**TABLE 2.2**

Boolean Functions of Two Variables

| Function | AB = 00 | AB = 01 | AB = 10 | AB = 11 |
|---|---|---|---|---|
| $F_0 = 0$ | 0 | 0 | 0 | 0 |
| $F_1 = AB$ | 0 | 0 | 0 | 1 |
| $F_2 = A\bar{B}$ | 0 | 0 | 1 | 0 |
| $F_3 = A$ | 0 | 0 | 1 | 1 |
| $F_4 = \bar{A}B$ | 0 | 1 | 0 | 0 |
| $F_5 = B$ | 0 | 1 | 0 | 1 |
| $F_6 = A\bar{B} + \bar{A}B$ | 0 | 1 | 1 | 0 |
| $F_7 = A + B$ | 0 | 1 | 1 | 1 |
| $F_8 = \overline{A + B}$ | 1 | 0 | 0 | 0 |
| $F_9 = AB + \bar{A}\bar{B}$ | 1 | 0 | 0 | 1 |
| $F_{10} = \bar{B}$ | 1 | 0 | 1 | 0 |
| $F_{11} = A + \bar{B}$ | 1 | 0 | 1 | 1 |
| $F_{12} = \bar{A}$ | 1 | 1 | 0 | 0 |
| $F_{13} = \bar{A} + B$ | 1 | 1 | 0 | 1 |
| $F_{14} = \overline{AB}$ | 1 | 1 | 1 | 0 |
| $F_{15} = 1$ | 1 | 1 | 1 | 1 |

## 2.4 Boolean Algebra

As shown in the last section, a canonical form of a logic function can usually be reduced to a simpler form that leads to a less expensive, less complicated and faster logic circuit. Such function complexity reduction is accomplished often by means of Boolean algebra, although it was not initially developed specifically for the purpose of function simplification. For a set of logic elements, Boolean algebra can be defined in terms of a set of operators and a number of axioms or postulates that are taken as true without the need for any proof. A binary operator defined on the set of elements is a rule that assigns to each pair of elements in the set another unique element that is also included in the set.

To appreciate the usefulness of Boolean algebra, we need to briefly consider the principles of ordinary algebra first. Some of the more common postulates used in defining formalisms for ordinary algebra are as follows:

*Closure*: A set is closed with respect to an operator $\nabla$ if and only if for every $A$ and $B$ in set $S$, $A \nabla B$ is also a member of set $S$.

*Associativity*: An operator $\nabla$ defined on a set $S$ is said to be associative if and only if for all $A$, $B$, and $C$ in set $S$, $(A \nabla B) \nabla C = A \nabla (B \nabla C)$.

*Commutativity*: An operator $\nabla$ defined on a set $S$ is said to be commutative if and only if for all $A$ and $B$ in set $S$, $A \nabla B = B \nabla A$.

*Distributivity*: If $\nabla$ and $\square$ are two operators on a set $S$, the operator $\nabla$ is said to be distributive over $\square$ if for all $A$, $B$, and $C$ in set $S$, $A \nabla (B \square C) = (A \square B) \nabla (A \square C)$.

*Identity*: With an operator $\nabla$ defined on set $S$, the set $S$ is said to have an identity element if and only if for an element $A$ in set $S$, $A \nabla B = B \nabla A = B$ for every $B$ in set $S$.

*Inverse*: With an operator $\nabla$ and the identity element $I$ defined on set $S$, the set $S$ is said to have an inverse element if and only if for every $A$ in set $S$, there exists an element $B$ in set $S$ such that $A \nabla B = I$.

Just as the case with ordinary algebra, the formulation of Boolean algebra is based on a set of postulates, known commonly as *Huntington's postulates*. Boolean algebraic structure is defined on a set of elements $S = \{0, 1\}$ with two binary operators, $+$ and $\cdot$, and satisfies the following postulates:

*Postulate 1.* The set $S$ is closed with respect to the operators: $+$ (i.e., OR) and $\cdot$ (i.e., AND). Closure is self-evident in tables of Figures 2.1b and 2.2b. Both the AND and OR output are elements of set $S$.

