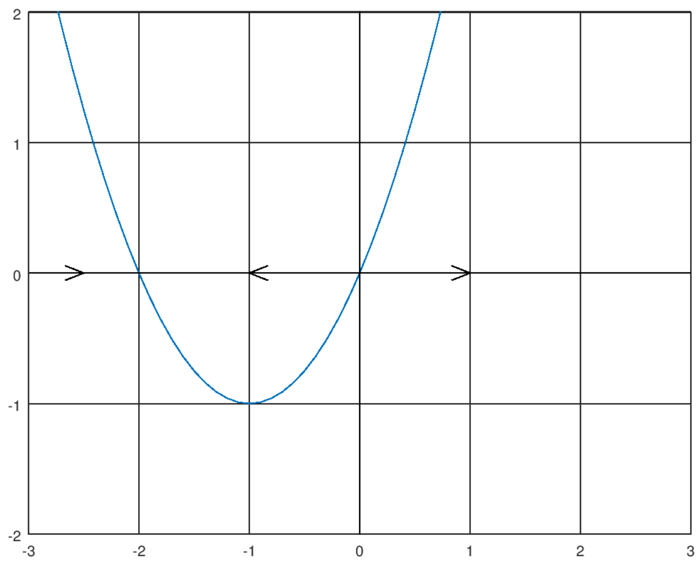
18.03

Pset 3a

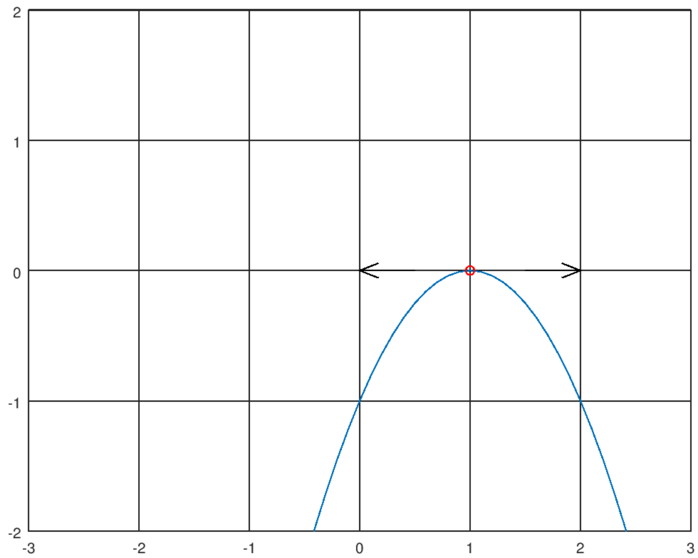
[Problem Source]

Section 1

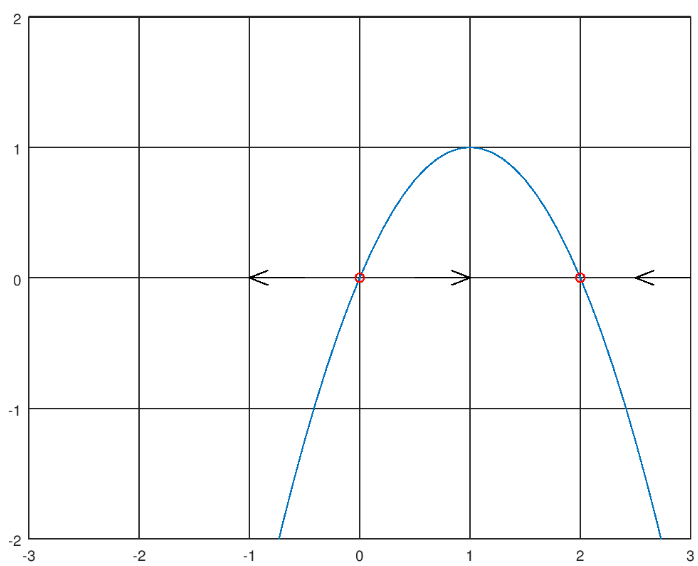
1E-1a)



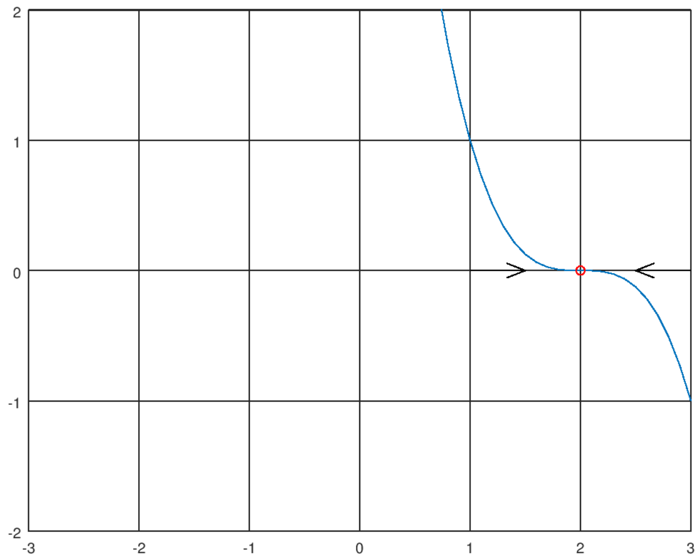
1E-1b)



