What Does f' Say About f?

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To see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1.

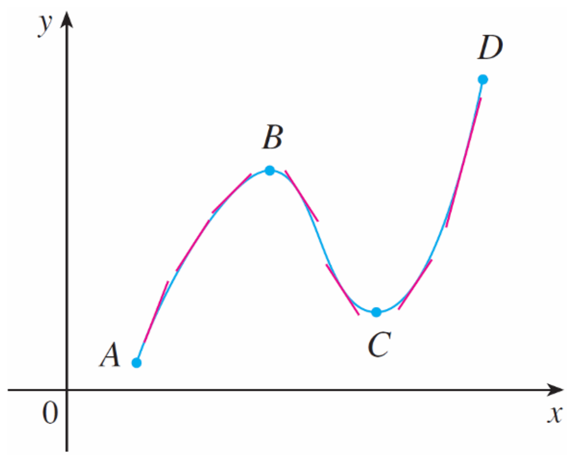


Figure 1

Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$.

What Does f' Say About f ?

Between B and C the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

To prove that this is always the case, we use the Mean Value Theorem.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 1

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

To use the I/D Test we have to know where $f'(x) > 0$ and where $f'(x) < 0$.

This depends on the signs of the three factors of $f'(x)$, namely, $12x$, $x - 2$, and $x + 1$.

Example 1 – *Solution* cont’d

We divide the real line into intervals whose endpoints are the critical numbers –1, 0 and 2 and arrange our work in a chart.

A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the I/D Test.

For instance, $f'(x) < 0$ for $0 < x < 2$, so f is decreasing on $(0, 2)$. (It would also be true to say that f is decreasing on the closed interval $[0, 2]$.)

Example 1 – *Solution* cont'd

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

The graph of *f* shown in Figure 2 confirms the information in the chart.

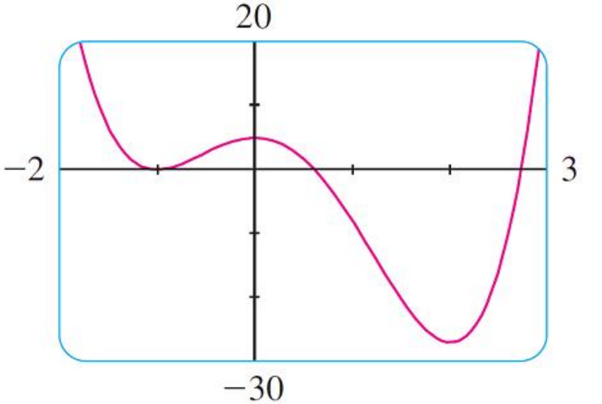


Figure 2

What Does f' Say About f ?

You can see from Figure 2 that $f(0) = 5$ is a local maximum value of f because f increases on $(–1, 0)$ and decreases on $(0, 2)$. Or, in terms of derivatives, $f'(x) > 0$ for $–1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$.

In other words, the sign of $f'(x)$ changes from positive to negative at 0. This observation is the basis of the following test.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

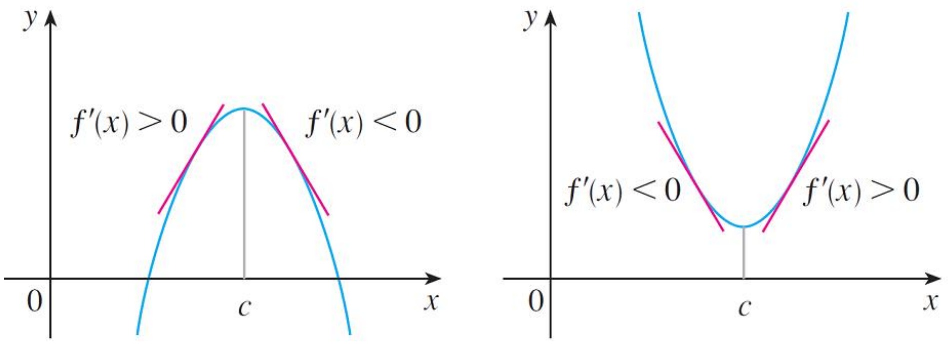
(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

What Does f' Say About f ?

