Module - 9:

Vector Integration

Line Integral

Any integral which is evaluated along a curve is called a line integral.

Let $\overline{F}$ be a continuous vectorpoint function, defined at each point of a curve C. Then the line integral $\overline{F}$ along C is given by

$$\int_C \overline{F}\cdot d\overline{r} = \int_C F_1 dx + F_2 dy + F_3 dz ,$$

when $\overline{F} = F_1\overline{i} + F_2\overline{j} + F_3\overline{k}$ and $d\overline{r} = \overline{i}dx + \overline{j}dy + \overline{k}dz$

Circulation:

If $\overline{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \overline{v}\cdot d\overline{r}$ is called the circulation of $\overline{v}$ around C.

Work done by a force:

If $\overline{F}$ represents the force vector acting on a particle moving along an arc AB, then workdone during a small displacement $\delta\overline{r}$ is $\overline{F}\cdot d\overline{r}$.

Hence, the total workdone by $\overline{F}$ during the displacement from A to B is given by the line integral $\int_A^B \overline{F}\cdot d\overline{r}$.

Problems:

1. If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in XY-plane $y = x^3$ from the point $(1,1)$ to $(2,8)$

Sol. Given curve C is $y = x^3$

$$\bar{F}\cdot d\bar{r} = \left[(5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}\right]\cdot(dx\,\bar{i} + dy\,\bar{j})$$

$$= (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

Along $y = x^3$. we have $dy = 3x^2\,dx$

and x varies from 1 to 2.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{(1,1)}^{(2,8)} \left[(5xy - 6x^2)\,dx + (2y - 4x)\,dy\right]$$

$$= \int_{x=1}^{2} \left[(5x^4 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2\,dx\right]$$

$$= \int_1^2 \left[5x^4 - 6x^2 + 6x^5 - 12x^3\right] dx$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2$$

$$= (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3)$$

$$= 35$$

2. Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the straight line from (0,0,0) to (2,1,3)

Sol. Given $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$

$$\therefore \bar{F}\cdot d\bar{r} = 3x^2dx + (2xz-y)dy + z\,dz$$

The equation of the straight line joining (0,0,0) to (2,1,3) is

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

Then $x = 2t$, $y = t$, $z = 3t$. t varies from 0 to 1.

$$\text{workdone by } \bar{F} = \int_{(0,0,0)}^{(2,1,3)} \bar{F}\cdot d\bar{r}$$

$$= \int_{t=0}^{1} [3(2t)^2 2dt + (2.2t.3t-t)dt + 3t.3dt]$$

$$= \int_0^1 (36t^2 + 8t)\,dt$$

$$= (12t^3 + 4t^2)_0^1$$

$$= 12 + 4$$

$$= 16$$

3. If $\bar{F} = (y-2x)\bar{i} + (3x+2y)\bar{j}$, calculate the circulation of $\bar{F}$ about the circle C in the plane $x^2+y^2=4$ oriented in the anticlockwise direction.

Sol. The curve is $C: x^2+y^2=4$ in XY plane

Then the parametric equations are

$$x = 2\cos t,\ y = 2\sin t, \quad t \text{ varies from } 0 \text{ to } 2\pi$$

$$\Rightarrow dx = -2\sin t\,dt,\ dy = 2\cos t\,dt$$

$$\bar{F}\cdot d\bar{r} = (y-2x)dx + (3x+2y)dy$$

$$= (2\sin t - 4\cos t)(-2\sin t\,dt) + (6\cos t + 4\sin t)(2\cos t\,dt)$$

$$= (-4\sin^2 t + 8\sin t\cos t + 12\cos^2 t + 8\sin t\cos t)dt$$

$$= (16\sin t\cos t + 16\cos^2 t - 4)dt.$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_0^{2\pi} (16\sin t\cos t + 16\cos^2 t - 4)dt$$

$$= \int_0^{2\pi} \left[8\sin 2t + 16\left(\frac{1+\cos 2t}{2}\right) - 4\right]dt$$

$$= \left[8\left(-\frac{\cos 2t}{2}\right) + 8\left(t + \frac{\sin 2t}{2}\right) - 4t\right]_0^{2\pi}$$

$$= -4 + 8(2\pi + 0) - 8\pi - (-4)$$

$$= 16\pi - 8\pi = 8\pi$$

Surface integrals

Any integral which is to be evaluated over a surface is called a surface integral.

