Phase and Group velocity of EM waves

Tools required:

http://demonstrations.wolfram.com/GroupAndPhaseVelocity/

Objective:

To understand the nature of EM waves travelling in a medium with the help of Phase and Group velocities.

Theory: Any real signal consists of travelling-waves of many different frequencies, which travel together as a group, at a speed that will always be less than or equal to the speed of light in vacuum. To gain some insight into what may happen when a real signal travels through a dispersive medium, we consider adding two waves of equal amplitude. When two travelling waves with unit amplitude $f_1(z,t) = \cos(k_1 z - \omega_1 t)$ and $f_2(z,t) = \cos(k_2 z - \omega_2 t)$ are added, we get

$$f_1(z,t) + f_2(z,t) = \cos(k_1 z - \omega_1 t) + \cos(k_2 z - \omega_2 t)$$

$$= 2\cos\left(\frac{\Delta k}{2} z - \frac{\Delta\omega}{2} t\right)\cos\left(\bar{k}\cdot z - \bar{\omega}\cdot t\right)$$

Where, $\frac{\Delta k}{2} \equiv \frac{k_1 - k_2}{2}$, $\frac{\Delta\omega}{2} \equiv \frac{\omega_1 - \omega_2}{2}$, $\bar{k} \equiv \frac{k_1 + k_2}{2}$ and $\bar{\omega} \equiv \frac{\omega_1 + \omega_2}{2}$

The result is a fast oscillating wave that travels with a phase velocity $v_p = \frac{\bar{\omega}}{\bar{k}}$ and the amplitude of this wave is being modulated in space and time by $2\cos\left(\frac{\Delta k}{2} z - \frac{\Delta\omega}{2} t\right)$. This modulated wave travels at the group velocity given by $v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k}$

Observation table

S. No	Δω	Δk	Wave pattern of the resultant waves	V_g
1	0.02	0.02		
2	0.04	0.04		
3	0.06	0.06		
4	0.08	0.08		
5	0.1	0.1		
6	0.2	0.2		
7	0.3	0.3		
8	0.4	0.4		
9	0.5	0.5		

Inferences:

1) Are the wave patterns for various values of Δω and Δk same? If not, why?
2) Comment on the Phase velocity (Vp) of the waves for increased values of Δω and Δk.
3) When do we see V_p and V_g being the same.
4) Draw a typical dispersion relation curve (ω-k curve) for $V_p = V_g$ and $V_p \neq V_g$ cases.