Module - 4

Applications of Multivariable Calculus

Taylor's expansion for two Variables

We know that Taylor's theorem for a function of one Variable is

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

Now we can state the Taylor's theorem for a function of two variables.

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$
$$+ \dots$$

where $h = x-a$ &

$k = y-b$.

① (i) Expand $e^x \cos y$ abt $(0, \frac{\pi}{2})$ upto the third term using Taylor's series

ii) $e^x \cos y$ in powers of x & y as far as the terms of the third degree.

Soln:-

function	Value at $(0, \frac{\pi}{2})$	Value at $(0,0)$
$f(x,y) = e^x \cos y$	$f = 0$	$f = 1$
$f_x = e^x \cos y$	$f_x = 0$	1
$f_y = -e^x \sin y$	$f_y = -1$	0

$f_{xx} = e^x \cos y$ $\qquad f_{xx} = 0 \qquad 1$

$f_{xy} = -e^x \sin y$ $\qquad f_{xy} = -1 \qquad 0$

$f_{yy} = -e^x \cos y$ $\qquad f_{yy} = 0 \qquad -1$

$f_{xxx} = e^x \cos y \qquad 0 \qquad 1$

$f_{xxy} = -e^x \sin y \qquad -1 \qquad 0$

$f_{xyy} = -e^x \cos y \qquad 0 \qquad -1$

$f_{yyy} = e^x \sin y \qquad 1 \qquad 0$

(i) $a = 0, \quad b = \frac{\pi}{2}$

$h = x - a = x, \quad k = y - b = y - \frac{\pi}{2}$

$$\therefore f(x,y) = 0 + \left[(x)(0) + \left(y - \tfrac{\pi}{2}\right)(-1)\right]$$
$$+ \frac{1}{2!}\left[(x^2)(0) + (2x)\left(y-\tfrac{\pi}{2}\right)(-1) + \left(y-\tfrac{\pi}{2}\right)^2(0)\right]$$
$$+ \frac{1}{3!}\left[(x^3)(0) + (3x^2)\left(y-\tfrac{\pi}{2}\right)(-1) + (3x)\left(y-\tfrac{\pi}{2}\right)^2(0) + \left(y-\tfrac{\pi}{2}\right)^3(1)\right] + \dots$$

$$= -y + \frac{\pi}{2} + \frac{1}{2!}\left[-2xy + 2x\frac{\pi}{2}\right] + \frac{1}{3!}\left[-3x^2y + \frac{3\pi}{2}x^2 + \left(y - \frac{\pi}{2}\right)^3\right]$$

(ii) $a = 0, \ b = 0, \ h = x - a = x - 0 = x$

$k = y - b = y - 0 = y$

$$f(x,y) = 1 + \frac{x}{1!} + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2) + \dots$$

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right]$$
$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] +$$
$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \dots$$

(2) Expand the function $\sin xy$ in powers of $x-1$ and $y-\frac{\pi}{2}$ upto second degree terms.

Soln:-

Function	Value at $(1, \frac{\pi}{2})$
$f(x,y) = \sin xy$	$f = 1$
$f_x = y\cos(xy)$	$f_x = 0$
$f_y = x\cos(xy)$	$f_y = 0$
$f_{xx} = -y^2 \sin(xy)$	$f_{xx} = -\frac{\pi^2}{4}$
$f_{xy} = -xy\sin(xy) + \cos(xy)$	$f_{xy} = -\frac{\pi}{2}$
$f_{yy} = -x^2 \sin(xy)$	$f_{yy} = -1.$

Taylor's series expansion is

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \dots$$

Here $h = x-a = x-1 \Rightarrow a = 1$

$k = y-b = y-\frac{\pi}{2} \Rightarrow b = \frac{\pi}{2}$

$$f(x,y) = 1 + \left[(x-1)0 + (y-\tfrac{\pi}{2})0\right] + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)(y-\tfrac{\pi}{2})\left(\frac{-\pi}{2}\right) + (y-\tfrac{\pi}{2})^2(-1)\right] + \dots$$

$$= 1 + \frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2 - \pi(x-1)(y-\tfrac{\pi}{2}) - (y-\tfrac{\pi}{2})^2\right] + \dots$$

H.W

(1) $e^x \log(1+x)$ in powers of x & y upto terms of third degree.