Let $\bar{F}$ be a continuous vector point function. Then the surface integral of $\bar{F}$ over a surface S is given by $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{n}$ is the outward drawn unit normal vector at any point of S.

If $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$, then we have

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint F_1\,dy\,dz + \iint F_2\,dz\,dx + \iint F_3\,dx\,dy$$

Note : Let R_1, R_2, R_3 be the projections of the surface S on XY plane, YZ plane, ZX planes respectively. Then

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{i}|}\,dy\,dz$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{j}|}\,dx\,dz.$$

Problems:

1. Evaluate $\iint \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = (x^2+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

Sol. The normal to the surface S is given by

$$\nabla(2x+y+2z) = 2\bar{i}+\bar{j}+\bar{k}$$

$\therefore$ unit normal is $\bar{n} = \dfrac{2\bar{i}+\bar{j}+2\bar{k}}{\sqrt{4+1+4}}$

$$= \frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k})$$

$$\Rightarrow \bar{n}\cdot\bar{k} = \frac{2}{3}.$$

Also $\bar{F}\cdot\bar{n} = [(x^2+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}]\cdot\frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k})$

$$= \frac{2}{3}(x^2+y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$= \frac{4y}{3}(3-x) \qquad \left(\because \text{ since on S, } z = \frac{1}{2}(6-2x-y)\right)$$

$\therefore \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \dfrac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\,dx\,dy$, where R is projection of S on xy plane

$$= \iint_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}\,dx\,dy$$

$$= \int_{x=0}^{3} \int_{y=0}^{6-2x} 2y(3-x)\,dy\,dx$$

$$= \int_{0}^{3} 2(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= 4\int_{0}^{3} (3-x)^3\,dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3$$

$$= -[0-81] = 81$$

2. Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$, where $\bar{F} = z\bar{i} + x\bar{j} + 3y^2 z\bar{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Sol. Given that S is the surface of $x^2+y^2=16$

normal to the surface $= \nabla S$

$$= \bar{i}\frac{\partial S}{\partial x} + \bar{j}\frac{\partial S}{\partial y} + \bar{k}\frac{\partial S}{\partial z}$$

$$= 2x\bar{i} + 2y\bar{j}$$

$$= 2(x\bar{i} + y\bar{j})$$

$\therefore$ unit normal $\bar{n} = \dfrac{2(x\bar{i}+y\bar{j})}{2\sqrt{x^2+y^2}}$

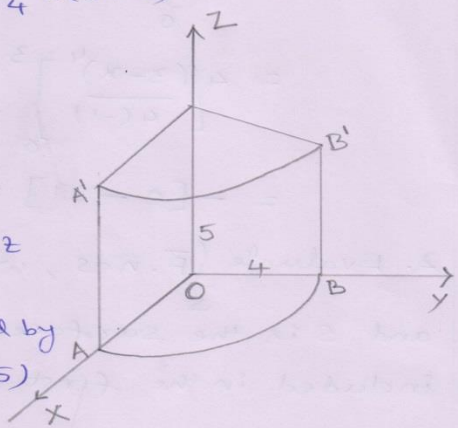
i.e $\bar{n} = \dfrac{2(x\bar{i}+y\bar{j})}{2\sqrt{16}} = \dfrac{1}{4}(x\bar{i}+y\bar{j})$

Let R be the projection of S on ~~X~~YZ plane

$\bar{F}\cdot\bar{n} = (z\bar{i}+x\bar{j}+3y^2z\bar{k})\cdot\frac{1}{4}(x\bar{i}+y\bar{j})$

