UNIT - 7

Vector Integral Theorems

- We have studied the Line integral, Surface integral & Volume integral (not in syllabus) for vector valued functions.

Now we study the connections between the surface & Volume integral and also between line integral and surface integral.

Vector integral Theorems

The following theorems are called vector integral theorems, establish the equivalence relation among the line, surface and volume integrals of vectors.

(1) Gauss divergence theorem (surface & volume)

(2) Stokes theorem (line & surface).

(3). Green's theorem in the plane.

Green's Theorem (Statement)

(i) If $M(x,y)$ & $N(x,y)$ are continuous functions with continuous partial derivatives in a region R of the xy plane bounded by a simple closed curve C, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Example:- Show that the area bounded by a closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

Soln:- Green's theorem states

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Take $M = -y$, $N = x$

Then
$$\oint_C -y\,dx + x\,dy = \iint_R (1+1)\,dx\,dy$$

$$= \iint_R 2\,dx\,dy = 2A = 2(\text{Area of the closed curve } C)$$

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Note:- Take $x = r\cos\theta$, $y = r\sin\theta$

$dx = -r\sin\theta\,d\theta$, $dy = r\cos\theta\,d\theta$

Area in polar coordinates $= \frac{1}{2}\oint_C r\cos\theta\, r\cos\theta\,d\theta + r\sin\theta\, r\sin\theta\,d\theta$

$$= \frac{1}{2}\oint_C (r^2\cos^2\theta + r^2\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\oint_C r^2\,d\theta$$

(2) Find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ (or) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\text{Area} = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$x = a\cos\theta$

$dx = -a\sin\theta\,d\theta$

$y = b\sin\theta$

$dy = b\cos\theta\,d\theta$

$\theta : 0 \text{ to } 2\pi$

$$= \frac{1}{2}\int_0^{2\pi} a\cos\theta\, b\cos\theta\,d\theta + b\sin\theta\, a\sin\theta\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta = \frac{1}{2}ab\,(\theta)_0^{2\pi}$$

$$= \frac{1}{2}ab\times 2\pi = \pi ab \text{ sq. units}$$

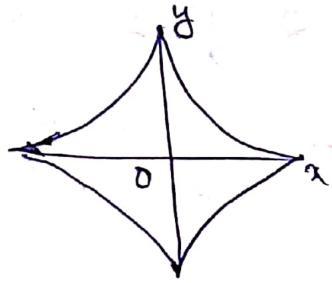
H.W. Find the area of a circle of radius 'a' units using Green's Thm.
$x = a\cos\theta$, $y = a\sin\theta$
$\theta : 0 \text{ to } 2\pi$
Ans: πa^2 sq. units.

(3) Find the area of the ~~[illegible]~~ curve $x^{2/3} + y^{2/3} = a^{2/3}$ (Hypocycloid), $a > 0$ using Green's theorem.

Soln:- The parametric equations of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ are $x = a\cos^3\theta$, $y = a\sin^3\theta$.

$dx = -3a\cos^2\theta\sin\theta\,d\theta$

$dy = 3a\sin^2\theta\cos\theta\,d\theta$



By Green's Theorem

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\oint_C a\cos^3\theta\, 3a\sin^2\theta\cos\theta\,d\theta + a\sin^3\theta\, 3a\cos^2\theta\sin\theta\,d\theta$$

$$= \frac{1}{2}3a^2\oint_C \cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta\,d\theta$$

$$= \frac{1}{2} 3a^2 \cdot \oint_C \cos^2\theta \sin^2\theta \, d\theta$$

$$= \frac{1}{2} 3a^2 \cdot 4 \int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta$$

$$= 6a^2 \int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta$$

$$= 6a^2 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3a^2 \pi}{8}.$$

Reduction formula:

$$\int_0^{\pi/2} \sin^n x \, dx \text{ or } \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 3} & \text{if } n \text{ is odd} \\ \frac{(n-1)(n-3)\ldots}{n(n-2)(n-4)\ldots} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

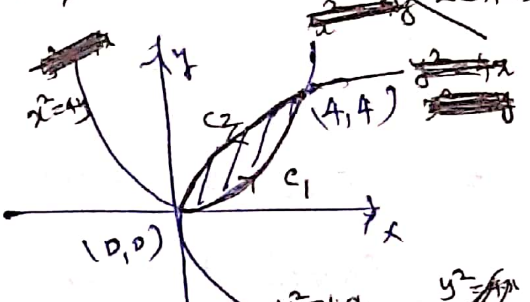
$$\int_0^{\pi/2} \cos^m x \sin^n x \, dx = \frac{1 \cdot 3 \cdot 5 \ldots (m-1) \; 1 \cdot 3 \cdot 5 \ldots (n-1)}{2 \cdot 4 \cdot 6 \ldots (m+n)} \cdot \frac{\pi}{2}$$

m & n are even integers.

4. Find the area between the parabola $y^2 = 4x$ & $x^2 = 4y$.

Soln :-

Area $= \frac{1}{2} \oint_C (x\,dy - y\,dx)$, The pts of intersection is $(0,0)$ & $(4,4)$.



$$= \frac{1}{2}\int_{C_1} + \frac{1}{2}\int_{C_2}$$

$$= \frac{1}{2}\int (x\,dy - y\,dx) + \frac{1}{2}\int (x\,dy - y\,dx)$$

$x^2 = 4y$ $\qquad$ $y^2 = 4x$

$$= \frac{1}{2}\left[\int_0^4 - + \int_4^0 -\right].$$

$$= \frac{1}{2}\int_0^4 x \frac{x}{2} dx - \frac{x^2}{4} dx - \frac{1}{2}\int_0^4 \frac{y^2}{4} dy - \frac{y^2}{2} dy$$

$$= \frac{1}{2}\int_0^4 \left(\frac{x^2}{2} - \frac{x^2}{4}\right) dx - \frac{1}{2}\int_0^4 \left(\frac{y^2}{4} - \frac{y^2}{2}\right) dy$$

$$= \frac{1}{2}\int_0^4 \frac{x^2}{4} dx + \frac{1}{2}\int_0^4 \frac{y^2}{4} dy$$

$x^2 = 4y$, $2x\,dx = 4\,dy$, $dy = \frac{2x}{4} dx$

$$= \frac{1}{2}\left(\frac{x^3}{12}\right)_0^4 + \frac{1}{2}\left(\frac{y^3}{12}\right)_0^4$$

$$= \frac{1}{2} \cdot \frac{64}{12} + \frac{1}{2} \cdot \frac{64}{12} = \frac{64}{12} = \frac{16}{3} \text{ sq. units.}$$

$y^2 = 4x$, $2y\,dy = 4\,dx$, $dy = \frac{2}{y} dx$

~~$= \frac{1}{2}\int_0^4 \frac{y^2}{4} dy - \frac{y^2}{2} dy + \frac{1}{2}\int_4^0 \frac{x^2}{2} dx - \frac{x^2}{4} dx$~~

~~$= \frac{1}{2}\int_0^4 -\frac{y^2}{4} dy - \frac{1}{2}\int_0^4 \frac{x^2}{4} dx$~~

~~$= \frac{1}{8}\left(\frac{y^3}{3} - \frac{x^3}{3}\right)_0^4 = -\frac{1}{24}(64) \times 2$~~

~~$= -\frac{16}{3}$ sq. units~~

5. ~~Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using Green's theorem.~~

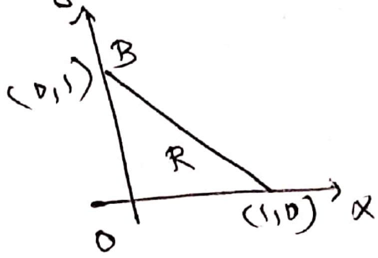
5. Use Green's Theorem to evaluate

Tutorial

$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the ~~circle~~ triangle formed by the pts $(0,0)$, $(1,0)$, $(0,1)$. (or) where C is the boundary of the region defined by $x\ge 0, y\ge 0, x+y=1$.

