Applications of Integration

Area of a Region Between Two Curves

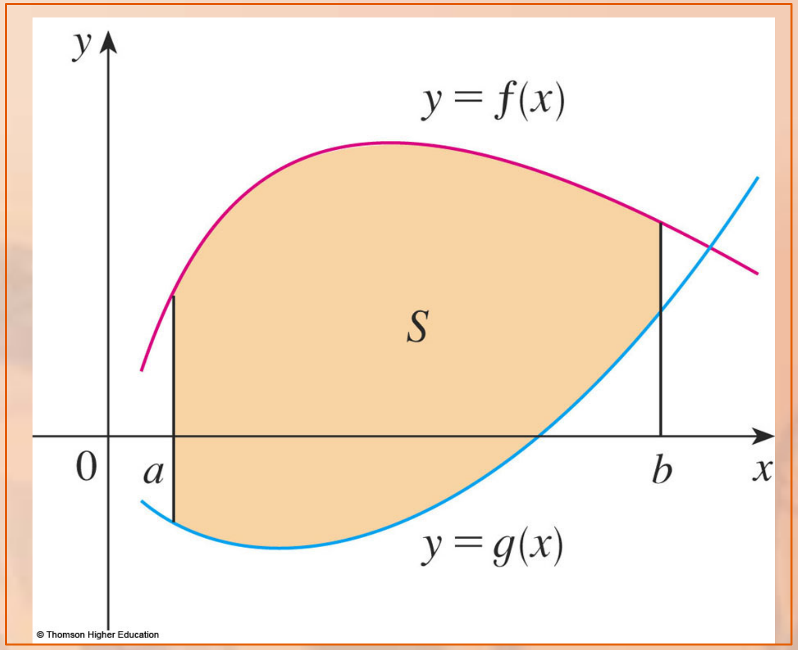
Objectives

- Find the area of a region between two curves using integration.
- Find the area of a region between intersecting curves using integration.
- Describe integration as an accumulation process.

AREAS BETWEEN CURVES

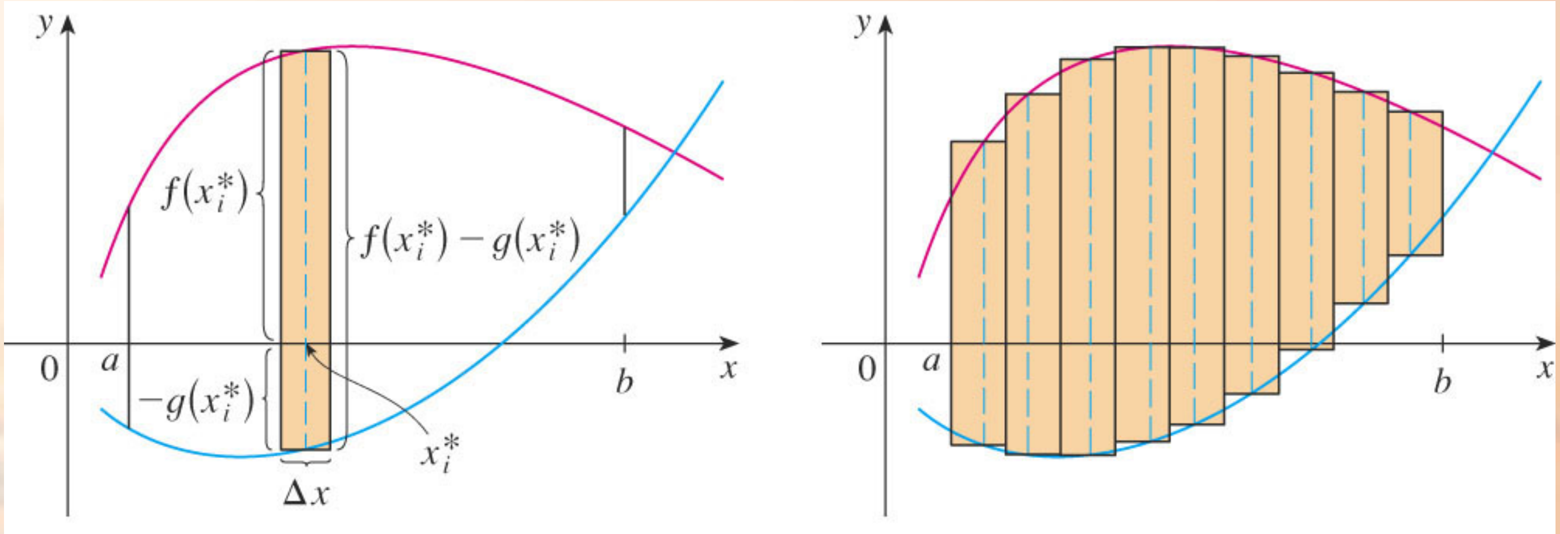
Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$.

