MULTIPLE INTEGRALS

Triple Integrals

In this section, we will learn about:

Triple integrals and their applications.

TRIPLE INTEGRALS

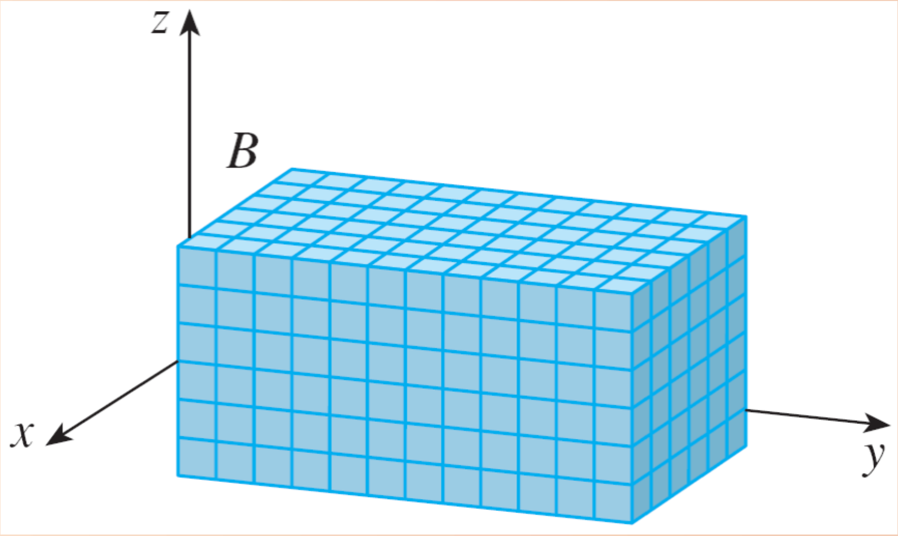
Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

Equation 1

Let's first deal with the simplest case where *f* is defined on a rectangular box:

$$B = \left\{ \left(x, y, z \right) \middle| a \le x \le b, c \le y \le d, r \le z \le s \right\}$$

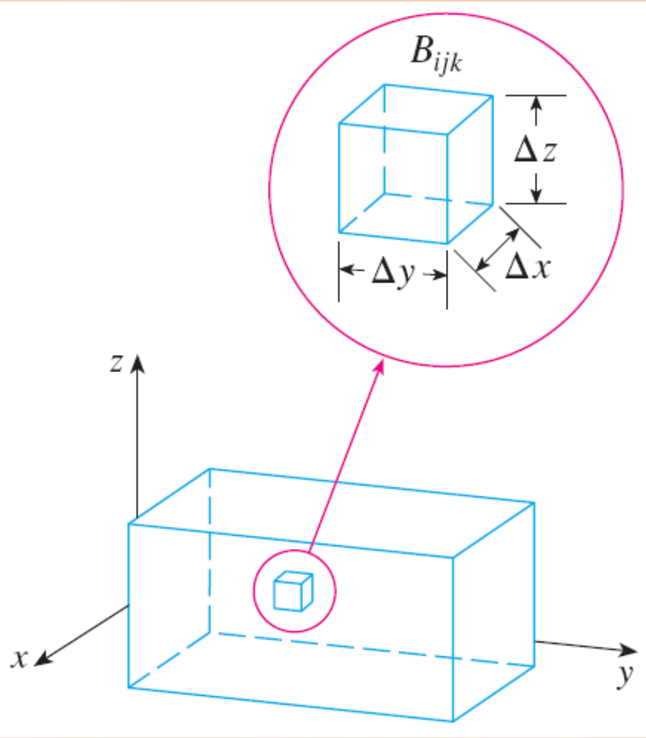
The first step is to divide *B* into sub-boxes—by dividing:



- The interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx.
- $[c, d]$ into m subintervals of width Δy.
- $[r, s]$ into n subintervals of width Δz.

TRIPLE INTEGRALS

The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box *B* into *lmn* sub-boxes



$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

- Each sub-box has volume $\Delta V = \Delta x \, \Delta y \, \Delta z$

 Equation 2

Then, we form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*}\right) \Delta V$$

where the sample point $\left(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*}\right)$ is in B_{ijk}.

By analogy with the definition of a double integral (Definition 5 in Section 15.1, Thomas Calculus),

we define the triple integral as the limit of the triple Riemann sums in Equation 2.

Definition 3

The triple integral of *f* over the box *B* is:

$$\iiint_B f(x, y, z)\,dV$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V$$

if this limit exists.

- Again, the triple integral always exists if *f* is continuous.

We can choose the sample point to be any point in the sub-box.

However, if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression:

$$\iiint_B f(x, y, z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta V$$

TRIPLE INTEGRALS

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals, as follows.

FUBINI'S TH. (TRIPLE INTEGRALS) **Theorem 4**

If f is continuous on the rectangular box

$B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV$$

$$= \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

FUBINI’S TH. (TRIPLE INTEGRALS)

The iterated integral on the right side of Fubini’s Theorem means that we integrate in the following order:

1. With respect to x (keeping y and z fixed)
2. With respect to y (keeping z fixed)
3. With respect to z

FUBINI'S TH. (TRIPLE INTEGRALS)

There are five other possible orders in which we can integrate, all of which give the same value.

