18.03

Recitation 1 -

Introduction, natural growth and decay, review of logarithm.

[Problem Source]

1. A model of the oryx population can be as follows:

$$f(t+\Delta t) = kf(t)\Delta t + f(t) - a\Delta t$$

$$\frac{f(t+\Delta t) - f(t)}{\Delta t} = kf(t) - a$$

As $\Delta t \to 0$:

$$\boxed{\frac{df(t)}{dt} = kf(t) - a}$$

This model is consistent if we choose the following units:

$$f(t+\Delta t) => oryxes$$

$$k => \frac{oryxes}{oryxes * year}$$

$$t => year$$

$$a => oryxes/year$$

$$\Delta t => year$$

$$f(t) => oryxes$$

Dimensional Analysis:

$$oryxes = \frac{\cancel{oryxes}}{\cancel{oryxes} * \cancel{year}} * oryxes * \cancel{year} - \frac{oryxes}{\cancel{year}} * \cancel{year} + oryxes$$

which results in

$$oryxes = oryxes$$

2. In (1), we gave a model for modeling oryx populations at time $t + \Delta t$. Now, the problem revolves around finding the value of t that doubles the population. As in, we are looking for

   $$f(t + \Delta t) = 2f(t)$$

   We have a bit of a differential equation, as we can rewrite the original model to have a differential form, reproduced below with $a = 0$:

   $$\frac{d}{dt} f(t) = kf(t)$$

   If we had the function $f(t)$, then we could solve the desired equation by plugging in t and Δt. Rewrite the function as:

   $$\frac{1}{f(t)} \frac{d}{dt} f(t) = k$$

   Take integral with respect to dt on both sides:

   $$\int \frac{1}{f(t)} \frac{d}{dt} f(t) \; dt = \int k \; dt$$

   Now we can notice that the left side can be expressed as:

   $$\frac{d}{df(t)} g(f(t)) \frac{d}{dt} f(t) = \frac{d}{dt} g(f(t))$$

   If we make the substitution:

   $$\frac{d}{df(t)} g(f(t)) = \frac{1}{f(t)}$$

   we can rewrite the left integral as:

   $$\int \frac{d}{dt} g(f(t)) \; dt$$

   By the first fundamental theorem of calculus this reduces to:

   $$g(f(t))$$

   To solve for $g(f(t))$, we have:

$$\frac{d}{df(t)}g(f(t)) = \frac{1}{f(t)}$$

$$\int \frac{d}{df(t)}g(f(t))\ df(t) = \int \frac{1}{f(t)}\ df(t)$$

Again using the first fundamental theorem of calculus:

$$g(f(t)) = ln(f(t)) + C$$

Which is the solution for the left side integral, so we have:

$$ln(f(t)) = kt + C$$

Solving for f(t):

$$f(t) = e^C e^{kt}$$

We want a factor of 2 between the f(t)s:

$$\frac{f(t + \Delta t)}{f(t)} = \frac{e^C e^{k(t+\Delta t)}}{e^C e^{kt}}$$

$$2 = e^{k(t+\Delta t)-kt}$$

$$2 = e^{k\Delta t}$$

$$\boxed{\frac{ln(2)}{k} = \Delta t}$$

3. The general solution of this problem is the solution of the following differential equation:

$$\frac{d}{dt}f(t) = kf(t) - a$$

Let's substitute $y(t) = f(t) - \frac{a}{k}$ to get:

$$\frac{dy(t)}{dt} = \frac{df(t)}{dt} = ky(t)$$

because the $\frac{a}{k}$ is a constant and disappears during differentiation.

$$\int \frac{1}{y(t)} \frac{dy(t)}{dt}\ dt = \int k\ dt$$

Solving as before we end up with

$$ln(y(t)) = kt + C$$

Substituting the y(t) back for f(t) we receive:

$$ln(f(t) - \frac{a}{k}) = kt + C$$

$$f(t) - \frac{a}{k} = e^{kt+C}$$

$$\boxed{f(t) = Ce^{kt} + \frac{a}{k}}$$

4. We can take the derivative of (3) and compare it to plugging (3) into (1).

$$f(t) = Ce^{kt} + \frac{a}{k} \qquad (3)$$

$$\frac{df(t)}{dt} = \frac{d}{dt}\left(Ce^{kt} + \frac{a}{k}\right)$$

$$\frac{df(t)}{dt} = Cke^{kt}$$

Combining Ck into C:

$$\boxed{\frac{df(t)}{dt} = Ce^{kt} \quad \checkmark}$$

From (1) we have:

$$\frac{df(t)}{dt} = kf(t) - a$$

Plugging in (3) we end up with:

$$\frac{df(t)}{dt} = k\left(Ce^{kt} + \frac{a}{k}\right) - a$$

$$\frac{df(t)}{dt} = kCe^{kt} + k\frac{a}{k} - a$$

$$\boxed{\frac{df(t)}{dt} = Ce^{kt} \quad \checkmark}$$

5. Steady state happens when derivative is zero.

$$0 = kf(t) - a$$

$$\boxed{f(t) = \frac{a}{k}}$$

Using the general solution from (3) for $f(t)$, we can even find out t:

$$Ce^{kt} + \frac{a}{k} = \frac{k}{a}$$

$$Ce^{kt} = \frac{k}{a} - \frac{a}{k}$$

$$Ce^{kt} = \frac{\left(k^2 - a^2\right)}{ak}$$

$$kt = ln(k^2 - a^2) - ln(C)$$

$$t = \frac{ln(k^2 - a^2) - ln(C)}{k}$$