- Here, f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.



AREAS BETWEEN CURVES

As we did for areas under curves, we divide *S* into *n* strips of equal width and approximate the *i* th strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$.



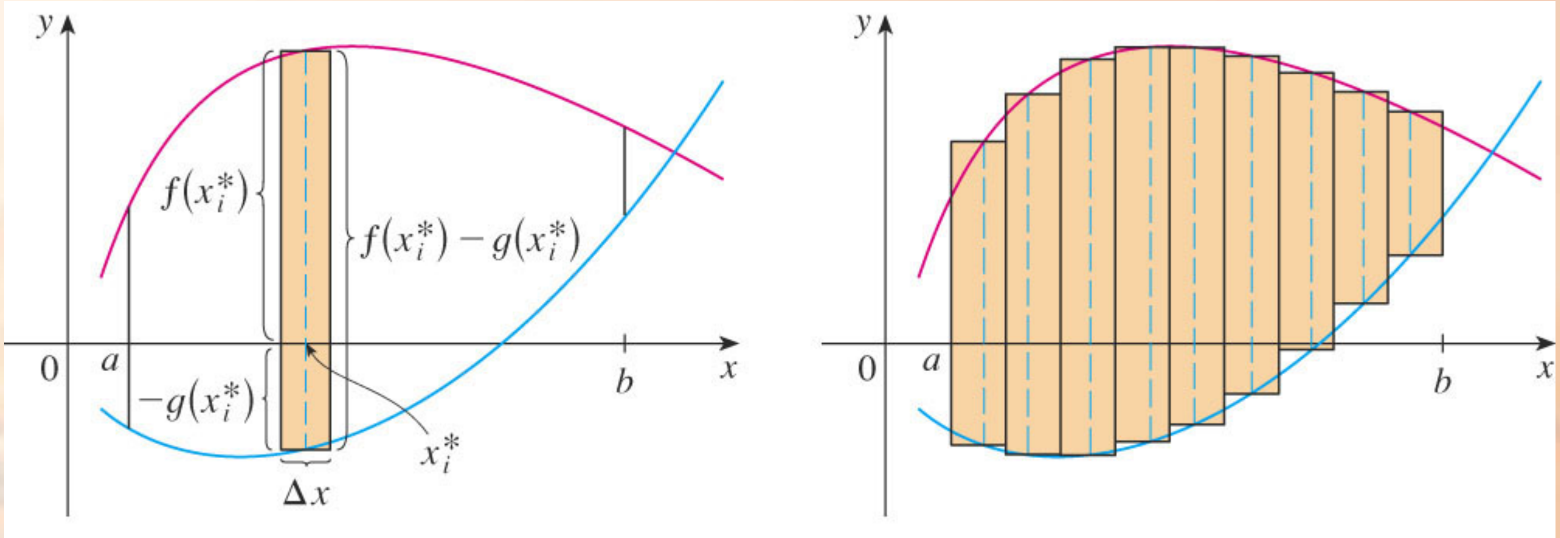
(a) Typical rectangle

(b) Approximating rectangles

AREAS BETWEEN CURVES

We could also take all the sample points to be right endpoints—in which case

$$x_i^* = x_i \, .$$



(a) Typical rectangle (b) Approximating rectangles

AREAS BETWEEN CURVES

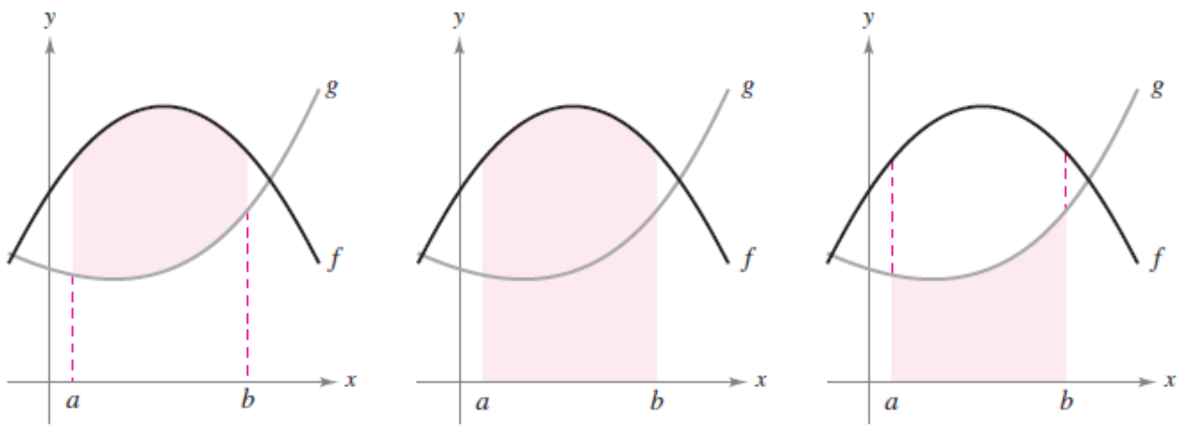
The Riemann sum $$\sum_{i=1}^{n}\left[f(x_i^*)-g(x_i^*)\right]\Delta x$$

is therefore an approximation to what we intuitively think of as the area of *S*.

- This approximation appears to become better and better as $n \to \infty$.

Area of a Region Between Two Curves

