Convolution

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Convolution of two functions.

Definition

The *convolution* of piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Remarks:

- $f * g$ is also called the generalized product of f and g.

Example-1

$$f(t) = e^{3t} \qquad \text{and} \qquad g(t) = e^{7t} \quad .$$

Since we will use $f(x)$ and $g(t-x)$ *in computing the convolution, let us note that*

$$f(x) = e^{3x} \qquad \text{and} \qquad g(t-x) = e^{7(t-x)} \quad .$$

So,

$$\begin{aligned} f * g(t) &= \int_{x=0}^{t} f(x)g(t-x)\,dx \\ &= \int_{x=0}^{t} e^{3x}e^{7(t-x)}\,dx \\ &= \int_{x=0}^{t} e^{3x}e^{7t}e^{-7x}\,dx \end{aligned}$$

$$= e^{7t} \int_{x=0}^{t} e^{-4x}\, dx$$

$$= e^{7t} \cdot \frac{-1}{4} e^{-4x} \Big|_{x=0}^{t}$$

$$= \frac{-1}{4} e^{7t} e^{-4t} - \frac{-1}{4} e^{7t} e^{-4 \cdot 0} = \frac{-1}{4} e^{3t} + \frac{1}{4} e^{7t} \quad .$$

Now Applying the Linear Transform

$$\mathscr{L}\left[(f * g)(t)\right] = \frac{1}{4}\frac{1}{s-7} - \frac{1}{4}\frac{1}{s-3}$$

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Example

Find the convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: By definition: $(f * g)(t) = \displaystyle\int_0^t e^{-\tau} \sin(t - \tau)\, d\tau$.

Integrate by parts twice: $\displaystyle\int_0^t e^{-\tau} \sin(t-\tau)\, d\tau =$

$$\Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t - \int_0^t e^{-\tau}\sin(t-\tau)\, d\tau,$$

$$2\int_0^t e^{-\tau}\sin(t-\tau)\, d\tau = \Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t,$$

$$2(f * g)(t) = e^{-t} - \cos(t) - 0 + \sin(t).$$

We conclude: $(f * g)(t) = \dfrac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big]$. $\triangleleft$

Laplace Transform

$$\mathscr{L}\left[(f * g)(t)\right] = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s^2+1} - \frac{s}{2}\frac{1}{s^2+1}$$