*Postulate 2.* The set $S$ has identity elements (a) 1 with respect to $+$ (i.e., OR) and (b) 0 with respect to $\cdot$ (i.e., AND). Using the tables of Figures 2.1b and 2.2b, we note that

(a) $A + 0 = A$ and $A \cdot 1 = A$

(b) $A \cdot 0 = 0$ and $A + 1 = 1$

which shows that both 0 and 1 are identity elements.

*Postulate 3.* The set $S$ is commutative with respect to the operators: $+$ (i.e., OR) and $\cdot$ (i.e., AND). It follows directly from the symmetry of the tables in Figures 2.1b and 2.2b since $A + B = B + A$ and $A \cdot B = B \cdot A$.

*Postulate 4.* The operator $+$ (i.e., OR) is distributive over $\cdot$ (i.e., AND) and similarly $\cdot$ (i.e., AND) is distributive over $+$ (i.e., OR). This can be demonstrated by verifying both sides of the logical equation $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$ for all possible cases of the variables.

| $A\ B\ C$ | $B + C$ | $A \cdot (B + C)$ | $A \cdot B$ | $A \cdot C$ | $A \cdot B + A \cdot C$ |
|---|---|---|---|---|---|
| 0 0 0 | 0 | 0 | 0 | 0 | 0 |
| 0 0 1 | 1 | 0 | 0 | 0 | 0 |
| 0 1 0 | 1 | 0 | 0 | 0 | 0 |
| 0 1 1 | 1 | 0 | 0 | 0 | 0 |
| 1 0 0 | 0 | 0 | 0 | 0 | 0 |
| 1 0 1 | 1 | 1 | 0 | 1 | 1 |
| 1 1 0 | 1 | 1 | 1 | 0 | 1 |
| 1 1 1 | 1 | 1 | 1 | 1 | 1 |

Similarly, we can also show by tabular method that $A + B \cdot C = (A + B) \cdot (A + C)$. Note that this is not the case in ordinary algebra.

| $A\ B\ C$ | $B \cdot C$ | $A + B \cdot C$ | $f_1 = A + B$ | $f_2 = A + C$ | $f_1 \cdot f_2$ |
|---|---|---|---|---|---|
| 0 0 0 | 0 | 0 | 0 | 0 | 0 |
| 0 0 1 | 0 | 0 | 0 | 1 | 0 |
| 0 1 0 | 0 | 0 | 1 | 0 | 0 |
| 0 1 1 | 1 | 1 | 1 | 1 | 1 |
| 1 0 0 | 0 | 1 | 1 | 1 | 1 |
| 1 0 1 | 0 | 1 | 1 | 1 | 1 |
| 1 1 0 | 0 | 1 | 1 | 1 | 1 |
| 1 1 1 | 1 | 1 | 1 | 1 | 1 |

*Postulate 5.* For every $A$ in set $S$, there exists a complement element $\bar{A}$ in set $S$ such that $A + \bar{A} = 1$ and $A \cdot \bar{A} = 0$.

There are several differences between ordinary algebra and Boolean algebra. In ordinary algebra, for example, + (i.e., OR) is not distributive over · (i.e., AND). Ordinary algebra, on the other hand, applies to the set of infinite real numbers while Boolean algebra applies to a set of finite elements. In Boolean algebra there is no room for subtraction or division operations since there are no inverses with respect to + or ·. On the plus side, however, complements are available in Boolean algebra but not in ordinary algebra. Although there are major differences between the two, Boolean algebra and ordinary algebra are alike in many respects. Note that Huntington's list of postulates does not include associativity since it can be derived from other postulates. Many in the field often include associativity among the postulates but that is unnecessary.