1E-1c)



1E-1d)



2ai)

$$\frac{d}{dt} + 2x(t) = 1$$

$$\frac{d}{dt} = 1 - 2x(t)$$

$$\int \frac{1}{1-2x}\, dx = \int 1\, dt$$

$$-\frac{1}{2} ln(1 - 2x(t)) = t + C_0$$

$$ln(1 - 2x(t)) = -2t + C_0$$

$$1 - 2x(t) = Ce^{-2t}$$

$$-2x(t) = Ce^{-2t} - 1$$

$$x(t) = \boxed{Ce^{-2t} + \frac{1}{2}}$$

2aii)

$$
\begin{aligned}
\frac{d}{dt}x(t) + 2x(t) &= 1 \\
\frac{d}{dt}x(t)e^{2t} + 2x(t)e^{2t} &= e^{2t} \\
\frac{d}{dt}\left(x(t)e^{2t}\right) &= e^{2t} \\
\int \frac{d}{dt}\left(x(t)e^{2t}\right)\, dt &= \int e^{2t}\, dt \\
x(t)e^{2t} &= \frac{1}{2}e^{2t} + C \\
x(t) &= \boxed{\frac{1}{2} + Ce^{-2t}}
\end{aligned}
$$

2aiii)

Regard RHS as e^{0t}, and assume solution takes the form Ae^{0t}:

$$
\begin{aligned}
\frac{d}{dt}x(t) + 2x(t) &= e^{0t} \\
\frac{d}{dt}Ae^{0t} + 2Ae^{0t} &= e^{0t} \\
0 + 2A &= 1 \\
A &= \frac{1}{2}
\end{aligned}
$$

$$\therefore x_p = \frac{1}{2}$$

Because we know that the solution for an equation of the form $\frac{d}{dt}x(t) + kx(t) = 0$ is $x(t) = Ce^{-kt}$, we can add in a transient, and write the solution as:

$$\boxed{\frac{1}{2} + Ce^{-2t}}$$

2b)

This equilibrium is semi-stable.

2c)

$$x(t_{n+1}) = h(1 - 2x(t_n)) + x(t_n)$$

n	t_n	$x(t_n)$	$1-2x(t_n)$	$h(1-2x(t_n))$
0	0	0	1	0.3333
1	0.3333	0.3333	0.3333	0.1111
2	0.6666	0.4444	0.1112	0.0371
3	0.9999	0.4815		

Section 2

1a)

Seems reasonable because $\frac{d}{dt}y(t) = (1-y(t))y(t)$ indicates that the population count stops changing in two places: in the event that the current population is the stable population (1 kilo-oryx) or zero (there are no oryxes). In the event that the population is greater than one kilo-oryx, we would expect the population to decline to the carrying capacity of 1 kilo-oryx, and in the event that the population is less than one kilo-oryx, we would expect the population to rise to the carrying capacity.

1b)

$$
\begin{aligned}
y(1-y) - a &= 0 \\
y(1-y) &= a \\
y - y^2 &= a \\
-y^2 + y - a &= 0 \\
\therefore \frac{-1 \pm \sqrt{1-4a}}{-2}
\end{aligned}
$$

We can see that:

$$
\begin{aligned}
&\text{0 roots:} \quad a > \frac{1}{4} \\
&\text{1 root:} \quad a = \frac{1}{4} \\
&\text{2 roots:} \quad a < \frac{1}{4}
\end{aligned}
$$

In the 1 root case, the critical point is $\frac{1}{2}$ and this point is semi-stable. In the two root case, the critical point occurs at $\frac{-1 \pm \sqrt{1-4a}}{-2}$ where $a < \frac{1}{4}$. The critical point at $\frac{-1+\sqrt{1-4a}}{-2}$ is unstable, while the critical point at $\frac{-1-\sqrt{1-4a}}{-2}$ is stable.

