Calculus for Engineers

LAPLACE TRANSFORM

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INTRODUCTION

- **Introduced by a french mathematician "Pierre Simmon Marquis De Laplace (1749-1827)".**
- **Laplace Transformations is a powerful Technique; it replaces operations of calculus by operations of Algebra.**
- **Suppose an Ordinary (or) Partial Differential Equation together with Initial conditions is reduced to a problem of solving an Algebraic Equation.**
- **The Laplace transform takes a function of time $f(t)$ and transforms it to a function of a complex variable $F(s)$.**

Idea

the Laplace transform converts *integral* and *differential* equations into *algebraic* equations

this is like phasors, but

- applies to general signals, not just sinusoids
- handles non-steady-state conditions

allows us to analyze

- LCCODEs
- complicated circuits with sources, Ls, Rs, and Cs
- complicated systems with integrators, differentiators, gains

- Let f be a function. Its Laplace transform (function) is denoted by the corresponding capitol letter F. Another notation is $\mathcal{L}(f)$.
- Input to the given function f is denoted by t; input to its Laplace transform F is denoted by s.
- By default, the domain of the function $f=f(t)$ is the set of all non-negative real numbers. The domain of its Laplace transform depends on f and can vary from a function to a function.

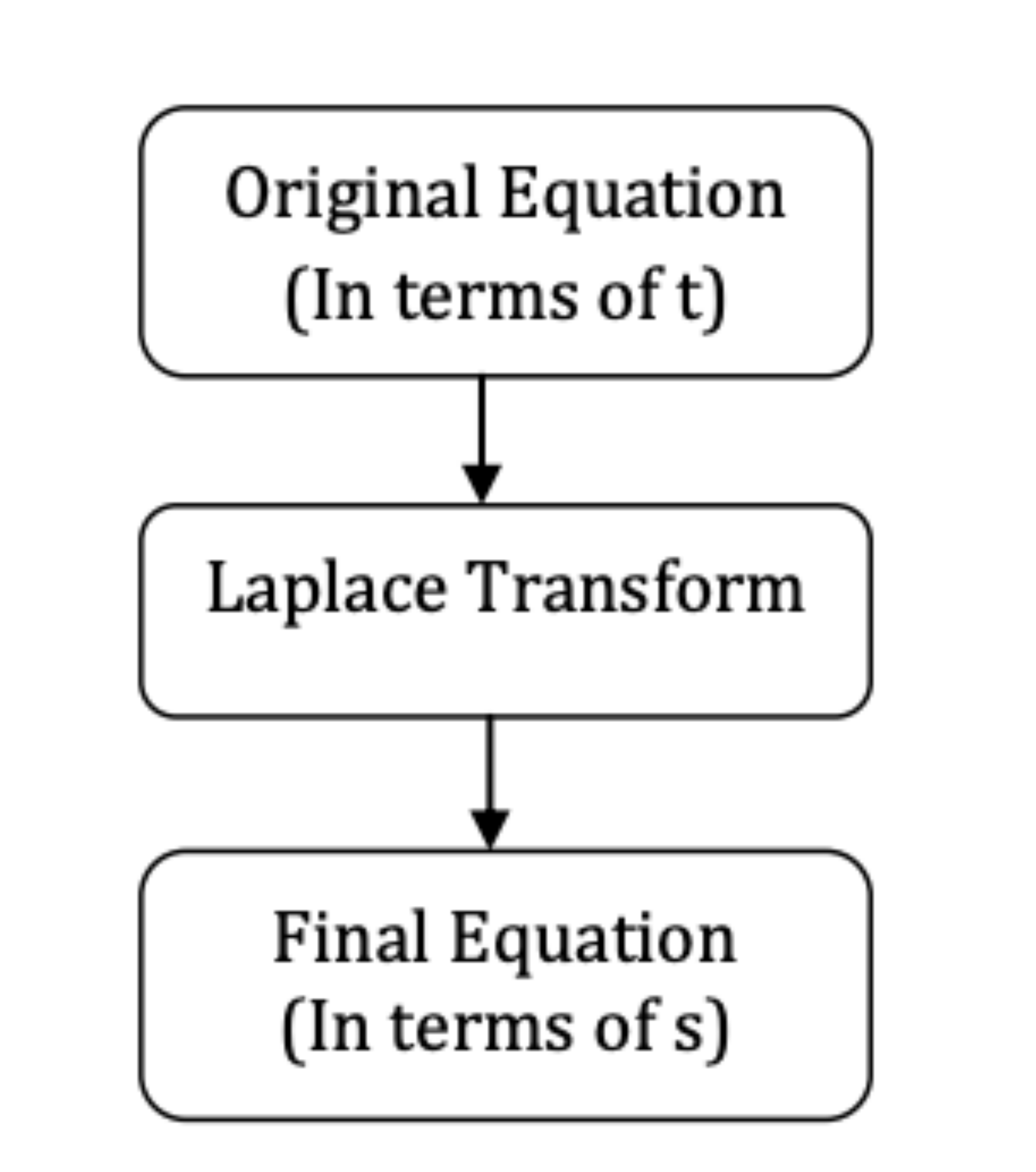
Definition of the Laplace Transform

- The Laplace transform ***F*=*F*(*s*)** of a function ***f*=*f*(*t*)** is defined by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-ts} f(t)\,dt.$$

- The integral is evaluated with respect to *t*, hence once the limits are substituted, what is left are in terms of *s*.

PROCEDURE



Original Equation
(In terms of t)
Laplace Transform
Final Equation
(In terms of s)

Not all functions $f(t)$, where t is any variable, are Laplace transformable. For a function $f(t)$ to be Laplace transformable, it must satisfy the Dirichlet conditions — a set of sufficient but not necessary conditions. These are

1. $f(t)$ must be piecewise continuous; that is, it must be single valued but can have a finite number of finite isolated discontinuities for $t > 0$.
2. $f(t)$ must be of exponential order; that is, $f(t)$ must remain less than $Me^{-a_o t}$ as t approaches ∞, where M is a positive constant and a_o is a real positive number.

