MAT 1011

Calculus for Engineers

Module - 1

Applications of Single Variable Calculus

Basic Concepts

★ **Sets**

A collection of well-defined objects is called a set.

★ The objects that comprises of the set are called elements.

★ Number of elts in a set can be finite or infinite.

Example:-
1) $\{x \mid x$ is an even integers$\}$
2) $\{x \mid x$ is an even prime numbers$\}$
3) Set of all integers $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$..

★ **function**

A relation f from a set A to a set B is said to be function if every elt of set A has one and only image in set B. Notation $f: A \to B$

$x \in A \to y \in B$, write $y = f(x)$. y is the image of x under f.

(or)

★ A function $f: A \to B$ is a rule s.t every elt of A has a unique elt in B.

Identity fn $\mathbb{R}$ set of all real numbers

A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \ \forall x \in \mathbb{R}$ is called an identity fn.

Let $f: A \to B$ be a function

(i) f is called onto or surjective if Codom f = range f.

(ii) f is called one-one or injective if for all $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$.

(iii) f is called bijective if f is onto and one-one.

Limits of a function

If $f(x)$ becomes arbitrarily close to a single number L as x approaches a from either side, then

$$\lim_{x \to a} f(x) = L.$$

Ex:- $\lim_{x \to 5} x^2+5x = 50$ //.

Continuous function

Let f be a real function on a subset of the real numbers and let a be a point in the domain of f, then f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

i.e) $f: A \to \mathbb{R}$
$a \in A$

$f: D \to \mathbb{R}$, $D \subset \mathbb{R}$ and $a \in D$

if $\lim_{x \to a} f(x) = f(a)$.

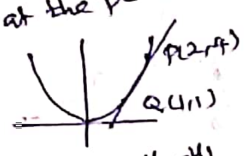
Topic - 1 Differentiation

Definition

★ The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

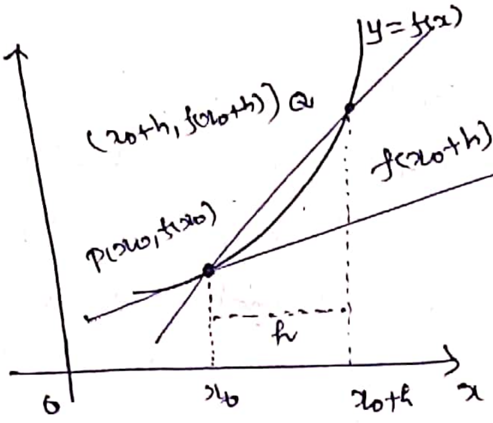
$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

Ex:- Find the slope of $y = x^2$ at the pt $(2,4)$.



slope of $PQ = \frac{y_2 - y_1}{x_2 - x_1}$

$= 3$ //.



★ Slope of a curve $y = f(x)$ at the pt P means the slope of the tangent at the point P.

Slope $= m =$ $\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$

★ The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{provided this limits exists.}$$

★ The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{provided the limits exists.}$$

★ Alternative formula for the derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Example: a) find the derivative of $f(x) = \sqrt{x}$ for $x > 0$

$$f'(x) = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} = \lim_{z \to x} \frac{\cancel{\sqrt{z} - \sqrt{x}}}{\cancel{(\sqrt{z} - \sqrt{x})}(\sqrt{z} + \sqrt{x})} = \frac{1}{2\sqrt{x}} \;//$$

b) H.W Try $f(x) = \frac{x}{x-1}$ Find $f'(x) = ?$ Ans: $-\frac{1}{(x-1)^2}$

Rolle's Theorem

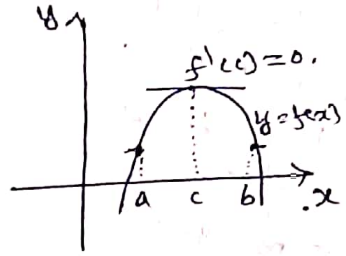
Let f be a real valued function that satisfies the following three conditions:

(i) f is defined and continuous on the closed interval $[a,b]$

(ii) f is differentiable on the open interval (a,b)

(iii) $f(a) = f(b)$

Then there exists atleast one point $c \in (a,b)$ s.t $f'(c) = 0$.



Note:

1. If the object is in the same place at two different instants $t=a$ and $t=b$ then $f(a) = f(b)$ satisfying hypothesis of Rolle's thm. Therefore the theorem says that there is some instant of time $t = c$ b/w a and b where $f'(c) = 0$. i.e., the velocity is 0 at $t = c$.

2. The converse of Rolle's theorem is not true.

Examples

① Find the value of c using the Rolle's thm.

$$f(x) = \sqrt{1-x^2} \quad , \quad -1 \le x \le 1.$$

Soln:- Given fn $f(x) = \sqrt{1-x^2}$ is continuous in $[-1,1]$ and differentiable in $(-1,1)$.

Also $f(-1) = f(1) = 0$ all 3 conditions are satisfied.

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$f'(x) = 0 \Rightarrow x = 0$$

thus the suitable pt is $c = 0$.

② $f(x) = 2x^3 - 5x^2 - 4x + 3 \quad , \quad \frac{1}{2} \le x \le 3.$

f is continuous $[\frac{1}{2}, 3]$, f is diff in $(\frac{1}{2}, 3)$.

$$f(1/2) = 0 = f(3).$$

$$f'(x) = 6x^2 - 10x - 4$$

$f'(x) = 0 \Rightarrow 3x^2 - 5x - 2 = 0 \Rightarrow x = -\frac{1}{3}$ & $x = 2$.

clearly $-1/3$ out side of $\frac{1}{2} \le x \le 3$, $\therefore \boxed{x = 2}$ is the suitable point.

