MULTIPLE INTEGRALS

MULTIPLE INTEGRALS

Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral.

- However, the Fundamental Theorem of Calculus (FTC) provides a much easier method.

The evaluation of double integrals from first principles is even more difficult.

Iterated Integrals

In this section, we will learn how to:

Express double integrals as iterated integrals.

INTRODUCTION

Once we have expressed a double integral as an iterated integral, we can then evaluate it by calculating two single integrals.

INTRODUCTION

Suppose that *f* is a function of two variables that is integrable on the rectangle

$$R = [a, b] \times [c, d]$$

INTRODUCTION

We use the notation $\int_c^d f(x,y)\,dy$ to mean:

- x is held fixed
- $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$

PARTIAL INTEGRATION

This procedure is called partial integration with respect to *y*.

- Notice its similarity to partial differentiation.

PARTIAL INTEGRATION

Now, $\int_c^d f(x, y)\, dy$ is a number that depends on the value of *x*.

So, it defines a function of *x*:

$$A(x) = \int_c^d f(x, y)\, dy$$

PARTIAL INTEGRATION

Equation 1

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

ITERATED INTEGRAL

The integral on the right side of Equation 1 is called an iterated integral.

- Usually, the brackets are omitted.

Equation 2

Thus,

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

means that:

- First, we integrate with respect to *y* from *c* to *d.*
- Then, we integrate with respect to *x* from *a* to *b*.

Similarly, the iterated integral

$$\int_c^d \int_a^b f(x, y)\, dy\, dx = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that:

- First, we integrate with respect to *x* (holding *y* fixed) from *x* = *a* to *x* = *b*.
- Then, we integrate the resulting function of *y* with respect to *y* from *y* = *c* to *y* = *d*.

Notice that, in both Equations 2 and 3, we work from the inside out.

Evaluate the iterated integrals.

a. $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

b. $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Regarding *x* as a constant, we obtain:

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2}$$

$$= x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right)$$

$$= \tfrac{3}{2} x^2$$

Thus, the function *A* in the preceding discussion is given by

$$A(x) = \tfrac{3}{2}x^2$$

in this example.

We now integrate this function of *x* from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2\, dx = \frac{x^3}{2} \Big]_0^3$$

$$= \frac{27}{2}$$

Here, we first integrate with respect to *x*:

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy = \int_1^2 \left[\int_0^3 x^2 y\, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y\, dy = 9 \frac{y^2}{2} \Big]_1^2 = \frac{27}{2}$$

Notice that, in Example 1, we obtained the same answer whether we integrated with respect to y or x first.

ITERATED INTEGRALS

In general, it turns out (see Theorem 4, Thomas Calculus, Chapter 15) that the two iterated integrals in Equations 2 and 3 are always equal.

- That is, the order of integration does not matter.
- This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.

ITERATED INTEGRALS

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

FUBUNI'S THEOREM

Theorem 4

If f is continuous on the rectangle

$$R = \{(x, y) \,|a \le x \le b,\ c \le y \le d$$

then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that:

- *f* is bounded on *R*.
- *f* is discontinuous only on a finite number of smooth curves.
- The iterated integrals exist.

FUBUNI’S THEOREM

Theorem 4 is named after the Italian mathematician Guido Fubini (1879–1943), who proved a very general version of this theorem in 1907.

- However, the version for continuous functions was known to the French mathematician Augustin-Louis Cauchy almost a century earlier.

FUBUNI'S THEOREM

The proof of Fubini's Theorem is too difficult to include in this book.

However, we can at least give an intuitive indication of why it is true for the case where $f(x, y) \geq 0$.

FUBUNI'S THEOREM

Recall that, if *f* is positive, then we can interpret the double integral

$$\iint_R f(x, y)\, dA$$

as:

- The volume *V* of the solid *S* that lies above *R* and under the surface $z = f(x, y)$.

FUBUNI'S THEOREM

However, we have another formula that we used for volume namely,

$$V = \int_a^b A(x)\, dx$$

where:

- $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis.

