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1. *Does the graph of an arbitrary* 1 *variable real function,* $f : R \to R$*, always have Lebesgue measure* 0 *in* R^2*? The graph of* $f := (t, f(t))$ *in* R^2 $|$ *for* t *in* R*. If not, under what conditions might it not be?*

   Solution: Let $\mathbb{Q}$ denote the set of rational numbers. A coset of $\mathbb{Q}$ in $\mathbb{R}$ is defined as any set of the form

   $$x + \mathbb{Q} = \{x + q | q \in \mathbb{Q}\} \tag{1}$$

   The collection of all the cosets of $\mathbb{Q}$ in $\mathbb{R}$ is denoted as $\mathbb{R}/\mathbb{Q}$. Note that there exists a canonical surjection $p : \mathbb{R} \to R/\mathbb{Q}$ defined by

   $$p(x) = x + \mathbb{Q} \tag{2}$$

   for all $x \in \mathbb{R}$. The equality $|\mathbb{R}/\mathbb{Q}||\mathbb{Q}| = |\mathbb{R}|$ implies $|\mathbb{R}/\mathbb{Q}| = |\mathbb{R}|$ (Use a section of the canonical projection $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ to prove $|\mathbb{R}/\mathbb{Q}||\mathbb{Q}| = |\mathbb{R}|$. Therefore there exists a bijection $g : \mathbb{R}/\mathbb{Q} \to \mathbb{R}$. For a non-constructive solution let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ be the canonicial surjection, and let $f = g \circ p$. The image of every open interval (a, b) is all of R (i.e. unbounded). Hence the graph of f is dense in R^2.

   A constructive example is given as below. Let $\mathfrak{C} = (\mathbb{R}/\mathbb{Q}) \setminus \{\mathbb{Q}\}$, and let $h : \mathfrak{C} \to \mathbb{R}$ be a bijection. Each coset $C \in \mathfrak{C}$ is dense in $\mathbb{R}$, so $C \cap (a, b) \neq \varnothing$ for each open interval (a, b) in $\mathbb{R}$ and $C \in \mathfrak{C}$. Now define

   $$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} h(x + \mathbb{Q}), & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ x, & \text{if } x \in \mathbb{Q}\ ; \end{cases}$$

   then $f[I] = \mathbb{R}$ for every non-empty open interval I in $\mathbb{R}$, and $f[\mathbb{Q}] = \mathbb{Q}$.

2. *Does the image of a continuous path in the plane,* $f : R \to R^2$ *with* $f(t) = (x(t), y(t))$*, always have Lebesgue measure 0 in* R^2*? The image of* $f := \{(x(t), y(t))$ *in* R^2 $|$ *for* t *in* $R\}$*.*

   Solution: It is well-known that there are continuous space filling curves whose image have positive measure (e.g peano curve). The Cantor function $F' : [0, 1] \to [0, 1]$ gives you a surjective continous map $F'|_{\mathcal{C}} : \mathcal{C} \to [0, 1]$ from the Cantor set to the unit interval. Thus you can construct a surjective continuous map g

   $$\mathcal{C} \xrightarrow{\cong} \mathcal{C} \times \mathcal{C} \xrightarrow{F'|_{\mathcal{C}} \times F'|_{\mathcal{C}}} [0, 1] \times [0, 1].$$

The homeomorphism can be proved using Jonsson-Tarski algebra. The basic idea is to show that $\mathcal{C}$ is homeomorphic to the product space $2^{\mathbb{N}}$. Since $2^{\mathbb{N}} \xrightarrow{\cong} 2^{\mathbb{N}+\mathbb{N}} \xrightarrow{\cong} 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, a "pairing function" pair $\mathcal{C} \xrightarrow{\cong} \mathcal{C} \times \mathcal{C}$ is a homeomorphism. We have the so-called Cantor function $F'|_{\mathcal{C}} : \mathcal{C} \to [0,1]$ which is continuous and surjective. Therefore taking compositions between the homeomorphism $\mathcal{C} \xrightarrow{\cong} \mathcal{C} \times \mathcal{C}$ and the continuous $F'|_{\mathcal{C}} \times F'|_{\mathcal{C}}$, we can get a continuous map g from $\mathcal{C}$ onto $[0,1] \times [0,1]$. The functions g can be extend to $[0,1]$ by linear interpolation explicitly. If $x \in I$ belongs to one of the open intervals (a,b) say $x = ta + (1-t)b$ for some $t \in (0,1)$, then define $f(x) = tf(a) + (1-t)f(b)$. It can be easily shown that this function is f surjective and continuous by using the properties of g.

In fact it can be proved that f is rectifiable (i.e. continuous and of bounded variation), then its image is of measure zero. Let $R \subset \mathbb{R}^2$ be a square such that the image of $f \subset R$. Assume w.l.g that its sides have length 2. Divide R into $4\,n^2$ equal squares of side $1/n^2$. Classify them into four classes A_n, B_n, C_n y D_n, each one containing n^2 squares, such that if R_1, R_2 are two different squares in the same group and $p_i \in R_i$, then $|p_1 - p_2| \geq 1/n$. Suppose that for some n, the image of f has non empty intersection with at least $n\sqrt{n}$ squares in A_n. Then, considering a polygonal with a point in each of the squares, we have

$$\text{length}(f) \geq n\sqrt{n}\,\frac{1}{n} = \sqrt{n}\,.$$

Since f is rectifiable, there exists n_A such that the image of f has non empty intersection with less than $n\,\sqrt{n}$ squares in A_n for all $n \geq n_A$. The sum of the areas of all those squares is bounded by

$$n\,\sqrt{n}\,\frac{1}{n^2} = \frac{1}{\sqrt{n}}\,.$$

Repeating the argument with B_n, C_n and D_n we see that if $n \geq \max(n_A, n_B, n_C, n_D)$, the image of f is contained in a set of squares of total area bounded by $4/\sqrt{n}$.