Soln:- Comparing $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

with $\int_C M\,dx + N\,dy$, we get



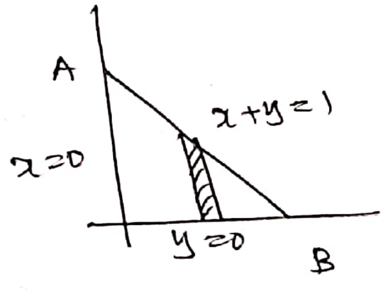
$M = 3x^2-8y^2$, $N = 4y-6xy$

By Green's Theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (-6y+16y)\,dx\,dy$$

where R is the region bdd by the lines $x=0$, $y=0$, $x+y=1$ (equation of AB).



$$\iint_R 10y\,dx\,dy = \int_0^1 \int_0^{1-x} 10y\,dy\,dx$$

$$= 10 \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2\,dx$$

$$= -\frac{5}{3}\left((1-x)^3\right)_0^1$$

$$= -5/3\,(0-1)$$

$$= 5/3.$$

$\cancel{1+x^2-2x}$

$\cancel{\left(x+\frac{x^3}{3}-x^2\right)_0^1}$

$\cancel{1+\frac{1}{3}-1}$

6. Evaluate using Green's theorem in the plane for

$\int_C (xy+y^2)\,dx + x^2\,dy$ where C is the closed curve

of the region bdd by $y=x$ & $y=x^2$.

Soln:-

Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = xy + y^2$

$N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$

$\frac{\partial N}{\partial x} = 2x$

$$\therefore \int_C M\,dx + N\,dy = \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x - 2y)\,dx\,dy$$

$x : 0$ to 1

$y : x^2$ to x

$\int_0^1 \int_{x^2}^{x} (x-2y)\,dx\,dy$

$= \int_0^1 \left(xy - \frac{2y^2}{2}\right)_{x^2}^{x} dx$

$= \int_0^1 \cancel{x^2} - \cancel{x^2} - x^3 + x^4\,dx$

$\left(-\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1$

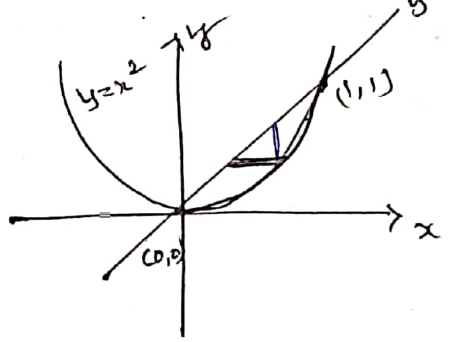
$= -\frac{1}{4} + \frac{1}{5}$

$= \frac{-5+4}{20} = -\frac{1}{20}$.

Given the region bdd by $y=x$ & $y=x^2$

x varies from y to $\sqrt{y}$

y varies from 0 to 1.



$$\therefore \int_C M\,dx + N\,dy = \int_0^1 \int_y^{\sqrt{y}} (x - 2y)\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2}{2} - 2xy\right)_y^{\sqrt{y}} dy = \int_0^1 \left(\frac{y}{2} - 2y^{3/2} - \frac{y^2}{2} + 2y^2\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} + \frac{3}{2}y^2 - 2y^{3/2}\right) dy = \left.\frac{y^2}{4} + \frac{3}{2}\frac{y^3}{3} - \frac{2\times 2y^{5/2}}{5}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{4}{5} = \frac{3}{4} - \frac{4}{5} = \frac{15-16}{20} = \frac{-1}{20} \,//.$$

$$\therefore \int_C (xy+y^2)\,dx + x^2\,dy = -\frac{1}{20} \,//.$$

7. Verify Green's theorem in plane for

$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the boundary of the region defined by $x=0,\ y=0,\ x+y=1$

(or) where C is the triangle formed by the pts $(0,0)$ $(1,0)$ & $(0,1)$

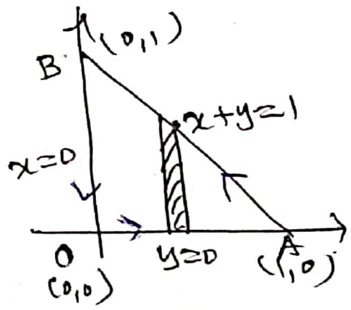
Soln:- we have to prove that

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = 3x^2 - 8y^2$ $\qquad \frac{\partial N}{\partial x} = -6y$

$N = 4y - 6xy$ $\qquad \frac{\partial M}{\partial y} = -16y$

where R is the region bdd by the lines $x=0,\ y=0$ & $x+y=1$.



RHS

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R 10y\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx$$

$$= 10\int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx = \frac{10}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{5}{3}\left((1-x)^3\right)_0^1$$

$$= -\frac{5}{3}(0-1)$$

$$= \frac{5}{3}.$$

L.H.S

$$\int_C M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$$

Along OA, $y = 0$, x varies 0 to 1

$dy = 0$.

$$\int_{OA} M\,dx + N\,dy = \int_0^1 3x^2\,dx = 1.$$

Along AB, y varies from 0 to 1.

$x = 1 - y$.

$dx = -dy$

$$\int_{AB} M\,dx + N\,dy = \int_0^1 (3(1-y)^2 - 8y^2)(-dy) + 4y - 6(1-y)y\,dy$$

$$= \int_0^1 3(1+y^2-2y) - 8y^2)(-dy) + 4y - 6y + 6y^2)\,dy$$

$$= \int_0^1 (11y^2 + 4y - 3)\,dy = \left.\frac{11y^3}{3} + \frac{4y^2}{2} - 3y\right)_0^1$$

$$= \frac{11}{3} + 2 - 3 = \frac{11}{3} - 1 = \frac{8}{3}.$$

Along BO, y varies from 1 to 0 & $x = 0$, $dx = 0$

$$\int_{BO} M\,dx + N\,dy = \int_1^0 4y\,dy = \left.\frac{4y^2}{2}\right)_1^0 = 0 - 2$$

$$= -2.$$

$$\therefore \int_C M\,dx + N\,dy = 1 - 2 + \frac{8}{3} = -1 + \frac{8}{3} = \frac{5}{3}.$$

$\therefore$ L.H.S = R.H.S

Hence Green's thm verified.

8. Verify Green's theorem in plane for $\int_C x^2(1+y)\,dx + (y^3+x^3)\,dy$ where C is the square bdd by $x = \pm a,\ y = \pm a$.

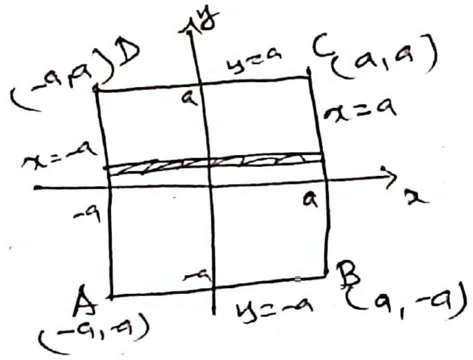
Soln:- Given $\int_C x^2(1+y)\,dx + (y^3+x^3)\,dy$

By Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = x^2(1+y) \qquad \frac{\partial M}{\partial y} = x^2$

$N = y^3 + x^3 \qquad \frac{\partial N}{\partial x} = 3x^2$



R.H.S

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (3x^2 - x^2)\,dx\,dy$$

$$= \int_{-a}^{a}\int_{-a}^{a} 2x^2\,dx\,dy = \int_{-a}^{a} \left.\frac{2x^3}{3}\right)_{-a}^{a} dy$$

$$= \int_{-a}^{a} \frac{2a^3}{3} + \frac{2a^3}{3}\,dy = \frac{4a^3}{3} \int_{-a}^{a} dy$$

$$= \frac{4a^3}{3}(2a) = \frac{8a^4}{3}.$$

L.H.S

$$\int_C M\,dx + N\,dy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB, $y = -a$, $dy = 0$, x varies $-a$ to a.

$$\int_{AB} M\,dx + N\,dy = \int_{-a}^{a} \cancel{x^2\,dx}\ x^2(1-a)\,dx = (1-a) \left.\frac{x^3}{3}\right)_{-a}^{a}$$

$$= (1-a)\left(\frac{a^3}{3} + \frac{a^3}{3}\right) = \frac{2a^3(1-a)}{3} = \frac{2a^3}{3} - \frac{2a^4}{3}.$$

