Compton Scattering

The photoelectric effect and Einstein's theories about light having a particle nature caused a lot of scientists to start to reexamine some basic ideas, as well as come up with some new ones

- Based on a lot of Einstein's work (including his Special Theory of Relativity), physicists predicted that these photons should have momentum, just like other particles do.
- The momentum that the light photons have must be very small, and not based on the common way of calculating momentum using p = mv (since light has no rest mass).
- Instead the formula was based on the wavelength and frequency of the light, just like Planck’s formula.

$$p = \frac{h}{\lambda} \quad \text{or} \quad p = \frac{h\nu}{c}$$

p = momentum (kgm/s)
h = Planck's Constant
λ = wavelength (m)
ν = frequency (Hz)
c = speed of light

In 1923 A.H. Compton started shooting high frequency X-rays at various materials and found that his results seemed to support the idea of photons having momentum. In one setup he shot the high frequency X-rays at a piece of graphite.

- ❑ If light was a wave, we would expect the X-rays to come out the other side with their wavelength smaller.

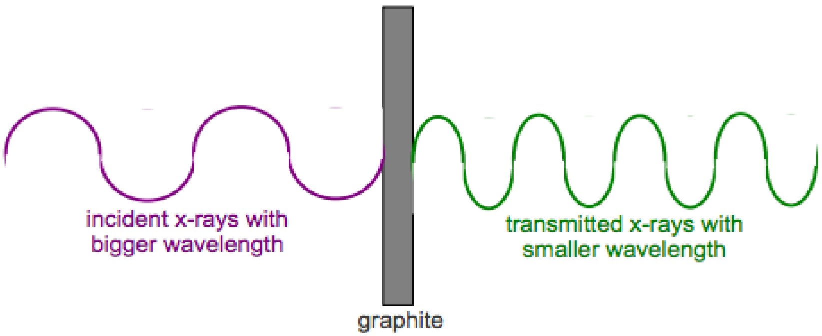


Illustration 1: If light was a wave, we'd expect results like this.

- ❑ Instead, Compton found that the X-rays scattered after hitting the target, changing the direction they were moving and actually getting a longer wavelength.
- Remember, longer wavelength means smaller frequency.
- Since $E = h\nu$, the scattered photons had less energy! Somehow, the X-ray photons were losing energy going through the graphite. So where'd the energy go?

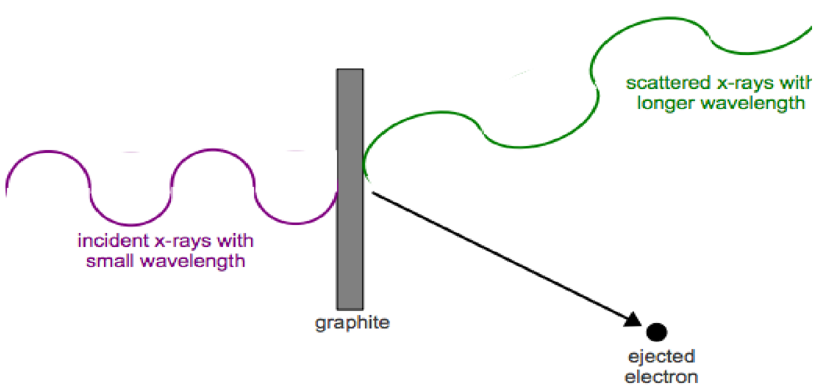
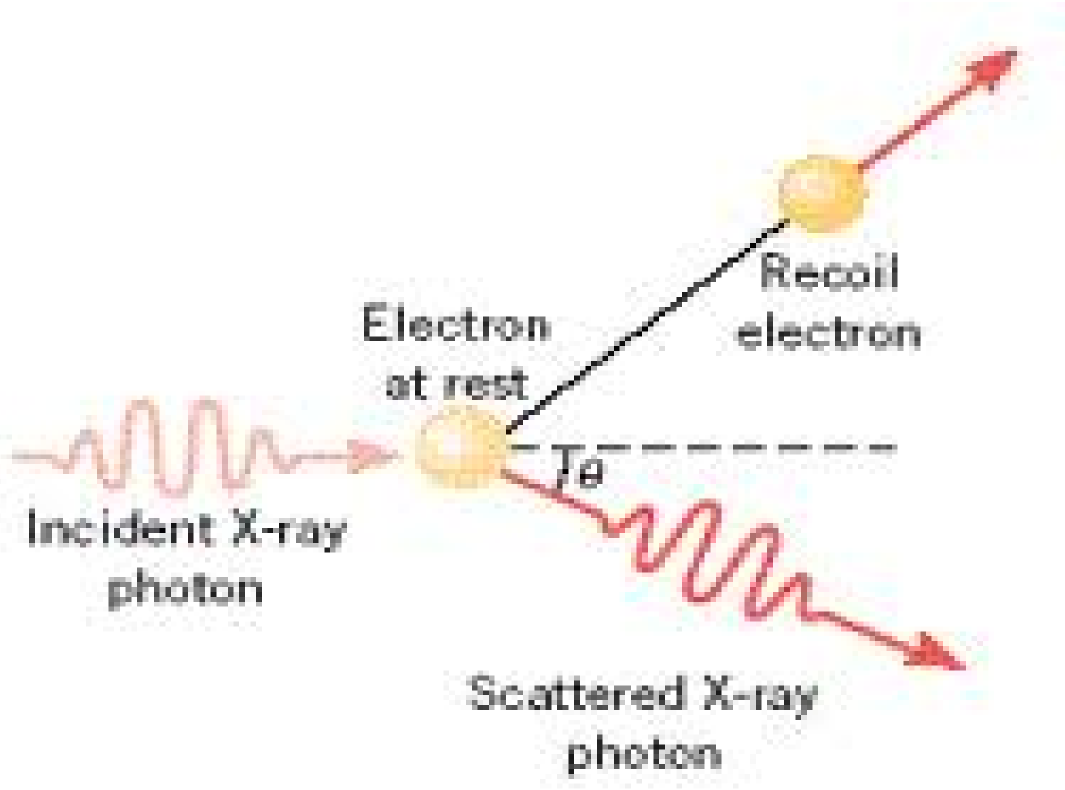


