Laplace Transform

Professor Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Properties of Laplace transforms- I - Part 2

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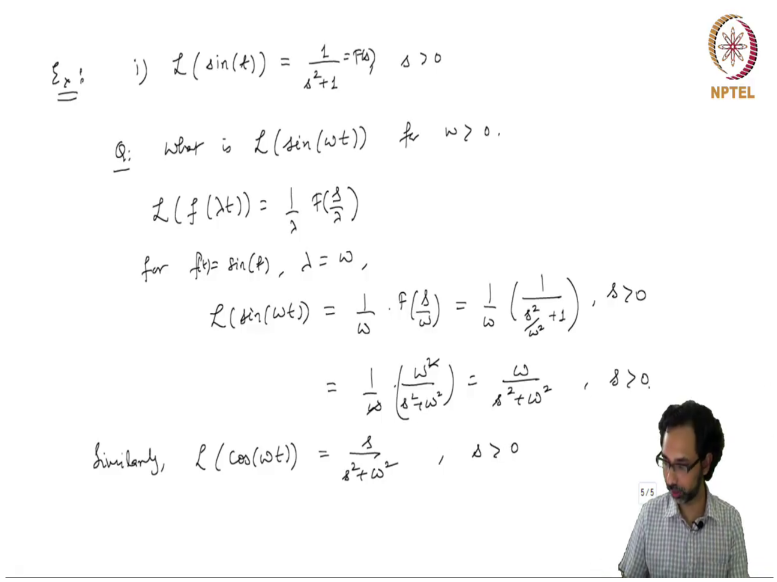

Let us see an example where this is used.

So we have already computed the Laplace transform of $\sin t$, and $\mathscr{L}(\sin t) = \frac{1}{s^2+1}$, and this is defined in the region $s > 0$.

Question: What is $\mathscr{L}(\sin \omega t)$ for $\omega > 0$?

Ans: We know, $\mathscr{L}(f(\lambda t)) = \frac{1}{\lambda} F\left(\frac{s}{\lambda}\right)$.

For $f(t) = \sin t$, and $\lambda = \omega$, we get,

$$\mathscr{L}(\sin \omega t) = \frac{1}{\omega}\left(\frac{1}{\frac{s^2}{\omega^2}+1}\right) = \frac{\omega}{s^2+\omega^2} .$$

And this is defined for $s > 0$ again, because $\omega .0 = 0$.

Similarly, one can compute, the Laplace transform of $\cos \omega t$, and $\mathscr{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$, and the region of convergence is again $s > 0$.

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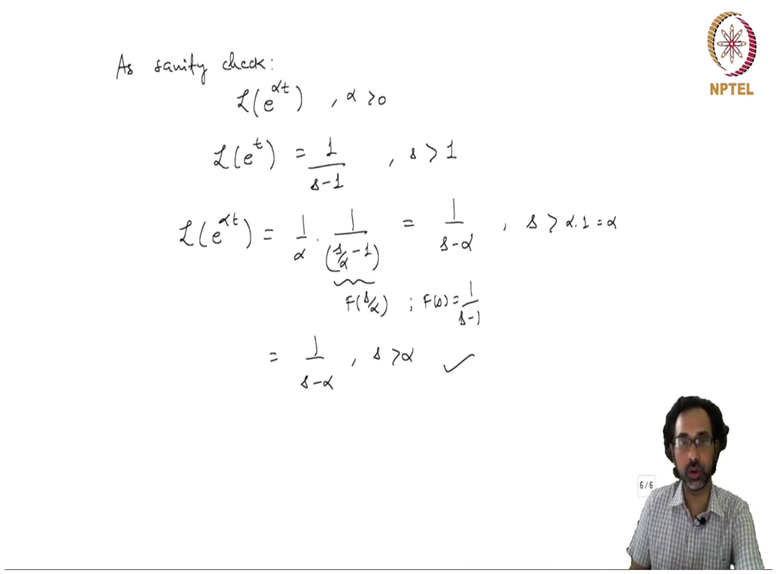