Problems

Verify Rolle's theorem for the functions.

(1) $f(x) = x^3 - 3x + 3, \quad 0 \le x \le 1$

$\because$ f is continuous $[0,1]$, diff in $(0,1)$

$f(0) = 3$ & $f(1) = 1 \quad \therefore f(a) \neq f(b)$ ।

$\therefore$ Rolle's thm is not true.

(2) $f(x) = \sin^2 x, \quad 0 \le x \le \pi$

f is continuous in $[0,\pi]$, diff in $(0,\pi)$.

$f(0) = f(\pi) = 0$.

$\therefore \quad f'(x) = 2\sin x\cos x = \sin 2x$

$\exists \; c \in (0,\pi)$ s.t $f'(c) = 0 \Rightarrow \sin 2c = 0$

$\Rightarrow 2c = 0, \pi, 2\pi, 3\pi, \ldots$

$\Rightarrow c = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$

Since $c = \frac{\pi}{2} \in (0,\pi)$. Hence $c = \pi/2$ is the suitable pt.

H.W

(3) $f(x) = x(x-1)(x-2), \quad 0 \le x \le 2$.

$f(x) = \tan x, \quad 0 \le x \le \pi$.

Mean Value Theorem (law of the mean due to Lagrange).

Theorem:- Let $f(x)$ be a real valued function that satisfies the following conditions:

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

Then there exists at least one point $c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Example:- Verify Lagrange's law of the mean for $f(x) = x^3$ on $[-2,2]$.

Soln:- f is a polynomial, hence is continuous on $[-2,2]$ and diff on $(-2,2)$.

$f(2) = 2^3 = 8, \quad f(-2) = -8$.

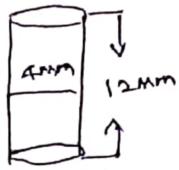
$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$.

By MVT, $\exists$ elt $c \in (-2,2)$ s.t $f'(c) = \frac{f(2) - f(-2)}{4} = \frac{8+8}{4} = 4$.

$\Rightarrow 3c^2 = 4 \Rightarrow c = \pm 2/\sqrt{3}$. The required $c = \pm 2/\sqrt{3}$ as both lies in $[-2,2]$.

2) A cylindrical hole 4 mm in diameter and 12 mm deep in a metal block is rebored to increase the diameter to 4.12 mm. Estimate the amt of metal removed.

Soln:- The volume of cylindrical hole of radius x mm and depth 12mm is given by



$$V = f(x) = 12\pi x^2$$

$$\Rightarrow \quad f'(c) = 24\pi c$$

To estimate $f(2.06) - f(2)$:

By Law of Mean $\quad f(2.06) - f(2) = 0.06\, f'(c)$

$$= 0.06\,(24\pi c), \quad 2 < c < 2.06$$

Take $c = 2.01$

$$f(2.06) - f(2) = 0.06\,(24\pi)(2.01)$$

$$= 2.89\pi \text{ cubic mm.}$$

H.W ① Verify MVT for (i) $f(x) = 1 - x^2, \; [0,3]$

(ii) $f(x) = 2x^3 + x^2 - x - 1 \; [0,2]$

(iii) $f(x) = \dfrac{1}{x} \; [1,2]$.

Monotonic functions :-

Increasing and Decreasing functions

Definition:- A function f is called increasing on an interval I if $f(x_1) \le f(x_2)$ whenever $x_1 < x_2$ in I. It is called decreasing on I if $f(x_1) \ge f(x_2)$ whenever $x_1 < x_2$ in I.

A function that is completely increasing or completely decreasing on I is called monotonic on I.

Note:-
1. $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ $\Rightarrow$ f is preserves the order.
2. $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ $\Rightarrow$ f is reverse the order.

Examples:-

(1) Every constant function is an increasing function.

(2) Every identity function is an increasing fn.

(3) The function $f(x) = \sin x$ is not an increasing function on $\mathbb{R}$; but $f(x) = \sin x$ is increasing on $[0, \frac{\pi}{2}]$.

(4) $f(x) = 4 - 2x$ is decreasing.

Properties of Increasing and Decreasing functions

Corollary:- Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) > 0$ at each point $x \in (a,b)$, then f is increasing on $[a,b]$. If $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$.

Theorem:- Let I be an open interval. Let $f: I \to \mathbb{R}$ be differentiable. Then (i) f is increasing iff $f'(x) \ge 0 \; \forall x \in I$.
(ii) f is decreasing iff $f'(x) \le 0 \; \forall x \in I$.

Definition:- $f: I \to \mathbb{R}$ is said to be strictly increasing if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$. Similarly f is strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Extreme Values of functions

Global Maxima and Global Minima

Definition:- A function f has an absolute maximum at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called maximum value of f on D. Similarly, f has an absolute minimum at c if $f(c) \leq f(x)$ $\forall x$ in D and the number $f(c)$ is called the minimum value of f on D.

