Moment about the origin.

The r^{th} moment about the origin of the random variable 'x' is given by

$$\mu_r' = E(x^r) = \begin{cases} \sum x^r p(x), & \text{if } x \text{ is discrete r.v} \\ \int_{-\infty}^{\infty} x^r f(x)\,dx, & \text{if } x \text{ is continuous r.v} \end{cases}$$

Moment generating function (M.G.F.)

The m.g.f of the random variable 'x' is given by $E(e^{tx})$ and is denoted by $M_x(t)$. Then

$$M_x(t) = E(e^{tx}) = \begin{cases} \sum e^{tx} p(x) & \text{if } x \text{ is discrete r.v} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } x \text{ is continuous r.v} \end{cases}$$

where 't' is real parameter and the integration or summation being extended to the entire range of x. (assuming that the integration or summation is absolutely converges for some positive number 'h' such that $|t| < h$)

To find the r^{th} moment of x about origin

$$M_x(t) = E(e^{tx})$$
$$= E\left\{1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \cdots + \frac{(tx)^n}{n!} + \cdots\right)$$

$$M_x(t) = 1 + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \cdots + \frac{t^n}{n!}E(x^n) + \cdots$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

Since $M_x(t)$ generates moments, it is known as moment generating function.

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Diff. w.r.to 't'.

$$M'_x(t) = E(x) + \frac{t}{1!}E(x^2) + \frac{3t^2}{3\cdot2\cdot1}E(x^3) + \cdots$$

$$M'_x(0) = E(x)$$

$$M''_x(t) = E(x^2) + tE(x^3) + \cdots$$

$$M''_x(0) = E(x^2).$$

Illy $M'''_x(0) = E(x^3)$

$$\therefore \quad M^n_x(0) = E(x^n)$$

$$\boxed{\mu'_r = E(x^r) = \frac{d^r}{dt^r}\left[M_x(t)\right]_{t=0}}$$

Problem - 1

Find the m.g.f of the random variable with the probability law $P(X=x) = q^{x-1}p$, $x = 1, 2, 3, \ldots$

Find the mean and variance.

Solution :- $M_x(t) = E(e^{tx})$

$$= \sum_{x=1}^{\infty} e^{tx}.p(x) = \sum_{x=1} e^{tx}.q^{x-1}.p$$

$$= \sum_{x=1}^{\infty} (qe^t)^x \frac{p}{q}$$

$$= \frac{p}{q}.qe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= pe^t\left[1 + qe^t + (qe^t)^2 + \cdots\right]$$

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$= pe^t (1-qe^t)^{-1}$

$$M_x(t) = \frac{pe^t}{1-qe^t}$$

Diff. w.r.to 't'

$$M'_x(t) = \frac{(1-qe^t)\cdot pe^t - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$M'_x(t) = \frac{pe^t}{(1-qe^t)^2}$$

$$\mu'_1 = M'_x(0) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Diff. w.r.to 't', we have

$$M''_x(t) = \frac{(1-qe^t)^2 \cdot pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$M''_x(0) = \frac{p(1+q)}{(1-q)^3} = \frac{p(1+q)}{p^3} \qquad \begin{aligned} p+q &= 1 \\ p &= 1-q \end{aligned}$$

$$\mu'_2 \quad M''_x(0) = \frac{1+q}{p^2}$$

$$\text{mean} = \mu'_1 = E(x) = \frac{1}{p}$$

$$\text{Variance} = \mu'_2 - (\mu'_1)^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

Problem - 2

Find the m.g.f for the distribution where

$$f(x) = \begin{cases} 2/3, & \text{at } x=1 \\ 1/3, & \text{at } x=2 \\ 0, & \text{otherwise} \end{cases}$$

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Given that $f(1) = 2/3$, $f(2) = 1/3$,

$f(3) = f(4) = \cdots = 0$

The m.g.f of x is

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= e^{0} \cdot p(0) + e^{t} p(1) + e^{2t} p(2) + \cdots$$

$$= 0 + e^{t} \cdot \frac{2}{3} + e^{2t} \cdot \frac{1}{3} + \cdots$$

$$M_x(t) = \frac{e^{t}}{3}(2 + e^{2t})$$

Problem - 3.