Now as a sanity check, we try to compute the Laplace transform of $e^{\alpha t}$ for some α positive. We know that the Laplace transform of e^t, that is

$$\mathscr{L}(e^t) = \frac{1}{s-1}, \quad \text{for } s > 1.$$

So, in terms of our change of scale formula, the Laplace transform of $e^{\alpha t}$ should be,

$$\mathscr{L}(e^{\alpha t}) = \frac{1}{\alpha} \left(\frac{1}{\frac{s}{\alpha} - 1} \right) = \frac{1}{s - \alpha} \text{, for } s > \alpha.1 = \alpha.$$

This is exactly what we got from direct computation. So, this formula is correct.

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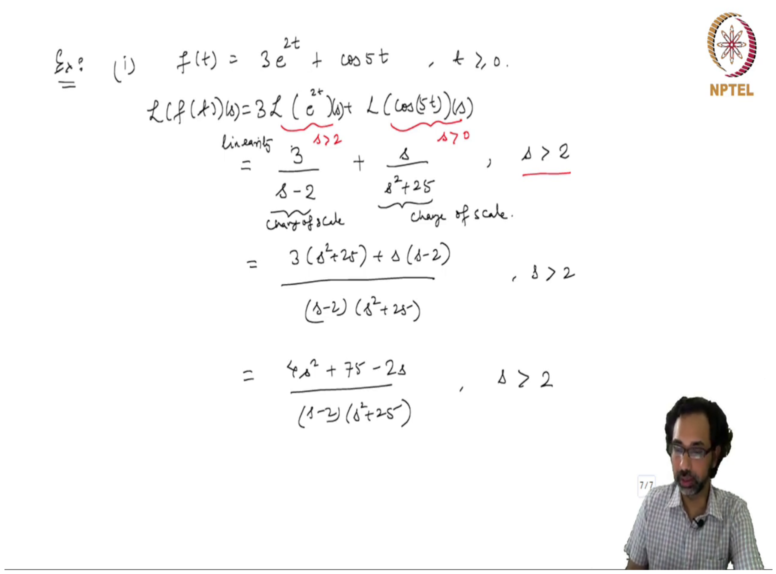

Now let us try to compute an example using both linearity as well as change of scale formula.

Question Let $f(t) = 3e^{2t} + cos(5t)$ for $t \geq 0$. What is the Laplace transform of $f(t)$?

Ans: Using linearity, as well as change of scale, we evaluate the Laplace transform of sums of functions and that gives,

$$\mathscr{L}(f(t))(s) = 3\mathscr{L}(e^{2t})(s) + \mathscr{L}(cos(5t))(s)$$

$$= \frac{3}{s-2} + \frac{s}{s^2+25} \quad = \frac{4s^2 - 2s + 75}{(s-2)(s^2+25)}\,.$$

The region of convergence is the intersection of the two regions of convergence which is here simply $s > 2$ since $3\mathscr{L}(e^{2t})(s)$ is defined for $s > 2$, and $\mathscr{L}(cos(5t))(s)$ is defined $s > 0$.

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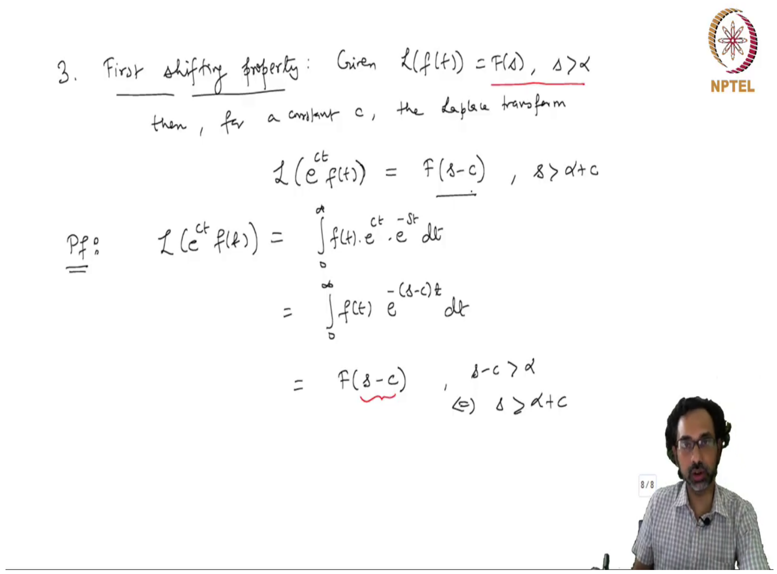

Now, let us come to the properties that I mentioned before which is the first and second shifting properties. So first let us look at the first shifting property.