You can geometrically interpret the area of the region between the graphs as the area of the region under the graph of *g* subtracted from the area of the region under the graph of *f*, as shown in the below Figure.

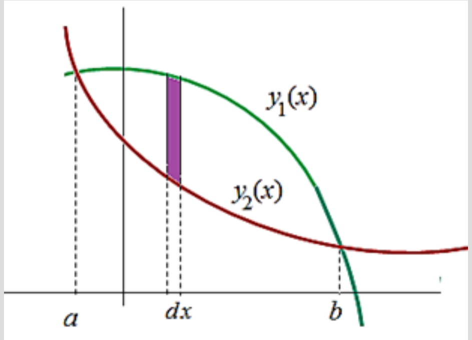


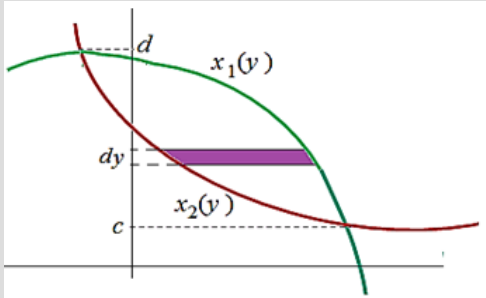
Area of region between *f* and *g*	=	Area of region under *f*	−	Area of region under *g*

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

AREA BETWEEN CURVES

$$dA = \left\{ \begin{pmatrix} outer \\ function \end{pmatrix} - \begin{pmatrix} inner \\ function \end{pmatrix} \right\} \text{dx}$$





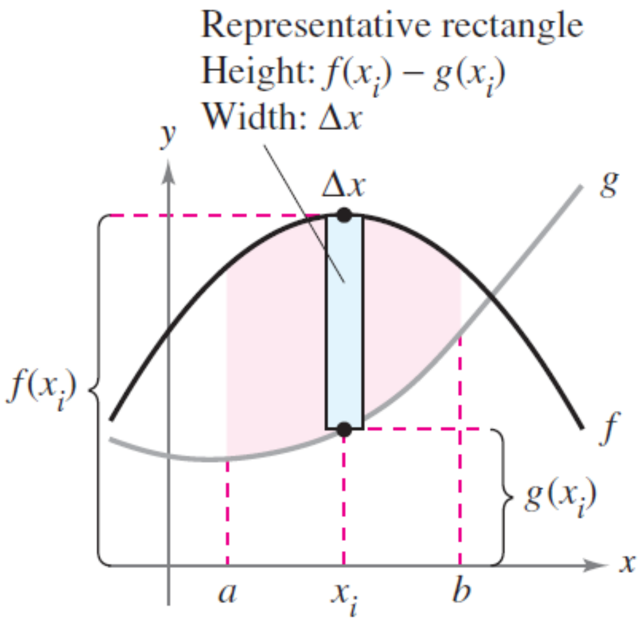
$$A = \int_a^b dA = \int_a^b \left[y_1(x) - y_2(x)\right] dx$$

$$A = \int_c^d dA = \int_c^d \left[x_1(y) - x_2(y)\right] dy$$

Area of a Region Between Two Curves

To verify the reasonableness of the result shown in Figure 2, you can partition the interval [a, b] into n subintervals, each of width Δx.

