Properties of Cumulative Distribution Function

1. *The Cumulative distribution function $F(X)$ is smooth.*
2. *It is non-decreasing function that increases from 0 to 1.*
3. *The expected value or mean of a continuous random variable X, is denoted by $E(X)$.*
4. *Its Variane is denoted by $V(X)$.*

MATHEMATICAL EXPECTATION

...ATHEMATICAL EXPECTATION OR EXPECTED VALUES

***Mathematical expectation** of a random variable is obtained by multiplying each probable of the variable by its corresponding probability and then adding these products.*

Let a random variable X assumes the values $x_1, x_2, \ldots\ldots, x_n$ with probabilities $p_1, p_2, \ldots\ldots, p_n$ ...vely. Then the mathematical expectation of the variable X is defined as:

$$E(X) = x_1 p_1 + x_2 p_2 + \ldots\ldots + x_n p_n = \Sigma x_i p_i.$$

Note: ***1. Sometimes $E(X)$ is also known as Expected value of X.***
2. Expected value of X is a population mean. If population mean is μ then $E(X) = \mu$.

THEOREMS ON MATHEMATICAL EXPECTATION

Theorem 1: *Expected value of constant term is constant, that is, if C is constant, then*

$$E(C) = C.$$

Theorem 2: *If C is constant, then:*

$$E(CX) = C \cdot E(X).$$

Theorem 3: *If a and b are constants, then:*

$$E(aX \pm b) = a\,E(X) \pm b.$$

Theorem 4: *If a, b and c are constants, then:*

$$E\left(\frac{aX + b}{c}\right) = \frac{1}{c}[a\,E(X) + b].$$

Theorem 5: *If X and Y are any two random variables, then:*

$$E(X + Y) = E(X) + E(Y).$$

Theorem 6: *If X and Y are two independent random variables, then*

$$E(X \cdot Y) = E(X) \cdot E(Y).$$

Remarks: ***1. If $g(x)$ is any function of random variable X and $f(x)$ is probability density function then: $E[g(x)] = \Sigma\{g(x) \cdot f(x)\}$***

2. $E[X - E(x)] = 0$, that is, if $E(X) = \mu$, then: $E(X - \mu) = 0$.

3. $E\left(\frac{1}{x}\right)$ and $\left(\frac{1}{E(x)}\right)$ are not same.

VARIANCE

Variance of the probability distribution of a random variable X is the mathematical expectation of $[X - E(X)]^2$. Then

$$\begin{aligned} Var(X) &= E[X - E(X)]^2 \\ &= [x_1 - E(X)^2] \cdot p(x_1) + [x_2 - E(X)]^2 \cdot p(x_2) + \ldots + [x_n - E(X)]^2 \cdot p(x_n) \\ &= \sum_{i=1}^{n} \left\{[x_i - E(X)]^2 \times p(x_i)\right\}. \end{aligned}$$

Hence, $\quad Var(X) = E[X - E(X)]^2$

If we put $E(X) = \mu$, then $Var(X) = E[(X - \mu)^2]$.

Another Form of Variance

Variance: If X be a random variable with first two moments $E(X) = u$ and $E(X^2) = u_2$, then the mathematical expectation of $(X - \mu)^2$ is defined to be the variance of the random variable X. Then

$$Var(X) = E[(X - u)^2] = E(X^2 - 2Xu + u^2) = E(X^2) - 2E(X)u + u^2$$

$$= u_2 - 2uu + u^2 = u_2 - 2u^2 + u^2 = u_2 - u^2 = E(X^2) - [E(X)]^2$$

Hence, $\quad Var(X) = E(X^2) - [E(X)]^2$

OR $Var(X) = E(X^2) - \mu^2$

STANDARD DEVIATION: ***The standard deviation of the probability distribution of a random variable X is the positive square-root of the variance of that random variable.***

$\therefore$ **Standard Deviation:** $\quad \sigma = \sqrt{E(X^2) - [E(X)]^2}$. **OR** $\sigma = \sqrt{E(X^2) - \mu^2}$

or **Standard Deviation:** $\quad \sigma = \sqrt{E[X - E(X)]^2}$.

Note: The variance of the random variable X is also denoted by $\mathbf{V(X)}$.

12.10 THEOREMS ON VARIANCE OF A RANDOM VARIABLE

Theorem 7: *If C is a constant, then:*

$$V(CX) = C^2 V(X).$$

Theorem 8: *Variance of constant is zero, i.e.,*

$$V(C) = 0.$$

Theorem 9: If X is a random variable and C is a constant, then:

$$V(X + C) = V(X).$$

Theorem 10: *If a and b are constants, then:*

$$V(aX + b) = a^2 V(X).$$

Theorem 11: *If X and Y are two independent random variables, then:*

(i) $\quad V(X + Y) = V(X) + V(Y)$.

(ii) $\quad V(X - Y) = V(X) + V(Y)$.

12.11 MEAN AND VARIANCE OF A LINEAR COMBINATION

If $Z = aX + bY$ be a linear combination of two random of two random variables X and Y, then

Mean : $\mu_Z = a\,E(X) + b\,E(Y) = a\,\mu_X + b\,\mu_Y$

Variance : $V(Z)$ **OR** $\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$.

