Characteristic function

The characteristic function of a random variable X (discrete or continuous) is defined as $E(e^{i\omega x})$ and denoted as $\phi(\omega)$. $\phi(\omega) = E(e^{i\omega x})$

If x is discrete r.v that can take the values $x_1, x_2, \ldots$ such that $P(X = x_r) = P_r$ then

$$\phi(\omega) = \sum_r e^{i\omega x_r} P(X = x_r)$$

$$\boxed{\phi(\omega) = \begin{cases} \sum_x e^{i\omega x} p(x) & \text{if } x \text{ is discrete r.v} \\ \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx & \text{if } x \text{ is continuous r.v} \end{cases}}$$

If x is continuous r.v with density fun. $f(x)$ then

$$\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx.$$

Properties of characteristic function

1. $\mu_n' = E(x^n) =$ the co-efficient of $\frac{i^n \omega^n}{n!}$ in the expansion of $\phi(\omega)$ in series of ascending powers of $i\omega$.

proof:-
$$\phi(\omega) = E(e^{i\omega x})$$
$$= E\left(1 + \frac{i\omega x}{1!} + \frac{(i\omega x)^2}{2!} + \cdots \frac{(i\omega x)^n}{n!} + \cdots\right)$$
$$= 1 + \frac{i\omega}{1!}E(x) + \frac{i^2\omega^2}{2!}E(x^2) + \cdots + \frac{i^n\omega^n}{n!}E(x^n) + \cdots$$
$$= 1 + \mu_1' \frac{i\omega}{1!} + \mu_2' \frac{i^2\omega^2}{2!} + \cdots + \mu_n' \frac{i^n\omega^n}{n!} + \cdots$$
$$\phi(\omega) = \sum_{n=0}^{\infty} \mu_n' \frac{i^n\omega^n}{n!}$$

$\therefore$ The co-eff. of $\frac{i^n\omega^n}{n!} = \mu_n' = E(x^n)$

2. $\mu_n' = \frac{1}{i^n}\left[\frac{d^n}{d\omega^n}\phi(\omega)\right]_{\omega=0}$.

proof:- we know that $\phi(\omega) = \sum_{n=0}^{\infty} \mu_n' \frac{i^n \omega^n}{n!}$

Diff. w.r.to ω $= 1 + \mu_1' \frac{i\omega}{1!} + \mu_2' \frac{i^2\omega^2}{2!} + \mu_3' \frac{i^3\omega^3}{3!} + \dots + \mu_n' \frac{i^n\omega^n}{n!} + \dots$

$\phi'(\omega) = \frac{i}{1!}\mu_1' + \cancel{2}\omega\, \mu_2' \frac{i^2}{\cancel{2}!} + \frac{\omega^2}{2} i^3 \mu_3' + \dots + \mu_n' \frac{i^n \omega^{n-1}}{(n-1)!} + \dots$

$[\phi'(\omega)]_{\omega=0} = \frac{i}{1!}\mu_1' = i\mu_1'$

$\phi''(\omega) = \cancel{2}\mu_2' \frac{i^2}{\cancel{2}!} + \frac{2\omega}{2} i^3 \mu_3' + \dots + \mu_n' i^n \frac{\omega^{n-2}}{(n-2)!} + \dots$

$[\phi''(\omega)]_{\omega=0} = i^2 \mu_2'$

$(\phi'''(\omega))_{\omega=0} = i^3 \mu_3'$

...

$[\phi^n(\omega)]_{\omega=0} = i^n \mu_n' \Rightarrow \mu_n' = \frac{1}{i^n}[\phi^n(\omega)]_{\omega=0}$

3. If the characteristic fn. of a r.v x is $\phi_x(\omega)$ and if $y = ax + b$ then

$$\phi_y(\omega) = e^{ib\omega}\phi_x(a\omega)$$

proof:- $\phi_y(\omega) = E(e^{i\omega y}) = E(e^{i\omega(ax+b)})$

$= E(e^{i\omega a x} \cdot e^{i\omega b})$

$= e^{i\omega b} E(e^{i\omega a x}) = e^{i\omega b} E(e^{i(a\omega)x})$

$= e^{i\omega b}\phi_x(a\omega)$

4) If X & Y are independent r.v's then

$$\phi_{X+Y}(\omega) = \phi_X(\omega) \times \phi_Y(\omega)$$

Proof :- $\phi_{X+Y}(\omega) = E(e^{i\omega(X+Y)})$

$$= E(e^{i\omega X} \cdot e^{i\omega Y})$$

$$= E(e^{i\omega X}) \cdot E(e^{i\omega Y}) \quad \because X \& Y \text{ are independent variables}$$

$$\phi_{X+Y}(\omega) = \phi_X(\omega) \cdot \phi_Y(\omega)$$

5) If the characteristic function of continuous r.v X with density function $f(x)$ is $\phi(\omega)$,

then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\omega)\, e^{-ix\omega}\, d\omega.$

Problem

Obtain the characteristic function of the Poisson distribution and also find its mean and variance.