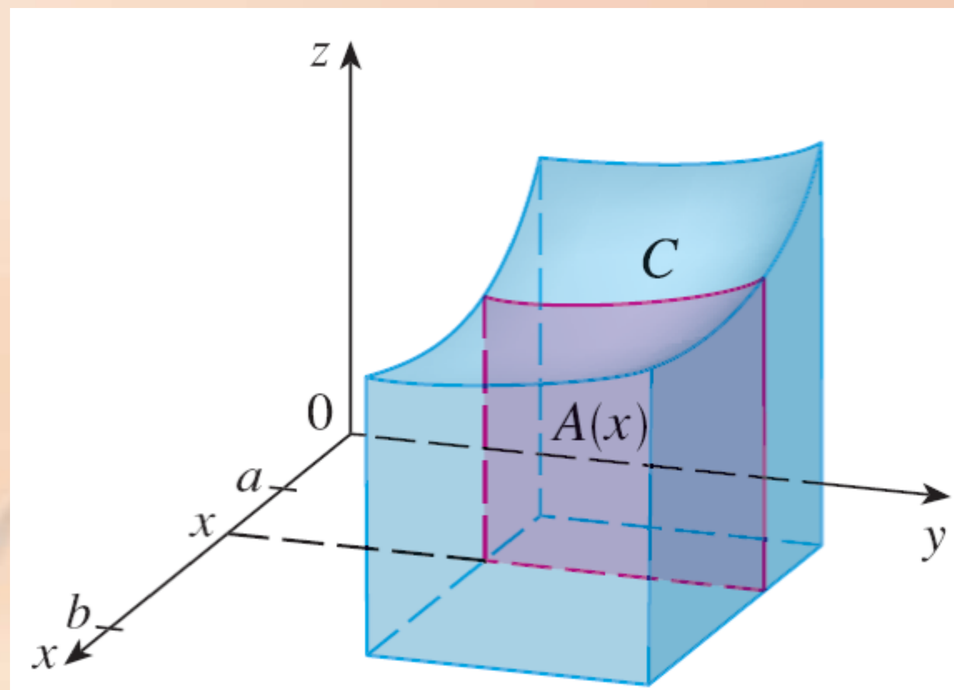
FUBUNI'S THEOREM

From the figure, you can see that $A(x)$ is the area under the curve C whose equation is

$$z = f(x, y)$$

where:

- x is held constant
- $c \leq y \leq d$



FUBUNI'S THEOREM

Therefore,

$$A(x) = \int_c^d f(x, y)\, dy$$

Then, we have:

$$\iint_R f(x, y)\, dA = V = \int_a^b A(x)\, dx$$

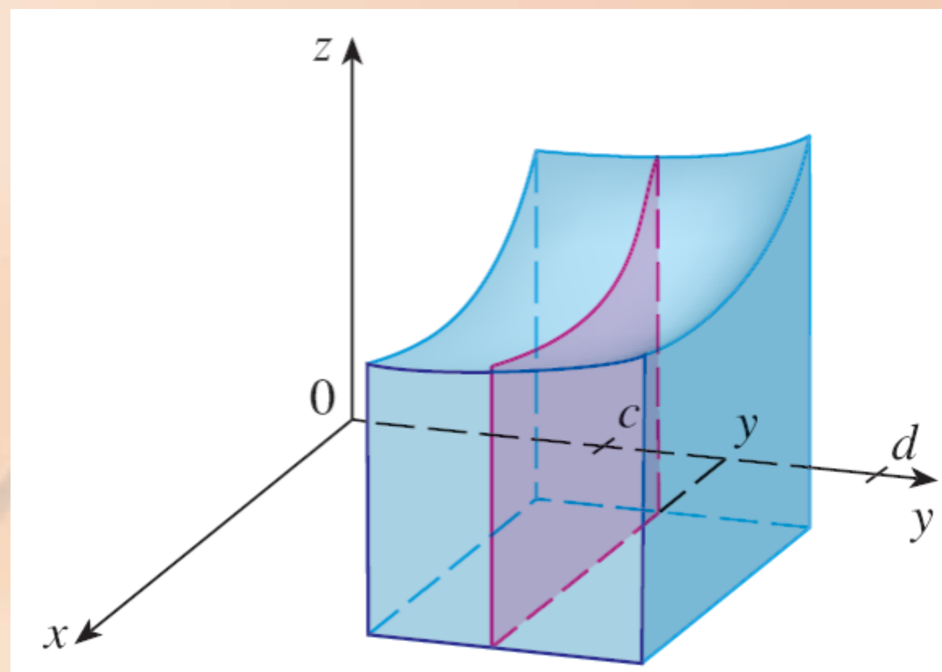
$$= \int_a^b \int_c^d f(x, y)\, dy\, dx$$

FUBUNI’S THEOREM

A similar argument, using cross-sections perpendicular to the *y*-axis,

shows that:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$



Example 2

Evaluate the double integral

$$\iint_R (x - 3y^2)\, dA$$

where

$$R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$$

- Compare with Example 3 in Section 15.1

Fubini's Theorem gives:

$$\iint_R (x - 3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx$$

$$= \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\,dx = \frac{x^2}{2} - 7x \Big]_0^2$$

$$= -12$$

This time, we first integrate with respect to *x*:

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3 \Big]_1^2$$

$$= -12$$

FUBUNI’S THEOREM

Notice the negative answer in Example 2.

