(i) Verify that $\int_0^1 \int_0^1 f(x, y)\, dx dy = 1.$

(ii) Find $P(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2)$, $P(X + Y < 1)$, $P(X > Y)$ *and* $P(X < 1 \mid Y < 2)$.

Solution. (i)

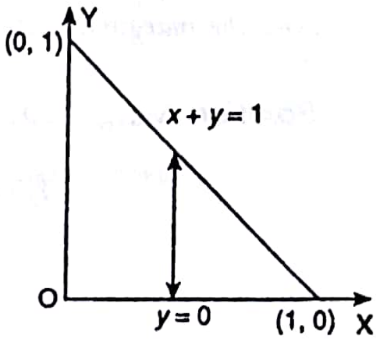
$$\int_0^1 \int_0^1 f(x, y)\, dxdy = \int_0^1 \int_0^1 6x^2y\, dxdy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^1 dx = \int_0^1 3x^2 dx = \left|x^3\right|_0^1 = 1$$

$$(ii)\ P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right) = \int_0^{3/4} \int_{1/3}^{1} 6x^2y\, dxdy + \int_0^{3/4} \int_1^2 0.\, dxdy$$

$$= \int_0^{3/4} 6x^2 \left|\frac{y^2}{2}\right|_{1/3}^1 dx = \frac{8}{9} \int_0^{3/4} 3x^2\, dx = \frac{8}{9}\left|x^3\right|_0^{3/4} = \frac{3}{8}.$$

$$P(X + Y < 1) = \int_0^1 \int_0^{1-x} 6x^2y\, dxdy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^{1-x} dx$$

$$= \int_0^1 3x^2(1-x)^2 dx = \frac{1}{10} \text{ [See Fig.]}$$



$$P(X > Y) = \int_0^1 \int_0^x 6x^2y\, dxdy = \int_0^1 3x^2 \left|y^2\right|_0^x dx$$

$$= \int_0^1 3x^4 dx = \frac{3}{5}.$$

$$P(X < 1 \mid Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)}$$

where $P(X < 1 \cap Y < 2) = \int_0^1 \int_0^1 6x^2\, y\, dxdy + \int_0^1 \int_1^2 0.\, dxdy = 1$

and $P(Y < 2) = \int_0^1 \int_0^2 f(x,y)\, dxdy = \int_0^1 \int_0^1 6x^2\, y\, dxdy + \int_0^1 \int_1^2 0.\, dxdy = 1$

$\therefore$ $P(X < 1 \mid Y < 2) = \dfrac{P(X < 1 \cap Y < 2)}{P(Y < 2)} = 1.$

Example 5·39. *The joint probability density function of a two-dimensional random variable (X, Y) is given by :*

$$f(x, y) = \begin{cases} 2\,;\, 0 < x < 1,\, 0 < y < x\,; \\ 0, \text{ elsewhere} \end{cases}$$

(i) Find the marginal density functions of X and Y.

(ii) Find the conditional density function of Y given X = x and conditional density function of X given Y = y.

(iii) Check for independence of X and Y.

Solution. Evidently $f(x,y) \geq 0$ and $\int_0^1 \int_0^x 2\, dx\, dy = 2\int_0^1 x\, dx = 1.$

(i) The marginal *p.d.f.'s* of X and Y are given by :

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^x 2dy = 2x, 0 < x < 1 \\ 0, \text{ elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_y^1 2dx = 2(1-y),\ 0 < y < 1 \\ 0 \text{ , elsewhere} \end{cases}$$

(*ii*) The conditional density function of Y given X , $(0 < x < 1)$ is :

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x},\ 0 < y < x.$$

The conditional density function of X given Y, $(0 < y < 1)$ is :

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)},\ y < x < 1$$

(*iii*) Since $f_X(x) f_Y(y) = 2(2x)(1-y) \neq f_{XY}(x, y)$, X and Y are not independent.

Example 3.15: A candy company distributes boxes of chocolates with a mixture of creams, toffees, and nuts coated in both light and dark chocolate. For a randomly selected box, let X and Y, respectively, be the proportions of the light and dark chocolates that are creams and suppose that the joint density function is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify condition 2 of Definition 3.9.

(b) Find $P[(X,Y \in A]$, where $A = \{(x,y) | 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

***Solution*:** (a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$$

$$= \int_0^1 \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5}\right) dy = \left(\frac{2y}{5} + \frac{3y^2}{5}\right)\Big|_0^1 = \frac{2}{5} + \frac{3}{5} = 1.$$

(b)

$$P[(X,Y) \in A] = P(0 < X < \frac{1}{2}, \frac{1}{4} < Y < \frac{1}{2})$$

$$= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\,dx\,dy = \int_{1/4}^{1/2} \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Big|_{x=0}^{x=1/2} dy$$

$$= \int_{1/4}^{1/2} \left(\frac{1}{10} + \frac{3y}{5}\right) dy = \left(\frac{y}{10} + \frac{3y^2}{10}\right)\Big|_{1/4}^{1/2}$$

$$= \frac{1}{10}\left[\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4} + \frac{3}{16}\right)\right] = \frac{13}{160}.$$

Given the joint probability distribution $f(x, y)$ of the ...

Example 3.19: The joint density for the random variables (X, Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f(x, y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities $g(x)$, $h(y)$, and the conditional density $f(y|x)$.

(b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased to 0.25 unit.

***Solution*:** (a) By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{x}^{1} 10xy^2\, dy$$

$$= \frac{10}{3} xy^3 \Big|_{y=x}^{y=1} = \frac{10}{3} x(1 - x^3),\ 0 < x < 1,$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{0}^{y} 10xy^2\, dx = 5x^2y^2 \Big|_{x=0}^{x=y} = 5y^4,\ 0 < y < 1.$$

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Now

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3},\ 0 < x < y < 1.$$

(b) Therefore,

$$P\left(Y > \frac{1}{2}\middle| X = 0.25\right) = \int_{1/2}^{1} f(y|x = 0.25)dy$$
$$= \int_{1/2}^{1} \frac{3y^2}{1-0.25^3}\, dy = \frac{8}{9}.$$

Example 3.20: Given the joint density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find $g(x)$, $h(y)$, $f(x|y)$, and evaluate $P(\frac{1}{4} < X < \frac{1}{2}|Y = \frac{1}{3})$.

Solution: By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 \frac{x(1+3y^2)}{4}dy$$
$$= \left(\frac{xy}{4} + \frac{xy^3}{4}\right)\Bigg|_{y=0}^{y=1} = \frac{x}{2},\quad 0 < x < 2,$$

and

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^2 \frac{x(1+3y^2)}{4}dx$$
$$= \left(\frac{x^2}{8} + \frac{3x^2y^2}{8}\right)\Bigg|_{x=0}^{x=2} = \frac{1+3y^2}{2},\quad 0 < y < 1.$$

Therefore,

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2},\quad 0 < x < 2,$$

and

$$P\left(\frac{1}{4} < X < \frac{1}{2}\middle| Y = \frac{1}{3}\right) = \int_{1/4}^{1/2} \frac{x}{2}dx = \frac{3}{64}.$$