18.03

Pset 2

[Problem Source]

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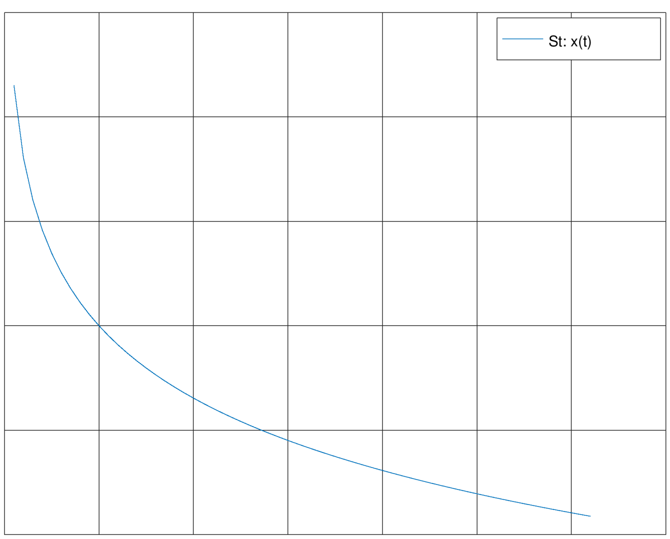
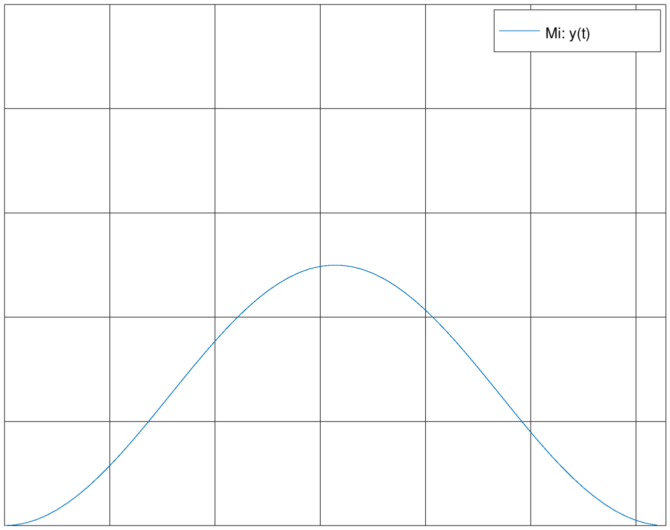
Part II.

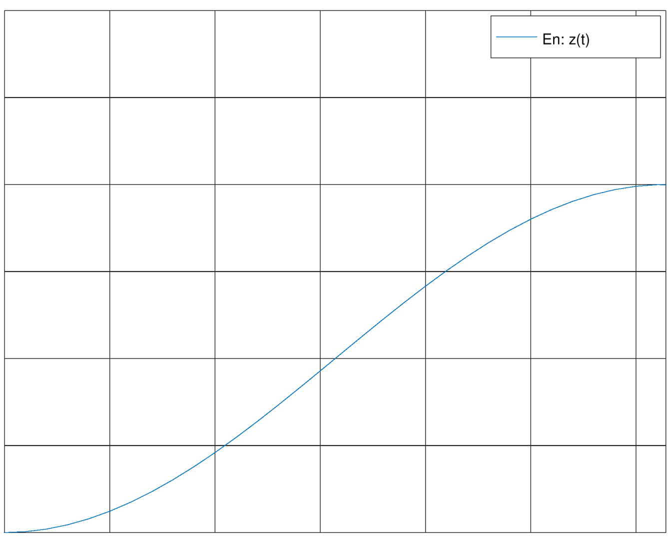
Section 1

1a)



1b) (continuation, figures)





The limiting values as $t \longrightarrow \infty$ should be:

$$x(\infty) \longrightarrow 0$$
$$y(\infty) \longrightarrow 0$$
$$z(\infty) \longrightarrow x(t_0)$$

1b)

$$\frac{dx}{dt} = -\frac{1}{2t_s}x(t)$$

$$\frac{dx}{dt} = -\sigma x(t),\ \sigma = \frac{1}{2t_s}$$

$$\frac{dy}{dt} = -\frac{1}{2}\frac{dx}{dt} - \frac{1}{2t_M}y(t)$$

$$\frac{dy}{dt} = \frac{1}{2}\sigma x(t) - \mu y(t),\ \mu = \frac{1}{2t_M}$$

$$\frac{dz}{dt} = \frac{1}{2}\sigma x(t) + \mu y(t)$$

1c)

Solving for $x(t)$:

$$\frac{dx}{dt} = -\sigma x(t)$$
$$\int \frac{1}{x(t)}dx = \int -\sigma dt$$
$$ln(x(t)) = -\sigma t + C$$
$$x(t) = Ce^{-\sigma t}$$