$= \frac{1}{4}(xz+xy) = \frac{1}{4}x(y+z)$

and $\bar{n}\cdot\bar{i} = \frac{1}{4}x$



$\therefore \int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \dfrac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{i}|}\,dy\,dz$

$= \int_{z=0}^{5}\int_{y=0}^{4} \frac{1}{4}x(y+z)\cdot\frac{4}{x}\,dy\,dz$

(since the region R is enclosed by $y=0$ to $y=4$ and $z=0$ to $z=5$)

$= \int_{z=0}^{5}\left(\frac{y^2}{2}+yz\right)_{y=0}^{4} dz$

$= \int_0^5 (8+4z)\,dz$

$= (8z+2z^2)_0^5$

$= 40+50$

$= 90$

Volume Integrals

Any integral which is evaluated over a volume is called a volume integral.

If V is the volume bounded by a surface S, then $\iiint_V \bar{F}\, dV$ is called the volume integral of $\bar{F}$

Problems

1. If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$, evaluate $\int_V \bar{F}\, dV$, where V is the volume of the region bounded by the surfaces $x=0$, $x=1$, $y=0$, $y=6$, $z=x^2$, $z=4$.

Sol. $\int_V \bar{F}\, dV = \int_{x=0}^{1}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz\bar{i} - x\bar{j} + y^2\bar{k})\, dx\, dy\, dz$

$= \bar{i}\int_0^1\int_0^6\int_{x^2}^4 2xz\, dx\, dy\, dz - \bar{j}\int_0^1\int_0^6\int_{x^2}^4 x\, dx\, dy\, dz + \bar{k}\int_0^1\int_0^6\int_{x^2}^4 y^2\, dx\, dy\, dz \longrightarrow 1)$

Consider, $\int_0^1\int_0^6\int_{x^2}^4 2xz\, dx\, dy\, dz$

$= \int_0^1\int_0^6 x\left(z^2\right)_{x^2}^{4} dx\, dy$

$$= \int_{x=0}^{1} \int_{y=0}^{6} x(16-x^4)\, dy\, dx$$

$$= \int_{0}^{1} x(16-x^4) [y]_0^6\, dx$$

$$= \int_{0}^{1} 6x(16-x^4)\, dx$$

$$= 6\int_{0}^{1} (16x - x^5)\, dx$$

$$= 6\left[8x^2 - \frac{x^6}{6}\right]_0^1$$

$$= 6\left[8 - \frac{1}{6}\right] = 47.$$

Now $\displaystyle\int_{x=0}^{1} \int_{y=0}^{6} \int_{z=x^2}^{4} x\, dx\, dy\, dz = \int_0^1 \int_0^6 x\,(z)_{x^2}^{4}\, dy\, dx$

$$= \int_0^1 \int_0^6 x(4-x^2)\, dy\, dx$$

$$= \int_0^1 (4x - x^3) [y]_0^6\, dx$$

$$= 6\int_0^1 (4x - x^3)\, dx$$

$$= 6\left[2x^2 - \frac{x^4}{4}\right]_0^1$$

$$= 6\left[2 - \frac{1}{4}\right]$$

$$= 21/2.$$

Finally, consider $\int_{x=0}^{1}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$

$$= \int_0^1\int_0^6 y^2 (z)_{x^2}^{4}\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^3}{3}\right)_0^6 (4-x^2)\,dx$$

$$= \frac{6^3}{3}\int_0^1 (4-x^2)\,dx$$

$$= 72\left[4x-\frac{x^3}{3}\right]_0^1$$

$$= 72\left(4-\frac{1}{3}\right)$$

$$= 72\left(\frac{11}{3}\right)$$

$$= 264$$

substitute these values in eq 1), we get

$$\int_V \bar{F}\,dV = 47\bar{i} - \frac{21}{2}\bar{j} + 264\bar{k}$$

2. If $\bar{F} = (2x^2-3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$, evaluate $\iiint_V \text{curl}\,\bar{F}\,dV$, where V is the closed region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$.

