Greens Theorem:- If R is a closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

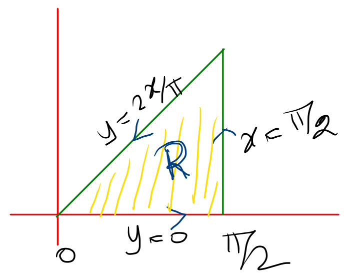
where C is travesed in anticlockwise direction.

Pb: 1 Using Green's theorem Evaluate $\int_C [y-\sin x]\,dx + \cos x\,dy$.

where $c$ is the triangle enclosed by the lines

$y=0$, $x=\pi/2$, $y=\dfrac{2x}{\pi}$.

Soln:-




Let $R$ be the region enclosed by the given curve $C$.

From Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dy\,dx.$$

We need to calculate $\int_C (y-\sin x)\,dx + \cos x\,dy$.

Here $M = y - \sin x$; $N = \cos x$ $\therefore \dfrac{\partial N}{\partial x} = -\sin x$; $\dfrac{\partial M}{\partial y} = 1$

$$\therefore \int_C (y-\sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dy\,dx.$$

$\therefore$ The limits of double integrations are

$$0 \le x \le \pi/2; \quad 0 \le y \le 2x/\pi.$$

$$\therefore \iint_R (-\sin x - 1)\,dy\,dx = \int_0^{\pi/2}\int_0^{2x/\pi} (-\sin x - 1)\,dy\,dx$$

$$= \int_0^{\pi/2} \frac{2x}{\pi}(-\sin x - 1)\,dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(1+\sin x)\,dx = -\pi/4 + 2/\pi.$$

**Stokes Theorem:-**

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\vec{F}$ be a vector field whose components have continuous first order partial derivatives on an open region containing S then

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dS = \int_C (\nabla\times\vec{F})\cdot\vec{N}\,dS$$

$\vec{N}$ represents the normal vector.

Pb: 1 verify Stoke's theorem for the vector field $\vec{F} = y\vec{i} - x\vec{j}$ for the hemisphere

$S: x^2+y^2+z^2 = 9, \quad z \geq 0$ its bounding circle

$C: x^2+y^2 = 9, \quad z = 0.$

Soln:- we need to verify $\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl} \vec{F} \cdot \vec{N} \, ds$

Left Hand side:- The parameterization of given $C$ is

$x = 3\cos t \; ; \; y = 3\sin t \quad z = 0 \quad 0 \leq t \leq 2\pi$

$\vec{F} = y\vec{i} - x\vec{j} = 3\sin t \,\vec{i} - 3\cos t \,\vec{j}$ we know

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} = -3\sin t \, dt \,\vec{i} + 3\cos t \, dt \,\vec{j}$

$\therefore \vec{F} \cdot d\vec{r} = [3\sin t \,\vec{i} - 3\cos t \,\vec{j}]\,[-3\sin t \, dt \,\vec{i} + 3\cos t \, dt \,\vec{j}]$

$= -9\sin^2 t - 9\cos^2 t = -9.$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -9 \, dt = -18\pi.$

Right hand side:- Surface is $S: x^2+y^2+z^2=9$, $z \geq 0$

$\therefore$ A parametrization of $S$ is

$$X(s,t) = (3\cos s \sin t,\ 3\sin s \sin t,\ 3\cos t) \quad 0 \leq s \leq 2\pi,\ 0 \leq t \leq \pi/2.$$

We calculate $\int_S \text{curl}\,\vec{F} \cdot \vec{N}(s,t)\,dt$

Now,

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \vec{i}[0] - \vec{j}[0] + \vec{k}[-1-1]$$

$$= -2\vec{k}$$

$$\vec{N}(s,t) = \frac{\partial(y,z)}{\partial(s,t)}\vec{i} - \frac{\partial(x,z)}{\partial(s,t)}\vec{j} + \frac{\partial(x,y)}{\partial(s,t)}\vec{k}$$

$$= \begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix}\vec{i} - \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}\vec{k}$$

$$= -9\cos s \sin^2 t\,\vec{i} - 9\sin s \sin^2 t\,\vec{j} - 9\sin t \cos t\,\vec{k}.$$

$$\therefore \text{curl}\,\vec{F} \cdot \vec{N}(s,t) = 18\sin t\cos t$$

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS = \int_0^{\pi/2} \int_0^{2\pi} 18 \sin t \cos t \, ds\, dt$$

$$= \int_0^{\pi/2} \int_0^{2\pi} 9 \sin 2t \, ds\, dt$$

$$= \int_0^{\pi/2} 9 \sin 2t \,(2\pi)\, dt$$

$$= 18\pi \int_0^{\pi/2} \sin 2t \, dt$$

$$= 18\pi \left[ -\frac{\cos 2t}{2} \right]_0^{\pi/2} = 18\pi \left[ \frac{-\cos 2(\pi/2) + \cos 0}{2} \right]$$

$$= 18\pi.$$

Pb: 2 Using Stoke's theorem show that the line integral $\oint_C yz\,dx + xz\,dy + xy\,dz = 0$ along any closed curve C.