Using initial condition x(0) = 1:

$$x(0) = Ce^{-\sigma(0)}$$

$$1 = C$$

$$\therefore \boxed{x(t) = e^{-\sigma t}}$$

Solving for $y(t)$:

$$\frac{dy}{dt} = -\frac{1}{2}\sigma x(t) - \mu y(t)$$
$$\frac{dy}{dt} = -\frac{1}{2}\sigma e^{-\sigma t} - \mu y(t)$$

Using the method of integrating factors:

$$\begin{aligned}
\frac{dy}{dt} + \mu y(t) &= -\frac{1}{2}\sigma e^{-\sigma t} \\
\frac{dy}{dt}e^{\mu t} + \mu y(t)e^{\mu t} &= -\frac{1}{2}\sigma e^{-\sigma t}e^{\mu t} \\
\frac{dy}{dt}\left(e^{\mu t}y(t)\right) &= -\frac{1}{2}\sigma e^{t(\mu-\sigma)} \\
\int \frac{d}{dt}\left(e^{\mu t}y(t)\right)\,dt &= \int -\frac{1}{2}\sigma e^{t(\mu-\sigma)}\,dt \\
e^{\mu t}y(t) &= \frac{-\frac{1}{2}\sigma}{\mu-\sigma}e^{t(\mu-\sigma)} + C \\
y(t) &= -\frac{\sigma}{2(\mu-\sigma)}e^{t(\mu-\sigma)}e^{-\mu t} + C(e^{-\mu t}) \\
y(t) &= -\frac{\sigma}{2(\mu-\sigma)}e^{-\sigma t} + Ce^{-\mu t}
\end{aligned}$$

Using the initial condition y(0) = 0:

$$y(0) = -\frac{\sigma}{2(\mu-\sigma)}e^{0(-\sigma)} + Ce^{-\mu 0}$$

$$0 = -\frac{\sigma}{2(\mu-\sigma)} + C$$

$$\frac{\sigma}{2(\mu-\sigma)} = C$$

$$\begin{aligned}
y(t) &= -\frac{\sigma}{2(\mu-\sigma)}e^{-\sigma t} + Ce^{-\mu t} \\
y(t) &= -\frac{\sigma}{2(\mu-\sigma)}e^{-\sigma t} + \frac{\sigma}{2(\mu-\sigma)}e^{-\mu t}
\end{aligned}$$

$$\therefore \boxed{y(t) = \frac{\sigma}{2(\mu-\sigma)}\left(e^{-\mu t} - e^{-\sigma t}\right)}$$

Solving for $z(t)$:

$$\frac{dz}{dt} = \frac{1}{2}\sigma x(t) + \mu y(t)$$
$$\frac{dz}{dt} = \frac{1}{2}\sigma e^{-\sigma t} + \mu\frac{\sigma}{2(\mu-\sigma)}\left(e^{-\mu t} - e^{-\sigma t}\right)$$
$$\frac{dz}{dt} = \frac{1}{2}\sigma e^{-\sigma t} + \mu\frac{\sigma}{2(\mu-\sigma)}e^{-\mu t} - \mu\frac{\sigma}{2(\mu-\sigma)}e^{-\sigma t}$$
$$\frac{dz}{dt} = \sigma\left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right)e^{-\sigma t} + \mu\frac{\sigma}{2(\mu-\sigma)}e^{-\mu t}$$
$$\int dz = \int \sigma\left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right)e^{-\sigma t} + \mu\frac{\sigma}{2(\mu-\sigma)}e^{-\mu t}\, dt$$
$$z(t) = \sigma\left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right)\frac{1}{-\sigma}e^{-\sigma t} + \mu\frac{\sigma}{2(\mu-\sigma)}\frac{1}{-\mu}e^{-\mu t} + C$$
$$z(t) = -\left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right)e^{-\sigma t} - \frac{\sigma}{2(\mu-\sigma)}e^{-\mu t} + C$$

Using the initial condition z(0) = 0:

$$z(0) = -\left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right)e^{-\sigma 0} - \frac{\sigma}{2(\mu-\sigma)}e^{-\mu 0} + C$$
$$0 = -\left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right) - \frac{\sigma}{2(\mu-\sigma)} + C$$
$$C = \left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right) + \frac{\sigma}{2(\mu-\sigma)}$$
$$C = \frac{1}{2} + \frac{\sigma-\mu}{2(\mu-\sigma)}$$
$$C = \frac{1}{2} + \frac{-(\mu-\sigma)}{2(\mu-\sigma)}$$
$$C = \frac{1}{2} + \frac{-1}{2}$$
$$C = 0$$

$$\therefore \boxed{z(t) = -\left(\frac{1}{2} - \frac{\mu}{2(\mu-\sigma)}\right)e^{-\sigma t} - \frac{\sigma}{2(\mu-\sigma)}e^{-\mu t}}$$

