Vector Calculus

UNIT - 6

Scalar and Vector point functions

If to each point (x, y, z) of a region R in space there corresponds a number or a scalar $\phi(x, y, z)$ then ϕ is called a Scalar point function.

ϕ : From pts in the space (x,y,z) → To real numbers $\phi(x,y,z)$.

Example :-

1. The temperature at any point within or on the surface of earth at a given time defines a scalar field.

2. pressure, Density, potential are also examples of Scalar point functions

Vector point functions :-

If to each point (x, y, z) of a region R in space there corresponds a vector $\vec{F}(x, y, z)$ then $\vec{F}$ is called a vector point function.

Example :-

1. The velocity at any point (x, y, z) within a moving field at time t is a vector pt function.

2. Force, electric intensity etc are also examples of vector pt functions.

Gradient

∇ (del) The Vector Differential operator

The vector differential operator ∇ is defined as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes OX, OY and OZ respectively.

Gradient of a scalar point function

or Slope of a scalar point function

Let $\phi(x,y,z)$ be a scalar point function and is continuously differentiable then the vector

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

grad ϕ

i.e., $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$ ~~or $\nabla\phi = \sum\vec{i}\frac{\partial\phi}{\partial x}$~~

Note:-

1) ∇ is an operator and vector
2) $\nabla \equiv \sum \vec{i}\frac{\partial}{\partial x}$
3) $\nabla\phi$ is also a vector
4) $\nabla\phi$ should not be written as $\phi\nabla$

~~5) $\vec{i}, \vec{j}, \vec{k}$ are~~

Properties If ϕ and ψ are scalar pt functions and c is a constant then

(1) $\nabla c = 0$ ~~where c is a constant~~

(2) $\nabla(c\phi) = c\nabla\phi$

(3) $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$

(4) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(5) $\nabla\frac{\phi}{\psi} = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$.

Proof:-

(1) $\nabla c = \sum \vec{i}\frac{\partial}{\partial x} c = \vec{i}\frac{\partial c}{\partial x} + \vec{j}\frac{\partial c}{\partial y} + \vec{k}\frac{\partial c}{\partial y}$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial y} = \frac{\partial c}{\partial z} = 0$$

$$\therefore \nabla c = 0.$$

(2) $\nabla(c\phi) = \sum \vec{i}\frac{\partial}{\partial x}(c\phi)$

$= \sum c\vec{i}\frac{\partial\phi}{\partial x}$ $\because$ c is constant

$= c\sum\vec{i}\frac{\partial\phi}{\partial x} = c\nabla\phi.$

(3) $\nabla(\phi+\psi) = \sum\vec{i}\frac{\partial}{\partial x}(\phi+\psi)$

$= \sum\vec{i}\left(\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial x}\right)$

$= \sum\vec{i}\frac{\partial\phi}{\partial x} + \sum\vec{i}\frac{\partial\psi}{\partial x}$

$= \nabla\phi + \nabla\psi.$

4) $$\nabla(\phi\psi) = \sum \vec{i}\,\frac{\partial}{\partial x}(\phi\psi)$$

$$= \sum \vec{i}\left(\phi\frac{\partial \psi}{\partial x} + \psi\frac{\partial \phi}{\partial x}\right)$$

$$= \sum \vec{i}\,\phi\frac{\partial \psi}{\partial x} + \sum \vec{i}\,\psi\frac{\partial \phi}{\partial x}$$

$$= \phi\sum \vec{i}\,\frac{\partial \psi}{\partial x} + \psi\sum \vec{i}\,\frac{\partial \phi}{\partial x}$$

$$= \phi\nabla\psi + \psi\nabla\phi.$$

5) $$\nabla\left(\frac{\phi}{\psi}\right) = \sum \vec{i}\,\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right)$$

$$= \sum \vec{i}\left(\frac{\psi\frac{\partial \phi}{\partial x} - \phi\frac{\partial \psi}{\partial x}}{\psi^2}\right)$$

$$= \frac{\psi\sum \vec{i}\,\frac{\partial \phi}{\partial x} - \phi\sum \vec{i}\,\frac{\partial \psi}{\partial x}}{\psi^2} = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}.$$

Problems

1. If $\phi = xyz$ find $\nabla\phi$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz;\quad \frac{\partial\phi}{\partial y} = xz;\quad \frac{\partial\phi}{\partial z} = xy$$

$$\therefore\ \nabla\phi = \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy.$$

2. If $\phi(x,y,z) = x^2y + y^2x + z^2$ find $\nabla\phi$ at the pt $(1,1,1)$.

$$\phi(x,y,z) = x^2y + y^2x + z^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(2xy + y^2) + \vec{j}(x^2 + 2xy) + \vec{k}(2z)$$

$$\nabla\phi_{(1,1,1)} = 3\vec{i} + 3\vec{j} + 2\vec{k}.$$

3. If $\phi(x,y,z) = x^3 - y^3 + xz^2$, find grad ϕ at $(1,-1,-2)$.

Ans: $\vec{i}(3x^2+z^2) + \vec{j}(-3y^2) + \vec{k}(2xz)$.

$$\nabla\phi_{(1,-1,-2)} = 7\vec{i} - 3\vec{j} - 4\vec{k}$$

4. If $\phi = \log(x^2+y^2+z^2)$ find $\nabla\phi$

$$\nabla\phi = \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}.$$

Note:- $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is called the position vector

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2.$$

1. Find ∇r.

$$\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$r^2 = x^2+y^2+z^2$$

Diff p.w.r.to $x, y \& z$

$$2r\frac{\partial r}{\partial x} = 2x; \quad 2r\frac{\partial r}{\partial y} = 2y; \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$r\frac{\partial r}{\partial x} = x; \quad r\frac{\partial r}{\partial y} = y; \quad r\frac{\partial r}{\partial z} = z$$

$$\therefore \nabla r = \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} = \frac{\vec{r}}{r}$$

2. $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$.

$$\vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \qquad \vec{i}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\right)$$

$$= \frac{-\vec{i}}{r^2}\frac{x}{r}$$

$$= -\frac{\vec{r}}{r^3}.$$

P.T $\nabla r^n = n r^{n-2} \vec{r}$

3) $$\nabla r^n = \vec{i} \frac{\partial}{\partial x}(r^n) + \vec{j} \frac{\partial}{\partial y}(r^n) + \vec{k} \frac{\partial}{\partial z}(r^n)$$

$$= n r^{n-1} \sum \vec{i} \frac{\partial r}{\partial x}$$

$$= n r^{n-1} \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$$

$$= n r^{n-2} \vec{r}$$

H.W 4. $\nabla f(r) = f'(r) \nabla(r)$

H.W 5. $\nabla(\log r) = \frac{\vec{r}}{r^2}$

Intro class - 1.

