



Maxima and Minima of the Function of Several Variables

PARTIAL DERIVATIVES

As we saw in Chapter 4 in Thomas Calculus, one of the main uses of ordinary derivatives is in finding maximum and minimum values.

14.7 (Thomas Calculus)

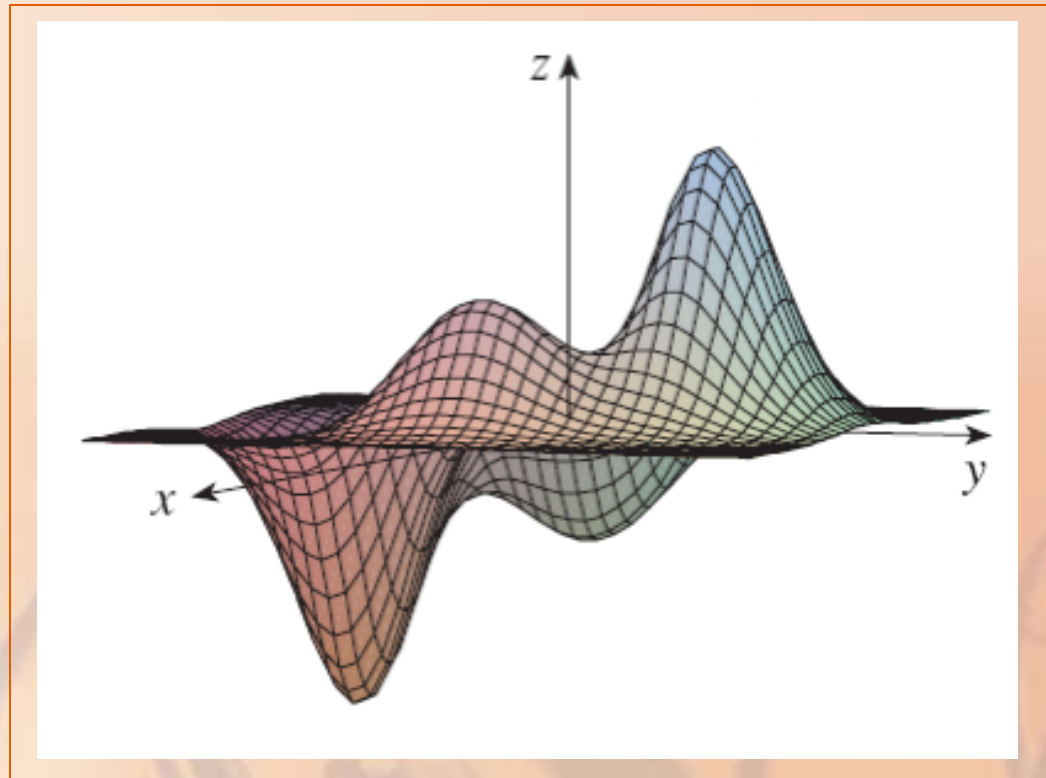
Maximum and Minimum Values

In this section, we will learn how to:

Use partial derivatives to locate
maxima and minima of functions of two variables.

MAXIMUM & MINIMUM VALUES

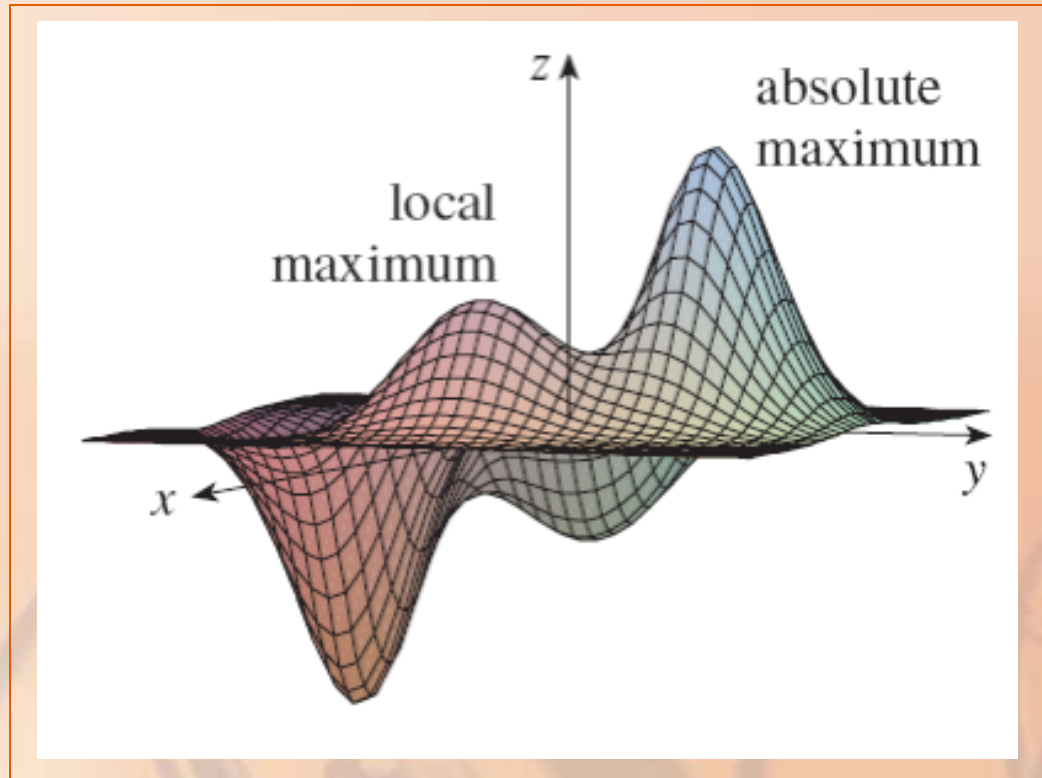
Look at the hills and valleys in the graph of f shown here.



ABSOLUTE MAXIMUM

There are two points (a, b) where f has a local maximum—that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.

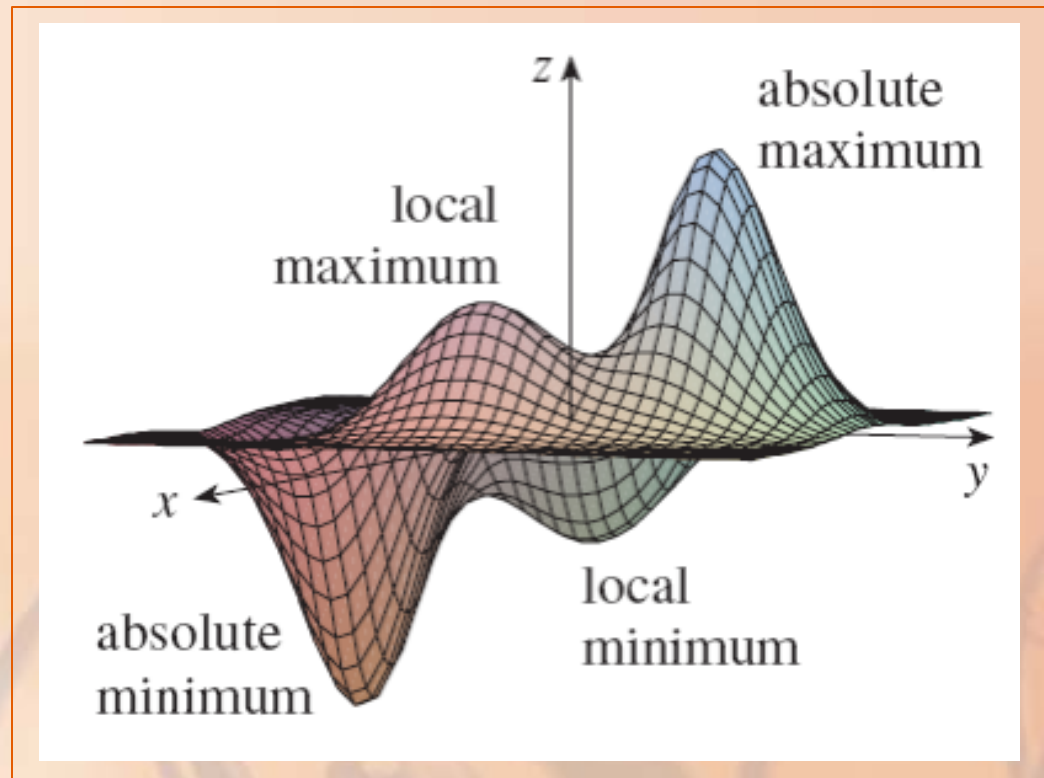
- The larger of these two values is the absolute maximum.



ABSOLUTE MINIMUM

Likewise, f has two local minima—where $f(a, b)$ is smaller than nearby values.