1d)

$$\frac{d}{dt}y(t) = 0$$

$$\frac{1}{2}\sigma x(t) - \mu y(t) = 0$$

$$\frac{1}{2}\sigma x(t) = \mu y(t)$$

$$\frac{1}{2}\sigma\left(e^{-\sigma t}\right) = \mu\left(\frac{\sigma}{2(\mu - \sigma)}\left(e^{-\mu t} - e^{-\sigma t}\right)\right)$$

$$e^{-\sigma t} = \left(\frac{\mu}{\mu - \sigma}\left(e^{-\mu t} - e^{-\sigma t}\right)\right)$$

$$e^{-\sigma t} = \frac{\mu}{\mu - \sigma}e^{-\mu t} - \frac{\mu}{\mu - \sigma}e^{-\sigma t}$$

$$e^{-\sigma t} + \frac{\mu}{\mu - \sigma}e^{-\sigma t} = \frac{\mu}{\mu - \sigma}e^{-\mu t}$$

$$e^{-\sigma t}\left(1 + \frac{\mu}{\mu - \sigma}\right) = \frac{\mu}{\mu - \sigma}e^{-\mu t}$$

$$e^{t(\mu - \sigma)}\left(1 + \frac{\mu}{\mu - \sigma}\right) = \frac{\mu}{\mu - \sigma}$$

$$e^{t(\mu - \sigma)} = \frac{\frac{\mu}{\mu - \sigma}}{1 + \frac{\mu}{\mu - \sigma}}$$

$$e^{t(\mu - \sigma)} = \frac{\mu}{2\mu - \sigma}$$

$$e^{t} = \frac{\mu}{2\mu - \sigma}e^{-(\mu - \sigma)}$$

$$t = ln\left(\frac{\mu}{2\mu - \sigma}e^{-(\mu - \sigma)}\right)$$

$$t = \boxed{ln(\mu) - ln(2\mu - \sigma) + ln(\sigma - \mu)}$$

1e)

Analytically, with $x(0) = 2$:
For x(t):

$$x(0) = Ce^{-\sigma(0)}$$

$$2 = C$$

$$\boxed{x(t) = 2e^{-\sigma t}}$$

For y(t):

$$\boxed{y(t) = \frac{\sigma}{\mu - \sigma}\left(e^{-\mu t} - e^{-\sigma t}\right)}$$

For z(t):

$$\boxed{z(t) = -\left(1 - \frac{\mu}{\mu - \sigma}\right)e^{-\sigma t} - \frac{\sigma}{\mu - \sigma}e^{-\mu t}}$$

It appears as if doubling $x(0)$ doubles the values of $x(t)$, $y(t)$, and $z(t)$.

1f)

$$t\frac{dx(t)}{dt} + 2x(t) = q(t)$$

Given the particular solution:

$$x(t) = e^t$$

We can write:

$$\frac{dx(t)}{dt} = te^t$$

$$t(te^t) + 2e^t = q(t)$$

$$\therefore \boxed{q(t) = e^t(t^2 + 2)}$$

To find the general solution we can write:

$$x_p + Cx_h$$

Where x_p denotes the particular solution (e.g. $x(t) = e^t$ and x_h denotes the homogeneous solution (e.g. the solution when $q(t) = 0$.

Finding the homogeneous solution:

$$t\frac{dx_h(t)}{dt} + 2x(t) = 0$$

$$\frac{dx_h(t)}{dt} = -\frac{2}{t}x(t)$$

$$\int \frac{1}{x_h(t)}\,dx(t) = \int -\frac{2}{t}\,dt$$

$$ln(x(t)) = -2ln(t) + C$$

$$x_h(t) = Ce^{ln(\frac{1}{t^2})}$$

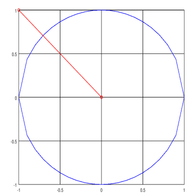
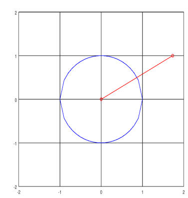
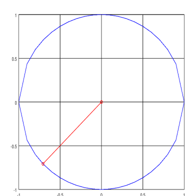
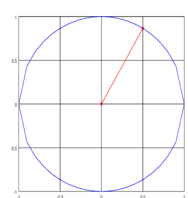
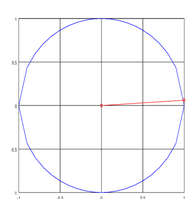
$$x_h(t) = C\frac{1}{t^2}$$

Substituting the particular solution and homogeneous solutions in:

$$\boxed{e^t + \frac{C}{t^2}}$$

Section 2

a)

n	$a + bi$	$\|z\|$	$arg(z)$	img
i	$i - 1$	$\sqrt{2}$	$\frac{-\pi}{4}$	
ii	$\sqrt{3} + i$	2	$\frac{\pi}{6}$	
iii	$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$	1	$\frac{5\pi}{4}$	
iv	$\frac{1}{2} + \frac{\sqrt{3}}{2}i$	1	$\frac{\pi}{3}$	
v	$0.9982 + 0.0604i$	1	$\frac{\pi}{52}$	

b)

$$
\begin{aligned}
z^4 + 4 &= 0 \\
z^4 &= -4 \\
z^4 &= 4e^{\pi i} \\
z &= 4^{\frac{1}{4}} e^{\frac{\pi}{4} i} \\
z &= \boxed{\sqrt{2} e^{\frac{\pi}{4} + \frac{\pi k}{2}}, \ k = 0, 1, 2, 3}
\end{aligned}
$$

