Extrema (Maxima and Minima) of a function

EXTREME VALUES OF FUNCTIONS

we are going to learn how derivatives affect the shape of
a graph of a function and, in particular, how they help us
to locate maximum and minimum values of functions.

Some of the most important applications of differential calculus are
optimization problems.

In these, we are required to find the optimal (best)
way of doing something.

Many practical problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation.

- What is the shape of a can that minimizes manufacturing costs?

- What is the maximum acceleration of a space shuttle?
  (This is an important question to the astronauts who have to withstand the effects of acceleration.)

- What is the radius of a contracted windpipe that expels air most rapidly during a cough?

- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function.

1. We say that *f(x)* has an **absolute (or global) maximum** at $x = c$
   if $f(x) \leq f(c)$ for every *x* in the domain we are working on.

2. We say that *f(x)* has a **relative (or local) maximum** at $x = c$
   if $f(x) \leq f(c)$ for every *x* in some **open interval** around $x = c$.

3. We say that *f(x)* has an **absolute (or global) minimum** at $x = c$
   if $f(x) \geq f(c)$ for every *x* in the domain we are working on.

4. We say that *f(x)* has a **relative (or local) minimum** at $x = c$
   if $f(x) \geq f(c)$ for every *x* in some **open interval** around $x = c$.

- Global/absolute extrema
   either maximum or minimum points on a whole curve.

- Local/relative extrema
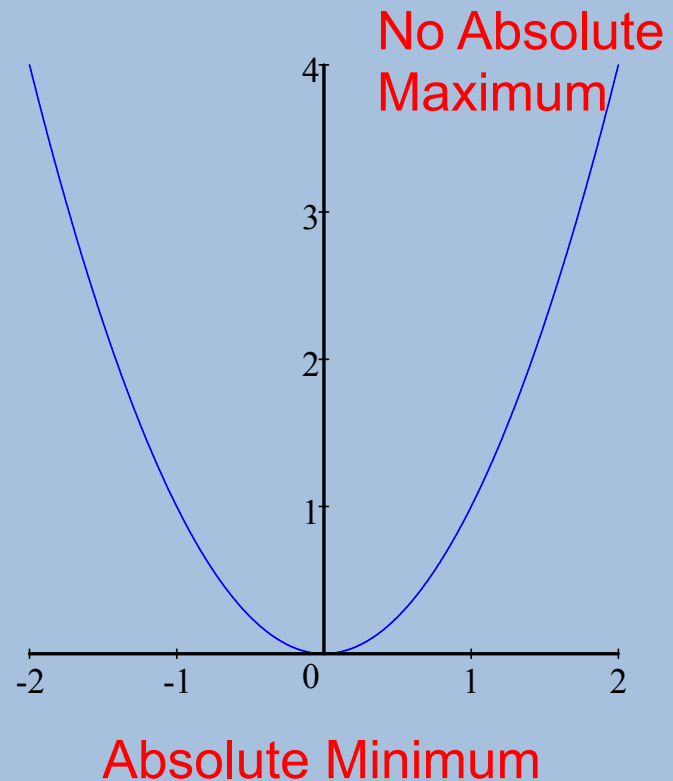   minimum/maximum value within some open interval
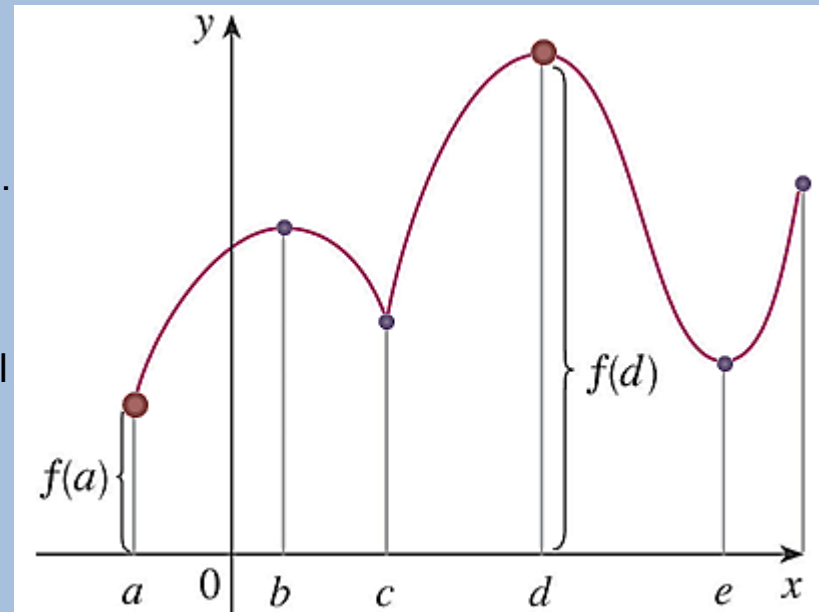
The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times. The same goes for minimum value of -1.

