LAPLACE TRANSFORM

PROF. INDRAVA ROY

Lecture 3: Improper Riemann integrals: Definition and Existence Part I

In the last lecture we saw the definition of the Laplace Transform and we saw that it is a special case of an integral transform. We also saw that it is defined using the concept of an improper Riemann integral because, the limits of integration for the Laplace transform is from 0 to ∞. So, the ordinary Riemann integral can no longer be defined and one uses a limit formula to define the so called improper Riemann integral.

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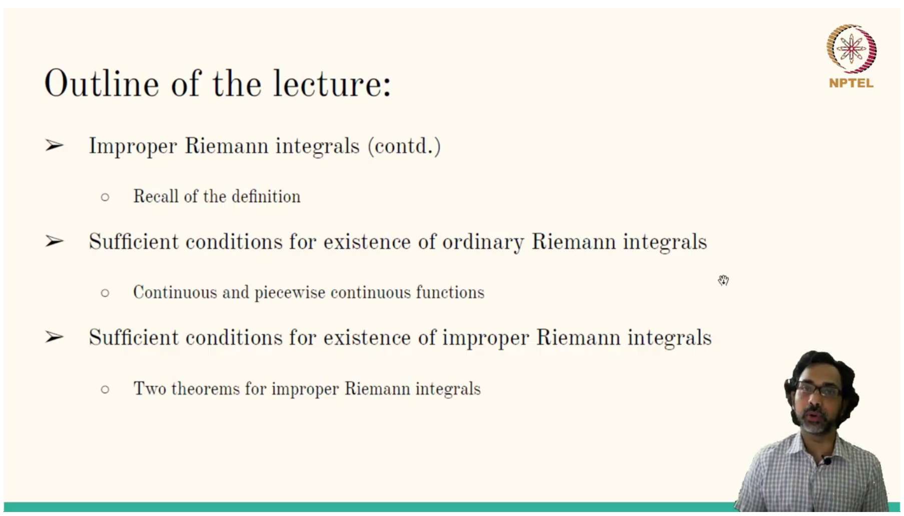

So, in this lecture we will review the definition of the improper Riemann integral using limits. And we will also see examples where this limit may feel to exist and so, we should study the sufficient conditions under which these kinds of integrals are guarantee to exist so, these limits will be finite.

But, before we do that we will also see the sufficient conditions for ordinary Riemann integrals and we will cover the definitions of what are called continuous and as well as piecewise continuous functions and then after we have done that we will see sufficient two theorems on sufficient conditions for the existence of improper Riemann integrals.

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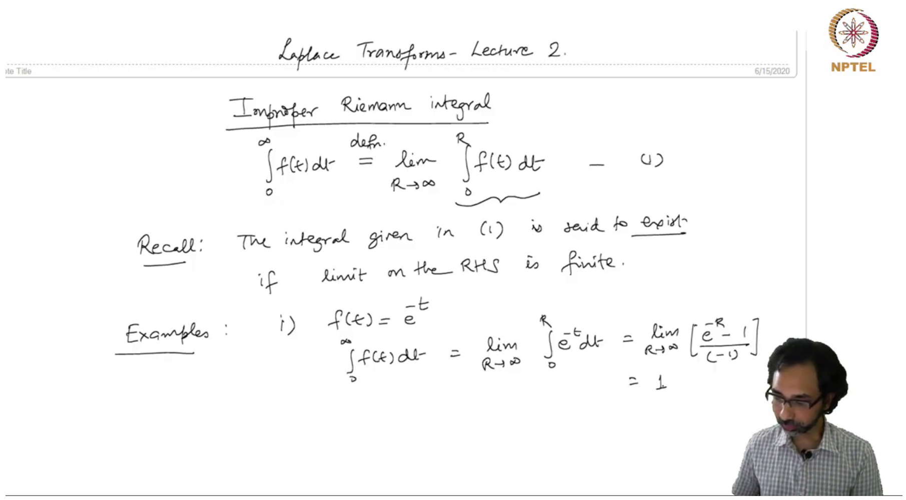

So, in the last class, we saw the notion of an improper Riemann integral and this was used to define an integral of the form $\int_0^\infty f(t)dt$ and this was by definition $\lim\limits_{R\to\infty} \int_0^R f(t)dt$. Now first of all for this definition to make sense this integral within the limits this $\int_0^R f(t)dt$ must be finite and then we say that the improper Riemann integral exists. If the limit as R goes to infinity of this evaluation of this inner integral is also finite.