For

$$z^2 + 2z + 2 = 0$$

Apply quadratic formula:

$$\frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2}$$

$$\frac{-2 \pm \sqrt{-4}}{2}$$

$$-1 \pm i$$

$$\therefore z = \boxed{-1 + i, \ -1 - i}$$

Section 3

a)

n	$Ae^{i\theta}$
i	$\sqrt{2} e^{\frac{-\pi}{4} i}$
ii	$2e^{\frac{\pi}{6} i}$
iii	$e^{\frac{5\pi}{4} i}$
iv	$e^{\frac{\pi}{3} i}$
v	$e^{\frac{\pi}{52} i}$

b)

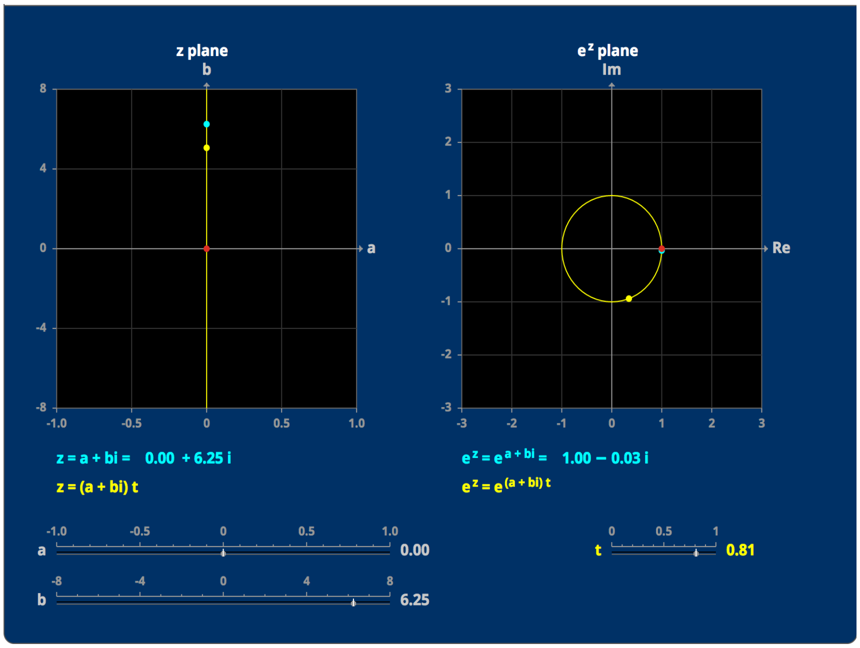
$$e^z = -2$$
$$e^{a+bi} = e^a(\cos(b) + i\sin(b))$$
$$-2 = e^a(\cos(b) + i\sin(b))$$
$$\therefore \boxed{z = ln(-2) + 0i,\ z = ln(2) + \pi i}$$

c)

$$cos(4t) = Re\left\{e^{4it}\right\}$$
$$Re\left\{e^{4it}\right\} = Re\left\{\left(e^{it}\right)^4\right\}$$
$$Re\left\{\left(e^{it}\right)^4\right\} = Re\left\{(cos(t) + isin(t))^4\right\}$$
$$Re\left\{(cos(t) + isin(t))^4\right\} = \cos^4 t + 4\cos^3 t \cdot i\sin t + 6\cos^2 t \cdot i^2 \sin^2 t + 4\cos t \cdot i^3 \sin^3 t + i^4 \sin^4 t$$
$$= \cos^4 t - 6\cos^2 t \cdot \sin^2 t + sin^4 t$$
$$\therefore \cos(4t) = \boxed{\cos^4 t - 6\cos^2 t \cdot \sin^2 t + \sin^4 t}$$

di)