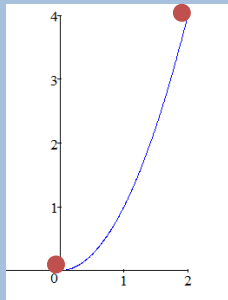If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x. Therefore, $f(0) = 0$ is the absolute (and local) minimum value of f.

It has neither a local maximum value nor an absolute maximum value

No Absolute Maximum

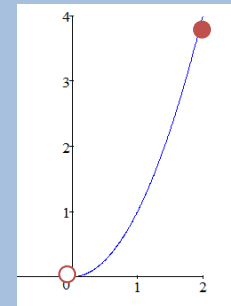$$D = \left(-\infty, \infty\right)$$

Absolute Minimum

Extreme values can be in the interior or the end points of a function.

$$D = \begin{bmatrix} 0, 2 \end{bmatrix}$$

Absolute Minimum
Absolute Maximum

$$D = \left(0, 2 \right]$$
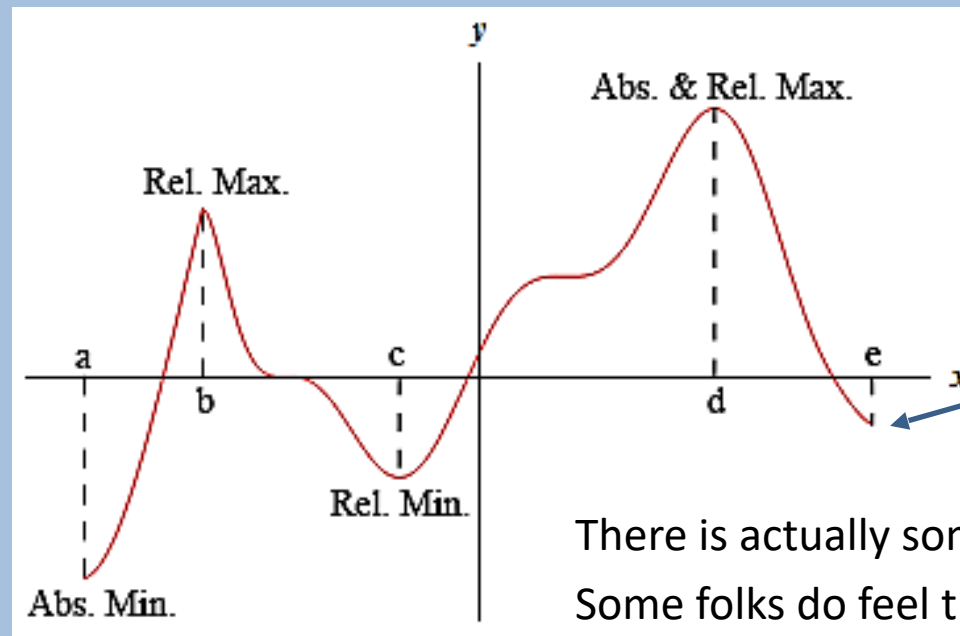
No Minimum
Absolute Maximum
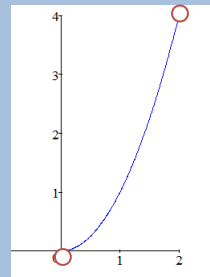
$$D = \left(0, 2 \right)$$