Sol. $\text{Curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$

$= \bar{i}(0) - \bar{j}(-4+3) + \bar{k}(-2y-0)$

$= \bar{j} - 2y\bar{k}$

The limits are $z=0$ to $z = 4-2x-2y$

$y=0$ to $y = 2-x$

$x=0$ to $x=2$

$$\therefore \iiint \nabla\times\bar{F}\,dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\bar{j}-2y\bar{k})\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x} (\bar{j}-2y\bar{k})\,[z]_0^{4-2x-2y}\,dx\,dy$$

$$= \int_0^2\int_0^{2-x} (\bar{j}-2y\bar{k})(4-2x-2y)\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} [\bar{j}(4-2x-2y) - 2y\bar{k}(4-2x-2y)]\,dy\,dx$$

$$= \int_0^2 \bar{j}\left[4y-2xy-y^2\right]_{y=0}^{2-x} - 2\bar{k}\left[2y^2-xy^2-\frac{2y^3}{3}\right]_{y=0}^{2-x} dx$$

$$= \int_0^2 \{\bar{j}\,[4(2-x)-2x(2-x)-(2-x)^2]$$

$$-\frac{2}{3}\bar{k}\left[6(2-x)^2-3x(2-x)^2-2(2-x)^3\right]\Big\}\,dx$$

$$=\int_{x=0}^{2}\left[(2-x)^2\bar{j}-\frac{2}{3}(2-x)^3\bar{k}\right]dx$$

$$=\left[\frac{(2-x)^3}{-3}\bar{j}+\frac{2}{3}\frac{(2-x)^4}{4}\bar{k}\right]_{x=0}^{2}$$

$$=\frac{8}{3}\bar{j}-\frac{8}{3}\bar{k}=\frac{8}{3}(\bar{j}-\bar{k})$$

Vector integral Theorems

Gauss Divergence theorem

If $\bar{F}$ is a vector point function having continuous first order partial derivatives over a closed surface S enclosing a volume V, then

$$\iiint_V \nabla\cdot\bar{F}\,dV=\iint_S \bar{F}\cdot\bar{n}\,ds,$$

where $\bar{n}$ is the outward drawn unit normal vector to the surface S.

Problems

1. Verify Gauss divergence theorem for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = 2$ and the coordinate planes.

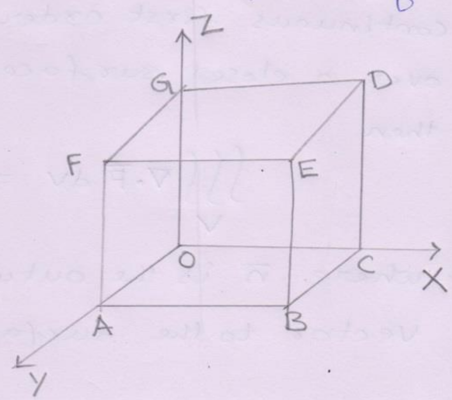
Sol. By Gauss divergence theorem,

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \nabla\cdot\bar{F}\,dv$$

consider the surface integral

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_{S_1} \bar{F}\cdot\bar{n}\,ds + \int_{S_2} \bar{F}\cdot\bar{n}\,ds + \int_{S_3} \bar{F}\cdot\bar{n}\,ds + \int_{S_4} \bar{F}\cdot\bar{n}\,ds + \int_{S_5} \bar{F}\cdot\bar{n}\,ds + \int_{S_6} \bar{F}\cdot\bar{n}\,ds \longrightarrow 1)$$

where S_1, S_2, S_3 are the six faces of the cube



Over the face S_1 i.e BCDE

$x = 2$

$\bar{n} = \bar{i}$ and $ds = dy\,dz$

$0 \le y \le 2$ and $0 \le z \le 2$

$\bar{F} \cdot \bar{n} = ((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}) \cdot \bar{i}$

