Some Special Failure Distributions

Some of the probability distributions describe the failure process and reliability of a component or a system more satisfactorily than others. They are the exponential, Weibull, normal and lognormal distributions. We shall now derive the reliability characteristics relating to these failure distributions using the formulas derived above.

(1) The exponential distribution

If the time to failure T follows an exponential distribution with parameter λ, then its *pdf* is given by

$$f(t) = \lambda e^{-\lambda t},\ t \geq 0 \tag{15}$$

Then, from (2),

$$R(t) = \int_t^{\infty} \lambda e^{-\lambda t}\, dt = [-e^{-\lambda t}]_t^{\infty} = e^{-\lambda t} \tag{16}$$

From (4)

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \tag{17}$$

This means that when the failure distribution is an exponential distribution with parameter λ, the failure rate at any time is a constant, equal to λ. Conversely when $\lambda(t)$ = a constant λ, we get from (7),

$$f(t) = \lambda \cdot e^{-\int_0^t \lambda\, dt} = \lambda e^{-\lambda t},\ t \geq 0$$

Due to this property, the exponential distribution is often referred to as *constant failure rate distribution* in reliability contexts. We have already derived in the earlier chapters, that

$$\text{MTTF} = E(T) = \frac{1}{\lambda} \tag{18}$$

and
$$\text{Var}(T) = \sigma_T^2 = \frac{1}{\lambda^2} \tag{19}$$

Also
$$R(t/T_0) = \frac{R(T_0+t)}{R(T_0)} = \frac{e^{-\lambda(T_0+t)}}{e^{-\lambda T_0}} \text{ by (16)}$$

$$= e^{-\lambda t} \tag{20}$$

This means that the time to failure of a component is not dependent on how long the component has been functioning. In other words the reliability of the component for the next 1000 hours, say, is the same regardless of whether the component is brand new or has been operating for several hours.This property is known as the *memoryless property* of the constant failure rate distribution.

(2) The Weibull distribution

The *pdf* of the Weibull distribution was defined as

$$f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}, t \ge 0 \tag{21}$$

An alternative form of Weibull's *pdf* is

$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right], \theta > 0, \beta > 0, t \ge 0 \tag{22}$$

(22) is obtained from (21) by putting $\alpha = \frac{1}{\theta^{\beta}}$. β is called the *shape parameter* and θ is called the *characteristic life* or *scale parameter* of the Weibull's distribution (22). If T follows Weibull's distribution (22),

then
$$R(t) = \int_t^{\infty} \frac{\beta}{\theta}\cdot\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right] dt$$

$$= \int_x^{\infty} e^{-x}\, dx, \text{ on putting } \left(\frac{t}{\theta}\right)^{\beta} = x$$

$$= e^{-x} = \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right] \tag{23}$$

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\theta}\cdot\left(\frac{t}{\theta}\right)^{\beta-1} \tag{24}$$

As derived in the chapter 'Some special probability distributions', we have

$$\text{MTTF} = E(T) = \theta\,\Gamma\left(1+\frac{1}{\beta}\right) \tag{25}$$

and
$$\operatorname{Var}(T) = \sigma_T^2 = \theta^2 \left\{ \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\} \tag{26}$$

Now
$$R(t/T_0) = \frac{R(t + T_0)}{R(T_0)}$$

$$= \frac{\exp\left[-\left(\frac{t + T_0}{\theta}\right)^\beta\right]}{\exp\left[-\left(\frac{T_0}{\theta}\right)^\beta\right]}$$

$$= \exp\left[-\left(\frac{t + T_0}{\theta}\right)^\beta + \left(\frac{T_0}{\theta}\right)^\beta\right] \tag{27}$$

(3) The normal distribution

If the time to failure T follows a normal distribution $N(\mu, \sigma)$ its *pdf* is given by

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(t - \mu)^2}{2\sigma^2}\right], -\infty < t < \infty$$

In this case, $MTTF = E(T) = \mu$ and

$$\operatorname{Var}(T) = \sigma_T^2 = \sigma^2.$$

$R(t) = \int_t^\infty f(t)\, dt$ is found out by expressing the integral in terms of standard normal integral and using the normal tables.

$\lambda(t)$ is then found out by using (4).

The density function of the time to failure in years of the gizmos (for use on widgets) manufactured by a certain company is given by $f(t) = \dfrac{200}{(t+10)^3}$, $t \geq 0$.

(a) Derive the reliability function and determine the reliability for the first year of operation.

(b) Compute the MTTF.

(c) What is the design life for a reliability 0.95?

(d) Will a one-year burn-in period improve the reliability in part (*a*)? If so, what is the new reliability?

(a) $$f(t) = \frac{200}{(t+10)^3}, \; t \geq 0$$

$$R(t) = \int_t^\infty f(t)\,dt = \left[\frac{-100}{(t+10)^2}\right]_t^\infty = \frac{100}{(t+10)^2}$$

$\therefore$ $$R(1) = \frac{100}{(1+10)^2} = 0.8264.$$

(b) $$\text{MTTF} = \int_0^\infty R(t)\,dt = \int_0^\infty \frac{100}{(t+10)^2}\,dt$$

$$= \left(\frac{-100}{t+10} \right)_0^\infty = 10 \text{ years.}$$

(c) *Design life* is the time to failure (t_D) that corresponds to a specified reliability. Now it is required to find t_D corresponding to $R = 0.95$

$$\therefore \qquad \frac{100}{(t_D + 10)^2} = 0.95$$

i.e., $\quad (t_D + 10)^2 = 100.2632$

$$\therefore \qquad t_D = 0.2598 \text{ year or 95 days}$$

(d) $$R(t/1) = \frac{R(t+1)}{R(1)} = \frac{100}{(t+11)^2} \div \frac{100}{11^2} = \frac{121}{(t+11)^2}$$

Now $\quad R(t/1) > R(t)$, if $\dfrac{121}{(t+11)^2} > \dfrac{100}{(t+10)^2}$

i.e., $\quad$ if $\quad \dfrac{(t+10)^2}{(t+11)^2} > \dfrac{100}{121}$

i.e., $\quad$ if $\quad \dfrac{t+10}{t+11} > \dfrac{10}{11}$

i.e., $\quad 11t > 10t$, which is true, as $t \geq 0$

$\therefore$ One year burn-in period will improve the reliability.

Now $R(1/1) = \dfrac{121}{(1+11)^2} = 0.8403 > 0.8264.$