1c)

We can write the new equation like:

$$\frac{d}{dt}y(t) = y(1-y) - 0.1875$$

Solving for the zeroes we have:

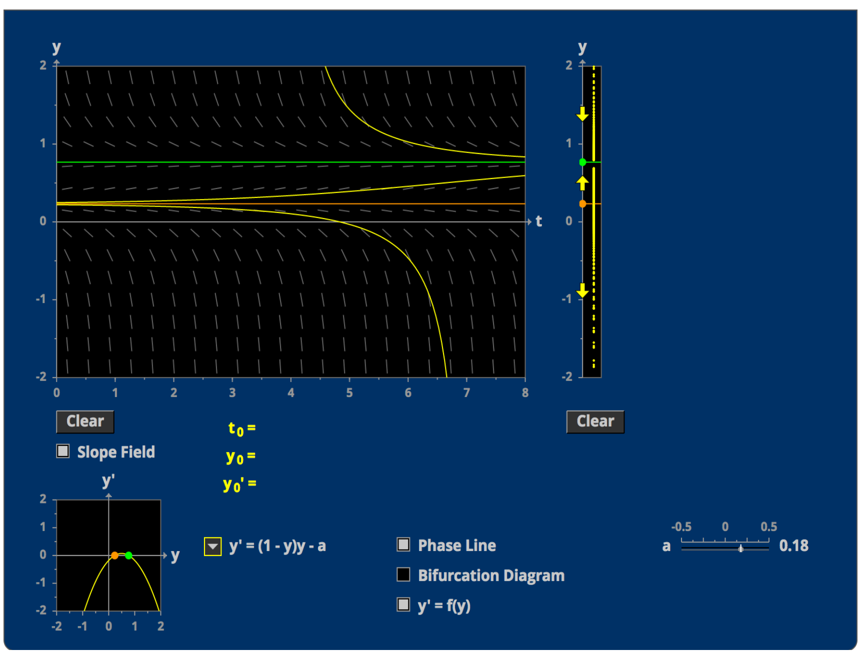
$$-y^2 + y - 0.1875 = 0$$

$$\frac{-1 \pm \sqrt{1 - 4 \cdot 0.1875}}{-2}$$

$$\boxed{0.25, 0.75}$$

And therefore from (1b) we can conclude that the stable population is 750 oryx and the population below which the oryx population will crash is 250 oryx.

1d)



1e)

The equation is simply the equation for a:

$$\boxed{-y^2 + y = a}$$

2a)

We can write the roots in a new equation:

$$\frac{d}{dt}y(t) = -\left(y - \frac{3}{4}\right)\left(y - \frac{1}{4}\right)$$

Substituting in terms of $u = y - \frac{3}{4}$:

$$\begin{aligned}\frac{d}{dt}u(t) &= \frac{d}{dt}\left(y(t) - \frac{3}{4}\right)\\ &= \frac{d}{dt}y(t)\end{aligned}$$

So we can directly substitute:

$$\begin{aligned}\frac{d}{dt}u(t) &= -u(t)\left(u(t) + \frac{1}{2}\right)\\ &= \boxed{-u(t)^2 - \frac{1}{2}u(t)}\end{aligned}$$

By inspection, this is most certainly autonomous, as we can write:

$$-u(t)^2 - \frac{1}{2}u(t) = \frac{d}{dt}u(t)$$

which is in the form:

$$\frac{d}{dt}u(t) = f(u(t)) \quad \checkmark$$

If $u(t) = 0$

:

$$-0^2 - \frac{1}{2}(0) = 0 \quad \checkmark$$

2b)

Linearize the equation to get

$$\frac{d}{dt}u(t) = -\frac{1}{2}u(t)$$

Solving for $u(t)$

:

$$\begin{aligned}
\frac{d}{dt}u(t) &= -\frac{1}{2}u(t) \\
\int \frac{1}{u(t)}\, du(t) &= \int -\frac{1}{2}\, dt \\
ln(u(t)) &= -\frac{1}{2}t + C_0 \\
u(t) &= \boxed{Ce^{-\frac{1}{2}t}}
\end{aligned}$$

2c)

$$\begin{aligned}
y(10) - \frac{3}{4} &= Ce^{\frac{-1}{2}10} \\
&= b \\
Ce^{-5} &= b \\
C &= be^{5} \\
u(t) &= be^{5}e^{-\frac{1}{2}t} \\
&= be^{-\frac{1}{2}t+5}
\end{aligned}$$

Translating $u(t)$

back to $y(t) - y_0$

:

$$\begin{aligned}
u(t) &= be^{-\frac{1}{2}t+5} \\
y(t) - \frac{3}{4} &= be^{-\frac{1}{2}t+5} \\
y(t) &= \boxed{be^{-\frac{1}{2}t+5} + \frac{3}{4}}
\end{aligned}$$

For the specific (11, 12) case:

$$y(11) = \boxed{be^{-\frac{11}{2}+5} + \frac{3}{4}}$$

$$y(12) = \boxed{be^{-\frac{12}{2}+5} + \frac{3}{4}}$$

2d)

$p(t),\ q(t)$
must not contain powers of t
greater than $\$t^0$
(must be constant). This is because in order to write $\frac{d}{dt}x(t) = f(x(t))$
, we cannot have factors of t
in the RHS.