- For instance, if we integrate with respect to *y*, then *z*, and then *x*, we have:

$$\iiint_B f(x, y, z)\,dV$$

$$= \int_a^b \int_r^s \int_c^d f(x, y, z)\,dy\,dz\,dx$$

Evaluate the triple integral

$$\iiint_B xyz^2\,dV$$

where *B* is the rectangular box

$$B = \{(x, y, z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$$

We could use any of the six possible orders of integration.

If we choose to integrate with respect to x, then y, and then z, we obtain the following result.

FUBINI'S TH. (TRIPLE INTEGRALS) **Example 1**

$$\iiint_B xyz^2\,dV = \int_0^3\int_{-1}^2\int_0^1 xyz^2\,dx\,dy\,dz$$

$$= \int_0^3\int_{-1}^2\left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dy\,dz$$

$$= \int_0^3\int_{-1}^2 \frac{yz^2}{2}\,dy\,dz$$

$$= \int_0^3\left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=1} dz = \int_0^3 \frac{3z^2}{4}\,dz = \frac{z^3}{4}\Big]_0^3 = \frac{27}{4}$$

INTEGRAL OVER BOUNDED REGION

Now, we define the triple integral over a general bounded region E in three-dimensional space (a solid) by much the same procedure that we used for double integrals.

-

INTEGRAL OVER BOUNDED REGION

We enclose E in a box B of the type given by Equation 1.

Then, we define a function F so that it agrees with f on E but is 0 for points in B that are outside E.

INTEGRAL OVER BOUNDED REGION

By definition,

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

- This integral exists if *f* is continuous and the boundary of *E* is "reasonably smooth."
- The triple integral has essentially the same properties as the double integral (Properties 6–9 in Section 15.3).

We restrict our attention to:

- Continuous functions *f*
- Certain simple types of regions

TYPE 1 REGION

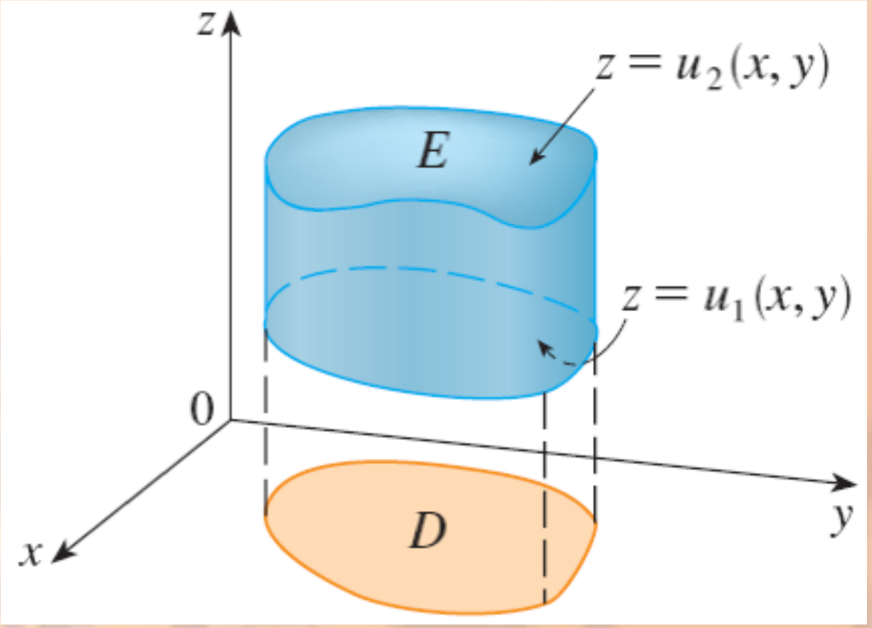
A solid region is said to be of type 1 if it lies between the graphs of two continuous functions of x and y.

Equation 5

That is,

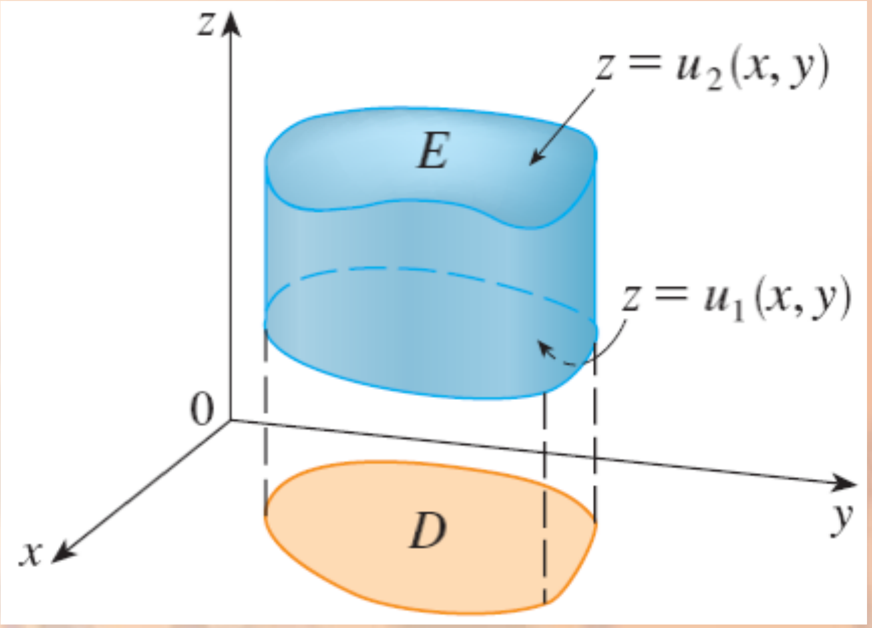
$$E=\{(x,y,z)\,|\,(x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\}$$