For example, such functions as: $\tan \beta t$, $\cot \beta t$, e^{t^2} are not Laplace transformable. Given a function $f(t)$ that satisfies the Dirichlet conditions, then

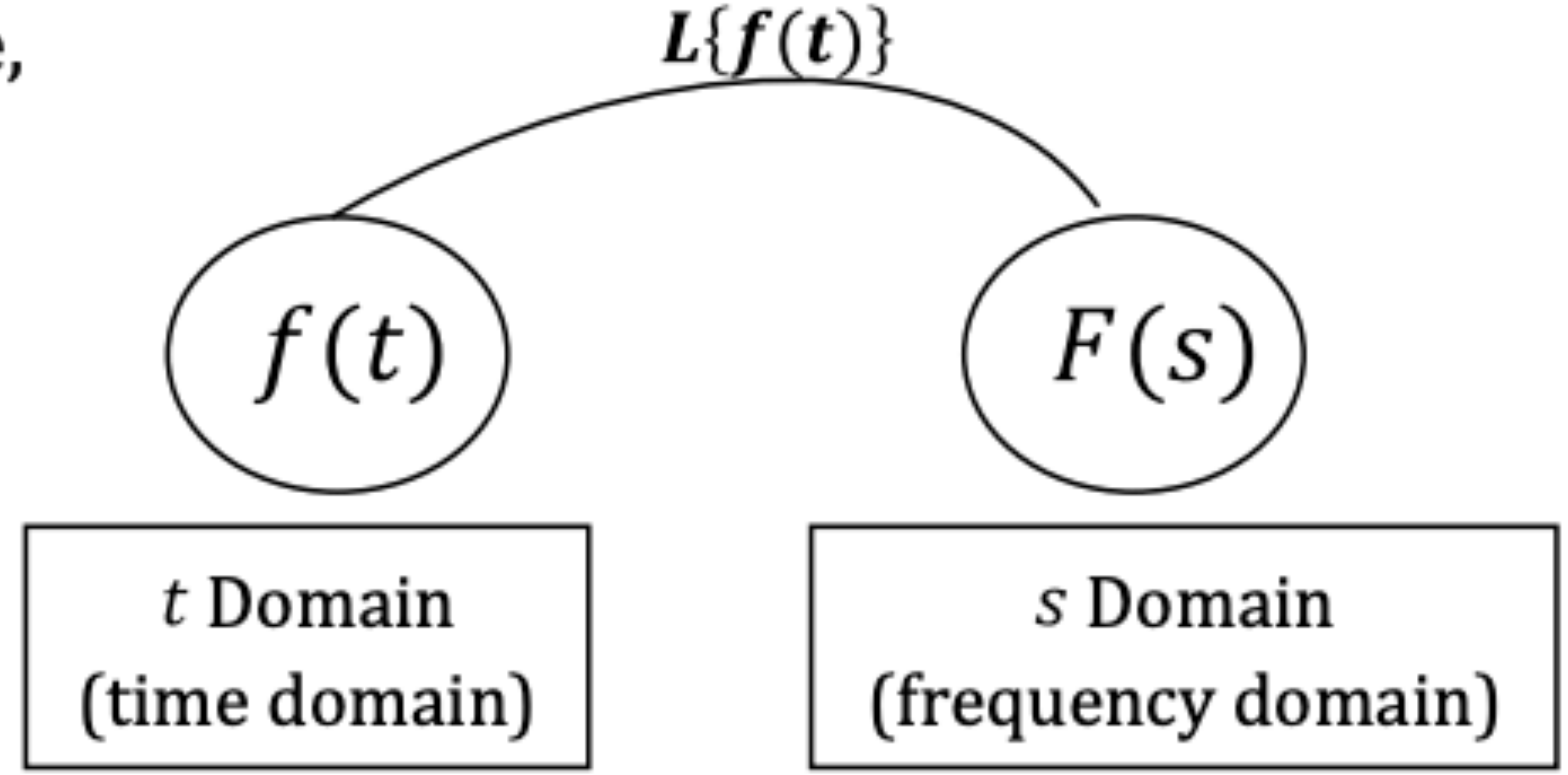
$$F(s) = \int_0^\infty f(t) e^{-st} dt \quad \text{written } \mathscr{L}\{f(t)\} \tag{1.1}$$

is called the Laplace transformation of $f(t)$. Here s can be either a real variable or a complex quantity. Observe the shorthand notation $\mathscr{L}\{f(t)\}$ to denote the Laplace transformation of $f(t)$. Observe also that only ordinary integration is involved in this integral.

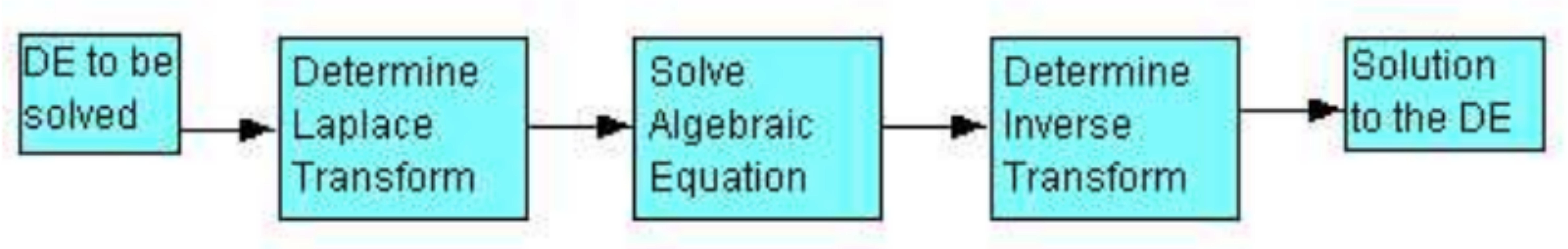
IMPROPER-INTEGRAL

Because the Upper limit in the Integral is Infinite, the domain of Integration is Infinite. Thus the Integral is an example of an Improper Integral.

$$\int_0^\infty e^{-st} f(t)\ dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t)\, dt$$



The Laplace Transformation of $f(t)$ is said to exist if the Integral $\int_0^\infty e^{-st} f(t)\ dt$ Converges for some values of s, Otherwise it does not exist.