$= x^3 - yz$

$= 8 - yz \quad (\because x = 2$

$\therefore \int_{S_1} \bar{F} \cdot \bar{n}\, ds = \int_{z=0}^{2}\int_{y=0}^{2} (8 - yz)\, dy\, dz$

$= \int_{z=0}^{2} \left(8y - \frac{y^2}{2} z\right)_0^2 dz$

$= \int_0^2 (16 - 2z)\, dz$

$= (16z - z^2)_0^2$

$= 32 - 4 = 28$

Over the face S_2 i.e AOGF

we have $x = 0$

$\bar{n} = -\bar{i}$ and $ds = dy\,dz$

$0 \le y \le 2, \quad 0 \le z \le 2$

$\bar{F} \cdot \bar{n} = -(x^3 - yz) = yz \quad (\because x = 0$

$$\therefore \int_{S_2} \bar{F}\cdot\bar{n}\,ds = \int_{z=0}^{2}\int_{y=0}^{2} yz\,dy\,dz$$

$$= \int_0^2 z\,dz \int_0^2 y\,dy$$

$$= \left(\frac{z^2}{2}\right)_0^2 \left(\frac{y^2}{2}\right)_0^2$$

$$= (2)(2) = 4.$$

over the face S_3 i.e FEDG

$z = 2 \Rightarrow dz = 0$

$\bar{n} = \bar{k}$ and $ds = dx\,dy$

$0 \le x \le 2$ and $0 \le y \le 2$

$\bar{F}\cdot\bar{n} = z = 2$

$$\therefore \int_{S_3} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{2}\int_{x=0}^{2} 2\,dx\,dy$$

$$= 2\,(x)_0^2\,(y)_0^2$$

$$= 2(2)(2) = 8$$

over the face S_4 i.e OABC

$z = 0$

$\bar{n} = -\bar{k}$ and $ds = dx\,dy$

$0 \le x \le 2, \quad 0 \le y \le 2$

$\bar{F}\cdot\bar{n} = -z = 0 \quad (\because z = 0$

$$\therefore \int_{S_4} \bar{F}\cdot\bar{n}\,ds = \int_{y=0}^{2}\int_{x=0}^{2} 0\,dx\,dy = 0$$

over the face S_5 i.e ABEF

we have $y = 2$

$\bar{n} = \bar{j}$ and $ds = dx\,dz$

$0 \le x \le 2$ and $0 \le z \le 2$.

$$\int_{S_5} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{2}\int_{z=0}^{2} (-2x^2 y)\,dx\,dz$$

$$= -4\int_{x=0}^{2}\int_{z=0}^{2} x^2\,dx\,dz$$

$$= -4\left(\frac{x^3}{3}\right)_0^2 (z)_0^2$$

$$= -4\left(\frac{8}{3}\right)(2)$$

$$= -\frac{64}{3}$$

Over the face S_6 i.e OCDG

$y = 0$

$\bar{n} = -\bar{j}$ and $ds = dx\,dz$

$$\int_{S_6} \bar{F}\cdot\bar{n}\,ds = \int_{x=0}^{2}\int_{z=0}^{2} 2x^2 y\,dx\,dz$$

$$= 0 \quad (\because y = 0$$

$\therefore$ from 1), we get

$$\int_S \bar{F}\cdot\bar{n}\,ds = 28+4+8+0-\frac{64}{3}+0 = \frac{56}{3} \longrightarrow 2)$$

Consider the volume integral

$$\int_V \nabla\cdot\bar{F}\,dV = \int_V (3x^2-2x^2+1)\,dV$$

$$= \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2}(x^2+1)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 \left(\frac{x^3}{3}+x\right)_0^2 dy\,dz$$

$$= \int_0^2\int_0^2 \left(\frac{8}{3}+2\right) dy\,dz$$

$$= \frac{14}{3}\int_0^2 dy\int_0^2 dz$$

$$= \frac{14}{3}(2)(2)$$

$$= \frac{56}{3} \longrightarrow 3)$$

$\therefore$ from 2) and 3), we have

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \nabla\cdot\bar{F}\,dV$$

Hence Gauss divergence theorem is ~~verf~~ verified

2. Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ by Gauss divergence theorem where $\bar{F} = 2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ taken over the region $y^2+z^2=9$, $x=2$ in the first octant.