Solution:- The Poisson distribution is given by $P(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$, $x = 0, 1, 2, \ldots$

Its characteristic function is

$$\phi_x(\omega) = E(e^{i\omega x}) = \sum e^{ix\omega} P(x)$$

$$= \sum_{x=0}^{\infty} e^{ix\omega} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(\lambda e^{i\omega})^x}{x!}$$

$$\mu_1' = \frac{1}{i} \frac{d}{d\omega}\left(\phi(\omega)\right)_{\omega=0}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^{i\omega}}{1!} + \frac{(\lambda e^{i\omega})^2}{2!} + \cdots\right]$$

$$\mu_1' = \frac{\phi'(0)}{i}$$

$$\mu_2' = \frac{\phi''(0)}{i^2}$$

$$\phi(\omega) = e^{-\lambda} \cdot e^{\lambda e^{i\omega}} = e^{\lambda(e^{i\omega}-1)}$$

$$\phi'(\omega) = e^{-\lambda} \cdot e^{\lambda e^{i\omega}} \cdot \lambda e^{i\omega} \cdot i = e^{\lambda(e^{i\omega}-1)} \lambda i e^{i\omega}$$

$$\phi'(0) = e^{-\lambda} \cdot e^{\lambda} \cdot \lambda \cdot i = \lambda i$$

$$\frac{\phi'(0)}{i} = \lambda = \mu_1'$$

$$\phi''(\omega) = e^{\lambda(e^{i\omega}-1)} \cdot (\lambda e^{i\omega})^2 \cdot i^2 + e^{\lambda(e^{i\omega}-1)} \cdot \lambda i e^{i\omega} i$$

$$\phi''(0) = \lambda^2 \cdot i^2 + \lambda i^2 = i^2(\lambda^2 + \lambda)$$

$$\frac{\phi''(0)}{i^2} = \lambda^2 + \lambda = \mu_2'$$

$$\mu_1' = \frac{(\phi'(\omega))_{\omega=0}}{i} = \lambda$$

$$\mu_2' = \frac{(\phi''(\omega))_{\omega=0}}{i^2} = \lambda^2+\lambda$$

$$\text{mean} = \mu_1' = E(x) = \lambda$$

$$\text{variance} = \mu_2' - \mu_1'^2$$

$$\text{Variance} = \lambda^2+\lambda-\lambda^2 = \underline{\lambda}$$

② Find the characteristic fn. of the uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

Soln:- The characteristic fn. is

$$\phi(\omega) = E(e^{i\omega x})$$

$$= \int_a^b e^{i\omega x} \frac{1}{(b-a)} dx.$$

$$= \frac{1}{b-a} \int_a^b e^{i\omega x} dx.$$

$$= \frac{1}{b-a} \left[\frac{e^{i\omega x}}{i\omega} \right]_a^b$$

$$= \frac{1}{i\omega(b-a)} \left[e^{i\omega b} - e^{i\omega a} \right] \quad \text{for } \omega \neq 0.$$

problem

Find the characteristic function of the geometric function given by $P(X=x) = q^x p$, $x = 0, 1, 2, \ldots$, $p+q=1$ and hence find its mean and variance.

Soln.:

$$\phi(\omega) = E(e^{i\omega x}) = \sum e^{i\omega x} p(x)$$

$$= \sum_{x=0}^{\infty} e^{i\omega x} \cdot q^x p$$

$$= p \sum_{x=0}^{\infty} (q e^{i\omega})^x$$

$$= p\left[1 + q e^{i\omega} + (q e^{i\omega})^2 + \cdots\right]$$

$$\phi(\omega) = p\left[1 - q e^{i\omega}\right]^{-1} = \frac{p}{1 - q e^{i\omega}}$$

$$\phi'(\omega) = p(-1)(1 - q e^{i\omega})^{-2}(-q e^{i\omega}) \cdot i$$

$$\phi'(0) = p(-1)(1-q)^{-2}(-q) \cdot i = \frac{pq\, i}{(1-q)^2} = \frac{q}{p} i$$