THINGS TO REMEMBER

The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of f, which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T \to \infty} \int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

The integral which defined a Laplace transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. When the improper integral in convergent then we say that the function $f(t)$ possesses a Laplace transform. So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral.

PROBLEMS

Find the Laplace transform, if it exists, of each of the following functions

$$(a)\ f(t) = e^{at} \quad (b)\ f(t) = 1 \quad (c)\ f(t) = t \quad (d)\ f(t) = e^{t^2}$$

Example: Find the Laplace transform of the constant function

$$f(t) = 1, \ 0 \le t < \infty.$$

Solution:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-ts} f(t)\, dt = \int_0^\infty e^{-ts}(1)\, dt \\
&= \lim_{b\to+\infty} \int_0^b e^{-ts}\, dt \\
&= \lim_{b\to+\infty} \left[\frac{e^{-ts}}{-s}\right]_0^b \quad \text{provided } s \neq 0. \\
&= \lim_{b\to+\infty} \left[\frac{e^{-bs}}{-s} - \frac{e^0}{-s}\right] \\
&= \lim_{b\to+\infty} \left[\frac{e^{-bs}}{-s} - \frac{1}{-s}\right]
\end{aligned}$$

At this stage we need to recall a limit from Cal 1:

$$e^{-x} \to \begin{cases} 0 & \text{if } x \to +\infty \\ +\infty & \text{if } x \to -\infty \end{cases}.$$

Hence,

$$\lim_{b \to +\infty} \frac{e^{-bs}}{-s} = \begin{cases} 0 & \text{if } s > 0 \\ +\infty & \text{if } s < 0 \end{cases}.$$

Thus,

$$F(s) = \frac{1}{s}, \ s > 0.$$

In this case the domain of the transform is the set of all positive real numbers.

Example 2. $f(t) = e^t$.

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} = \lim_{A\to\infty}\int_0^A e^{-st}e^{at}dt = \lim_{A\to\infty}\int_0^A e^{-(s-a)t}dt = \lim_{A\to\infty} -\frac{1}{s-a}e^{-(s-a)t}\Big|_0^A \\ &= \lim_{A\to\infty} -\frac{1}{s-a}\left(e^{-(s-a)A}-1\right) = \frac{1}{s-a}, \quad (s>a) \end{aligned}$$

Example 3. $f(t) = t^n$, for $n \geq 1$ integer.

$$
\begin{aligned}
F(s) &= \lim_{A\to\infty} \int_0^A e^{-st} t^n dt = \lim_{A\to\infty} \left\{ t^n \left. \frac{e^{-st}}{-s} \right|_0^A - \int_0^A \frac{nt^{n-1}e^{-st}}{-s} dt \right\} \\
&= 0 + \frac{n}{s} \lim_{A\to\infty} \int_0^A e^{-st} t^{n-1} dt = \frac{n}{s} \mathcal{L}\{t^{n-1}\}.
\end{aligned}
$$

So we get a recursive relation

$$
\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \quad \forall n,
$$

which means

$$
\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}, \quad \mathcal{L}\{t^{n-2}\} = \frac{n-2}{s}\mathcal{L}\{t^{n-3}\}, \cdots
$$

By induction, we get

$$
\begin{aligned}
\mathcal{L}\{t^n\} &= \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n}{s}\frac{(n-1)}{s}\mathcal{L}\{t^{n-2}\} = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\mathcal{L}\{t^{n-3}\} \\
&= \cdots = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\cdots\frac{1}{s}\mathcal{L}\{1\} = \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}, \quad (s > 0)
\end{aligned}
$$

Example 4. Find the Laplace transform of $\sin at$ and $\cos at$.

Method 1. Compute by definition, with integration-by-parts, twice. (lots of work...)

Method 2. Use the Euler's formula

$$e^{iat} = \cos at + i \sin at, \qquad \Rightarrow \qquad \mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}.$$

By Example 2 we have

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s - ia} = \frac{1(s+ia)}{(s-ia)(s+ia)} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}.$$

Comparing the real and imaginary parts, we get

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad (s > 0).$$

Remark: Now we will use $\int_0^\infty$ instead of $\lim_{A\to\infty} \int_0^A$, without causing confusion.

For piecewise continuous functions, Laplace transform can be computed by integrating each integral and add up at the end.

EXAMPLE FOR PIECEWISE CONTINUOUS FUNCTIONS

Example 5. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ t-2, & 2 \le t. \end{cases}$$

We do this by definition:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st} dt + \int_2^\infty (t-2)e^{-st} dt \\ &= \frac{1}{-s} e^{-st}\Big|_{t=0}^{2} + (t-2)\frac{1}{-s} e^{-st}\Big|_{t=2}^{\infty} - \int_2^A \frac{1}{-s} e^{-st} dt \\ &= \frac{1}{-s}(e^{-2s} - 1) + (0-0) + \frac{1}{s}\,\frac{1}{-s} e^{-st}\Big|_{t=2}^{\infty} = \frac{1}{-s}(e^{-2s} - 1) + \frac{1}{s^2} e^{-2s} \end{aligned}$$