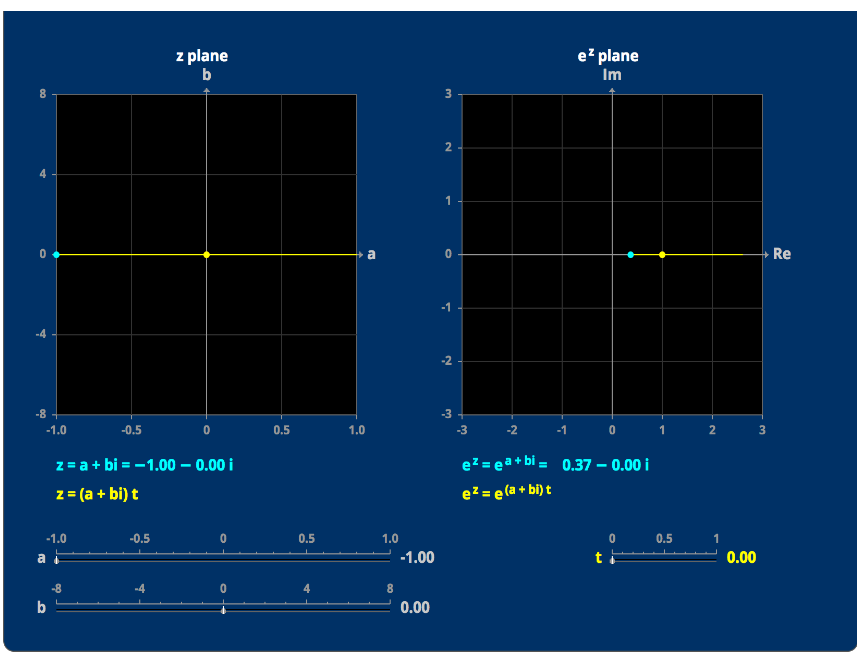
$$f(t) = \cos(2\pi t)$$
$$f(t) = Re\left\{e^{2\pi it}\right\}$$



dii)

$$f(t) = e^{-t}$$

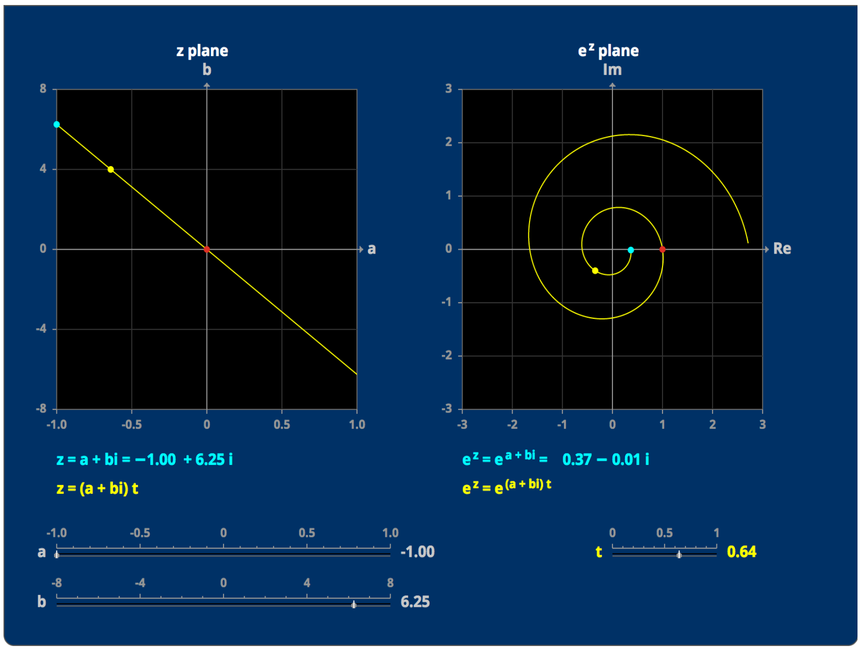
$$f(t) = e^{(-1+0i)t}$$



diii)

$$f(t) = e^{-t}cos(2\pi t)$$

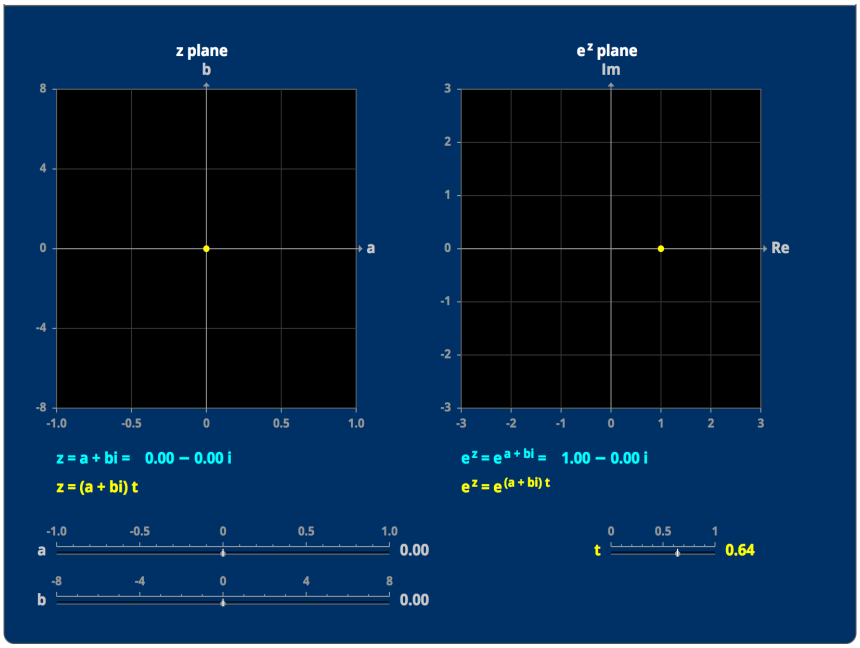
$$f(t) = e^{(-1+2\pi i)t}$$



div)

$$f(t) = 1$$

$$f(t) = e^{(0+0i)t}$$



Section 4

a)