First Shifting Property: Suppose the Laplace transform of a function $f(t)$ is $F(s)$ for $s > \alpha$. Then for a constant c, the Laplace transform of the function $e^{ct} f(t)$ is given by $F(s-c)$ and the region of convergence should be $s > \alpha + c$.

Proof: So, we can simply write down the definition of the Laplace transform and the proof will follow from there.

$$\mathscr{L}(e^{ct} f(t)) = \int_0^\infty f(t) e^{ct} e^{-st} dt$$

$$= \int_0^\infty f(t) e^{-(s-c)t} dt$$

$$= F(s-c)\,,$$

and this is nothing but the Laplace transform of the function $f(t)$ evaluated at the point $(s-c)$.

So, the only thing now remains is to compute the region of convergence. But we need $(s-c)$ to be greater than α because our initial function $F(s)$ is defined for $s > \alpha$. Since now we have the argument $(s-c)$, it should be greater than α, and this is nothing but $s > \alpha + c$.

So, we have seen that the Laplace transform when you multiply by an exponential function with constant c, then the Laplace transform on the right hand side gets shifted by c horizontally. This is why it is called the first shifting property.

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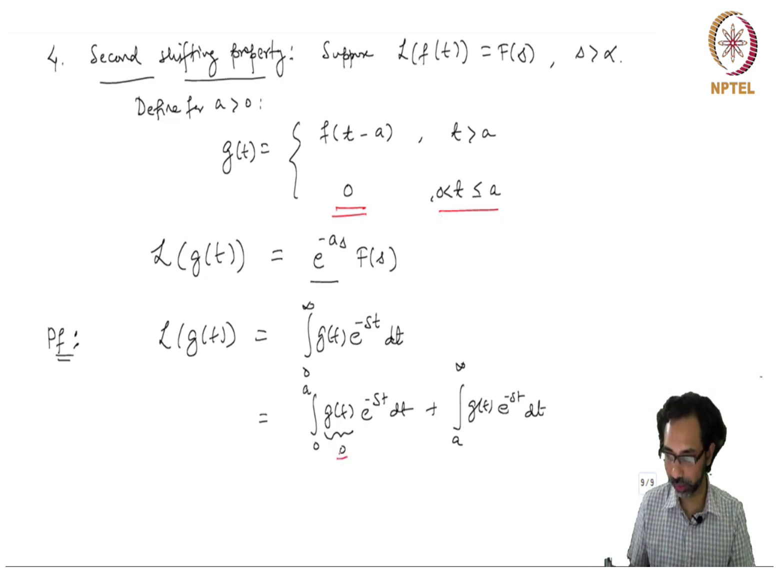

Second shifting property: Suppose again that the Laplace transform of a function $f(t)$ is $F(s)$ for $s > \alpha$. Now define a new function $g(t)$ for a positive constant a such that,

$$g(t) = f(t-a), \quad \text{for } t > a$$

$$= 0, \qquad \text{for } 0 < t \le a.$$

That is, we have shifted the whole function $f(t)$ to the right by defining this function $g(t)$ to be $f(t-a)$ for $t > a$, and this means, when t approaches to a, then $g(t)$ approaches to $f(0)$. Then

$$\mathscr{L}(g(t)) = e^{-as} F(s).$$

So our domain gets shifted to the right, and this is why it is called again the shifting property.