Or $\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2$ **, If** X **and** Y **are independent.**

Example 1. *The probability function of a random variable X is* $p(x) = \dfrac{2x+1}{48}$, $x = 1, 2, 3, 4, 5, 6$. *Verify whether p (x) is probability function?*

Solution: $p(x)$ will have the following values for different values of x.

x	:	1	2	3	4	5	6
$p(x) = \dfrac{2x+1}{48}$	:	$\dfrac{3}{48}$	$\dfrac{5}{48}$	$\dfrac{7}{48}$	$\dfrac{9}{48}$	$\dfrac{11}{48}$	$\dfrac{13}{48}$

$$\Sigma p(x) = \frac{1}{48}[3+5+7+9+11+13] = \frac{48}{48} = 1.$$

$\therefore$ $\Sigma p(x) = 1$ **and** $p(x) > 0$ **for all** x.

Now

Hence $p(x)$ is probability function.

Example 2. *For a random variable X, $p(x) = \frac{x}{x+1}$, where $x = 1, 2, 3$. Is $p(x)$ a probability density function?*

Solution: Here $p(x) = \frac{x}{x+1}$.

$\therefore$ For $x = 1, 2, 3$, $p(x)$ will take the values $\frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$.

$$\Sigma p(x) = \frac{1}{2}+\frac{2}{3}+\frac{3}{4} = \frac{23}{12} > 1. \quad \text{Now } \Sigma p(x) > 1.$$

$\therefore$

Hence, p (x) is not a probability function.

Example 3. *The probability distribution of a random variable x is given below. Find (i) E (x) (ii) V (x) (iii) E (2x – 3) and (iv) V (2x – 3).*

x	– 2	– 1	0	1	2
$p(x)$	0.2	0.1	0.3	0.3	0.1

Solution:

TABLE: Computalton of E (X) and V (X)

x	$p(x)$	$xp(x)$	x^2	$x^2p(x)$
– 2	0.2	– 0.4	4	0.8
– 1	0.1	– 0.1	1	0.1
0	0.3	0.0	0	0.0
1	0.3	0.3	1	0.3
2	0.1	0.2	4	0.4
		$\Sigma xp(x) = 0$		$\Sigma x^2p(x) = 1.6$

(i) E (x): From the above table: $E(x) = \Sigma x\,p(x) = 0$.

(ii) From the above table: $E(x)^2 = \Sigma x^2p(x) = 1.6$.

$V(x) = E(x)^2 - [E(x)]^2 = 1.6 - 0 = 1.6.$

(iii) E (2x – 3): $E(2x-3) = 2E(x) - 3 = 2 \times 0 - 3 = -3.$

(iv) V (2x – 3): $V(2x-3) = (2)^2 V(x) + V(-3) = 4(1.6) + 0 = 6.4.$

Example 4. *Amit plays a game of tossing a die. If the number less than 3 appears, he is getting Rs. a, otherwise he has to pay Rs. 10. If the game is fair, find a.*

Solution: Let x = gain of Amit.

For number less than 3, *i.e.*, 1 or 2, then $x = a$, $p(x) = \frac{2}{6}$.

Number 3 or more, *i.e.*, for 3, 4, 5 or 6; then $x = -10$, $p(x) = \frac{4}{6}$.

Hence,

Example 7. *Daily demand for transistors is having the following probability distribution:*

Demand :	*1*	*2*	*3*	*4*	*5*	*6*
Probability :	*0.10*	*0.15*	*0.20*	*0.25*	*0.18*	*0.12*

Determine the expected daily demand for transistors. Also obtain the variance of the demand.

Solution: If $p(x)$ be the probability for daily demand of x transistors, we may write expected daily demand as:

$$\mathbf{E(x) = \Sigma x \cdot p(x)} = 0.10 + 2 \times 0.15 + 3 \times 0.20 + 4 \times 0.25 + 5 \times 0.18 + 6 \times 0.12$$

$$= 0.10 + 0.30 + 0.60 + 1.00 + 0.90 + 0.72 = \mathbf{3.62.}$$

$$\mathbf{E(x^2) = \Sigma x^2 \cdot p(x)} = 1 \times 0.10 + 4 \times 0.15 + 9 \times 0.20 + 16 \times 0.25 + 25 \times 0.18 + 36 \times 0.12$$

$$= 0.10 + 0.60 + 1.80 + 4.00 + 4.50 + 4.32 = \mathbf{15.32.}$$

$$\mathbf{Var(x)} = \mathbf{E(x^2) - \{E(x)\}^2} = 15.32 - (3.62)^2 = 15.32 - 13.10 = \mathbf{2.22.}$$

Example 8. *A random variable has the following probability distribution:*

Value of X :	*0*	*1*	*2*	*3*
P [X = x] :	*1/3*	*1/2*	*0*	*1/6.*

Find $E[\{X - E(X)\}^2]$ *and hence obtain* $V(Y)$*, where* $Y = 2X - 1$.