Class - 2

——— x ——— x ———

Unit Vector normal to the Surface

Let $\phi(x, y, z)$ be any scalar point function. Then the unit normal vector at any point (x, y, z) is given by

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

Pblm : 1 Find the unit vector normal to the surface $\phi = x^3 + y^3 + z^3 - 3xyz$ at point $(1, -1, 2)$.

Soln :- Given $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$$

$$\nabla \phi_{(1,-1,2)} = \vec{i}\,9 + \vec{j}(-3) + \vec{k}(15)$$

$$|\nabla \phi| = \sqrt{9^2 + 3^2 + 15^2} = \sqrt{315}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{9\vec{i} - 3\vec{j} + 15\vec{k}}{\sqrt{315}}$$

2. Find the unit vector normal to the surface $\phi = xyz - xy^2z^3$ at any point $(1, 2, -1)$.

<u>Soln:-</u>

$$\nabla\phi = \vec{i}(yz - y^2z^3) + \vec{j}(xz - 2xyz^3) + \vec{k}(xy - 3xy^2z^2)$$

$$\nabla\phi_{(1,2,-1)} = \vec{i}(-2+4) + \vec{j}(-1+4) + \vec{k}(2-12)$$

$$= 2\vec{i} + 3\vec{j} - 10\vec{k}$$

$$|\nabla\phi| = \sqrt{4+9+100} = \sqrt{113}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 3\vec{j} - 10\vec{k}}{\sqrt{113}}.$$

3. $\phi = 5x^2 - 2yz - 9x$ at $(1, -1, 2)$.

$\vec{i} - 4\vec{j} + 2\vec{k} \,/\, \sqrt{21}$

4. $x^2y + 2xz^2 = 8$ at $(1, 0, 2)$ $\quad \dfrac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{129}}$

<u>Directional Derivative</u>

The directional derivative of a scalar point function $\phi(x,y,z)$ at a given point (x,y,z) in the direction of a given vector $\vec{a}$ is given by

$$D.D = \nabla\phi \cdot \hat{a} \quad \text{where} \quad \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

i.e., $$\boxed{D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}}$$

Example: 1 Find the directional derivative of the function $2xy+z^2$ in the direction of $\vec{i}+2\vec{j}+2\vec{k}$ at $(1,-1,3)$.

Soln:- Given $\phi = 2xy+z^2$

$$\vec{a} = \vec{i}+2\vec{j}+2\vec{k}$$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla\phi = \vec{i}\,(2y)+\vec{j}\,(2x)+\vec{k}\,2z$$

$$\nabla\phi_{(1,-1,3)} = -2\vec{i}+2\vec{j}+6\vec{k}$$

$$\hat{a} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{\sqrt{1+4+4}} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{\sqrt{9}=3}$$

$$D.D = (-2\vec{i}+2\vec{j}+6\vec{k}) \cdot \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$$

$$= \frac{-2+4+12}{3} = \frac{14}{3}.$$

H.W Example: 2 Find the D.D of $\phi = x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction of $2\vec{i}-\vec{j}-2\vec{k}$

Soln:- $\phi = x^2yz+4xz^2$

$$\vec{a} = 2\vec{i}-\vec{j}-2\vec{k}$$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla\phi = (2xyz+4z^2)\vec{i}+(x^2z)\vec{j}+(x^2y+8xz)\vec{k}$$

$$\nabla\phi_{(1,-2,-1)} = (4+4)\vec{i}-\vec{j}+(-2-8)\vec{k}$$

$$= 8\vec{i}-\vec{j}-10\vec{k}$$

$$|\vec{a}| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\therefore D.D = (8\vec{i}-\vec{j}-10\vec{k}) \cdot \frac{(2\vec{i}-\vec{j}-2\vec{k})}{3}$$

$$= \frac{16+1+20}{3}$$

$$= \frac{37}{3} \;//.$$

3. What is the directional derivative of $\phi = xy^2 + yz^3$ at the pt $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at the pt $(-1,2,1)$.

Soln:- Given $\phi = xy^2 + yz^3$

$$\nabla\phi = \vec{i}(y^2) + \vec{j}(2xy + z^3) + \vec{k}(3yz^2)$$

$$\nabla\phi_{(2,-1,1)} = \vec{i} + \vec{j}(-4+1) + \vec{k}(-3)$$

$$= \vec{i} - 3\vec{j} - 3\vec{k}$$

$$\varphi = x\log z - y^2 + 4$$

$$\nabla\varphi = \vec{i}(\log z) + \vec{j}(-2y) + \vec{k}(x/z)$$

$$\nabla\varphi_{(-1,2,1)} = -4\vec{j} - \vec{k} \quad (\vec{a} \text{ say})$$

$$|\vec{a}| = \sqrt{16+1} = \sqrt{17}$$

$$\therefore D.D = \frac{(\vec{i} - 3\vec{j} - 3\vec{k}) \cdot (-4\vec{j} - \vec{k})}{\sqrt{17}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}.$$

4. If the D.D of $f = x^2 + y^2 + z^2 - 14$ at the point $(1,2,3)$ in the direction of $\alpha\vec{i} + \vec{j} + \vec{k}$ is 2 find α.

$$f = x^2 + y^2 + z^2 - 14$$

$$\nabla f = 2(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla f_{(1,2,3)} = 2(\vec{i} + 2\vec{j} + 3\vec{k})$$

$$\vec{a} = \alpha\vec{i} + \vec{j} + \vec{k}$$

$$|\vec{a}| = \sqrt{\alpha^2 + 2}$$

$$D.D = \frac{2(\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (\alpha\vec{i} + \vec{j} + \vec{k})}{\sqrt{\alpha^2 + 2}}$$

$$\Rightarrow \frac{2(\alpha + 2 + 3)}{\sqrt{\alpha^2 + 2}} = 2$$

$$\alpha + 5 = \sqrt{\alpha^2 + 2}$$

$$\alpha^2 + 25 + 10\alpha - \alpha^2 - 2 = 0$$

$$10\alpha + 23 = 0 \Rightarrow \alpha = -23/10.$$

H.W. $\phi = xyz - xy^2z^3$ at $(1, 2, -1)$

$\vec{a} = \vec{i} - \vec{j} - 3\vec{k}$

Ans: $\frac{29}{\sqrt{11}}$.

Angle between two surfaces

The angle between the surfaces $\phi_1(x,y,z)$ and $\phi_2(x,y,z)$ at a given point (x,y,z) is given by

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

Note:- If the two surfaces ϕ_1 and ϕ_2 cut orthogonally, then $\nabla\phi_1 \cdot \nabla\phi_2 = 0$.

Example:- 1. Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1,1,1)$.

