Example 1

Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint probability distribution of (X, Y).

As there are only 2 white balls in the box, X can take the values 0, 1 and 2 and Y can take the values 0, 1, 2 and 3.

$P(X=0, Y=0)$ = P(drawing 3 balls none of which is white or red)

= P(all the 3 balls drawn are black)

$$= 4C_3/9C_3 = \frac{1}{21}$$

$P(X=0, Y=1)$ = P(drawing 1 red and 2 black balls)

$$= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

Similarly, $P(X=0, Y=2) = \dfrac{3C_2 \times 4C_1}{9C_3} = \dfrac{1}{7}$; $P(X=0, Y=3) = \dfrac{1}{84}$

$P(X=1, Y=0) = \dfrac{1}{7}$; $P(X=1, Y=1) = \dfrac{2}{7}$; $P(X=1, Y=2) = \dfrac{1}{14}$;

$P(X=1, Y=3) = 0$ (since only 3 balls are drawn)

$P(X=2, Y=0) = \dfrac{1}{21}$; $P(X=2, Y=1) = \dfrac{1}{28}$; $P(X=2, Y=2) = 0$;

$P(X=2, Y=3) = 0$

The joint probability distribution of (X, Y) may be represented in the form of a table as given below:

X	Y			
	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Example 2

For the bivariate probability distribution of (X, Y) given below, find $P(X \le 1)$, $P(Y \le 3)$, $P(X \le 1, Y \le 3)$, $P(X \le 1/Y \le 3)$, $P(Y \le 3/X \le 1)$ and $P(X + Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

$$P(X \le 1) = P(X = 0) + P(X = 1)$$

$$= \sum_{j=1}^{6} P(X = 0, Y = j) + \sum_{j=1}^{6} P(X = 1, Y = j)$$

$$= \left(0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$$

$$P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= \sum_{i=0}^{2} P(X = i, Y = 1) + \sum_{i=0}^{2} P(X = i, Y = 2)$$

$$+ \sum_{i=0}^{2} P(X = i, Y = 3)$$

$$= \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(\frac{1}{32} + \frac{1}{8} + \frac{1}{64}\right)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

$$P(X \le 1, Y \le 3) = \sum_{j=1}^{3} P(X = 0, Y = j) + \sum_{j=1}^{3} P(X = 1, Y = j)$$

$$= \left(0+0+\frac{1}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right)=\frac{9}{32}$$

$$P(X \le 1/Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)} = \frac{9/32}{23/64} = \frac{18}{23}$$

$$P(Y \le 3/X \le 1) = \frac{P(X \le 1, Y \le 3)}{P(X \le 1)} = \frac{9/32}{7/8} = \frac{9}{28}$$

$$P(X+Y \le 4) = \sum_{j=1}^{4} P(X=0, Y=j) + \sum_{j=1}^{3} P(X=1, Y=j) + \sum_{j=1}^{2} P(X=2, Y=j)$$

$$= \frac{3}{32}+\frac{1}{4}+\frac{1}{16}=\frac{13}{32}$$

Example 3

The joint probability mass function of (X, Y) is given by $p(x, y) = k(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $(X + Y)$.

The joint probability distribution of (X, Y) is given below. The relevant probabilities have been computed by using the given law.

X	Y		
	1	2	3
0	$3k$	$6k$	$9k$
1	$5k$	$8k$	$11k$
2	$7k$	$10k$	$13k$

$$\sum_{j=1}^{3} \sum_{i=0}^{2} p(x_i, y_j) = 1$$

i.e., the sum of all the probabilities in the table is equal to 1.

i.e., $72k = 1$.

$$\therefore \quad k = \frac{1}{72}$$

Marginal Probability Distribution of X: $\{i, p_{i*}\}$

$X = i$	$p_{i*} = \sum_{j=1}^{3} p_{ij}$
0	$p_{01} + p_{02} + p_{03} = \frac{18}{72}$
1	$p_{11} + p_{12} + p_{13} = \frac{24}{72}$
2	$p_{21} + p_{22} + p_{23} = \frac{30}{72}$
	Total = 1