Then, as shown in this Figure, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith subinterval.



Area of a Region Between Two Curves

The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $||\Delta|| \to 0$ ($n \to \infty$), you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x.$$

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x$$

$$= \int_a^b [f(x) - g(x)]\,dx.$$

Area of a Region Between Two Curves

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Area of a Region Between Two Curves

In this Figure, the graphs of f and g are shown above the x-axis. This, however, is not necessary.

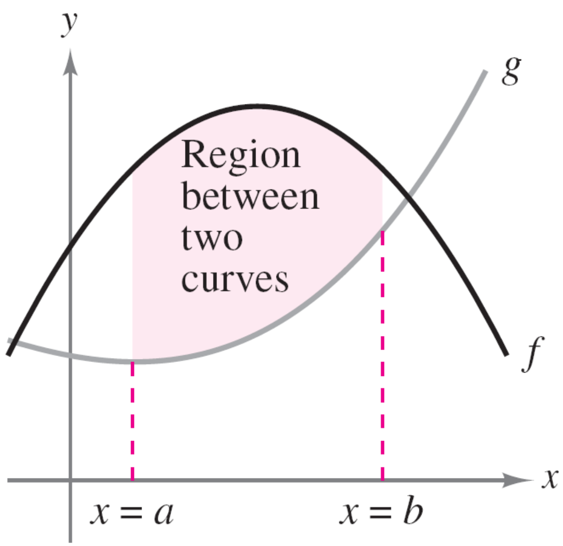


Figure-1

The same integrand $[f(x) – g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ for all x in the interval $[a, b]$.

Area of a Region Between Two Curves

This is summarized graphically in Figure 4.

Notice in Figure-4 that the height of a representative rectangle is *f*(*x*) – *g*(*x*) regardless of the relative position of the *x*-axis.