CONTRADICTORY EXAMPLE-$f(t) = e^{t^2}$

$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2-st}dt.$$

If $s \leq 0$ then $t^2 - st \geq 0$ so that $e^{t^2-st} \geq 1$ and this implies that $\int_0^\infty e^{t^2-st}dt \geq \int_0^\infty$. Since the integral on the right is divergent, by the comparison theorem of improper integrals (see Theorem 43.1 below) the integral on the left is also divergent. Now, if $s > 0$ then $\int_0^\infty e^{t(t-s)}dt \geq \int_s^\infty dt$. By the same reasoning the integral on the left is divergent. This shows that the function $f(t) = e^{t^2}$ does not possess a Laplace transform ■

TABLE OF TRANSFORMS

$f(t) = 1,\ t \geq 0$	$F(s) = \frac{1}{s},\ s \geq 0$
$f(t) = t^n,\ t \geq 0$	$F(s) = \frac{n!}{s^{n+1}},\ s \geq 0$
$f(t) = e^{at},\ t \geq 0$	$F(s) = \frac{1}{s-a},\ s > a$
$f(t) = \sin(kt),\ t \geq 0$	$F(s) = \frac{k}{s^2+k^2}$
$f(t) = \cos(kt),\ t \geq 0$	$F(s) = \frac{s}{s^2+k^2}$
$f(t) = \sinh(kt),\ t \geq 0$	$F(s) = \frac{k}{s^2-k^2},\ s > \lvert k \rvert$
$f(t) = \cosh(kt),\ t \geq 0$	$F(s) = \frac{s}{s^2-k^2},\ s > \lvert k \rvert$

DEFINITION OF GAMMA FUNCTION

- Definition of Gama Function

$$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt\,, n \geq 0$$

(OR)

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dt\,, n \geq 0\,.$$

Note: i) $\Gamma(n+1) = n\Gamma(n) = n!$

ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

USING GAMMA FUNCTION

- The Laplace Transformation of t^n , where n is a non-negative Real number.

Sol: We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$\Rightarrow L\{f(t)\} = \int_0^\infty e^{-st} t^n\, dt$$

$$Put\ st = x \ \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{1}{s}\, dx$$

$$As\ t \to 0\ to\ \infty \quad \Rightarrow x \to 0\ to\ \infty$$

$$\Rightarrow L\{t^n\} = \int_{x=0}^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{1}{s} dx$$

$$\Rightarrow L\{t^n\} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \qquad \left[\because\ \Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dt\, , n \geq 0\right]$$

$$\Rightarrow L\{t^n\} = \frac{n!}{s^{n+1}} \qquad [\because\ \Gamma(n+1) = n!]$$

COMPARISON

Example 3. $f(t) = t^n$, for $n \geq 1$ integer.

$$F(s) = \lim_{A\to\infty}\int_0^A e^{-st}t^n dt = \lim_{A\to\infty}\left\{ t^n \left.\frac{e^{-st}}{-s}\right|_0^A - \int_0^A \frac{nt^{n-1}e^{-st}}{-s}dt \right\}$$
$$= 0 + \frac{n}{s}\lim_{A\to\infty}\int_0^A e^{-st}t^{n-1}dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\}.$$

So we get a recursive relation

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \quad \forall n,$$

which means

$$\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s}\mathcal{L}\{t^{n-2}\}, \quad \mathcal{L}\{t^{n-2}\} = \frac{n-2}{s}\mathcal{L}\{t^{n-3}\}, \cdots$$

By induction, we get

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n}{s}\frac{(n-1)}{s}\mathcal{L}\{t^{n-2}\} = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\mathcal{L}\{t^{n-3}\}$$
$$= \cdots = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\cdots\frac{1}{s}\mathcal{L}\{1\} = \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}, \quad (s > 0)$$

- The Laplace Transformation of t^n , where n is a non-negative Real numb

Sol: We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$$\Rightarrow L\{f(t)\} = \int_0^\infty e^{-st} t^n \, dt$$

Put $st = x \Rightarrow t = \frac{x}{s}$

$dt = \frac{1}{s}\, dx$

As $t \to 0$ *to* $\infty \quad \Rightarrow x \to 0$ *to* ∞

$$\Rightarrow L\{t^n\} = \int_{x=0}^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{1}{s} dx$$

$$\Rightarrow L\{t^n\} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n \, dx$$

$$= \frac{1}{s^{n+1}}\Gamma(n+1) \qquad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dt, n \geq 0\right]$$

$$\Rightarrow L\{t^n\} = \frac{n!}{s^{n+1}} \qquad [\because \Gamma(n+1) = n!]$$

USING GAMMA FUNCTION FOR NEGATIVE REAL UMBER

Find the Laplace transform of $t^{1/2}$

Sol: $f(t) = t^{1/2}$ and by formula

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt = F(s)$$

Substitute f(t) in the above formula

$$\mathscr{L}(t^{1/2}) = \int_0^\infty e^{-st} t^{1/2} dt = F(s)$$

Here $n = \frac{1}{2}$ and by formula $\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$.

$$\mathscr{L}(t^{\frac{1}{2}}) = \frac{\frac{1}{2}!}{s^{1/2+1}} \quad Note: n! = \Gamma(n+1) \text{ or } n\Gamma(n) = n!.$$

So $\frac{1}{2}! = \frac{1}{2}\Gamma(\frac{1}{2})$ and also $\Gamma(1/2) = \sqrt(\pi)$

Finally

We are having $\mathscr{L}(t^{\frac{1}{2}}) = \frac{1/2 * \sqrt(\pi)}{s^{1/2+1}}$

After Simplification

$$\mathscr{L}(t^{\frac{1}{2}}) = \frac{\sqrt(\pi)}{2s^{3/2}}$$

Find the Laplace transform of $t^{-1/2}$

Sol: $f(t) = t^{-1/2}$ and by formula

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt = F(s)$$

Substitute f(t) in the above formula

$$\mathscr{L}(t^{1/2}) = \int_0^\infty e^{-st} t^{-1/2} dt = F(s)$$

Here $n = \frac{1}{2}$ and by formula $\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$.