Soln:- Given $\phi_1 = x\log z - y^2 + 1$ $\quad \phi_2 = x^2y - 2 + z$

$\nabla\phi_1 = \vec{i}(\log z) + \vec{j}(-2y) + \vec{k}\,\frac{x}{z}$ $\quad \nabla\phi_2 = \vec{i}(2xy) + \vec{j}(x^2) + \vec{k}$

$(\nabla\phi_1)_{1,1,1} = -2\vec{j} + \vec{k}$ $\quad (\nabla\phi_2)_{(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}$

$|\nabla\phi_1| = \sqrt{5}$ $\quad |\nabla\phi_2| = \sqrt{6}$.

$$\cos\theta = \frac{(-2\vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{30}} = \frac{-1}{\sqrt{30}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right).$$

HW Q. $\phi_1 = x^2+y^2-z$ $\phi_2 = x^2+y^2+z^2-9$

at $(2,-1,2)$

Ans: $\nabla\phi_1 = 4\vec{i}-2\vec{j}-\vec{k}$ $\nabla\phi_2 = 4\vec{i}-2\vec{j}+4\vec{k}$

$|\nabla\phi_1| = \sqrt{21}$ $|\nabla\phi_2| = \sqrt{36}$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

3. Find the constants a and b so that the surfaces $5x^2-2yz-9x=0$ and $ax^2y+bz^3=4$ may cut orthogonally at the point $(1,-1,2)$.

Soln:- Let $\phi_1 = 5x^2-2yz-9x$ $\phi_2 = ax^2y+bz^3-4$.

$\nabla\phi_1 = \vec{i}(10x-9)+\vec{j}(-2z)+\vec{k}(-2y)$

$\nabla\phi_1{}_{(1,-1,2)} = \vec{i}-4\vec{j}+2\vec{k}$ $\nabla\phi_2 = \vec{i}(2axy)+\vec{j}(ax^2)+\vec{k}(3bz^2)$

$\nabla\phi_2(1,-1,2) = -2a\vec{i}+a\vec{j}+12b\vec{k}$

Since ϕ_1 and ϕ_2 are cut orthogonally so

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$(\vec{i}-4\vec{j}+2\vec{k})\cdot(-2a\vec{i}+a\vec{j}+12b\vec{k}) = 0$

$-2a-4a+24b = 0 \Rightarrow -6a+24b = 0$ ——(1)

Since $(1,-1,2)$ is a pt of intersection of the surfaces, it lies on $ax^2y+bz^3=4$. Hence we get $-a+8b=4$. ——(2)

Solve (1) & (2) we get $\boxed{a=4 \quad b=1}$

Divergence and Curl of a Vector function

Divergence

Let $\vec{F}$ be any differentiable vector point function.

Then the divergence of $\vec{F}$ denoted as div $\vec{F}$ or

$\nabla \cdot \vec{F}$ is defined as

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \vec{F}$$

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\text{div } \vec{F} \text{ (or) } \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$\therefore$ $\nabla \cdot \vec{F}$ is scalar.

Solenoidal Vector

A vector $\vec{F}$ is said to be Solenoidal if

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = 0.$$

Example 1) Find div $\vec{F}$ where $\vec{F} = (x^2+yz)\vec{i} + (y^2+zx)\vec{j} + (z^2+xy)\vec{k}$.

$F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Soln:-

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z)$$

Example: 2 Find div $\vec{F}$ where $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ at the pt $(1,-1,1)$

Soln:- $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= y^2 + 2x^2z - 6yz$$

$$(\text{div}\,\vec{F})_{(1,-1,1)} = (-1)^2 + 2(1)^2(1) - 6(-1)(1) = 9.$$

Example: 3 Find div $\vec{F}$ where $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$.

$$\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$$

$$= \underbrace{\vec{i}(3x^2-3yz)}_{F_1} + \underbrace{\vec{j}(3y^2-3xz)}_{F_2} + \underbrace{\vec{k}(3z^2-3xy)}_{F_3}.$$

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 6x + 6y + 6z.$$

Example: 4 S.T the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Soln:- $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 0 + 0 + 0$$

$$= 0.$$

Hence $\vec{F}$ is a solenoidal.

5. If $\vec{F} = (ax+3y+4z)\vec{i} + (x-2y+3z)\vec{j} + (3x+2y-z)\vec{k}$ is

Solenoidal, find the value of a.

Soln. Given $\nabla.\vec{F} = 0$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$a - 2 - 1 = 0$$

$$a = 3.$$

H.W

1. Find div $\vec{F}$ if $\vec{F} = (y^2+z^2-x^2)\vec{i} + (x^2+z^2-y^2)\vec{j} + (x^2+y^2-z^2)\vec{k}$.

Ans $-2(x+y+z)$.

2. $\vec{F} = (x^2-y^2)\vec{i} + (2xy)\vec{j} + (y^2-xy)\vec{k}$

Ans: $4x$.

3. If $\vec{F} = (2x^2y+yz)\vec{i} + (xy^2-xz^2)\vec{j} + (axyz-2x^2y^2)\vec{k}$

is solenoidal find the value of a? $a=-6$.

<u>Curl:-</u> Let $\vec{F}$ be any differentiable vector pt function. Then the curl of $\vec{F}$ denoted as curl $\vec{F}$ or. $\nabla\times\vec{F}$ is defined as

$$\nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{F}$$

i.e., If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ then

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

<u>Irrotational vector</u>

A vector $\vec{F}$ is said to be irrotational if curl $\vec{F} = \nabla\times\vec{F} = \vec{0}$.

Example :1 Find $\nabla \times \vec{F}$ if $\vec{F} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$

Soln:- Given $\vec{F} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \vec{i}\,(2z + 2x) - \vec{j}\,(0 - 0) + \vec{k}\,(-2z - x^2).$$

$$= (2z+2x)\vec{i} - (2z+x^2)\vec{k}.$$

2. Find curl $\vec{F}$ if $\vec{F} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$ at $(1,-1,1)$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \vec{i}\,(-2yz) - \vec{j}\,(z^2 - xy) + \vec{k}\,(6xy - xz)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = 2\vec{i} - 2\vec{j} - 7\vec{k}$$

3. S.T $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is irrotational.

soln:- $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\,(0)\ \vec{j}\,(0) + \vec{k}\,(0)$$

$$= \vec{0}.$$

Hence $\vec{F}$ is irrotational.

4. S.T the vector $\vec{F} = (\sin y + z)\vec{i} + (x\cos y - z)\vec{j} + (x-y)\vec{k}$ is irrotational.

<u>Soln:-</u>

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= \vec{i}\,(-1-(-1)) - \vec{j}\,(1-1) + \vec{k}\,(\cos y - \cos y)$$

$$= \vec{0}.$$

5. Find the constants a, b, c s.t $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

<u>Soln:-</u> Let $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z) + (4x+cy+2z)\vec{k}$

Given that $\vec{F}$ is irrotational.

i.e., $\nabla \times \vec{F} = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\vec{i}\,(c+1) - \vec{j}\,(4-a) + \vec{k}\,(b-2) = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$\Rightarrow \quad c+1 = 0 \quad , \quad 4-a = 0 \quad , \quad b-2 = 0$

$\Rightarrow c = -1 \; ; \quad a = 4, \quad b = 2.$

<u>H.W</u>

1. $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$ is irrotational

2. $\vec{F} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$ is irrotational then find a.