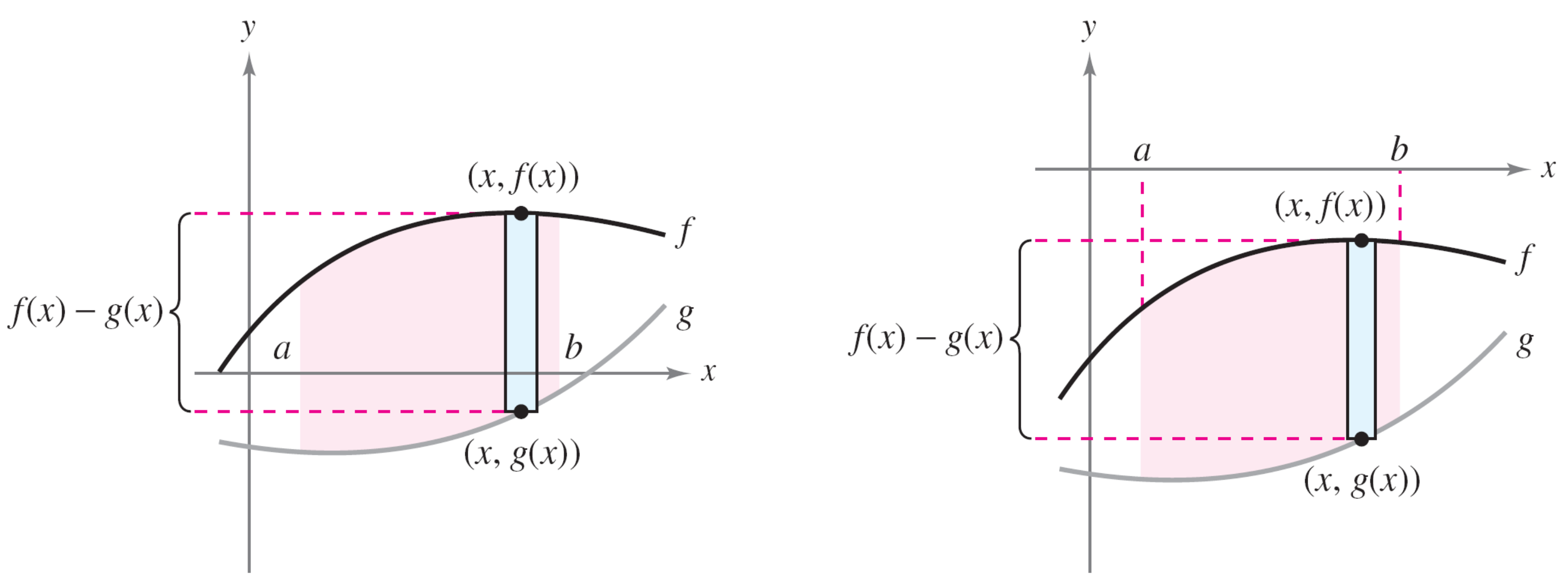


Figure-4

Area of a Region Between Two Curves

A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

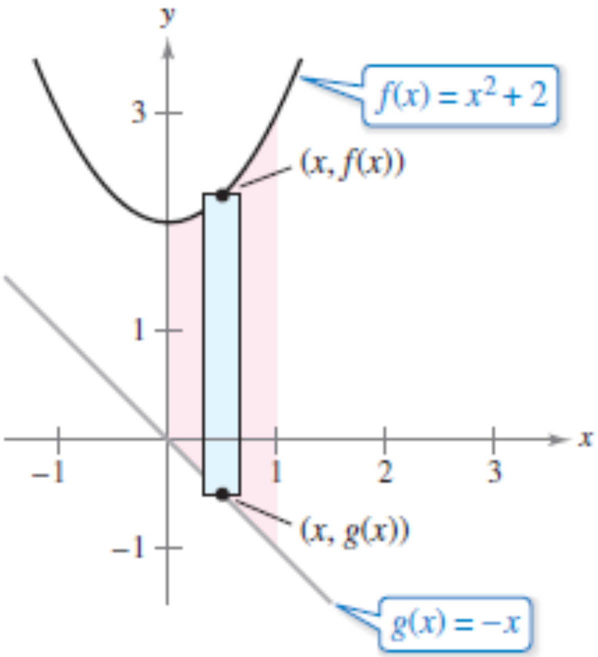
Example 1 – *Finding the Area of a Region Between Two Curves*

Find the area of the region bounded by the graphs of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

Solution:

Let $g(x) = -x$ and $f(x) = x^2 + 2$.

Then $g(x) \le f(x)$ for all x in $[0, 1]$, as shown in Figure 5.



Region bounded by the graph of f, the graph of g, $x = 0$, and $x = 1$

Figure-5

Example 1 – *Solution*

cont'd

So, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\Delta x$$
$$= [(x^2 + 2) - (-x)]\Delta x$$

and the area of the region is

$$A = \int_a^b [f(x) - g(x)]\,dx$$
$$= \int_0^1 [(x^2 + 2) - (-x)]\,dx$$
$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1$$
$$= \frac{1}{3} + \frac{1}{2} + 2$$
$$= \frac{17}{6}.$$

Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of *a* and *b* are given explicitly.

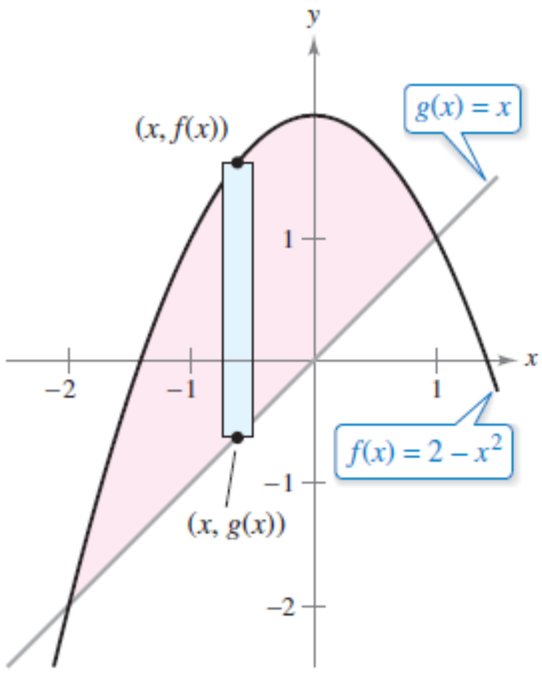
A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of *a* and *b* must be calculated.