First way:

$$Re\left\{\frac{e^{3it}}{\sqrt{3}+i}\right\} = Re\left\{\frac{\sqrt{3}-i}{4}(\cos(3t)+i\sin(3t))\right\}$$

$$= \frac{\sqrt{3}}{4}\cos(3t)+\frac{1}{4}\sin(3t)$$

$$= \frac{1}{2}\cos\left(3t-\frac{\pi}{6}\right)$$

Other way:

$$\frac{e^{3it}}{\sqrt{3}+i} = Be^{i(3t-\phi)}$$

$$\frac{e^{3it}}{2e^{\frac{\pi i}{6}}} = \frac{1}{2}e^{i(3t-\frac{\pi}{6})}$$

$$Re\left\{\frac{1}{2}e^{i(3t-\frac{\pi}{6})}\right\} = \frac{1}{2}cos\left(3t-\frac{\pi}{6}\right)$$

$$\therefore \boxed{\frac{1}{2}cos\left(3t-\frac{\pi}{6}\right) = \frac{1}{2}cos\left(3t-\frac{\pi}{6}\right)} \quad \checkmark$$

b)

Particular solution:

$$\frac{d}{dt}z(t) + 3z(t) = e^{2it}$$

$$\frac{d}{dt}\omega e^{2it} + 3\omega e^{2it} = e^{2it}$$

$$2i\omega e^{2it} + 3\omega e^{2it} = e^{2it}$$

$$\omega(3+2i)e^{2it} = e^{2it}$$

$$\omega(3+2i) = 1$$

$$\omega = \frac{1}{3+2i}$$

$$\boxed{\frac{1}{3+2i}e^{2it}}$$

Homogeneous solution:

$$\frac{d}{dt}z(t) + 3z(t) = 0$$

$$\frac{d}{dt}z(t) = -3z(t)$$

$$\int \frac{1}{z(t)}dz(t) = \int -3dt$$

$$ln(z(t)) = -3t + C$$

$$z(t) = Ce^{-3t}$$

$$\boxed{e^{-3t}}$$

General solution:

$$\boxed{\frac{1}{3+2i}e^{2it} + Ce^{-3t}}$$

c)

$$\frac{d}{dt}x(t) + 3x(t) = cos(2t)$$

Knowing the equation from (b) and realizing that $cos(2t) = Re\left\{e^{2it}\right\}$:

$$\begin{aligned} Re\left\{\frac{1}{3+2i}e^{2it}\right\} &= Re\left\{\frac{3-2i}{(3+2i)(3-2i)}e^{2it}\right\} \\ &= Re\left\{\frac{3-2i}{13}(\cos(2t) + i\sin(2t))\right\} \\ &= \frac{1}{13}(3\cos(2t) + 2\sin(2t)) \end{aligned}$$

$$\therefore x(t) = \boxed{\frac{1}{13}(3\cos(2t) + 2\sin(2t))}$$

The general solution follows the solution found in (b):

$$\boxed{\frac{1}{13}(3\cos(2t) + 2\sin(2t)) + ce^{-3t}}$$

Part I.

Section 1

1.5 #1)

$$\frac{dy}{dx} + y(x) = 2,\ y(0) = 0$$

This is actually separable, and can be solved with:

$$\frac{dy}{dx} = 2 - y(x)$$

$$\int \frac{1}{2-y(x)}\,dy = \int dx$$

$$ln(2 - y(x)) = t + C$$

$$2 - y(x) = e^{t + C_0}$$

$$y(x) = Ce^{x} - 2$$

Using the initial condition $y(0) = 0$:

$$y(0) = Ce^{0} - 2$$

$$0 = C - 2$$

$$C = 2$$

$$\boxed{y(x) = 2e^{x} - 2}$$

1.5 #2)

$$\frac{dy}{dx} - 2y(x) = 3e^{2x}$$

This is unseparable. Proceed by method of integrating factors.

$$\frac{d}{dx}a(x) = -2a(x)$$

Separate variables:

$$\int \frac{1}{a(x)} da(x) = \int -2\ dx$$

$$ln(a(x)) = -2x + C_0$$

$$a(x) = Ce^{-2x}$$

Multiplying this back into the original equation:

$$\frac{dy}{dx}e^{-2x} - 2y(x)e^{-2x} = 3e^{2x}e^{-2x}$$

$$\frac{d}{dx}\left(y(x)e^{-2x}\right) = 3$$

$$\int \frac{d}{dx}\left(y(x)e^{-2x}\right) dx = \int 3\ dx$$

$$y(x)e^{-2x} = 3x$$

$$\boxed{y(x) = 3xe^{2x}}$$

1.5 #5)

$$x\frac{dy}{dx} + 2y(x) = 3x, y(1) = 5$$

$$\frac{dy}{dx} + \frac{2}{x}y(x) = 3$$

This equation is not separable. We can write this in standard form though, so proceed by method of integrating factors.