So, let me remind you that improper Riemann integral please read out. So, recall that the integral given in equation 1 in the definition 1 is said to exist if this limit on the right side is finite. Now, to see some examples and non-examples of this improper Riemann integral. Let us take this following example, which we have already seen first one is $f(t) = e^{-t}$.

Now, we have already computed this and it turns out that first of all this so, if you want compute

$$\int_0^\infty f(t)dt = \lim_{R\to\infty} e^{-t}dt = \lim_{R\to\infty}\left[\frac{e^{-R}-1}{-1}\right] = 1, \tag{1}$$

and this final limit exist because, the exponential function raised to a negative power will decrease as R goes to infinity to 0 so, this is in fact 1.

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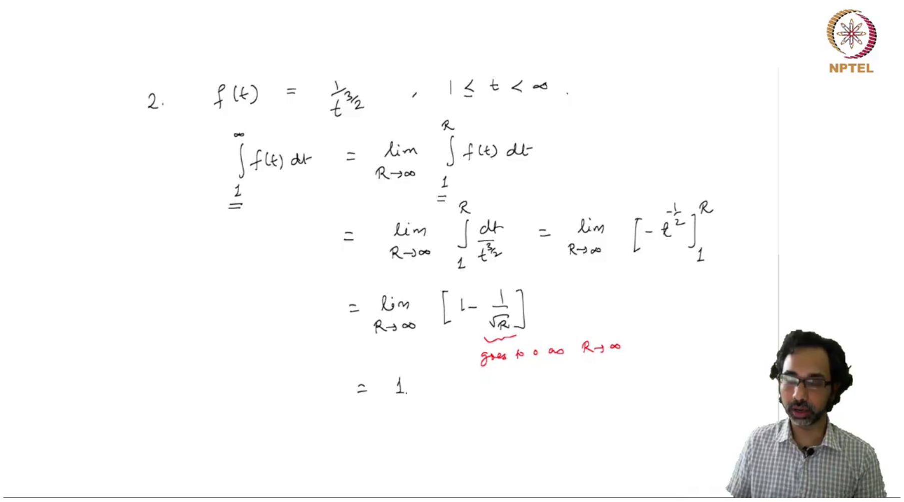

A second example where the integral exist is the following. So, the following example $f(t) = \frac{1}{t^{\frac{3}{2}}}$ define in the domain $1 \leq t < \infty$. Now, let us see this $\int_1^{\infty} f(t)dt$ exists as an improper Riemann integral or not. So, we write it in the same way $\lim\limits_{R\to\infty} f(t)dt$.

So, here note that the lower limit is not 0 is now equal to 1 but, we are using the same definition for the improper Riemann integral. So in fact 1 can have any finite real number on the lower limit and if the positive if the upper limit is positive infinity then we always take this definition as limit as R goes to ∞ with the integral from the lower limit to R. So, the lower limit can be 1 could be 0 it could be any other finite real number. So, in this case the lower limit is 1.

Now, let us see if this integral exists so, we integrate

$$\begin{aligned}\int_1^{\infty} f(t)dt &= \lim_{R\to\infty}\int_1^{R} f(t)dt = \lim_{R\to\infty}\int_1^{R}\frac{dt}{t^{\frac{3}{2}}} \\ &= \lim_{R\to\infty}\left[-t^{\frac{-1}{2}}\right]_1^{R} = \lim_{R\to\infty}\left[1-\frac{1}{\sqrt{R}}\right]\end{aligned}$$

Now, again this 1 over square root of R term as R goes to infinity goes to 0 as R goes to infinity therefore, in the limit you will simply get 1 so, the integral 1 to infinity $f(t)dt$ where $f(t)$ is $\frac{1}{t^{\frac{3}{2}}}$ is in fact finite and it exists as an improper Riemann integral.