Illustration 2: What Compton actually observed.

Compton scattering is an inelastic scattering of a photon by a free charged particle, usually an electron. It results in a decrease in energy (increase in wavelength) of the photon (which may be an X-ray or gamma ray photon), called the Compton effect. Part of the energy of the photon is transferred to the recoiling electron.



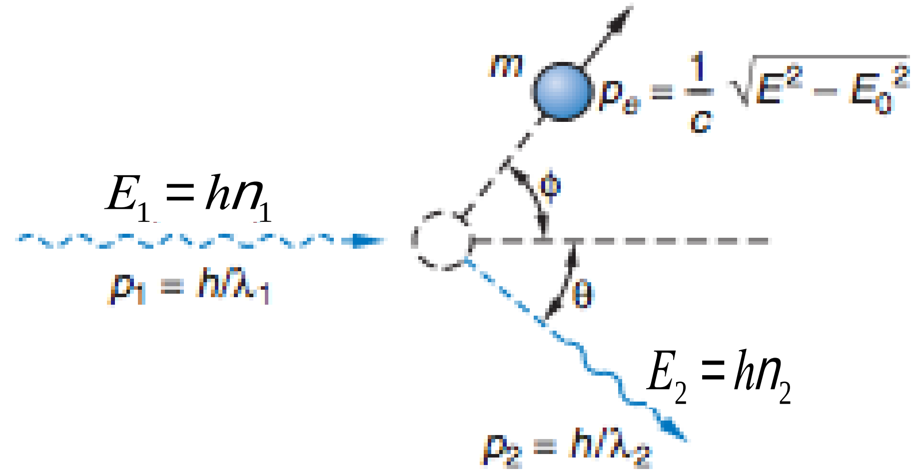
Let $\nu_1(\lambda_1)$ and $\nu_2(\lambda_2)$ be the frequencies (wavelengths) of the incident and scattered x rays, respectively, as shown in Figure. The corresponding momenta are

For incoming photon

$$p_1 = \frac{hn_1}{c} = \frac{E_1}{c} = \frac{h}{l_1}$$

For scattering photon

$$p_2 = \frac{hn_2}{c} = \frac{E_2}{c} = \frac{h}{l_2}$$



Energy of electron at rest

$$E_0 = mc^2$$

Energy of electron after scattering

$$E_e = \sqrt{m^2c^4 + p_e^2c^2}$$

From the conservation of *Momentum*

$$\vec{p}_1 + 0 = \vec{p}_2 + \vec{p}_e$$

$$\Rightarrow \vec{p}_e = \vec{p}_1 - \vec{p}_2$$

$$\Rightarrow (\vec{p}_e)^2 = (\vec{p}_1 - \vec{p}_2)^2$$

$$\Rightarrow \vec{p}_e^{\,2} = p_1^2 + p_2^2 - 2\vec{p}_1 \bullet \vec{p}_2$$

$$\Rightarrow \vec{p}_e^{\,2} = p_1^2 + p_2^2 - 2\vec{p}_1\vec{p}_2\cos\theta \rightarrow (1)$$