Marginal Probability Distribution of Y: $\{j, p_{*j}\}$

$Y = j$	$p_{*j} = \sum_{i=0}^{2} p_{ij}$
1	15/27
2	24/72
3	33/72
	Total = 1

Conditional distribution of X, given $Y = 1$, is given by $\{i, P(X = i/Y = 1)\} = \{i, P(X = i, Y = 1)/P(Y= 1)\}$
$= \{i, p_{i1}/p_{*1}\}$, $i = 0, 1, 2$.
The tabular representation is given below:

$X = i$	p_{i1}/p_{*1}
0	$3k/15k = \frac{1}{5}$
1	$5k/15k = \frac{1}{3}$
2	$7k/15k = \frac{7}{15}$
	Total = 1

The other conditional distributions are given below:

C.P.D. of X, given $Y=2$	
$X=i$	p_{i2}/p_{*2}
0	$\frac{6k}{24k}=\frac{1}{4}$
1	$\frac{8k}{24k}=\frac{1}{3}$
2	$\frac{10k}{24k}=\frac{5}{12}$
	Total = 1

C.P.D. of X, given $Y=3$	
$X=i$	p_{i3}/p_{*3}
0	$\frac{9k}{33k}=\frac{3}{11}$
1	$\frac{11k}{33k}=\frac{1}{3}$
2	$\frac{13k}{33k}=\frac{13}{33}$
	Total = 1

C.P.D. of Y, given $X=0$	
$Y=j$	p_{oj}/p_{o*}
1	$\frac{3k}{18k}=\frac{1}{6}$
2	$\frac{6k}{18k}=\frac{1}{3}$
3	$\frac{9k}{18k}=\frac{1}{2}$
	Total = 1

C.P.D. of Y, given $X=1$	
$Y=j$	p_{1j}/p_{1*}
1	$\frac{5k}{24k}=\frac{5}{24}$
2	$\frac{8k}{24k}=\frac{1}{3}$
3	$\frac{11k}{24k}=\frac{11}{24}$
	Total = 1

C.P.D. of Y, given $X=2$	
$Y=j$	p_{2j}/p_{2*}
1	$\frac{7k}{30k}=\frac{7}{30}$
2	$\frac{10k}{30k}=\frac{1}{3}$
3	$\frac{13k}{30k}=\frac{13}{30}$
	Total = 1

Probability distribution of $(X+Y)$	
$(X+Y)$	P
1	$p_{01} = \frac{3}{72}$
2	$p_{02} + p_{11} = \frac{11}{72}$
3	$p_{03} + p_{12} + p_{21} = \frac{24}{72}$
4	$p_{13} + p_{22} = \frac{21}{72}$
5	$p_{23} = \frac{13}{72}$
	Total = 1

Example 4

A machine is used for a particular job in the forenoon and for a different job in the afternoon. The joint probability distribution of (X, Y), where X and Y represent the number of times the machine breaks down in the forenoon and in the afternoon respectively, is given in the following table. Examine if X and Y are independent RVs.

X	Y		
	0	1	2
0	0.1	0.04	0.06
1	0.2	0.08	0.12
2	0.2	0.08	0.12

X and Y are independent, if $P_{i*} \times P_{*j} = P_{ij}$ for all i and j. So, let us find P_{i*} P_{*j} for all i and j.

$$P_{0*} = 0.1 + 0.04 + 0.06 = 0.2;\ P_{1*} = 0.4;\ P_{2*} = 0.4$$

$$P_{*0} = 0.5;\ P_{*1} = 0.2;\ P_{*2} = 0.3$$

Now $P_{0*} \times P_{*0} = 0.2 \times 0.5 = 0.1 = P_{00}$

$$P_{0*} \times P_{*1} = 0.2 \times 0.2 = 0.04 = P_{01}$$

$$P_{0*} \times P_{*2} = 0.2 \times 0.3 = 0.06 = P_{02}$$

Similarly we can verify that

$$P_{1*} \times P_{*0} = P_{10};\ P_{1*} \times P_{*1} = P_{11};\ P_{1*} \times P_{*2} = P_{12};$$

$$P_{2*} \times P_{*0} = P_{20};\ P_{2*} \times P_{*1} = P_{21};\ P_{2*} \times P_{*2} = P_{22}$$

Hence the RVs X and Y are independent.