(2) $x^2 y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$.

(3) $f(x,y) = \cos x \cos y$ at $(0,0)$ (2 degree)

Maxima and minima for functions of two Variables

Maximum Value :-

A function $f(x,y)$ is said to have a maximum at a pt (a,b) if $f(a,b) > f(a+h, b+k)$ for small positive or negative values of h and k.

Minimum Value :-

$f(a,b)$ is called minimum value of $f(x,y)$ if

$$f(a,b) < f(a+h, b+k).$$

Note :-

$f(a+h, b+k) - f(a,b)$ = negative for Maxi value

$f(a+h, b+k) - f(a,b)$ = positive for Mini value

Extremum :-

$f(a,b)$ is said to be an extremum value of $f(x,y)$ if it is either a maximum or a minimum.

Saddle point :- (or) minimax is a point where function is neither maximum nor minimum at such pt f is maximum in one direction while minimum in other direction.

Note : Local Maxima and Local Minima for the functions of (X) Two variables is self study.

Sufficient conditions

If $p = \frac{\partial f(a,b)}{\partial x}$, $q = \frac{\partial f(a,b)}{\partial y}$ and $p=0, q=0$ is called the stationary points.

Let $r = \frac{\partial^2 f(a,b)}{\partial x^2}$, $s = \frac{\partial^2 f(a,b)}{\partial x \partial y}$, $t = \frac{\partial^2 f(a,b)}{\partial y^2}$

and $\Delta = rt - s^2$

(i) If $\Delta = rt - s^2 > 0$ and $r < 0$, then $f(x,y)$ has Maxi value at (a,b)

(ii) If $\Delta = rt - s^2 > 0$ & $r > 0$, then $f(x,y)$ has Mini value at (a,b)

(iii) If $\Delta = rt - s^2 < 0$, then $f(x,y)$ has ~~a~~ neither a Maxi nor mini.

(iv) If $\Delta = 0$, then case fail and investigate more for the nature of function.

Problems

(1) Examine the extrema of $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$

Soln:-

$$f_x = 2x + y - \frac{1}{x^2}, \quad f_y = x + 2y - \frac{1}{y^2}$$

To find stationary pt $(f_x, f_y) = 0$

$$\Rightarrow \quad 2x + y - \frac{1}{x^2} = 0 \quad \text{--- (1)}$$

$$x + 2y - \frac{1}{y^2} = 0 \quad \text{--- (2)}$$

(1) - (2) $\Rightarrow$

$$x - y + \frac{1}{y^2} - \frac{1}{x^2} = 0$$

$$x-y + \frac{x^2-y^2}{x^2y^2} = 0$$

$$(x-y)(x^2y^2) + (x+y)(x-y) = 0$$

$$(x-y)(x^2y^2+x+y) = 0$$

$$\Rightarrow x = y \quad \text{—} (3)$$

$\therefore (1) \Rightarrow$ $3x - \frac{1}{x^2} = 0$

$$\Rightarrow 3x^3 - 1 = 0$$

$$\Rightarrow x = \left(\frac{1}{3}\right)^{1/3} = y .$$

$\therefore$ $\left((1/3)^{1/3}, (1/3)^{1/3}\right)$ is the stationay pts.