The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, since the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has a local maximum at c.

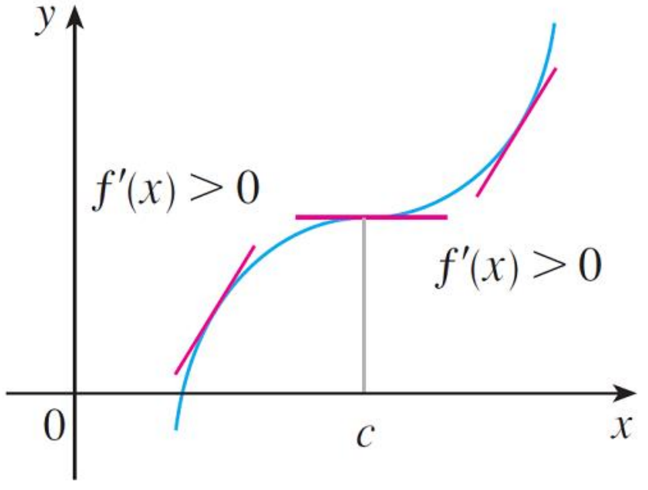
It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.



Local maximum
Figure 3(a)

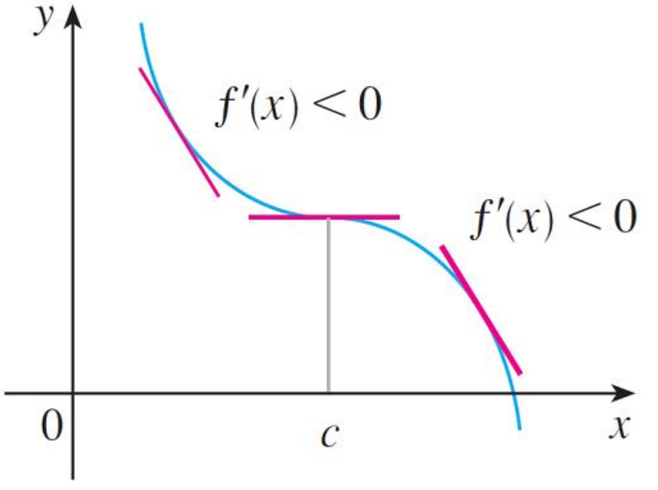
Local minimum
Figure 3(b)

What Does f' Say About f ?



No maximum or minimum

Figure 3(c)



No maximum or minimum

Figure 3(d)

Example 3

Find the local maximum and minimum values of the function

$$g(x) = x + 2 \sin x \qquad 0 \leq x \leq 2\pi$$

Solution:

To find the critical numbers of *g*, we differentiate:

$$g'(x) = 1 + 2 \cos x$$

So $g'(x) = 0$ when $x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ and $4\pi/3$.

Example 3 – *Solution*

cont'd

Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and $4\pi/3$ and so we analyze g in the following table.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

Example 3 – *Solution*

cont'd

Because $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}$$

$$= \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3} + \sqrt{3}$$

$$\approx 3.83$$

Example 3 – *Solution*

cont'd

Likewise $g'(x)$, changes from negative to positive at $4\pi/3$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3}$$

$$= \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

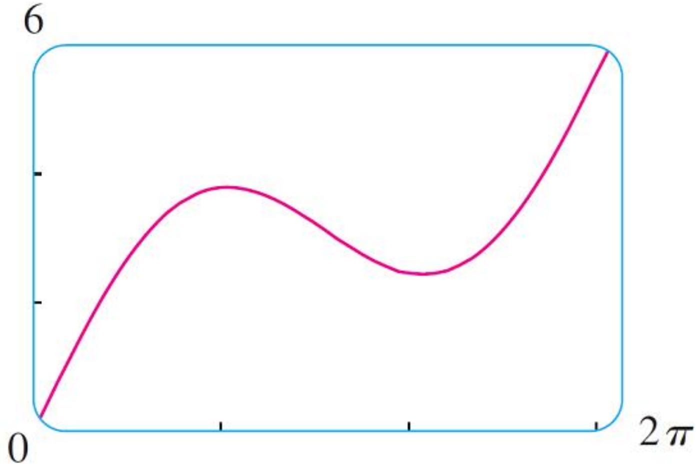
$$\approx 2.46$$

is a local minimum value.