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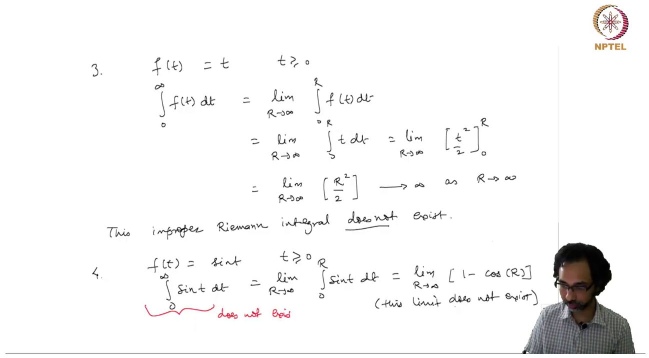

Now, let us see an example where the improper Riemann integral does not exist. So, the easiest example is $f(t) = t$, $t \geq 0$ and if you take the integral $\int_0^\infty f(t)dt$ then you by definition this is again $\lim\limits_{R\to\infty} \int_0^R f(t)dt$ which is again $\lim\limits_{R\to\infty} \int_0^R tdt$ and you get simply $\left[\frac{t^2}{2}\right]_0^R$. So, you get $\lim\limits_{R\to\infty} \left[\frac{R^2}{2}\right]$ but, now this goes to infinity as R goes to ∞. So, in fact, this limit is now, not finite and therefore, this improper Riemann integral does not exist just because, this is not a finite.

Another example is $f(t) = \sin t$ for $t \geq 0$. So, if you compute $\int_0^\infty \sin t dt$ dt this is $\lim\limits_{R\to\infty} \int_0^R \sin t dt$ but, this is limit R goes to ∞ you will get $1 - \cos R$. But, here in the second term $\cos R$ does not converge to any finite limit as R goes to ∞ because, the *cos* function will oscillate between -1 and 1 infinitely often as you take $R \to \infty$. So, this limit does not exist this limit also does not exist and so, we also say that this improper Riemann integral of $f(t) = \sin t$ does not exist.

So, we have seen that improper Riemann integrals may or may not exist now it will be useful to have some sufficient conditions which will guarantee the existence of such improper Riemann integrals and so, this is what we will study now.

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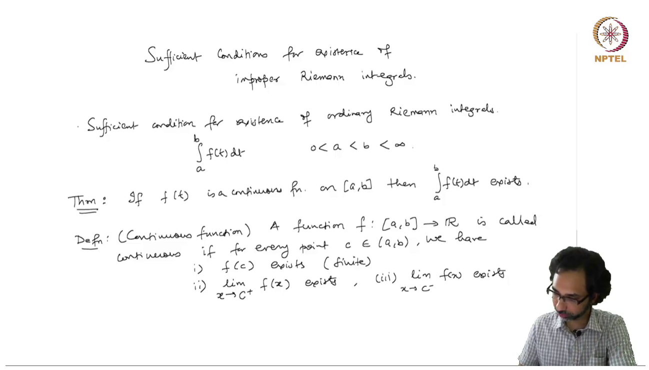

So, these are sufficient conditions for existence of improper Riemann integrals. First of all before talking about sufficient conditions for existence of improper Riemann integrals let us see a sufficient condition for existence of ordinary Riemann integrals by an ordinary Riemann integrals I mean an integral of the form a to b $f(t)dt$ where, a and b are finite values. So, without loss of generality I am taking a to b positive you can also have a negative but, since all our competitions will be on the positive real line I am here taking a to b positive.

So, this is a theorem which gives the sufficient condition for the existence of ordinary Riemann integrals this is that if $f(t)$ is a continuous function on the close interval $[a, b]$ then, $\int_a^b f(t)dt$ exists meaning that this is a finite value. So, recall the definition of a continuous function.

So, a function f which takes real values is called continuous if for any point c in this interval a to b so, first of all let us take it in the open interval (a, b) we have first is that $f(c)$ exists again exist means it is finite and secondly $\lim_{x \to c^+} f(x)$ exists third $\lim_{x \to c^-} f(x)$ exists.

