Average Value of a Function

Average Value of a Function

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible?

Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

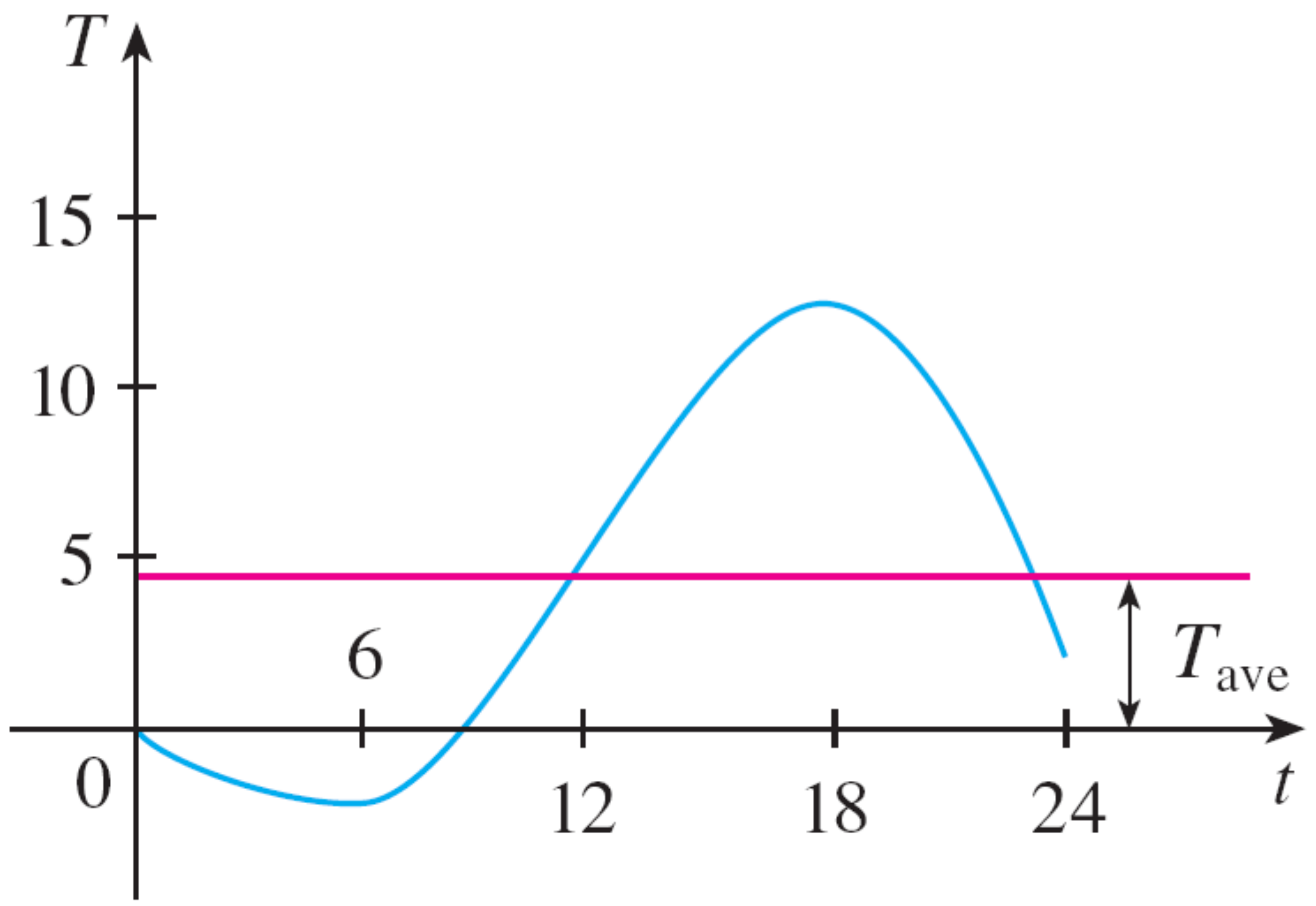


Figure 1

Average Value of a Function

In general, let's try to compute the average value of a function $y = f(x)$, $a \leq x \leq b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length
$\Delta x = (b - a)/n$.

Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and calculate the average of the numbers
$f(x_1^*), \ldots, f(x_n^*)$:

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

(For example, if f represents a temperature function and $n = 24$, this means that we take temperature readings every hour and then average them.)

Average Value of a Function

Since $\Delta x = (b - a)/n$, we can write $n = (b - a)/\Delta x$ and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\dfrac{b-a}{\Delta x}} = \frac{1}{b-a}\left[f(x_1^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\right]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

If we let n increase, we would be computing the average value of a large number of closely spaced values.

Average Value of a Function

The limiting value is

$$\lim_{n \to \infty} \frac{1}{b - a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b - a} \int_a^b f(x)\,dx$$

by the definition of a definite integral.