Along BC, $x = a$ $dx = 0$ y : $-a$ to a

$$\int_{BC} Mdx + Ndy = \int_{-a}^{a} (a^3+y^3)\,dy = a^3y + \frac{y^4}{4}\Big]_{-a}^{a}$$

$$= a^4 + \cancel{\frac{a^4}{4}} + a^4 - \cancel{\frac{a^4}{4}}$$

$$= 2a^4.$$

Along CD $y = a$, $dy = 0$ x : a to $-a$

$$\int_{CD} Mdx + Ndy = \int_{a}^{-a} (1+a)\,x^2\,dx = (1+a)\frac{x^3}{3}\Big]_{a}^{-a}$$

$$= (1+a)\left(\frac{-a^3}{3}\right) - (1+a)\frac{a^3}{3} = (1+a)\left(-\frac{2a^3}{3}\right)$$

$$= -\frac{2a^3}{3} - \frac{2a^4}{3}.$$

Along DA, $x = -a$, $dx = 0$, y : a to $-a$.

$$\int_{DA} Mdx + Ndy = \int_{a}^{-a} (y^3 - a^3)\,dy = \frac{y^4}{4} - a^3y\Big]_{a}^{-a}$$

$$= \frac{a^4}{4} + a^4 - \frac{a^4}{4} + a^4$$

$$= 2a^4$$

$$\int_C Mdx + Ndy = \cancel{\frac{2a^3}{3}} - \frac{2a^4}{3} + 2a^4 + 2a^4 - \cancel{\frac{2a^3}{3}} - \frac{2a^4}{3}$$

$$= -\frac{4a^4}{3} + 4a^4 = \frac{8a^4}{3}.$$

L.HS = RHS

Hence Green's thm is verified,

$\oint_C (x^2-y^2)dx + 2xy\,dy$
$x=0, x=a, y=0$ & $y=b$.
ab^2

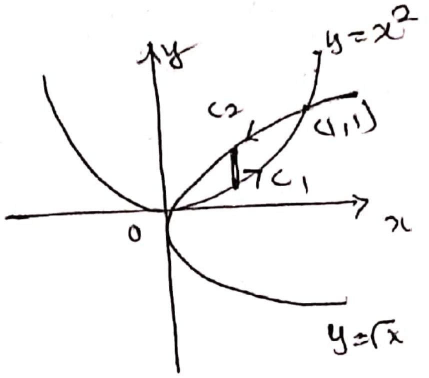
9. H.W Verify Green's Thm for $\int_C (xy+x^2)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the lines $x=-1$, $x=1$, $y=-1$ & $y=1$.

$\boxed{\text{Ans: } 0}$

Q. Verify Green's theorem in the plane for

$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the boundary

of the region defined by $y=\sqrt{x}$ and $y=x^2$.



Soln:-

$y=\sqrt{x}$ & $y=x^2$ are the parabolas whose points of intersection are $(0,0)$ & $(1,1)$.

Greens Thm

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = 3x^2 - 8y^2 \qquad \frac{\partial M}{\partial y} = -16y$

$N = 4y - 6xy \qquad \frac{\partial N}{\partial y} = -6y$

R.H.S

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dy\,dx$$

$$= \int_0^1 10 \left.\frac{y^2}{2}\right|_{x^2}^{\sqrt{x}} dx = 5\int_0^1 (x - x^4)\,dx$$

$$= \cancel{5\int_0 x^3} = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$= 5\left(\frac{3}{10}\right) = \frac{3}{2}$$

L.H.S

$$\int_C M\,dx + N\,dy = \int_{C_1} + \int_{C_2}$$

On C_1 $\quad y = x^2 \Rightarrow dy = 2x\,dx \qquad x: 0$ to 1.

$$\int_{C_1} M\,dx + N\,dy = \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x\,x^2)\,2x\,dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\, dx = \left. \frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5} \right)_0^1$$

$$= 1 + 2 - 4 = -1.$$

on C_2 $y^2 = x$, $2y\,dy = dx$, y varies from 1 to 0

$$\int_{C_2} M\,dx + N\,dy = \int_1^0 (3y^4 - 8y^2)\cdot dy \cdot 2y + (4y - 6y^2 y)\, dy$$

$$= \int_1^0 6y^5 - 16y^3 + 4y - 6y^3 \,) dy$$

$$= \left. 6\frac{y^6}{6} - 16\frac{y^4}{4} + 4\frac{y^2}{2} - 6\frac{y^4}{4} \right)_1^0$$

$$= 0 - (1 - 4 + 2 - 3/2)$$

$$= -(-1 - 3/2) = 5/2 .$$

$$\therefore \int_C M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2}.$$

$\therefore$ L.H.S = R.H.S

Hence Green's thm verified.

10. Verify Green's Theorem in a plane for the integral $\int_C (x - 2y)\, dx + x\, dy$ taken around the circle $x^2 + y^2 = 4$.

Soln:- By Green's Theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Here $M = x - 2y$; $\frac{\partial M}{\partial y} = -2$

$N = x$; $\frac{\partial N}{\partial x} = 1.$

R.H.S

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1+2)\, dx\,dy$$

$$= 3 \iint_R dx\,dy$$

$= 3$(Area of the closed curve C ie, circle)

$$= 3\pi r^2$$

$$= 3 \times \pi \times 4 \qquad \because r = 2$$

$$= 12\pi.$$

L.H.S $\int_C M dx + N dy$

Parametric eqn of the circle $x^2+y^2=4$ is

$x = 2\cos\theta$; $y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta$; $dy = 2\cos\theta\, d\theta$

$$\int_C M dx + N dy = \int_0^{2\pi} (x-2y)\,dx + x\,dy$$

$$= \int_0^{2\pi} (2\cos\theta - 2\times 2\sin\theta)(-2\sin\theta\, d\theta) + 2\cos\theta\, 2\cos\theta\, d\theta$$

$$= \int_0^{2\pi} (-4\sin\theta\cos\theta\, d\theta + 8\sin^2\theta + 4\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} -2\sin 2\theta + 4 + 4\sin^2\theta\, d\theta$$

$$= \int_0^{2\pi} -2\sin 2\theta + 4 + 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \left(+\frac{\cancel{2}\cos 2\theta}{\cancel{2}} + 4\theta + \frac{4}{2}\theta - \frac{\cancel{2}\sin 2\theta}{\cancel{2}}\right)_0^{2\pi}$$

$$= \cos 4\pi + 8\pi + 4\pi - \sin 4\pi - 1$$

$$= 1 + 12\pi - 1 = 12\pi.$$

$\therefore$ L.H.S = R.H.S.

H.W

1. Verify Green's thm for $\int_C y^2 dx - x^2 dy$ where C is the bounded by the triangle whose vertices are $(1,0)$, $(0,1)$ & $(-1,0)$

Ans: $-2/3$.

2. Apply Green's thm to evaluate

$\int_C (y - \sin x)\, dx + \cos x\, dy$ where C is the plane triangle enclosed by the lines $y=0$, $x=\pi/2$ & $y = \frac{2x}{\pi}$.

Ans $-\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$

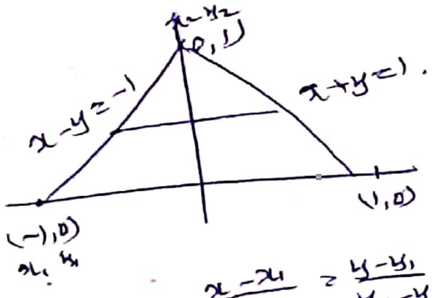
③ Using Green's Theorem in a plane to evaluate $\int_C (2x-y)\,dx + (x+y)\,dy$, where C is the boundary of the circle $x^2+y^2=a^2$ in the xoy-plane.

$M = y^2$

$N = x^2$

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = -2x$



$\int_0^1 \int_{-1+y}^{1-y} \cancel{y^2 dx - x^2 dy}\ (-2x-2y)\, dx\, dy$

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$

$\frac{x+1}{1} = \frac{y-0}{1}$

$x - y = -1$.

