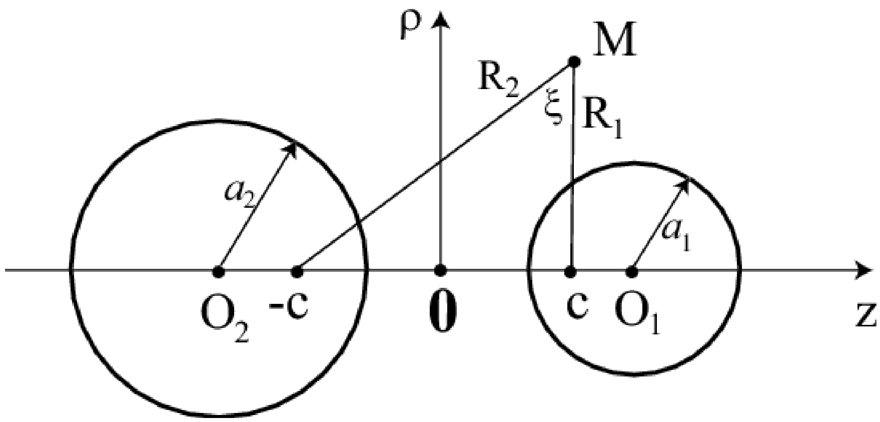

Let two points (poles of bipolar coordinate system) be separated by an arbitrary distance 2c. Let the origin O be the midpoint between these two poles. Let R1 and R2 be the distances from an observation point M to these two poles , and ξ be the angle between those connecting lines (as shown in the figure); this angle is from 0 to pi. By definition, bipolar (=bispherical) coordinates η, ξ for an observation point M are defined as $\eta = \ln(R_2 / R_1),$ and ξ as already defined above. Simple algebra (do it!) relates the bispherical and cylindrical coordinates of point M :

$$z = \frac{c\,\mathrm{sh}(\eta)}{\mathrm{ch}(\eta) - \mu}, \quad \rho = \frac{c\,\sin(\xi)}{\mathrm{ch}(\eta) - \mu}, \quad \mu = \cos(\xi). \quad (1)$$

[I am using simplified Russian notations sh and ch instead of common sinh and cosh]. Coordinate lines $\eta = const$ are circles of radius $c / |\mathrm{sh}\,\eta|$ centered at z=$c\ \mathrm{cth}\,\eta$ (where cth= cosh/sinh). [Lines $\xi = const$ are arcs of circles between the poles]. These arcs are orthogonal to lines $\eta = const$. Appendix A19 of Happel and Brenner ("Low Re hydrodynamics")…has the necessary details.

Now, for two given spheres of given radii a1 and a2, with centers O1 and O2 and given separation, you can make them the coordinate surfaces $\eta = \eta_1 = const > 0$ and $\eta = \eta_2 = const < 0$, which gives the eqs for eta1, eta2 and c:

$$sh\eta_2 / sh\eta_1 = -k = -a_1 / a_2, \quad c(cth\eta_1 - cth\eta_2) = a_1(1 + \varepsilon + 1/k), \quad (2)$$

Where εa_1 is the surface gap. The last eqn (2) simplifies to

$$ch(\eta_1) + ch(\eta_2)/k = 1 + \varepsilon + 1/k. \quad (3)$$

From (3), we get

$$ch^2(\eta_2) = 1 + sh^2(\eta_2) = [1 + k + k\varepsilon - kch(\eta_2)]^2. \qquad (4)$$

Also,

$$sh^2(\eta_2) = k^2[ch^2(\eta_1) - 1] \qquad (5)$$

Substituting (5) into (4) gives an equation for $X = ch(\eta_1)$,

$$1 + k^2(X^2 - 1) = [1 + k + k\varepsilon - kX]^2 \quad ,$$

which happens to be linear after simplification, giving eventually

$$ch(\eta_1) = \frac{(1+\varepsilon)(1+k) + k\varepsilon^2/2}{1 + k + k\varepsilon} \quad (6)$$

[I do not know if there is a simpler derivation], which confirms (1.3) from my PMM paper.

Of course, eqn for $ch(\eta_2)$ would not need a separate derivation, but can be obtained from (6) by replacing k with 1/k, and ε with $k\varepsilon$.