Therefore we define the **average value of *f*** on the interval [*a*, *b*] as

$$\boxed{f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\,dx}$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Solution:

With $a = -1$ and $b = 2$ we have

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{2-(-1)}\int_{-1}^{2} (1 + x^2)\,dx \\ &= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} \\ &= 2 \end{aligned}$$

Practice Problems-for finding the Average value of a function

(a) $2x^3 - 3x^2 + 4x - 1, [-1, 1]$

(b) $\sqrt{5x + 1}, [0, 3]$

(c) $2/(x + 1)^2, [3, 5]$

(d) $\cos 2x, [3, \pi/4]$

(e) $x^{2/3} - x^{-2/3}, [1, 4]$

(f) $f(x) = x\sqrt{x^2 + 16}, [0, 3]$

(g) $f(x) = |x| - 1$ on $[-1, 3]$

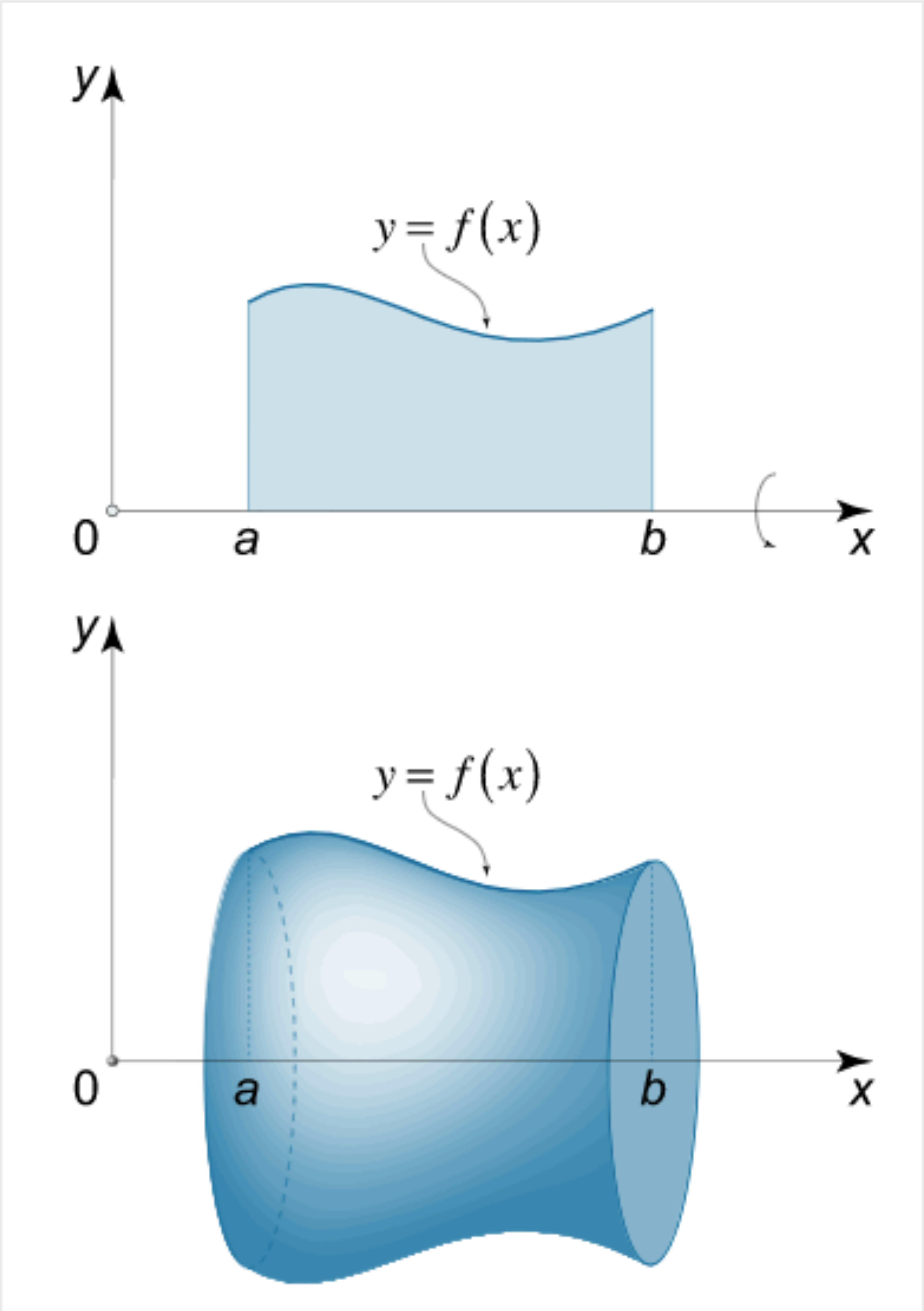
Volume of a solid of Revolution

Disk and Washer Method.

Prakash and 25/01

Disk Method-Introduction

The disk method is used when we rotate a single curve y=f(x) around the x– (or y–) axis.