$= -2\int_0^1 \left(\frac{x^2}{2} + xy\right)_{-1+y}^{1-y} dy$

$= -2\int_0^1 \frac{(1-y)^2}{2} + y(1-y) - \left(\frac{(-1+y)^2}{2} + y(-1+y)\right) dy$

$= -2\int_0^1 \frac{(1-y)^2}{2} + y - y^2 - \left(\frac{(-1+y)^2}{2} - y + y^2\right) dy$

$= -2\int_0^1 \left(\frac{(1-y)^2}{2} + 2y - 2y^2 - \frac{(-1+y)^2}{2}\right) dy$

$= -2\left(\frac{(1-y)^3}{-3\times 2} + \frac{2y^2}{2} - \frac{2y^3}{3} - \frac{(-1+y)^3}{2\times 3}\right)_0^1$

$= -2\left(1 - \frac{2}{3} - \left(\cancel{\tfrac{1}{6}} + \cancel{\tfrac{1}{6}}\right)\right)$

$= -2\left(\frac{3-2}{3}\right) = -2\left(\frac{1}{3}\right) = -2/3$.

Stoke's Theorem

Statement:- Let S be an open surface bounded by a closed curve C. Let $\vec{F}$ be a vector point function defined on the surface S and $\hat{n}$ be a unit outward drawn normal at any point P on S.

Thus

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

Problems

1. Prove that $\int_C \vec{r} \cdot \vec{dr} = \vec{0}$.

By Stoke's thm,

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

Take $\vec{F} = \vec{r}$

$$\nabla \times \vec{F} = \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{0}$$

$$\therefore \int_C \vec{r} \cdot \vec{dr} = \iint_R \vec{0} \cdot \hat{n}\, ds$$

$$= \vec{0}.$$

2. Verify Stoke's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

where S is the upper half of the sphere $x^2+y^2+z^2=1$.

Soln:- By Stoke's Thm

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

R.H.S

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}(-2yz+2yz) - \vec{j}(0-0) + \vec{k}(0+1)$$

$$= \vec{k}$$

$$\phi = x^2+y^2+z^2-1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2} = 2.$$

$$\hat{n} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2}$$

$$= x\vec{i} + y\vec{j} + z\vec{k}$$

$$\nabla\times\vec{F}\cdot\hat{n} = \vec{k}\cdot x\vec{i}+y\vec{j}+z\vec{k}$$

$$= z.$$

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds = \iint_R z\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \iint_R z\,\frac{dx\,dy}{z} = \iint_R dx\,dy$$

where R is the region bdd by the projection of $x^2+y^2+z^2=1$ on the xy plane, i.e, R is the region bdd by the circle $x^2+y^2=1$.

$\therefore \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$ = Area of the circle $x^2+y^2=1$

$= \pi$ —— ①

L.H.S

C is the circle $x^2+y^2=1$.

The parametric eqns are

$x=\cos\theta, \quad y=\sin\theta$

θ: 0 to 2π

$$\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\vec{F}\cdot d\vec{r} = (2x-y)\,dx - yz^2\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}(2\cos\theta - \sin\theta)(-\sin\theta) - 0$$

$$= \int_0^{2\pi}(-2\sin\theta\cos\theta + \sin^2\theta)\,d\theta = \int_0^{2\pi} -\sin 2\theta + \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \left(\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$= \frac{\cos 4\pi}{2} + \pi - \frac{\sin 4\pi}{2} - \frac{\cos 0}{2} + 0 - 0$$

$$= \frac{1}{2} + \pi - 0 - \frac{1}{2}$$

$= \pi$ —— ②

from ① & ②, R.H.S = L.H.S

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \pi \quad //.$$

8. Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken round the rectangle bdd by the lines $x = \pm a$, $y=0$, $y=b$.

Soln:- Stoke's theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

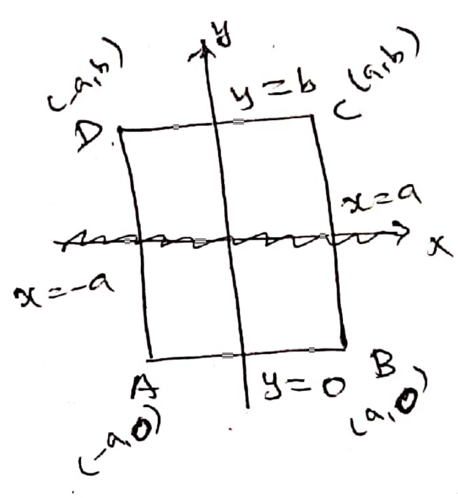
R.H.S

$$\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-2y-2y)$$

$$= -4y\vec{k}$$



Since the rectangle lies in the xy plane, we have $\hat{n} = \vec{k}$,

$ds = dx\,dy$

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds = \int_0^b\int_{-a}^{a} -4y\, dx\, dy$$

$$= -4\int_0^b y(x)_{-a}^{a}\, dy$$

$$= -4\times 2a \int_0^b y\, dy$$

$$= -8a\left(\frac{y^2}{2}\right)_0^b$$

$$= -4ab^2. \quad \text{—— (1)}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

LHS

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB, $y=0$, $dy=0$

x : $-a$ to a

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx = \left.\frac{x^3}{3}\right)_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}.$$

Along BC, $x=a$, $dx=0$, y : 0 to b

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^b -2ay\,dy = -\cancel{2}a\left.\frac{y^2}{\cancel{2}}\right)_0^b$$

$$= -ab^2$$

Along CD, $y=b$, $dy=0$ x : a to $-a$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)\,dx = \left.\frac{x^3}{3} + b^2x\right)_a^{-a}$$

$$= -\frac{a^3}{3} - b^2a - \frac{a^3}{3} - b^2a$$

$$= -\frac{2a^3}{3} - 2b^2a$$

Along DA, $x=-a$, $dx=0$, y : b to 0

$$\int_{DA} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy = \left.\cancel{2}a\frac{y^2}{\cancel{2}}\right)_b^0 = 0 - ab^2$$

$$= -ab^2$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \frac{\cancel{2a^3}}{\cancel{3}} - ab^2 - \frac{\cancel{2a^3}}{\cancel{3}} - 2ab^2 - ab^2$$

$$= -4ab^2 \quad \text{——} (2)$$

$\therefore$ from (1) & (2)

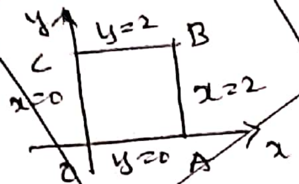
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \vec{F}\cdot\hat{n}\,ds = -4ab^2 \,//.$$

H.W

(3) (4) Verify stoke's theorem for

check: $\vec{F} = (y - z + 2)\vec{i} + (4 + yz)\vec{j} - xz\vec{k}$ where S is the surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy plane.

Hint:



xy plane.

$\hat{n} = \vec{k}$

$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_R -dx\, dy$

$= -$ (Area of the square of side 2 units)

$= -4.$ or $\int_0^2 \int_0^2 -dx\, dy = -4$

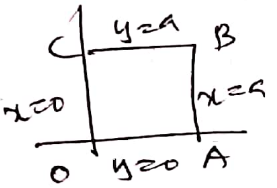
$\int_C \vec{F} \cdot d\vec{r} = -4.$

Along OA, 4, AB 8, BC −8, CO −8

(6). Verify stoke's theorem for $\vec{F} = x^2\vec{i} + xy\vec{j}$ taken

H.W round the square in the xy plane whose sides are $x = 0, x = a, y = 0, y = a$.

Hint:



xy plane

$\hat{n} = \vec{k}$

$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^a \int_0^a y\, dx\, dy$

$= \frac{a^3}{2}.$

$\int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{2}$

Along OA, $\frac{a^3}{3}$, AB $\frac{a^3}{2}$, BC, $-\frac{a^3}{3}$, CO $= 0$

~~(6a) Verify Stoke's thm for $\vec{F} = (y - z + 2)\vec{i} + (4 + yz)\vec{j} - xz\vec{k}$ where S is the surface of the cube~~

(6a) Verify Stoke's thm for $\vec{F} = xy\vec{i} + xy^2\vec{j}$ where S is the surface of the square formed by points $(1,1)$ $(-1,1)$ $(-1,-1)$ & $(1,-1)$. &

(6b) $(1,0)$ $(-1,0)$ $(0,1)$ & $(0,-1)$.