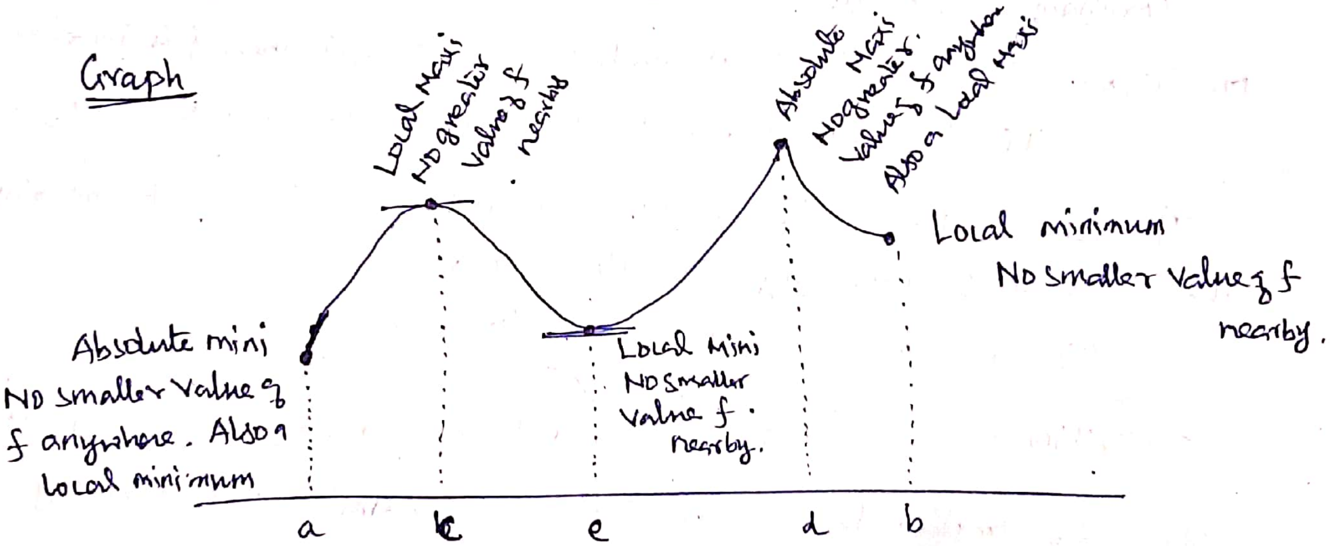
* The maximum and minimum values of f are called extreme values of f.
* Absolute maxima and minima are also referred to as global maxima & minima.

Local Maxima and Local minima

Defn:- A function f has a Local maximum (or relative maxi) at c if there is an open interval I containing c s.t $f(c) \geq f(x)$ $\forall x$ in I.

Similarly, f has a local minimum at c if there is an open interval I containing c s.t $f(c) \leq f(x)$ $\forall x$ in I.

Graph



$y = f(x)$

Critical points

A critical number of a function f is a number c in the domain of f s.t either $f'(c)=0$ or $f'(c)$ does not exist.

Stationary points

Stationary Points are critical numbers c in the domain of f, for which $f'(c)=0$.

Concavity (convexity)

Defn:- If the graph of f lies above all of its tangents on an interval I, then it is called concave upward (convex downward) on I. If the graph of f lies below all of its tangents on I, it is called concave downward (convex upward) on I.

Defn:- The graph of a differentiable function $y=f(x)$ is

(a) Concave up on an open interval I if f' is increasing on I;

(b) Concave down on an open interval I if f' is decreasing on I.

Second derivative Test for concavity (convexity)

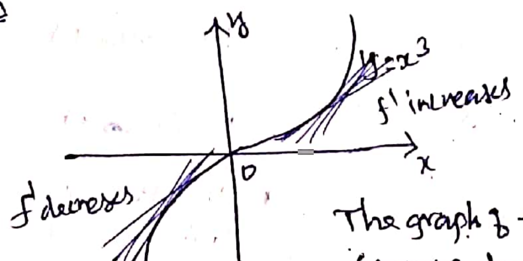
The test for concavity (convexity)

Suppose f is twice differentiable on an interval I.

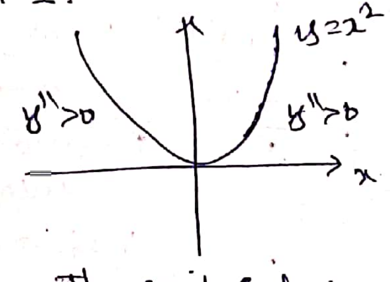
(i) If $f''(x)>0$ for all x in I, then the graph of f is concave upward (convex downward) on I.

(ii) If $f''(x)<0$ $\forall\, x \in I$, then the graph of f is concave downward (convex upward) on I.

Graph



The graph of $f(x)=x^3$ is concave down on $(-\infty,0)$ & concave up on $(0,\infty)$

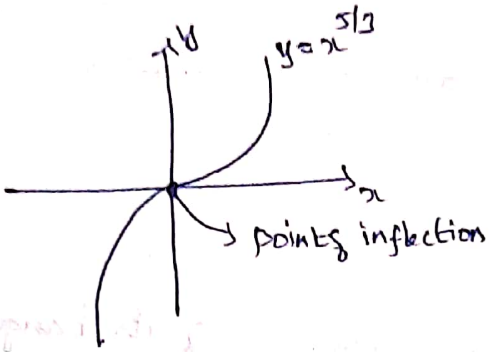


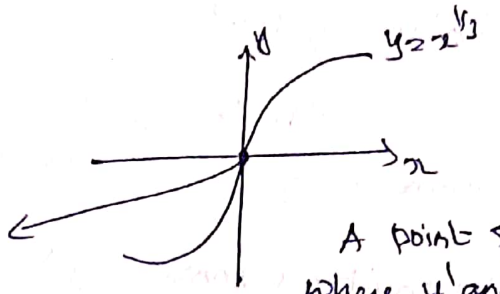
The graph of $f(x)=x^2$ is concave up on every interval.