- The smaller of these two values is the absolute minimum.



LOCAL MAX. & LOCAL MAX. VAL. Definition 1

A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) .

This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) .

- The number $f(a, b)$ is called a local maximum value.

LOCAL MIN. & LOCAL MIN. VALUE Definition 1

If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) , then f has a local minimum at (a, b) .

$f(a, b)$ is a local minimum value.

ABSOLUTE MAXIMUM & MINIMUM

If the inequalities in Definition 1 hold for all points (x, y) in the domain of f , then f has an absolute maximum (or absolute minimum) at (a, b) .

LOCAL MAXIMUM & MINIMUM

Theorem 2

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

Let $g(x) = f(x, b)$.

- If f has a local maximum (or minimum) at (a, b) , then g has a local maximum (or minimum) at a .
- So, $g'(a) = 0$ by Fermat's Theorem.

However, $g'(a) = f_x(a, b)$

- See Equation 1 in Section 14.3
- So, $f_x(a, b) = 0$.

LOCAL MAXIMUM & MINIMUM

Proof

Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain:

$$f_y(a, b) = 0$$

LOCAL MAXIMUM & MINIMUM

If we put $f_x(a, b) = 0$ and $f_y(a, b) = 0$ in the equation of a tangent plane (Equation 2 in Section 14.4), we get:

$$z = z_0$$

THEOREM 2—GEOMETRIC INTERPRETATION

Thus, the geometric interpretation of Theorem 2 is:

- If the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

CRITICAL POINT

A point (a, b) is called a critical point (or stationary point) of f if either:

- $f_x(a, b) = 0$ and $f_y(a, b) = 0$
- One of these partial derivatives does not exist.

CRITICAL POINTS

Theorem 2 says that, if f has a local maximum or minimum at (a, b) , then (a, b) is a critical point of f .

CRITICAL POINTS

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

- At a critical point, a function could have a local maximum or a local minimum or neither.

LOCAL MINIMUM

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

Then, $f_x(x, y) = 2x - 2$

$$f_y(x, y) = 2y - 6$$

- These partial derivatives are equal to 0 when $x = 1$ and $y = 3$.
- So, the only critical point is $(1, 3)$.

LOCAL MINIMUM

Example 1

By completing the square, we find:

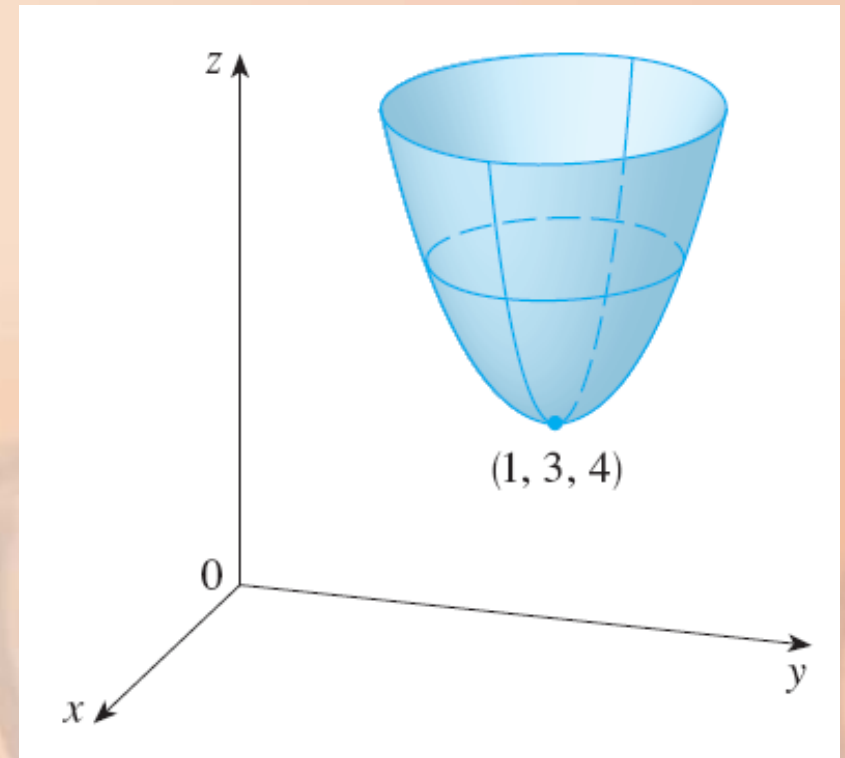
$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

- Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y .
- So, $f(1, 3) = 4$ is a local minimum.
- In fact, it is the absolute minimum of f .

LOCAL MINIMUM

Example 1

This can be confirmed geometrically from the graph of f , which is the elliptic paraboloid with vertex $(1, 3, 4)$.



Find the extreme values of

$$f(x, y) = y^2 - x^2$$

- Since $f_x = -2x$ and $f_y = 2y$,
the only critical point is $(0, 0)$.

EXTREME VALUES

Example 2

Notice that, for points on the x -axis, we have $y = 0$.

- So, $f(x, y) = -x^2 < 0$ (if $x \neq 0$).

For points on the y -axis, we have $x = 0$.

- So, $f(x, y) = y^2 > 0$ (if $y \neq 0$).

EXTREME VALUES

Example 2

Thus, every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values.

- So, $f(0, 0) = 0$ can't be an extreme value for f .
- Hence, f has no extreme value.

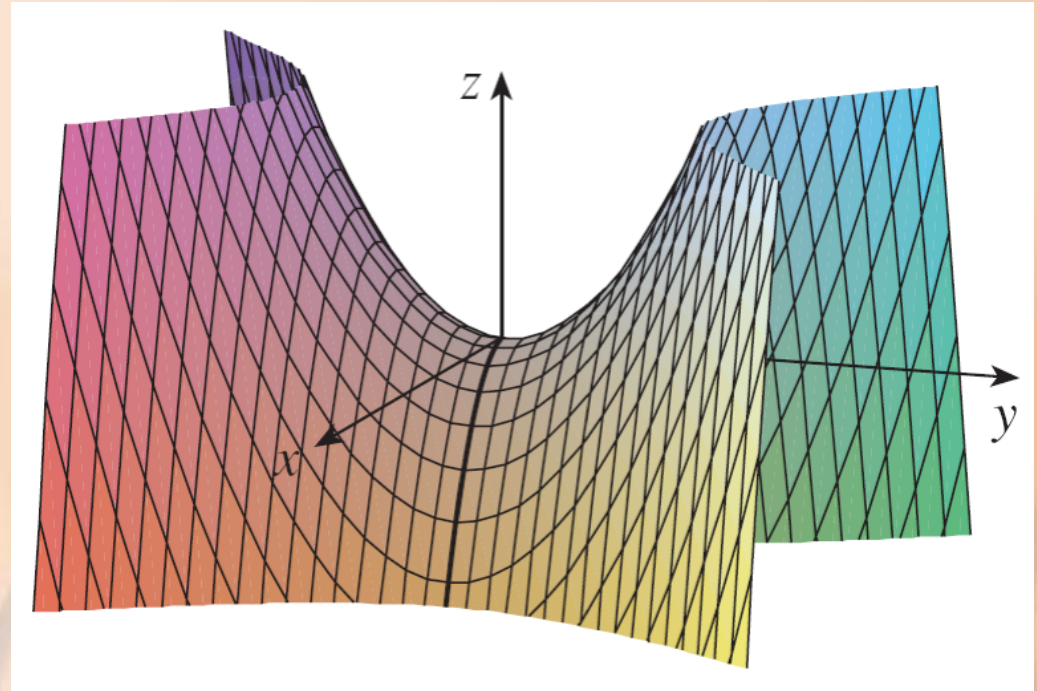
MAXIMUM & MINIMUM VALUES

Example 2 illustrates the fact that a function need not have a maximum or minimum value at a critical point.

MAXIMUM & MINIMUM VALUES

The figure shows how this is possible.