Ans: $a = 2$

1. Find $\nabla \cdot \vec{r}$ and $\nabla \times \vec{r}$

Soln:- $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= 1+1+1 = 3$$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}.$$

2. Prove that curl (grad ϕ) $= \vec{0}$ (or) $\nabla \times \nabla\phi = \vec{0}$.

$$\text{grad}\,\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$\text{curl}(\text{grad}\,\phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) - \vec{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + \vec{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right)$$

$$= \vec{0}.$$

3. Prove div (curl $\vec{F}$) $= 0$ (or) $\nabla \cdot (\nabla \times \vec{F}) = 0$.

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\text{div}(\text{curl}\,\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x \partial y} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y \partial z}$$

$$= 0.$$

4. Prove $\vec{F} = (x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x-y+2z)\vec{k}$ is irrotational. Hence find the scalar potential ϕ s.t $\vec{F} = \nabla\phi$.

Soln:- $\vec{F} = (x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x-y+2z)\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+4z & 2x-3y-z & 4x-y+2z \end{vmatrix}$$

$$= \vec{i}\,(-1+1) - \vec{j}\,(4-4) + \vec{k}\,(2-2)$$

$$= \vec{0}.$$

$\therefore \vec{F}$ is an irrotational vector. (or) conservative field.

Given $\vec{F} = \nabla\phi$

$$(x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x-y+2z)\vec{k}$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

equating coefficients

$$\frac{\partial\phi}{\partial x} = x+2y+4z;$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz + f_1(y,z) \quad \text{--- (1)}$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = 2xy - \frac{3y^2}{2} - yz + f_2(x,z). \quad \text{--- (2)}$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xz - yz + \frac{2z^2}{2} + f_3(x,y)$$

from (1), (2) & (3) $\phi(x,y,z) = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + c$

where c is constant.

H.W

5. P.T $\vec{F} = (z - e^{-x}\sin y)\vec{i} + (1 + e^{-x}\cos y)\vec{j} + (x - 8z)\vec{k}$ is irrotational. Hence ϕ s.t $\vec{F} = \nabla\phi$, Scalar potential.

$\nabla \times \vec{F} = 0$

$\phi(x,y,z) = xz + e^{-x}\sin y + y - 4z^2 + c.$ conservative vector field, and hence find ϕ?

6. $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2 z - y + 2z)\vec{k}$ represents a ...

Example:- If $\vec{r}$ is the position vector of a point in the plane.

Prove

(i) $\nabla(r^n) = n r^{n-2}\vec{r}$

(ii) $\nabla.\nabla(r^n) = n(n+1) r^{n-2}$

(iii) $\nabla.(r^n\vec{r}) = (n+3) r^n$

(iv) $\nabla \times (r^n \vec{r}) = \vec{0}$

Soln:- (i) $\nabla(r^n) = n r^{n-2}\vec{r}$; $\nabla(r^{n-2}) = (n-2) r^{n-4}\vec{r}$

(ii) $\nabla.\nabla(r^n) = \nabla.(\underbrace{n r^{n-2}}_{\phi}\vec{r})$

$= n r^{n-2}\,\nabla.\vec{r} + n\vec{r}.\nabla r^{n-2}$

$= n r^{n-2} \times 3 + n\vec{r}.(n-2) r^{n-4}\vec{r}$

$= 3n r^{n-2} + n(n-2) r^{n-4}(\vec{r}.\vec{r})$

$= 3n r^{n-2} + n(n-2) r^{n-4} r^2$

$= n r^{n-2}(3 + (n-2))$

$= n(n+1) r^{n-2}.$

Vector identities

$\nabla.(\phi\vec{u}) = \nabla\phi.\vec{u} + \phi\nabla.\vec{u}$

$\nabla \times (\phi\vec{u}) = \nabla\phi \times \vec{u} + \phi\nabla \times \vec{u}$

(iii) $\nabla.(r^n\vec{r}) = r^n\,\nabla.\vec{r} + \vec{r}.\nabla(r^n)$

$= r^n.3 + \vec{r}.n r^{n-2}\vec{r}$

$= 3r^n + n r^{n-2}(\vec{r}.\vec{r})$

$= 3r^n + n r^{n-2} r^2$

$= r^n(n+3).$

Note: when $n = -3$, Then $r^n\vec{r}$ is Solenoidal vector.

(iv) $\nabla \times (r^n \vec{r}) = r^n (\nabla \times \vec{r}) + (\nabla r^n) \times \vec{r}$

$= r^n (0) + n r^{n-2} \vec{r} \times \vec{r}$

this also correct.

$= 0 + n r^{n-2} (\vec{r} \times \vec{r})$

$= 0$

$\nabla \times (r^n \vec{r}) = 0 \Rightarrow r^n \vec{r}$ is irrotational vector.

Note:- $r^n \vec{r}$ is irrotational vector for all values of n. but it is solenoidal for $n=-3$ only.

(or)

(iv) P.T $\nabla \times (r^n \vec{r}) = \vec{0}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad r^2 = x^2+y^2+z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}$ $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$ $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n \vec{r} = r^n(x\vec{i} + y\vec{j} + z\vec{k}) = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$= \sum \vec{i} \left(z n r^{n-1} \frac{\partial r}{\partial y} - y n r^{n-1} \frac{\partial y}{\partial z} \right)$

$= \sum \vec{i} \left(z n r^{n-1} \frac{y}{r} - y n r^{n-1} \frac{z}{r} \right)$

$\nabla \times (r^n \vec{r}) = \vec{0}$

6. If $\vec{F}$ is a vector pt function then

$\text{curl}(\text{curl} \vec{F}) = \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

(7) Laplacian operator ∇^2.

$\nabla^2 \vec{F}$

$\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}$

soln:-

$$\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$$\text{curl}(\text{curl} \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & " & " \end{vmatrix}$$

$= \sum \vec{i} \left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \right]$

$= \sum \vec{i} \left(\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} \right)$

$= \sum \vec{i} \left(\left(\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} \right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right)$

$= \sum \vec{i} \left[\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} \right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right]$

$= \sum \vec{i} \left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) - \nabla^2 F_1 \right]$

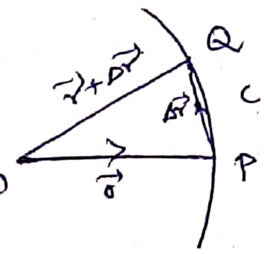
$= \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ //.

