LAPLACE TRANSFORM

PROF. INDRAVA ROY

Lecture 4: Improper Riemann integrals: Definition and Existence Part 2

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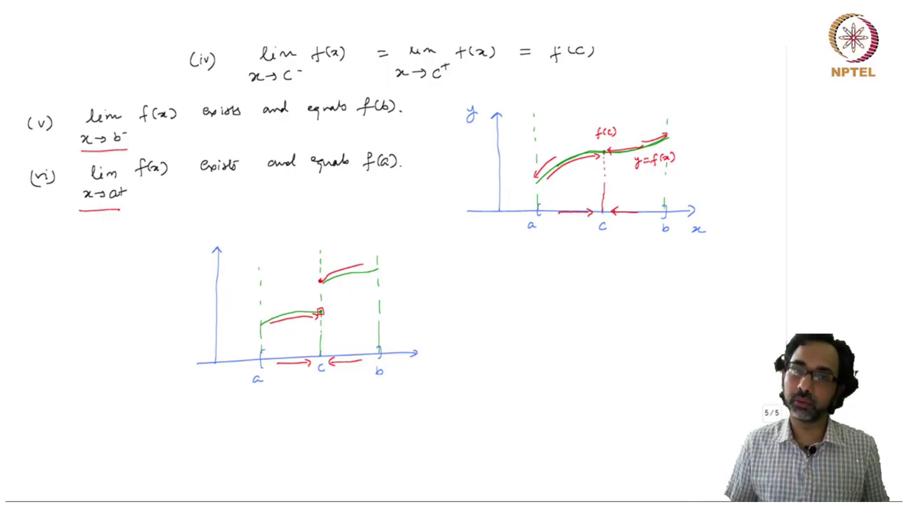

Now, another sufficient condition is when the function may not be continuous but it can be broken into finitely many pieces where it is continuous.

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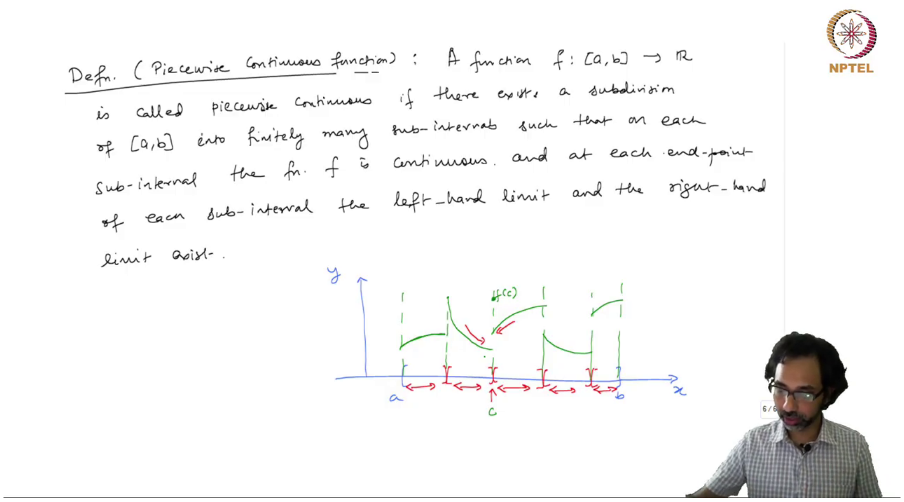
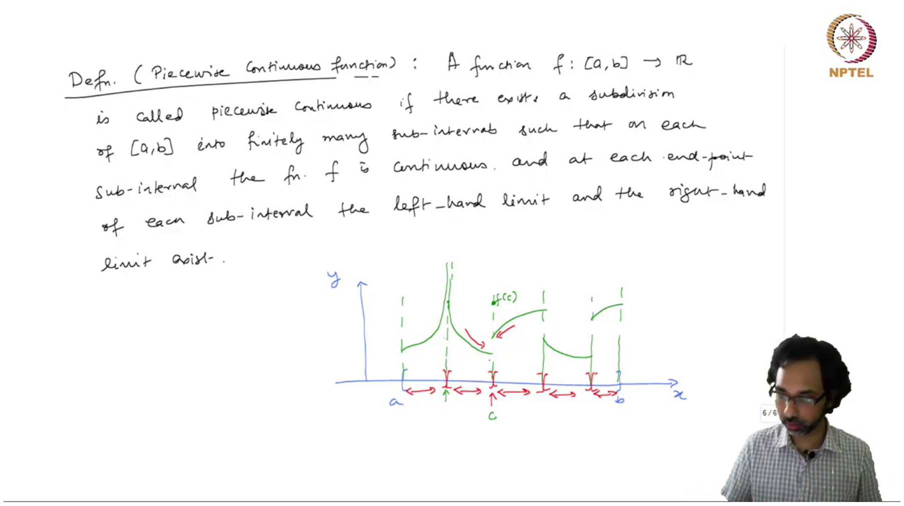