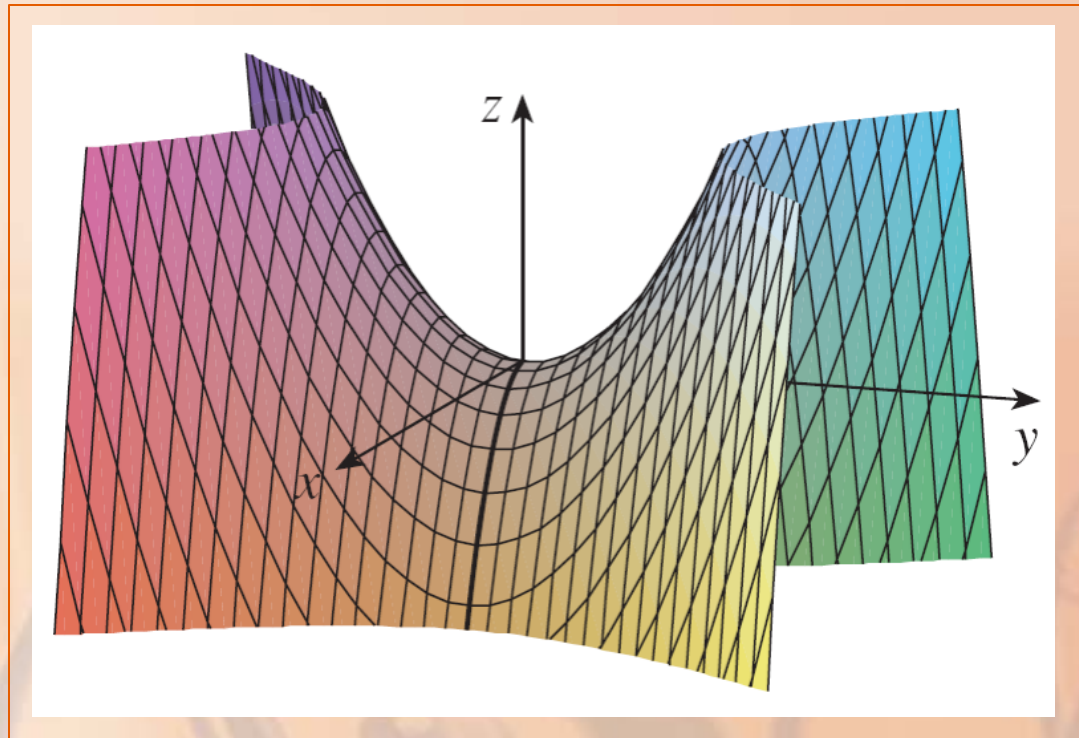
- The graph of f is the hyperbolic paraboloid $z = y^2 - x^2$.
- It has a horizontal tangent plane ($z = 0$) at the origin.



MAXIMUM & MINIMUM VALUES

You can see that $f(0, 0) = 0$ is:

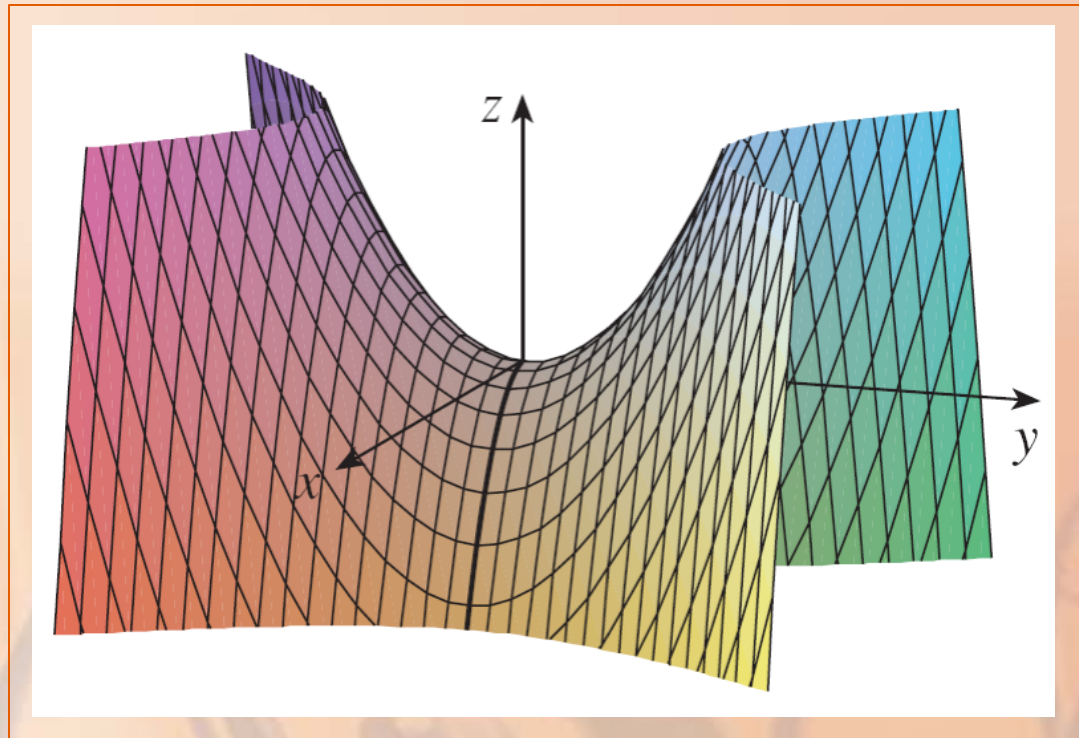
- A maximum in the direction of the x -axis.
- A minimum in the direction of the y -axis.



SADDLE POINT

Near the origin, the graph has the shape of a saddle.

- So, $(0, 0)$ is called a saddle point of f .



EXTREME VALUE AT CRITICAL POINT

We need to be able to determine whether or not a function has an extreme value at a critical point.

The following test is analogous to the Second Derivative Test for functions of one variable.

Suppose that:

- The second partial derivatives of f are continuous on a disk with center (a, b) .
- $f_x(a, b) = 0$ and $f_y(a, b) = 0$
[that is, (a, b) is a critical point of f].

SECOND DERIVATIVES TEST

Theorem 3

Let $D = D(a, b)$

$$= f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- a) If $D > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum.
- b) If $D > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum.
- c) If $D < 0$, $f(a, b)$ is not a local maximum or minimum.

In case c,

- The point (a, b) is called a saddle point of f .
- The graph of f crosses its tangent plane at (a, b) .

If $D = 0$, the test gives no information:

- f could have a local maximum or local minimum at (a, b) , or (a, b) could be a saddle point of f .

To remember the formula for D ,
it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

SECOND DERIVATIVES TEST

Example 3

Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

SECOND DERIVATIVES TEST

Example 3

We first locate the critical points:

$$f_x = 4x^3 - 4y$$

$$f_y = 4y^3 - 4x$$

SECOND DERIVATIVES TEST

Example 3

Setting these partial derivatives equal to 0, we obtain:

$$x^3 - y = 0$$

$$y^3 - x = 0$$

- To solve these equations, we substitute $y = x^3$ from the first equation into the second one.

This gives:

$$\begin{aligned} 0 &= x^9 - x \\ &= x(x^8 - 1) \\ &= x(x^4 - 1)(x^4 + 1) \\ &= x(x^2 - 1)(x^2 + 1)(x^4 + 1) \end{aligned}$$

So, there are three real roots:

$$x = 0, 1, -1$$

- The three critical points are:

$$(0, 0), (1, 1), (-1, -1)$$

SECOND DERIVATIVES TEST

Example 3

Next, we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2$$

$$f_{xy} = -4$$

$$f_{yy} = 12y^2$$

$$\begin{aligned} D(x, y) &= f_{xx} f_{yy} - (f_{xy})^2 \\ &= 144x^2y^2 - 16 \end{aligned}$$

SECOND DERIVATIVES TEST

Example 3

As $D(0, 0) = -16 < 0$, it follows from case c of the Second Derivatives Test that the origin is a saddle point.

- That is, f has no local maximum or minimum at $(0, 0)$.

SECOND DERIVATIVES TEST

Example 3

As $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$,
we see from case a of the test that $f(1, 1) = -1$
is a local minimum.