As is true for other axiomatic systems, each Huntington's postulate has a corresponding dual. Briefly explained, the *principle of duality* states that if and when a given logic expression is valid, the dual of the same logic expression is also valid. The Huntington's dual or mirror image is obtained by replacing each 0 with a 1, each 1 with an 0, each AND with an OR, and each OR with an AND. For example, the dual expression of the democracy function given by

$$\bar{A}BC + A\bar{B}C + AB\bar{C} + ABC = AB + BC + CA$$

is

$$(\bar{A} + B + C) \cdot (A + \bar{B} + C) \cdot (A + B + \bar{C}) \cdot (A + B + C) = (A + B) \cdot (B + C) \cdot (A + C)$$

This principle of duality along with the idea of taking a complement will be used later in the development of DeMorgan's theorems.

Boolean algebra is established on a set of two binary operators, AND and OR. However, since one of the postulates (i.e., Postulate 5) establishes the complement operator, Boolean algebra is frequently defined in terms of two or more elements subject to an equivalence relation "=" and three binary operators OR, AND, and NOT. Boolean theorems may now be derived using the Huntington's postulates. These theorems can be proven easily using the postulates stated above. While some of these proofs are left as an end-of-the-chapter problem (see Problem 1), a few of them are worked out so that you may get used to the mechanics of Boolean reductions. The theorems are listed as follows:

*THEOREM 1*
*The Law of Idempotency.*

For all $A$ in set $S$,

(a) $A + A = A$ and
(b) $A \cdot A = A$

The part (a) follows directly from Postulate 2(b) since

$$A + A = A \cdot (1 + 1) = A \cdot 1 = A$$

Alternatively, we can verify this theorem by proceeding as follows:

$$\begin{aligned} A + A &= (A + A) \cdot 1 && \text{using Postulate 2(a)} \\ &= (A + A) \cdot (A + \bar{A}) && \text{using Postulate 5} \\ &= A \cdot A + A \cdot \bar{A} && \text{using Postulate 4} \\ &= A + 0 && \text{using Postulate 5} \\ &= A && \text{using Postulate 2(a)} \end{aligned}$$

Part (b), on the other hand, can be justified as follows:

$$\begin{aligned} A \cdot A &= A \cdot A + 0 && \text{using Postulate 2(a)} \\ &= A \cdot A + A \cdot \bar{A} && \text{using Postulate 5} \\ &= A \cdot (A + \bar{A}) && \text{using Postulate 4} \\ &= A \cdot 1 && \text{using Postulate 5} \\ &= A && \text{using Postulate 2(a)} \end{aligned}$$

*THEOREM 2*
*The Law of Absorption.*

For all $A$ and $B$ in set $S$,

(a) $A + (A \cdot B) = A$ and
(b) $A \cdot (A + B) = A$

Part (a) of this theorem can be proven as follows:

$$\begin{aligned} A + AB &= A \cdot 1 + AB && \text{using Postulate 2(a)} \\ &= A \cdot (1 + B) && \text{using Postulate 4} \\ &= A \cdot 1 && \text{using Postulate 2(b)} \\ &= A && \text{using Postulate 2(a)} \end{aligned}$$

Part (b) of the theorem likewise can be justified as follows:

$$\begin{aligned} A \cdot (A + B) &= A \cdot A + A \cdot B && \text{using Postulate 4} \\ &= A + A \cdot B && \text{using Theorem 1(b)} \\ &= A && \text{using Theorem 2(a)} \end{aligned}$$

*THEOREM 3*
*The Law of Identity.*

For all $A$ and $B$ in set $S$, if (a) $A + B = B$ and (b) $A \cdot B = B$ then $A = B$.

By substituting condition (b) into the left-hand side of condition (a), we get

$$A + A \cdot B = B$$

But according to Theorem 2(a)

$$A + A \cdot B = A$$

Therefore, $A = B$.

