MAXWELL'S EQUATION

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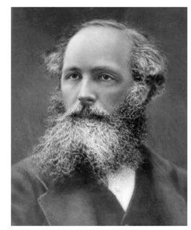
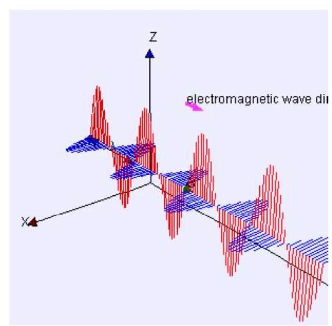
Maxwell's Equations

- The equations are named after the physicist and mathematician James Clerk Maxwell, who between 1861 and 1862 published an early form of the equations that included the Lorentz force law.
- Maxwell's field equations are a set of partial differential equations that describe how electric and magnetic fields are generated by charges, currents, and changing fields.
- One important consequence of these equations is that they demonstrate how fluctuating electric and magnetic fields propagate at the speed of light.
- This eventually led to the identification of light waves as electromagnetic in nature.

Electromagnetic theory of light



James Clerk Maxwell



Maxwell's Equations [instantaneous, differential form]

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \text{(Ampere's law)}$$

$$\nabla \cdot \boldsymbol{D} = \rho \qquad \text{(Gauss's law)}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{(Gauss's law for magnetic fields)}$$

Electromagnetic Fields

$\vec{E}$: Electric Field Intensity (V/m)

$\vec{D}$: Electric Flux Density (C/m^2)

$\vec{H}$: Magentic Field Intensity (A/m)

$\vec{B}$: Magnetic Flux Density $(T) or (Wb/m^2)$

$\vec{J}$: Current Density (A/m^2)

V : Electric Potential (V)

ρ_v : Volume Charge Density (C/m^3)

Permeability of free space- $\mu_0 = 4\pi \times 10^{-7} (N/A^2)$

Permittivity of free space- $\varepsilon_0 = 8.85 \times 10^{-12} (C^2/Nm^2)$

Physical Significance of Maxwell Equation

Gauss's Law of Electrostatic	$\oint_S D \cdot ds = \oint_V \rho_v \, dv = Q$	Electric flux through a closed surface is proportional to the charged enclosed
Gauss's Law of magneto static	$\oint_S B \cdot ds = 0$	The total magnetic flux through a closed surface is zero
Faraday's law of Induction	$\oint_L E \cdot dl = -\frac{\partial}{\partial t} \iint_S B \cdot ds$	Changing magnetic flux produces an electric field
Maxwell's modified Ampere's Circuital Law	$\oint_L H \cdot dl = \iint_S \left(J + \frac{\partial D}{\partial t} \right) \cdot ds$	Electric current and changing electric flux produces a magnetic field

4.2 Electromagnetic Wave Equation

The application of Maxwell's equations is the prediction of existence of electromagnetic wave. Electromagnetic wave equation can be obtained from Maxwell's equations.

The Maxwell's equation from Faraday's law is given by,

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$H = \frac{B}{\mu} \longrightarrow$ *Magnetic field strength*

$$= -\mu \frac{\partial H}{\partial t}$$

Take curl on both sides,

$$\nabla \times \nabla \times E = -\mu \nabla \times \frac{\partial H}{\partial t} \qquad \text{... (4.1)}$$

But Maxwell's equation from Ampere's law is

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$D = \varepsilon E \longrightarrow$ *Electric Displacement vector*

J = σ **E** of Ohm's law

$$= \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

Differentiating

$$\nabla \times \frac{\partial H}{\partial t} = \frac{\partial}{\partial t}\left(\sigma E + \varepsilon \frac{\partial E}{\partial t}\right)$$

$$\nabla \times \frac{\partial H}{\partial t} = \sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2} \quad \text{... (4.2)}$$

Substituting the equation (4.2) in equation (4.1)

$$\nabla \times \nabla \times E = -\mu\left[\sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2}\right]$$

$$= -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \quad \text{... (4.3)}$$

But according to the identity

$$\nabla \times \nabla \times E = \nabla(\nabla \bullet E) - \nabla^2 E \quad \text{... (4.4)}$$

But $\nabla \bullet E = \frac{1}{\varepsilon} \nabla \bullet D$

Since there is not net charge within the conductor, the charge density $\rho = 0$.

$$\nabla \bullet D = 0$$

$$\nabla \bullet E = 0$$

Then equation (4.4) becomes

$$\nabla \times \nabla \times E = -\nabla^2 E \quad \text{... (4.5)}$$

$$\nabla \times \nabla \times E = -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \quad \text{... (4.3)}$$

Comparing the equations (4.3) and (4.5)

$$\nabla^2 E = -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

$$\boxed{\nabla^2 E - \mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0} \quad \text{... (4.6)}$$

This is the wave equation for electric field E.

The wave equation for Magnetic field H is obtained in a similar manner as follows.

The Maxwell's equation from Ampere's law is given by,

$$\nabla \times H = \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

Take curl on both sides,

$$\nabla \times \nabla \times H = \sigma \nabla \times E + \varepsilon \nabla \times \frac{\partial E}{\partial t} \quad \text{... (4.7)}$$

But Maxwell's equation from Faraday's law

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

Differentiating, $\nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial H^2}{\partial t^2}$

Substituting the values of $\nabla \times E$ and $\nabla \times \frac{\partial E}{\partial t}$ in equation (4.7)

$$\nabla \times \nabla \times H = -\mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} \quad \text{... (4.8)}$$

But the identity is

$$\nabla \times \nabla \times H = \nabla(\nabla \bullet H) - \nabla^2 H$$

But $\nabla \bullet B = \mu \nabla \bullet H = 0$

Then, $$\nabla \times \nabla \times H = \nabla^2 H \quad \ldots (4.9)$$

Comparing the equations (4.8) and (4.9)

$$\nabla \times \nabla \times H = -\mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} \quad \ldots (4.8)$$

$$\nabla^2 H = -\mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2}$$

$$\boxed{\nabla^2 H - \mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0} \quad \ldots (4.10)$$

This is the wave equation for magnetic field H.

4.3 Wave Equation for Free Space

For free space (dielectric medium) the conductivity of the medium is zero. (i.e., $\sigma = 0$) and there is no charge containing in it (i.e, $\rho = 0$). The electromagnetic wave equations for free space can be obtained from Maxwell's equations.

$$\boxed{\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0} \quad \ldots (4.11)$$

This is the wave equation for free space interms of Electric field.

The wave equation for free space in terms of magnetic field H is obtained in a similar manner as follows.

$$\boxed{\nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0} \quad \ldots (4.12)$$

For free space $\mu_r = 1$ and $\varepsilon_r = 1$ (air)

Then the wave equation becomes

$$\boxed{\nabla^2 H - \frac{1}{c^2} \frac{\partial^2 H}{\partial t^2} = 0} \quad \text{or} \quad \boxed{\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0}$$