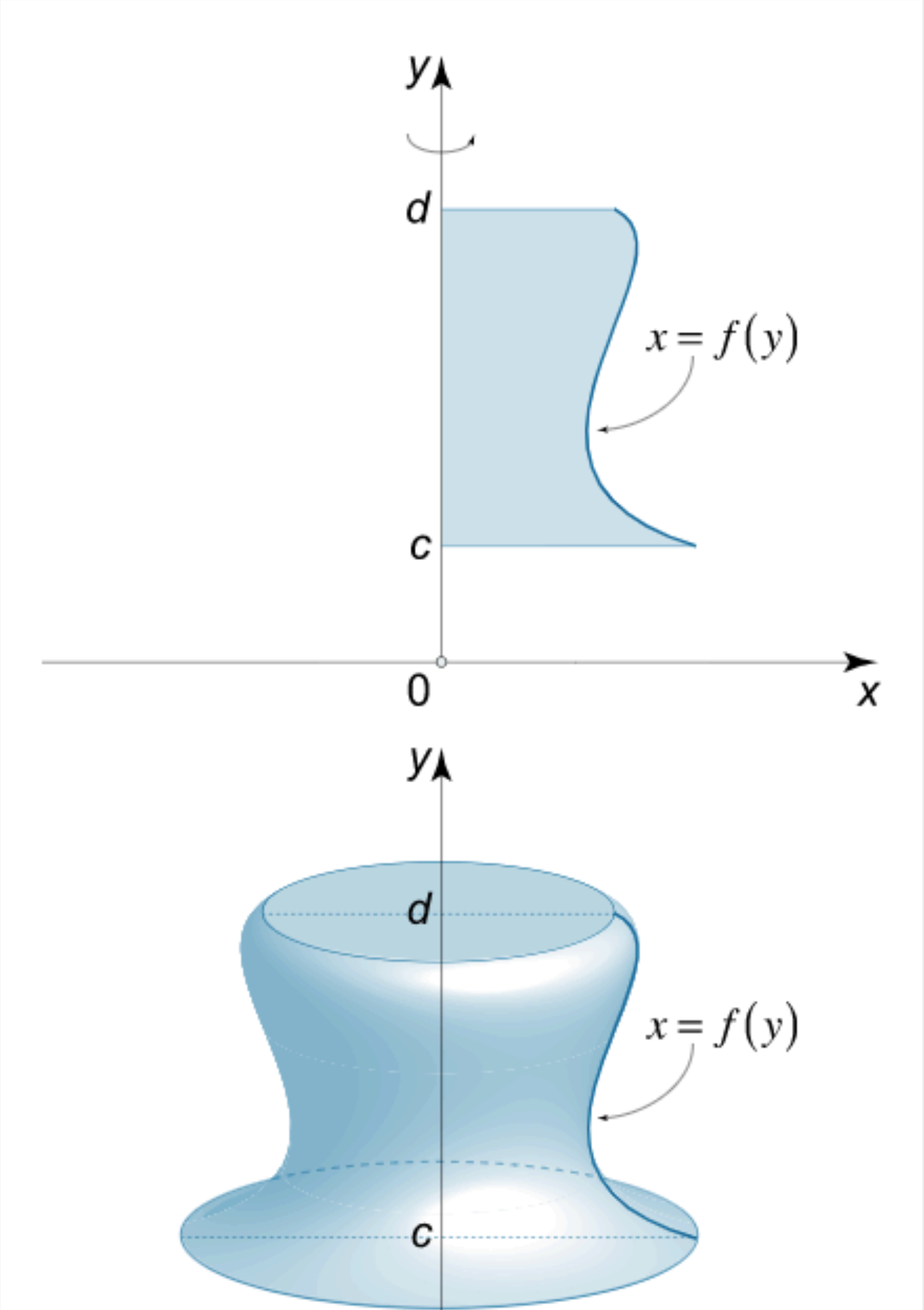


The volume of the solid formed by revolving the region bounded by the curve $y = f\left(x\right)$ and the $x-$axis between $x = a$ and $x = b$ about the $x-$axis is given by

$$V = \pi \int\limits_a^b {{{\left[{f\left(x \right)} \right]}^2}dx} .$$

The cross section perpendicular to the axis of revolution has the form of a disk of radius $R = f\left(x\right).$

Similarly, we can find the volume of the solid when the region is bounded by the curve $x = f(y)$ and the y—axis between $y = c$ and $y = d$, and is rotated about the y—axis.