(3) Using Stoke's theorem find $\iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$ for

$\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$ where S is the upper half of the sphere $x^2+y^2+z^2=9$, and C is its boundary.

(or)

Verify stoke's theorem for the function

$\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$ over the surface of the hemisphere $x^2+y^2+z^2=9$ above the xy plane.

Soln:-

$$\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$$

$$\vec{F}\cdot d\vec{r} = 2y\,dx + 3x\,dy - z^2 dz$$

Since projection is on xy plane, we get $x^2+y^2=9$

$$\therefore \vec{F}\cdot d\vec{r} = 2y\,dx + 3x\,dy \qquad \because z=0 \ \& \ dz=0.$$

Stokes thm $\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$

L.HS $\int_C \vec{F}\cdot d\vec{r} = \int_C 2y\,dx + 3x\,dy.$

the parametric form of $x^2+y^2=9$ are

$x = 3\cos\theta \qquad y = 3\sin\theta$

$dx = -3\sin\theta\, d\theta \qquad dy = 3\cos\theta\, d\theta$

$$\therefore \int_0^{2\pi} 2\times 3\sin\theta \times -3\sin\theta\, d\theta + 3\times 3\cos\theta\ 3\cos\theta\, d\theta$$

$$= \int_0^{2\pi} -18\sin^2\theta + 27\cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} -18\left(\frac{1-\cos 2\theta}{2}\right) + 27\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= -9\int_0^{2\pi}(1-\cos 2\theta)\, d\theta + \frac{27}{2}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta$$

$$= -9\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{2\pi} + \frac{27}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$= -9\left(2\pi - \frac{\sin 4\pi}{2}\right) + \frac{27}{2}\left(2\pi + \frac{\sin 4\pi}{2}\right)$$

$$= -18\pi + 27\pi \quad = 9\pi \quad //.$$

R.H.S

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(3-2)$$

$$= \vec{k}$$

$\nabla\times\vec{F} = \vec{k}$, $\quad \hat{n} = \vec{k}$ $\quad \because$ xy plane

$(\nabla\times\vec{F})\cdot\hat{n} = 1.$

$$\therefore \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \iint_R 1 \cdot \frac{dx\, dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \iint_R dx\, dy$$

$=$ Area of the circle $x^2+y^2=9$

$$= \pi\, 3^2$$

$$= 9\pi \quad //.$$

$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$|\nabla\phi| = 2\times 3 = 6$

$\hat{n} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{3}$

$\hat{n}\cdot\vec{k} = \frac{z}{3}$

$\iint_R (\nabla\times\vec{F})\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$

$= \iint_R \frac{z/3}{z/3}\, dx\, dy$

$= \iint_R dx\,dy =$ Area of the circle $x^2+y^2=9$

$= 9\pi //.$

$\phi = x^2+y^2+z^2-5$

$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\therefore$ L.H.S $=$ R.H.S

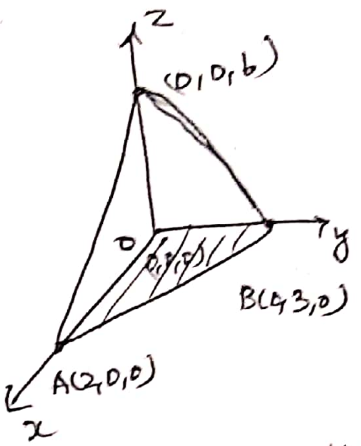
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = 9\pi \quad //.$$

H.W. ④. Verify Stokes thm for $\vec{F} = (x^2+y-4)\vec{i} + 3xy\vec{j} + (2xz+z^2)\vec{k}$ S is the surface of the hemisphere $x^2+y^2+z^2=16$ above xy plane. (Doubt).

(7) Evaluate the integral $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$ where C is the boundary of the Δ^{le} with vertices $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$ using Stokes thm.

Soln:- Stokes thm states

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$



R.H.S

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}(1+1) - \vec{j}(0-0) + \vec{k}(2-1)$$

$$= 2\vec{i} + \vec{k}$$

If the pts a, b, c intersects at x, y & z then the eqn is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

The eqn of the plane is $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1.$

i.e, $3x+2y+z=6$

$$\phi = 3x+2y+z-6$$

$$\nabla\phi = 3\vec{i}+2\vec{j}+\vec{k}$$

$$|\nabla\phi| = \sqrt{9+4+1} = \sqrt{14}$$

$$\hat{n} = \frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}$$

$$\therefore \nabla\times\vec{F}\cdot\hat{n} = 2\vec{i}+\vec{k}\cdot\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}} = \frac{6+1}{\sqrt{14}}$$

$$= \frac{7}{\sqrt{14}}$$

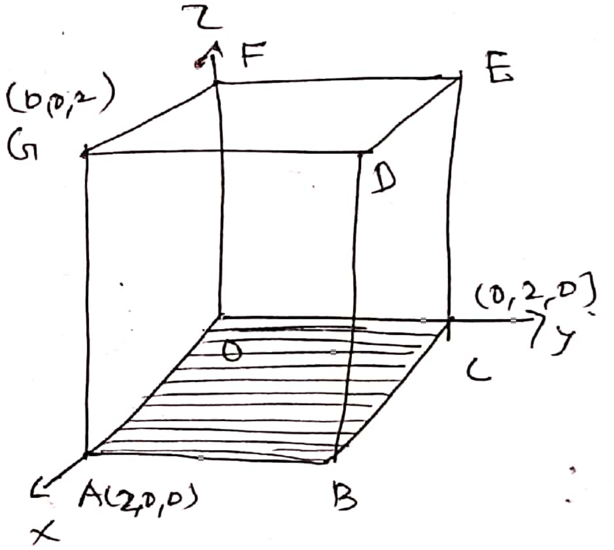
$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_R \frac{7}{\sqrt{14}}\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \iint \frac{7}{\sqrt{14}}\frac{dx\,dy}{\frac{1}{\sqrt{14}}}$$

$$= 7\iint_R dx\,dy$$

$= 7$ (Area of the Δ^{le} ABC)

$= 7\cdot\frac{1}{2}(2)(3) = 21$ sq. units.

(8) Verify Stoke's theorem for $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$ over the surface of a cube $x=0$, $y=0$, & $x=2$, $y=2$, $z=2$. above xy plane (open at the bottom).



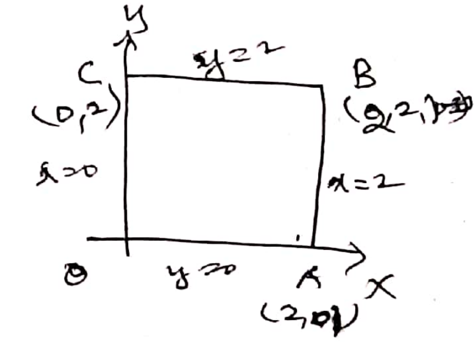
soln:- $\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$.

$\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

L.H.S $\int_C \vec{F}\cdot d\vec{r} = \int_C (y-z+2)dx + (yz+4)dy - xz\,dz$.

C is the bounding curve OABC in the plane $z=0$.



$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C (y+2)dx + 4dy$.

Along OA, $y=0$, $dy=0$, $x: 0$ to 2

$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^2 2dx = 4$.

Along AB, $x=2$, $dx=0$, $y: 0$ to 2.

$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^2 4dy = 8$

Along BC, $y=2$, $dy=0$ $x: (2, 0)$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_2^0 4\,dx = -8$$

Along CO, $x=0$, $dx=0$; $y: 2$ to 0

$$\int_{CO} \vec{F}\cdot d\vec{r} = \int_2^0 4\,dy = -8$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = 4+8-8-8 = -4 \quad \text{—— (1)}$$

<u>R.H.S</u>

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= \vec{i}(0-y) - \vec{j}(-z+1) + \vec{k}(0-1)$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

$$\therefore \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_{GDEF} + \iint_{OCEF} + \iint_{GABD} + \iint_{OAGF} + \iint_{BDEC}.$$

$\because$ cube above the xy plane so we omit OABC.

