Weibull Distribution

A random variable 'x' is said to have Weibull distribution with parameters α and β, if its p.d.f is given by

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, & x>0,\ \alpha>0,\ \beta>0 \\ 0, & \text{otherwise} \end{cases}$$

Mean and variance

$$\text{Mean} = E(X) = \left(\frac{1}{\alpha}\right)^{1/\beta} \Gamma\left(\frac{1}{\beta}+1\right)$$

$$\text{Variance} = Var(X) = (1/\alpha)^{2/\beta}\left[\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2\right]$$

Note:- when $\beta=1$, in the Weibull distribution, then the p.d.f becomes $f(x) = \alpha e^{-\alpha x}$, $x>0, \alpha>0$. which is the p.d.f of exponential distribution.

ie Exponential distribution is the special case of Weibull distribution.

CDF

The cumulative distribution function (c.d.f) of Weibull distribution is given by

$$P(X \le x) = F(x) = 1 - e^{-\alpha x^{\beta}}, \quad x \geq 0$$

$$\alpha, \beta > 0.$$

Problem 1

The length of life 'x' is hrs of an item in the machine shop has a weibull distribution with $\alpha = 0.01$, $\beta = 2$. What is the probability that it fails before eight hours of usage

Soln:- $P(x<8) = F(8) = 1 - e^{-0.001 \times 8^2} = 1 - 0.527$

$= 0.473$

Prob. 2 : Suppose that the lifetime of a certain kind of an emergency backup battery (in hrs) is a random variable 'x' having the weibull distribution with $\alpha = 0.1$ and $\beta = 0.5$. Find

i) the mean lifetime of these batteries

ii) the probability that such a battery will last more than 300 hrs.

Soln:- Let x denote the lifetime of emergency backup battery. (in hrs.)

i) mean $= \mu = \alpha^{-1/\beta} \Gamma(1 + \frac{1}{\beta}) = (0.1)^{-1/0.5} \Gamma(1 + \frac{1}{1/2})$

$= (0.1)^{-1/0.5} \Gamma(3)$

$= 200$ hours.

ii) $P(x > 300) = 1 - P(x \le 300)$

$F(x) = P(x \le x) = 1 - e^{-\alpha x^{\beta}}$

$P(x \le 300) = 1 - e^{-(0.1) \times 300^{1/2}} = 1 - e^{-1.7321}$

$= 1 - (1 - e^{-1.7321})$

$F(x) = e^{-1.7321} = 0.177$

prob-3

If the life x (in yrs) of a certain type of car has a Weibull distribution with parameter $\beta=2$, find the value of 'α', given that probability that the life of the car exceeds 5 years is $e^{-0.25}$. For these values of α and β, find the mean and variance of x.

Soln:- The density function of 'x' is given by

$$f(x) = \alpha\beta\, x^{\beta-1} e^{-\alpha x^2}, \quad x>0$$

$$f(x) = 2\alpha\, x\, e^{-\alpha x^2}, \quad x>0$$

Given that $P(X>5) = e^{-0.25} \rightarrow (1)$

By defn $P(X>5) = \int_5^\infty 2\alpha x\, e^{-\alpha x^2}\, dx$

$$= 2\alpha \int_5^\infty x\, e^{-\alpha x^2}\, dx.$$

Let $t = \alpha x^2$, $\frac{t}{\alpha} = x^2 \Rightarrow x = \frac{t^{1/2}}{\alpha^{1/2}}$

$dt = 2\alpha x\, dx$, $dx = \frac{dt}{2\alpha \cdot \frac{t^{1/2}}{\alpha^{1/2}}} = \frac{\alpha^{1/2} dt}{2\alpha t^{1/2}}$

when $x=5$, $t = 25\alpha$

when $x=\infty$, $t=\infty$.

$$= 2\alpha \int_{25\alpha}^\infty \frac{t^{1/2}}{\alpha^{1/2}} \cdot e^{-t} \cdot \frac{\alpha^{1/2}\, dt}{2\alpha t^{1/2}}$$

$$P(X>5) = \int_{25\alpha}^\infty e^{-t}\, dt = e^{-25\alpha} \rightarrow (2)$$

Using CDF

(or) $P(X>5) = 1 - P(X\le 5) = 1 - F(5)$

$F(x) = 1 - e^{-\alpha x^\beta}$; $F(5) = 1 - e^{-\alpha\times 5^2} = 1 - e^{-25\alpha}$

$P(X>5) = 1 - (1 - e^{-25\alpha}) = e^{-25\alpha}$

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$$e^{-25\alpha} = e^{-0.25}$$

$$\Rightarrow \alpha = \frac{1}{100}$$

For the Weibull distribution, with parameters α and β

$$\text{Mean} = E(x) = (1/\alpha)^{1/\beta}\, \Gamma\left(\frac{1}{\beta}+1\right)$$

$$= (100)^{1/2}\, \Gamma(1/2+1) \qquad \Gamma(n+1) = n\Gamma(n)$$
$$\Gamma(1/2+1) = 1/2\,\Gamma(1/2)$$

$$E(x) = 10 \times \frac{1}{2}\, \Gamma(1/2) = 5\sqrt{\pi}$$

$$E(x^2) = \left(\frac{1}{\alpha}\right)^{2/\beta} \Gamma\left(\frac{2}{\beta}+1\right)$$

$$Var(x) = (1/\alpha)^{2/\beta}\left[\Gamma\left(\frac{2}{\beta}+1\right) - \left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2\right]$$

$$= \left(\frac{1}{1/100}\right)^{2/2}\left[\Gamma\left(\frac{2}{2}+1\right) - \left(\Gamma\left(\frac{1}{2}+1\right)\right)^2\right]$$

$$= 100\left[\Gamma(2) - 1/4\left(\Gamma(1/2)\right)^2\right]$$

$$= 100\left(1 - 1/4 \times \pi\right) = 100\left(1 - \pi/4\right)$$

Prob-4

Suppose that the lifetime of a certain kind of an emergency backup battery (in hrs) is a random variable 'x' having the Weibull distribution with $\alpha = 0.1$ and $\beta = 0.5$. Find $\qquad F(x) = 1 - e^{-\alpha x^\beta}$

i) The mean lifetime of these batteries.

ii) The probability that such a battery will last more than 300 hours.

Soln:- i) $\text{Mean} = \mu = \alpha^{-1/\beta}\,\Gamma\left(\frac{1}{\beta}+1\right) = (0.1)^{-1/0.5}\,\Gamma(2+1)$
$$= (0.1)^{-2}\,\Gamma(3)$$
$$= 200 \text{ hours.}$$

ii) $P(x>300) = 1 - P(x \le 300) = 1 - F(300)$
$$= 1 - \left(1 - e^{-(0.1)\times(300)^{1/2}}\right) = e^{-1.7321} = 0.177$$