At $\left((1/3)^{1/3}, (1/3)^{1/3}\right)$

$r = f_{xx} = 2 + \frac{2}{x^3}$ $\quad t = 2 + \frac{2}{y^3}$

$s = f_{xy} = 1.$

$$\Delta = rt - s^2 = 2 + \frac{2}{(1/3)^{\cancel{1/3 \times 3}}} \cdot 2 + \frac{2}{\left(\frac{1}{3}\right)^{1/3 \times 3}} - 1$$

$$= (2+6)(2+6) - 1 = 64 - 1 = 63 > 0 .$$

$\therefore$ $\Delta > 0$ & $r > 0$

$\therefore$ $f(x,y)$ attains its minimum value at $\left((1/3)^{1/3}, (1/3)^{1/3}\right)$

Min Value

$f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$ $\quad \because x = y$

$$= x^2 + x^2 + x^2 + \frac{x+y}{xy} = 3x^2 + \frac{2x}{x^2}$$

$$= 3x^2 + 2x^{-1}$$

$$= 3(1/3)^{2/3} + 2(1/3)^{-1/3} = 3(3)^{-2/3} + 2(3)^{1/3}$$

$$= 3^{1/3} + 2(3)^{1/3} = (1+2)3^{1/3} = 3 \times 3^{1/3}$$

$$= 3^{3/3} \cdot 3^{1/3} = 3^{4/3} \quad //.$$

① $f(x,y) = x^2 - xy + y^2 - 2x + y$. Find the extreme values of the function.

$p = \frac{\partial f}{\partial x} = 2x - y - 2$ —— ①

$q = \frac{\partial f}{\partial y} = -x + 2y + 1$ —— ②.

To find the stationary points put $p, q = 0$.

① - ② $\Rightarrow$ $3x - 3y - 3 = 0$

$\Rightarrow x = y + 1$

② $\Rightarrow$ $-y - \cancel{1} + 2y + \cancel{1} = 0$

$y = 0$

$\Rightarrow x = 1$.

$\therefore (1, 0)$ is the stationary pts.

$r = f_{xx} = 2 > 0 \qquad s = -1 \qquad t = 2$.

$\Delta = rt - s^2 = 4 - 1 = 3 > 0$.

$\therefore$ $\Delta > 0$ & $r > 0$, $f(x,y)$ attains its min at $(1,0)$.

The min value is $f(1,0) = 1 - 0 + 0 - 2 + 0$

$= -1$ //.

② $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

$p = f_x = 3x^2 + 3y^2 - 30x + 72$ —— ①

$q = f_y = 6xy - 30y$ —— ②

$r = f_{xx} = 6x - 30$

$s = f_{xy} = 6y$

$t = f_{yy} = 6x - 30$

To find stationary points put, $p = q = 0$.

$\therefore$ $3x^2 + 3y^2 - 30x + 72 = 0$

$\Rightarrow 3(x^2 + y^2 - 10x + 24) = 0$

$\Rightarrow$ $x^2 + y^2 - 10x + 24 = 0$ —— ③.

$6xy - 30y = 0$

$xy - 5y = 0$

$y(x-5) = 0$

$\Rightarrow y = 0$ & $x = 5$.

For $y=0$, ① $\Rightarrow$ $x^2 - 10x + 24 = 0$

$x = 4$ & 6.

$$\overset{24}{\widehat{-6\ -4}}$$

For $x=5$, ③ $\Rightarrow$ $25 + y^2 - 50 + 24 = 0$

$y^2 - 1 = 0$

$y^2 = 1$

$y = \pm 1$.

∴ ~~x=5~~ The stationary pts are.

$(5,1)$ $(5,-1)$

$(4,0)$ & $(6,0)$.

(1) At the point $(5,1)$.

$\Delta = rt - s^2 = (6x-30)(6x-30) - 36y^2$

$= 0 - 36 < 0$ & $r = 0$

(2) Ill'y At the point $(5,-1)$

$\Delta = -36 < 0$ & $r = 0$.

∴ The pts $(5,1)$ & $(5,-1)$ is neither a maximum or a minimum pt.

(3) At the pt $(4,0)$

$\Delta = rt - s^2 = 36 > 0$

$r = 6x - 30 = -6 < 0$.

$f(x)$ attains maximum at $(4,0)$.

$f = 64 - 240 + 288 = 112$.

(4) At the pt $(6,0)$

$\Delta = 36 > 0$ $r = 6 > 0$ mini

$f = 108$.