Example 3 – *Solution*

cont'd

The graph of *g* in Figure 4 supports our conclusion.



$g(x) = x + 2 \sin x$

Figure 4

What Does f'' Say About f?

What Does f'' Say About f ?

Figure 5 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions.

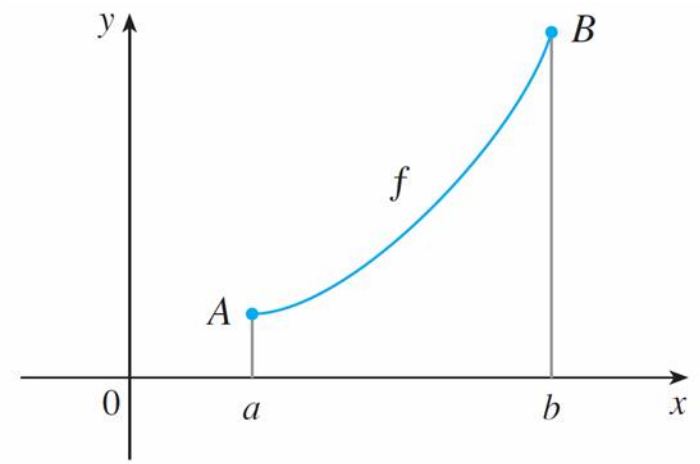


Figure 5(a)

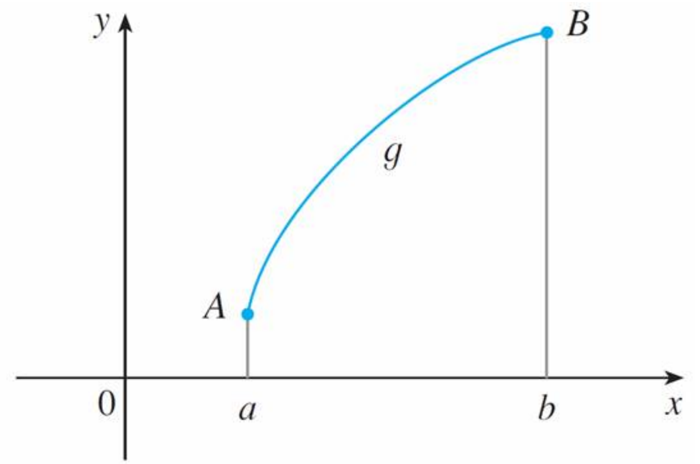
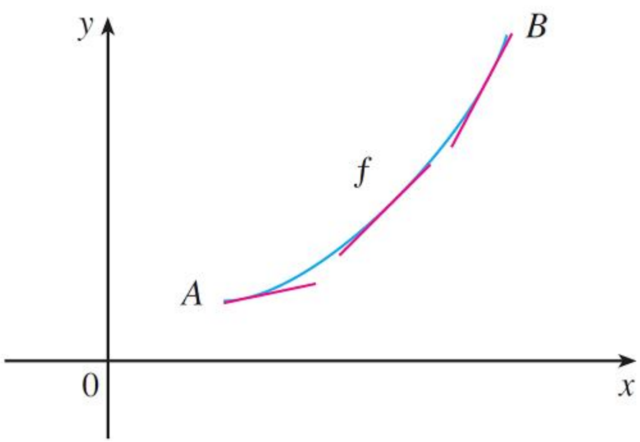


Figure 5(b)