Example 2 – *A Region Lying Between Two Intersecting Graphs*

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution:

In Figure 6, notice that the graphs of *f* and *g* have two points of intersection.



Region bounded by the graph of *f* and the graph of *g*

Figure 7.6

Example 2 – *Solution*

cont'd

To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$2 - x^2 = x$	Set $f(x)$ equal to $g(x)$
$-x^2 - x + 2 = 0$	Write in general form.
$-(x + 2)(x - 1) = 0$	Factor
$x = -2$ or 1	Solve for x.

So, $a = -2$ and $b = 1$.

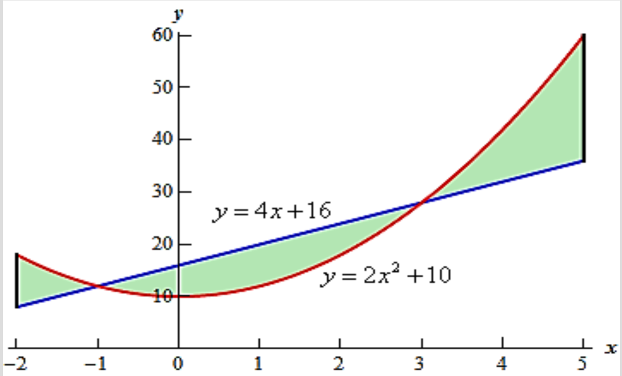
Determine the area of the region bounded by y = 2x² +10 and y = 4x +16 between x = -2 and x = 5

$$A=\int_a^b dA=\int_a^b\left[\begin{pmatrix}outer\\function\end{pmatrix}-\begin{pmatrix}inner\\function\end{pmatrix}\right]dx$$

$$A=\int_{-2}^{-1}\left[2x^2+10-(4x+16)\right]dx+\int_{-1}^{3}\left[4x+16-(2x^2+10)\right]dx+\int_{3}^{5}\left[2x^2-4x-6\right]dx$$

$$=\left(\frac{2}{3}x^3-2x^2-6x\right)\Big|_{-2}^{-1}+\left(-\frac{2}{3}x^3+2x^2+6x\right)\Big|_{-1}^{3}+\left(\frac{2}{3}x^3-2x^2-6x\right)\Big|_{3}^{5}$$

$$=\frac{14}{3}+\frac{64}{3}+\frac{64}{3}=\frac{142}{3}$$

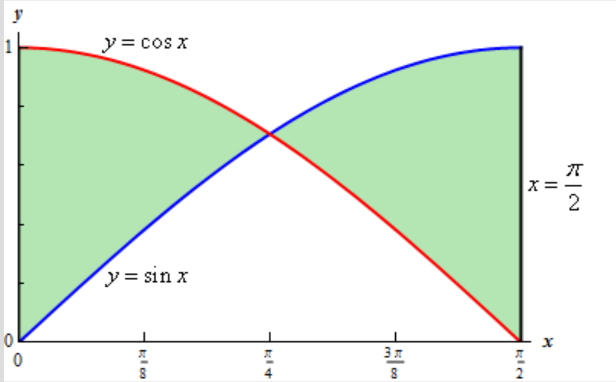


Determine the area of the region enclosed by y = sin x and y = cos x and the *y* -axis for $0 \le x \le \frac{\pi}{2}$.

$$A=\int_0^{\frac{\pi}{4}}\left[\cos x-\sin x\right]dx+\int_{\pi/4}^{\pi/2}\left[\sin x-\cos x\right]dx$$

$$=\left(\sin x+\cos x\right)\Big|_0^{\frac{\pi}{4}}+\left(-\cos x-\sin x\right)\Big|_{\pi/4}^{\pi/2}$$

$$=\sqrt{2}-1+\left(\sqrt{2}-1\right)=2\sqrt{2}-2=0.828427$$



Example 2 – *Solution*

cont'd

Because $g(x) \leq f(x)$ for all x in the interval [–2, 1], the representative rectangle has an area of

$$\Delta A = [f(x) - g(x)]\Delta x$$

$$= [(2 - x^2) - x]\Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\, dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1}$$

$$= \frac{9}{2}.$$