No Maximum

No Minimum

All that's required for a point to be a relative maximum or minimum is for that point to be a maximum or minimum in some interval of x's around $x = c$

Note as well that in order for a point to be a relative extrema we must be able to look at function values on both sides of $x = c$ to see if it really is a maximum or minimum at that point. This means that relative extrema do not occur at the end points of a domain. They can only occur interior to the domain.

It is NOT relative minimum

There is actually some debate on the preceding point. Some folks do feel that relative extrema can occur on the end points of a domain. However, in this class we will be using the definition that says that they can't occur at the end points of a domain.

EX: Identify the absolute extrema and relative extrema for $f(x) = x^2$ on [-1, 2]
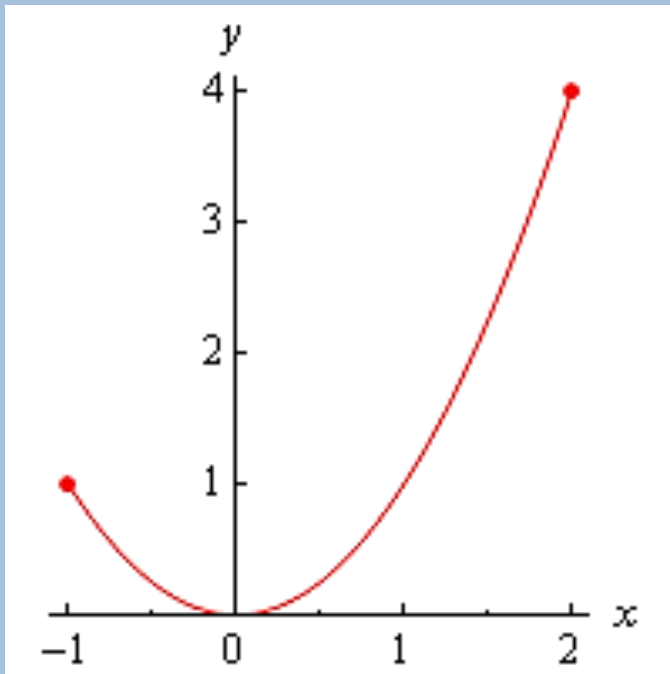
relative and absolute minimum: x=0
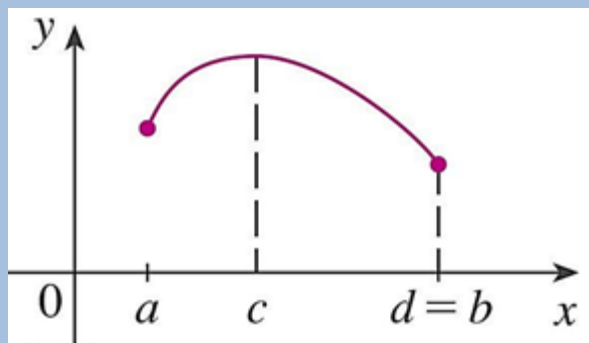
absolute maximum: x=2

Note that $x = -1$ is not a relative maximum sice it is at the end point of the interval. This function doesn't have any relative maximums.
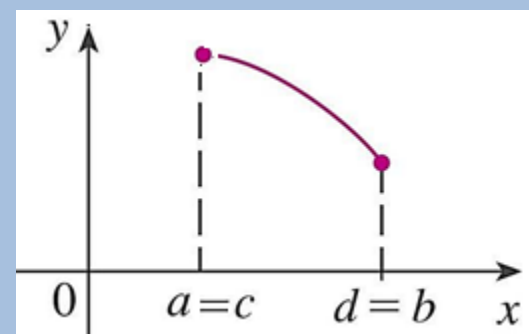
Extreme Value Theorem:

If f is continuous on a closed interval $[a,\ b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,\ b]$.
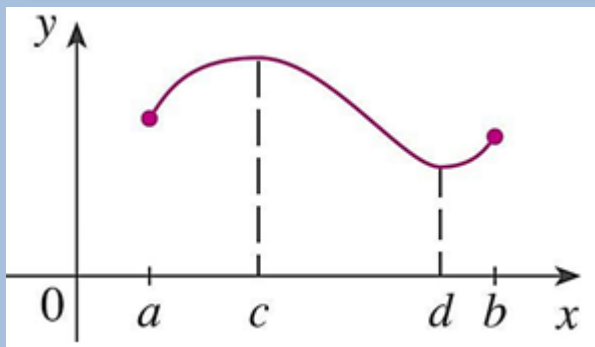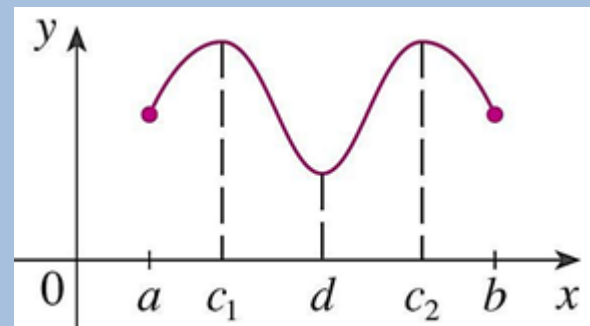

Maximum & minimum at interior points

Maximum at interior point, minimum at endpoint

Maximum & minimum at endpoints

Note that an extreme value can be taken on more than once.

Just because a **function is not continuous** at a point that doesn't mean that it won't have local/absolute extreme in an interval that contains that point. Below is the graph of a function that is not continuous at a point $x = c$ in the given interval and yet it does have both an absolute maximum ($x = b$) and an absolute minimum ($x = c$).
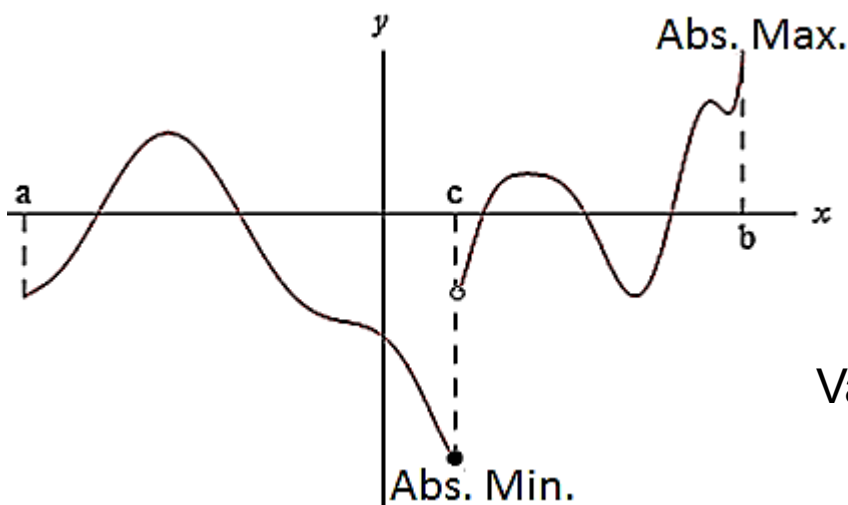
The point is that *f(x)* has a **local minimum** at $x = c$ if $f(x) \geq f(c)$ for every *x* in some open interval around $x = c$