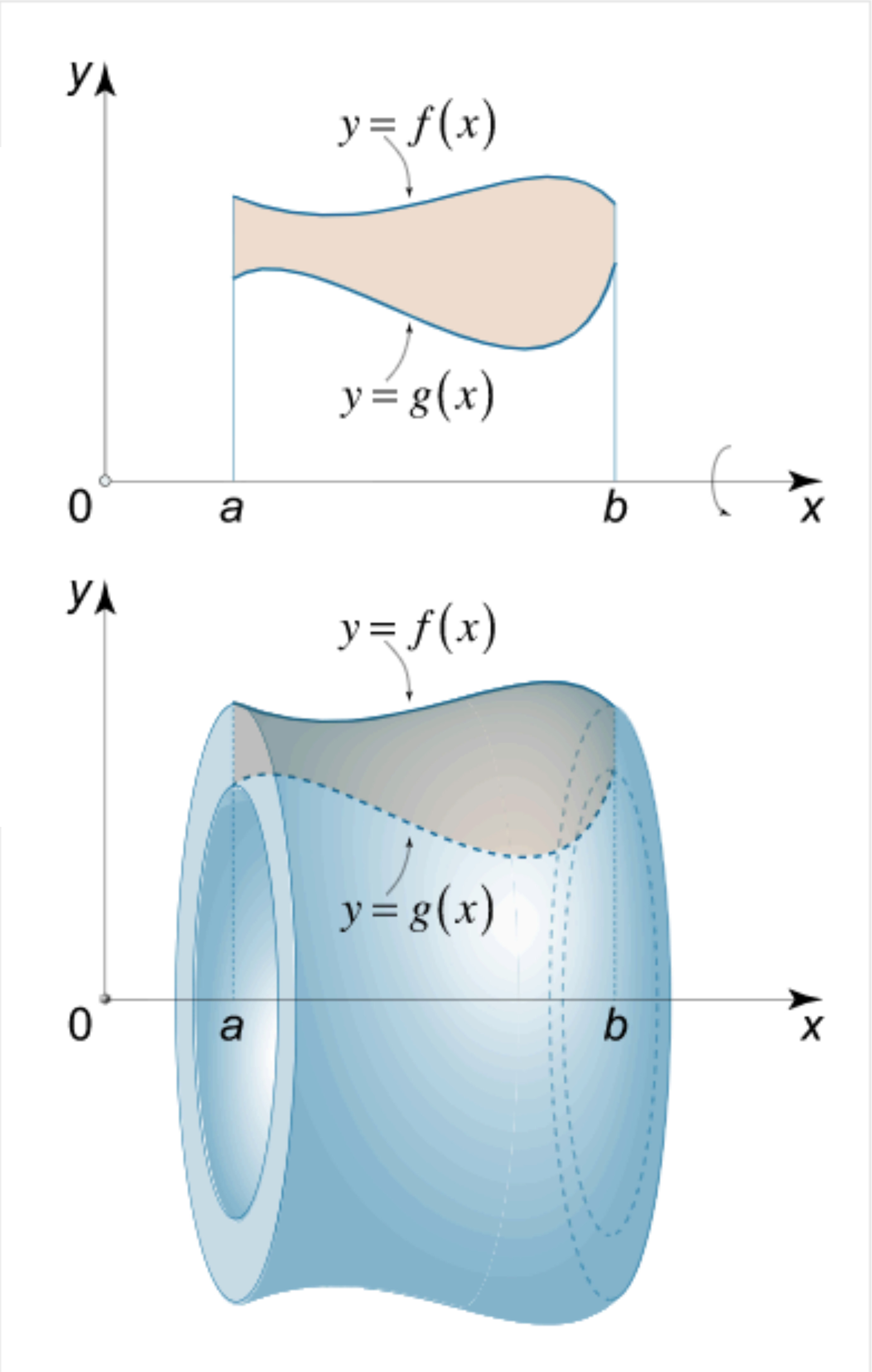


The resulting formula is

$$V = \pi \int_{c}^{d} [f(y)]^2 dy.$$

Washer method

Assuming that the functions $f(x)$ and $g(x)$ are continuous and non-negative on the interval $[a, b]$ and $g(x) \leq f(x)$, consider a region that is bounded by two curves $y = f(x)$ and $y = g(x)$, between $x = a$ and $x = b$.



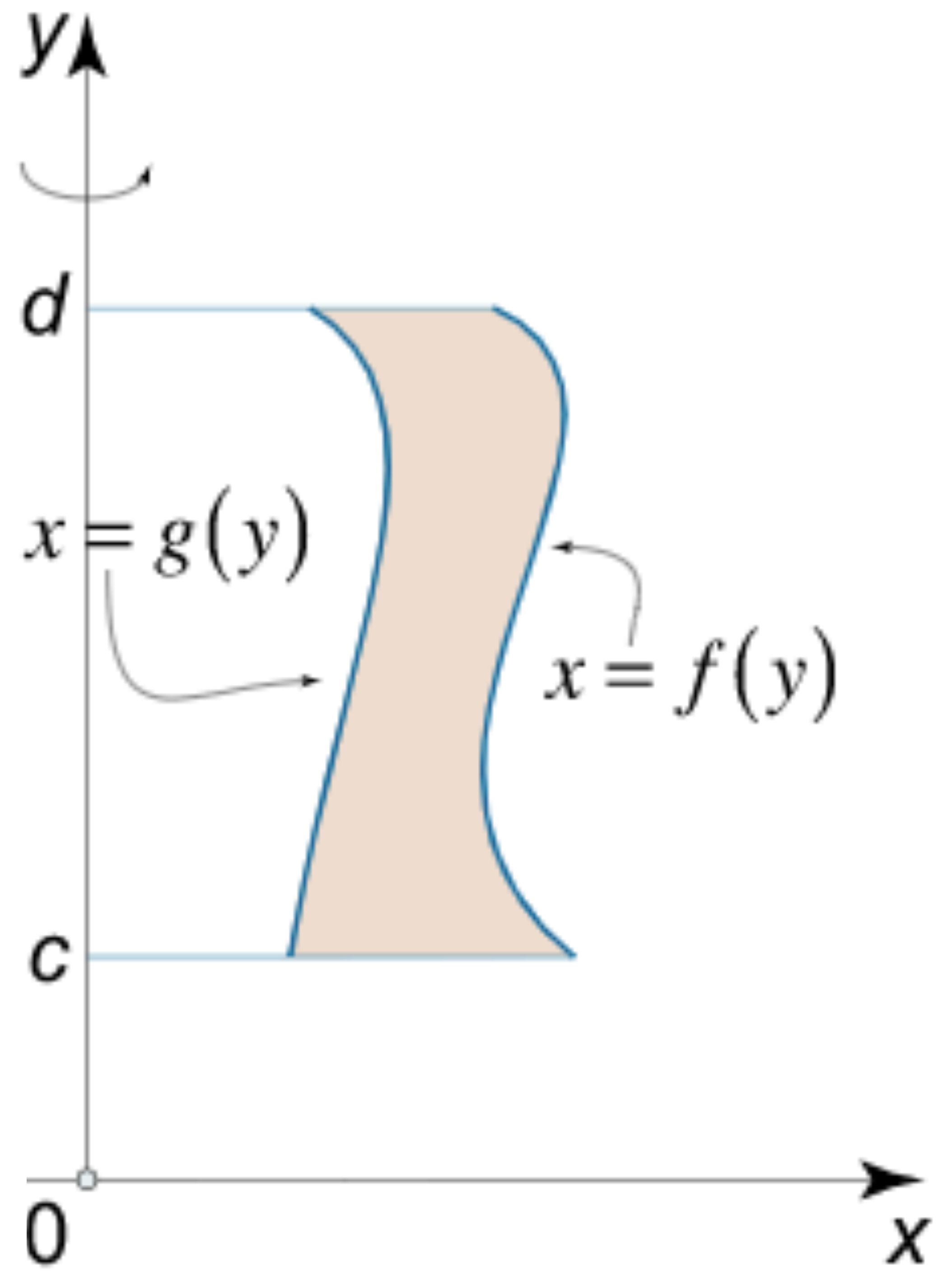
The volume of the solid formed by revolving the region about the $x-$axis is