$$\mathscr{L}(t^{\frac{-1}{2}}) = \frac{\frac{-1}{2}!}{s^{-1/2+1}} \quad Note: n! = \Gamma(n+1) \text{ or } n\Gamma(n) = n!.$$

So $\frac{-1}{2}! = \Gamma(\frac{-1}{2} + 1)$ and also $\Gamma(1/2) = \sqrt(\pi)$

Finally

We are having $\mathscr{L}(t^{\frac{-1}{2}}) = \frac{\sqrt(\pi)}{s^{1/2}}$

After Simplification

$$\mathscr{L}(t^{\frac{-1}{2}}) = \frac{\sqrt{\pi}}{\sqrt{s}}$$

PROPERTY-1

The Laplace Transform is Linear

If a is a constant and f and g are functions, then

$$\mathcal{L}(af) = a\mathcal{L}(f) \tag{1}$$

$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g) \tag{2}$$

For example, by the above property (1)

$$\mathcal{L}(3t^5) = 3\mathcal{L}(t^5) = 3\left(\frac{5!}{s^6}\right) = \frac{360}{s^6}, \ s > 0.$$

As an another example, by property (2)

$$\mathcal{L}(e^{5t} + \cos(3t)) = \mathcal{L}(e^{5t}) + \mathcal{L}(\cos(3t)) = \frac{1}{s-5} + \frac{s}{s^2+9}, \ s > 5.$$

Properties of Laplace transform:

1. Linearity: $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$.
2. First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.
3. Second derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
4. Higher order derivative:
$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$
5. $\mathcal{L}\{-tf(t)\} = F'(s)$ where $F(s) = \mathcal{L}\{f(t)\}$. This also implies $\mathcal{L}\{tf(t)\} = -F'(s)$.
6. $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $F(s) = \mathcal{L}\{f(t)\}$. This implies $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$.

Property-6

FIRST SHIFTING PROPERTY (or) FIRST TRANSLATION PROPERTY

Statement: If $L\{f(t)\} = F(s)$ then $L\{e^{at}f(t)\} = F(s-a)$

Proof: We know that

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

$$\Rightarrow L\{e^{at}f(t)\} = \int_0^{\infty} e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt$$

(Here we have taken exponential quantity as negative, but not positive, because as $t \to \infty \Rightarrow e^{t} \to \infty \Rightarrow e^{-t} \to 0$)

Put $s - a = p,\ p > 0$

$$\Rightarrow L\{e^{at}f(t)\} = \int_0^{\infty} e^{-pt} f(t) dt$$

$$= F(p)$$

$$= F(s-a)$$

Hence, If $L\{f(t)\} = F(s)$ *then* $L\{e^{at}f(t)\} = F(s-a)$

- whenever we want to evaluate $\boldsymbol{L\{e^{at}f(t)\}}$, first evaluate $\boldsymbol{L\{f(t)\}}$ which is equal to $\boldsymbol{F(s)}$ and then evaluate $\boldsymbol{L\{e^{at}f(t)\}}$, which will be obtained simply, by substituting $\boldsymbol{s-a}$ in place of $\boldsymbol{a}$ in $\boldsymbol{F(s)}$.

Example 1.

$$\text{From} \qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

Example 2.

$$\text{From} \qquad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

Example 3.

$$\text{From} \qquad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, \qquad \text{we get} \qquad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}.$$

Example 4.

$$\mathcal{L}\{t^3+5t-2\} = \mathcal{L}\{t^3\} + 5\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = \frac{3!}{s^4} + 5\frac{1}{s^2} - 2\frac{1}{s}.$$

EXAMPLES-CONT..

Example 5.

$$\mathcal{L}\{e^{2t}(t^3+5t-2)\} = \frac{3!}{(s-2)^4} + 5\frac{1}{(s-2)^2} - 2\frac{1}{s-2}.$$

Example 6.

$$\mathcal{L}\{(t^2+4)e^{2t} - e^{-t}\cos t\} = \frac{2}{(s-2)^3} + \frac{4}{s-2} - \frac{s+1}{(s+1)^2+1},$$

because

$$\mathcal{L}\{t^2+4\} = \frac{2}{s^3} + \frac{4}{s}, \qquad \Rightarrow \mathcal{L}\{(t^2+4)e^{2t}\} = \frac{2}{(s-2)^3} + \frac{4}{s-2}.$$

MULTIPLICATION PROPERTY

$$\mathcal{L}(tf(t);s) = -F'(s)$$

$$\mathcal{L}(t^n f(t);s) = (-1)^n F^{(n)}(s)$$

$$
\begin{aligned}
\textstyle\int_0^\infty (1)e^{-st}dt &= -(1/s)e^{-st}\Big|_{t=0}^{t=\infty} && \text{Laplace integral of } g(t)=1. \\
&= 1/s && \text{Assumed } s>0. \\
\textstyle\int_0^\infty (t)e^{-st}dt &= \textstyle\int_0^\infty -\frac{d}{ds}(e^{-st})dt && \text{Laplace integral of } g(t)=t. \\
&= -\tfrac{d}{ds}\textstyle\int_0^\infty (1)e^{-st}dt && \text{Use } \textstyle\int \frac{d}{ds}F(t,s)dt = \frac{d}{ds}\int F(t,s)dt. \\
&= -\tfrac{d}{ds}(1/s) && \text{Use } \mathcal{L}(1)=1/s. \\
&= 1/s^2 && \text{Differentiate.} \\
\textstyle\int_0^\infty (t^2)e^{-st}dt &= \textstyle\int_0^\infty -\frac{d}{ds}(te^{-st})dt && \text{Laplace integral of } g(t)=t^2. \\
&= -\tfrac{d}{ds}\textstyle\int_0^\infty (t)e^{-st}dt && \\
&= -\tfrac{d}{ds}(1/s^2) && \text{Use } \mathcal{L}(t)=1/s^2. \\
&= 2/s^3 &&
\end{aligned}
$$

PROBLEMS IN MULTIPLICATION PROPERTY

Next are a few examples for Property 5.