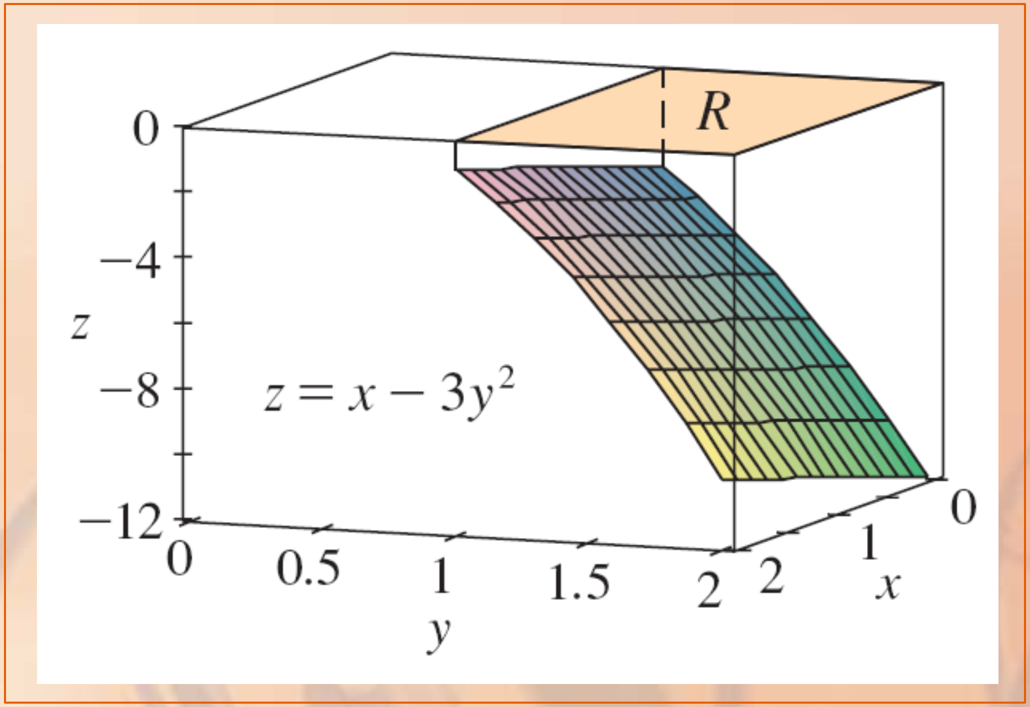
Nothing is wrong with that.

- The function *f* in the example is not a positive function.
- So, its integral doesn’t represent a volume.

FUBUNI'S THEOREM

From the figure, we see that *f* is always negative on *R*.

- Thus, the value of the integral is the negative of the volume that lies above the graph of *f* and below *R*.



Evaluate

$$\iint_R y \sin(xy)\, dA$$

where

$$R = [1, 2] \times [0, \pi]$$

If we first integrate with respect to *x*, we get:

$$\iint_R y\sin(xy)\,dA = \int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$

$$= \int_0^{\pi}[-\cos(xy)]_{x=1}^{x=2}\,dy$$

$$= \int_0^{\pi}(-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\,\Big]_0^{\pi} = 0$$

If we reverse the order of integration, we get:

$$\iint_R y\sin(xy)\,dA = \int_1^2 \int_0^{\pi} y\sin(xy)\,dy\,dx$$

To evaluate the inner integral, we use integration by parts with:

$$u = y \qquad dv = \sin(xy)\,dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

Thus,

$$\int_0^{\pi} y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^{\pi}\cos(xy)\,dy$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\Big[\sin(xy)\Big]_{y=0}^{y=\pi}$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

If we now integrate the first term by parts with $u = -1/x$ and $dv = \pi \cos \pi x \, dx$, we get:

$$du = dx/x^2$$

$$v = \sin \pi x$$

and

$$\int \left(-\frac{\pi \cos \pi x}{x} \right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2} dx$$

Therefore,

$$\int \left(-\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2} \right) dx = -\frac{\sin \pi x}{x}$$

Thus,

$$\int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx = \left[-\frac{\sin \pi x}{x} \right]_1^2$$

$$= -\frac{\sin 2\pi}{2} + \sin \pi = 0$$

In Example 2, Solutions 1 and 2 are equally straightforward.