Along GDEF, $\hat{n} = \vec{k}$, $ds = dx\,dy$

$$\iint_{GDEF} (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_0^2\int_0^2 -dx\,dy = -4$$

Along OCEF, $\hat{n} = -\vec{i}$, $ds = dy\,dz$

$$\iint_{OCEF} (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_0^2\int_0^2 y\,dy\,dz$$

$$= 2\left.\frac{y^2}{2}\right)_0^2 = 4.$$

<u>H.W.</u>

$\vec{F} = (x^2-y^2)dx + 2xy\,dy$

$x=0, x=a, y=0, y=b,$

$z=0, z=c$ above the xy plane.

Along the GABD, $ds = dy\,dz$, $\hat{n} = \vec{j}$

$$\iint_{GABD} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^2 \int_0^2 -y\, dy\, dz = -4,$$

Along the OAGF, $\hat{n} = -\vec{j}$ $ds = dx\,dz$

$$\iint_{OAGF} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^2 \int_0^2 (z-1)\,(-dx\,dz)$$

$$= \int_0^2 \left[-\frac{z^2}{2} + z\right]_0^2 dx = 0.$$

Along the face BDEC, $\hat{n} = \vec{j}$, $ds = dx\,dz$

$$\iint_{BDEC} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \int_0^2 \int_0^2 (z-1)\, dx\, dz$$

$$= \left(\frac{z-1}{2}\right)^2\Big]_0^2 \quad \frac{1}{2} - \frac{1}{2} = 0$$

$$= 0.$$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = -4 + 4 - 4 + 0 + 0$$

$$= -4. \quad \text{--- (2)}$$

$\therefore$ R.H.S = L.H.S

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = -4$$

Hence Stoke's theorem verified.

~ . ~ . ~

H.W.

1. Verify stokes theorem for $\vec{F} = y^2 z \vec{i} + z^2 x \vec{j} + x^2 y \vec{k}$ where S is the open surface of the cube formed by the plane $x = \pm a$, $y = \pm a$ & $z = \pm a$ in which the plane $z = -a$ is cut. $\boxed{\text{Ans: } 4a^2}$

Gauss Divergence Theorem

Statement :

The surface integral of the normal component of a vector point function $\vec{F}$ over a closed surface 'S' is equal to volume integral of divergence of $\vec{F}$ taken over the volume 'V' enclosed by a surface S.

i.e., $$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV$$

where $\hat{n}$ is the unit vector along the outward drawn normal at any point P on the surface S.

Example : 1 If S is a closed surface enclosing a volume V and $\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$ show that $\iint_S \vec{F}\cdot\hat{n}\, ds = 6V$.

Soln:- By GDT

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV$$

$$\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$$

$$\nabla\cdot\vec{F} = 1+2+3 = 6$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV = \iiint_V 6\, dV = 6V$$

2. If $\vec{F} = \text{curl}\,\vec{g}$, prove that $\iint_S \vec{F}\cdot\hat{n}\,ds = 0$ for any closed surface S.

Soln:- For a closed surface S, by GDT, we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dV$$

$$= \iiint_V \nabla\cdot(\nabla\times\vec{g})\,dV$$

$$= 0 \qquad \because \nabla\cdot(\nabla\times\vec{F}) = 0.$$

div curl $\vec{F} = 0$.

3. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ for $\vec{F} = (ax-by)\vec{i} + (bx+cy)\vec{j} + (cy^2+dz)\vec{k}$ and S is the surface of the sphere of radius K units with centre at $(1,2,3)$

Soln:- $\vec{F} = (ax-by)\vec{i} + (bx+cy)\vec{j} + (cy^2+dz)\vec{k}$

$\nabla\cdot\vec{F} = a+c+d$

By GDT

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$= \iiint_V (a+c+d)\,dV$$

$$= (a+c+d)\iiint_V dV$$

$$= (a+c+d)(\text{Volume of the sphere of radius } K)$$

$$= (a+c+d)\left(\frac{4}{3}\pi K^3\right) \quad //.$$

4. Show that $\iint_S \vec{r}\cdot\hat{n}\,ds = 3V$ where V is the Volume enclosed by the closed surface S.

Soln:- By GDT,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

here $\vec{F} = \vec{r}$

$$\nabla\cdot\vec{r} = 3.$$

$$\therefore \iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V 3\,dv = 3V.$$

5. Using GDT, evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bdd by the planes $x=0,\ y=0,\ z=0,\ x=2,\ y=2\ \&\ z=2.$

Soln:- By GDT $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv$

$$\nabla\cdot\vec{F} = 4z - 2y + y = 4z - y$$

$$\therefore \int_0^2\int_0^2\int_0^2 (4z-y)\,dx\,dy\,dz = \int_0^2\int_0^2 \left(\frac{\cancel{4}z^2}{\cancel{2}} - yz\right)_0^2 dy\,dx$$

$$= \int_0^2\int_0^2 (8-2y)\,dy\,dx$$

$$= \int_0^2 \left(8y - \frac{\cancel{2}y^2}{\cancel{2}}\right)_0^2 dx = \int_0^2 (16-4)\,dx$$

$$= 12\int_0^2 dx$$

$$= 12(x)_0^2 = 24.$$

(6) Verify GDT for $\vec{F} = x\vec{i}+y\vec{j}+z\vec{k}$ taken over the region bdd by the planes $x=0, x=a, y=0, y=a, z=0, z=a$.
Hw

Soln:-

By GDT $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$

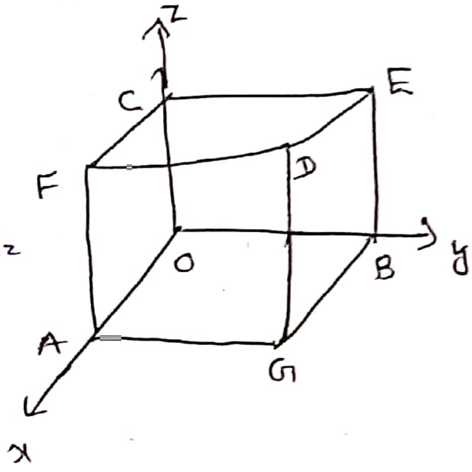
R.H.S

$\vec{F} = x\vec{i}+y\vec{j}+z\vec{k}$

$\nabla\cdot\vec{F} = 1+1+1 = 3$

$\iiint_V \nabla\cdot\vec{F}\,dV = \int_0^a\int_0^a\int_0^a 3\,dV =$ $dv = dx\,dy\,dz$

$= 3a^3$



L.H.S $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

Along the face AGDF, $x=a$, $\hat{n} = \vec{i}$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a x\,dy\,dz = \int_0^a\int_0^a a\,dy\,dz$

$= a^3$.

Along the face, OBEC, $x=0$, $\hat{n} = -\vec{i}$

$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a -x\,dy\,dz = 0$.

Along the face, BEDG, $y=a$, $\hat{n} = \vec{j}$

$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a y\,dx\,dz = \int_0^a\int_0^a a\,dx\,dz = a^3$

Along the face, ADFC, $y=0$, $\hat{n} = -\vec{j}$

$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a -y\,dx\,dz = 0$.

Along the face FDEC, $z=a$, $\hat{n}=\vec{k}$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a z\,dx\,dy = \int_0^a\int_0^a a\,dx\,dy = a^3$$

Along the face AGBO, $z=0$, $\hat{n}=-\vec{k}$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a -z\,dx\,dy = 0.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = a^3+0+a^3+0+a^3+0$$

$$= 3a^3$$

$\therefore$ L.HS = R.HS

Hence GDT is verified.

7) ~~Verify GDT for $\vec{F} = (x^2-yz)\vec{i} + (y^2-3x)\vec{j} + (z^2-xy)\vec{k}$ taken over the rectangular parallelopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$.~~

7) Verify GDT for $\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$ & $z=a$.