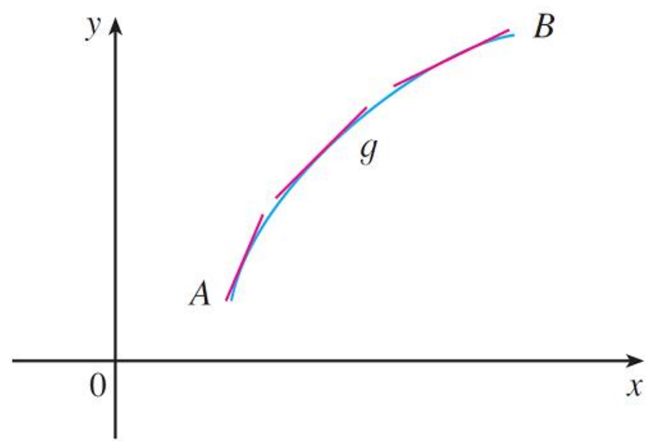
What Does f'' Say About f ?

In Figure 6 tangents to these curves have been drawn at several points. In (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In (b) the curve lies below the tangents and g is called *concave downward* on (a, b).



Concave upward

Figure 6(a)



Concave downward

Figure 6(b)

What Does f'' Say About f?

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).

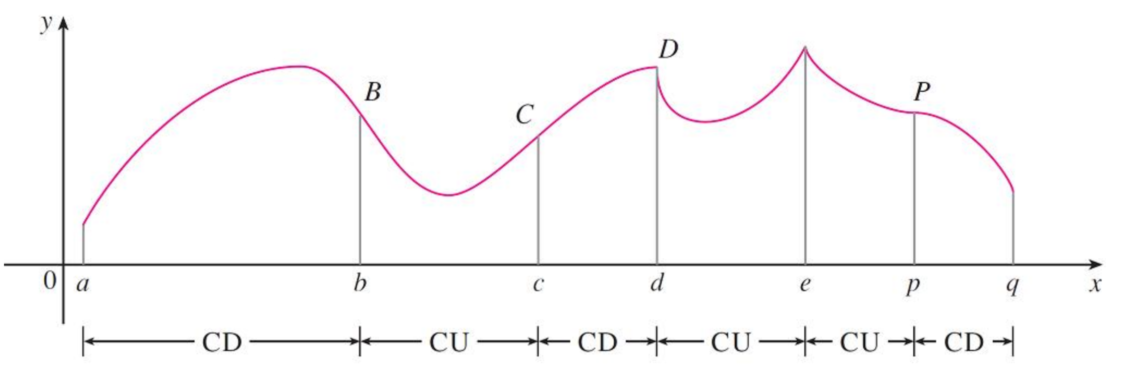
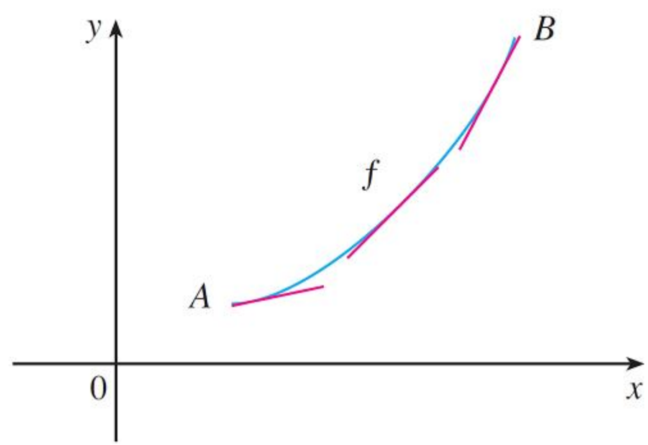


Figure 7

What Does f'' Say About f ?

Let's see how the second derivative helps determine the intervals of concavity. Looking at Figure 6(a), you can see that, going from left to right, the slope of the tangent increases.