However, in Example 3, the first solution is much easier than the second one.

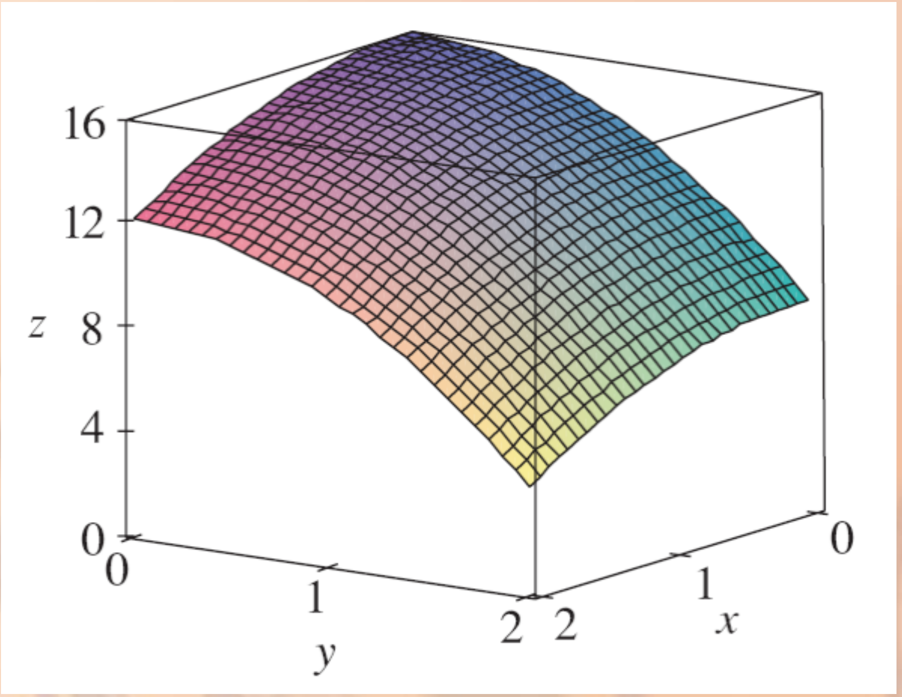
- Thus, when we evaluate double integrals, it is wise to choose the order of integration that gives simpler integrals.

Find the volume of the solid S that is bounded by:

- The elliptic paraboloid $x^2 + 2y^2 + z = 16$
- The planes $x = 2$ and $y = 2$
- The three coordinate planes

We first observe that S is the solid that lies:

- Under the surface $z = 16 - x^2 - 2y^2$
- Above the square $R = [0, 2] \times [0, 2]$



This solid was considered in Example 1 in Section 15.1.

Now, however, we are in a position to evaluate the double integral using Fubini's Theorem.

Thus,

$$V = \iint_R (16 - x^2 - 2y^2)\, dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \tfrac{1}{3}x^3 - 2y^2 x\right]_{x=0}^{x=2} dy$$

$$= \int_0^2 \left(\tfrac{88}{3} - 4y^2\right) dy$$

$$= \left[\tfrac{88}{3}y - \tfrac{4}{3}y^3\right]_0^2 = 48$$

ITERATED INTEGRALS

Consider the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only.

- Then, the double integral of f can be written in a particularly simple form.

To be specific, suppose that:

- $f(x, y) = g(x)h(y)$
- $R = [a, b] \times [c, d]$

Then, Fubini's Theorem gives:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy$$

$$= \int_c^d \left[\int_a^b g(x)h(y)\,dx \right] dy$$

In the inner integral, *y* is a constant.

So, *h*(*y*) is a constant and we can write:

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx\right]dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx\right)\right]dy$$

$$= \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant.

Equation 5

Hence, in this case, the double integral of *f* can be written as the product of two single integrals:

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

where $R = [a, b] \times [c, d]$

If $R = [0, \pi/2] \times [0, \pi/2]$, then,

by Equation 5,

$$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= \left[-\cos x\right]_0^{\pi/2} \left[\sin y\right]_0^{\pi/2}$$

$$= 1 \cdot 1 = 1$$

The function $f(x, y) = \sin x \cos y$ in Example 5 is positive on R.

So, the integral represents the volume of the solid that lies above *R* and below the graph of *f* shown here.