where *D* is the projection of *E* onto the *xy*-plane.



Notice that:

- The upper boundary of the solid E is the surface with equation $z = u_2(x, y)$.
- The lower boundary is the surface $z = u_1(x, y)$.



By the same sort of argument that led to Formula 3 in Section 15.3, it can be shown that, if *E* is a type 1 region given by Equation 5, then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

TYPE 1 REGIONS

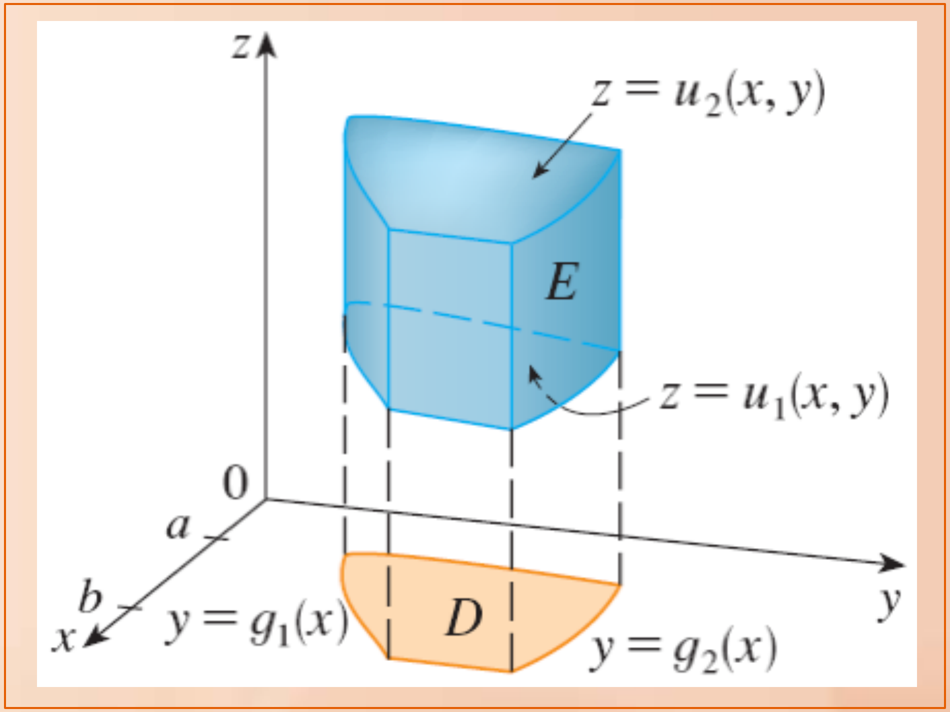
The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed.

Therefore,

- $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants.
- $f(x, y, z)$ is integrated with respect to z.

TYPE 1 REGIONS

In particular, if the projection *D* of *E* onto the *xy*-plane is a type I plane region, then



$$E = \left\{ (x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y) \right\}$$

Equation 7

Thus, Equation 6 becomes:

$$\iiint_E f(x, y, z)\,dV$$

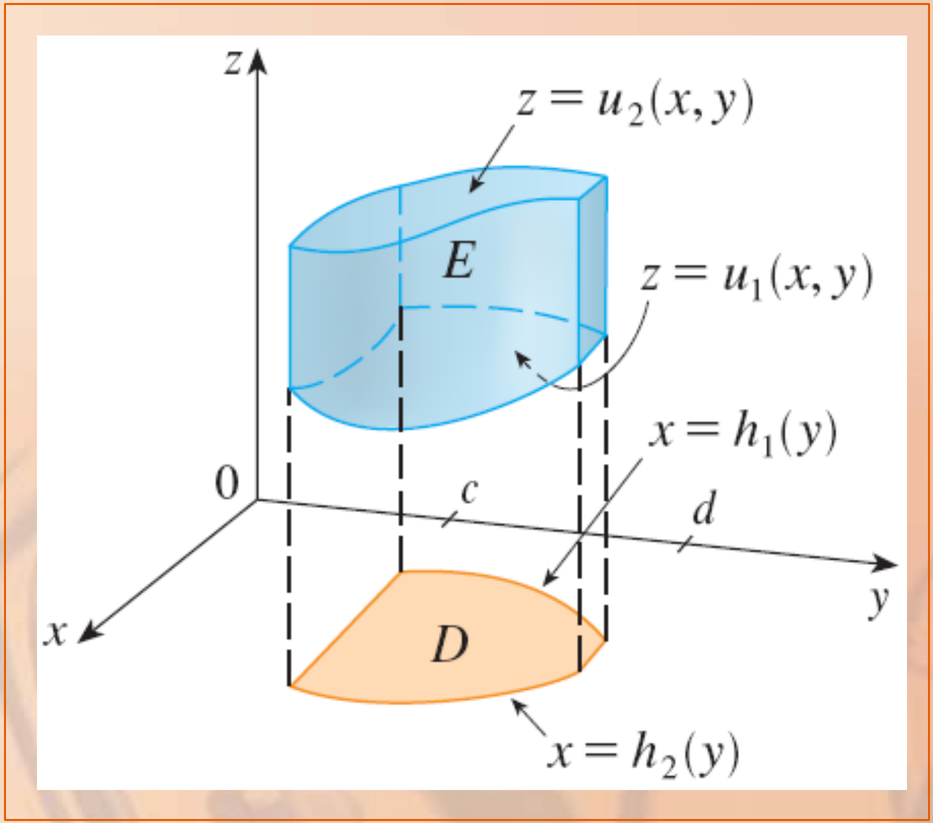
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\,dy\,dx$$

TYPE 1 REGIONS

If, instead, *D* is a type II plane region, then

$E =$

$$\{(x, y, z) \mid c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x, y) \le z \le u_2(x, y)\}$$



Equation 8

Then, Equation 6 becomes:

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Evaluate

$$\iiint_E z\,dV$$

where E is the solid tetrahedron bounded by the four planes

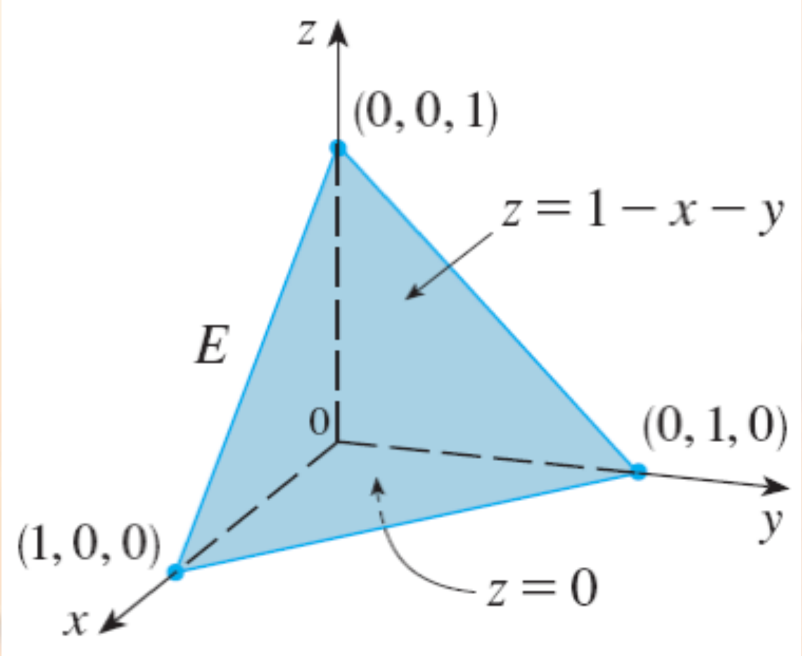
$$x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$$

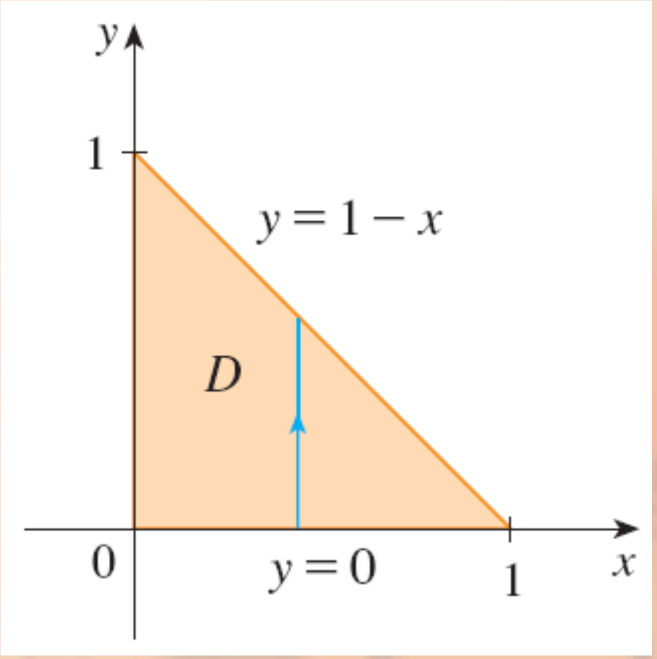
TYPE 1 REGIONS

Example 2

When we set up a triple integral, it's wise to draw two diagrams:

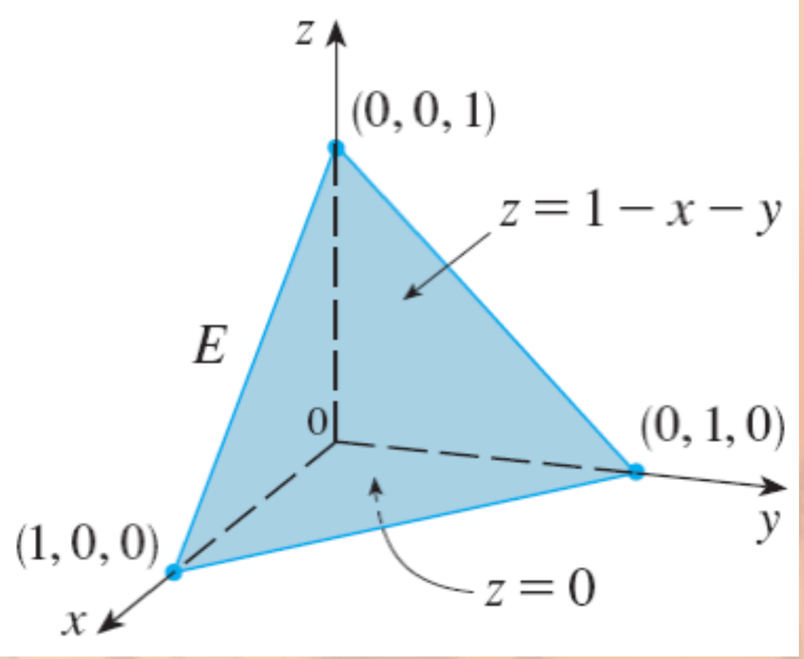
- The solid region E
- Its projection *D* on the *xy*-plane





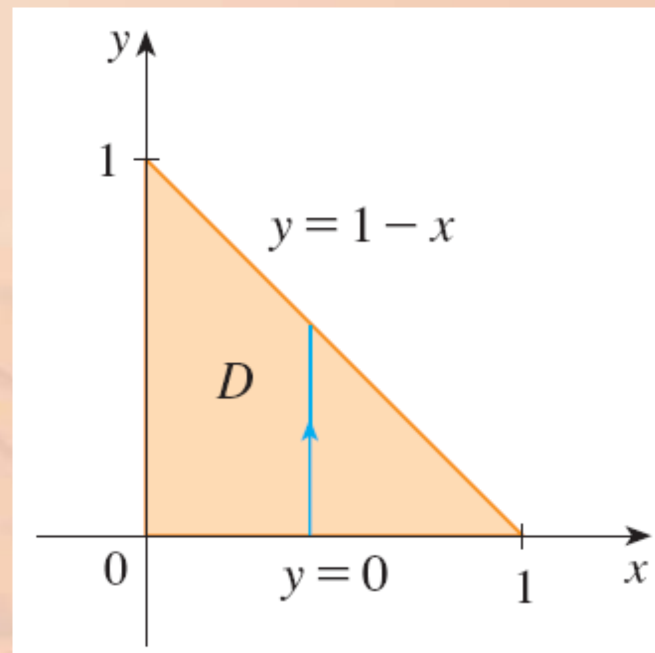
The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$ (or $z = 1 - x - y$).

- So, we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$ in Formula 7.



Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ (or $y = 1 - x$) in the *xy*-plane.

- So, the projection of E is the triangular region shown here, and we have the following equation.



TYPE 1 REGIONS

E. g. 2—Equation 9

$$E = \left\{ (x, y, z) \middle| 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y \right\}$$

- This description of *E* as a type 1 region enables us to evaluate the integral as follows.

TYPE 1 REGIONS

Example 2

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\,dx$$

$$= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2\,dy\,dx$$

$$= \tfrac{1}{2} \int_0^1 \left[-\frac{(1-x-y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6} \int_0^1 (1-x)^3\,dx$$