Sol. By Gauss divergence theorem,

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \nabla\cdot\bar{F}\,dv$$

Now $\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$

$= 4xy - 2y + 8xz$

The limits are $z = 0$ to $\sqrt{9-y^2}$

$y = 0$ to 3 $\quad$ ($\because$ In first quadrant)

$x = 0$ to 2

$$\int_V \nabla\cdot\bar{F}\,dv = \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}\int_{x=0}^{2}(4xy - 2y + 8xz)\,dx\,dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}\left[2x^2y - 2xy + 4x^2z\right]_0^2 dz\,dy$$

$$= \int_0^3\int_{z=0}^{\sqrt{9-y^2}}(8y - 4y + 16z)\,dz\,dy$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}}(4y + 16z)\,dz\,dy$$

$$= \int_0^3 \left[4yz + 8z^2\right]_0^{\sqrt{9-y^2}} dy$$

$$= 4\int_0^3 \left[y\sqrt{9-y^2} + 2(9-y^2)\right] dy$$

$$= 4\left[-\frac{1}{2}(9-y^2)^{3/2}\cdot\frac{2}{3} + 2\left(9y - \frac{y^3}{3}\right)\right]_0^3$$

$$= 4\left[0 + 9 + 36 - 0\right]$$

$$= 180$$

Green's theorem in a plane

If R is a closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in positive (anticlockwise) direction.

Problems

1. Verify Green's theorem for $\int [(3x^2-8y^2)dx+(4y-6xy)dy]$ where C is the boundary of the region bounded by $x=0, y=0$ and $x+y=1$.

Sol. Given $\int [(3x^2-8y^2)dx+(4y-6xy)]dy = \int_C \bar{F}\cdot d\bar{r}$

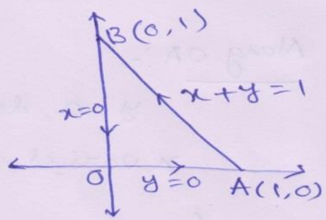
where $\bar{F} = (3x^2-8y^2)\bar{i} + (4y-6xy)\bar{j}$

and $d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where R is the region bounded by $x=0, y=0$ and $x+y=1$



Here $M = 3x^2 - 8y^2$

$N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$, $\frac{\partial N}{\partial x} = -6y$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (+16y - 6y)\,dx\,dy$

$$= 10\int_{x=0}^{1}\int_{y=0}^{1-x} y\,dy\,dx$$

$$= 10 \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2 dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}(0-1)$$

$$= \frac{5}{3}. \longrightarrow 1)$$

Now $\int_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy$

$$+ \int_{BO} Mdx + Ndy \longrightarrow 2)$$

Along OA :-

$y = 0$, $dy = 0$

x varies from 0 to 1

$$\therefore \int_{OA} Mdx + Ndy = \int_{x=0}^{1} 3x^2 dx \quad (\because y = 0 \; \& \; dy = 0$$

$$= (x^3)_0^1$$

$$= 1$$

Along AB

we have $x+y=1 \Rightarrow y=1-x$

$\Rightarrow dy = -dx$

x varies from 1 to 0

$$\int_{AB} M\,dx + N\,dy = \int_{OA} [(3x^2-8y^2)dx + (4y-6xy)dy]$$

$$= \int_{x=1}^{0} [(3x^2 - 8(1-x)^2)dx + (4(1-x) - 6x(1-x))(-dx)]$$

$$= \int_{1}^{0} [3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2]\,dx$$

$$= \int_{1}^{0} (-11x^2 + 26x - 12)\,dx$$

$$= \left(-11\frac{x^3}{3} + 26\frac{x^2}{2} - 12x\right)_1^0$$

$$= 0 - \left(-\frac{11}{3} + 13 - 12\right)$$

$$= \frac{11}{3} - 1$$

$$= \frac{8}{3} \longrightarrow 3)$$

Along BO,

$x=0$, $dx=0$

and y varies from 1 to 0.