<u>Soln</u>:- By GDT

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$$

$$\nabla\cdot\vec{F} = 2 + x^2 - 2xz$$

$$\iiint_V \nabla\cdot\vec{F}\,dV = \int_0^a\int_0^a\int_0^a (2+x^2-2xz)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a \left[2x + \frac{x^3}{3} - \frac{2x^2}{2}z\right]_0^a dy\,dz$$

$$= \int_0^a\int_0^a \left[2a + \frac{a^3}{3} - a^2z\right] dy\,dz$$

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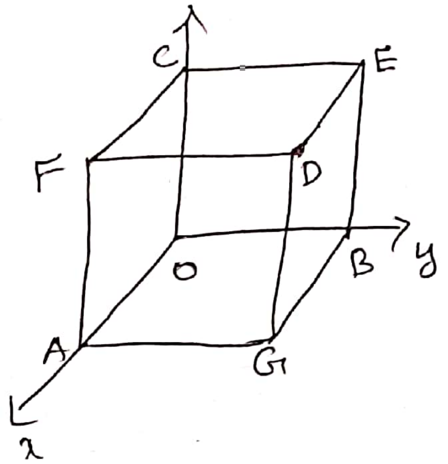
$$= \int_0^a \left(2ay + \frac{a^3}{3}y - a^3 zy\right]_0^a dz$$

$$= \int_0^a 2a^2 + \frac{a^4}{3} - a^4 z \, dz = 2a^2 z + \frac{a^4 z}{3} - a^3 \frac{z^2}{2}\Big)_0^a$$

$$= \cancel{\frac{2a^3}{3}} + \cancel{\frac{a^5}{15}} - \cancel{a^5} = 2a^3 + \frac{a^5}{3} - \frac{a^5}{2}$$

$$\iiint_V \nabla \cdot \vec{F} \, dV = 2a^3 - \frac{a^5}{6}.$$

<u>L.HS</u> $\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S_1} + \iint_{S_2} + \cdots + \iint_{S_6}$



Along the face AGDF, $x = a$, $\hat{n} = \vec{i}$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^a (2a - z) \, dy \, dz$$

$$= \int_0^a 2ay - zy\Big)_0^a dz$$

$$= \int_0^a (2a^2 - az) dz = 2a^2 z - a\frac{z^2}{2}\Big)_0^a$$

$$= 2a^3 - \frac{a^3}{2} = \frac{3a^3}{2}$$

Along the face, OBEC, $x = 0$, $\hat{n} = -\vec{i}$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^a (z - 2x) \, dy \, dz = \int_0^a \int_0^a z \, dy \, dz$$

$$= \int_0^a az \, dz = a\frac{z^2}{2}\Big)_0^a = \frac{a^3}{2}$$

Along the face, BEDG, $y = a$, $\hat{n} = \vec{j}$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^a x^2 y \, dx \, dz = \int_0^a \int_0^a x^2 a \, dx \, dz$$

$$= a^2 \frac{x^3}{3}\Big)_0^a = \frac{a^5}{3}$$

Along the face, AOFC, $y=0$, $\hat{n} = -\vec{j}$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a -x^2y\,dx\,dz = 0 \quad \because y=0.$$

Along the face, FDEC, $z=a$, $\hat{n} = \vec{k}$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a -xz^2\,dx\,dy = -a^2\int_0^a\int_0^a x\,dx\,dy$$

$$= -a^2\int_0^a x(y)_0^a\,dx = -a^3\left.\frac{x^2}{2}\right)_0^a = -\frac{a^5}{2}.$$

Along the face, AGBO, $z=0$, $\hat{n} = -\vec{k}$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a xz^2\,dx\,dy = 0 \quad \because z=0.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3a^3}{2} + \frac{a^3}{2} + \frac{a^5}{3} - \frac{a^5}{2}$$

$$= 2a^3 - \frac{a^5}{6}.$$

L.HS = R.H.S

Hence GDT is verified.

⑧ H.W

Verify Gauss divergence theorem for $\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$

Over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, & $z=1$.

Ans: $\frac{11}{6}$

$$\int_0^1\int_0^1\int_0^1 (2 + x^2 - 2xz)\,dx\,dy\,dz$$

$$2 - \frac{1}{6} = \frac{12-1}{6} = \frac{11}{6}.$$

(9) Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b$ & $0 \le z \le c$.

Soln:-

By Gauss divergence theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dV$$

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z$$

R.H.S.

$$\iiint_V \nabla \cdot \vec{F}\, dV = \int_0^a \int_0^b \int_0^c (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_0^b \int_0^c \left. 2\frac{x^2}{2} + 2xy + 2xz \right)_0^a dy\, dz$$

$$= \int_0^b \int_0^c (a^2 + 2ay + 2az)\, dy\, dz$$

$$= \int_0^c \left. \left(a^2 y + 2a\frac{y^2}{2} + 2ayz\right) \right|_0^b dz$$

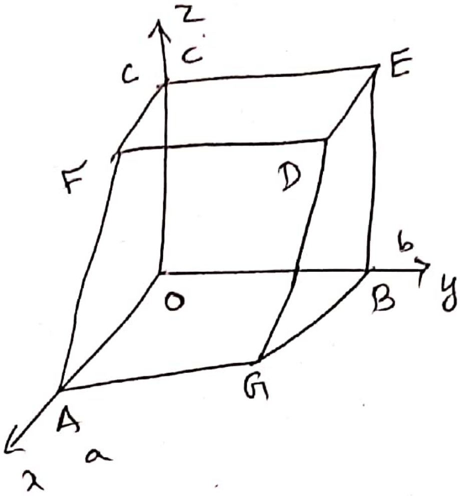
$$= \int_0^c a^2 b + ab^2 + 2abz\, dz$$

$$= \left. a^2 bz + ab^2 z + 2ab\frac{z^2}{2} \right)_0^c$$

$$= a^2 bc + ab^2 c + abc^2$$

$$= abc(a + b + c) \,//$$

L.H.S



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \cdots + \iint_{S_6}$$

Along the face AGDF, $x=a$, $\hat{n}=\vec{i}$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^c x^2-yz)\,dy\,dz$$

$$= \int_0^b\int_0^c a^2-yz)\,dy\,dz$$

$$= \int_0^c a^2 y - \frac{y^2 z}{2}\Big)_0^b dz = \int_0^c a^2 b - \frac{b^2 z}{2}\Big)dz$$

$$= a^2 bz - \frac{b^2 z^2}{4}\Big)_0^c = a^2 bc - \frac{b^2 c^2}{4}$$

Along the face OBEC, $x=0$, $\hat{n}=-\vec{i}$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^c (yz-x^2)\,dy\,dz = \int_0^b\int_0^c yz\,dy\,dz$$

$$= \int_0^c \frac{y^2}{2}\Big)_0^b z\,dz = \int_0^c \frac{b^2}{2} z\,dz = \frac{c^2 b^2}{4}$$

Along the face BEDG, $y=b$, $\hat{n}=\vec{j}$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^c (y^2-zx)\,dx\,dz = \int_0^a\int_0^c (b^2-zx)\,dx\,dz$$

$$= \int_0^c b^2 x - \frac{z x^2}{2}\Big)_0^a dz = \int_0^c b^2 a - \frac{z a^2}{2}\Big)dz$$

$$= b^2 a z - \frac{z^2 a^2}{4}\Big)_0^c$$

$$= b^2 ac - \frac{c^2 a^2}{4}$$

Along the face, AOFC, $y=0$, $\hat{n}=-\vec{j}$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^c zx-y^2)\,dx\,dz = \int_0^a\int_0^c zx\,dx\,dz$$

$$= \int_0^c z\frac{x^2}{2}\Big)_0^a dz = \frac{a^2}{2}\frac{z^2}{2}\Big)_0^c$$

$$= \frac{a^2c^2}{4}$$

Along the face, FDEC, $z=c$, $\hat{n}=\vec{k}$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^b z^2-xy)\,dx\,dy = \int_0^a\int_0^b (c^2-xy)\,dx\,dy$$

$$= \int_0^b c^2x-\frac{x^2}{2}y\Big)_0^a dy = \int_0^b c^2a-\frac{a^2}{2}y\Big)dy$$

$$= c^2ay - \frac{a^2}{2}\frac{y^2}{2}\Big)_0^b$$

$$= c^2ab - \frac{a^2b^2}{4}$$

Along the face, AGBO, $z=0$, $\hat{n}=-\vec{k}$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^b xy-z^2\,dx\,dy = \int_0^a\int_0^b xy\,dx\,dy$$

$$= \int_0^b \frac{x^2}{2}y\Big)_0^a dy = \frac{a^2}{2}\frac{y^2}{2}\Big)_0^b$$

$$= \frac{a^2b^2}{4}.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + ab^2c - \frac{c^2a^2}{4} + \frac{c^2a^2}{4}$$

$$+ c^2ab - \frac{a^2b^2}{4} - \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc\,(a+b+c).$$

L.H.S $=$ R.H.S.