From the conservation of *Energy*

$$E_1 + E_0 = E_2 + E_e$$

$$\Rightarrow hn_1 + mc^2 = hn_2 + \sqrt{m^2c^4 + p_e^2c^2}$$

$$\Rightarrow hn_1 - hn_2 + mc^2 = \sqrt{m^2c^4 + p_e^2c^2}$$

$$\Rightarrow p_1c - p_2c + mc^2 = \sqrt{m^2c^4 + p_e^2c^2}$$

$$\Rightarrow (p_1 - p_2)c + mc^2 = \sqrt{m^2c^4 + p_e^2c^2}$$

$$\Rightarrow \left[(p_1 - p_2)c + E_0\right]^2 = E_0^2 + p_e^2c^2$$

$$\Rightarrow p_e^2 = (p_1 - p_2)^2 + \frac{2E_0(p_1 - p_2)}{c} \quad \rightarrow (2)$$

From eq 1 & 2

$$p_1^2 + p_2^2 - 2p_1p_2\cos\theta = (p_1 - p_2)^2 + \frac{2E_0(p_1 - p_2)}{c}$$

$$\Rightarrow p_1^2 + p_2^2 - 2p_1p_2\cos\theta = p_1^2 + p_2^2 - 2p_1p_2 + \frac{2E_0(p_1 - p_2)}{c}$$

$$\Rightarrow \frac{E_0(p_1 - p_2)}{c} = p_1p_2(1 - \cos\theta)$$

$$Again\ E_0 = mc^2\ \&\ p_1 = \frac{h}{\lambda_1}\ \&\ p_2 = \frac{h}{\lambda_2}$$

$$\Rightarrow \frac{mc^2}{c}\left[\frac{h}{\lambda_1} - \frac{h}{\lambda_2}\right] = \frac{h}{\lambda_1}\frac{h}{\lambda_2}(1 - \cos\theta)$$

$$\boxed{\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos\theta)} \rightarrow Compton's\ Equation$$