Find the m.g.f of a random variable 'x' having the density function

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Soln:-

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{0}^{2} e^{tx} \cdot \frac{x}{2}\, dx$$

$$= \frac{1}{2}\left[x \cdot \frac{e^{tx}}{t} - 1 \cdot \frac{e^{tx}}{t^2} \right]_0^2$$

$$= \frac{1}{2}\left[\left(2 \cdot \frac{e^{2t}}{t} - \frac{e^{2t}}{t^2} \right) - \left(0 - \frac{1}{t^2} \right) \right]$$

$$M_x(t) = \frac{1}{2}\left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2} \right]$$

Problem - 4

Let x be a random variable with pdf

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find a) $P(X>3)$, b) $M_x(t)$

c) $E(X)$ and $var(X)$

Soln:- Given $f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

a) $P(X>3) = \int_3^{\infty} f(x)\,dx = \int_3^{\infty} \frac{1}{3} e^{-x/3}\,dx$

$$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_3^{\infty} = -\left[0 - e^{-1}\right] = \frac{1}{e}$$

b) $M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$$= \int_0^{\infty} e^{tx} \cdot \frac{1}{3} e^{-\frac{x}{3}}\,dx$$

$$= \frac{1}{3}\int_0^{\infty} e^{-(\frac{1}{3}-t)x}\,dx$$

$$= \frac{1}{3}\left[\frac{e^{-(\frac{1}{3}-t)x}}{-(\frac{1}{3}-t)}\right]_0^{\infty}$$

$$M_x(t) = \frac{1}{3}\left[0 + \frac{1}{\frac{1}{3}-t}\right] = \frac{1}{1-3t}$$

c) $M_x(t) = (1-3t)^{-1}$

$$M'_x(t) = -(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$$

$E(x) = M_x'(0) = 3$

$M_x''(t) = (-6)(1-3t)^{-3}(-3) = 18(1-3t)^{-3}$

$E(x^2) = M_x''(0) = 18$

$Var(x) = E(x^2) - (E(x))^2 = 18 - 9 = 9$

problem - 5

Find the m.g.f of the random variable 'x' whose moments are $\mu_r' = (r+1)!\, 2^r$.

Soln:-

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)!\, 2^r$$

$$= \sum_{r=0}^{\infty} \frac{(2t)^r (r+1)!}{r!}$$

$$M_x(t) = 1 + 2(2t) + 3(2t)^2 + \cdots$$

$$M_x(t) = (1-2t)^{-2} = \frac{1}{(1-2t)^2}$$

$\left((1-x)^{-2} = 1 + 2x + 3x^2 + \cdots\right)$

problem - 6 :- If a random variable 'x' has the m.g.f $M_x(t) = \frac{3}{3-t}$, find the S.D of x

Soln:-

$$M_x(t) = \frac{3}{3-t} = 3(3-t)^{-1}$$

$$M_x'(t) = -3(3-t)^{-2}(-1) = 3(3-t)^{-2}$$

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$$E(X) = M'_X(0) = \frac{3}{9} = \frac{1}{3}$$

$$M''_X(t) = (-6)(3-t)^{-3}(-1) = 6(3-t)^{-3}$$

$$E(X^2) = M''_X(0) = \frac{6}{27} = \frac{2}{9}.$$

$$Var(X) = \frac{2}{9} - (1/3)^2 = \frac{1}{9}$$

$$S.D = \sqrt{Var(X)} = \sqrt{1/9} = 1/3 \,//.$$

Problem 7: A random variable 'x' has the pdf given by $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Find a) the m.g.f b) the first four moments about the origin.

Soln:-

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \cdot 2e^{-2x}\,dx$$

$$= 2\int_0^{\infty} e^{-(2-t)x}\,dx$$

$$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{-2}{2-t}\left[e^{-\infty} - e^{0}\right]$$

$$M_X(t) = 2(2-t)^{-1}$$

$$M'_X(t) = 2(-1)(2-t)^{-2}(-1)$$

$$M'_X(t) = 2(2-t)^{-2}$$

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First moment $= E(x) = M'_x(0) = \frac{2}{4} = \frac{1}{2}$.

$$M''_x(t) = 2(-2)(2-t)^{-3}(-1)$$

$$M''_x(t) = 4(2-t)^{-3}$$

$$E(x^2) = M''_x(0) = \frac{4}{8} = \frac{1}{2}$$

$$M'''_x(t) = 4(-3)(2-t)^{-4}(-1)$$

$$M'''_x(t) = 12(2-t)^{-4}$$

$$E(x^3) = M'''_x(0) = \frac{12}{2^4} = \frac{3}{4}$$

$$M''''_x(t) = 12(-4)(2-t)^{-5}(-1)$$

$$M''''_x(t) = 48(2-t)^{-5}$$

$$E(x^4) = M''''_x(0) = \frac{48}{2^5} = \frac{48}{32} = \frac{3}{2}$$

Problem

1. Find the m.g.f of the distribution given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and hence find the fourth moment.