H.W

(10) Verify Gauss divergence theorem for

$\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ and S is the rectangular

Parallelopiped bdd by $x = z = y = 0$ & $x = 2, y = 1, z = 3$.

Ans : 20

(11) $\vec{F} = x^3y\vec{i} + x^2y^2\vec{j} + x^2yz\vec{k}$. Find $\iint_S \vec{F}.\hat{n}\,ds$

where S is the Surface of the region in the first

octant for which $x+y+z \leq 1$.

z : 0 to $1-x-y$

y : 0 to $1-x$

x : 0 to 1.

Soln:- By GDT

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V (\nabla.\vec{F})\,dV$$

$$\nabla.\vec{F} = 3x^2y + 2x^2y + x^2y = 6x^2y$$

$$\iiint_V (\nabla.\vec{F})\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 6x^2y \; dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-x} 6x^2y\,(z)_0^{1-x-y}\; dy\,dx$$

$$= \int_0^1\int_0^{1-x} 6x^2y\,(1-x-y)\; dy\,dx$$

$$= \int_0^1\int_0^{1-x} 6x^2y - 6x^3y - 6x^2y^2)\; dy\,dx$$

$$= \int_0^1 6x^2\frac{y^2}{2} - 6x^3\frac{y^2}{2} - 6x^2\frac{y^3}{3}\Big)_0^{1-x}\; dx$$

$$= \int_0^1 3x^2(1-x)^2 - 3x^3(1-x)^2 - 2x^2(1-x)^3\Big)\; dx$$

$$= \int_0^1 x^2(1-x)^2 \left(3-3x-2(1-x)\right) dx$$

$$= \int_0^1 x^2(1-x)^2 (3-3x-2+2x)\, dx$$

$$= \int_0^1 x^2(1-x)^2 (1-x)\, dx$$

$$= \int_0^1 x^2(1-x)^3\, dx = \int_0^1 x^2(1-x^3-3x+3x^2)\, dx$$

$$= \int_0^1 (x^2-x^5-3x^3+3x^4)\, dx$$

$$= \left(\frac{x^3}{3} - \frac{x^6}{6} - \frac{3x^4}{4} + \frac{3x^5}{5}\right)_0^1$$

$$= \frac{1}{3} - \frac{1}{6} - \frac{3}{4} + \frac{3}{5} = \frac{40-90+72-20}{120}$$

$$= \frac{2}{120} = \frac{1}{60}.$$

(12) Use divergence theorem to evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$.

Soln:- By GDT

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

$$\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

$$\nabla\cdot\vec{F} = 3x^2+3y^2+3z^2$$

$$= 3(x^2+y^2+z^2).$$

By transforming Cartesian coordinates into spherical polar coordinates, we have $x = r\sin\theta\cos\phi$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\therefore\ x^2+y^2+z^2 = r^2$$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 3r^2 . r^2 \sin\theta\,dr\,d\theta\,d\phi$$

Here r varies from 0 to a

θ " " 0 to π

ϕ " " 0 to 2π.

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 3\int_0^{\pi}\int_0^{2\pi} \frac{r^5}{5}\sin\theta\Big)_0^{\theta} d\theta\,d\phi$$

$$= \frac{3a^5}{5}\int_0^{\pi}\int_0^{2\pi} \sin\theta\,d\theta\,d\phi$$

$$= \frac{3a^5}{5}\int_0^{2\pi} -\cos\theta\Big)_0^{\pi} d\phi$$

$$= \frac{3a^5}{5}\int_0^{2\pi} (-\cos\pi+\cos 0)\,d\phi$$

$$= \frac{6a^5}{5}(\phi)_0^{2\pi} = \frac{6a^5}{5}.2\pi$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{12\pi a^5}{5}\ //.$$

(13) Verify divergence theorem for the function

$\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface region

bdd by the cylinder $x^2+y^2=4$, $z=0$ & $z=3$.

Soln:- By GDT

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

Here $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$\nabla\cdot\vec{F} = 4 - 4y + 2z$

R.H.S

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V (4-4y+2z)\,dV$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3} (4-4y+2z)\,dV$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left.4z-4yz+z^2\right)_0^3 dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\,dx\,dy = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\,dx\,dy$$

$$= \int_{-2}^{2} \left.21y - \frac{12y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \int_{-2}^{2}\left(21\sqrt{4-x^2} - 6(4-x^2) + 21\sqrt{4-x^2} + 6(4-x^2)\right)dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\,dx = 42\times 2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$\boxed{\int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}}$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

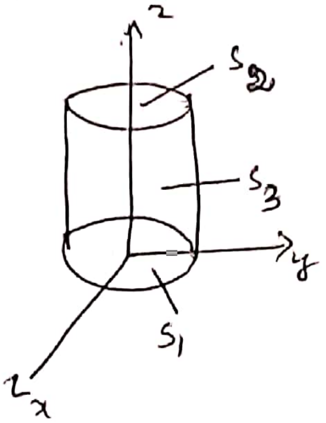
$$= 84\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) = 84\pi$$

$$42\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= 42\left[0 + \frac{2\pi}{2} - 0 + \frac{2\pi}{\cancel{2}}\right]$$

$= 84\pi$ //.

To find $\iint_S \vec{F}\cdot\hat{n}\,ds$



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

S_1 is a circle & $z=0$ $\quad \hat{n} = -\vec{k}$

$\vec{F}\cdot\hat{n} = -z^2 = 0$

$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = 0.$

on S_2, $\quad \hat{n} = \vec{k}$, $z=3$ $\quad ds = dx\,dy$

$\vec{F}\cdot\hat{n} = z^2 = 9$

$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = 9\iint_R dx\,dy = 9 \times \pi 2^2 = 36\pi$

circle with radius 2.

on S_3 Consider the S_3 is the curved surface of the cylinder $x^2+y^2=4$. So let $\phi = x^2+y^2-4$

$\nabla\phi = 2x\vec{i} + 2y\vec{j}$

$|\nabla\phi| = \sqrt{2^2x^2+2^2y^2} = 2\sqrt{4} = 4.$

$\hat{n} = \dfrac{x\vec{i}+y\vec{j}}{2}$

$\vec{F}\cdot\hat{n} = 2x^2 - y^3$

The parametric co-ordinates for the circle $x^2+y^2=4$

is $2\cos\theta$, $2\sin\theta$, $ds = 2\,d\theta\,dz$

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^3\int_0^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\, 2\,d\theta\,dz$$

$$= 16 \int_0^3\int_0^{2\pi} \left(\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{3\sin\theta - \sin 3\theta}{4}\right)\right) d\theta\,dz$$

$$= \frac{16}{4}\int_0^3\int_0^{2\pi} 2+2\cos 2\theta - 3\sin\theta + \sin 3\theta)\, d\theta\,dz$$

$$= 4\int_0^3 \left[2\theta + \frac{\cancel{2}\sin 2\theta}{\cancel{2}} + 3\cos\theta - \frac{\cos 3\theta}{3}\right]_0^{2\pi} dz$$

$$= 4\int_0^3 4\pi + 0 + \cancel{3} - \cancel{\tfrac{1}{3}} - \cancel{3} + \cancel{\tfrac{1}{3}}\,)\,dz$$

$$= 16\pi\int_0^3 dz = 16\pi\times 3 = 48\pi.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 0 + 36\pi + 48\pi$$

$$= 84\pi$$

$\therefore$ L.HS = R.HS

Hence GDT is verified,

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$