Example 7.

$$\text{Given} \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad \text{we get} \qquad \mathcal{L}\{te^{at}\} = -\left(\frac{1}{s-a}\right)' = \frac{1}{(s-a)^2}$$

Example 8.

$$\mathcal{L}\{t \sin bt\} = -\left(\frac{b}{s^2+b^2}\right)' = \frac{-2bs}{(s^2+b^2)^2}.$$

Example 9.

$$\mathcal{L}\{t \cos bt\} = -\left(\frac{s}{s^2+b^2}\right)' = \cdots = \frac{s^2-b^2}{(s^2+b^2)^2}.$$

DIVISION PROPERTY

$$L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(s)\,ds \quad \text{where } F(s) = L(f(t))$$

PROBLEMS ON DIVISION PROPERTY

Problems

- Find the $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol: Here $f(t) = e^{-at} - e^{-bt}$

$$\Rightarrow F(s) = L\{f(t)\} = L\{e^{-at} - e^{-bt}\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

We know that $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$

$$\Rightarrow L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

$$= [\log|s+a| - \log|s+b|]_s^{\infty}$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_{s=s}^{\infty} \qquad log\left(\frac{s+b}{s+a}\right)$$

$$= \log 1 - \log\left(\frac{s+a}{s+b}\right) = \cancel{\log\left(\frac{s+a}{s+b}\right)}$$

INVERSE LAPLACE

Inverse Laplace transform. Definition:

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \text{if} \qquad F(s) = \mathcal{L}\{f(t)\}.$$

Technique: find the way back.

Some simple examples:

Example 10.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\sin 2t.$$

Example 11.

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{6}{(s+5)^4}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{(s+5)^4}\right\} = \frac{1}{3}e^{-5t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3}e^{-5t}t^3.$$

Example 12.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \cos 2t\frac{1}{2}\sin 2t.$$

Let $f(t)$ be a periodic function with period T

$(f(t+T) = f(t))$. Then

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

Put $t = u+T$ in 2^{nd} integral, $t = u+2T$ in 3^{rd} integral ....

$$L(f(t)) = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-s2T}\int_0^T e^{-su} f(u)\,du + \cdots$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right)\int_0^T e^{-st} f(t)\,dt$$

$$\boxed{L(f(t)) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt}$$

If $f(t)$ is a periodic function with period T, i.e., $f(t+T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

We have $$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \ldots$$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$, and so on. Then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \ldots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT} \int_0^T e^{-su} f(u)du + e^{-2sT} \int_0^T e^{-su} f(u)du + \ldots$$

$$[\because \quad f(u) = f(u+T) = f(u+2T) \text{ etc.}]$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT} \int_0^T e^{-st} f(t)dt + e^{-2sT} \int_0^T e^{-st} f(t)dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots \infty) \int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)dt.$$

(V.T.U., 2008 ; Mumbai, 2006)

PROBLEM

Pb:1 Find Laplace transform for

$$f(t) = \begin{cases} \sin \omega t, & \text{for } 0 < t < \pi/\omega \\ 0 & \text{for } \pi/\omega < t < \frac{2\pi}{\omega} \end{cases}$$

with $f\left(t + \frac{2\pi}{\omega}\right) = f(t) \quad \forall\, t \geq 2\pi/3.$

Example 21.9. *Find the Laplace transform of the function*

$$f(t) = \sin \omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \pi/\omega < t < 2\pi/\omega.$$

(Kurukshetra, 2005 ; Madras, 2003)

Solution. Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\therefore \qquad L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} . 0 \, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left| \frac{e^{-st}(-s \sin \omega t - \omega \cos \omega t)}{s^2 + \omega^2} \right|_0^{\pi/\omega} = \frac{\omega e^{-\pi s/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)} = \frac{\omega}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}.$$

Example 21.10. *Draw the graph of the periodic function*

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$$

and find its Laplace transform. *(U.P.T.U., 2003)*

Solution. Here the period of $f(t) = 2\pi$ and its graph is as in Fig. 21.1.

