Total Derivative

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Laplace eqn.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$v = f(x, y, z)$

Problem:

1) If $v = (x^2+y^2+z^2)^{-1/2}$ P.T

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0}$$

Soln: $\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)$ | $v = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial v}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2-1}\,\cancel{2}x$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\Big(\underbrace{-x}_{u}\underbrace{(x^2+y^2+z^2)^{-3/2}}_{v}\Big)$$

$$= -\left[1\cdot(x^2+y^2+z^2)^{-3/2} + x\left[-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}\,\cancel{2}x\right]\right]$$

$$= -\left[(x^2+y^2+z^2)^{-3/2} - 3x^2(x^2+y^2+z^2)^{-5/2}\right]$$

$$= (x^2+y^2+z^2)^{-5/2}\left[+3x^2 - \frac{(x^2+y^2+z^2)^{-3/2}}{(x^2+y^2+z^2)^{-5/2}}\right]$$

$$= (x^2+y^2+z^2)^{-5/2}\left[3x^2 - (x^2+y^2+z^2)^{-3/2+5/2}\right]$$

$$= (x^2+y^2+z^2)^{-5/2}\left[3x^2 - (x^2+y^2+z^2)\right]$$

$$\frac{\partial^2 v}{\partial x^2} = (x^2+y^2+z^2)^{-5/2}\left[2x^2 - y^2 - z^2\right]$$

lly $\frac{\partial^2 v}{\partial y^2}$ & $\frac{\partial^2 v}{\partial z^2}$ | $v = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial^2 v}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}\left[2y^2 - x^2 - z^2\right]$$

$$\frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}\left[2z^2 - y^2 - x^2\right]$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$\Rightarrow (x^2+y^2+z^2)^{-5/2}\left[0\right] = 0$$

A function v satisfies the Laplace equation $\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0\right)$ then is called as harmonic function

Harmonic

Practice

$$u = \log(x^3+y^3+z^3-3xyz)$$

P.T $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

Wave-equation:

$$\frac{\partial^2 z}{\partial t^2}=c^2\cdot\frac{\partial^2 z}{\partial x^2}$$

polar-co-ordinates

$$x=r\cos\theta,\quad y=r\sin\theta$$

$$\theta=\tan^{-1}(y/x)$$

$$r=\sqrt{x^2+y^2}$$

Problem:

$$x=e^{r\cos\theta}\cdot\cos(r\sin\theta)$$

$$y=e^{r\cos\theta}\cdot\sin(r\sin\theta)$$

P.T

$$\frac{\partial x}{\partial\theta}=-r\cdot\frac{\partial y}{\partial r} \quad (1)$$

$$\frac{\partial y}{\partial\theta}=+r\cdot\frac{\partial x}{\partial r} \quad (2)$$

Soln:

$$x=e^{r}\cos\theta\cdot\cos(r\sin\theta)$$

Statement-problem

If x increases at the rate of 2 cm/sec at the instant $x=3$cm and $y=1$ cm. at what rate must 'y' be changing in order that the function $2xy-3x^2y$ shall be neither increasing (or) decreasing?

$$\frac{dx}{dt}=2$$

Rate of change:

$$\boxed{y=1,\quad x=3}$$

$$u=2xy-3x^2y \rightarrow \text{neither decreasing nor increasing}$$

$$\boxed{\frac{du}{dt}=0}$$

To find $\boxed{\frac{dy}{dt}=\,?}$ $\quad u=2xy-3x^2y$

To find $\boxed{\dfrac{dy}{dt} = ?}$ $u = 2xy - 3x^2y$

Sol'n $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\boxed{\dfrac{dy}{dt}}$

$$0 = (2y - 6xy)(2) + (2x - 3x^2)\frac{dy}{dt}$$

$$\frac{dy}{dx} = \frac{-(2y - 6xy)(2)}{2x - 3x^2}$$

Sub $x = 3$, $y = 1$ | Present values

$$= \frac{-(2(1) - 6(3)(1))(2)}{2(3) - 3(3)^2}$$

$$= \frac{-(2 - 18)(2)}{6 - 27}$$

$$= \frac{+(-16)(2)}{+21}$$

$$\boxed{\frac{dy}{dt} = \frac{-32}{21}\ cm}$$

Implicit differentiation

Ex. $u = x \log xy$. $(u = f(x, y)$

$x^3 + y^3 + 3xy = 1$ $\boxed{f(x, y) = c}$

$$\boxed{\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{+\frac{\partial f}{\partial y}}}$$

$$x^3 + y^3 + 3xy - 1 = 0$$

$$f(x, y) = 0$$

$$\boxed{f(x, y) = x^3 + y^3 + 3xy - 1}$$

To find $\dfrac{du}{}$

To find $\boxed{\frac{du}{dx}}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\cancel{\frac{dx}{dx}} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$\boxed{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}}$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{+\frac{\partial f}{\partial y}} = \frac{-(3x^2+3y)}{3y^2+3x} = \frac{-\cancel{3}(x^2+y)}{\cancel{3}(y^2+x)}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2+y}{y^2+x}}$$

$$u = x\log xy \qquad \frac{dy}{dx} = \frac{x^2+y}{y^2+x}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= (\log x\cdot y) + \cancel{x}\frac{1}{\cancel{xy}}\cdot\cancel{y} + \left(x\cdot\frac{1}{\cancel{x}y}\cdot\cancel{x}\right)\left(\frac{x^2+y}{x+y^2}\right)$$

$$= 1 + \log(xy) + \left(\frac{x}{y}\right)\left[\frac{x^2+y}{x+y^2}\right]$$

$$u = f(x,y) \quad \& \quad f(x,y) = c$$

$$\left.\frac{y}{{}^{2}}\right)$$

$u = f(x, y)$

To find $\frac{du}{dx}$ (or) $\frac{du}{dy}$