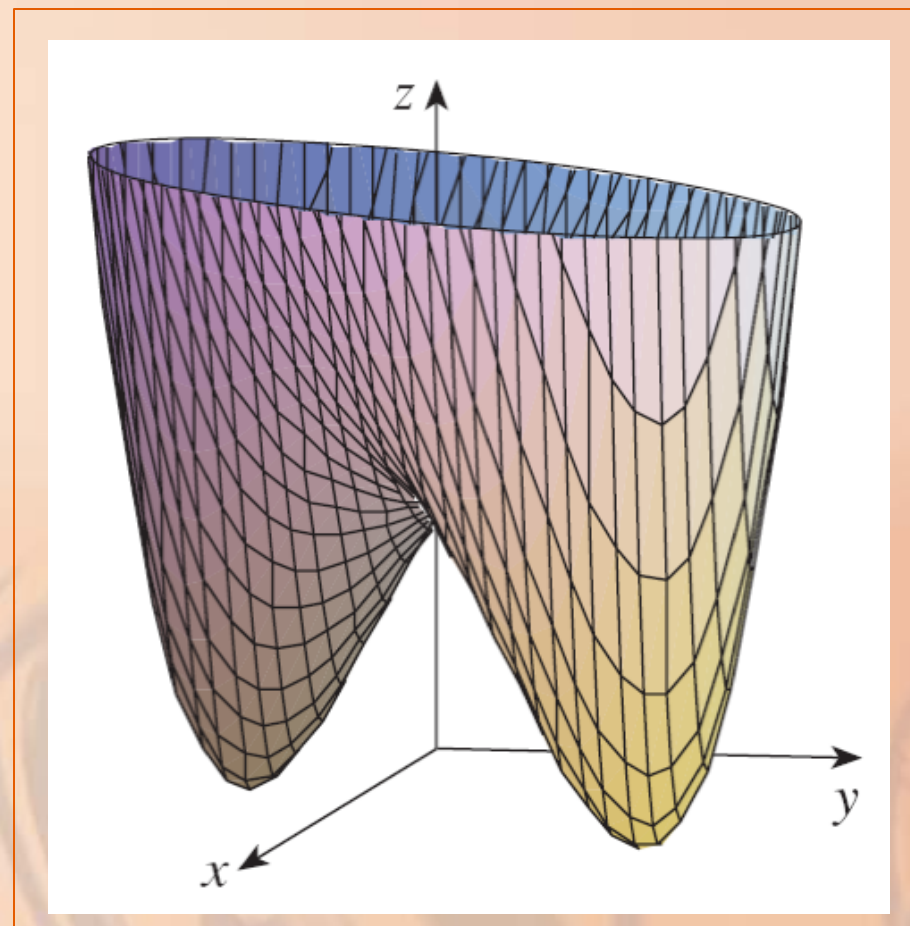
Similarly, we have $D(-1, -1) = 128 > 0$
and $f_{xx}(-1, -1) = 12 > 0$.

- So $f(-1, -1) = -1$ is also a local minimum.

SECOND DERIVATIVES TEST

Example 3

The graph of f is shown here.



MAXIMUM & MINIMUM VALUES

Example 4

Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also, find the highest point on the graph of f .

MAXIMUM & MINIMUM VALUES

Example 4

The first-order partial derivatives are:

$$f_x = 20xy - 10x - 4x^3$$

$$f_y = 10x^2 - 8y - 8y^3$$

MAXIMUM & MINIMUM VALUES

E. g. 4—Eqns. 4 & 5

So, to find the critical points, we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0$$

$$5x^2 - 4y - 4y^3 = 0$$

From Equation 4, we see that either:

- $x = 0$
- $10y - 5 - 2x^2 = 0$

MAXIMUM & MINIMUM VALUES

Example 4

In the first case ($x = 0$), Equation 5 becomes:

$$-4y(1 + y^2) = 0$$

So, $y = 0$, and we have the critical point $(0, 0)$.

MAXIMUM & MINIMUM VALUES

E. g. 4—Equation 6

In the second case ($10y - 5 - 2x^2 = 0$),
we get:

$$x^2 = 5y - 2.5$$

- Putting this in Equation 5, we have:

$$25y - 12.5 - 4y - 4y^3 = 0$$

MAXIMUM & MINIMUM VALUES

E. g. 4—Equation 7

So, we have to solve the cubic equation

$$4y^3 - 21y + 12.5 = 0$$

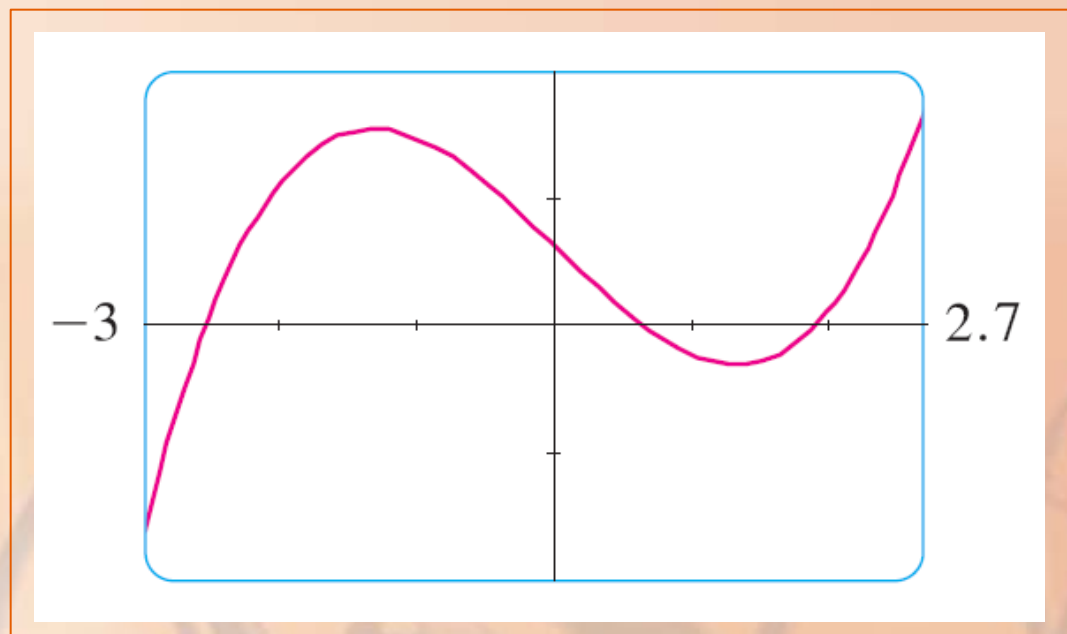
MAXIMUM & MINIMUM VALUES

Example 4

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$

we see Equation 7
has three real
roots.



MAXIMUM & MINIMUM VALUES

Example 4

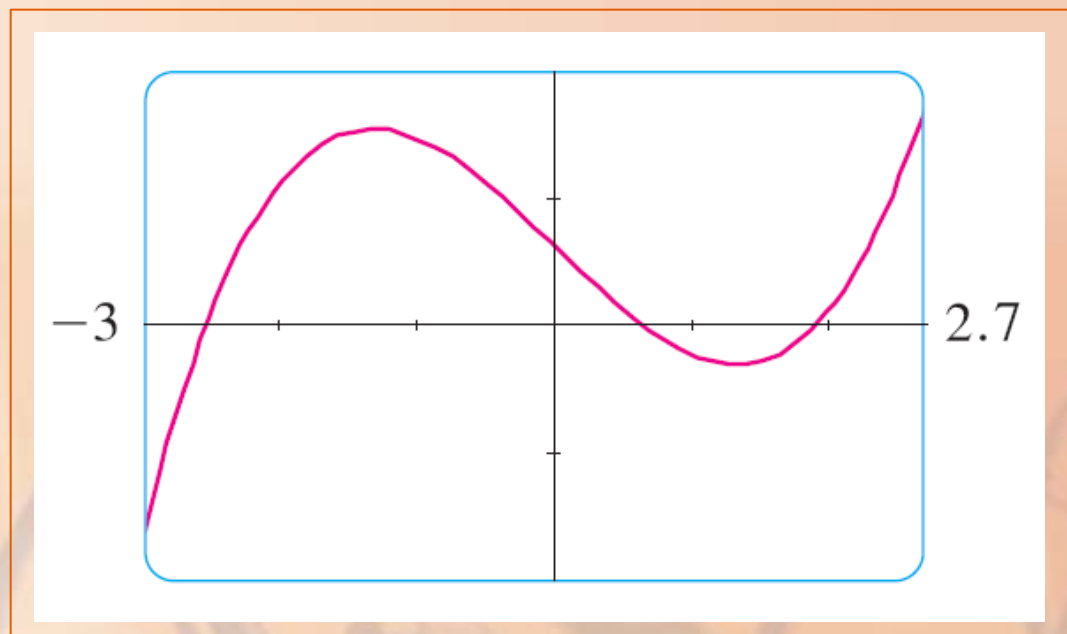
Zooming in, we can find the roots to four decimal places:

$$y \approx -2.5452$$

$$y \approx 0.6468$$

$$y \approx 1.8984$$

- Alternatively, we could have used Newton's method or a rootfinder to locate these roots.



MAXIMUM & MINIMUM VALUES

Example 4

From Equation 6, the corresponding x -values are given by:

$$x = \pm\sqrt{5y - 2.5}$$

- If $y \approx -2.5452$, x has no corresponding real values.
- If $y \approx 0.6468$, $x \approx \pm 0.8567$
- If $y \approx 1.8984$, $x \approx \pm 2.6442$

MAXIMUM & MINIMUM VALUES

Example 4

So, we have a total of five critical points, which are analyzed in the chart.

- All quantities are rounded to two decimal places.