6

5

(3) H.W

$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$.

$f_x = 3x^2 + 3y^2 - 6x$

$f_y = 6xy - 6y$

$f_y = 0 \Rightarrow 6xy - 6y = 0$

$xy - y = 0$

$y(x-1) = 0$

$\Rightarrow y = 0 \;\&\; x = 1$

$3x^2 + 3y^2 - 6x = 0 \Rightarrow x^2 + y^2 - 2x = 0$ —— (1)

$y = 0 \Rightarrow$ (1) $\Rightarrow$ $x^2 - 2x = 0$

$x(x-2) = 0$

$\Rightarrow x = 0 \;\&\; x = 2$.

$\therefore$ $(0,0)$ & $(2,0)$ stationary pts

$x = 1$, (1) $\Rightarrow$ $1 + y^2 - 2 = 0$

$y^2 = 1$

$y = \pm 1$.

$(1,1)$ & $(1,-1)$ are stationary pts.

Max 7 min 3.

(4). $f = \sin x + \sin y + \sin(x+y)$. $0 \le x, y \le \pi$.

$f_x = \cos x + \cos(x+y)$ —— (1)

$f_y = \cos y + \cos(x+y)$ —— (2)

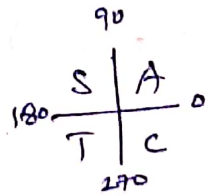
$f_x = f_y = 0 \Rightarrow$ $\cos x + \cos(x+y) = 0$

$\cos y + \cos(x+y) = 0$

$\Rightarrow \cos x + \cancel{\cos(x+y)} = \cos y + \cancel{\cos(x+y)}$

$\Rightarrow \cos x = \cos y$

$\Rightarrow x = y$



Put $y = x$ in (1), $\cos x + \cos(x+x) = 0$

$\cos x + \cos 2x = 0$

$\cos x = -\cos 2x$

$= \cos(\pi - 2x)$ or $\cos(\pi + 2x)$

$x = \pi - 2x$ or $\pi + 2x$

$x = \frac{\pi}{3}, -\pi$

$y = \frac{\pi}{3}, -\pi$.

$\therefore$ stationary pts are $(\frac{\pi}{3}, \frac{\pi}{3})$ $(-\pi, -\pi)$.

$r = -\sin x - \sin(x+y)$

$s = -\sin(x+y)$

$t = -\sin y - \sin(x+y)$

$\Delta = rt - s^2 = (\sin x + \sin(x+y))(\sin y + \sin(x+y)) - \sin^2(x+y)$

$= \sin x \sin y + \sin x \sin(x+y) + \sin y \sin(x+y)$

At $(\frac{\pi}{3}, \frac{\pi}{3})$ ~~$\Delta = \sin\frac{\pi}{3}\sin\frac{\pi}{3} + \sin\frac{\pi}{3}\sin(\frac{2\pi}{3}) + \sin\frac{\pi}{3}\sin(\frac{2\pi}{3})$~~

$=$

$r_{(\pi/3, \pi/3)} = -\sin(\pi/3) - \sin(\frac{2\pi}{3}) = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$. < 0

$s_{(\pi/3, \pi/3)} = -\frac{\sqrt{3}}{2}$.

$t = -\sin(\pi/3) - \sin(\frac{2\pi}{3}) = -\sqrt{3}$.

$\Delta = rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$.

$\therefore (\pi/3, \pi/3)$ is maxi. Value is $\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$ //.

$(-\pi, \pi)$ $r = -\sin\pi - \sin(2\pi) = 0$

$t = 0$

$s = 0$

$\Delta = 0$, No conclusion //.

H.W ① $f(x,y) = \sin x \sin y \sin(x+y)$, $0 < x, y < \pi$

② $f(x,y) = x^3 y^2 (1-x-y)$ Ans $(\frac{1}{2}, \frac{1}{3})$ Maxi Value $= \frac{1}{432}$.

