Exponential Distribution

A continuous random variable 'x' is said to follow an exponential distribution with parameter $\lambda > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Note:-

The p.d.f $f(x)$ is a legitimate density function.

$$\int_0^\infty f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty$$

$$= -[0-1] = 1$$

Mean and variance of the Exponential distribution

Moments about the origin μ_r' of the exponential distribution is given by

$$\mu_r' = E(x^r) = \int_0^\infty x^r \lambda e^{-\lambda x}\,dx$$

Let $y = \lambda x$, $dy = \lambda\,dx$

$$= \int_0^\infty \frac{y^r}{\lambda^r} \cdot \lambda e^{-y} \cdot \frac{dy}{\lambda}$$

$$= \frac{1}{\lambda^r} \int_0^\infty e^{-y} \cdot y^r\,dy$$

$$\mu_r' = E(x^r) = \frac{\Gamma(r+1)}{\lambda^r} = \frac{r!}{\lambda^r}$$

$$\mu_1' = E(x) = \frac{1}{\lambda}$$

$$\mu_2' = E(x^2) = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}$$

$$Var(x) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

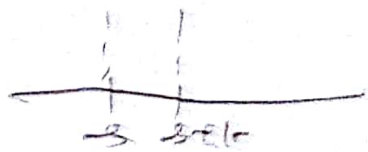
Memoryless property of the Exponential distribution

If X is Exponentially distributed then

$$P(X > s+t \,/\, X > s) = P(X > t) \quad \text{for any } s, t > 0$$

Proof

$$P(X > t) = \int_t^\infty f(x)\,dx$$



$$= \int_t^\infty \lambda e^{-\lambda x}\,dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_t^\infty$$

$$P(X > t) = -\left[0 - e^{-\lambda t}\right] = e^{-\lambda t}$$

$$P(X > s+t \,/\, X > s) = \frac{P(X > s+t \text{ and } X > s)}{P(X > s)}$$

$$=$$

$$= \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$P(X > s+t \,/\, X > s) = e^{-\lambda t} = P(X > t)$$

Problems

(1) The mileage which car owners get with a certain kind of radial tire is a random variable having an exponential distribution with mean 40,000 k.m. Find the probabilities that one of these tires will last

(i) atleast 20,000 km

(ii) atmost 30,000 km.

Soln:- Let X denote the mileage obtained with the tire.

Given mean = 40,000

$\frac{1}{\lambda} = 40,000$

$\Rightarrow \lambda = \frac{1}{40,000}$

$$f(x) = \lambda e^{-\lambda x}$$

$$= \frac{1}{40,000} e^{\frac{-x}{40,000}}$$

i) $P(X \geq 20,000) = \int_{20,000}^{\infty} \frac{1}{40,000} e^{\frac{-x}{40,000}} dx.$

$$= \frac{1}{40,000} \left[\frac{e^{-x/40,000}}{-1/40,000} \right]_{20,000}^{\infty} = -1 \left[0 - e^{-1/2} \right]$$

$$= e^{-0.5} = 0.6065$$

(ii) $P(X \leq 30,000) = \int_{0}^{30,000} \frac{1}{40,000} e^{\frac{-x}{40,000}} dx = 1 - e^{-3/4}$

$$= 1 - e^{-0.75} = 0.5270$$

(7) The time (in hrs) required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$.

a) what is the probability that the repair time exceeds 2 hr.?

b) what is the conditional probability that a repair takes atleast 10 hr. given that its duration exceeds 9 hr.?

Problem:- If X represents the time to repair the machine, the density fn. of X is given by

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-x/2}, \quad x \geq 0.$$

a) $$P(X>2) = \frac{1}{2}\int_2^{\infty} e^{-x/2}\,dx = \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_2^{\infty}$$

$$= -\left[0 - e^{-1}\right] = 1/e = 0.3679.$$

b) $$P(X \geq 10 / X > 9) = P(X \geq 9+1 / X > 9) = P(X > 1)$$

$$= \int_1^{\infty} \frac{1}{2} e^{-\frac{x}{2}}\,dx = \frac{1}{2}\left(\frac{e^{-x/2}}{-1/2}\right)_1^{\infty}$$

$$= -\left[0 - e^{-1/2}\right] = e^{-0.5}$$

$$= 0.6065$$