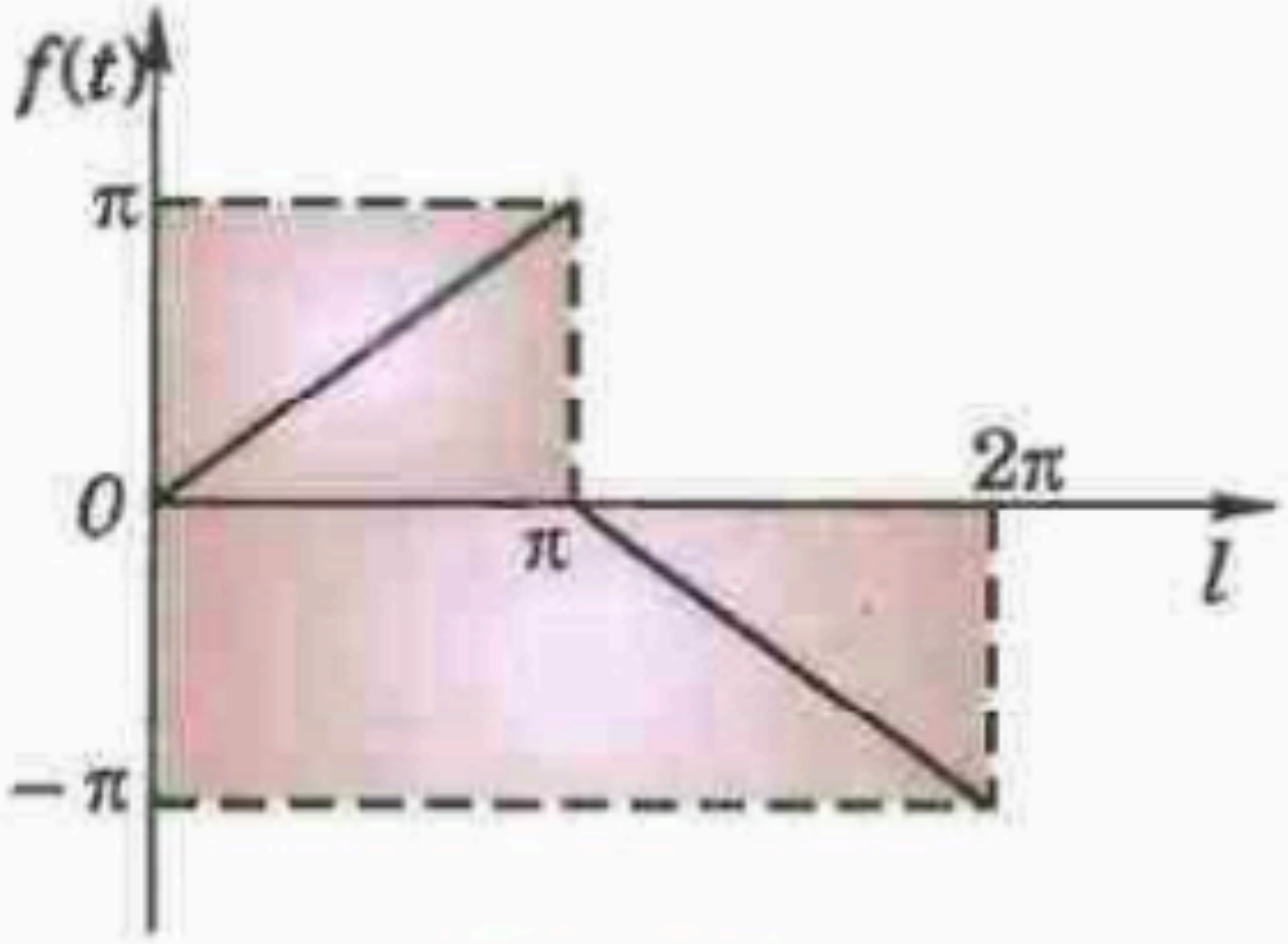


Fig. 21.1

$$\therefore \quad L f(t) = \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st}\, t\, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \left| t\left(\frac{e^{-st}}{-s}\right) - 1 . \left(\frac{e^{-st}}{s^2}\right) \right|_0^{\pi} + \left| (\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right|_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(1 + e^{-2\pi s} - 2e^{-\pi s}\right) \right\}.$$

RECALL

$\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$ Linearity.

The Laplace of a sum is the sum of the Laplaces.

$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$ Linearity.

Constants move through the $\mathcal{L}$-symbol.

$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$ The t-derivative rule.

Derivatives $\mathcal{L}(y')$ are replaced in transformed equations.

$\mathcal{L}\left(\int_0^t g(x)dx\right) = \frac{1}{s}\mathcal{L}(g(t))$ The t-integral rule.

$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))$ The s-differentiation rule.

Multiplying f by t applies $-d/ds$ to the transform of f.

$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))|_{s\to(s-a)}$ First shifting rule.

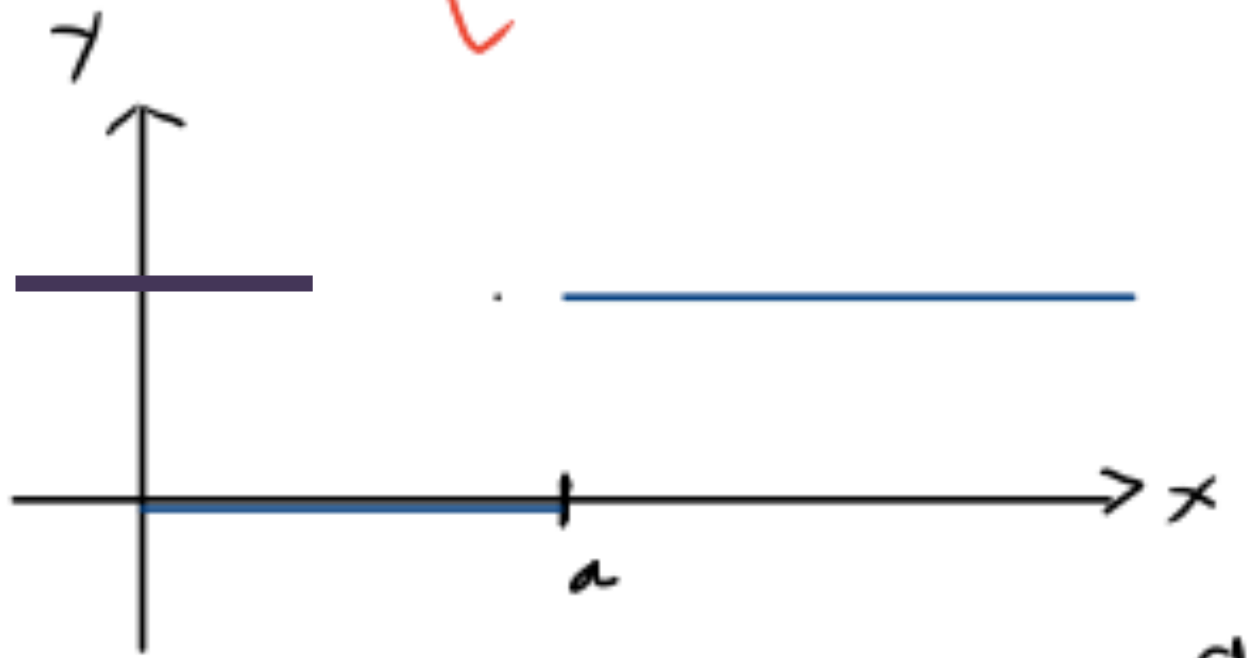
$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\,ds$ where $F(s) = L(f(t))$

HEAVY-SIDE FUNCTION OR UNIT STEP FUNCTION

Laplace transform for unit step function (or) Heaviside function

The unit step function is defined as $H:[0,\infty)\to\mathbb{R}$ by

$$H(t-a)=\begin{cases}1 & \text{if } t>a\\ 0 & \text{if } 0\le t\le a.\end{cases}$$



$$L(H(t-a)) = \int_0^\infty e^{-st}H(t-a)\,dt = \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$

$$L(H(t-a)) = \frac{e^{-as}}{s}$$

Example (Heaviside) Find the Laplace transform of $f(t)$ in Figure 1.