We need to be careful to only use the Extreme Value Theorem when the conditions of the theorem are met and not misinterpret the results if the conditions aren't met.
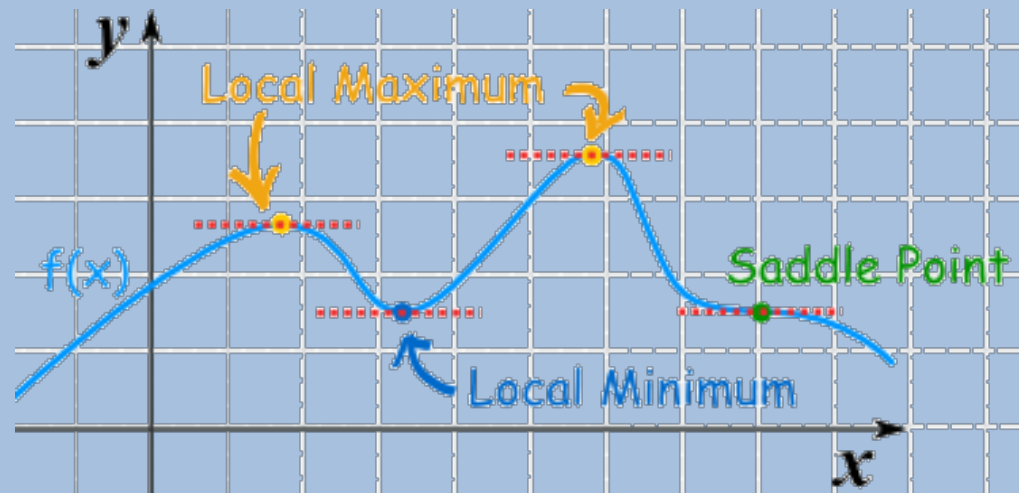
In order to use the Extreme Value Theorem we must have an interval and the function must be continuous on that interval. If we don't have an interval and/or the function isn't continuous on the interval then the function may or may not have absolute extrema.

Absolute maximum

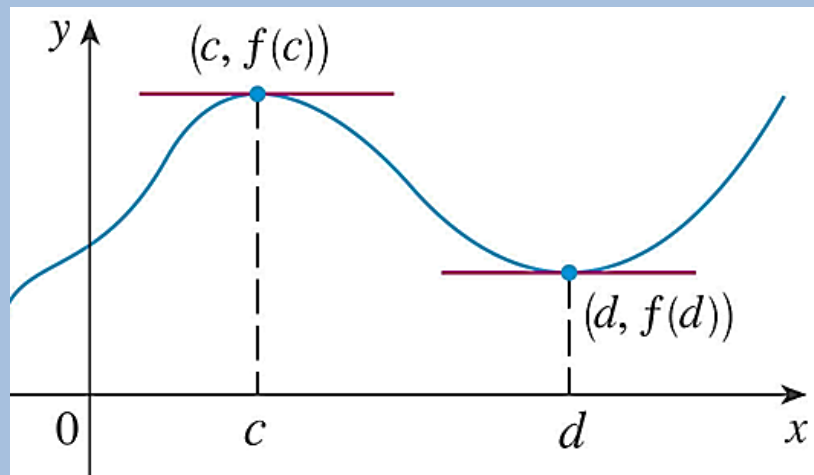(also local maximum)

Local maximum

Local minimum

Absolute minimum

The theorem says that a continuous function on a closed interval has a maximum value and a minimum value.
However, it does not tell us **how to find** these extreme values.
We start by looking for local extreme values.

The figure shows the graph of a function f with a local maximum at c and a local minimum at d.

It appears that, at the maximum and minimum points, the tangent lines are horizontal and therefore each has slope 0.
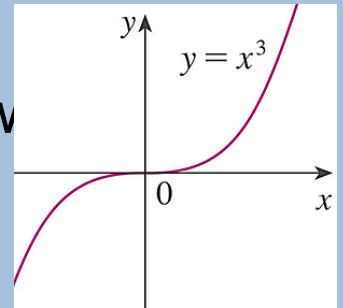

So, it appears that $f'(c) = 0$ and $f'(d) = 0$.

