Normal Distribution (or) Gaussian Distribution.

Normal distribution is a limiting case of the binomial distribution under the following conditions

i) n, the number of trials is indefinitely large i.e $n \to \infty$

ii) Neither p nor q is very small.

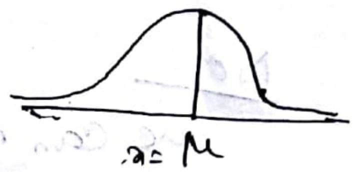
Definition:-

A continuous random variable 'x' is said to follow a normal distribution with parameters μ and σ, if its probability density function is given by $\boxed{N(\mu, \sigma)}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0.$$

Properties of normal distribution

i) The normal curve is bell shaped and symmetrical about the line $x = \mu$



$x = \mu$

ii) mean, median and mode of the distribution coincide.

iii) The total area under the normal curve is unity.
(we can show that $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$)

(iv) $\beta_1 = 0$ and $\beta_2 = 3$

(v) x-axis is an asymptote to the curve.

vi) As x increases numerically, $f(x)$ decreases rapidly, the maximum (frequency) probability occurs at $x = \mu$.

M.G.F of normal distribution

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \cdot \left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right) dx.$$

let $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$, $x = \mu + \sigma z$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{\frac{-z^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

$$M_X(t) = e^{\mu t + t^2\sigma^2/2}$$

Standard Normal Distribution

The normal distribution $N(0,1)$ is called the standardised or simply the standard normal distribution, whose density function is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}, \quad -\infty < z < \infty.$$

Note

We can transform all the observations of any random variable x to a new set of observations of a normal variable z with mean 0 and variance 1. This can be done by the transformation $z = \frac{x-\mu}{\sigma}$.

$$P(x_1 < x < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

let $z = \frac{x-\mu}{\sigma}$, $\sigma dz = dx$.

$$= \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{\frac{-z^2}{2}} dz = \int_{z_1}^{z_2} \varphi(z)\,dz = P(z_1 < z < z_2)$$

15.2

1000 light bulbs with a mean life of 120 days are installed in a new factory. Their length of life is normally distributed with a s.d of 20 days.

i) How many bulbs will expire in less than 90 days?

ii) If it is decided to replace all bulbs together, what interval should be allowed between replacement if not more than 10% should expire before replacement.

Soln:- Let x be a random variable of life of light bulbs. Given that mean $= \mu = 120$ and s.d $= 20$.

$$z = \frac{x-\mu}{\sigma} = \frac{x-120}{20}.$$

i) When $x = 90$, $z = \frac{90-120}{20} = -1.5$

$$P(X<90) = P(z<-1.5) = 0.0668$$

$\therefore$ No. of bulbs expected to expire less than 90 days out of 1000 bulbs $= 1000 \times 0.0668 = 66.8 \approx 67$

ii) Since not more than 10% or 0.1 expire before replacement so the value of the standard normal variate z to an area $0.5 - 0.1 = 0.4$ is 1.28.

Thus the value of z is less than -1.28

$$z = \frac{x-120}{20} = -1.28 \Rightarrow x = 120 - 25.6$$
$$= 94.4 \text{ or } 94 \text{ days}.$$

The bulbs may be replaced after 94 days.

Normal curve is a bell shaped curve which describe approximately many phenomena that occur in nature, industry and research.

Physical measurements in areas such as meteoro-logical experiments, rainfall studies and measurements of manufactured parts are more than adequately explained with a normal distribution.

Problem:-

Given a random variable 'x' having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that x assumes a value between 45 and 62.

Here $z = \frac{x-50}{10}$

Soln.:- The z value corresponding to $x_1 = 45$ and $x_2 = 62$

are $z_1 = \frac{45-50}{10} = -0.5$, $z_2 = \frac{62-50}{10} = 1.2$.

$$z_2 = \frac{62-50}{10} = \frac{12}{10} = 1.2.$$

$$P(45 < x < 62) = P(-0.5 < z < 1.2)$$

$$= P(-0.5 < z < 0) + P(0 < z < 1.2)$$

$$= P(0 < z < 0.5) + P(0 < z < 1.2)$$

$$= 0.1915 + 0.3849 = 0.5764$$

$$= P(z < 1.2) - P(z < -0.5)$$

$$= 0.8849 - 0.3085 = 0.5764$$

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