NPTEL
Defn. (Piecewise Continuous function) : A function f : [a,b] → ℝ is called piecewise continuous if there exists a subdivision of [a,b] into finitely many sub-intervals such that on each sub-interval the fn. f is continuous. and at each end-point of each sub-interval the left-hand limit and the right-hand limit exist.
y
f(c)
a
c
b
x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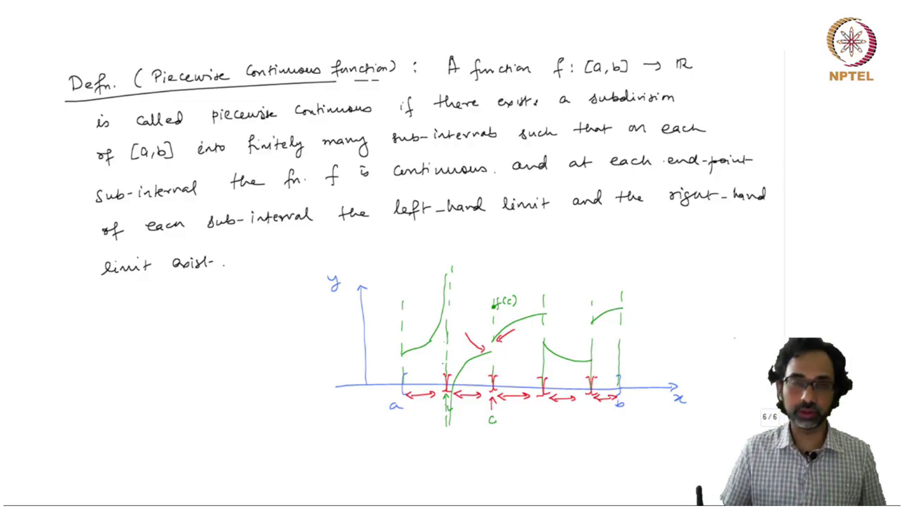
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y
f(c)
a
c
b
x

So, this is the definition of a piecewise continuous function. So, a function let us call it $f : [a, b] \to \mathbb{R}$ is called piecewise continuous if, there exists a subdivision of this interval $[a, b]$ into finitely many sub intervals such that, on each such sub interval the function f is continuous and at each end point of this set of all sub interval the left-hand limit and the right-hand limit exists but, they may not be equal.

So, we have already seen an example so, let me redraw so this is our a, this is our b. So, if we can divide this interval $[a, b]$ into finitely many pieces like this so, each such sub interval is a part of the subdivision. Now, our graph of this function f which is now a piecewise continuous function will look like something like this. So, on each piece it is a continuous function and the left-hand side and the right-hand side limit exist which mean that they are finite limits.

So, here the first chunk looks like this but, the second chunk may look like this, the third chunk may look like this, the fourth chunk may look like this, and so on. So, this is on each subdivision it is a continuous function and when you approach at any end point let us, say this one at any end point approach it from the left and you approach it from the right the limit exist but, they may not be equal in case of a continuous function they were equal and there were equal also the left and the right hand side limit were also equal to the value of the function at this end point but, here also they may not be equal.

So, first of all the left-hand and the right-hand side limits may not be equal and they may not be equal to the value of the function at this point which may lie here so, this is if this point is c, this is can be $f(c)$ and this is the left limit and this is the right limit and all three are different still, the left-hand and the right-hand side limits exist and this is all we want for a piecewise continuous function.

