0$

$$M=\sup_{Q} c+1$$

则有 $Lv=Lu+\varepsilon(-Me^{-Mt}+ce^{-Mt})$

$$=Lu+\varepsilon(-M+c)e^{-Mt}\leq -\varepsilon e^{-Mt}<0$$

$\max\limits_{\overline{Q}} v\geq\max\limits_{\overline{Q}} u\geq 0$ 由(i)得 $\max\limits_{\overline{Q}} v=\max\limits_{\Gamma} v$

另外有

$$\max_{\Gamma} u\leq\max_{\overline{Q}} u\leq\max_{\overline{Q}} v=\max_{\Gamma} v\leq\max_{\Gamma} u+\varepsilon$$

令 $\varepsilon\to 0$，得 $\max\limits_{\overline{Q}} u=\max\limits_{\Gamma} u$

推论：若 $Lu\leq 0$ 则 $u|_{\overline{Q}}\leq 0$ $u|_{\Gamma}\leq 0$

证明：若 $\exists p(x_0,t_0)\in\bar{Q}$, s.t $u|_p>0$ 则 $\max\limits_{\bar{Q}} u>0$

故有 $\max\limits_{\bar{Q}} u=\max\limits_{\Gamma} u\leq 0$ 与上面矛盾.

推论. 若 $Lu\geq 0$, 则 $u|_{\bar{Q}}\geq 0$. $u|_{\Gamma}\geq 0$

19. 令 $u=e^{Mt}v$

因为 $Lu=Me^{Mt}v+e^{Mt}v_t-a^2e^{Mt}v_{xx}+ce^{Mt}v$

$$=e^{Mt}[v_t-a^2v_{xx}+(M+c)v]\leq 0$$

故 $v_t-a^2v_{xx}+(M+c)v\leq 0$

取 $M=c_0$ 则 $M+c\geq c_0-c_0=0$

由 $u|_{\Gamma}\leq 0 \Rightarrow v|_{\Gamma}\leq 0$ 由上题推论得

$v|_{\bar{Q}}\leq 0 \Rightarrow u|_{\bar{Q}}\leq 0$

20. 证明: $Lu=u_t-u_{xx}+(u-a)u=0 \quad (x,t)\in Q$

$u|_{\Gamma}\geq 0 \quad u-a\geq\min(u-a)$

由上题推论 知 $u|_{\bar{Q}}\geq 0$

记 $\Phi=\max\limits_{[0,l]}\varphi(x)$, $v=\Phi$ 则 $u|_{\Gamma}\leq v|_{\Gamma}$

$C=\max\limits_{\bar{Q}}|a(x,t)|$ 令 $u=e^{ct}w$. 则

$$Lw=w_t-w_{xx}+(c-a(x,t))w$$
$$=-ce^{-ct}u+e^{-ct}u_t-e^{-ct}w_{xx}+(c-a(x,t))e^{-ct}u$$
$$=e^{-ct}(u_t-u_{xx}-a(x,t)u)$$
$$=-e^{-ct}u\leq 0$$

$$\begin{cases} Lv=v_t-v_{xx}+(c-a(x,t))v=(c-a(x,t))\Phi\geq 0\geq Lw \\ v|_{\Gamma}\geq u|_{\Gamma}\geq w|_{\Gamma} \end{cases}$$

18题推论 $v\geq w$ in $\bar{Q}$ $\quad v\geq e^{-ct}u \Rightarrow$

$$u\leq e^{ct}v\leq e^{cT}\max_{[0,l]}\varphi(x)$$

另法：设 $v=M\Phi$, $M>1$ 为常数. 则有 $v|_{\Gamma}\geq u|_{\Gamma}$ 且

$$L(u-v)=v_t-u_{xx}+(u-v-a)(u-v)$$
$$=v_t-u_{xx}+(u-a)u-vu-uv-v^2+av$$
$$=-2uv-v^2+av=-v(v-a+2u)$$
$$\leq -v^2+av=-v(v-a)\quad (u\geq 0,\ v\geq 0)$$

