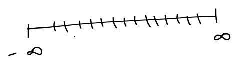
Maxima and minim

Monday, 15 March 2021 12:00 PM

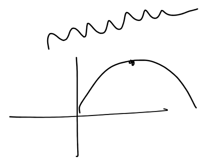
Recall: $f(x)$ on $\mathbb{R}^1$



① $f(x) = 2x$ on $\mathbb{R}$

$f(a) \geq f(x)$

maximum



$z = f(x, y)$ $\quad x, y \in D(\mathbb{R}^2)$

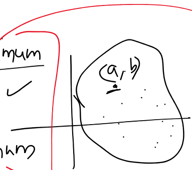
Def: Let $f(x,y)$ defined on a region D containing a point (a,b) then

(i) $f(a,b)$ is a local maximum

↳ $\boxed{f(a,b)} \geq f(x,y)$ ✓

$f(a,b)$ is a local minimum

↳ $f(a,b) \leq f(x,y)$



$f(a,b) \geq f(x,y)$

$f(a,b) \leq f(x,y)$

Facts:

local extrema are based

(i) Boundary points. end-points. $[a, b]$ ✓

(ii) critical points

Problem:

Find local extreme values of $f(x,y) = x^2 + y^2 - 4y + 9$

$$\frac{\partial f(x,y)}{\partial x} = 2x$$

$$0 = 2x$$

$$\boxed{x = 0}$$

$$\frac{\partial f(x,y)}{\partial y} = 2y - 4$$

$$0 = 2y - 4$$

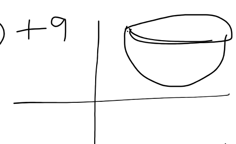
$$y = 4/2 = 2$$

$$y = 2$$

The critical point $(0, 2)$

$$f(0,2) = 0^2 + 2^2 - 4(2) + 9$$

$$\boxed{f(0,2) = 5}$$



$(x, y, f(x,y))$

$$f(x,y) = x^2 + y^2 - 4y + 9$$

$$f(1,1) = 1 + 1 - 4 + 9$$

$$= 2 - 4 + 9$$

$$= 7 > f(0,2)$$

$$5$$

Second

$$f(x,y) = x^2 + y^2 - 4y + 9 \checkmark$$

$$= x^2 + y^2 - 4y + 4 - 4 + 9$$

$$= x^2 + (y-2)^2 + 5$$

$$f(0,2) = 5 \le f(x,y) \le f(1,1)$$

$$5 \le 7$$

② Find local extreme values (if any)

$$f(x,y) = y^2 - x^2$$ [No domain]

Step 1:

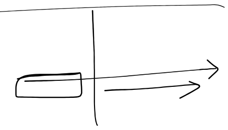
$$\frac{\partial f(x,y)}{\partial x} = -2x \qquad \frac{\partial f(x,y)}{\partial y} = 2y$$

$$\boxed{x = 0} \qquad \boxed{y = 0}$$

The critical point $(0,0)$.

$$f(x,0) = -x^2 < 0$$

$$f(0,y) = y^2 > 0$$

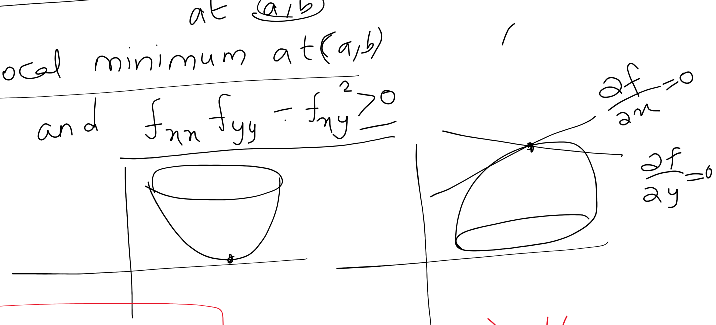


(i) $f(x,y)$ has local maximum at (a,b)

↳ $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b)

(ii) $f(x,y)$ has local minimum at (a,b)

↳ $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$



(iii) $\boxed{f_{xx} f_{yy} - f_{xy}^2 < 0}$ at (a,b) then point (a,b) is saddle.

(iv) $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b) then the test is inconclusive.

Problem:

(1) $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Step 1: find critical points.

$$\frac{\partial f}{\partial x} = -2x + y - 2$$

$$0 = 2x - 2 + y \quad (1)$$

$$\frac{\partial f}{\partial y} = x - 2y - 2$$

$$0 = x - 2y - 2 \quad (2)$$

$$x = -2 \qquad y = -2$$

$\therefore$ The critical point is $(-2, -2)$.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(-2x + y - 2)$$

$$= -2 \quad (f_{xx})$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x - 2y - 2)$$

$$= -2 \quad (f_{yy})$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x - 2y - 2)$$

$$= 1 \quad (f_{xy})$$

The discriminant $(f_{xx} f_{yy} - f_{xy}^2)$.

$$\therefore f_{xx} = -2 < 0$$

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2$$

$$= 4 - 1$$

$$= 3 > 0$$

$$\therefore \boxed{f_{xx} < 0 \ \& \ f_{xx} f_{yy} - f_{xy}^2 > 0}$$

local maximum.

$2(-2)+4$

local maximum.

$$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$
$$= 8$$

H.W

Find local extreme values

$$f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$$

Sol: ① Find critical points

$(0,0)$ and $(2,2)$

② $f_{xx} f_{yy} - f_{xy}^2 = 72(y-1)$

$f_x = -6x + 6y$ | $f_y = 6y - 6y^2 + 6x$

$f_{xx} = -6$ | $f_{yy} = 6 - 12y$

$$\frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(6y - 6y^2 + 6x)$$
$$= 6.$$

Discriminant $= f_{xx} \cdot f_{yy} - f_{xy}^2$

$$= (-6)(6-12y) - 6^2$$
$$= -36 + 72y - 36$$
$$= -72 + 72y$$

$$\boxed{f_{xx} f_{yy} - f_{xy}^2 = 72[y-1]}$$

At critical point $(0,0)$

At critical point $(0,0)$

$$\boxed{f_{xx}f_{yy}-f_{xy}^2 = 72(0-1) = -72 < 0}$$

Saddle point

At critical point $(2,2)$

$$f_{xx}\cdot f_{yy}-f_{xy}^2 = 72(y-1)$$
$$= 72(2-1)$$
$$= 72 > 0$$

$f_{xx} < 0$ and $f_{xx}f_{yy}-f_{xy}^2 > 0$

Then it has local maximum.

$$f(x,y) = 3y^2-2y^3-3x^2+6xy$$

$$\frac{\partial f(x,y)}{\partial x} = -6x+6y$$
$$0 = -6x+6y \quad -(1) \longrightarrow \quad 6x = 6y$$
$$x = y$$

$$\frac{\partial f(x,y)}{\partial y} = 6y-6y^2+6x$$
$$0 = -6y^2+6y+6x \quad -(2)$$