Point of inflection

A point P on a curve is called a point of inflection if the curve changes from concave upward (convex downward) to concave downward (convex upward) and vise versa.

Ex:-





A point of inflection where y' and y'' fails to exist.

First derivative test for local Extrema

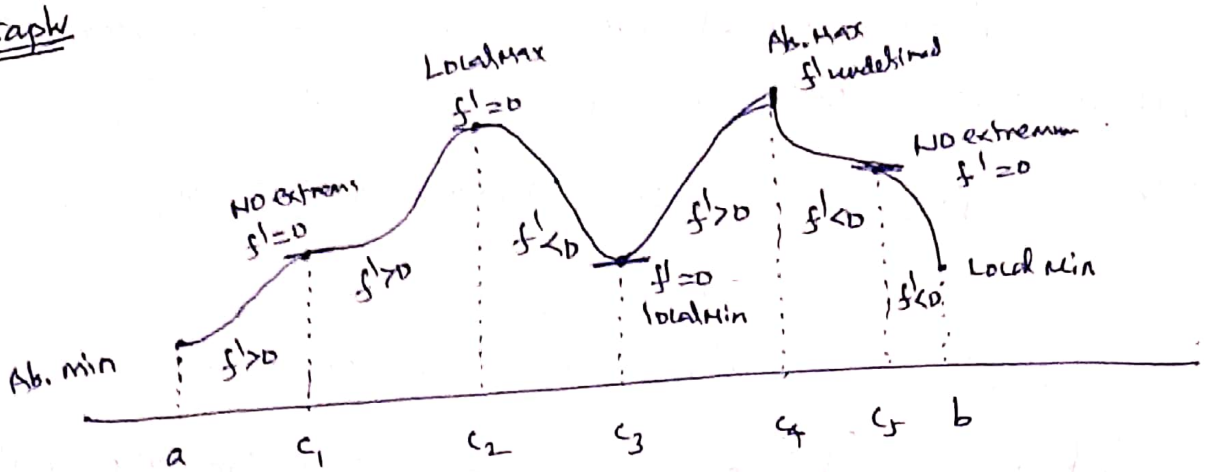
Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c.
2. if f' changes from positive to negative at c, then f has a local maximum at c.
3. if f' does not change sign at c (i.e., f' is positive on both sides of c or negative on both sides) then f has no local extremum at c.

Table:

Signs of derivative $f'(x)$ when passing through critical point x_0			Character of critical point
$x < x_0$	$x = x_0$	$x > 0$	
+	$f'(x_0) = 0$ or is discont	−	Maximum pt
−	$f'(x_0) = 0$ or is discont	+	Minimum pt
+	$f'(x_0) = 0$ or is disconti	+	Neither Maxi nor mini (increase) but is a pt of inflection
−	"	−	" (decreases)

Graph



Second derivative test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ & $f''(c) < 0$, then f has a local Maxi at $x = c$.
2. If $f'(c) = 0$ & $f''(c) > 0$, then f has a local Min at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The fn f may have a local maxi, a local min, or neither.

Table

Signs of derivative $f''(x)$ at the critical point of $f(x)$ or $f'(x)$				Character of the Point
		$x = x_0$		
	$f'(x_0)$	$f''(x_0)$		
	0	−	Critical pt of f	Maxi point
	0	+	"	Mini point
$x < x_0$		$f''(x_0)$	$x > x_0$	
+	0 or ≠ 0	0	−	Pt of inflection
−	0 or ≠ 0	0	+	"
+	0 or ≠ 0	0	+	Unknown
−	0 or ≠ 0	0	−	"

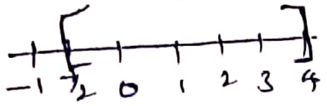
Example: Find the absolute maximum and minimum values of the

① function. $f(x) = x^3 - 3x^2 + 1, \; -\frac{1}{2} \le x \le 4.$

Soln:- Note that f is continuous on $[-\frac{1}{2}, 4]$

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x-2).$$



Since $f'(x)$ exists $\forall x$, the only critical values of f are $x=0$ & $x=2$.

Both $x=0$ & $x=2$ lies in the interval $[-\frac{1}{2}, 4]$.

$\therefore$ Value of f at these critical numbers are

$$f(0) = 1 \quad \& \quad f(2) = -3.$$

The values of f at the end points of the interval are

$$f(-\tfrac{1}{2}) = (-\tfrac{1}{2})^3 - 3(-\tfrac{1}{2})^2 + 1 = \tfrac{1}{8} \quad \&$$

$$f(4) = 17.$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the ab. minimum value is $f(2) = -3$.

Note that in this example the ab. maxi occurs at an end point, whereas the ab. min occurs at a critical number.