So, what is not allowed here is if you want to have say the if I delete this first chunk here now let us, say it starts here but, at this end point it goes to $+\infty$ as an asymptote, it has an asymptotic value plus infinity at this end point. So, this kind of behavior is not allowed. So, similarly you cannot have $-\infty$ as a vertical asymptote going to $-\infty$ at this end point or at any end points like this. So, this kind of behavior is not allowed.

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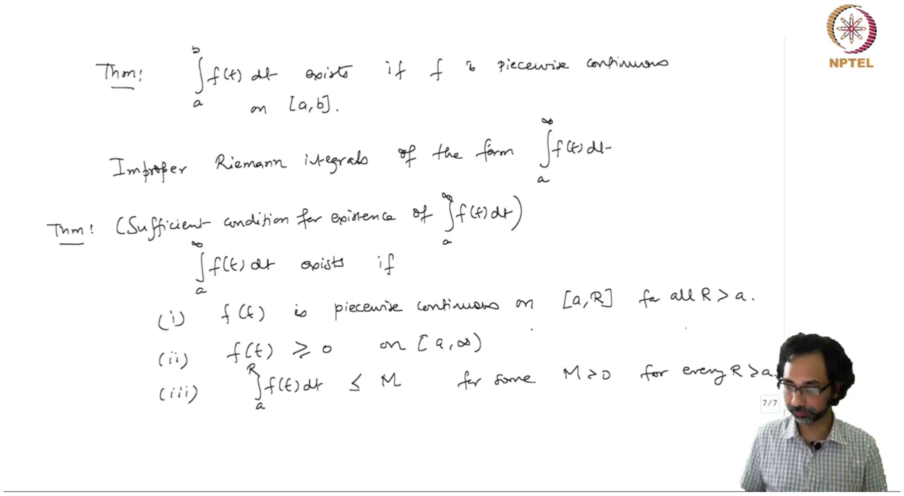

Once, we know the definition of piecewise continuous function we can state a sufficient condition that integral $\int_a^b f(t)dt$ exists if f is piecewise continuous on this interval $[a, b]$. So, this gives us sufficient conditions for the existence of Riemann integrals on closed and bounded intervals.

Now, for improper Riemann integrals we can state a sufficient condition. Improper Riemann integrals of the form $\int_a^\infty f(t)dt$ so, we can state this theorem which is a sufficient condition for existence of this integral. So, we can write that $\int_a^\infty f(t)dt$ exists if, first condition the function $f(t)$ is piecewise continuous on the interval $[a, R]$ for all $R > a$.

Secondly, we suppose that $f(t) \geq 0$ on $[a, \infty)$ so, we suppose that the function is positive and thirdly, we suppose that $\int_a^R f(t)dt \leq M$, for some $M > 0$ every $R > a$. So, if these three conditions are met then we can assured that our improper integral $\int_a^\infty f(t)dt$ will exist. So, this is one sufficient condition.

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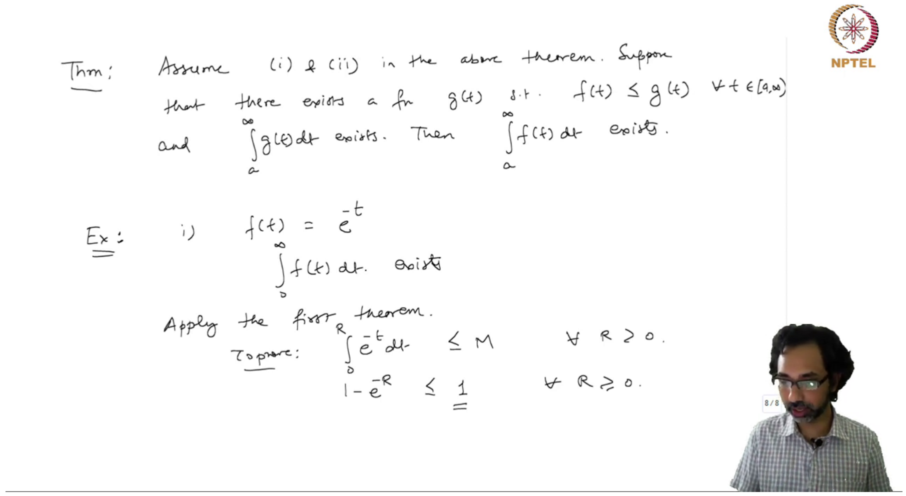