Line integral

If the line integral is along the curve C is

denoted by $\int_C \vec{F}\cdot d\vec{r}$.

If C is closed curve then the line integral is $\oint_C \vec{F}\cdot d\vec{r}$.



Consider the function $F(x,y,z)$ defined throughout some region of space and let C be any curve in that region.

Let P and Q be two neighbouring points on C with position vectors

$\vec{r}$ and $\vec{r}+\Delta\vec{r}$ so that $\vec{PQ} = \vec{OQ}-\vec{OP}$

$= \vec{r}+\Delta\vec{r}-\vec{r} = \Delta\vec{r}$.

Let $\vec{F}$ act at P, making an angle θ with PQ.

$\vec{F}\cdot\Delta\vec{r}$ in the limiting process becomes $\vec{F}\cdot d\vec{r}$ and this gives the work done by the force $\vec{F}$ in moving a particle through a distance $d\vec{r}$. $\int_C \vec{F}\cdot d\vec{r}$ is defined as the line integral of $\vec{F}$ along the curve C.

✓ * If C is a closed curve then we denote the integral as $\oint_C \vec{F}\cdot d\vec{r}$.

✓ * $\int_C \vec{F}\cdot d\vec{r}$ gives the total work done by the force $\vec{F}$.

* $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

* If the equation of the curve is given in parametric form say $x = x(t)$, $y = y(t)$, and $z = z(t)$ and the parametric values at A and B are $t = t_1$ and $t = t_2$ then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

Example: 1

If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve on the xy plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Soln:-

$$\vec{F} = 3xy\vec{i} - y^2\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot d\vec{r} = 3xy\,dx - y^2 dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\,dx - y^2 dy)$$

$$= \int_0^1 3x\,2x^2 dx - 4x^4\,4x\,dx$$

$$= \int_0^1 (6x^3 - 16x^5)dx = \left(\frac{6x^4}{4} - \frac{16x^6}{6}\right)_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = -\frac{7}{6}$$

Q. If $\vec{F} = 3xy\vec{i} + z\vec{j} + x\vec{k}$. Find the value of $\int_c \vec{F}\cdot d\vec{r}$ along the curve $x = 1+t^2$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

Soln:- $\vec{F} = 3xy\vec{i} + z\vec{j} + x\vec{k}$; $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + dz\,\vec{k}$

$\vec{F}\cdot d\vec{r} = 3xy\,dx + z\,dy + x\,dz$

$x = 1+t^2 \Rightarrow dx = 2t\,dt$

$y = 2t^2 \Rightarrow dy = 4t\,dt$

$z = t^3 \Rightarrow dz = 3t^2\,dt$

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 3(1+t^2)2t^2\,2t\,dt + t^3\,4t\,dt + (1+t^2)\,3t^2\,dt$$

$$= \int_1^2 (12t^5 + 12t^3 + 4t^4 + 3t^2 + 3t^4)\,dt$$

$$= \int_1^2 (12t^5 + 7t^4 + 12t^3 + 3t^2)\,dt$$

$$= \left(12\left(\frac{t^6}{6}\right) + 7\frac{t^5}{5} + 12\frac{t^4}{4} + \frac{3t^3}{3}\right)_1^2$$

$$= \frac{1107}{5}.$$

H.W 1. ~~Find $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ along the arc of the curve $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ from $t=0$ to $t=2\pi$.~~

Ans 3π

1. If $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$, find $\int_c \vec{F}\cdot d\vec{r}$ where C is given by $x=t$, $y=t^2$, $z=t^3$ from $P(0,0,0)$ to $Q(2,4,8)$.

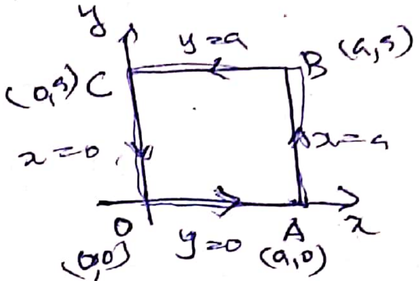
Ans 0.

2. If $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and C is the curve $y^2 = 4x$ connecting the pts $(0,0)$ and $(4,4)$ find $\int_c \vec{F}\cdot d\vec{r}$.

Ans: 264

3. Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ and C is the square bounded by the coordinate axes and the lines $x=a$ and $y=a$.

Soln:- $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$



$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F}\cdot d\vec{r} = (x^2-y^2)dx + 2xy\,dy$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r} \quad \text{--- (1)}$$

Along OA, x varies from 0 to a and $y=0$ & $dy=0$

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3} \quad \text{--- (2)}$$

Along AB, y varies 0 to a and $x=a$; $dx=0$.

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^a 2ay\,dy = \left(2a\frac{y^2}{2}\right)_0^a = a^3 \quad \text{--- (3)}.$$

Along BC, x varies from a to 0 and $y=a$, $dy=0$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 (x^2-a^2)\,dx = \left(\frac{x^3}{3}-a^2x\right)_a^0 = -\frac{a^3}{3}+a^3 = \frac{2a^3}{3} \quad \text{--- (4)}$$

$\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$x=0, x=4$

$y=k, y=0.$

Along CO, y varies from a to 0 and $x=0$; $dx=0$

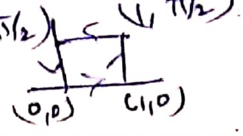
$$\int_{CO} \vec{F}\cdot d\vec{r} = \int_a^0 \cancel{2ay\,dy} = \cancel{\left(\frac{2ay^2}{2}\right)_a^0 = -a^3}$$

$$= 0. \quad \text{--- (5)}$$

$$\therefore (1) \Rightarrow \int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + a^3 + \frac{2a^3}{3} = 2a^3 \,//.$$

H.W

1. Find the circulation of $\vec{F}$ around the curve C where $\vec{F} = e^x\sin y\,\vec{i} + e^x\cos y\,\vec{j}$ and C is the rectangle whose vertices are $(0,0)$, $(1,0)$, $(1,\pi/2)$ & $(0,\pi/2)$.



Ans : 0

(13)

4. Find the work done in moving a particle once around a circle C in the xy plane if the circle has centre at the origin and radius 2 units and if the force field is given by

$$\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y - 5z)\vec{k}.$$

Soln:- $d\vec{r} = dx\vec{i} + dy\vec{j}$ for any pt along the circle $x^2+y^2=4$.

The parametric form of the curve is $x = 2\cos t$, $y = 2\sin t$, $z=0$.