$$E(X) = \frac{\phi'(0)}{i} = \frac{q}{p}$$

$p+q=1$
$p = 1-q$

$$\phi'(\omega) = pq(1 - q e^{i\omega})^{-2}\, i e^{i\omega}$$

$$\phi''(\omega) = pq i\left[(-2)(1 - q e^{i\omega})^{-3}(-q e^{i\omega}) \cdot i\, e^{i\omega} + (1 - q e^{i\omega})^{-2} e^{i\omega} \cdot i\right]$$

$$\phi''(0) = pq i\left[(-2)(1-q)^{-3}(-q) i + (1-q)^{-2} i\right]$$

$$= pq i^2 \left[\frac{2q}{p^3} + \frac{1}{p^2}\right],$$

$$E(X^2) = \frac{\phi''(0)}{i^2} = \frac{2q^2}{p^2} + \frac{q}{p^2}, \quad Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{q}{p^2}(1+q)$$

Problem

Find the characteristic function of the density function

$$f(x) = \frac{\alpha}{2} e^{-\alpha|x|}, \quad -\infty < x < \infty$$

Hence find its mean and variance.

Soln:- $\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx = \int_{-\infty}^{\infty} e^{i\omega x} \cdot \frac{\alpha}{2} e^{-\alpha|x|}\,dx$

$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{-\alpha(-x)} e^{i\omega x}\,dx + \int_{0}^{\infty} e^{-\alpha(x)} e^{i\omega x}\,dx\right]$$

$|x| = \begin{cases} -x, & x<0 \\ x, & x>0 \end{cases}$

$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{(\alpha+i\omega)x}\,dx + \int_{0}^{\infty} e^{-(\alpha-i\omega)x}\,dx\right]$$

$$= \frac{\alpha}{2}\left[\left[\frac{e^{(\alpha+i\omega)x}}{\alpha+i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha-i\omega)x}}{-(\alpha-i\omega)}\right]_{0}^{\infty}\right]$$

$$= \frac{\alpha}{2}\left[\frac{1}{\alpha+i\omega} + \frac{1}{\alpha-i\omega}\right] = \frac{\alpha}{2} \cdot \frac{2\alpha}{\alpha^2+\omega^2}$$

$$\phi(\omega) = \frac{\alpha^2}{\alpha^2+\omega^2}$$

$$\phi(\omega) = \left(1+\frac{\omega^2}{\alpha^2}\right)^{-1} = 1 - \frac{\omega^2}{\alpha^2} + \left(\frac{\omega^2}{\alpha^2}\right)^2 - \left(\frac{\omega^2}{\alpha^2}\right)^3 + \cdots \rightarrow (1)$$

we know that

$$\phi(\omega) = 1 + \frac{i\omega}{1!}E(X) + \frac{i^2\omega^2}{2!}E(X^2) + \cdots + \frac{i^n\omega^n}{n!}E(X^n) \cdots$$

From (1) we have

$E(X) = 0$, since the co-efficients of $\frac{i\omega}{1!}$ is zero

$E(X^2) = \frac{2}{\alpha^2}$ since the co-effs of $\frac{i^2\omega^2}{2!}$ is $\frac{2}{\alpha^2}$

$$\mathrm{var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{2}{\alpha^2} - 0 = 2/\alpha^2$$

Problem
Find the density function $f(x)$ corresponding to the characteristic function defined as

$$\varphi(t) = \begin{cases} 1-|t|, & |t|\le 1 \\ 0, & |t|>1 \end{cases}$$

Soln.:-

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t)\, e^{-ixt}\, dt$$

$$= \frac{1}{2\pi}\int_{-1}^{1} (1-|t|)\, e^{-ixt}\, dt$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{1} (1-|t|)(\cos xt - i\sin xt)\, dt\right.$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{1} (1-|t|)\cos xt\, dt - i\int_{-1}^{1}(1-|t|)\sin xt\, dt\right]$$

$$= \frac{2}{2\pi}\int_{0}^{1}(1-t)\cos xt\, dt$$

$$= \frac{1}{\pi}\left[(1-t)\frac{\sin xt}{x} - (-1)\left(\frac{-\cos xt}{x^2}\right)\right]_0^1$$

$$= \frac{1}{\pi}\left[-\frac{\cos x}{x^2} - \left(0 - \frac{1}{x^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1-\cos x}{x^2}\right]$$