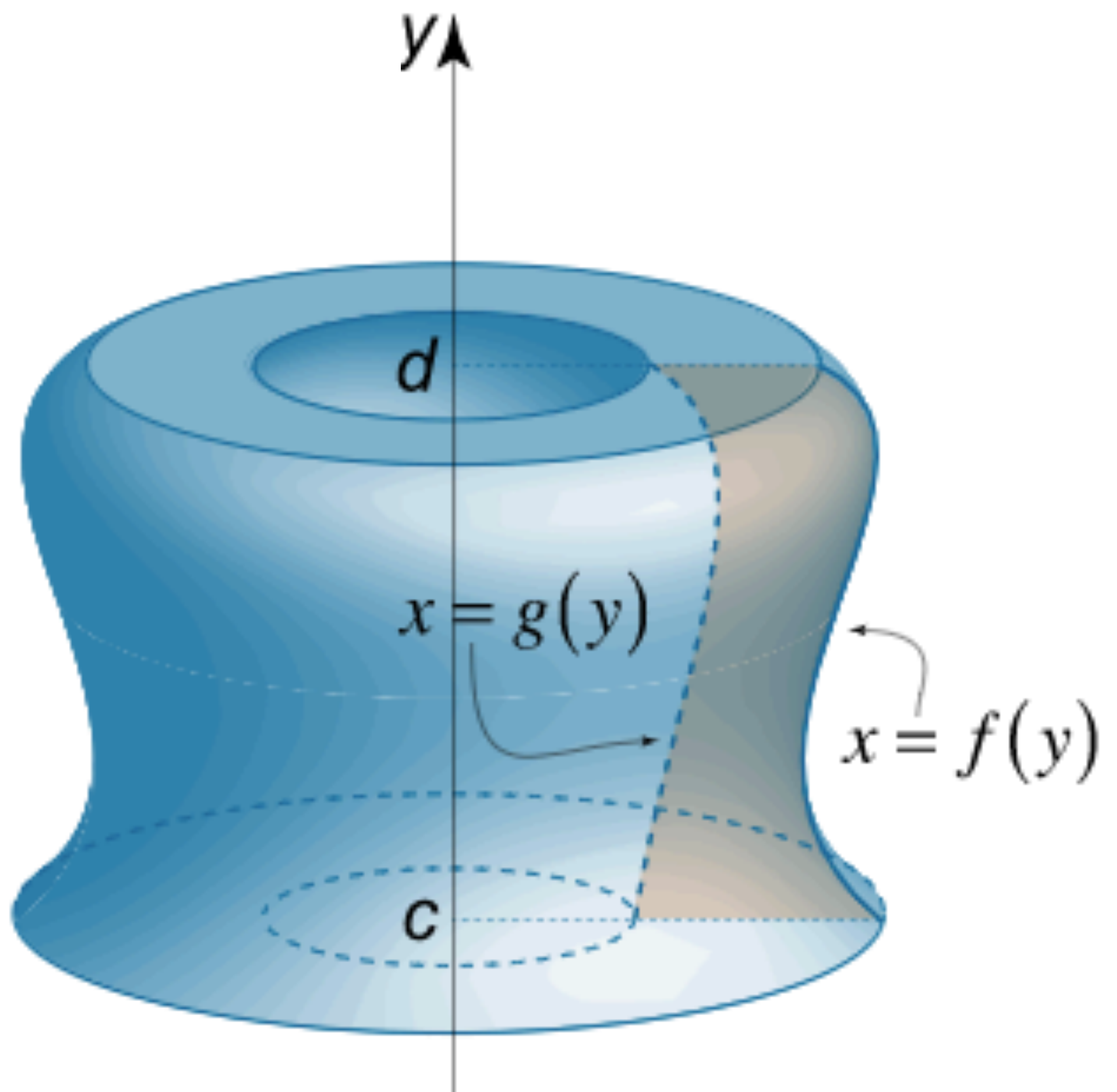
$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx.$$

At a point x on the $x-$axis, a perpendicular cross section of the solid is washer-shape with the inner radius $r = g(x)$ and the outer radius $R = f(x)$.

