Use of binomial distribution

p = probability that an item is defective = 0.05, $q = 0.95$ and n = No. of independent items (trials) considered = 20.

Let X denote the number of defectives in the n items considered.

$$P(X = r) = nC_r\, p^r\, q^{n-r}$$

(i) $\therefore\quad P(X = 2) = 20\,C_2\,(.05)^2\,(0.95)^{18}$

$$= 0.1887$$

If N is the number of sets (packets), each set (packet) containing 20 trials (items), then the number of sets containing exactly 2 successes (defectives) is given by

$$N(X = 2) = N \times P(X = 2)$$
$$= 1000 \times 0.1887 = 189, \text{ nearly}$$

(ii) $P(\text{at least 2 defectives}) = P(X \geq 2)$

$$= 1 - \{P(X = 0) + P(X = 1)\}$$
$$= 1 - [20C_0\,(0.05)^0\,(0.95)^{20} + 20C_1\,(0.05)^1\,(0.95)^{19}]$$
$$= 1 - [0.3585 + 0.3774]$$
$$= 0.2641$$

$$\therefore\quad N(X \geq 2) = N \times P\,(X \geq 2)$$
$$= 1000 \times 0.2641 = 264, \text{ nearly}$$

(iii) $P(\text{at most 2 defectives}) = P(X \leq 2)$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$
$$= \sum_{r=0}^{2} 20C_r\,(0.05)^r\,(0.95)^{20-r}$$
$$= 0.3585 + 0.3774 + 0.1887$$
$$= 0.9246$$

$$\therefore\quad N(X \leq 2) = N \times P(X \leq 2)$$
$$= 1000 \times 0.9246 = 925, \text{ nearly}$$

Use of Poisson distribution

Since $p = 0.05$ is very small and $n = 20$ is sufficiently large, binomial distribution may be approximated by Poisson distribution with parameter $\lambda = np = 1$.

$$\therefore\quad P(X = r) = \frac{e^{-\lambda}\,.\,\lambda^r}{r!} = \frac{e^{-1}}{r!}$$

(i) $P(X = 2) = \dfrac{e^{-1}}{2!} = 0.1839$

$$\therefore\ N\,(X = 2) = 1000 \times 0.1839 = 184, \text{ nearly}$$

(ii) $P(X \geq 2) = 1 - \{P(X = 0) + P(X = 1)\}$

$= 1 - \{e^{-1} + e^{-1}\} = 0.2642$

$\therefore N(X \geq 2) = 1000 \times 0.2642 = 264$, nearly.

(iii) $P(X \leq 2) = \sum_{r=0}^{2} P(X = r) = \sum_{r=0}^{2} \frac{e^{-1}}{r!}$

$= 0.9197$

$\therefore N(X \leq 2) = 920$, nearly.

Fit a Poisson distribution for the following distribution:

x:	0	1	2	3	4	5	Total
f:	142	156	69	27	5	1	400

Fitting a Poisson distribution for a given distribution means assuming that the given distribution is approximately Poisson and hence finding the probability mass function and then finding the theoretical frequencies.

To find the probability mass function

$$P\{X = r\} = \frac{e^{-\lambda} \cdot \lambda^r}{r!} \qquad r = 0, 1, 2, \ldots, \infty$$

of the approximate Poisson distribution, we require λ, which is the mean of the Poisson distribution.

We find the mean of the given distribution and assume it as λ.

x:	0	1	2	3	4	5	Total
f:	142	156	69	27	5	1	400
fx:	0	156	138	81	20	5	400

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{400}{400} = 1 = \lambda$$

The theoretical frequencies are given by

$$\frac{N e^{-\lambda} \cdot \lambda^r}{r!}$$ where $N = 400$, obtained from the given distribution.

$$= \frac{400\, e^{-1}}{r!}, \quad r = 0, 1, 2, \ldots, \infty$$

Thus, we get

	x:	0	1	2	3	4	5
Theoretical	f:	147.15	147.15	73.58	24.53	6.13	1.23

The theoretical frequencies for $x = 6, 7, 8, \ldots$ are very small and hence neglected.

Converting the theoretical frequencies into whole numbers consistent with the condition that the total frequency = 400, we get the following Poisson frequency distribution which fits the given distribution:

	x:	0	1	2	3	4	5
Theoretical	f:	147	147	74	25	6	1

Example 14

If the probability that an applicant for a driver's licence will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test (a) on the fourth trial and (b) in fewer than 4 trials?

Let X denote the number of trials required to achieve the first success. Then X follows a geometric distribution given by

$$P(X = r) = q^{r-1} p; \quad r = 1, 2, 3, \ldots, \infty$$

Here $p = 0.8$ and $q = 0.2$

(a) $P(X = 4) = 0.8 \times (0.2)^{4-1}$

$$= 0.8 \times 0.008 = 0.0064.$$

(b) $P(X < 4) = \sum_{r=1}^{3} 0.8 \times (0.2)^{r-1}$

$$= 0.8\,[(0.2)^0 + (0.2)^1 + (0.2)^2 + (0.2)^3]$$

$$= 0.9984.$$