18.03

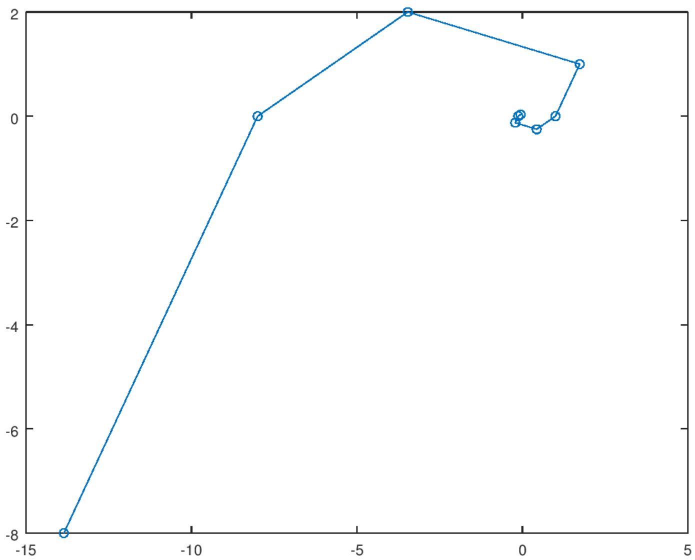
Recitation 5

[Problem Source]

Section 1

Polar coordinates: $(1, \sqrt{3}i)$

n	$a + bi$	$Ae^{i\theta}$
-4	$\frac{1}{16}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$	$\frac{1}{16}e^{\frac{-4\pi}{3}i}$
-3	$\frac{1}{8}(-1 + 0i)$	$\frac{1}{8}e^{-\pi i}$
-2	$\frac{1}{4}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$	$\frac{1}{4}e^{\frac{-2\pi}{3}i}$
-1	$\frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$	$\frac{1}{2}e^{\frac{-\pi}{3}i}$
0	$1 + 0i$	1
1	$2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$	$2e^{\frac{\pi}{3}i}$
2	$4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$	$4e^{\frac{2\pi}{3}i}$
3	$8(-1 + 0i)$	$8e^{\pi i}$
4	$16\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$	$16e^{\frac{4\pi}{3}i}$



Section 2

From (1) we know that one possible expression is:

$$2e^{i\frac{\pi}{3}}$$

Rewriting this in the form e^{a+bi} :

$$e^{ln(2)}e^{i\frac{\pi}{3}}$$

$$e^{ln(2)+i\frac{\pi}{3}}$$

$$\therefore a = ln(2),\ b = \frac{\pi}{3}$$

Fixing b to be as small as possible:

$$e^{ln(2)+i\frac{\pi}{3}}$$

It's clear that solving for $\left(e^{ln(2)+i\frac{\pi}{3}}\right)^n$ where n is $\{-4...4\}$ will result in the same answers as in (1).

Section 3

a)

Expression:

$$\cos(2t) + \sin(2t)$$

Writing in the form of $A\cos(\omega t - \phi)$:

$$A = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$

$$\phi = \arctan(\frac{1}{2}) = \frac{\pi}{4}$$

$$\therefore \cos(2t) + \sin(2t) = \boxed{\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)}$$

b)

Expression:

$$\cos(\pi t) - \sqrt{3}\sin(\pi t)$$

Writing in the form of $A\cos(\omega t - \phi)$:

$$A = \sqrt{1^2 + \sqrt{3}^2} = 2$$

$$\phi = -\arctan\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

$$\therefore \cos(\pi t) - \sqrt{3}\sin(\pi t) = \boxed{2\cos\left(\pi t + \frac{\pi}{3}\right)}$$

c)