The volume of the solid generated by revolving about the y—axis a region between the curves $x = f(y)$ and $x = g(y)$, where $g(y) \leq f(y)$ and $c \leq y \leq d$ is given by the formula

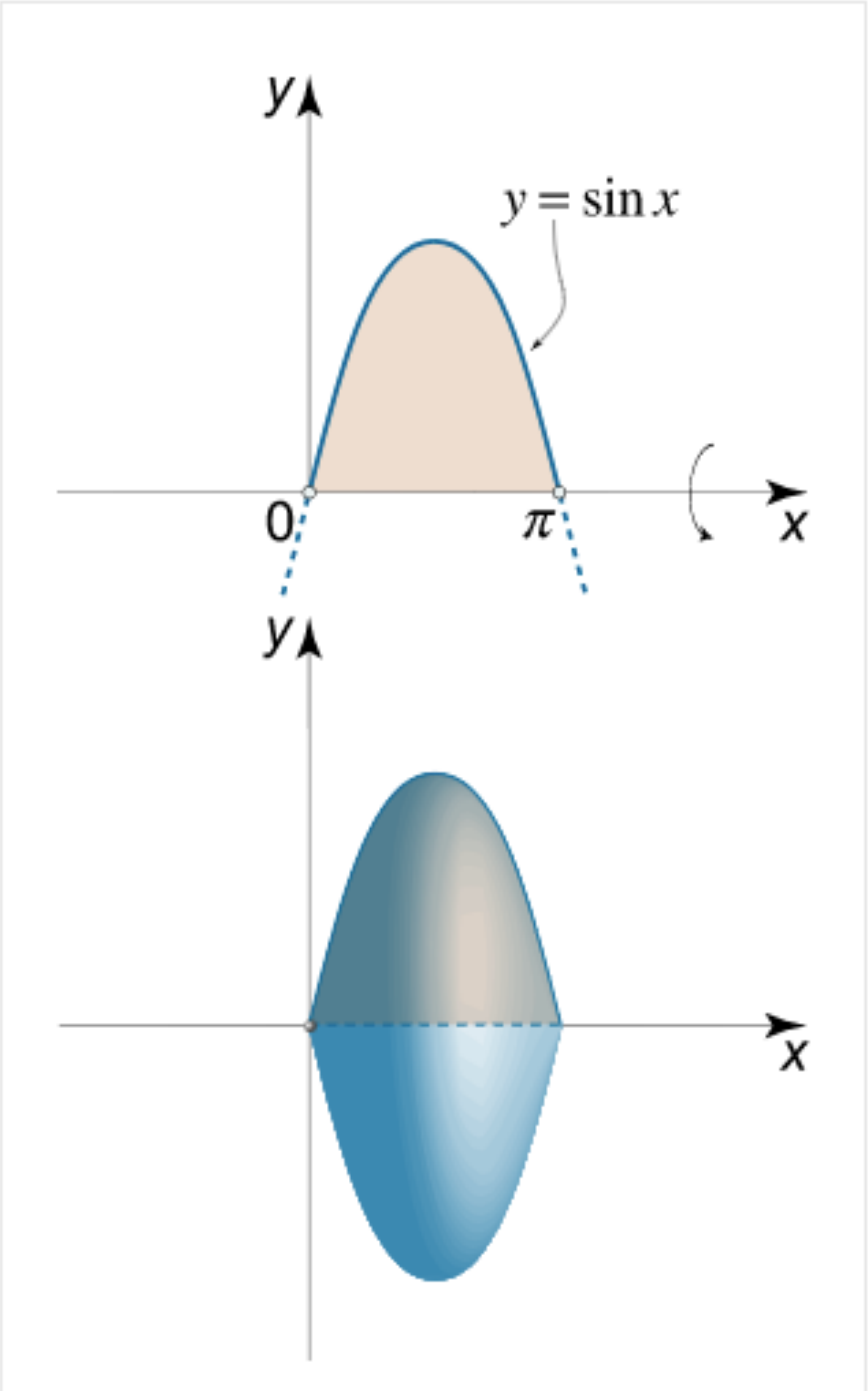
$$V = \pi \int_c^d \left([f(y)]^2 - [g(y)]^2 \right) dy.$$





Problems

Find the volume of the solid obtained by rotating the sine function between $x = 0$ and $x = \pi$ about the $x-$axis.