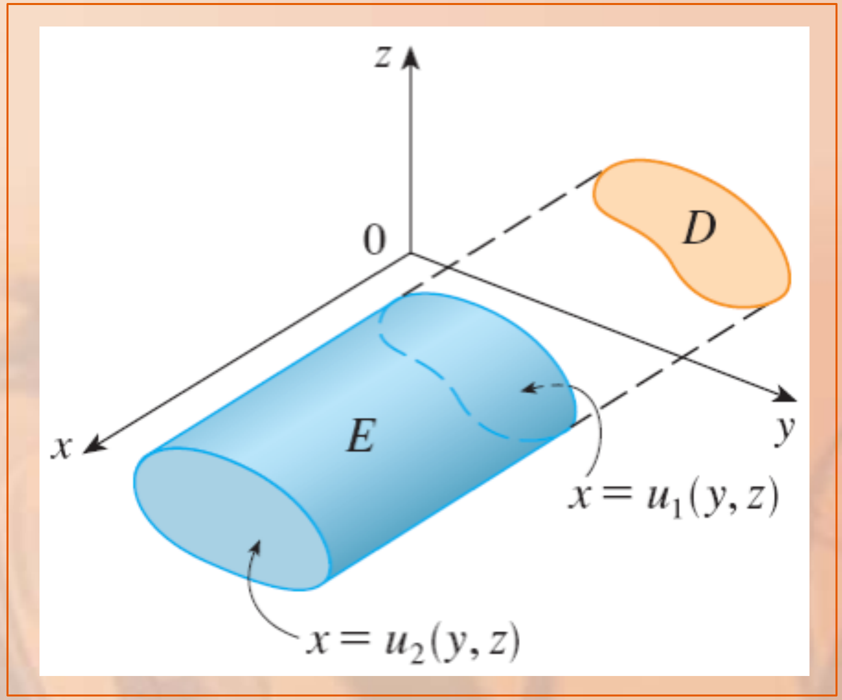
$$= \frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1 = \frac{1}{24}$$

TYPE 2 REGION

A solid region *E* is of type 2 if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$$

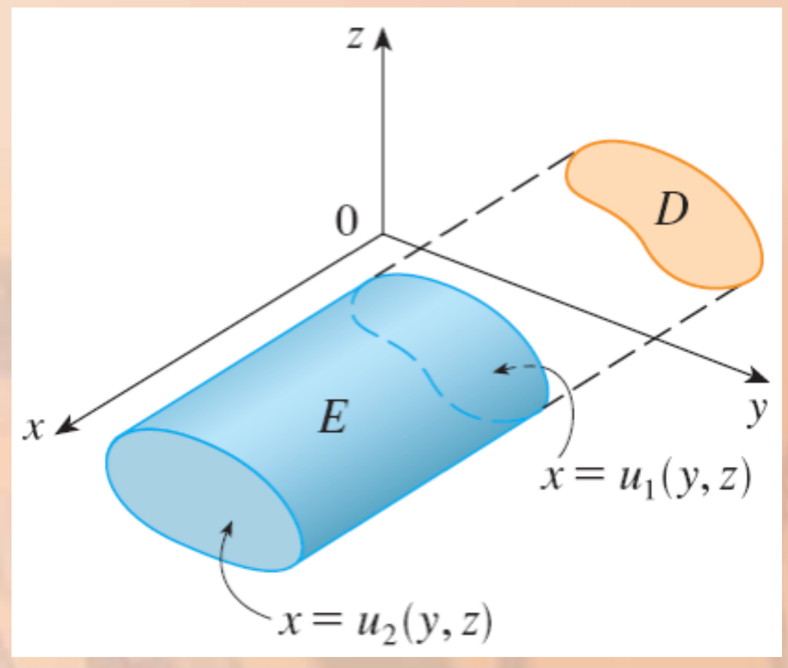
where *D* is the projection of *E* onto the *yz*-plane.



TYPE 2 REGION

The back surface is $x = u_1(y, z)$.

The front surface is $x = u_2(y, z)$.



TYPE 2 REGION

Equation 10

Thus, we have:

$$\iiint_E f(x, y, z)\,dV$$

$$= \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\,dx \right] dA$$

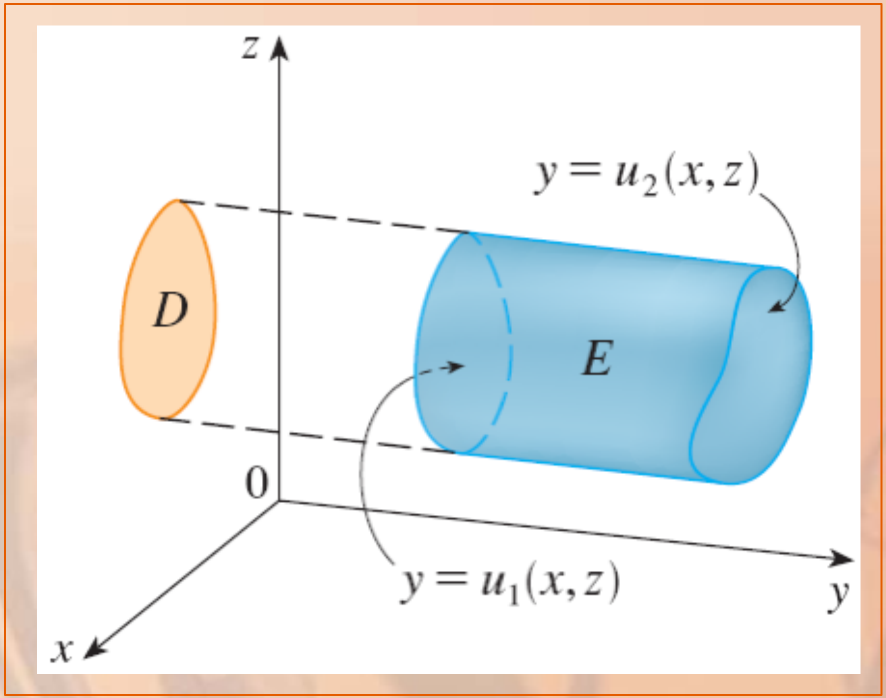
TYPE 3 REGION

Finally, a type 3 region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

where:

- D is the projection of E onto the xz-plane.
- $y = u_1(x, z)$ is the left surface.
- $y = u_2(x, z)$ is the right surface.