Theorem:

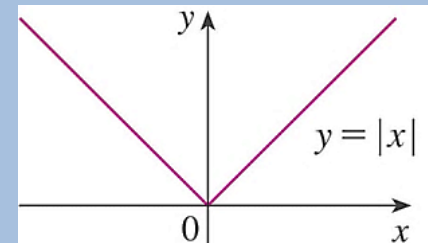If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

BUT…. if you know that $f'(c) = 0$ at point c, does it mean that point c is extreme?

In other words: If we set $f'(x) = 0$ and solve for x , are w̲ sure that we located ONLY extreme values,

or

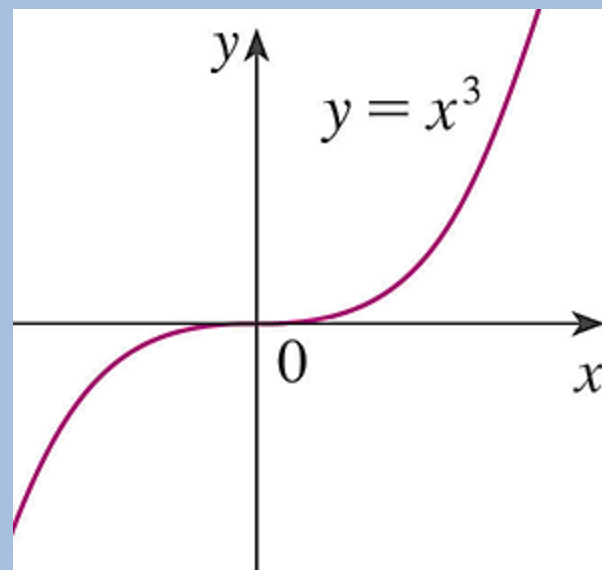Can we expect to locate ALL extreme values simply by setting $f'(x) = 0$ and solve for x

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.

However, f has no maximum or minimum at 0 as you can see from the graph.

Alternatively, observe that $x^3 > 0$ for $x > 0$ but $x^3 < 0$ for $x < 0$.

The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0).

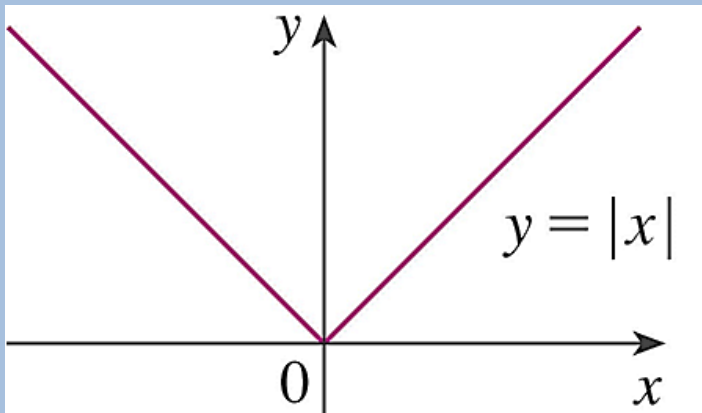The function $f(x) = |x|$ has its (local and absolute) minimum value at 0.

However, that value can't be found by setting $f'(x) = 0$.