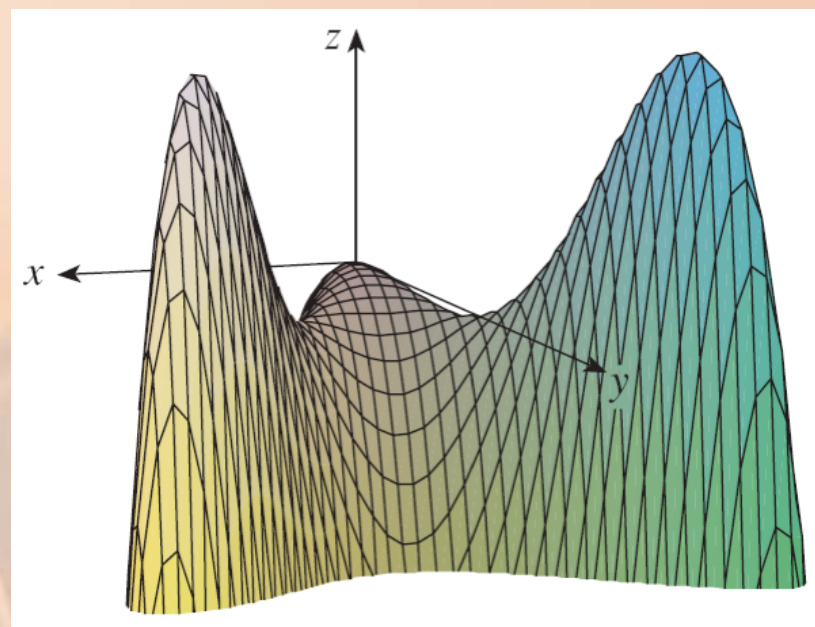
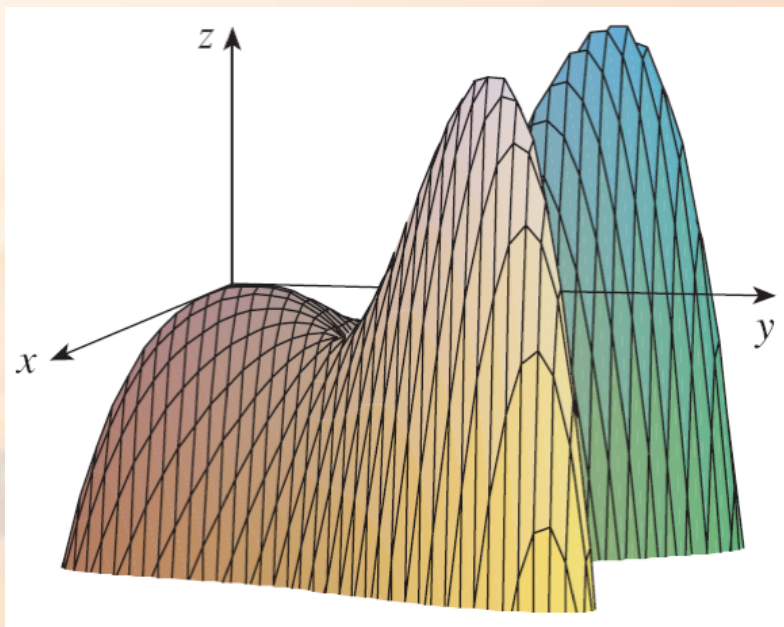
Critical point	Value of f	f_{xx}	D	Conclusion
(0, 0)	0.00	-10.00	80.00	local maximum
(± 2.64 , 1.90)	8.50	-55.93	2488.72	local maximum
(± 0.86 , 0.65)	-1.48	-5.87	-187.64	saddle point

MAXIMUM & MINIMUM VALUES

Example 4

These figures give two views of the graph of f .

- We see that the surface opens downward.

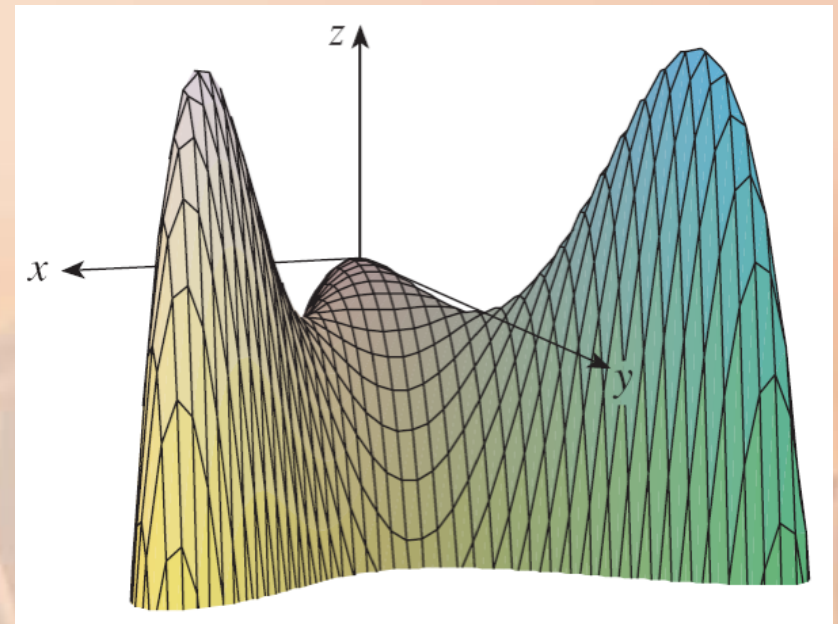
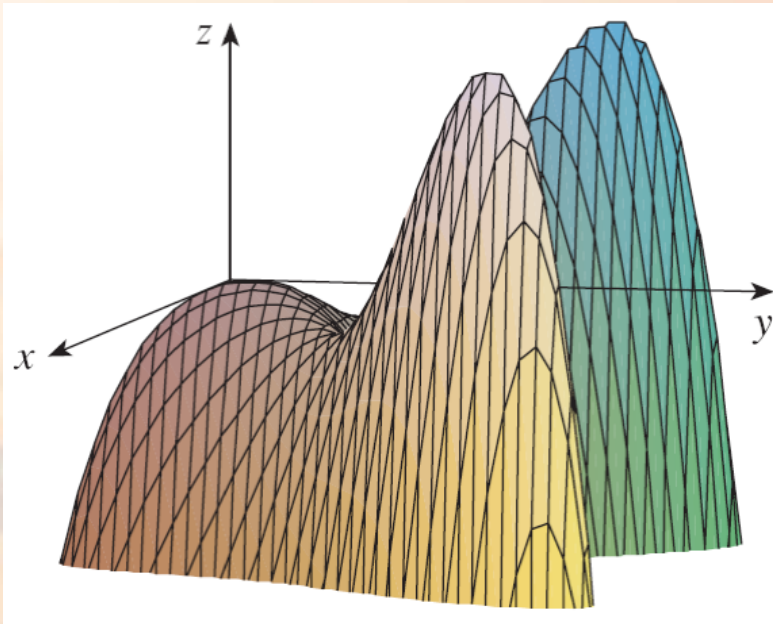


MAXIMUM & MINIMUM VALUES

Example 4

This can also be seen from the expression for $f(x, y)$:

- The dominant terms are $-x^4 - 2y^4$ when $|x|$ and $|y|$ are large.



MAXIMUM & MINIMUM VALUES

Example 4

Comparing the values of f at its local maximum points, we see that the absolute maximum value of f is:

$$f(\pm 2.64, 1.90) \approx 8.50$$

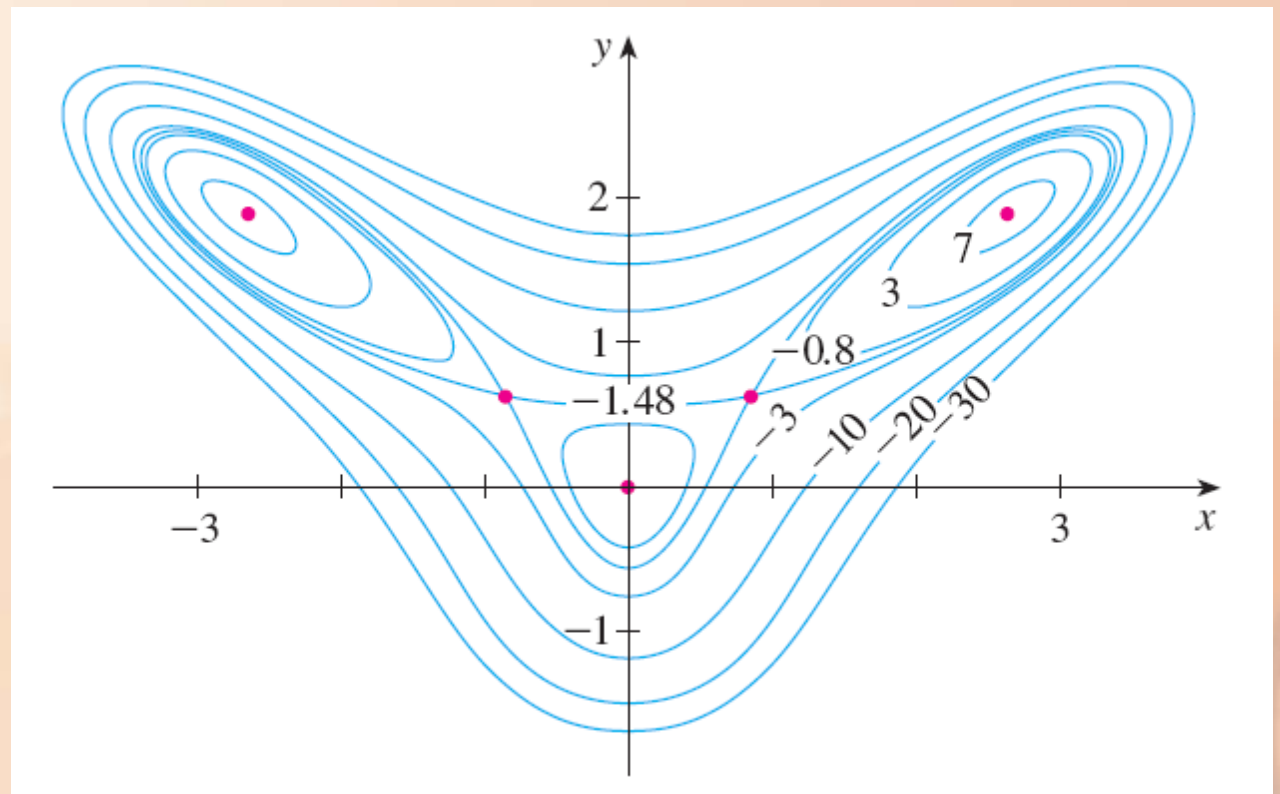
- That is, the highest points on the graph of f are:

$$(\pm 2.64, 1.90, 8.50)$$

MAXIMUM & MINIMUM VALUES

Example 4

The five critical points of the function f in Example 4 are shown in red in this contour map of f .



MAXIMUM & MINIMUM VALUES

Example 5

Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

- The distance from any point (x, y, z) to the point $(1, 0, -2)$ is:

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

MAXIMUM & MINIMUM VALUES

Example 5

However, if (x, y, z) lies on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$.

■ Thus, we have:

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

We can minimize d by minimizing
the simpler expression

$$\begin{aligned}d^2 &= f(x, y) \\ &= (x - 1)^2 + y^2 + (6 - x - 2y)^2\end{aligned}$$

MAXIMUM & MINIMUM VALUES

Example 5

By solving the equations

$$f_x = 2(x - 1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$.

MAXIMUM & MINIMUM VALUES

Example 5

Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$,

we have:

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0 \text{ and } f_{xx} > 0$$

- So, by the Second Derivatives Test, f has a local minimum at $(\frac{11}{6}, \frac{5}{3})$.

Intuitively, we can see that this local minimum is actually an absolute minimum:

- There must be a point on the given plane that is closest to $(1, 0, -2)$.

MAXIMUM & MINIMUM VALUES

Example 5

If $x = \frac{11}{6}$ and $y = \frac{5}{3}$, then

$$\begin{aligned} d &= \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} \\ &= \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{6} \end{aligned}$$

- The shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$.

MAXIMUM & MINIMUM VALUES

Example 6

A rectangular box without a lid is to be made from 12 m^2 of cardboard.

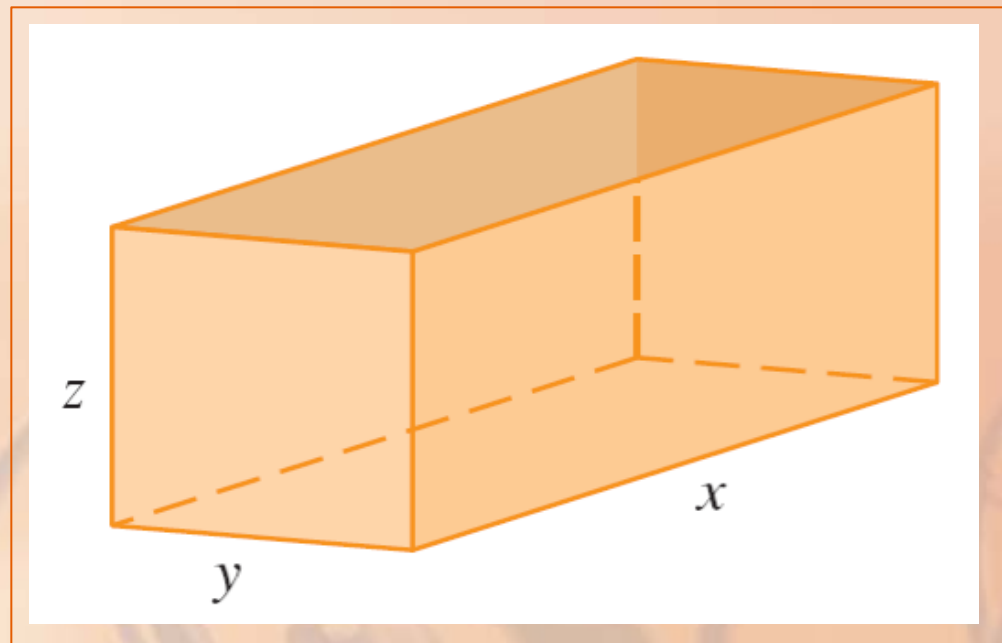
Find the maximum volume of such a box.

MAXIMUM & MINIMUM VALUES

Example 6

Let the length, width, and height of the box (in meters) be x , y , and z .

- Then, its volume is: $V = xyz$



MAXIMUM & MINIMUM VALUES

Example 6

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is:

$$2xz + 2yz + xy = 12$$

Solving this equation for z , we get:

$$z = (12 - xy)/[2(x + y)]$$

- So, the expression for V becomes:

$$V = xy \frac{12 - xy}{2(x + y)} = \frac{12xy - x^2 y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2}$$

$$\frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then

$$\partial V / \partial x = \partial V / \partial y = 0$$

- However, $x = 0$ or $y = 0$ gives $V = 0$.

So, we must solve:

$$12 - 2xy - x^2 = 0$$

$$12 - 2xy - y^2 = 0$$

- These imply that $x^2 = y^2$ and so $x = y$.
- Note that x and y must both be positive in this problem.

MAXIMUM & MINIMUM VALUES

Example 6

If we put $x = y$ in either equation,
we get:

$$12 - 3x^2 = 0$$

- This gives:

$$x = 2$$

$$y = 2$$

$$z = (12 - 2 \cdot 2) / [2(2 + 2)] = 1$$

MAXIMUM & MINIMUM VALUES

Example 6

We could use the Second Derivatives Test to show that this gives a local maximum of V .

MAXIMUM & MINIMUM VALUES

Example 6

Alternatively, we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V .

- So, it must occur when $x = 2$, $y = 2$, $z = 1$.

Then,

$$\begin{aligned} V &= 2 \cdot 2 \cdot 1 \\ &= 4 \end{aligned}$$

- Thus, the maximum volume of the box is 4 m³.

ABSOLUTE MAXIMUM & MINIMUM VALUES

For a function f of one variable,
the Extreme Value Theorem says that:

- If f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value.

ABSOLUTE MAXIMUM & MINIMUM VALUES

According to the Closed Interval Method in Section 4.1, we found these by evaluating f at both:

- The critical numbers
- The endpoints a and b

CLOSED SET

There is a similar situation for functions of two variables.

Just as a closed interval contains its endpoints, a closed set in \mathbb{R}^2 is one that contains all its boundary points.

BOUNDARY POINT

A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D .

