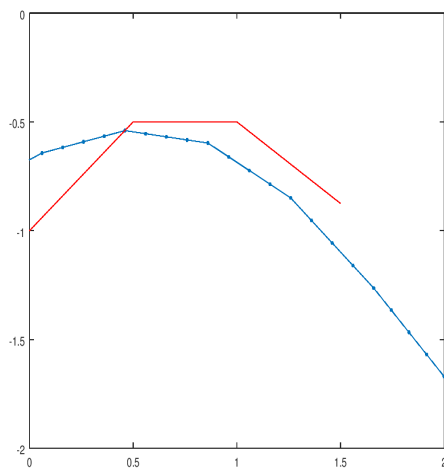


18.03

Recitation 3

1)

| n | x_n | y_n | $f(x_n, y_n)$ | $hf(x_n, y_n)$ |
|-----|-------|--------|---------------|----------------|
| 0 | 0 | -1 | 1 | 0.5 |
| 1 | 0.5 | -0.5 | 0 | 0 |
| 2 | 1 | -0.5 | -0.75 | -0.375 |
| 3 | 1.5 | -0.875 | 0 | 0 |



2)

To figure this out we can find out the double derivative at the target value of $x = 1.5$.

$$y'' = (y^2 - x^2)'$$

$$y'' = 2y' - 2x$$

$$y'' = 2(y^2 - x^2) - 2x$$

$$y'' = 2y^2 - x^2 - 2x$$

$$y'' = 2y^2 - (1.5)^2 - 2(1.5)$$

We can see that the second derivative is negative, indicating that the curve is concave-down, and therefore the euler's approximation going to be too large. This is confirmed by the image from problem (1) where we can see that the red line (euler's approximation) is above the approximated solution curve for the given differential equation.

3)

Equation:

$$\frac{d}{dt}x(t) = r \left(c - \frac{x(t)}{V} \right)$$

In standard form:

$$\boxed{\frac{d}{dt}x(t) + \frac{r}{V}x(t) = rc}$$

Checking this via dimensional reduction:

$$\frac{\text{gram}}{\text{minute}} = \frac{\text{liter}}{\text{minute}} \left(\frac{\text{gram}}{\text{liter}} - \frac{\text{gram}}{\text{liter}} \right)$$

$$\frac{\text{gram}}{\text{minute}} = \frac{\cancel{\text{liter}}}{\text{minute}} \cdot \frac{\text{gram}}{\cancel{\text{liter}}}$$

$$\frac{\text{gram}}{\text{minute}} = \frac{\text{gram}}{\text{minute}} \checkmark$$

4)

Givens:

$$V = 1, r = 2, x(0) = 0$$

From (3):

$$\frac{d}{dt}x(t) + \frac{r}{V}x(t) = rc$$

$$\frac{d}{dt}x(t) + \frac{2}{1}x(t) = 2c$$

Notice that the method of integrating factors can be applied here:

$$\frac{d}{dt}x(t)a(t) + 2x(t)a(t) = 2ca(t)$$

If we pick $2a(t) = \frac{d}{dt}a(t)$, we can write the LHS as $\frac{d}{dt}(x(t) \cdot a(t))$.

$$2a(t) = \frac{d}{dt}a(t)$$

Separate variables:

$$\int \frac{1}{a(t)} da(t) = \int 2 dt$$

$$\ln(a(t)) = 2t + C_1$$

$$a(t) = C_2 e^{2t}$$

We can drop the C as we are using this as an integrating factor. Multiplying over the differential equation we have:

$$\frac{d}{dt}x(t)e^{2t} + 2x(t)e^{2t} = 2ce^{2t}$$

$$\frac{d}{dt}(x(t)e^{2t}) = 2ce^{2t}$$

$$\int \frac{d}{dt}(x(t)e^{2t}) dt = \int 2ce^{2t} dt$$

$$x(t)e^{2t} = ce^{2t} + C$$

$$x(t) = c + Ce^{-2t}$$

Using the initial condition of $x(0) = 0$:

$$0 = c + Ce^0$$

$$-c = C$$

Plugging this back into the solution:

$$x(t) = c + -ce^{-2t}$$

$$\boxed{x(t) = c(1 - e^{-2t})}$$

The limiting amount of salt in the tank is c which makes sense, as the maximum concentration of salt cannot exceed the concentration of the salt entering the tank.

To figure out when the tank will contain half this amount, the equation for $x(t)$ can be solved for $x(t) = \frac{c}{2}$:

$$\frac{c}{2} = c(1 - e^{-2t})$$

$$\frac{1}{2} = (1 - e^{-2t})$$

$$-\frac{1}{2} = -e^{-2t}$$

$$\frac{1}{2} = e^{-2t}$$

$$\ln\left(\frac{1}{2}\right) = -2t$$

$$t = -\frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$t \approx 0.3466 \text{ minutes}$$

5)

Representing the amount of salt in the second tank as $y(t)$:

$$\frac{d}{dt}y(t) = r \left(\frac{x(t)}{V} - \frac{y(t)}{V} \right)$$

In english: the rate of change of salt in the second tank is the concentration of salt flowing out of the first tank minus the concentration of salt flowing out of the second tank multiplied by the flow rate.

Plugging in $x(t)$ from (4), and using the initial conditions of $V = 1$, $y(1) = 0$:

$$\frac{d}{dt}y(t) = 2 \left(c(1 - e^{-2t}) - y(t) \right)$$

6)

$$R \frac{d}{dt} I(t) + \frac{1}{C} I(t) = \frac{d}{dt} V(t)$$

Since $V(t)$ is constant, $\frac{d}{dt} V(t)$ is 0:

$$R \frac{d}{dt} I(t) + \frac{1}{C} I(t) = 0$$

$$R \frac{d}{dt} I(t) = -\frac{1}{C} I(t)$$

$$\frac{d}{dt} I(t) = -\frac{1}{RC} I(t)$$

We can solve this differential equation for $I(t)$:

$$\int \frac{1}{I(t)} dI(t) = \int -\frac{1}{RC} dt$$

$$\ln(I(t)) = -\frac{t}{RC} + K_1$$

$$I(t) = K e^{-\frac{t}{RC}}$$

With initial condition $I(0) = I_0$:

$$I(0) = K e^0$$

$$I(0) = K$$

$$\therefore I_0 = I(0) = K$$

$$I(t) = I_0 e^{-\frac{t}{RC}}$$

$$c = I_0, \tau = RC$$

$$I(t + \tau) = I_0 e^{-\frac{t+\tau}{RC}}$$

$$I(t + \tau) = I_0 e^{-\frac{t}{RC}} e^{-\frac{\tau}{RC}}$$

$$I(t + \tau) = I(t) e^{-\frac{\tau}{RC}}$$

$$I(t + \tau) = I(t) e^{-\frac{RC}{RC}}$$

$$I(t + \tau) = I(t)e^{-1}$$