5. In a plane triangle ABC, find the maximum value of $\cos x \cos y \cos z$.

Soln:-

$\cos x \cos y \cos z = \cos x \cos y \cos(\pi - (x+y))$

$\because$ In a triangle $x+y+z=\pi$

$= \cos x \cos y\,(-\cos(x+y))$

Let $f(x,y) = -\cos x \cos y \cos(x+y)$

$\frac{\partial f}{\partial x} = -\cos y\,[-\cos x \sin(x+y) - \sin x \cos(x+y)]$

$= \cos y\,[\sin x \cos(x+y) + \cos x \sin(x+y)]$

$= \cos y\,[\sin(x+x+y)] = \cos y \sin(2x+y)$ —— ①

$\frac{\partial f}{\partial y} = -\cos x[-\sin y \cos(x+y) - \cos y \sin(x+y)]$

$= \cos x\,[\sin y \cos(x+y) + \cos y \sin(x+y)]$ —— ②

$= \cos x \sin(x+2y)$

$r = \frac{\partial^2 f}{\partial x^2} = 2\cos(2x+y)\cos y$

$t = \frac{\partial^2 f}{\partial y^2} = 2\cos(x+2y)\cos x$

$s = \frac{\partial^2 f}{\partial x \partial y} = \cos x \cos(x+2y) + \sin(x+2y)(-\sin x)$

$= \cos x \cos(x+2y) - \sin x \sin(x+2y)$

$= \cos(x+x+2y) = \cos(2x+2y)$

①, ② $\Rightarrow 0$ $\Rightarrow$ $\cos y \sin(2x+y) = 0$, $\cos x \sin(x+2y) = 0$

$\Rightarrow \cos y = 0$ or $\sin(2x+y) = 0$

$\cos x = 0$ or $\sin(x+2y) = 0$.

ie, $y = \frac{\pi}{2}$ or $2x+y = 0$ or π

$x = \pi/2$ or $x+2y = 0$ or π

Here $x = \pi/2$, $y = \pi/2$ & $2x+y=0$, $x+2y=0$ are meaning less..

How?

Hence $\left.\begin{array}{l} 2x+y = \pi \\ x+2y = \pi \end{array}\right\}$

$\Rightarrow$ $x = \pi/3$, $y = \pi/3$.

at $(\pi/3, \pi/3)$ $rt - s^2 = \left(2x-1\times\frac{1}{2}\right)\left(2x-1\times\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2$

$= 1 - \frac{1}{4} = \frac{3}{4} > 0$

& $r = -1 < 0$

$\therefore$ f is Maxi at $(\pi/3, \pi/3)$

Also $x+y+z = \pi \Rightarrow z = \pi/3$

$\therefore$ Maxi value $= \cos\pi/3 \cos\pi/3 \cos\pi/3 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$ //.

Method of Lagrangian Multiplier

Suppose we require to find the maximum & minimum values of $f(x,y,z)$ where x,y,z are subject to a constraint equation $\phi(x,y,z) = 0$.

We define a function $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$ — ①

where λ is called Lagrangian multiplier which is independent of x,y,z.

The necessary conditions for a maxi or mini are

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0 \quad \text{—— ②}$$

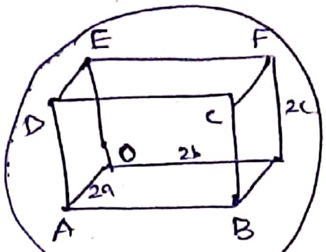
Solving ① & ② find λ x,y,z. The pt (x,y,z) may be a maxi, mini or neither which is decided by the physical consideration

Lagrangian Multiplier

1. Find the Volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution:-

The given ellipsoid is



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Given

$$\phi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,$$

The volume of the parallelopiped is $f(x,y,z)=8xyz$.

The auxillary function is $g=f+\lambda\phi$.

λ is Lagrangian multiplier.