② Find the local minimum and maximum values of

$$f(x) = x^4 - 3x^3 + 3x^2 - x.$$

Soln:-

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$f'(x) = 4x^3 - 9x^2 + 6x - 1$$

$$f'(x) = 0 \Rightarrow \quad 4x^3 - 9x^2 + 6x - 1 = 0$$

$$(x-1)^2(4x-1) = 0$$

$$\Rightarrow x = 1, 1, \tfrac{1}{4}.$$

when $x=1$, $f(1) = 0$ & when $x = \frac{1}{4}$, $f(\frac{1}{4}) = \frac{-27}{256}$

Hence the coordinates of the stationary pts are

$(1, 0)$ & $\left(\frac{1}{4}, \frac{-27}{256}\right)$

$f''(x) = 12x^2 - 18x + 6 = 6(2x^2 - 3x + 1) = 6(x-1)(2x-1)$

when $x = 1$, $f''(1) = 0$.

Thus the second derivative test gives no information about the extremum nature of f at $x = 1$.

when $x = 1/4$, $f''(1/4) = \frac{9}{4} > 0$, hence $\left(\frac{1}{4}, \frac{-27}{256}\right)$ is a minimum point.

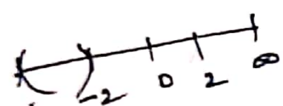
(3) Find the critical points of $f(x) = x^4 - 8x^2 + 16 = (x^2-4)^2$ and separate the intervals on which f is ↗ and on which f is ↘.

$f'(x) = 4x^3 - 16x$
$= 4x(x^2 - 4)$
$= 4x(x+2)(x-2)$

Soln:-

$f(x) = (x^2-4)^2$

$f'(x) = 2(x^2-4)(2x) = 4x(x+2)(x-2)$



∴ the critical pts of f are $x = 0, 2$ & -2.

x-Values	$4x$	$x-2$	$x+2$	$f'(x)$	$f(x)$	Interval
$x < -2$	$-$	$-$	$-$	$- < 0$	↘	$(-\infty, -2)$
$-2 < x < 0$	$-$	$-$	$+$	$+ > 0$	↗	$(-2, 0)$
$0 < x < 2$	$+$	$-$	$+$	$- < 0$	↘	$(0, 2)$
$x > 2$	$+$	$+$	$+$	$+ > 0$	↗	$(2, \infty)$

∴ f ↗ on $(-2, 0) \cup (2, \infty)$

f ↘ on $(-\infty, -2) \cup (0, 2)$.

(4) H.W (1) $f(x) = 2x^3 - 18x$ $\quad x = \pm\sqrt{3}$ $\quad$ ↗ on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

↘ on $(-\sqrt{3}, \sqrt{3})$

(2) $f(x) = x^4 - 4x^3 + 4x^2$

↳ $x = 0, 1, 2$

↗ $(0, 1)$ & $(2, \infty)$

↘ $(-\infty, 0) \cup (1, 2)$.

(4) Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is $\nearrow$ and on which f is $\searrow$.

<u>Soln:-</u>

$$f(x) = x^3 - 12x - 5$$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x+2)(x-2)$$

$$f'(x) = 0 \Rightarrow x = 2 \ \& \ -2$$

critical points are $x = 2, \ \& \ -2$.

$-\infty \qquad -2 \qquad 2 \qquad \infty$

Take some values b/w the intervals

$f'(-3) = +ve$, $f'(0) = -12$ $-ve$, $f'(3) = 15$ $+ve$

i.e,	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
$f'(x)$	$+$	$-$	$+$
Behaviour of f	$\nearrow$	$\searrow$	$\nearrow$

<u>Problems on ab. Maxi & mini</u>

(1) Find the ab. Maxima & ab. Minima of the following functions on the interval $f(x) = \frac{2x}{3} - 5, \quad -2 \le x \le 3.$

<u>Soln:-</u> $f(x) = \frac{2x}{3} - 5 \quad , [-2, 3]$

$f'(x) = \frac{2}{3}$. Therefore critical pt does not exist.

Now $f(-2) = -\frac{19}{3}$, $f(3) = -3$

$\therefore$ ab. Maxi at $x = +3$

ab. Mini at $x = -2$

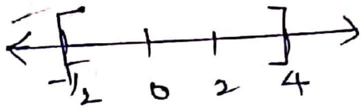
(2) $f(x) = x^3 - 3x^2 + 1, \quad -\frac{1}{2} \le x \le 4.$ Find ab. Min & ab. Max.

Soln:- $f(x) = x^3 - 3x^2 + 1, \quad [-\frac{1}{2}, 4]$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$\because f'(x) = 0 \Rightarrow x = 0$ & $x = 2$ are critical pts.

$\therefore$ we check



$f(0) = 1$

$f(-\frac{1}{2}) = \frac{1}{8}$

$f(2) = \boxed{-3}$

$f(4) = \boxed{17}$

$\therefore$ ab. Min at $x = 2$ & ab. Max at $x = 4$.

H/W (1) $f(x) = \dfrac{x}{x^2 - x + 1}, \quad 0 \le x \le 3$

(2) $f(x) = \cancel{x^3 - x^2 +} x\sqrt{5 - x^2}, \quad -2 \le x \le 1.$

(3) Consider $f(x) = x^a(1-x)^b$, where a and b are positive real numbers. To find the maximum value of f in $[0,1]$.

Soln:- $f(x) = x^a(1-x)^b$

$f'(x) = x^a \cdot b(1-x)^{b-1}(-1) + a x^{a-1}(1-x)^b$

$= x^{a-1}(1-x)^{b-1}[a^{(1-x)} - bx] \quad \forall x \in (0,1)$

$f(0) = f(1) = 0$. therefore $f'(x) = 0$ only if

$-bx + a(1-x) = 0$

$-bx + a - ax = 0$

$\Rightarrow x = \dfrac{a}{a+b}$ is the critical pt.