$$\frac{2}{x}a(x) = \frac{da(x)}{dx}$$

Separate variables:

$$\int \frac{1}{a(x)}\, da(x) = \int \frac{2}{x}\, dx$$

$$ln(a(x)) = 2ln(x) + C$$

$$a(x) = e^{2ln(x)+C}$$

$$a(x) = Cx^2$$

Substitute back into the original equation:

$$\frac{dy}{dx}x^2 + \frac{2}{x}y(x)x^2 = 3x^2$$

$$\frac{dy}{dx}x^2 + 2xy(x) = 3x^2$$

$$\frac{d}{dx}\left(y(x)x^2\right) = 3x^2$$

$$\int \frac{d}{dx}\left(y(x)x^2\right) dx = \int 3x^2\, dx$$

$$y(x)x^2 = x^3 + C$$

$$y(x) = \frac{x^3 + C}{x^2}$$

$$y(x) = x + Cx^{-2}$$

Using the initial condition of $y(1) = 5$:

$$y(1) = (1) + C(1)^{-2}$$

$$5 = 1 + C$$

$$C = 4$$

$$\boxed{\therefore\, y(x) = x + 4x^{-2}}$$

Section 2

2E-1a)

$$-1 + i$$

$$r = \sqrt{(-1)^2 + 1^2}, \quad tan(\theta) = \frac{1}{-1}$$

$$r = \sqrt{2}, \quad \theta = -\frac{\pi}{4}$$

$$\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$\boxed{\sqrt{2}e^{-\frac{\pi}{4}i}}$$

2E-1b)

$$\sqrt{3} - i$$

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2}, \quad tan(\theta) = \frac{-1}{\sqrt{3}}$$

$$r = 2, \quad \theta = -\frac{\pi}{6}$$

$$2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$\boxed{2e^{-\frac{\pi}{6}i}}$$

2E-2)

$$\frac{1-i}{1+i}$$

Rectangular:

$$\frac{1-i}{1+i} = \frac{(1-i)\cdot(1-i)}{(1+i)\cdot(1-i)}$$

$$\frac{\frac{1-i}{1+i} = -2i}{2}$$

$$\boxed{\frac{1-i}{1+i} = -i}$$

Polar:

$$1-i \rightarrow \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \rightarrow \sqrt{2}e^{-\frac{\pi}{4}i}$$

$$1+i \rightarrow \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \rightarrow \sqrt{2}e^{\frac{\pi}{4}i}$$

$$\frac{1-i}{1+i} = \frac{\sqrt{2}e^{-\frac{\pi}{4}i}}{\sqrt{2}e^{\frac{\pi}{4}i}}$$

$$\frac{1-i}{1+i} = e^{-\frac{\pi}{4}i}e^{-\frac{\pi}{4}i}$$

$$\frac{1-i}{1+i} = e^{-\frac{\pi}{2}i}$$

$$\frac{1-i}{1+i} = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})$$

$$\frac{1-i}{1+i} = 0 + i(-1)$$

$$\boxed{\frac{1-i}{1+i} = -i}$$

Comparing the answers from the rectangular method and the polar method we find that they are the same:

$$-i = -i \quad \checkmark$$

2E-7a)

$$(1-i)^4$$

De Moivre's formula:

$$1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}$$

$$(1-i)^4 = \left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^4$$

$$(1-i)^4 = 4e^{-\pi i}$$

$$(1-i)^4 = 4(\cos(-\pi) + i\sin(-\pi))$$

$$(1-i)^4 = 4(-1 + i(0))$$

$$\boxed{(1-i)^4 = -4}$$

Binomial theorem:

$$(1-i)^4 = 1 \cdot 1^4 + 4 \cdot 1^3(-i)^1 + 6 \cdot 1^2(-i)^2 + 4 \cdot 1(-i)^3 + 1 \cdot (-i)^4$$

$$(1-i)^4 = 1 \cdot 1 + 4 \cdot (-i) + 6 \cdot 1(-1) + 4 \cdot 1(-i) + 1 \cdot 1$$

$$(1-i)^4 = 1 - 4i - 6 + 4i + 1$$

$$\boxed{(1-i)^4 = -4}$$

Comparing the two solution methods we find that:

$$-4 = -4 \quad \checkmark$$

2E-7b)

$$(1+i\sqrt{3})^3$$

De Moivre's formula:

$$1+i\sqrt{3} = 2e^{\frac{\pi}{3}i}$$

$$(1+i\sqrt{3})^3 = 8e^{\pi i}$$

$$(1+i\sqrt{3})^3 = 8(\cos(\pi) + i\sin(\pi))$$

$$\boxed{(1+i\sqrt{3})^3 = -8}$$

Binomial theorem:

$$(1+i\sqrt{3})^3 = 1\cdot 1^3 + 3\cdot 1^2(i\sqrt{3}) + 3\cdot 1(i\sqrt{3})^2 + 1\cdot(i\sqrt{3})^3$$

$$(1+i\sqrt{3})^3 = 1 + 3i\sqrt{3} + 9(-1) - 3i\sqrt{3}$$

$$\boxed{(1+i\sqrt{3})^3 = -8}$$

Comparing the two solution methods we find that:

$$-8 = -8 \quad \checkmark$$

Section 3

2E-9)

$$1 = e^{2\pi i}$$

$$1^{\frac{1}{6}} = e^{\frac{2\pi i}{6}}$$

We can write down all the roots:

$$e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\pi i}, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{3}}, e^{2\pi i}$$

In a+bi format:

$$\boxed{\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2},\ i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i,\ 1}$$

2E-10)

$$x^4 + 16 = 0$$

$$x^4 = -16$$

$$x^4 = 16(\cos(\pi) + i\sin(\pi))$$

$$x^4 = 16e^{\pi i}$$

$$x = 16^{\frac{1}{4}} e^{\frac{\pi}{4} i}$$

There are four roots:

$$x = 2e^{\frac{\pi}{4} i},\ 2e^{\frac{3\pi}{4} i},\ 2e^{\frac{5\pi}{4} i},\ 2e^{\frac{7\pi}{4} i}$$

Written in the $a + bi$ format:

$$\boxed{x = \sqrt{2} + \sqrt{2}i,\ -\sqrt{2} + \sqrt{2}i,\ -\sqrt{2} - \sqrt{2}i,\ \sqrt{2} - \sqrt{2}i}$$

i)

$$Acos(\theta) + Bsin(\theta) = Ccos(\theta - \phi), C = \sqrt{A^2 + B^2}, \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

$$\boxed{\cos(2t) + \sin(2t) = \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)}$$

ii)

$$\boxed{\cos(\pi t) - \sqrt{3}\sin(\pi t) = 2\cos\left(\pi t - \frac{\pi}{3}\right)}$$

iii)

$$\boxed{\cos\left(t - \frac{\pi}{8}\right) + \sin\left(t - \frac{\pi}{8}\right) = \sqrt{2}\cos\left(t - \frac{3\pi}{8}\right)}$$

Section 4

2E-15a)

$$\int e^{2x}\sin x\,dx = Im\left\{\int e^{2x}e^{ix}dx\right\}$$

$$\begin{aligned}\int e^{2x}e^{ix}dx &= \frac{1}{(2+i)}e^{x(2+i)}\\ &= \frac{e^{2x}}{(2+i)}e^{ix}\\ &= \frac{e^{2x}(2-i)}{5}e^{ix}\\ &= \frac{e^{2x}(2-i)}{5}(\cos x + i\sin x)\end{aligned}$$

Note that we want only the imaginary component:

$$\begin{aligned}Im\left\{\frac{e^{2x}(2-i)}{5}(\cos x + i\sin x)\right\} &= Im\left\{\frac{2e^{2x}-ie^{2x}}{5}(\cos x + i\sin x)\right\}\\ &= Im\left\{i\left(\frac{-e^{2x}\cos x}{5}+\frac{2e^{2x}\sin x}{5}\right)\right\}\\ &= \boxed{\frac{-e^{2x}\cos x}{5}+\frac{2e^{2x}\sin x}{5}}\end{aligned}$$

bi)

$$\frac{d}{dt}x(t) + 2x(t) = e^{3t}$$

$$\frac{d}{dt}x(t)e^{2t} + 2x(t)e^{2t} = e^{3t}e^{2t}$$

$$\frac{d}{dt}\left(x(t)e^{2t}\right) = e^{5t}$$

$$\int \frac{d}{dt}\left(x(t)e^{2t}\right)dt = \int e^{5t}\,dt$$

$$x(t)e^{2t} = \frac{1}{5}e^{5t} + C$$

$$\boxed{x(t) = e^{-2t}\left[\frac{1}{5}e^{5t} + C\right]}$$

bii)

$$\frac{d}{dt}x(t) + 2x(t) = e^{3it}$$

$$\frac{d}{dt}x(t)e^{2t} + 2x(t)e^{2t} = e^{3it}e^{2t}$$

$$\frac{d}{dt}\left(x(t)e^{2t}\right) = e^{t(2+3i)}$$

$$\int \frac{d}{dt}\left(x(t)e^{2t}\right) dt = \int e^{t(2+3i)}\, dt$$

$$x(t)e^{2t} = \frac{1}{(2+3i)}e^{t(2+3i)} + C$$

$$\boxed{x(t) = e^{-2t}\left[\frac{1}{(2+3i)}e^{t(2+3i)} + C\right]}$$