*THEOREM 4*
*The Law of Complement.*

For all $A$ in set $S$, $\bar{A}$ is unique.
Assume that there are two distinct elements $\bar{a}_1$ and $\bar{a}_2$ that satisfy Postulate 5, that is,

$$A + \bar{a}_1 = 1, \quad A + \bar{a}_2 = 1, \quad A \cdot \bar{a}_1 = 0, \quad A \cdot \bar{a}_2 = 0$$

Then,

$$\begin{aligned}
\bar{a}_2 &= 1 \cdot \bar{a}_2 && \text{using Postulate 2(a)} \\
&= (A + \bar{a}_1) \cdot \bar{a}_2 && \text{using Postulate 5} \\
&= A \cdot \bar{a}_2 + \bar{a}_1 \cdot \bar{a}_2 && \text{using Postulate 4} \\
&= 0 + \bar{a}_1 \cdot \bar{a}_2 && \text{using Postulate 5} \\
&= A \cdot \bar{a}_1 + \bar{a}_1 \cdot \bar{a}_2 && \text{using Postulate 5} \\
&= (A + \bar{a}_2) \cdot \bar{a}_1 && \text{using Postulate 4} \\
&= 1 \cdot \bar{a}_1 && \text{using Postulate 5} \\
&= \bar{a}_1 && \text{using Postulate 2(a)}
\end{aligned}$$

Therefore, $A$ has a unique complement in set $S$.

*THEOREM 5*
*The Law of Involution.*

For all $A$ in set $S$, $(\bar{A})' = A$.
Since,

$$A + \bar{A} = 1 \quad \text{and} \quad A \cdot \bar{A} = 0 \qquad \text{(by Postulate 5)}$$

Then the complement of $\bar{A}$ is $A$. Therefore, $(\bar{A})' = A$.

*THEOREM 6*
*The Law of Association.*

For every $A$, $B$, and $C$ in set $S$,

(a) $A + (B + C) = (A + B) + C$ and
(b) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

Let

$$
\begin{aligned}
D &= \{A + (B + C)\} \cdot \{(A + B) + C\} && \text{using Postulate 4} \\
&= A\{(A + B) + C\} + (B + C)\{(A + B) + C\} && \text{using Postulate 4} \\
&= A(A + B) + AC + \{B[(A + B) + C] + C[(A + B) + C]\} && \text{using Theorem 2} \\
&= A + AC + \{B(A + B) + BC + C[(A + B) + C]\} && \text{using Theorem 2} \\
&= A + \{B + BC + C(A + B) + CC\} && \text{using Theorem 2} \\
&= A + \{B + C(A + B) + C\} && \text{using Theorem 2} \\
&= A + (B + C)
\end{aligned}
$$

But

$$
\begin{aligned}
D &= \{A + (B + C)\}(A + B) + \{A + (B + C)\}C && \text{using Postulate 4} \\
&= \{[A + (B + C)]A + [A + (B + C)]B\} + AC + (B + C)C && \text{using Postulate 4} \\
&= \{[A + (B + C)]A + [A + (B + C)]B\} + AC + C && \text{using Theorem 2} \\
&= \{[A + (B + C)]A + (AB + B)\} + C && \text{using Theorem 2} \\
&= \{AA + (B + C)A + B\} + C && \text{using Theorem 2} \\
&= \{A + (B + C)A + B\} + C && \text{using Theorem 2} \\
&= (A + B) + C && \text{using Theorem 2}
\end{aligned}
$$

Therefore, $A + (B + C) = (A + B) + C$

*THEOREM 7*
*Law of Elimination.*

For all $A$ and $B$ in set $S$,

(a) $A + \bar{A}B = A + B$
(b) $A \cdot (\bar{A} + B) = A \cdot B$

Part (a) of this theorem can be proved as follows:

$$
\begin{aligned}
A + \bar{A}B &= (A + \bar{A})(A + B) && \text{using Postulate 4} \\
&= 1 \cdot (A + B) && \text{using Postulate 5} \\
&= A + B && \text{using Postulate 2(a)}
\end{aligned}
$$

This theorem is very much like that of absorption in that it can be employed to eliminate extra literal from a Boolean function.