By the disk method,

$$V = \pi \int_0^\pi [\sin x]^2 dx = \frac{\pi}{2} \int_0^\pi (1 - \cos 2x)\, dx = \frac{\pi}{2} \left(x - \frac{\sin 2x}{2} \right) \Bigg|_0^\pi$$

$$= \frac{\pi}{2} [(\pi - 0) - (0 - 0)] = \frac{\pi^2}{2}.$$

Example-2

Calculate the volume of the solid obtained by rotating the region bounded by the parabola $y = x^2$ and the square root function $y = \sqrt{x}$ around the $x-$axis.

Both curves intersect at the points $x = 0$ and $x = 1$. Using the washer method, we have

$$V = \pi \int_0^1 \left(\left[\sqrt{x}\right]^2 - \left[x^2\right]^2 \right) dx = \pi \int_0^1 \left(x - x^4\right) dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right)\Bigg|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right)$$

$$= \frac{3\pi}{10}.$$

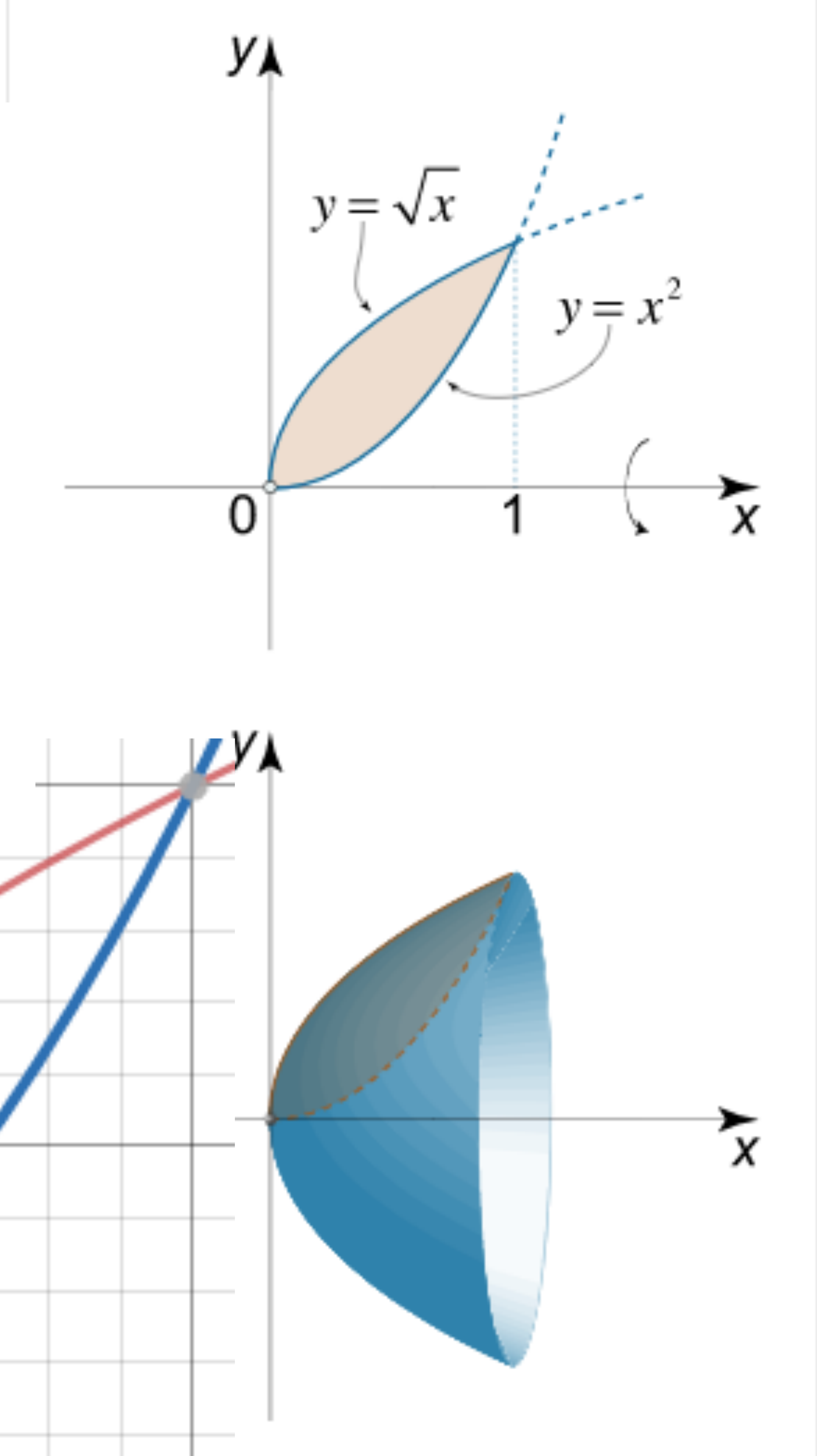
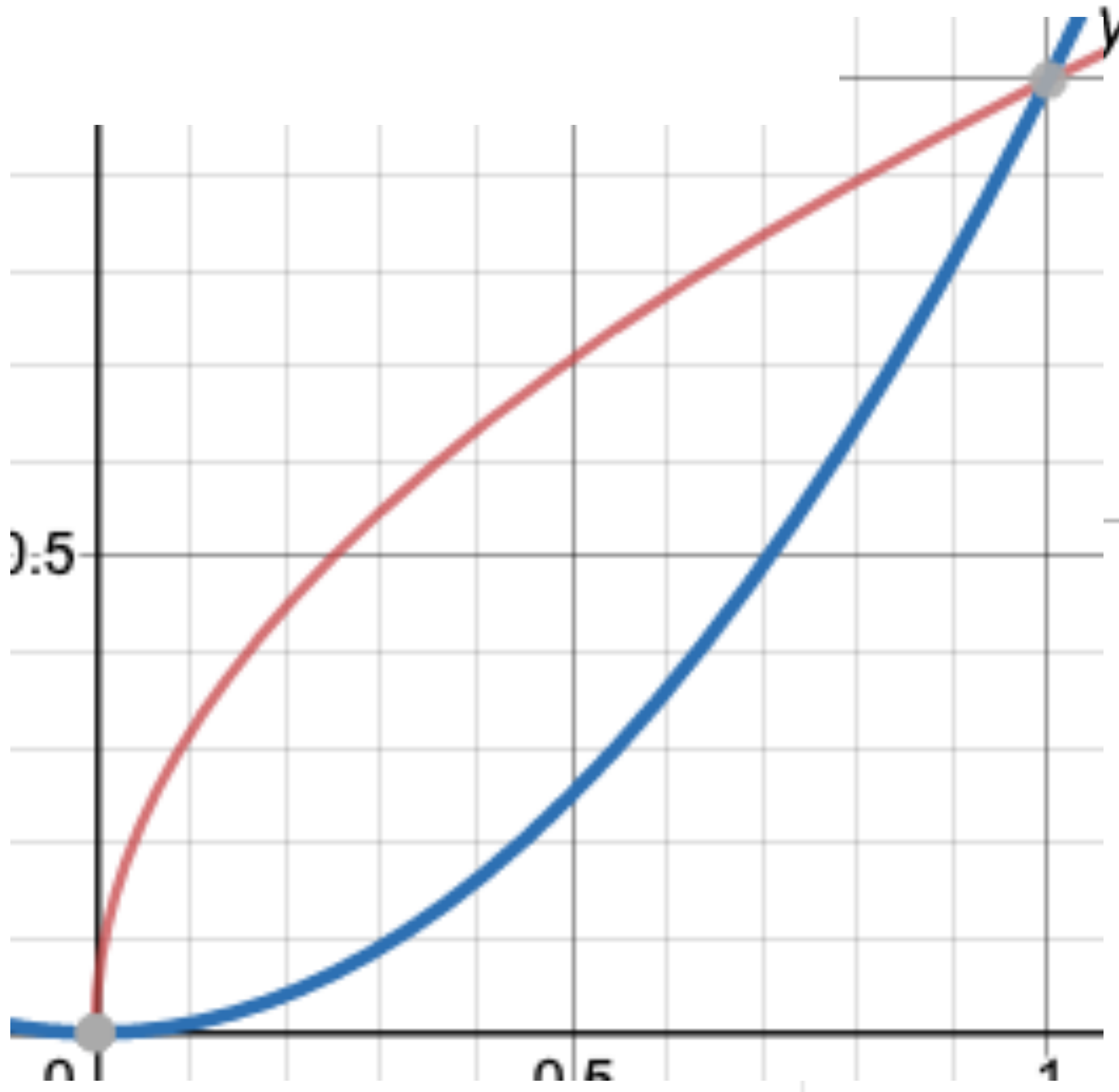