Proof: Now, let us see the proof of this property. We can simply write out the definition of the Laplace transform of $g(t)$, that is,

$$\mathcal{L}(g(t)) = \int_0^\infty g(t)e^{-st}dt$$

But we can now break this integral, and we get

$$\mathcal{L}(g(t)) = \int_0^a g(t)e^{-st}dt + \int_a^\infty g(t)e^{-st}dt$$

Now, by definition of the function $g(t)$, in the region 0 to a, this $g(t)$ is simply 0. So we are only left with the second integral. So let us see what is our second integral.

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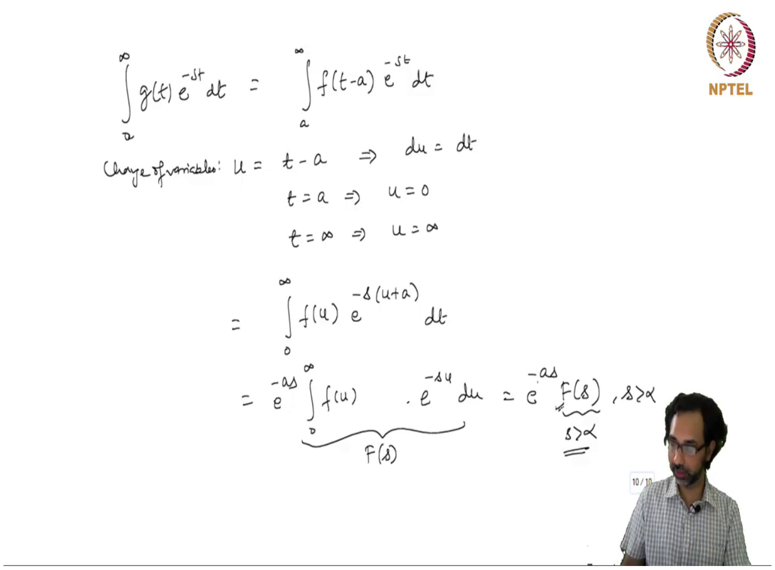

We have,

$$\int_a^\infty g(t)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt \, .$$

now we make a change of variables $u = (t-a)$, which means $du = dt$ since a is a constant. Let us see what are the limits of integration. When $t = a$, this means $u = 0$, and when t equal to infinity, this means u equal to infinity.

So we write this integral after the change of variables and we get,

$$\int_a^\infty g(t)e^{-st}dt = \int_0^\infty f(u)e^{-s(u+a)}du = e^{-sa}\int_0^\infty f(u)e^{-su}du = e^{-sa}F(s).$$

Therefore, at the end we get that, the Laplace transform of $g(t)$ is simply $e^{-sa}F(s)$. Now, note that, if $F(s)$ is defined over $s > \alpha$, then the Laplace transform $e^{-sa}F(s)$ is also defined for $s > \alpha$. So the region of convergence will not change because exponential function is defined everywhere. So the region of convergence will only depend on the region of convergence of this function $F(s)$ which is $s > \alpha$.

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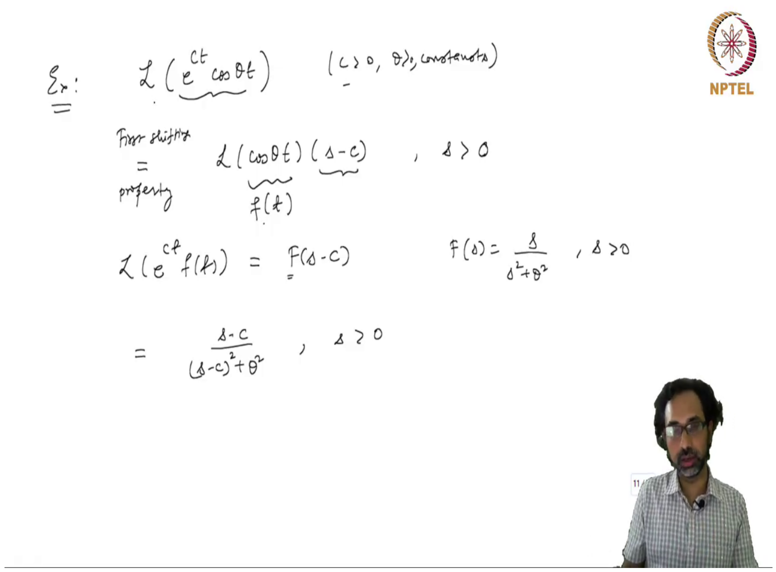

So let us see another example where we can use the shifting property.