$dx = -2\sin t\, dt$, $dy = 2\cos t\, dt$

$$\therefore \vec{F}\cdot d\vec{r} = [(4\cos t - 2\sin t)\vec{i} + (2\cos t + 2\sin t)\vec{j} + (6\cos t - 4\sin t)\vec{k} \cdot -2\sin t\,\vec{i} + 2\cos t\,\vec{j}]\,dt.$$

$$= (-8\sin t\cos t + 4\sin^2 t + 4\cos^2 t + 4\sin t\cos t)\,dt$$

$$= (4 - 4\sin t\cos t)\,dt$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (4 - 2\sin 2t)\,dt = (4t + \cos 2t)_0^{2\pi}$$

$$= 8\pi.$$

5. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ where C is the line joining the points $A(0,0,0)$ to $B(1,1,1)$.

Soln:- The equation of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

$\therefore$ Equation of AB is $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$

$\therefore dx = dy = dz$

$x = t$

$y = t$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$z = t$

$$\vec{F}\cdot d\vec{r} = (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

$$= (3x^2+6x)\,dx - 14x^2\,dx + 20x^3\,dx$$

$$= (20x^3 - 11x^2 + 6x)\,dx$$

6. If $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the straight line from A(0,0,0) to B(2,1,3).

soln next page.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (20x^3 - 11x^2 + 6x)\,dx$$

$$= \left(20\frac{x^4}{4} - 11\frac{x^3}{3} + 6\frac{x^2}{2}\right)_0^1 = \frac{13}{3}.$$

7. S.T $\vec{F} = y\sin 2x\,\vec{i} + \sin^2 x\,\vec{j}$ is a conservative field vector.

Find a scalar point function ϕ s.t $\vec{F} = \nabla\phi$ and hence find the work done in moving a particle from $(0,0,0)$ to $(\pi/2, 2, 1)$.

Soln:- $\vec{F} = y\sin 2x\,\vec{i} + \sin^2 x\,\vec{j}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin 2x & \sin^2 x & 0 \end{vmatrix}$$

$$= \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\,(2\sin x\cos x - \sin 2x)$$

$$= \vec{0}.$$

$\vec{F}$ is irrotational and hence conservative.

Let ϕ be a scalar function s.t $\vec{F} = \nabla\phi$.

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = y\sin 2x\,\vec{i} + \sin^2 x\,\vec{j}$$

$$\frac{\partial\phi}{\partial x} = y\sin 2x \quad ; \quad \frac{\partial\phi}{\partial y} = \sin^2 x$$

$$\phi = -\frac{y}{2}\cos 2x + f_1(y,z) \quad ; \quad \phi = \sin^2 x\cdot y + f_2(x,z)$$

$$= y\left(\frac{1-\cos 2x}{2}\right) + f_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 0 \Rightarrow \phi = c.$$

$$\therefore \phi = \frac{y}{2} - \frac{y}{2}\cos 2x + c$$

$$\text{workdone} = \phi(\pi/2, 2, 1) - \phi(0,0,0)$$

$$= (1 - \cos\pi + c) - (0 - 0 + c)$$

$$= 2 \,//$$

6. If $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where (14)

C is the straight line from $A(0,0,0)$ to $B(2,1,3)$.

Soln:- Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

Straight line formula

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$(0,0,0)$ $(2,1,3)$

$x_1\ y_1\ z_1$ $x_2\ y_2\ z_2$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

If $x = 0 \Rightarrow t = 0$

$y = 0 \Rightarrow t = 0$

$z = 0 \Rightarrow t = 0$

$x = 2t,\ y = t\ ;\ z = 3t$

$dx = 2dt,\ dy = dt\ ;\ dz = 3dt$

$x = 2$ $t = 1$

$y = 1$ $t = 1$

$z = 3$ $t = 1$

Limit $t = 0$ to $t = 1$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3\times 4t^2 + (2\times 2t\times 3t - t) + 3t)\,dt$$

$$= \int_0^1 12t^2 + 12t^2 - t + 3t\,dt = \int_0^1 24t^2 + 2t\,dt$$

$$= \left(24\frac{t^3}{3} + t^2\right)_0^1 = \frac{24}{3} + 1 = 27$$

$$= \int_0^1 3\times 4t^2\times 2\,dt + (2\times 2t\times 3t - t)\,dt + 3t\times 3\,dt$$

$$= \int_0^1 24t^2 + 12t^2 - t + 9t\,dt = \int_0^1 36t^2 + 8t\,dt$$

$$= \left(\frac{36t^3}{3} + \frac{8t^2}{2}\right)_0^1 = \frac{36}{3} + \frac{8}{2} = 12 + 4$$

$$\int_C \vec{F}\cdot d\vec{r} = 16.$$

8. Prove $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k}$ is a conservative vector field. Find the scalar potential and hence the work done by $\vec{F}$ in moving an object in this field from ~~(...)~~ to $(\pi/2, -1, 2)$
$(0, 1, -1)$

<u>Soln:-</u> $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2y\cos x)$$

$$= \vec{0}$$

$\therefore \vec{F}$ is irrotational and hence conservative.

Let ϕ be a scalar potential s.t $\vec{F} = \nabla\phi$

i) $\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k}$

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \; ; \quad \frac{\partial\phi}{\partial y} = 2y\sin x - 4 \; ; \quad \frac{\partial\phi}{\partial z} = 3xz^2 + 2$$

$$\phi = y^2\sin x + z^3 x + f_1(y,z) \qquad \phi = \frac{2y^2}{2}\sin x - 4y + f_2(x,z)$$

$$\phi = \frac{3xz^3}{3} + 2z + f_3(x,y)$$

$\therefore$ we get $\phi = y^2\sin x - 4y + xz^3 + 2z + C.$

work done $= \phi(\pi/2, -1, 2) - \phi(0, 1, -1)$

$= \cancel{(-1)^2\sin\pi/2 - 4(-1) + \pi/2(8) + 4}$

$= (-1)^2\sin\pi/2 - 4(-1) + \pi/2(8) + 4 - 0 + 4 - 2$

$= 1 + 4 + 4\pi + 4 + 4 - 2$

$= 4\pi + 15$ //.