Another sufficient condition is given as follows. So, assume 1 and 2 in the above theorem meaning that first that, $f(t)$ is piecewise continuous on each interval of the form $[a, R]$ for $R > a$ and it is positive on $[a, \infty)$. Now, we suppose that there exists a function $g(t)$ such that $f(t) \le g(t)$ for all $t \in [a, \infty)$ so, $f(t)$ is bounded above by this function $g(t)$ for all t and the improper integral $\int_a^\infty g(t)dt$ exists then, we can say that the improper integral $\int_a^\infty f(t)dt$ also exists.

So, this is a kind of comparison theorem where you have two functions and the right-hand side function you know that it is improper Riemann integral exists then the improper Riemann integral of the left-hand function is also finite and exists. So, these are two sufficient conditions that I wanted to know and they can be used to check whether some integrals are exists as an improper Riemann integral or not.

So, let me give a couple of examples. So, the first case is $f(t) = e^{-t}$. Now, we already know that this improper Riemann integral $\int_0^\infty f(t)dt$ exists we have computed it using the limits but, we can also apply the first theorem as follows. So, what we need to prove that $\int_0^\infty e^{-t}dt$ is bounded above by some M for all $R > 0$. So, this is what we want.

So, let us see if this is true what is this integral, we can evaluate it directly and we find that this is $1 - e^{-R}$. So, of course it is very easy to see that this is always less than or equal to 1 for all $R > 0$. So therefore, we have found this upper bound M which is equal to 1 in this case and therefore, we can say that this improper Riemann integral exists.

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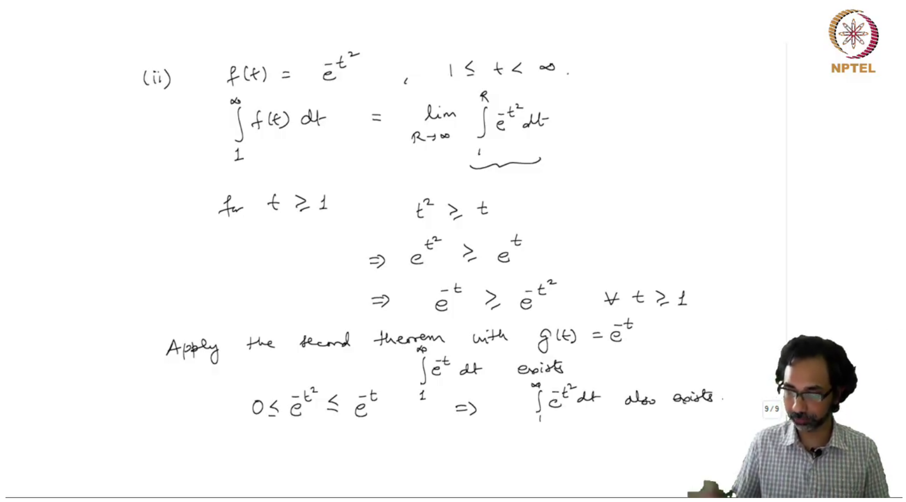

Another example is as follows, $f(t) = e^{-t^2}$, $1 \le t < \infty$. So, let us see what is whether this integral $\int_1^\infty f(t)dt$ exists or not. So, this is equal to $\lim_{R\to\infty} \int_1^R e^{-t^2} dt$ this is the definition of the improper Riemann integral. Now, note that in the first example e^{-t} we can evaluate this integral explicitly and we can answer the question using the first theorem but, here we do not know how to evaluate it explicitly so, we will use the second theorem here.

So, note that for $t \ge 1$, $t^2 \ge t$. Therefore, $e^{t^2} \ge e^t$ and this implies that $e^{-t} \ge e^{-t^2}$ for all $t \ge 1$.

Now, we can apply the second theorem with $g(t) = e^{-t}$ we know that, $\int_1^\infty e^{-t}dt$ exists, in the same fashion as we did for 0 to ∞ we can also evaluate it from 1 to ∞ and one can show that this exists, and we have shown that this is $e^{-t^2} \le e^{-t}$ and these are both positive therefore, by the second theorem $\int_1^\infty e^{-t^2}dt$ also exists.