若 $\Phi=0$ 则 $L(u\cdot v)=0$

若 $\Phi\neq 0$ 则取 $M=(\Phi+\max|a(x,t)|)/\Phi$,

则有 $L(u\cdot v)\leq 0$

故对 Lu 用19题 得 $u\leq v$ in $\bar{Q}$ 即

$$u\leq M\cdot\max_{[0,l]}\varphi(x)$$

21. 证明: $\begin{cases} Lu=u_t-a^2u_{xx}=0 \\ u|_{t=0}=0 \\ u|_{x=0}=0 \end{cases}$ 第一边值 的有界解只有零解

记 $Q_{LT}=\{(x,t)\,|\,0<x<L,\ 0<t\leq T\}$ u有界 故 $\exists k>0$.

$\sup\limits_{Q}|u|\leq k \quad Q=\{(x,t)\,|\,x\in(0,+\infty),\ t>0\}$

作辅助函数

$$W(x,t)=\frac{k}{L^2}(2a^2t+x^2)\pm u$$

则有 $LW=0$

$$W|_{t=0}=\frac{k}{L^2}x^2\pm 0\geq 0$$

$$W|_{x=0}=\frac{k}{L^2}(2a^2t)\pm 0\geq 0$$

$$W|_{x=L}=\frac{k}{L^2}(2a^2t+L^2)\pm u\geq 0$$

$W\geq 0$ in $\overline{Q}$ 即 $|u|\leq\frac{k}{L^2}(2a^2t+x_0^2)$

对 $\forall(x_0,t_0)\in Q$ 有 L 充分大，T 充分大 s.t.

$(x_0,t_0)\in Q_{TL}$ 则有 $|u(x,t)|\leq\frac{k}{L^2}(2a^2t_0+x_0^2)\to 0$

$L\to\infty$

故 $u(x,t)\equiv 0$ $\forall(x,t)\in Q$ 解唯一

若考虑第二，三边值问题，则证明问题

$$\begin{cases}Lu=u_t-a^2u_{xx}=0\\ u|_{t=0}=0\\ (u_x+\alpha u)|_{x=0}=0,\quad \alpha>0\end{cases}$$

的有界解为零

取 $W(x,t)=\frac{k}{L^2}(2a^2t+x^2)\pm u$ 则有

$$\begin{cases}LW=0\\ W|_{t=0}\geq 0\\ W|_{x=L}\geq 0\\ (-W_x+\alpha W)|_{x=0}=\alpha\cdot\frac{k}{L^2}(2a^2t)\pm 0\geq 0\end{cases}$$

22，23 同书上证明。

第四章

比较定理 $Lu=-\Delta u+c(x)u$，$c(x)\geq 0$ 则有

$$\begin{cases}Lu|_\Omega\leq 0\\ u|_{\partial\Omega}\leq 0\end{cases}\Rightarrow u|_\Omega\leq 0$$

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1. (1) 作辅助函数 $W(x)=\frac{F}{c_0}\pm u$，则

$$\begin{cases}LW=c(x)\cdot\frac{F}{c_0}\pm f(x)\geq F+f\geq 0\\ W|_{\partial\Omega}=\frac{F}{c_0}\geq 0\end{cases}$$

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$\Rightarrow W \geq 0$ in Ω $\Rightarrow |u| \leq \frac{F}{C_0} = C^{-1}\sup_{\Omega}|f(x)|$

(2) 作辅助函数 $W(x) = \frac{F}{2n}(d^2-|x|^2) \pm u$

$$d = \sup_{x,y\in\Omega}|x-y|$$

$$LW = -\frac{F}{2n}(-2n) + c(x)\frac{F}{2n}(d^2-|x|^2) \pm f$$

$$= F \pm f + c(x)\frac{F}{2n}(d^2-|x|^2) \geq 0$$

$$W|_{\partial\Omega} = \frac{F}{2n}(d^2-|x|^2)|_{\partial\Omega} \pm 0 \geq 0$$

$\Rightarrow W \geq 0$ in Ω $\Rightarrow |u| \leq \frac{F}{2n}(d^2-|x|^2)$

$$\leq \sup_{x\in\Omega}\left(\frac{d^2-|x|^2}{2n}\right)F \leq \frac{d^2}{2n}F$$