This is because $f'(0)$ does not exist.
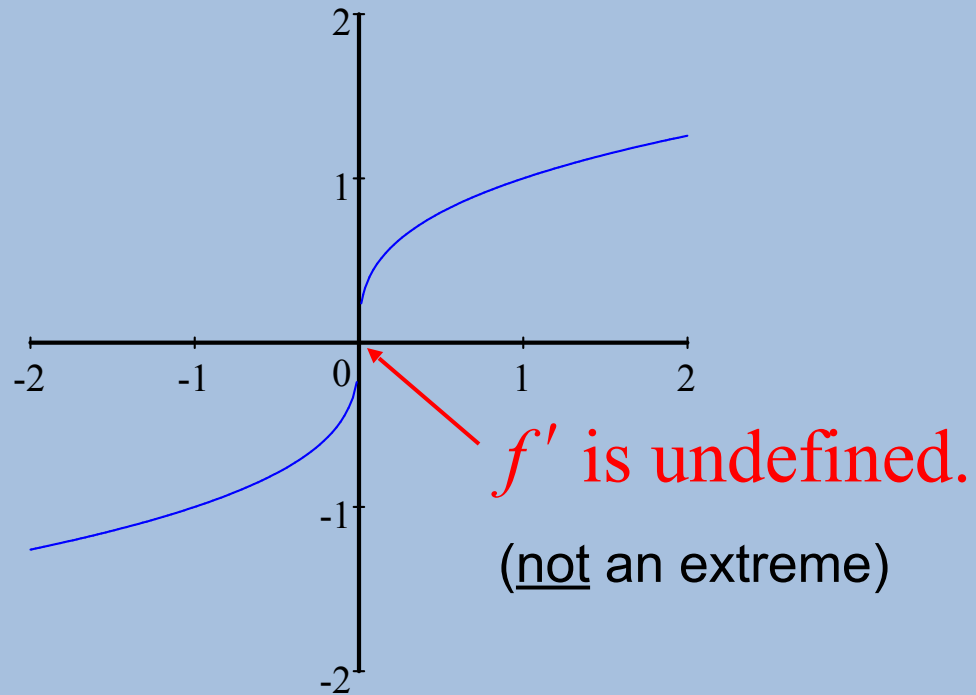
A function is differentiable at a point if the derivative of the function exists at that point.

$$\Rightarrow \quad \lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$$

$$y = x^{1/3}$$

f' is undefined.

(<u>not</u> an extreme)

p

Warnings 1, 2 and 3 show that we must be careful when using the theorem.

Warning 1 demonstrates that, even when $f'(c) = 0$,
there need not be a maximum or minimum at c.

In other words, the converse of the theorem is false in general.
If $f'(c) = 0$, it doesn't mean that c is extreme point.

Furthermore, there may be an extreme value even when
$f'(c)$ does not exist (as in Warning 2).

On the other hand, Warning 3 tells you that when
$f'(c)$ does not exist, there might be a simple vertical tangent.

The theorem does suggest that we should at least start looking
for extreme values of f at the numbers c where either:

$f'(c) = 0$

$f'(c)$ does not exist

Such points are given a special name — critical points.

Definition of Critical Point

A critical point of a function f is a point c in the domain of f such that either $f\,'(c)\,=\,0$ or $f\,'(c)$ does not exist.

These are **possible** extrema

example:

Find the critical points of $f(x) = x^{3/5}(4 - x) = 4x^{3/5} - x^{8/5}$

$$f'(x) = 4(3/5)\,x^{-2/5} - (8/5)x^{3/5}$$

critical points: $f\,'(x) = 0$ or *DNE (Does Not Exists)*

1. $f'(x)\,is\,DNE\,for\,x = 0$

2. $4(3/5)\,x^{-2/5} - (8/5)x^{3/5} = 0 \Rightarrow x^{-2/5}\,(4-3x) = 0\;x = 3/2$

the critical points are x=3/2 and x=0.

EXAMPLE FINDING ABSOLUTE EXTREMA

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2,3]$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(x) = \frac{2}{3\sqrt[3]{x}}$$

There are no values of x that will make the first derivative equal to zero.

The first derivative is undefined at $x = 0$, so (0,0) is a critical point.