The stationary points of g are given by

$$g_x=0,\quad g_y=0,\quad g_z=0 \ \& \ g_\lambda=0.$$

$$\therefore\ g=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right).$$

$$g_x=0 \Rightarrow 8yz+\frac{2\lambda x}{a^2}=0 \quad \text{———} \ (1)$$

$$g_y=0 \Rightarrow 8xz+\frac{2\lambda y}{b^2}=0 \quad \text{———} \ (2)$$

$$g_z=0 \Rightarrow 8xy+\frac{2\lambda z}{c^2}=0 \quad \text{———} \ (3)$$

$$g_\lambda=0 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{———} \ (4).$$

from ① $\lambda = \dfrac{-8yz\,a^2}{2x} = -\dfrac{4a^2yz}{x}$

from ② $\lambda = \dfrac{-4b^2xz}{y}$

from ③ $\lambda = \dfrac{-4c^2xy}{z}$

from these we get

$$\frac{-4a^2yz}{x} = \frac{-4b^2xz}{y} = \frac{-4c^2xy}{z}$$

$$\frac{a^2\cancel{yz}}{x.\cancel{yz}\,x} = \frac{b^2\cancel{xz}}{\cancel{xz}\,y\,y} = \frac{c^2\cancel{xy}}{\cancel{xy}z.z}$$

$$\frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{1+1+1} = \frac{1}{3}.$$

$\dfrac{x^2}{a^2} = \dfrac{1}{3}$ $\quad y = \dfrac{b}{\sqrt{3}}$, $z = \dfrac{c}{\sqrt{3}}$.

$x = \dfrac{a}{\sqrt{3}}$,

$\therefore$ Max Volume is $\dfrac{8abc}{3\sqrt{3}}$.

2. The temperature $u(x,y,z)$ at any pt in space $u = 400\,xyz^2$. Find the highest temperature on surface of the sphere $x^2+y^2+z^2=1$.

Soln:- Let $f = u = 400\,xyz^2$.

$\phi = x^2+y^2+z^2-1$.

The auxillary eqn is $g = f + \lambda\phi$

$= 400xyz^2 + \lambda(x^2+y^2+z^2-1)$

The stationary points of g are given by

$g_x = 0,\ g_y = 0,\ g_z = 0$ & $g_\lambda = 0$.

$\therefore g_x = 0 \Rightarrow$ $400yz^2 + 2x\lambda = 0$

$\lambda = -\frac{400yz^2}{2x} = \frac{-200yz^2}{x}$ —— ①

$g_y = 0 \Rightarrow$ $400xz^2 + 2\lambda y = 0$

$\lambda = \frac{-200xz^2}{y}$ —— ②

$g_z = 0 \Rightarrow$ $800xyz + 2z\lambda = 0$

$\Rightarrow \lambda = -400xy$ —— ③.

$g_\lambda = 0 \Rightarrow$ $x^2+y^2+z^2 = 1$ —— ④.

from ①, ② & ③

$$\frac{-200yz^2}{x} = \frac{-200xz^2}{y} = -400xy$$

$$-200y^2z^2 = -200x^2z^2 = -400x^2y^2$$

$\Rightarrow \frac{-1}{200x^2y^2z^2} \times$ $\Rightarrow$ $\frac{1}{x^2} = \frac{1}{y^2} = \frac{2}{z^2}$

$$x^2 = y^2 = \frac{z^2}{2} = \frac{x^2+y^2+z^2}{1+1+2} = \frac{1}{4}$$

$\therefore x^2+y^2+z^2 = 1 \Rightarrow x^2+x^2+2x^2 = 1$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

$\therefore y = \pm\frac{1}{2}$

$z = \pm\frac{1}{\sqrt{2}}$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

$y = \pm\frac{1}{2}$

$\frac{z^2}{2} = \frac{1}{4}$

$z^2 = \frac{1}{2}$

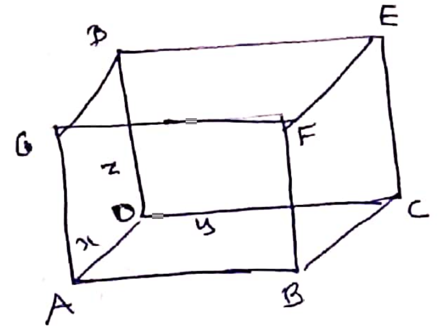
$z = \pm\frac{1}{\sqrt{2}}$.