Surface Integral

Defn

Consider a surface S. Let $\hat{n}$ denote the unit vector normal to the surface S. Let 'R' be the projection of the surface on the xy plane. Let $\vec{F}$ be a vector point function defined at all points of S. Then the surface integral $\vec{F}$ is defined to be

$$\iint_S \vec{F}\cdot\hat{n}\,ds \text{ (or) } \iint_S \vec{F}\cdot d\vec{s} = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy$$

IIIly projection on yz plane, or xz plane is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\,dy\,dz = \iint_{R_2} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|}\,dx\,dz.$$

① Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$ and S is the part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Soln:-

$$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}, \quad \phi = x^2+y^2+z^2-1$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i}+y\vec{j}+z\vec{k})}{\sqrt{2^2x^2+2^2y^2+2^2z^2}} = \frac{2(x\vec{i}+y\vec{j}+z\vec{k})}{2\sqrt{x^2+y^2+z^2}}$$

$$= x\vec{i} + y\vec{j} + z\vec{k} \qquad \because x^2+y^2+z^2=1.$$

~~Theorem ② $\vec{F} = 12x^2\vec{j} - 3yz\vec{j} + 2z\vec{k}$, $x+y+z=1$. (120)~~

~~② $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$, $2x+3y+6z=12$~~

$$\vec{F}\cdot\hat{n} = (yz\vec{i} + zx\vec{j} + xy\vec{k})\cdot(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= xyz + xyz + xyz$$

$$= 3xyz$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \iint_R 3xy\cancel{z}\,\frac{dx\,dy}{\cancel{z}}$$

$$= \iint_R 3xy\, dx\, dy$$

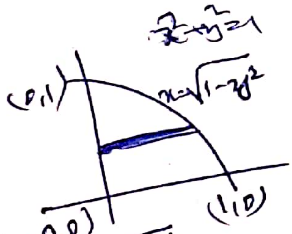
(or)

Let us convert this into polar ~~form~~ coordinates.

$x = r\cos\theta \qquad y = r\sin\theta \qquad dx\,dy = r\,d\theta\,dr$

$\therefore r = 0$ to $1 \qquad \theta = 0$ to $\pi/2$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \int_0^{\pi/2}\int_0^1 3r^2\cos\theta\sin\theta \cdot r\,dr\,d\theta$$



$$= \int_0^{\pi/2}\int_0^1 3r^3\sin\theta\cos\theta\, d\theta$$

$$= 3\int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^1 \sin\theta\cos\theta\, d\theta$$

$$\int_0^1\int_0^{\sqrt{1-y^2}} 3xy\,dx\,dy$$

$$= \frac{3}{4}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{3}{8}\int_0^{\pi/2}\sin 2\theta\, d\theta$$

$= \frac{3}{8}$ sq. units

$$= \frac{3}{8}\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi/2}$$

$$\int_0^1\int_0^{\sqrt{1-x^2}} 3xy\, dy\, dx$$

$$= \frac{3}{16}(-\cos\pi + \cos 0) = \frac{3}{16}(2)$$

$$= \int_0^1 3x\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 3x\frac{(1-x^2)}{2}\, dx$$

$$= \frac{3}{8} \text{ sq. units.}$$

$$= \frac{3}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^1$$

$$= \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}.$$

Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and S

(N) is the part of the plane $2x+3y+6z=12$ which is located in the first quadrant.

Soln:-

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy$$

where R is the projection of S in the xy-plane

Given $\phi = 2x+3y+6z-12$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}2+\vec{j}3+\vec{k}6}{\sqrt{2^2+3^2+6^2}} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}.$$

$$\vec{F}\cdot\hat{n} = (18z\vec{i} - 12\vec{j} + 3y\vec{k})\cdot\left(\frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}\right)$$

$$= \frac{36z - 36 + 18y}{7}.$$

$$|\hat{n}\cdot\vec{k}| = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}\cdot\vec{k}$$

$$= \frac{6}{7}.$$

$$\therefore \iint_R \frac{36z-36+18y}{\cancel{7}}\times\frac{\cancel{7}}{6}\,dx\,dy = \iint_R \frac{\cancel{18}^{3}(2z-2+y)}{\cancel{6}}\,dx\,dy$$

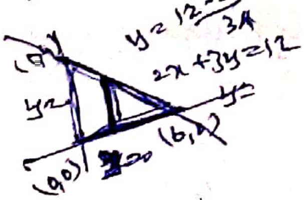
$$= 3\iint_R \left[2\left(\frac{12-3y-2x}{6}\right) - 2 + y\right]dx\,dy$$

2x+3y=12

x=0, y=4

y=0, x=6

$$= 3\iint_R \left(4 - \cancel{y} - \frac{2x}{3} - 2 + \cancel{y}\right)dx\,dy = 3\iint_R 2 - \frac{2x}{3}\,dx\,dy$$



$$= \iint_R (6-2x)\,dx\,dy = \int_{x=0}^{x=6}\int_0^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

$$= \int_0^6 (6-2x)\,y\,dx = \int_0^6 (6-2x)\left(\frac{12-2x}{3}\right)dx$$

$$= \frac{1}{3}\int_0^6 (72-36x+4x^2)\,dx = \frac{1}{3}\left(72x - \frac{36x^2}{2} + \frac{4x^3}{3}\right)_0^6 = 24 \text{ sq. units.}$$

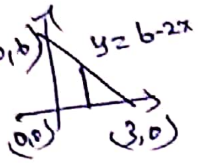
H.W

3) $\iint_S \vec{F}\cdot\hat{n}\,ds$ If $\vec{F} = \vec{i}(x+y^2) - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

$$\nabla\phi = 2\vec{i} + \vec{j} + 2\vec{k}$$

$$|\nabla\phi| = 3$$

$$\hat{n} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{3} \qquad \hat{n}\cdot\vec{k} = \frac{2}{3}$$



$$\vec{F}\cdot\hat{n} = \frac{2}{3}(x+y^2 - x + 2yz) = \frac{2}{3}(y^2+2yz) = \frac{4}{3}(3-x)y$$

put z value

x: 0 to 3

y: 0 to $6-2x$

$$= 2\int_0^3\int_0^{6-2x} (3-x)\,y\,dy\,dx = 81 \text{ units.}$$

4. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Soln:- Let $\phi = x^2+y^2-16$ be the given surface

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}.$$

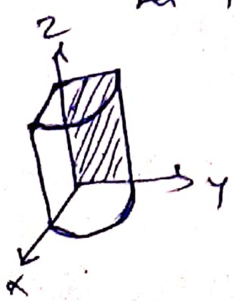
$$|\nabla\phi| = 2\sqrt{x^2+y^2} = 2\times 4 = 8$$

$$\hat{n} = \frac{2x\vec{i}+2y\vec{j}}{8} = \frac{x\vec{i}+y\vec{j}}{4}.$$

$$\vec{F}\cdot\hat{n} = (z\vec{i}+x\vec{j}-3y^2z)\cdot\left(\frac{x\vec{i}+y\vec{j}}{4}\right)$$