$$\therefore \int_{BO} (M\,dx + N\,dy) = \int_{y=1}^{0} 4y\,dy$$

$$= -2 \longrightarrow (4)$$

Hence

$$\int_{C} M\,dx + N\,dy = 1 + \frac{8}{3} - 2$$

$$= \frac{5}{3} \longrightarrow (5)$$

$\therefore$ from 1) and 5), we get

$$\int_{C} M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence, Green's theorem is verified.

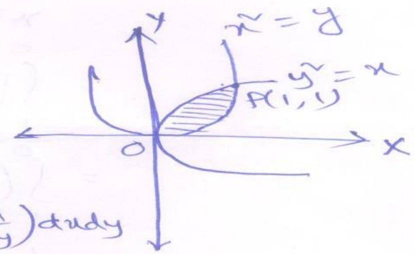
2. Using Green's theorem, evaluate the integral $\int_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$, where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Sol: Given that $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$

$$= \int_C M\,dx + N\,dy$$

where $M = 2xy - x^2$, $N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x$$



By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (2x - 2x)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 0\,dx\,dy$$

$$= 0$$

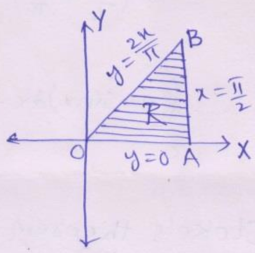
i.e $\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy = 0$

3. Evaluate by Green's theorem $\oint_C [(y-\sin x)dx+\cos x dy]$ where C is the triangle enclosed by the lines $y=0$, $x=\frac{\pi}{2}$, $y=\frac{2x}{\pi}$

Sol. Given that $\oint_C [(y-\sin x)dx+\cos x\, dy] = \int_C M dx + N dy$

where $M = y-\sin x$, $N=\cos x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$



By Green's theorem,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

$$= \int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1) dy dx$$

$$= -\int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2x}{\pi}} (1+\sin x) dy dx$$

$$= -\int_0^{\frac{\pi}{2}} (1+\sin x) \left[y\right]_0^{\frac{2x}{\pi}} dx$$

$$= -\int_0^{\frac{\pi}{2}} (1+\sin x)\left(\frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi} \int_0^{\frac{\pi}{2}} x(1+\sin x) dx$$

$$= -\frac{2}{\pi}\left[x(-\cos x + x) - 1.\left(-\sin x + \frac{x^2}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi}\left[\frac{\pi}{2}\left(0+\frac{\pi}{2}\right) + 1 - \frac{\pi^2}{8}\right]$$

$$= -\frac{2}{\pi}\left(\frac{\pi^2}{4} - \frac{\pi^2}{8} + 1\right)$$

$$= -\frac{2}{\pi}\left(\frac{\pi^2}{8} + 1\right)$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

$$\therefore \oint_C [(y - \sin x)\,dx + \cos x\,dy] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Stoke's theorem

If S is an open surface bounded by a closed curve C and if $\bar{F}$ is any continuously differentiable vector field, then

$$\oint_C \bar{F}.d\bar{r} = \iint_S \text{curl}\,\bar{F}.\bar{n}\,ds$$

where $\bar{n}$ is outward drawn unit normal at any point of S.

Problems :-

1. verify Stoke's theorem for $\bar{F} = (2x-y)\,\bar{i} + yz^2\,\bar{j} - y^2 z\,\bar{k}$ over the upper half of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy plane.