Consider positive sign, $x = \frac{1}{2}, y = \frac{1}{2}$ & $z = \frac{1}{\sqrt{2}}$.

$\therefore$ Highest temperature on the surface of the sphere $u = 400xyz^2 = 400(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = 50$.

(3) A rectangular box open at the top is to have a volume of 32 cubic feet. Find the dimensions of the box that requires the least materials for its construction.

Soln:-



Let x, y, z be the dimensions of the box.

Surface area of the box is

$$f = S = xy + 2xz + 2yz$$

Volume $xyz = 32$

Let $\phi = xyz - 32$

$$g = f + \lambda\phi$$

$$g = xy + 2xz + 2yz + \lambda(xyz - 32).$$

$g_x = 0 \Rightarrow \quad y + 2z + \lambda yz = 0.$

$$\lambda = -\frac{y+2z}{yz} = -\frac{1}{z} - \frac{2}{y} \quad \text{--- (1)}$$

$g_y = 0 \Rightarrow \quad x + 2z + \lambda(xz) = 0$

$$\Rightarrow \lambda = -\frac{1}{z} - \frac{2}{x} \quad \text{--- (2)}$$

$g_z = 0 \Rightarrow \quad 2x + 2y + \lambda(xy) = 0$

$$\lambda = -\frac{2}{y} - \frac{2}{x} \quad \text{--- (3)}$$

$g_\lambda = 0 \Rightarrow \quad xyz = 32. \quad \text{--- (4)}$

$$-\frac{1}{z} - \frac{2}{y} = -\frac{1}{z} - \frac{2}{x} = -\frac{2}{y} - \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x} \quad \& \quad \frac{1}{z} = \frac{2}{x}$$

$$y = x \quad \& \quad z = \frac{x}{2}$$

$\therefore$ (4) $\Rightarrow \quad x \cdot x \cdot \frac{x}{2} = 32 \quad \Rightarrow x^3 = 64$

$$x = 4$$

$\therefore y = 4$

$z = 2.$

$\therefore$ The dimensions are $4, 4$ & 2.

4. Find the minimum and maximum distance from the point $(1, 2, -1)$ to the sphere $x^2+y^2+z^2 = 24$.

Soln:- Let (x,y,z) be any point on the sphere. Then its distance D from the pt $(1, 2, -1)$ is

$$D = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$$

$$f = D^2 = (x-1)^2 + (y-2)^2 + (z+1)^2.$$

Subject to the condition $x^2+y^2+z^2-24=0$.

$\therefore$ The auxillary eqn $g = f + \lambda\phi$

$$g = (x-1)^2+(y-2)^2+(z+1)^2 + \lambda(x^2+y^2+z^2-24)$$

The stationary pts are

$$g_x=0,\ g_y=0,\ g_z=0 \ \& \ g_\lambda = 0.$$

$g_x = 0 \Rightarrow \quad 2(x-1) + 2x\lambda = 0 \Rightarrow 2x-2+2x\lambda=0 \Rightarrow x-1+x\lambda=0$

$$\Rightarrow \lambda = \frac{1-x}{x} = \frac{1}{x}-1 \quad \text{---} (1)$$

$g_y = 0 \Rightarrow \quad 2(y-2)+2\lambda y = 0 \Rightarrow y-2+\lambda y=0$

$$\lambda = \frac{2-y}{y} = \frac{2}{y}-1 \quad \text{---} (2)$$

$g_z = 0 \Rightarrow \quad 2(z+1)+2z\lambda = 0 \Rightarrow \lambda = \frac{-z-1}{z} = -1-\frac{1}{z} \quad \text{---} (3)$