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$$

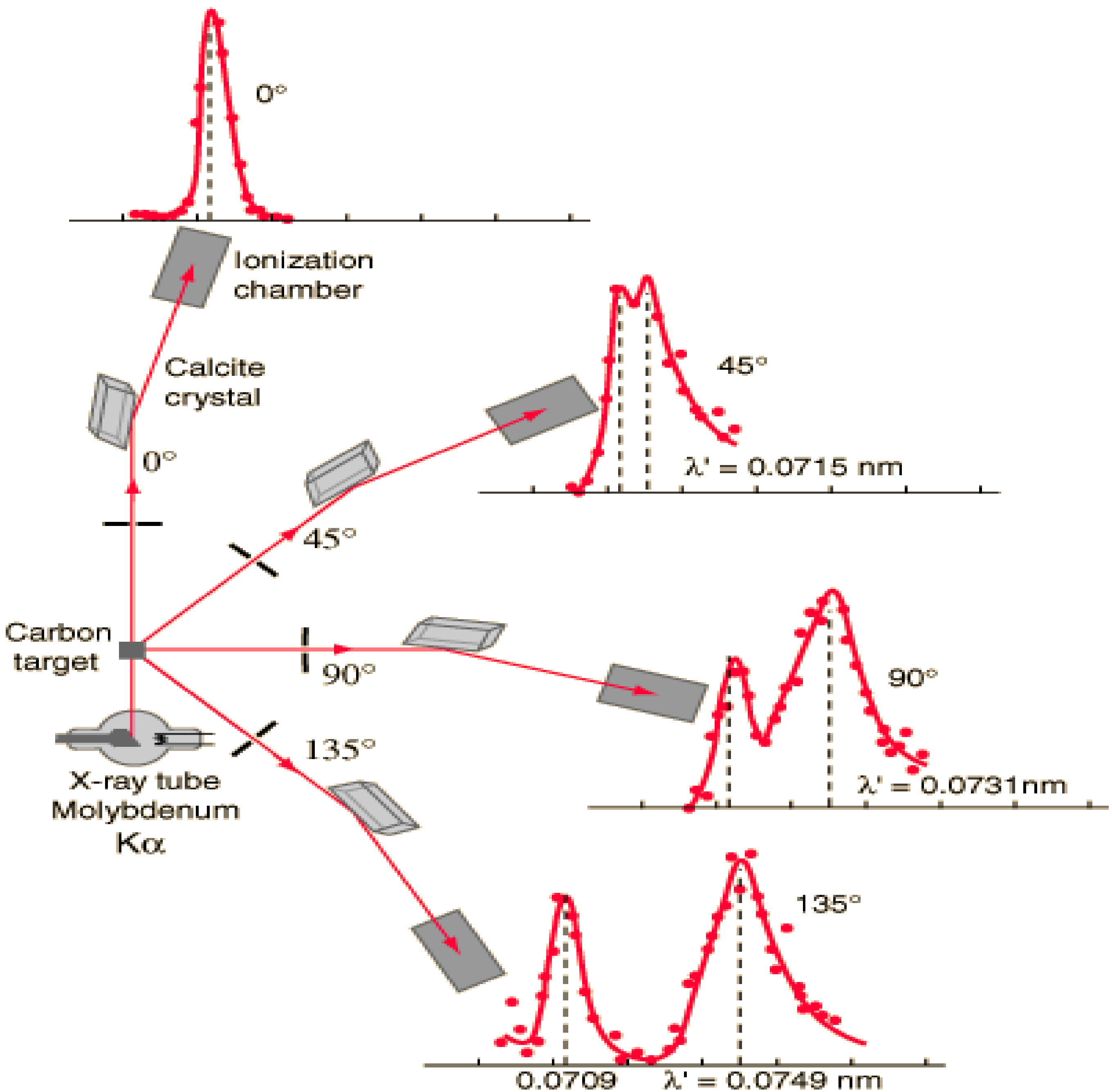
Δλ is change in wavelength

$$\frac{h}{mc} = \frac{6.626 \times 10^{-34}}{(9.11 \times 10^{-31})(3 \times 10^{8})}$$

$$\lambda_c = 2.426 \times 10^{-12}\, m = 0.02426\, \overset{o}{A}$$ → Compton Wavelength

If $\theta=0$, $\Delta\lambda=0$, i.e. no wavelength shift along the direction of incident radiation.
The shift increases with increase of angle of scattering.
If $\theta=\pi/2$, $\Delta\lambda=\lambda_c$ & If $\theta=\pi$, $\Delta\lambda=2\lambda_c$

Thus Compton shift Δλ and its dependence on the angle of scattering can be explained by treating X-rays as particles rather than waves.



0°
Ionization chamber
Calcite crystal
0°
45°
λ' = 0.0715 nm
45°
Carbon target
90°
90°
λ' = 0.0731nm
135°
X-ray tube Molybdenum Kα
135°
0.0709
λ' = 0.0749 nm

Recalling Einstein relativistic equation

$$Total\ \ Energy, \mathrm{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\mathrm{E}^2 = \frac{m^2c^4}{1-\frac{v^2}{c^2}}$$

$$momentum, P = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$P^2\mathrm{c}^2 = \frac{m^2v^2\mathrm{c}^2}{1-\frac{v^2}{c^2}}$$

$$\mathrm{E}^2\text{-}p^2\mathrm{c}^2 = \frac{m^2c^4}{\left[1-\frac{v^2}{c^2}\right]} - \frac{m^2v^2\mathrm{c}^2}{\left[1-\frac{v^2}{c^2}\right]}$$

$$\mathrm{E}^2\text{-}p^2\mathrm{c}^2 = \frac{m^2c^4 - m^2v^2\mathrm{c}^2}{\left[1-\frac{v^2}{c^2}\right]}$$

$$\mathrm{E}^2\text{-}p^2\mathrm{c}^2 = \frac{m^2c^4\left[1-\frac{v^2}{c^2}\right]}{\left[1-\frac{v^2}{c^2}\right]} = m^2c^4$$

$$\mathrm{E}^2 = m^2c^4 + p^2\mathrm{c}^2$$

$$\mathrm{E} = \sqrt{m^2c^4 + p^2\mathrm{c}^2}$$