Expression:

$$Re\left\{\frac{e^{it}}{2 + 2i}\right\}$$

Writing in the form of $A\cos(\omega t - \phi)$:

$$
\begin{aligned}
Re\left\{\frac{e^{it}}{2+2i}\right\} &= Re\left\{\frac{e^{it}}{2\sqrt{2}e^{\frac{\pi}{4}i}}\right\} \\
&= Re\left\{2\sqrt{2}e^{it}e^{-\frac{\pi}{4}i}\right\} \\
&= Re\left\{2\sqrt{2}e^{\left(t-\frac{\pi}{4}\right)i}\right\} \\
&= \boxed{2\sqrt{2}\cos\left(t-\frac{\pi}{4}\right)}
\end{aligned}
$$

Section 4

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a)

$$
\begin{aligned}
\frac{d}{dt}x(t) + 2x(t) &= e^{t} \\
\frac{d}{dt}x(t)e^{2t} + 2x(t)e^{2t} &= e^{3t} \\
\frac{d}{dt}\left(x(t)e^{2t}\right) &= e^{3t} \\
\int \frac{d}{dt}\left(x(t)e^{2t}\right)\ dt &= \int e^{3t}\ dt \\
x(t)e^{2t} &= \int e^{3t}\ dt \\
x(t)e^{2t} &= \frac{1}{3}e^{3t} + C \\
x(t) &= \boxed{\frac{1}{3}e^{t} + Ce^{-2t}}
\end{aligned}
$$

b)

$$
\begin{aligned}
\frac{d}{dt}z(t) + 2z(t) &= e^{2it} \\
e^{2t}\frac{d}{dt}z(t) + 2e^{2t}z(t) &= e^{2it}e^{t} \\
\int \frac{d}{dt}\left(e^{2t}z(t)\right)\, dt &= \int e^{2it}e^{2t}\, dt \\
e^{2t}z(t) &= \int e^{t(2+2i)}\, dt \\
e^{2t}z(t) &= \frac{1}{2+2i}e^{t(2+2i)} + C \\
z(t) &= \frac{1}{2+2i}e^{t(2+2i)-2t} + Ce^{-2t} \\
z(t) &= \boxed{\frac{1}{2+2i}e^{2it} + Ce^{-2t}}
\end{aligned}
$$

Section 5

$$
\begin{aligned}
\frac{d}{dt}x(t) + 2x(t) &= \cos(2t) \\
\frac{d}{dt}x(t) + 2x(t) &= Re\left\{e^{2it}\right\} \\
\frac{d}{dt}x(t)e^{2t} + 2x(t)e^{2t} &= Re\left\{e^{2it}\right\}e^{2t} \\
\frac{d}{dt}\left(x(t)e^{2t}\right) &= Re\left\{e^{(2+2i)t}\right\} \\
\int \frac{d}{dt}\left(x(t)e^{2t}\right)\, dt &= \int Re\left\{e^{(2+2i)t}\right\}\, dt \\
x(t)e^{2t} &= Re\left\{\frac{1}{2+2i}e^{(2+2i)t}\right\} + C \\
x(t) &= Re\left\{\frac{1}{2+2i}e^{(2+2i)t}\right\}e^{-2t} + Ce^{-2t} \\
&= Re\left\{\frac{1}{2+2i}\frac{2-2i}{2-2i}e^{2it}\right\} + Ce^{-2t} \\
&= Re\left\{\frac{2-2i}{8}e^{2it}\right\} + Ce^{-2t} \\
&= Re\left\{\frac{1-i}{4}\left(\cos(2t) + i\sin(2t)\right)\right\} + Ce^{-2t} \\
&= \frac{1}{4}Re\left\{\cos(2t) + \sin(2t) \cancel{-i\cos(2t)} + \cancel{i sin(2t)}\right\} + Ce^{-2t} \\
&= \boxed{\frac{1}{4}\left(\cos(2t) + \sin(2t)\right)} + Ce^{-2t}
\end{aligned}
$$

The solution for

$$\frac{d}{dt}x(t) + 2x(t) = sin(2t)$$

is simply the Imaginary part of the prior solution:

$$\boxed{\frac{1}{4}\left(-\cos(2t) + \sin(2t)\right) + Ce^{-2t}}$$