$$\frac{1}{x}\cancel{-1} = \frac{2}{y}\cancel{-1} = \cancel{-1}-\frac{1}{z}$$

$g_\lambda = 0 \Rightarrow x^2+y^2+z^2=24.$

$$\frac{1}{x} = \frac{2}{y} = -\frac{1}{z}$$

$$y = 2x \quad \& \quad z = -x$$

Sub y & z in (4) $\quad x^2+y^2+z^2=24$

$x^2+4x^2+x^2=24$

$6x^2=24$

$x^2=4$

$x = \pm 2$

$y = \pm 4 \quad z = \mp 2.$

$\therefore$ Min distance $= \sqrt{(2-1)^2+(4-2)^2+(-2+1)^2} = \sqrt{1+4+1} = \sqrt{6}.$

Max distance $= \sqrt{(-2-1)^2+(-4-2)^2+(2+1)^2} = \sqrt{9+36+9} = \sqrt{54}.$

5) Find the max & min values of $x^2+y^2+z^2$ subject to the condition $x+y+z=3a$.

$f = x^2+y^2+z^2 \qquad \phi = x+y+z-3a$

By Lagrange's $g = f+\lambda\phi$. (or) $F = f+\lambda g$

$x^2+y^2+z^2+\lambda(x+y+z-3a)=0$

$g_x = 2x+\lambda, \quad g_y = 2y+\lambda \quad g_z = 2z+\lambda \quad g_\lambda = x+y+z-3a$

To find stationary points $g_x = g_y = g_z = g_\lambda = 0$

$\Rightarrow \lambda = -2x, \quad \lambda = -2y, \quad \lambda = -2z \qquad x+y+z=3a$

$\Rightarrow x = -\frac{\lambda}{2} \quad y = -\frac{\lambda}{2} \quad z = -\frac{\lambda}{2}$

$\therefore x=y=z.$

$\therefore 3x = 3a \Rightarrow x = a$

$\therefore y = x = z = a.$

Max value $f(x,y) = x^2+y^2+z^2 = 3a^2$ //.

6) Find the maxi value of $x^m y^n z^p$ when $x+y+z=a$

$g = f+\lambda\phi \qquad g_x = 0 = g_y = g_z = g_\lambda$

$g = x^m y^n z^p + \lambda(x+y+z-a) \qquad \Rightarrow \boxed{x+y+z=a}$

$g_x = m x^{m-1} y^n z^p + \lambda \qquad \lambda = -m x^{m-1} y^n z^p \qquad \Rightarrow \frac{m x^m y^n z^p}{x} = \frac{n x^m y^n z^p}{y} = \frac{p x^m y^n z^p}{z}$

$g_y = n x^m y^{n-1} z^p + \lambda \qquad \lambda = -n x^m y^{n-1} z^p \qquad \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$

$g_z = p x^m y^n z^{p-1} + \lambda \qquad \lambda = -p x^m y^n z^{p-1}$

$g_\lambda = x+y+z-a$

$\therefore \frac{m}{x} = \frac{m+n+p}{a}$

$\Rightarrow x = \frac{ma}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p}$

$\therefore$ Max value $\frac{(ma)^m (na)^n (pa)^p}{(m+n+p)^{m+n+n}}$ //.

7) Find the mini value of $x^2+y^2+z^2$ subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

$g = (x^2+y^2+z^2) + \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$

$g_x = 2x - \frac{\lambda}{x^2} \qquad g_x = 0 \Rightarrow 2x - \frac{\lambda}{x^2} = 0$

$g_y = 2y - \frac{\lambda}{y^2} \qquad \Rightarrow x^3 = \frac{\lambda}{2}$

$g_z = 2z - \frac{\lambda}{z^2} \qquad x = \left(\frac{\lambda}{2}\right)^{1/3}$

$\Rightarrow y = (\lambda/2)^{1/3} = z$

$\Rightarrow x = y = z$

Given $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

$3\left(\frac{1}{x}\right)=1$

$\Rightarrow x = 3$

$\Rightarrow x=y=z=3$

$x^2+y^2+z^2 = 3^2+3^2+3^2$

Min value is $= 27$ //.