$f\left(\dfrac{a}{a+b}\right) = \left(\dfrac{a}{a+b}\right)^a\left(1 - \dfrac{a}{a+b}\right)^b = \dfrac{a^a b^b}{(a+b)^{a+b}} > 0.$

$\therefore f$ has an ab. Max value is $\dfrac{a^a b^b}{(a+b)^{a+b}}$ at $x = \dfrac{a}{a+b}$ & ab. Min at 0 & 1 Value is 0.

Local Maxima and Local Minima

(1) Using the first derivative test, identify the local extreme values of $f(x) = x^2 - 4x + 4 = (x-2)^2$ in $[1, \infty)$.

Soln:-

$$f(x) = x^2 - 4x + 4 \text{ in } [1, -\infty).$$

$$f'(x) = 2x - 4 = 0$$

$\Rightarrow x = 2$ is the critical point.

Note that $f(1) = 1$.

1 is the left end pt

$-\infty$ 1 2 ∞

left $\to$ right

$- \to +$

Local Mini

Also $f'(x) = \begin{cases} > 0 & \text{for } x > 2 \\ < 0 & x < 2 \end{cases}$

Therefore $x = 2$, is a point of Local minimum and the value is $f(2) = 0$.

Also since $f(x) \to \infty$ as $x \to \infty$, the is no ab. Maxi. But the left end point $x = 1$ is a pt of local Maxi and the value is $f(1) = 1$.

Using first derivative test

(2) $f(x) = \sqrt{25 - x^2}$ in $[-5, 5]$

Soln:-

$$f(x) = \sqrt{25 - x^2}, \quad [-5, 5]$$

$f(-5) = f(5) = 0$ (end points)

$f'(x) = -\dfrac{x}{\sqrt{25 - x^2}}$ for $x \in (-5, 5)$. The critical pt $x = 0$.

$f'(x) = \begin{cases} > 0 & \text{when } x < 0 \quad \cancel{(-5, 0)} \quad \cancel{x > 0} \\ < 0 & \cancel{(0, 5)} \quad \cancel{x < 0} \end{cases}$

x

-5 0 5

Left $\to$ right

$+ \to -$

Local Maxi

$\therefore x = 0$ is a pt of Local Maximum

at the value of local Maxi is 5.

and at the end point, f has the ab. min value is $f(x) = 0$.

(X)

1. Determine the critical points, points of local maxima & local minima of $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$, and then identify the intervals on which f is concave up and concave down. Also find the points of inflection of f.

Soln:-

$$f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$$

$$f'(x) = x^2 - x - 2$$

$$f''(x) = 2x - 1;$$

(i) $f'(x) = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = -1$ & 2 are critical points.

(ii) $-\infty \quad -1 \quad 2 \quad \infty$

	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
$f'(x)$	+	−	+
f is	↗	↘	↗

$f'(x) = \begin{cases} +ve & x < -1 \\ -ve & x > -1 \end{cases}$

+ve to −ve $\Rightarrow$ local ~~maxi~~ mini

$f'(x) = \begin{cases} -ve & x < 2 \\ +ve & x > 2 \end{cases}$

−ve to +ve $\Rightarrow$ local maxi

(iii) Using first derivative test,

(critical pts) $-1, 2$

$-\infty \quad -1 \quad 2 \quad \infty$

$f(-1) = \frac{3}{2}$ (Local maxi)

$f(2) = -3$ (local mini)

(iv) $f''(x) = 0 \Rightarrow x = 1/2$

$(-\infty, 1/2)$ concave ↘ $\quad (1/2, \infty)$ concave ↗

Concave up for $x > 1/2$ & concave down for $x < 1/2$.

(v) $x = 1/2$ is point of inflection.

(2) $f(x) = 3x^4 - 4x^3 - 12x^2 + 4$

★ critical pt occur when $f'(x) = 0 \Rightarrow x = -1, 0, 2.$

$f'(x) = 12x^3 - 12x^2 - 24x$

$12x(x^2 - x - 2) = 0$

$f''(x) = 36x^2 - 24x - 24$

$6x^2 - 4x - 4$

$3x^2 - 2x - 2 = 0$

$= \frac{2 \pm \sqrt{4+24}}{6}$

$= \frac{1 \pm \sqrt{7}}{3}$

★ $-\infty \quad -1 \quad 0 \quad 2 \quad \infty$

	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	↘	↗	↘	↗

★ $f''(-1) > 0 \Rightarrow f(x)$ is local mini at $x = -1$. mini value $f(-1) = -1$

$f''(0) < 0 \Rightarrow f(x)$ is local Maxi at $x = 0$ $\quad f(0) = 4$

$f''(2) > 0 \Rightarrow f(x)$ is local mini at $x = 2$. $\quad f(2) = -28.$

★ Concavity $f''(x) = 0 \Rightarrow x = \frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}.$

Interval	$\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$	$\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$	$\left(\frac{1+\sqrt{7}}{3}, \infty\right)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	concave ↗	concave ↘	concave ↗

★ Points of inflection at $x = \frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}.$

(3) H.W $f(x) = \frac{9x^{1/3}}{14}(x^2 - 7)$, $\quad f'(x) = \frac{3x^{-2/3}}{2}(x^2 - 1),$

$f''(x) = x^{-5/3}(2x^2 + 1)$

★ $f(x) = x^4 - 4x^3 + 10$

Ans ★ $x = 0, \pm 1$

★ f is ↗ $(-\infty, -1) \cup (1, \infty)$

↘ $(-1, 1)$

★ Local maxi $\frac{27}{7}$ at $x = +1$, Local Min $-\frac{27}{7}$ at $x = -1$.