2. 记 $\Phi_1 = \sup_{\Gamma_1}|\varphi_1|$, $\Phi_2 = \sup_{\Gamma_2}|\varphi_2|$, $F = \sup_{\Omega}|f|$

$W(x) = \frac{\Phi_1}{\alpha_0} + \Phi$ ——

F

3. $v(x) = |x|^{-a} - r^{-a} = |x|^{2\cdot(-\frac{a}{2})} - r^{-a}$

$$Lv = -\Delta v + c\cdot v$$

$$= -\left[\left(\frac{a^2}{4}+\frac{a}{2}\right)\cdot 4|x|^{-a-2} - an|x|^{-a-2}\right] + cv$$

$$= -[a^2+2a-an]|x|^{-a-2} + cv < 0$$

a 充分大. $x \in S^*$

$v|_{|x|=r} = 0$, $w|_{|x|=\frac{r}{2}} < 0$ (1(3)右上).

$$-\frac{\partial v}{\partial \nu}\Big|_{x=x^0} = -\frac{\partial v}{\partial n}\cos(\nu,n)\Big|_{x^0}$$

$$= a|x_0|^{-a-1}\cos(\nu,n) = a r^{-a-1}\cos(\nu,n)$$

> 0

4. 证明：记 $Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}$

$+ c(x)u$

先证若 $Lu = f < 0$ 则 $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$

设 u 在 $\bar\Omega$ 上为负，则 $\sup_{\Omega} u \leq 0 = \sup_{\partial\Omega} u^+$

若 $\max_{\bar\Omega} u \geq 0$，设 $u(x^0) = \max_{\bar\Omega} u \geq 0$ 则证 $x^0 \in \partial\Omega$

否则 $x^0 \in \Omega$ 有 $\frac{\partial u}{\partial x_i}\Big|_{x^0} = 0$

$J = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\Big|_{x^0} \leq 0$ $J = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}\Big|_{x^0}$

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}\Big|_{x^0} = tr(AJ) \le 0,$$

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$. 由条件 A 正定，

J 半负定.

$\Rightarrow AJ$ 半负定？

$\therefore tr(AJ) \le 0$.

$\left(-\sum_{i,j=1} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} + cu\right)\Big|_{x^0} \ge 0$ 与条件矛盾.

若 $f\le 0$，作 $w = u(x) + \varepsilon e^{ax_1}$，$Lw = Lu + \varepsilon[-a_{11}a^2 e^{ax_1} + b_1 a e^{ax_1} + c(x)e^{ax_1}] \le \varepsilon e^{ax_1}\underbrace{(-a_{11}a^2+b_1 a+c)}_{<0}$ （a充分大）

同理上.

5. 由 $\lim_{|x|\to\infty} u(x) = l$，

$\forall \varepsilon>0$，$\exists a>0$，当 $|x|>a$ 时，有

$$|u(x)-l|<\varepsilon$$

取 a 充分大，s.t. $B_{a+1} = \{x \mid |x| < a+1\}$ 包含 Ω_0. 记 $\Omega' = \Omega \cap B_{a+1}$，在 Ω' 上用最大模估计

$$\max_{\Omega'} |u(x)| \le \max\{\max_{\partial B_{a+1}}|u|, \max_{\partial\Omega}|u|\}$$

$$\le \max\{|l|+\varepsilon, \max_{\partial\Omega}|u|\}$$

$$\Longrightarrow \sup_{\Omega}|u(x)| \le \max\{|l|+\varepsilon, \max_{\partial\Omega}|u|\}$$

令 $\varepsilon\to 0$ 则有 $\sup_\Omega |u(x)| \le \max\{|l|, \max_{\partial\Omega}|\varphi(x)|\}$

6. 因为 $-\Delta u + u^2\cdot u = 0$

$$\left(\frac{\partial u}{\partial n} + \alpha(x) u\right)\Big|_{\partial\Omega} = \varphi$$

记 $c(x) = u^2(x) \ge 0$，又 $c(x)$ 有上界，由Th2.6得

$$\max_{\Omega}|u(x)| \le \frac{\Phi}{\alpha_0} = \frac{1}{\alpha_0}\cdot\max_{\partial\Omega}|\varphi(x)|$$