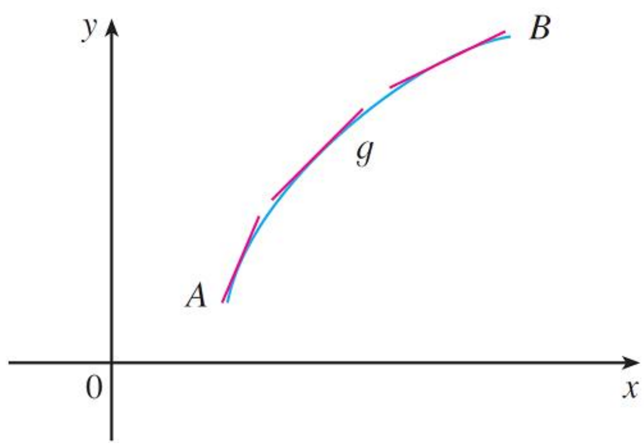
Concave upward

Figure 6(a)

What Does f'' Say About f ?

This means that the derivative f' is an increasing function and therefore its derivative f'' is positive.

Likewise, in Figure 6(b) the slope of the tangent decreases from left to right, so f' decreases and therefore f'' is negative.



Concave downward

Figure 6(b)

What Does f'' Say About f ?

This reasoning can be reversed and suggests that the following theorem is true.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

What Does f'' Say About f ?

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 6

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution:

If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Example 6 – *Solution*

cont'd

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$.

To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad\qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum.

Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0.

Example 6 – *Solution*

cont'd

But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.]

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

Example 6 – *Solution*

cont'd

The point (0, 0) is an inflection point since the curve changes from concave upward to concave downward there.

Also (2, –16) is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Figure 11.

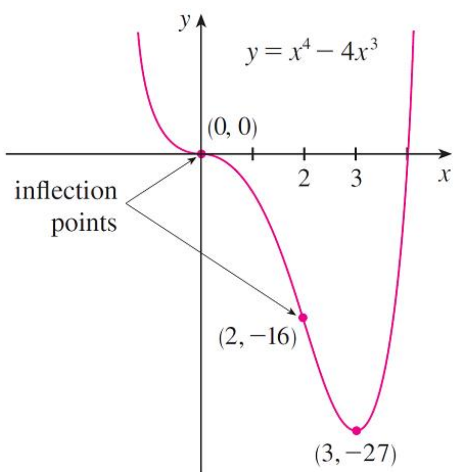


Figure 11

What Does f'' Say About f ?

Note:
The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 6).

This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

Example 7

Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

Solution:

Calculation of the first two derivatives gives

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4 and 6.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

Example 7 – *Solution*

cont'd

To find the local extreme values we use the First Derivative Test.

Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum.

Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum.

The sign of f' does not change at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4 but not at 0 or 6 since f'' does not exist at either of these numbers.)

Example 7 – *Solution*

cont'd

Looking at the expression $f''(x)$ for and noting that $x^{4/3} \geq 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for and $0 < x < 6$ for $x > 6$.

So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$.

Example 7 – *Solution*

cont'd

The graph is sketched in Figure 12.

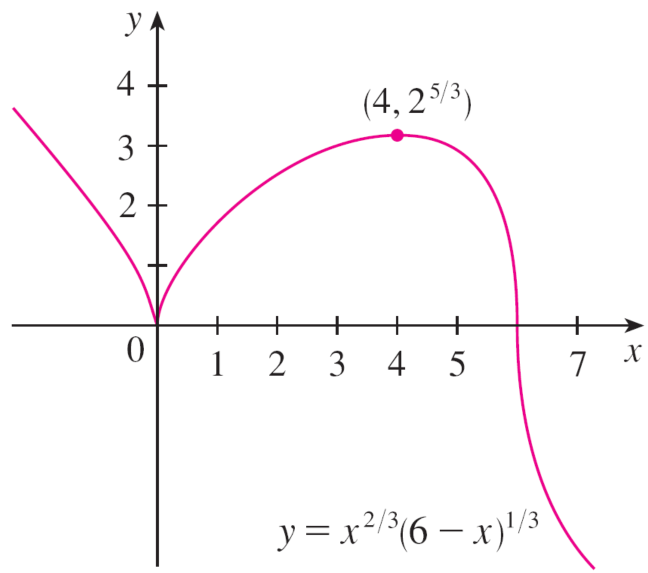


Figure 12

Note that the curve has vertical tangents at (0,0) and (6,0) because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.