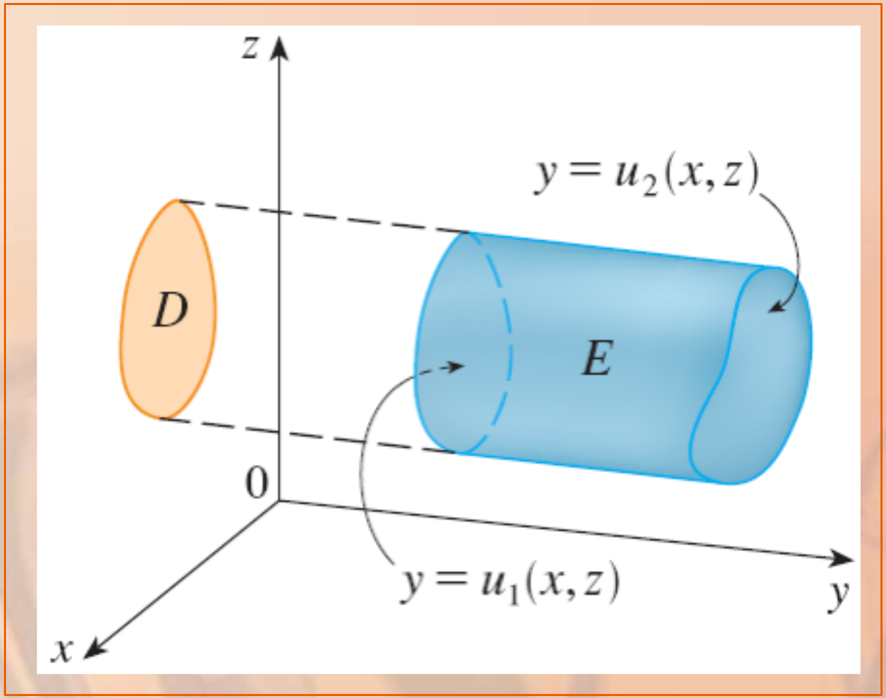


TYPE 3 REGION

Equation 11

For this type of region, we have:

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u2(x,z)} f(x,y,z)\,dy\right] dA$$



TYPE 2 & 3 REGIONS

In each of Equations 10 and 11, there may be two possible expressions for the integral depending on:

- Whether D is a type I or type II plane region (and corresponding to Equations 7 and 8).