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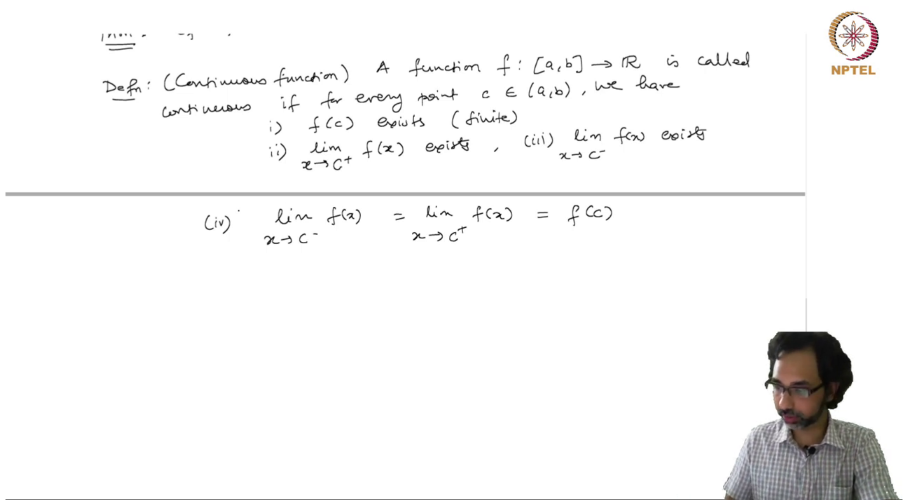

And lastly fourth condition is that $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$. So, we see that these four conditions give you a continuous function.

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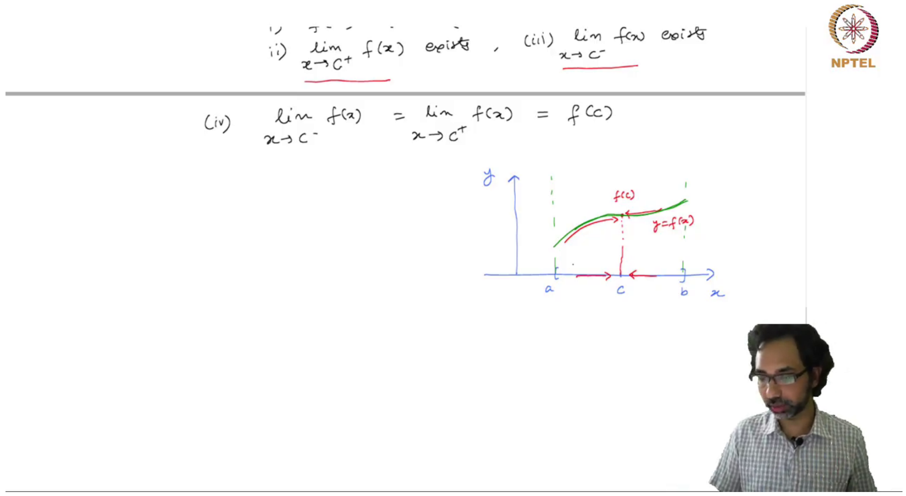

So, graphically let me draw a picture so, this is x this is $f(x)$, $y = f(x)$. So, let me draw a picture now, suppose that here is your a here is your b and you are considering some point c in between and your function looks and something like this so, this is $y = f(x)$ now, at c we have this value $f(c)$.

Now, the second condition says that $\lim_{x\to c^+} f(x)$ exists, which means that when you approach c from the right hand side from values greater than c then the corresponding limit as you traverse this graph of $f(x)$ also exists and the third condition says that $\lim_{x\to c^-} f(x)$ meaning that x tends to c from below from the left hand side then again your $\lim_{x\to c^+} f(x)$ exists and the fourth condition says that all these three values are the same.

So, this means that informally if you take a pen and you move your pain over the graph of $f(x)$ you can move this move your pen without taking it out of the paper this is an informal definition of a continuous function.