CLOSED SETS

For instance, the disk

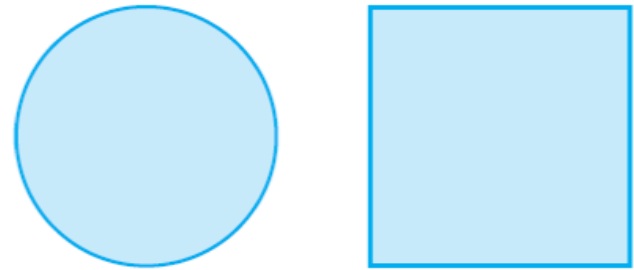
$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$ is a closed set because:

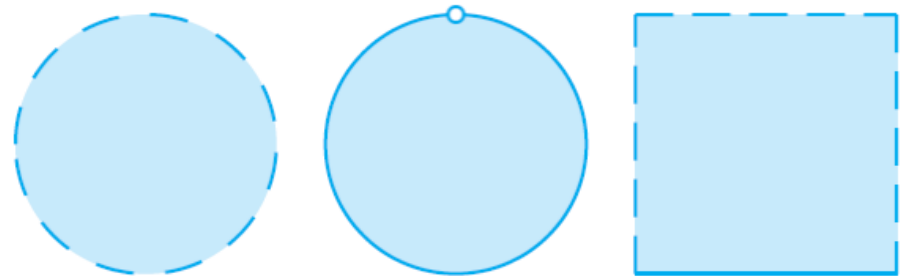
- It contains all of its boundary points (the points on the circle $x^2 + y^2 = 1$).

NON-CLOSED SETS

However, if even one point on the boundary curve were omitted, the set would not be closed.



(a) Closed sets



(b) Sets that are not closed

BOUNDED SET

A bounded set in \mathbb{R}^n is one that is contained within some disk.

- In other words, it is finite in extent.

CLOSED & BOUNDED SETS

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem (EVT) for functions of two variables in two dimensions.

EVT (TWO-VARIABLE FUNCTIONS) Theorem 8

If f is continuous on a closed, bounded set D in \mathbb{R}^2 , then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

EVT (TWO-VARIABLE FUNCTIONS)

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1) , then (x_1, y_1) is either:

- A critical point of f
- A boundary point of D

EVT (TWO-VARIABLE FUNCTIONS)

Thus, we have the following extension of the Closed Interval Method.

CLOSED INTVL. METHOD (EXTN.) Method 9

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .
2. Find the extreme values of f on the boundary of D .
3. The largest value from steps 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

CLOSED & BOUNDED SETS

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle

$$D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$$

As f is a polynomial, it is continuous on the closed, bounded rectangle D .

- So, Theorem 8 tells us there is both an absolute maximum and an absolute minimum.

CLOSED & BOUNDED SETS

Example 7

According to step 1 in Method 9, we first find the critical points.

- These occur when

$$f_x = 2x - 2y = 0$$

$$f_y = -2x + 2 = 0$$

- So, the only critical point is $(1, 1)$.
- The value of f there is $f(1, 1) = 1$.

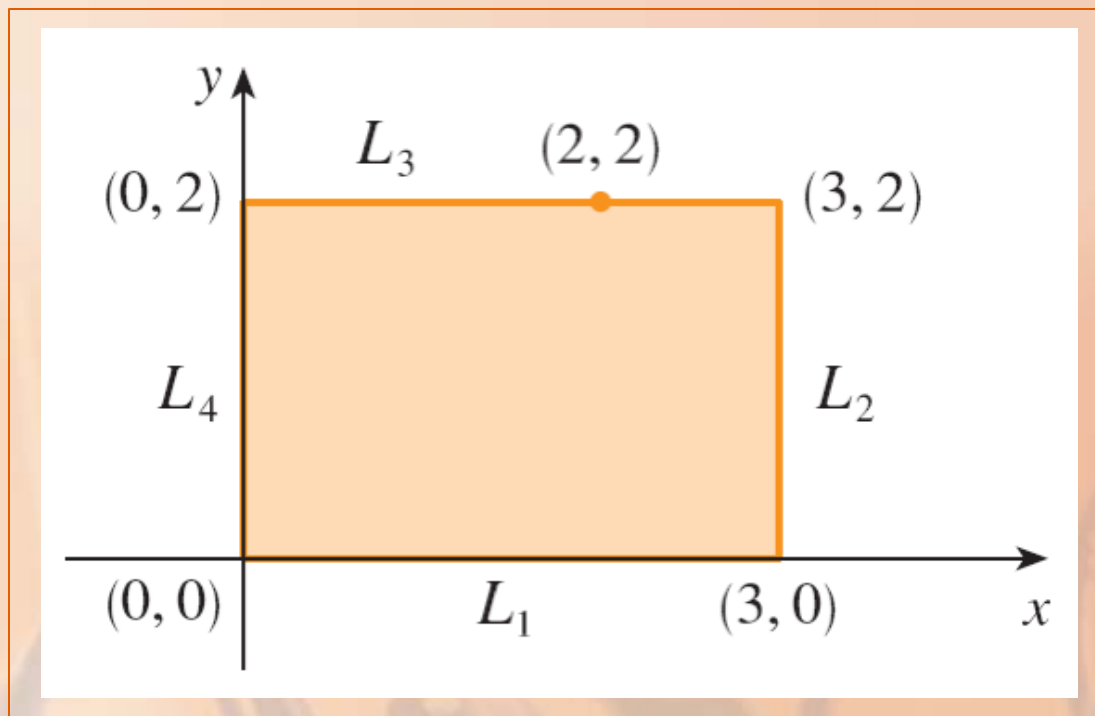
CLOSED & BOUNDED SETS

Example 7

In step 2, we look at the values of f on the boundary of D .

- This consists of the four line segments

$$L_1, L_2, L_3, L_4$$



CLOSED & BOUNDED SETS

Example 7

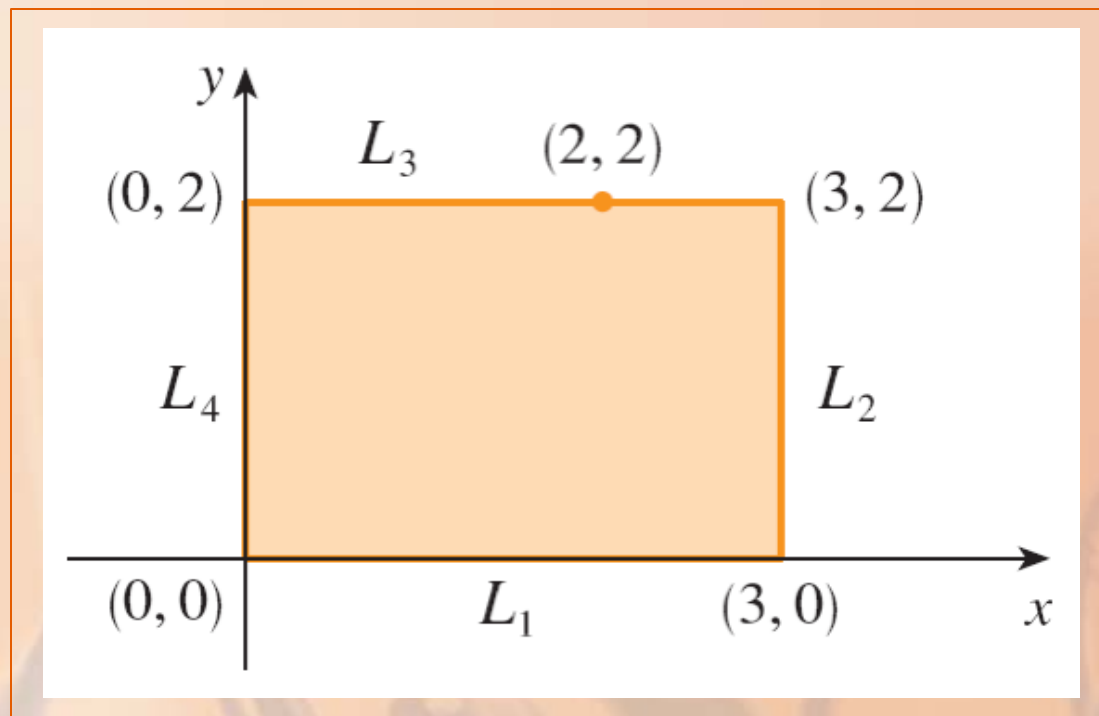
On L_1 , we have $y = 0$ and

$$f(x, 0) = x^2 \quad 0 \leq x \leq 3$$

This is an increasing function of x .