Sol. Given $\bar{F} = (2x-y)\,\bar{i} - yz^2\,\bar{j} - y^2 z\,\bar{k}$

The bound C of S is a circle in xy plane i.e $z=0$

By stoke's theorem

$$\int_C \bar{F}\cdot d\bar{r} = \int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$$

$$\operatorname{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \bar{i}(-2yz+2yz) - \bar{j}(0-0) + \bar{k}(0+1)$$

$$= \bar{k}$$

$$\therefore \int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds = \iint_S \bar{k}\cdot\bar{n}\,ds$$

$$= \iint_S dx\,dy$$

$= \pi$ (which is area of the circle $x^2+y^2=1$ i.e $\pi(1^2) = \pi$)

$\longrightarrow 1)$

Now, we have to evaluate the integral $\int_C \bar{F} \cdot d\bar{r}$ where C is the circle $x^2+y^2=1$

Then $x = \cos\theta$, $y = \sin\theta$ $(\because r=1)$

$\Rightarrow dx = -\sin\theta \, d\theta$, $dy = \cos\theta \, d\theta$

θ varies from 0 to 2π over the circle.

$$\int_C \bar{F} \cdot d\bar{r} = \int_C [(2x-y)\,dx - yz^2\,dy - y^2 z\,dz]$$

$$= \int_C (2x-y)\,dx, \quad \text{since } z=0,\ dz=0$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta)$$

$$= \int_0^{2\pi} (\sin^2\theta - 2\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1-\cos 2\theta}{2} - \sin 2\theta\right] d\theta$$

$$= \left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) + \frac{1}{2} - 0 - \frac{1}{2} \quad (\because \cos 4\pi = 1,\ \sin 4\pi = 0$$

$$= \pi \longrightarrow 2)$$

from 1) & 2): $\int_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl}\,\bar{F} \cdot \bar{n}\,ds$.

Hence, ~~Green's~~ stoke's theorem is verified.

2. Verify Stokes theorem for the function $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$, integrated round the rectangle in the plane $z=0$ and bounded by the lines $x=0$, $y=0$, $x=a$ and $y=b$.

Sol. By stoke's theorem

$$\oint_C \bar{F}.d\bar{r} = \int_S \text{curl}\,\bar{F}.\bar{n}\,ds$$

Here

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(2y+2y)$$

$$= 4y\bar{k}$$

$$\iint_S \text{curl}\,\bar{F}.\bar{n}\,ds = \iint_S 4y\bar{k}.\bar{n}\,ds$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dx\,dy$$

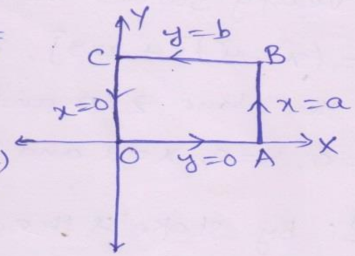
$$= \int_0^a (2y^2)_0^b\,dx$$

$$= 2b^2\int_0^a dx = 2b^2(x)_0^a$$

$$= 2ab^2 \longrightarrow 1)$$

Here C is the rectangle OABC

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r} \longrightarrow 2)$$



Here $\bar{F}\cdot d\bar{r} = (x^2 - y^2)\,dx + 2xy\,dy$

Along OA

$y = 0,\ dy = 0$

x varies from 0 to a

$$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

Along AB

$x = a,\ dx = 0$

y varies from 0 to b

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b 2ay\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b$$

$$= 2a\left(\frac{b^2}{2}\right)$$

$$= ab^2$$

Along BC :

$y = b$, $dy = 0$

x varies from a to 0

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_a^0 (x^2 - b^2)\, dx$$

$$= \left(\frac{x^3}{3} - xb^2\right)_a^0$$

$$= 0 - \left(\frac{a^3}{3} - ab^2\right)$$

$$= -\frac{a^3}{3} + ab^2$$

Along CO :

$x = 0$, $dx = 0$

y varies from b to 0

$$\int_{CO} \bar{F} \cdot d\bar{r} = \int_b^0 0\, dy$$

$$= 0$$

substitute all these in eq 2), we get

$$\oint_C \bar{F} \cdot d\bar{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \longrightarrow 3)$$

$\therefore$ from 1) and 3), $\oint_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl} \bar{F} \cdot \bar{n}\, ds$

Hence, Stokes theorem is verified.