Figure 7.



Practice Problems

(a) the semi-circular arc $x^2 + y^2 = a^2$ from $x = -a$ to $x = a$ about the x-axis;

(b) the arc of the curve $y = x^3$ from $y = 0$ to $y = 8$ about the y-axis;

(c) the hyperbola $y^2 - x^2 = 1$ from $x = -a$ to $x = a$ about the x-axis

(d) the hyperbola $xy = 2$ about the y-axis, between the limits $y = 1$ to $y = 8$

(e) the arc of the parabola $y = \sqrt{x}$ from $x = 0$ to $x = 1$ about the x-axis.

Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = 2x - x^2$ and the $x-$axis about the $y-$axis.

Find the volume of the solid obtained by rotating the region bounded by two parabolas $y = x^2 + 1$ and $y = 3 - x^2$ about the $x-$axis.

Exercise 1.8.4 (Self-check). Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the x-axis (between the given limits):

(a) $y = x^3$ and $y = x^2$

(b) $y^2 = 4(x - 1)$ and $y = x - 1$

(c) $y = x^2 + 2$ and $y = 10 - x^2$

(d) $y = 1/x$ and $2y = 5 - 2x$

(e) by the parabola $y = x^2$ and the line $y = x$.

Exercise 1.8.5 (Self-check). Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the y-axis:

(a) $y = x^{1/3}$ and $x = 4y$, $x, y \geq 0$

(b) $x^2 - 2x$ and $y = x$

(c) $y = 16 - x$ and $y = 3x + 2$

(d) $y = x^3$ and $y = x^{1/3}$

Reference for Practice Problems

https://tutorial.math.lamar.edu/classes/calci/volumewithrings.aspx