To determine if this critical point is actually a maximum or minimum, we try points on either side, without passing other critical points.

$$f(0) = 0 \qquad f(-1) = 1 \qquad f(1) = 1$$

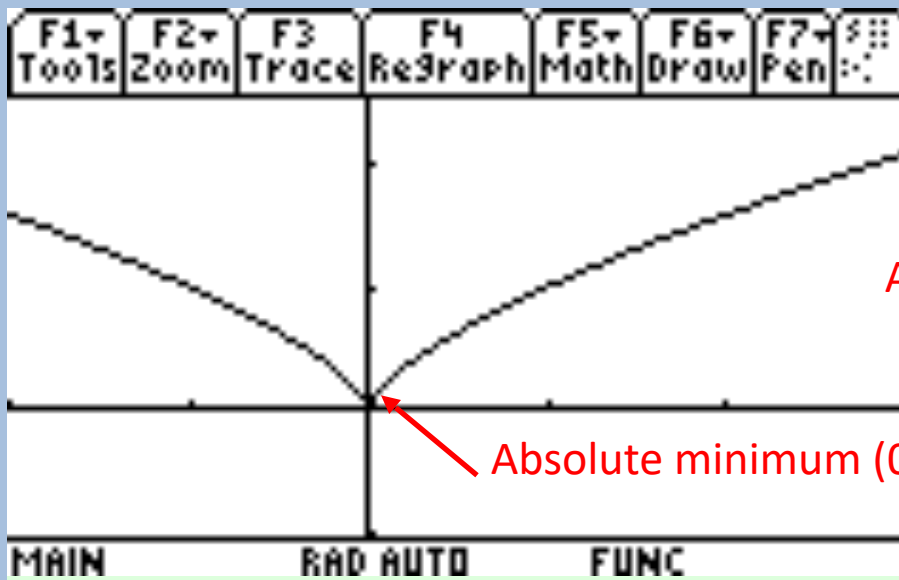Since 0<1, this must be at least a local minimum, and possibly a global minimum.

$$D = [-2,3]$$

Because the function is defined over a closed interval, we also must check the endpoints.

At: $x = -2$ $f(-2) = (-2)^{\frac{2}{3}} \approx 1.5874$

At: $x = 3$ $f(3) = (3)^{\frac{2}{3}} \approx 2.08008$

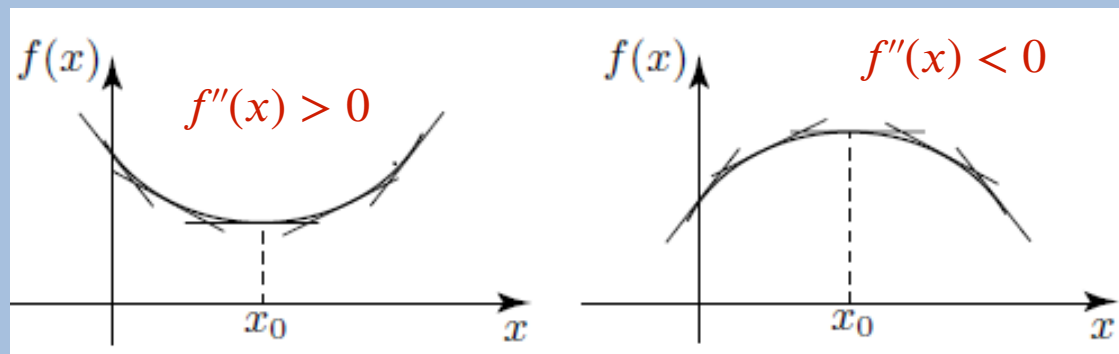Absolute minimum: $(0,0)$ Absolute maximum: $(3, 2.08)$

| F1▾ Tools | F2▾ Zoom | F3 Trace | F4 ReGraph | F5▾ Math | F6▾ Draw | F7▾ Pen | ✐ |

$f(x) = x^{2/3}$

Absolute maximum (3,2.08)

Absolute minimum (0,0)

MAIN   RAD AUTO   FUNC

Finding Maxima and Minima Analytically:

0   Check if the function is continuous on the interval

1   Find points for which $f'(x) = 0$.

nimum: $f'(x)$ *changes from negative to positive* *(concave up)*

     Maximum: $f'(x)$ *changes from positive to negative* *(concave down)*

OR

We check points on function f (x) from either side of critical point to see if the function is concave up or down.

3   For closed intervals, check the end points as well.