$$= \frac{xz}{4} + \frac{xy}{4}.$$

Let R be the projection of the surface S on yz plane



$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\,dy\,dz$$

$$|\hat{n}\cdot\vec{i}| = \frac{x}{4}$$

$$\iint_R \frac{\frac{x}{4}(z+y)}{\frac{x}{4}}\, dy\, dz$$

y : 0 to 4 & $z = 0$ to 5

yz plane $x=0$ $x^2+y^2=16$, $y^2=16$, $y=\pm 4$

$$\int_0^4 \int_0^5 (z+y)\, dy\, dz$$

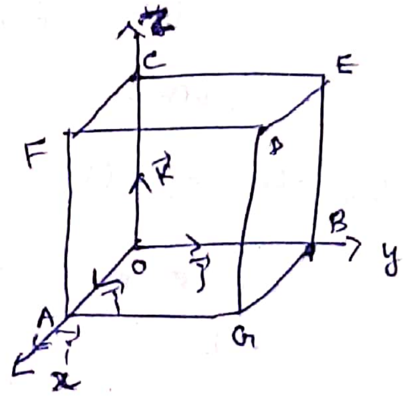
$$= \int_0^4 \left(\frac{z^2}{2}+yz\right)_0^5 dy = \int_0^4 \left(\frac{25}{2}+5y\right) dy$$

$$= \left.\frac{25y}{2}+\frac{5y^2}{2}\right]_0^4 = \frac{100}{2}+\frac{16\times 5}{2}$$

$$= 50 + 40 = 90 \,//.$$

(5) (8) If $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$, evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Soln:- $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$



S is the surface of the cube bdd by the planes $x=0$, $x=1$, $y=0$, $y=1$ & $z=0$, $z=1$ and it is a cube bdd by six faces.

Let it be S_1 : AGDF, S_2 OBEC, S_3 BEDG, S_4 OAFC, S_5 CFDE; S_6 OAGB.

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} \vec{F}\cdot\hat{n}\, ds + \iint_{S_2} \vec{F}\cdot\hat{n}\, ds + \iint_{S_3} \vec{F}\cdot\hat{n}\, ds$$

$$+ \iint_{S_4} \vec{F}\cdot\hat{n}\, ds + \iint_{S_5} \vec{F}\cdot\hat{n}\, ds + \iint_{S_6} \vec{F}\cdot\hat{n}\, ds.$$

For the face AGDF, $x=1$, $\vec{n}=\vec{i}$, $|\hat{n}\cdot\vec{i}|=1$.

$$\iint_{AGDF} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4z\,dy\,dz = \int_0^1 4z(y)_0^1\,dz = 4\left(\frac{z^2}{2}\right)_0^1 = 2.$$

For the face OBEC, $x=0$, and $\hat{n}=-\vec{i}$

$$\iint_{OBEC} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -(-y^2\vec{j}+yz\vec{k})\cdot\vec{i}\,dy\,dz$$

$$= 0.$$

For the face BEDG, $y=1$ and $\vec{n}=\vec{j}$

$$\therefore \iint_{BEDG} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\vec{i}-\vec{j}+z\vec{k})\cdot\vec{j}\,dx\,dz$$

$$= \int_0^1\int_0^1 -dx\,dz = -1$$

For the face OAFC, $y=0$ & $\vec{n}=-\vec{j}$

$$\therefore \iint_{OAFC} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4xz\vec{i}\cdot(-\vec{j})\,dx\,dz = 0.$$

For the face CFDE, $z=1$, and $\vec{n}=\vec{k}$

$$\therefore \iint_{CFDE} \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^1 (4x\vec{i}-y^2\vec{j}+y\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy = \frac{1}{2}.$$

For the face OAGB, $z=0$, $\hat{n}=-\vec{k}$

$$\iint_{OAGB} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (-y^2\vec{j})\cdot(-\vec{k})\,dx\,dy = 0.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 2+0-1+0+\frac{1}{2}+0 = \frac{3}{2}$$ //.

6. If $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ find $\iint (\nabla\times\vec{F})\cdot\hat{n}\, ds$ over the region bdd by the lines $x=0,\ y=0,\ x=1,\ y=1$.

Soln:-

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y)$$

$$= 4y\vec{k}$$

Since the region lies in the xy plane, we have $\hat{n} = \vec{k}$ and $ds = dx\,dy$.

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_0^1\int_0^1 4y\vec{k}\cdot\vec{k})\, dx\,dy$$

$$= \int_0^1\int_0^1 4y\, dx\,dy = \int_0^1 4y\,x]_0^1\, dy = \int_0^1 4y\,dy$$

$$= 4\,\frac{y^2}{2}\Big]_0^1 = 2.$$

H.W

1. Evaluate $\iint_S \vec{A}\cdot\hat{n}\, ds$ for $\vec{A} = 12x^2y\vec{i} - 3yz\vec{j} + 2z\vec{k}$ and S is the portion of the plane $x+y+z=1$ included in first octant. Ans $49/120$.

2. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = xy\vec{i} - x^2\vec{j} + (x+z)\vec{k}$ and S is the region of the plane $2x+2y+z=6$ bdd in the first octant. Ans: $\frac{27}{4}$.

17

$\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$

$2x+y+2z=6$ $\Rightarrow 2z = 6-2x-y$

$\phi = 2x+y+2z-6$

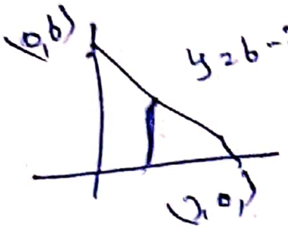
$\nabla\phi = 2\vec{i}+\vec{j}+2\vec{k}$ $\hat{n} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{3}$

$|\nabla\phi| = \sqrt{4+1+4} = 3$

$|\hat{n}\cdot\vec{k}| = \frac{2}{3}$

$2x+y=6$

$\iint_R \frac{\frac{2}{3}(x+y^2) - 2x + 4yz}{\frac{2}{3}}\, dx\,dy$



$= \iint_R x+y^2 - 2x + \cancel{4}y\,\cancel{x}\left(\frac{6-2x-y}{2}\right) dx\,dy$

$= \frac{1}{2}\iint_R \cancel{(x+y^2-2x+12y-4xy-2y^2)}\,dx\,dy$

$= \iint \cancel{x}+\cancel{y^2} - \cancel{x} + 6y - 2xy - \cancel{y^2}$

$= \iint_R (6y-2xy)\,dx\,dy$

$= \int_0^3\int_0^{6-2x} 6y - 2xy\, dx\, dy$

$= \int_0^3 \left(\frac{6y^2}{2} - 2x\frac{y^2}{2}\right)_0^{6-2x}$

$= \int_0^3 3(6-2x)^2 - x(6-2x)^2\,dx$

$= \int_0^3 3(36+4x^2-24x) - x(36+4x^2-24x)$

$= 3\left(36x+\frac{4x^3}{3}-\frac{24x^2}{2}\right)_0^3$

$\cancel{-6\left(36x+\frac{4x^3}{3}-\frac{24x^2}{2}\right)}$

$-36\frac{x^2}{2} - 4\frac{x^4}{4}+24\frac{x^3}{3}\Big|_0^3$

$= 3(36\times3 + \frac{4\times81}{3} + 2$

$= 81$