18.03

Recitation 4

[Problem Source]

Section 1

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a)

If k is the proportional constant, $y(t)$ is the height of the ocean, and $x(t)$ is the height of the bay, we can represent the fact that the change is proportional to the difference between ocean and bay with the following equation:

$$\frac{d}{dt}x(t) = k(y(t) - x(t))$$

Written in standard form:

$$\frac{d}{dt}x(t) = ky(t) - kx(t)$$

$$\boxed{\frac{d}{dt}x(t) + k\underbrace{x(t)}_{output} = k\underbrace{y(t)}_{input} \quad \underbrace{}_{system}}$$

b)

If we are modeling the tide using

$$y(t) = cos(\omega t)$$

and expecting it to have a period of 4π, which is double the normal period:

$$\boxed{\omega = \frac{1}{2}}$$

c)

$$
\begin{aligned}
\frac{d}{dt}x(t) + kx(t) &= ky(t) \\
\frac{d}{dt}x(t)e^{kt} + kx(t)e^{kt} &= ky(t)e^{kt} \\
\frac{d}{dt}\left(x(t)e^{kt}\right) &= ky(t)e^{kt} \\
\int \frac{d}{dt}\left(x(t)e^{kt}\right) dt &= \int ky(t)e^{kt}\, dt
\end{aligned}
$$

$$
\begin{aligned}
x(t)e^{kt} &= k\left(\frac{1}{k^2+\omega^2}e^{kt}(k\cos(\omega t) + \omega\sin(\omega t)) + C\right) \\
x(t) &= \boxed{\frac{k}{k^2+\omega^2}(k\cos(\omega t) + \omega\sin(\omega t)) + \frac{C}{e^{kt}}}
\end{aligned}
$$

d)

Assuming $x(t) = a\cos(\omega t) + b\sin(\omega t)$:

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt}(a\cos(\omega t) + b\sin(\omega t)) \\
&= -a\omega\sin(\omega t) + b\omega\cos(\omega t)
\end{aligned}
$$

Plugging this back into the original diffeq and substituting $y(t) = \cos(\omega t)$:

$$
\begin{aligned}
ky(t) &= \frac{d}{dt}x(t) + kx(t) \\
k\cos(\omega t) &= -a\omega\sin(\omega t) + b\omega\cos(\omega t) + k(a\cos(\omega t) + b\sin(\omega t)) \\
&= (ka + b\omega)\cos(\omega t) + (kb - \omega a)sin(\omega t)
\end{aligned}
$$

We can see that:

$$ka + b\omega = k$$

$$kb - \omega a = 0$$

Solving this system of equations:

$$
\begin{aligned}
kb - \omega a &= 0 \\
kb &= \omega a \\
b &= \frac{\omega a}{k}
\end{aligned}
$$

Plugging this b into the other equation:

$$ka + b\omega = k$$

$$ka + \frac{\omega a}{k}\omega = k$$

$$ka + \frac{\omega^2 a}{k} = k$$

$$a\left(\frac{k + \omega^2}{k}\right) = k$$

$$a = \frac{k^2}{k + \omega^2}$$

Plugging this back into the first equation to get an expression for b:

$$b = \frac{\omega a}{k}$$

$$b = \frac{\omega \frac{k^2}{k+\omega^2}}{k}$$

$$b = \omega \frac{k}{k + \omega^2}$$

$$\therefore \boxed{a = \frac{k^2}{k + \omega^2},\ b = \omega \frac{k}{k + \omega^2}}$$

Section 2

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a)

$$\frac{d}{dx}y(x) = x - 2y(x)$$

$$\frac{d}{dx}y(x) + 2y(x) = x$$

$$\frac{d}{dx}y(x)e^{2x} + 2y(x)e^{2x} = xe^{2x}$$

$$\frac{d}{dx}\left(y(x)e^{2x}\right) = xe^{2x}$$

$$\int \frac{d}{dx}\left(y(x)e^{2x}\right)\,dx = \int xe^{2x}\,dx$$

$$y(x)e^{2x} = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx + C$$

$$y(x)e^{2x} = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$

$$y(x) = \boxed{\frac{x}{2} - \frac{1}{4} + Ce^{-2x}}$$

b)

Evidently the straight line is at

$$y(x) = \boxed{\frac{x}{2} - \frac{1}{4}}$$

c)

Given the initial condition $y(0) = 1$
:

$$y(x) = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$$

$$y(0) = \frac{0}{2} - \frac{1}{4} + Ce^{0}$$

$$0 = -\frac{1}{4} + C$$

$$C = \frac{5}{4}$$

$$\therefore y_p(x) = \boxed{\frac{x}{2} - \frac{1}{4} + \frac{5}{4}e^{-2x}}$$

Section 3

$$x^2 \frac{d}{dx} y(x) + 2xy(x) = \sin(2x)$$

$$\frac{d}{dx}\left(x^2 y(x)\right) = \sin(2x)$$

$$\int \frac{d}{dx}\left(x^2 y(x)\right)\, dx = \int \sin(2x)\, dx$$

$$x^2 y(x) = -\frac{1}{2}\cos(2x) + C$$

$$y(x) = \frac{-\frac{1}{2}\cos(2x)}{x^2} + Cx^{-2}$$

$$y(x) = \boxed{\frac{-\cos(2x)}{2x^2} + Cx^{-2}}$$