★ concave up on $(0, \infty)$ ↘ on $(-\infty, 0)$, ★ $x = 0$ is pt of inflection.

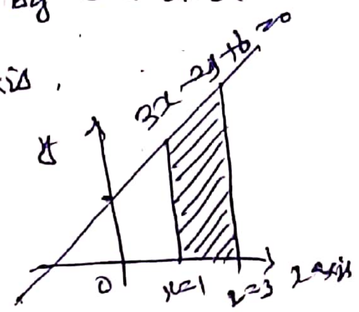
Integration

Area of the Region bounded by two plane curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region b/w the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f-g)$ from a to b:

$$A = \int_a^b (f(x) - g(x))\,dx.$$

① Find the area of the region bdd by the line $3x - 2y + 6 = 0$, $x = 1$ & $x = 3$ and x-axis.



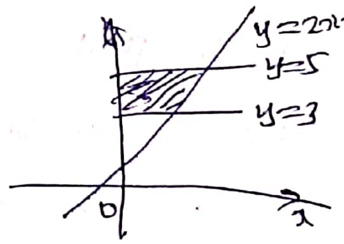
Soln:- $3x - 2y + 6 = 0$ (The line lies above the x-axis)

$\Rightarrow 2y = 3x + 6$

$y = \frac{3}{2}(x+2)$

The required Area $= \int_1^3 y\,dx = \frac{3}{2}\int_1^3 x+2\,dx = 12$ sq. units.

② Find the area of the region bdd by $y = 2x+1$, $y = 3$, $y = 5$ & y-axis.



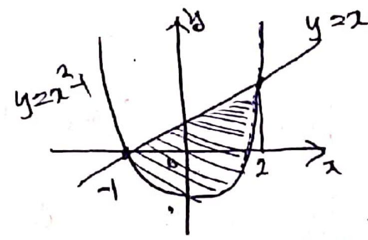
Soln:- $y = 2x + 1$

$\Rightarrow x = \frac{y-1}{2}$

The required area $A = \int_3^5 x\,dy$

$= \int_3^5 \frac{y-1}{2}\,dy = 3$ sq. units

③ Find the area b/w the line $y = x+1$, the curve $y = x^2 - 1$.



Soln:- To get the pt of intersection of the curves we should solve the equations

$y = x+1$, $y = x^2 - 1$.

$\Rightarrow x + 1 = x^2 - 1 \Rightarrow x^2 - x - 2 = 0$

$x = -1$ & 2.

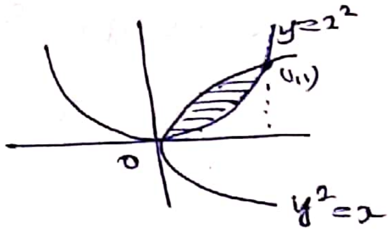
$\therefore$ Required Area $A = \int_{-1}^{2} (f(x) - g(x))\, dx$

$f(x)$: above, $g(x)$: below

$= \int_{-1}^{2} ((x+1) - (x^2-1))\, dx$

$= \int_{-1}^{2} (2+x-x^2)\, dx = \frac{9}{2}$ sq. units.

④. Determine the area of the region enclosed by $y=x^2$ & $y=\sqrt{x}$



Soln:- $y=x^2$, $y^2=x$

$y=y^4 \Rightarrow y^3=1$

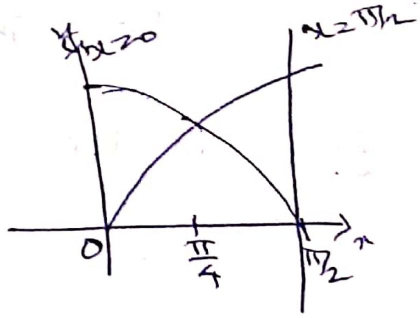
$y(y^3-1)=0 \Rightarrow y=1$

$\Rightarrow y=0$ & $\Rightarrow x=1$

$(0,0)$ & $(1,1)$ are pts of intersection.

$A = \int_{0}^{1} (\sqrt{x} - x^2)\, dx = \frac{1}{3}$

⑤ Find the area of the region enclosed by the curves $y=\sin x$ & $y=\cos x$ b/w the ordinates $x=0$ & $x=\pi/2$.



Soln:- The curves $y=\sin x$ & $y=\cos x$ intersect in the point $x=\pi/4$. Also,

$\cos x \geq \sin x$ for $0 \leq x \leq \pi/4$ &

$\sin x \geq \cos x$ for $\pi/4 \leq x \leq \pi/2$.

$\therefore A = \int_{0}^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = 2(\sqrt{2}-1)$.

⑥ (H.W) $y=\sin x$, $y=\sin 2x$, $x=0$, $x=\pi$. Pt of intersection $x=\pi/3$

$A = 5/2$

$\sin 2x \geq \sin x$, $0 \leq x \leq \pi/3$.

$\sin x \geq \sin 2x$, $\pi/3 \leq x \leq \pi$.