So,

- Its minimum value is:
 $f(0, 0) = 0$
- Its maximum value is:
 $f(3, 0) = 9$



CLOSED & BOUNDED SETS

Example 7

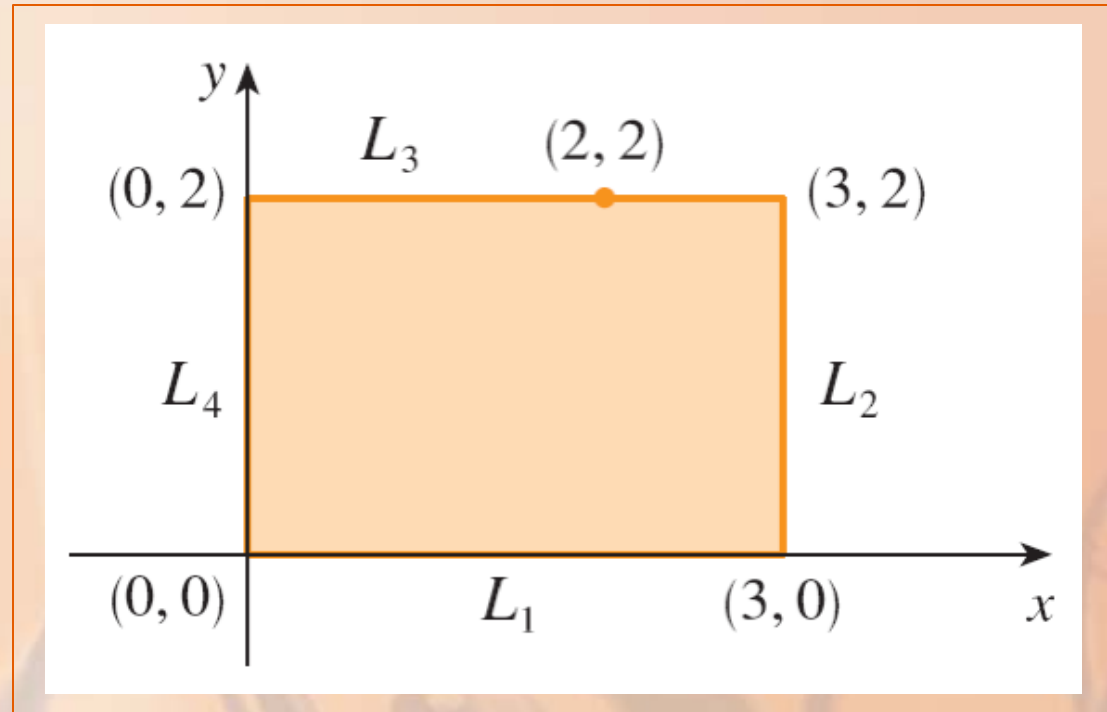
On L_2 , we have $x = 3$ and

$$f(3, y) = 9 - 4y \quad 0 \leq y \leq 2$$

This is a decreasing function of y .

So,

- Its maximum value is:
 $f(3, 0) = 9$
- Its minimum value is:
 $f(3, 2) = 1$

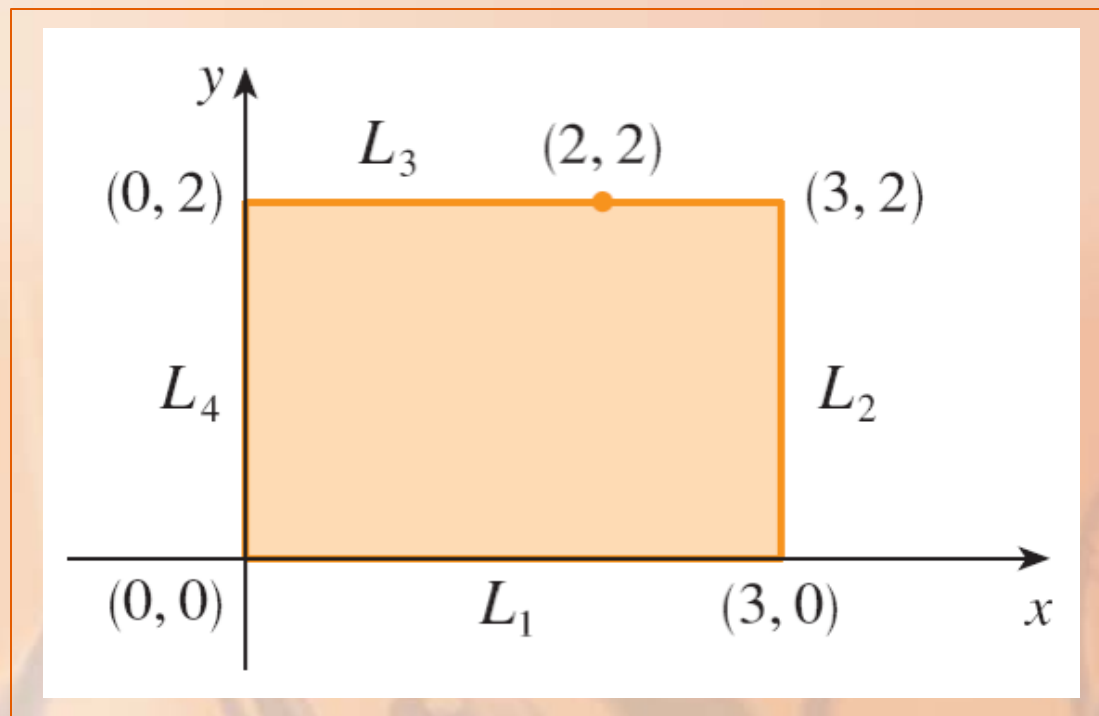


CLOSED & BOUNDED SETS

Example 7

On L_3 , we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \quad 0 \leq x \leq 3$$



CLOSED & BOUNDED SETS

Example 7

By the methods of Chapter 4, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that:

- The minimum value of the function is $f(2, 2) = 0$.
- The maximum value of the function is $f(0, 2) = 4$.

CLOSED & BOUNDED SETS

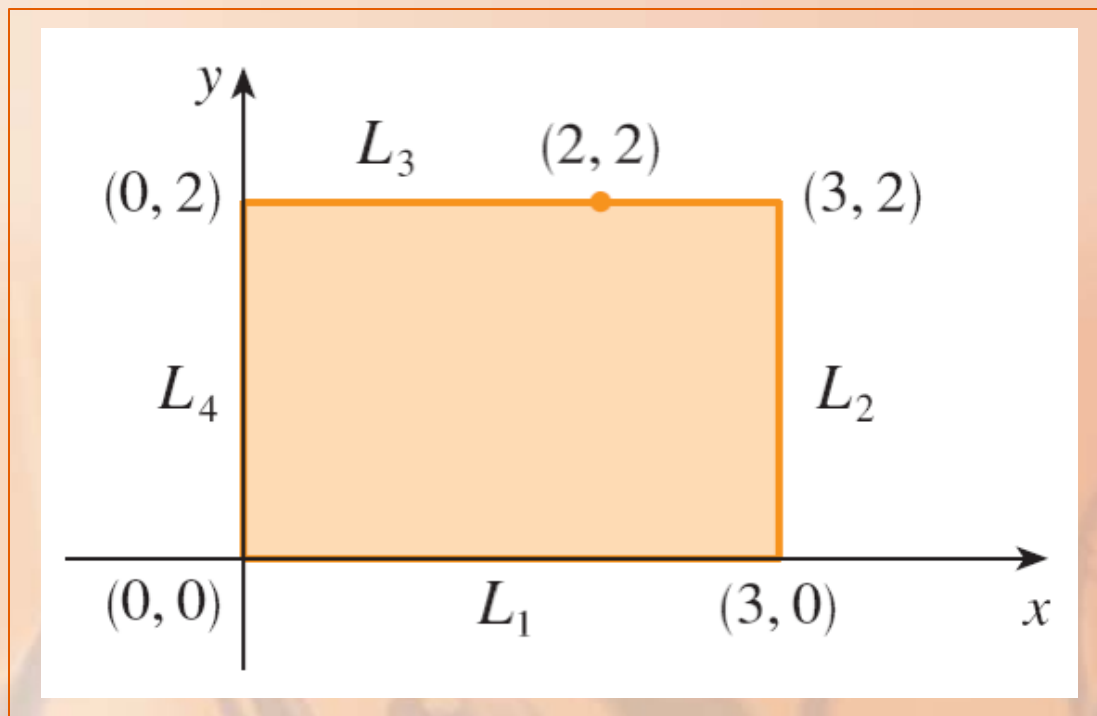
Example 7

Finally, on L_4 , we have $x = 0$ and

$$f(0, y) = 2y \quad 0 \leq y \leq 2$$

with:

- Maximum value
 $f(0, 2) = 4$
- Minimum value
 $f(0, 0) = 0$



Thus, on the boundary,

- The minimum value of f is 0.
- The maximum value of f is 9.

CLOSED & BOUNDED SETS

Example 7

In step 3, we compare these values with the value $f(1, 1) = 1$ at the critical point.

We conclude that:

- The absolute maximum value of f on D is $f(3, 0) = 9$.
- The absolute minimum value of f on D is $f(0, 0) = f(2, 2) = 0$.

CLOSED & BOUNDED SETS

Example 7

This figure shows the graph of f .