*THEOREM 8*
*DeMorgan's Law.*

For all $A$ and $B$ in set $S$,

(a) $\overline{A + B} = \bar{A} \cdot \bar{B}$
(b) $\overline{A \cdot B} = \bar{A} + \bar{B}$

This theorem implies that a function may be complemented by changing each OR to an AND, each AND to an OR, and by complementing each of the variables. Let us prove part (a) only, first, for a two variable function.

Let $x = A + B$, then $\bar{x} = \overline{A + B}$. If $x \cdot y = 0$ and $x + y = 1$, then $y = \bar{x}$ per Theorem 4. Thus to prove part (a) of DeMorgan's theorem, we need to set $y = \bar{A} \cdot \bar{B}$ and evaluate $x \cdot y$ and $x + y$.

$$
\begin{aligned}
x \cdot y &= (A + B) \cdot (\bar{A} \cdot \bar{B}) & \\
&= A\bar{A}\bar{B} + B\bar{A}\bar{B} & \text{using Postulate 4} \\
&= 0 + 0 & \text{using Postulate 5} \\
&= 0 & \text{using Postulate 2(a)} \\
x + y &= (A + B) + \bar{A} \cdot \bar{B} & \\
&= A + B + \bar{A}\bar{B} & \\
&= A + (B + \bar{A}\bar{B}) & \text{using Theorem 6} \\
&= A + (B + \bar{A}) & \text{using Theorem 7} \\
&= B + (A + \bar{A}) & \text{using Theorem 6} \\
&= B + 1 & \text{using Postulate 2(b)} \\
&= 1 &
\end{aligned}
$$

Therefore, $\overline{A + B} = \bar{A} \cdot \bar{B}$.

This theorem may be generalized for all $A, B, \ldots,$ and $Z$ in set $S$ as follows:

(a) $\overline{A + B + \cdots + Z} = \bar{A} \cdot \bar{B} \cdot \cdots \cdot \bar{Z}$
(b) $\overline{A \cdot B \cdot \cdots \cdot Z} = \bar{A} + \bar{B} + \cdots + \bar{Z}$

The consequence of these two theorems can be summarized as follows: The complement of any logic function is obtained by replacing each variable with its complement, each AND with an OR, and each OR with an AND, each 0 with an 1 and each 1 with a 0.

*THEOREM 9*
*The Law of Consensus.*

For all $A$, $B$, and $C$ in set $S$,

(a) $AB + \bar{A}C + BC = AB + \bar{A}C$
(b) $(A + B)(\bar{A} + C)(B + C) = (A + B)(\bar{A} + C)$

Part (a) of this theorem can be proved as follows:

$$
\begin{aligned}
AB + \bar{A}C + BC &= AB + \bar{A}C + 1 \cdot BC && \text{using Postulate 2(a)} \\
&= AB + \bar{A}C + (A + \bar{A}) \cdot BC && \text{using Postulate 5} \\
&= AB + \bar{A}C + ABC + \bar{A}BC && \text{using Postulate 4} \\
&= AB(1 + C) + \bar{A}C(1 + B) && \text{using Postulate 4} \\
&= AB + \bar{A}C && \text{using Postulate 2(b)}
\end{aligned}
$$

This theorem is used in both reduction and expansion of Boolean expressions. The key to using this theorem is to locate a literal in a term, its complement in another term, and associated literal or literal combination in both of these terms and only then the included term (the consensus term) that is composed of the associated literals can be eliminated.