⑦ $y^2 = 4ax$, $4ay = x^2$ in the first Quadrant. $A = 16a^2/3$.

⑧ $y=2x-x^2$, $y=x^2$ $A = 1/3$.

Volumes of Solids of Revolution

Disk Method

Let $f(x)$ be a continuous function on $[a,b]$. The Volume of solid of revolution obtained by revolving the arc of the plane curve $y=f(x)$ from $x=a$ to $x=b$ about the x-axis, is

$$V=\int_a^b \pi y^2\,dx$$

* Let $g(y)$ conti $[c,d]$

$x=g(y)$ from $y=c$ to $y=d$ abt y-axis is

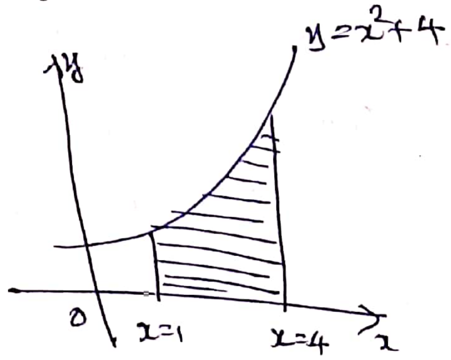
$$V=\int_c^d \pi x^2\,dy.$$

Question

(1) The curve $y=x^2+4$ is rotated one revolution about the x-axis between the limits $x=1$ and $x=4$. Determine the volume of the solid of revolution produced.

Soln:-

Revolving the shaded area shown in figure abt the x-axis 360° produces a solid of revolution given by



$$V=\int_1^4 \pi y^2\,dx=\int_1^4 \pi\,(x^2+4)^2\,dx$$

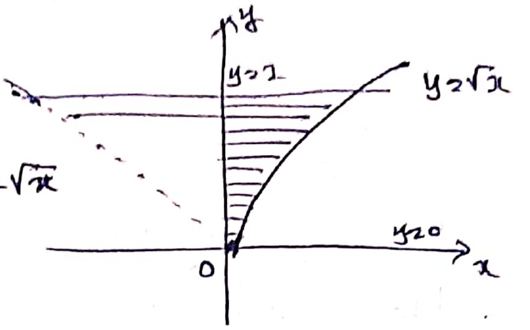
$$=420.6\pi \text{ cubic units.}$$

(2) $y=x^2-3x$, $y=0$, $x=3$.

② Find the volumn of the solid generated when the region enclosed by $y=\sqrt{x}$, $y=2$ and $x=0$ is revolved abt the y-axis

Soln:-

Since the solid is generated by revolving abt the y-axis, rewrite $y=\sqrt{x}$ as $x=y^2$.

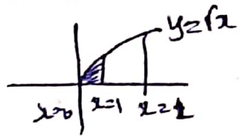


Limits $y=0$ & $y=2$

$$V=\int_c^d \pi x^2 dy = \int_0^2 \pi y^4 dy = \frac{32\pi}{5} \text{ cubic units.}$$

Washer Method

②A $y=\sqrt{x}$, $x=0$, $x=1$.

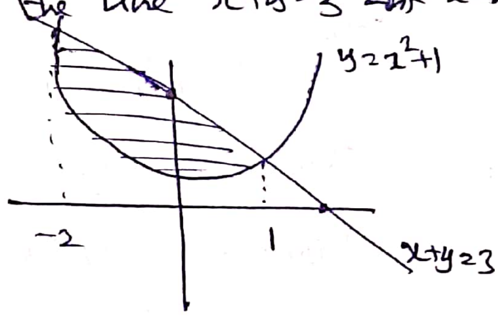


$$\int \pi y^2 dx = \int_0^1 \pi x dx = \pi/2$$

* The volume of solid of revolution of the region enclosed by the plane curves $y=f(x)$ and $y=g(x)$ with $f(x) \geq g(x)$ from $x=a$ to $x=b$ abt the x-axis is

$$V=\int_a^b \pi \left(f(x)^2 - g(x)^2\right) dx$$

① Find the volume of the solid of revolution of the region enclosed by the Parabola $y=x^2+1$ and the line $x+y=3$ abt x-axis



Soln:- The two curves intersect in the pts where $x=-2$ and $x=1$.

$x+x^2+1=3$

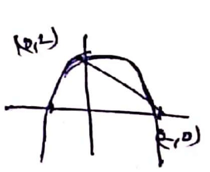
$x^2+x-2=0$

$x=1$ & -2.

Also $x^2+1 \leq \cancel{x}-3$

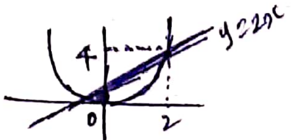
$$V=\int_{-2}^{1} \pi\left((x-3)^2-(x^2+1)^2\right) dx = \frac{117\pi}{5} \text{ cubic units.}$$

② $x+y=2$ & $x^2=4-y$ pts, -1 & 2



$$V=\int_{-1}^{2} \pi\left((4-x^2)^2-(2-x)^2\right) dx = \frac{108\pi}{5}$$

③ $y=x^2$, $y=2x$ abt y-axis



$f(y) \geq g(y)$

$x=\sqrt{y}$ $x^2-2x=0$ $x=0, x=2$ $x=\frac{y}{2}$

$$\int_0^4 \pi\left((\sqrt{y})^2-\left(\frac{y}{2}\right)^2\right) dy = 8/3\,\pi$$ //