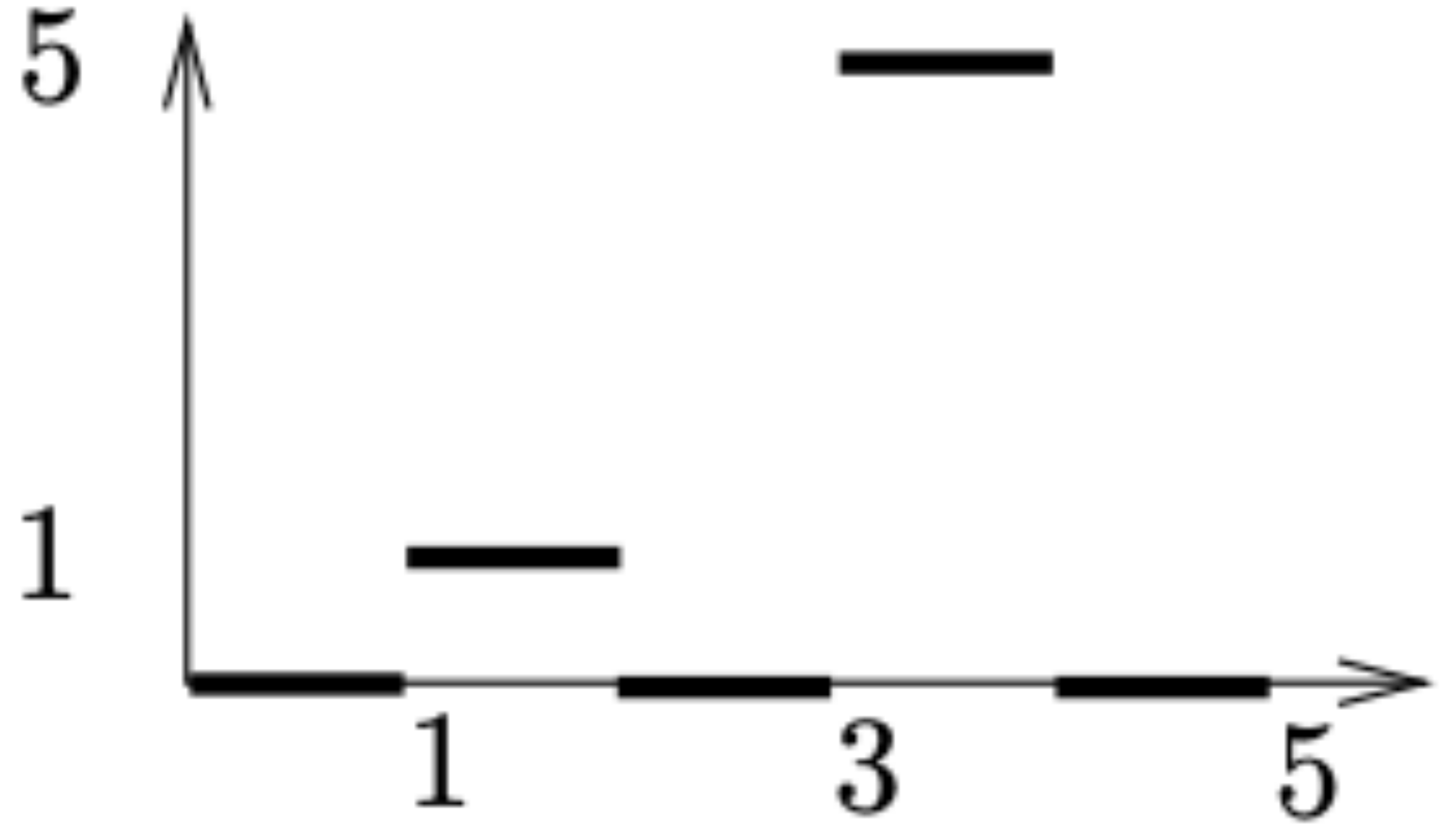


Figure 1. A piecewise defined function $f(t)$ on $0 \le t < \infty$: $f(t) = 0$ except for $1 \le t < 2$ and $3 \le t < 4$.

Solution: The details require the use of the Heaviside function formula

$$H(t-a) - H(t-b) = \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise.} \end{cases}$$

The formula for $f(t)$:

$$f(t) = \begin{cases} 1 & 1 \le t < 2, \\ 5 & 3 \le t < 4, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 1 \le t < 2, \\ 0 & \text{otherwise} \end{cases} + 5 \begin{cases} 1 & 3 \le t < 4, \\ 0 & \text{otherwise} \end{cases}$$

Then $f(t) = f_1(t) + 5f_2(t)$ where $f_1(t) = H(t-1) - H(t-2)$ and $f_2(t) = H(t-3) - H(t-4)$. The extended table gives

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(f_1(t)) + 5\mathcal{L}(f_2(t)) && \text{Linearity.} \\ &= \mathcal{L}(H(t-1)) - \mathcal{L}(H(t-2)) + 5\mathcal{L}(f_2(t)) && \text{Substitute for } f_1. \end{aligned}$$

$$= \frac{e^{-s} - e^{-2s}}{s} + 5\mathcal{L}(f_2(t))$$

Extended table used.

$$= \frac{e^{-s} - e^{-2s} + 5e^{-3s} - 5e^{-4s}}{s}$$

Similarly for f_2.