Example: Let us compute $\mathscr{L}(e^{ct}\cos\theta t)$, where $c > 0$, and $\theta > 0$ are real constants.

By the first shifting property,

$$\mathscr{L}(e^{ct}\cos\theta t) = \mathscr{L}(\cos\theta t)(s-c)$$

and this is defined for $s > c$.

So note that here we have taken the function $cos\theta t$ to be our $f(t)$ and so we are trying to compute the Laplace transform of $e^{ct}f(t)$, which by the first shifting theorem is $F(s-c)$, where $F(s)$ is the Laplace transform of $f(t)$.

Now, we know

$$\mathscr{L}(\cos\theta t)(s) = \frac{s}{s^2+\theta^2}\,, \qquad s > 0.$$

But here we are evaluating at $(s-c)$. So, wherever there is an s, we replace it by $(s-c)$.

So, we get,

$$\mathscr{L}(\cos\theta t)(s-c) = \frac{(s-c)}{(s-c)^2+\theta^2}\,, \qquad s > 0.$$

In this way, we can apply the first shifting theorem to compute the Laplace transform of any function which is multiplied by e^{ct} in the input function.

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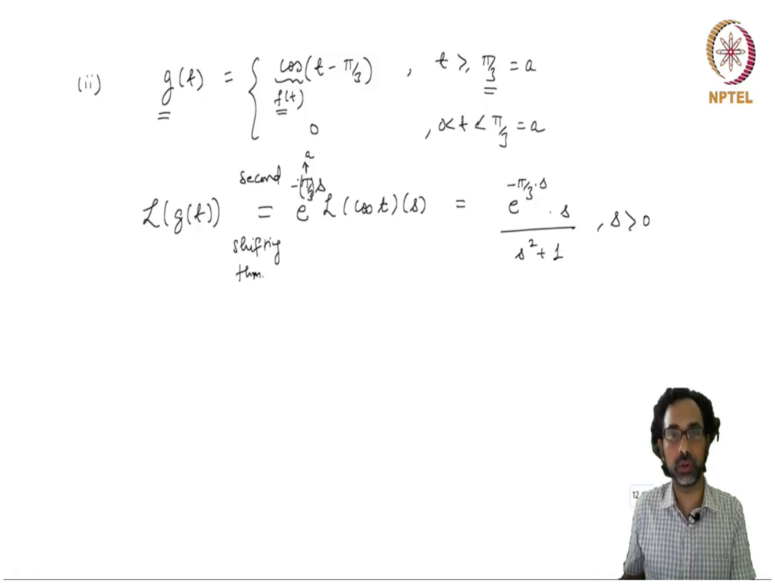

For the second shifting property, let us see another example.

Example: Let $g(t) = \cos(t-\frac{\pi}{3})\,, \quad$ for $t \geq \frac{\pi}{3}$

$$= 0, \qquad \text{for } 0 < t < \frac{\pi}{3}\,.$$

So, in the second shifting theorem or second shifting property, if we put $f(t) = \cos t$, and $a = \frac{\pi}{3}$, we get our function $g(t)$ simply the way we define the new function $g(t)$ in the theorem, using the shifting in the domain. So, by the second shifting property, the Laplace transform of $g(t)$ should be given by the Laplace transform of $(\cos t)$ evaluated at s and you have to multiply by $e^{-\frac{\pi}{3}s}$. That is,

$$\mathscr{L}(g(t)) = e^{-\frac{\pi}{3}s}\mathscr{L}(\cos t)(s) = \frac{se^{-\frac{\pi}{3}s}}{s^2+1}.$$

Here the region of convergence is simply $s > 0$. So, we have applied the second shifting property for this example.

In the next lectures, we will continue to develop more properties of the Laplace transform, which will help us compute the Laplace transform of more and more functions.