*THEOREM 10*
*The Law of Interchange.*

For all $A$, $B$, and $C$ in set $S$,

(a) $AB + \bar{A}C = (A + C) \cdot (\bar{A} + B)$
(b) $(A + B) \cdot (\bar{A} + C) = AC + \bar{A}B$

Part (a) of the theorem can be justified as follows:

$$
\begin{aligned}
AB + \bar{A}C &= (AB + \bar{A}) \cdot (AB + C) && \text{using Postulate 4} \\
&= (A + \bar{A}) \cdot (B + \bar{A}) \cdot (A + C) \cdot (B + C) && \text{using Postulate 4} \\
&= 1 \cdot (\bar{A} + B) \cdot (A + C) \cdot (B + C) && \text{using Postulate 5} \\
&= (\bar{A} + B) \cdot (A + C) \cdot (B + C) && \text{using Postulate 2(a)} \\
&= (A + C) \cdot (\bar{A} + B) && \text{using Theorem 9}
\end{aligned}
$$

*THEOREM 11*
*The Generalized Functional Laws.*

The AND/OR operation of a variable $A$ and a multivariable composite function that is also a function of $A$ is equivalent to AND/OR operation of $A$ with the composite function wherein $A$ is replaced by 0:

(a) $A + f(A, B, \ldots, Z) = A + f(0, B, \ldots, Z)$
(b) $A \cdot f(A, B, \ldots, Z) = A \cdot f(0, B, \ldots, Z)$

The basis of this theorem is Theorem 1 and Postulate 2(a). Since $A = A + A = A + 0$, the variable $A$ within the function in part (a) may be replaced by 0. For all $A, B, \ldots,$ and $Z$

in set $S$,

(a) $f(A, B, \ldots, Z) = A \cdot f(1, B, \ldots, Z) + \bar{A} \cdot f(0, B, \ldots, Z)$
(b) $f(A, B, \ldots, Z) = [A + f(0, B, \ldots, Z)] \cdot [\bar{A} + f(1, B, \ldots, Z)]$

The latter two versions of the theorem can be proved by making use of the other two versions, and Postulates 2(a) and 5(a).

The generalized functional laws become useful in designing a particular digital circuit called *multiplexer* or *data selector*. This circuit in turn can be used for implementing almost any Boolean function. These are amazingly powerful since they allow writing a Boolean function so that a selected variable and its complement appear only once. More will be said about this in Chapter 4. The following examples illustrate the use of the various Boolean theorems we have already introduced in this section.

**Example 2.1**
Obtain the overflow logic function in terms of (a) only sign bits and (b) only carries (in and out of sign bit).

**Solution**
(a) If the sign bit of the sum is different from that of both addend and augend, an overflow has occurred. Consider two $n$-bit numbers $A$ and $B$ when added together yields the sum $S$. Let $A_{n-1}$, $B_{n-1}$, and $S_{n-1}$ represent the sign bits of $A$, $B$, and $S$, respectively. The truth table for the overflow logic function $f$ is shown in Figure 2.11. Whenever the sign bit of both $A$ and $B$ are similar but different than that of $S$, the overflow has occurred.

Since there are only two ls in the output column, it is wise to obtain an SOP expression for $f$.

Therefore,

$$f = \bar{A}_{n-1}\bar{B}_{n-1}S_{n-1} + A_{n-1}B_{n-1}\bar{S}_{n-1}$$

(b) The second method involves an examination of both carry-in and carry-out of the sign bit. Presence of different values for these two carries is indicative of an overflow. Given that we denote carry-in as $C_{in}$ and carry-out as $C_{out}$, the truth table for $f$ is obtained as shown in Figure 2.12.

Accordingly, we can use the SOP form to obtain

$$f = \bar{C}_{in}C_{out} + C_{in}\bar{C}_{out} = C_{in} \oplus C_{out}$$

Alternatively, we can make use of the POS form as,

$$\begin{aligned} f &= (C_{in} + C_{out})(\bar{C}_{in} + \bar{C}_{out}) \\ &= C_{in}\bar{C}_{in} + C_{in}\bar{C}_{out} + C_{out}\bar{C}_{in} + C_{out}\bar{C}_{out} \\ &= C_{in} \oplus C_{out} \end{aligned}$$