Soln':- We observe the following

$$\oint_C yz\,dx + xz\,dy + xy\,dz = \oint_C \left[yz\vec{i} + xz\vec{j} + xy\vec{k}\right] \cdot \left[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right]$$

$$= \oint_C \vec{F} \cdot d\vec{r}$$

where $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$.

From Stoke's theorem we know $\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl}\,\vec{F} \cdot \vec{N} \cdot ds$

Now, $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \vec{i}\,[x - x] - \vec{j}\,[y - y] + \vec{k}\,[z - z]$

$= 0$.

Thus $\oint_C yz\,dx + xz\,dy + xy\,dz = 0$.

Gauss divergence Theorem:-

(Relates surface integral to triple integral)

Let $\vec{F}$ be a vector field whose components have continuous first order partial derivatives and let S be a piecewise smooth oriented closed surface. Then

$$\int_S \vec{F} \cdot \vec{N} \, ds = \iiint_D \operatorname{div} \vec{F} \, dV$$

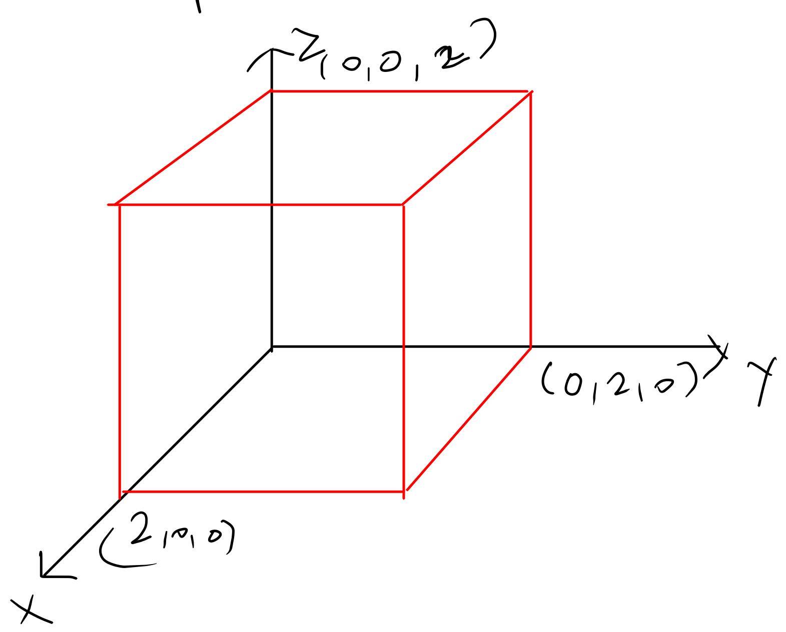
where D is the region enclosed by the surface S.

Pb: 1 Using Gauss divergence theorem Find

$$\int_S \vec{F} \cdot \vec{N} \, ds \quad \text{where} \quad \vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + z\vec{k}$$

and S is the surface of cube bounded by the planes $x = 2$; $y = 2$, $z = 2$. and the co-ordinate planes.

Soln:-



By Gauss divergence theorem

$$\int_S \vec{F}\,\vec{N}\,ds = \iiint_D \operatorname{div}\vec{F}\,dV$$

Given $\vec{F} = (x^3 - yz)\,\vec{i} - 2x^2y\,\vec{j} + z\vec{k}$

$$\operatorname{div}\vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\cdot\left((x^3-yz)\,\vec{i} - 2x^2y\,\vec{j} + z\vec{k}\right)$$

$$= 3x^2 - 2x^2 + 1 = x^2 + 1$$

$$\iiint_D \operatorname{div}\vec{F}\,dV = \int\int\int (x^2+1)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^2\int_0^2 (x^2+1)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^2 (x^2+1)\cdot 2\,dy\,dx = 4\int_0^2 (x^2+1)\,dx$$

$$= 4\left(\frac{x^3}{3} + x\right)_0^2 = 4\left(\frac{8}{3} + 2\right)$$

$$= 56/3$$

Pb. 2 Evaluate $\int_S \vec{F}\cdot\vec{N}\, ds$ using Gauss divergence theorem where $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ over $S: x^2+y^2+z^2 = 9$.

Soln:- By Gauss divergence theorem.

$$\int_S \vec{F}\cdot\vec{N}\, ds = \iiint_D \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= 3.$$

$$\therefore \int_S \vec{F}\cdot\vec{N}\, ds = \iiint_D 3\, dV = 3\iiint_D dz\,dy\,dx$$

[Recall:- $\iiint_D dz\,dy\,dx =$ volume of the solid D]

$= 3 \times$ (volume of sphere with centre $O$ and radius 3)

$$\int_S \vec{F}\cdot\vec{N}\, ds = 3\times\frac{4}{3}\pi\, 3^3 = 108\pi.$$

$\times$ ——————— $\times$.