18.03

Recitation 7

Problem Source

1a)

$$x = cos(\omega t)$$
$$\frac{d^2}{dt^2}x(t) + \omega^2 x(t) = 0$$
$$\frac{d^2}{dt^2}cos(\omega t) + \omega^2 cos(\omega t) = 0$$
$$-\omega^2 cos(\omega t) + \omega^2 cos(\omega t) = 0$$
$$0 = 0 \quad \checkmark$$

1b)

$$x = sin(\omega t)$$
$$\frac{d^2}{dt^2}x(t) + \omega^2 x(t) = 0$$
$$\frac{d^2}{dt^2}sin(\omega t) + \omega^2 sin(\omega t) = 0$$
$$-\omega^2 sin(\omega t) + \omega^2 sin(\omega t) = 0$$
$$0 = 0 \quad \checkmark$$

2)

$$x = Acos(\omega t - \phi)$$
$$\frac{d^2}{dt^2}x(t) + \omega^2 x(t) = 0$$
$$\frac{d^2}{dt^2}Acos(\omega t - \phi) + \omega^2 Acos(\omega t - \phi) = 0$$
$$-A\omega^2 cos(\omega t - \phi) + \omega^2 Acos(\omega t - \phi) = 0$$
$$0 = 0 \quad \checkmark$$

3)

The uniqueness theorem only applies to first order equations. The given equation is second order, and thus uniqueness doesn't apply.

4)

$$x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0$$
$$\ddot{x} + \omega^2 x = 0$$

The characteristic equation here is:

$$r^2 + \omega^2 = 0$$

with roots $\pm i\omega$.
We know that if the characteristic equation has complex root $a + bi$, $e^{at}cos(bt)$ and $e^{at}sin(bt)$ are solutions, and therefore we can write:

$$C_1 e^{at} cos(\omega t) + C_2 e^{at} sin(\omega t) = x(t)$$

Using the initial condition $x(0) = x_0$:

$$C_1 e^0 cos(0) + C_2 e^0 sin(0) = x(0)$$
$$C_1 = x_0$$

Using the initial condition $\dot{x}(0) = \dot{x}_0$:

$$\frac{d}{dt}\left(C_1 e^{at} cos(\omega t) + C_2 e^{at} sin(\omega t)\right) = \frac{d}{dt}x(t)$$
$$-C_1 e^{at} \omega sin(\omega t) + C_2 e^{at} \omega cos(\omega t) = \frac{d}{dt}x(t)$$
$$-C_1 e^0 \omega sin(0) + C_2 e^0 \omega cos(0) = \frac{d}{dt}x(0)$$
$$C_2 \omega = \dot{x}(0)$$
$$C_2 = \frac{\dot{x}(0)}{\omega}$$

Therefore we can write the solution as:

$$x(t) = \boxed{x_0 e^{at} cos(\omega t) + \frac{\dot{x}(0)}{\omega} e^{at} sin(\omega t)}$$

And there is only one solution.

5)

Assuming e^{rt} is a solution:

$$\begin{aligned} \ddot{x} + kx &= 0 \\ \frac{d^2}{dt^2}x(t) + kx(t) &= 0 \\ \frac{d^2}{dt^2}e^{rt} + ke^{rt} &= 0 \\ r^2e^{rt} + ke^{rt} &= 0 \\ r^2 &= -k \end{aligned}$$

$$\therefore \boxed{r = i\sqrt{k}}$$

6)

$$\begin{aligned} \ddot{x} - a^2x &= 0 \\ r^2 - a^2 &= 0 \\ r &= \pm a \\ C_1e^{at} + C_2e^{-at} &= x(t) \end{aligned}$$

.

With initial conditions $x_1(0) = 1,\ x_1(0) = 0$
:

$$\begin{aligned} C_1e^0 + C_2e^0 &= x_1(0) \\ C_1 + C_2 &= 1 \end{aligned}$$

.

$$\begin{aligned} C_1ae^0 + C_2(-a)e^0 &= x_1(0) \\ C_1a - C_2a &= 0 \\ a\left(C_1 - C_2\right) &= 0 \\ C_1 &= C_2 \end{aligned}$$

$$C_1 = \frac{1}{2}$$

$$C_2 = \frac{1}{2}$$

$$\therefore x_1(t) = \boxed{\frac{1}{2}e^{at} + \frac{1}{2}e^{-at}}$$

With initial conditions $x_2(0) = 0,\ \dot{x}_2(0) = 1$
:

$$C_1e^0 + C_2e^0 = x_1(0)$$

$$C_1 + C_2 = 0$$

$$C_1ae^0 + C_2(-a)e^0 = \dot{x}_1(0)$$

$$C_1a - C_2a = 1$$

$$a\left(C_1 - C_2\right) = 1$$

$$C_1 - C_2 = \frac{1}{a}$$

$$C_1 = \frac{1}{a} + C_2$$

$$\frac{1}{a} + C_2 + C_2 = 0$$

$$2C_2 = -\frac{1}{a}$$

$$C_2 = -\frac{1}{2a}$$

$$C_1 = -\frac{1}{2a} + \frac{1}{a}$$

$$C_1 = \frac{1}{2a}$$

$$\therefore x_2(t) = \boxed{\frac{1}{2a}e^{at} - \frac{1}{2a}e^{-at}}$$