Evaluate

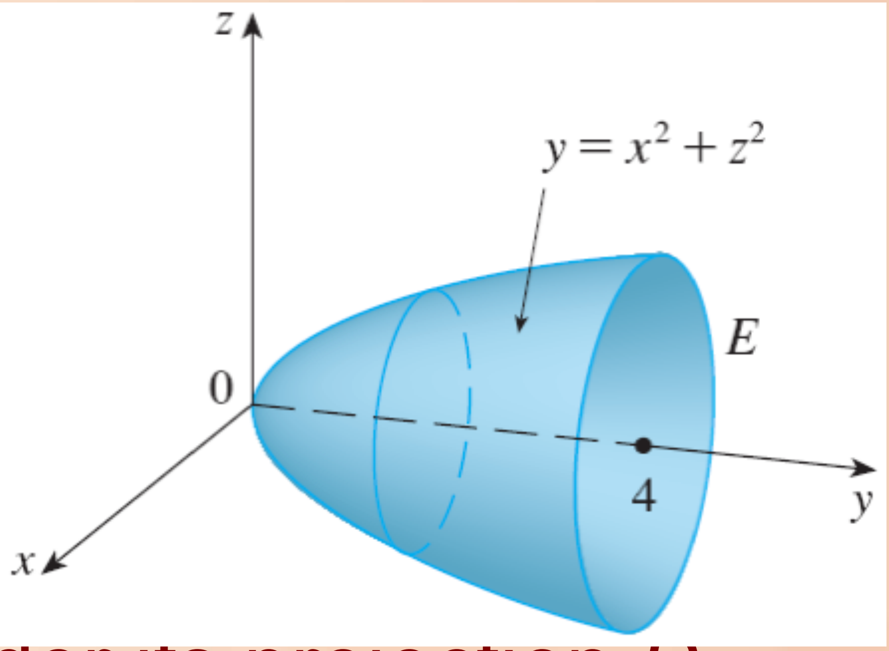
$$\iiint_E \sqrt{x^2 + z^2}\, dV$$

where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

TYPE 1 REGIONS

Example 3

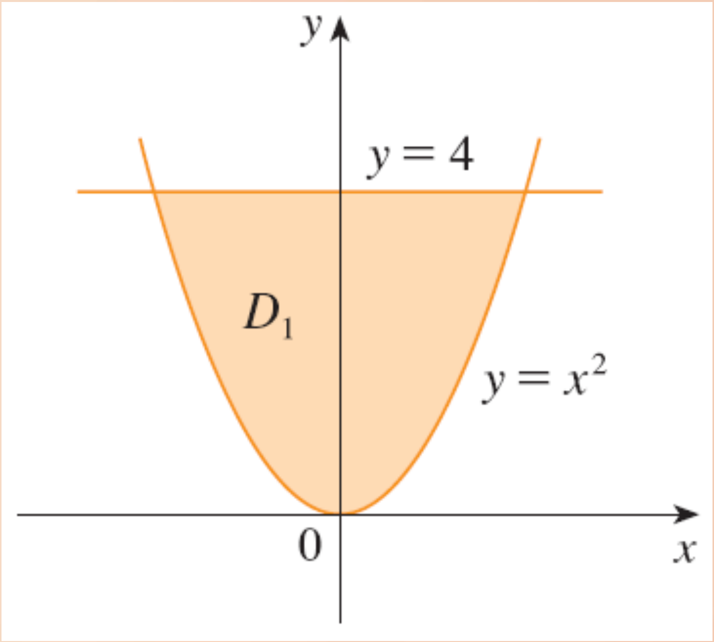
The solid E is shown here.



If we regard it as a type 1 region, then we need to consider its projection D_1 onto the xy-plane.

That is the parabolic region shown here.

- The trace of $y = x^2 + z^2$ in the plane $z = 0$ is the parabola $y = x^2$



From $y = x^2 + z^2$, we obtain:

$$z = \pm\sqrt{y - x^2}$$

- So, the lower boundary surface of *E* is:
- The upper surface is:

$$z = -\sqrt{y - x^2}$$

$$z = \sqrt{y - x^2}$$

Example 3

Therefore, the description of *E* as a type 1 region is:

$$E = \left\{ (x, y, z) \middle| -2 \le x \le 2, x^2 \le y \le 4, -\sqrt{y - x^2} \le z \le \sqrt{y - x^2} \right\}$$

Thus, we obtain:

$$\iiint_E \sqrt{x^2 + y^2}\, dV$$

$$= \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

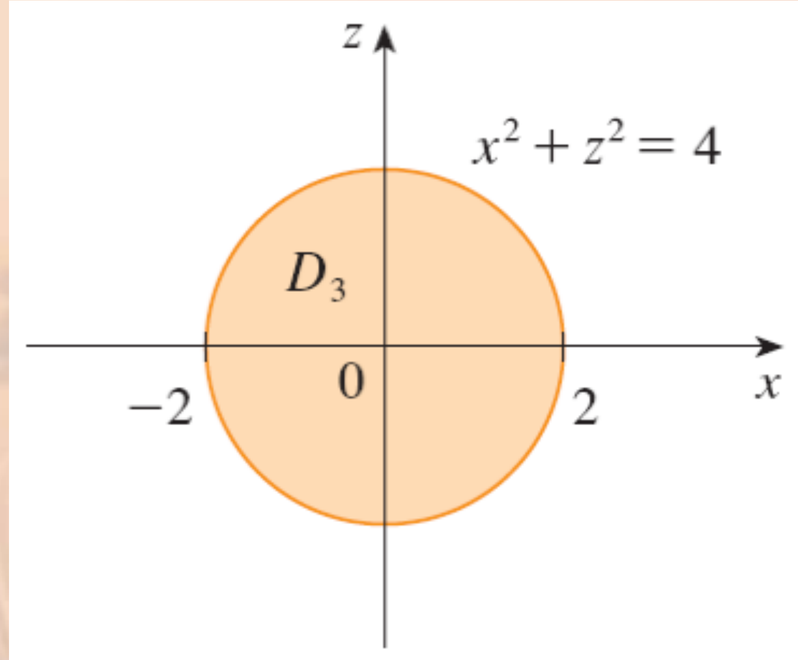
- Though this expression is correct, it is extremely difficult to evaluate.

TYPE 3 REGIONS

Example 3

So, let's instead consider *E* as a type 3 region.

- As such, its projection D_3 onto the *xz*-plane is the disk $x^2 + z^2 \leq 4$.



Then, the left boundary of E is the paraboloid $y = x^2 + z^2$.

The right boundary is the plane $y = 4$.

So, taking $u_1(x, z) = x^2 + z^2$ and $u_2(x, z) = 4$ in Equation 11, we have:

$$\iiint_E \sqrt{x^2 + y^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy \right] dA$$

$$= \iint_{D_3} \left(4 - x^2 - z^2\right) \sqrt{x^2 + z^2}\, dA$$

This integral could be written as:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4 - x^2 - z^2\right)\sqrt{x^2 + z^2}\, dz\, dx$$

However, it's easier to convert to polar coordinates in the *xz*-plane:

$x = r \cos \theta$, $z = r \sin \theta$

That gives:

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 \left(4-r^2\right) r\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(4r^2-r^4\right) dr$$

$$= 2\pi \left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$