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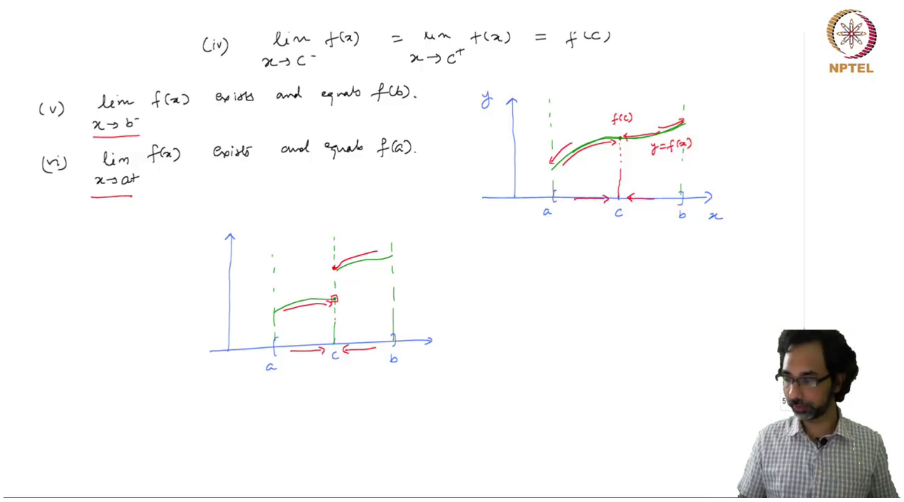

So, what would not be a continuous function maybe that is something we should need to see to understand continuity so, let us say a and b are here c is here and our $f(x)$ look something like this so, here is our line $x = c$ now, let see which part is getting violated.

So, here I am assuming that this is $f(c)$ so, in this graph we have that when you approach $f(c)$ from c from the left hand side you get this value $f(c)$ here but, when you approach c from the right hand side you get this value here therefore, these two values do not coincide and so there is a discontinuity at the point c. So, this is about the definition of continuity at an interior point in this $[a, b]$ in addition we should also say what happens at the end points a and b.

So, let me write the fifth condition which is $\lim_{x\to b^-} f(x)$ exists and equals $f(b)$ and the sixth condition is that $\lim_{x\to a^+} f(x)$ exists and equals $f(a)$ so, all these six conditions together make a continuous function on the interval $[a, b]$. So, note here, that when you talking about the upper end point then you are approaching from the left here and when you are talking about the lower in point a you are approaching from the right so, here you are approaching from the right and here you are approaching from the left wherever, the function is defined you can only approach from that side. So, this is how a continuous function is defined.

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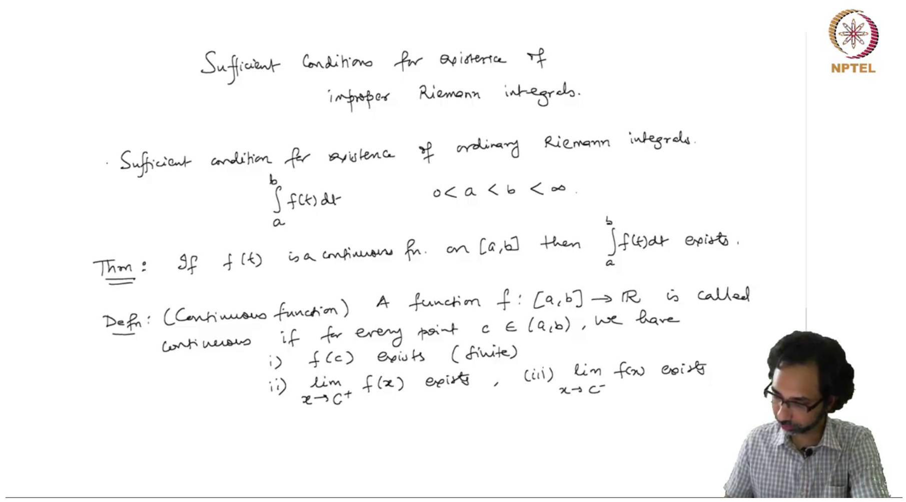

So, our theorem says that if this function is continuous on this entire interval $[a, b]$ then this ordinary Riemann integral $\int_a^b f(t)dt$ will exist.