LAPLACE TRANSFORM

PROF. INDRAVA ROY

Lecture 2: Introduction and Motivation for Laplace Transform Part 2

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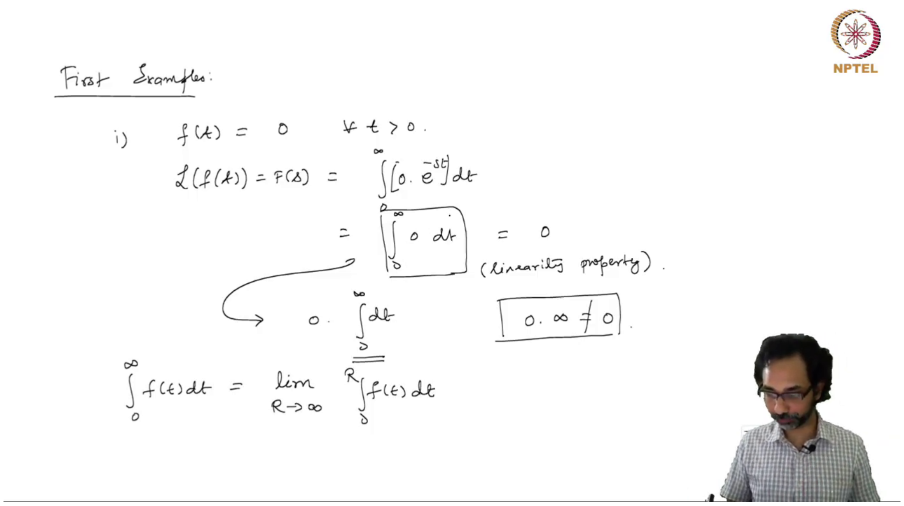

Now, let us compute some examples of Laplace Transform. So, the first example is $f(t) = 0$, for all $t > 0$. So, this is an easy example. So, here the Laplace Transform of $f(t)$ also that written as $F(s)$ this is simply $\int_0^\infty 0.e^{-st}dt = \int_0^\infty 0dt = 0$, using the linearity property of the integral.

Here, notice that we can apply the linearity property, but we have to be careful because, if we write this is as $0 \cdot \int_0^\infty dt$ then this is an undefined integral, this gives an infinite value. So, this will not be 0. I mean we cannot use $0.\infty = 0$. So, this is not a formula so, what we do here, is the following.

The any integral of this form $\int_0^\infty f(t)dt$ by definition $\int_0^\infty f(t)dt = \lim\limits_{R\to 0} \int_0^R f(t)dt$. So, now we will use this formula with the limit concept to evaluate the integral of the function 0 from 0 to ∞.

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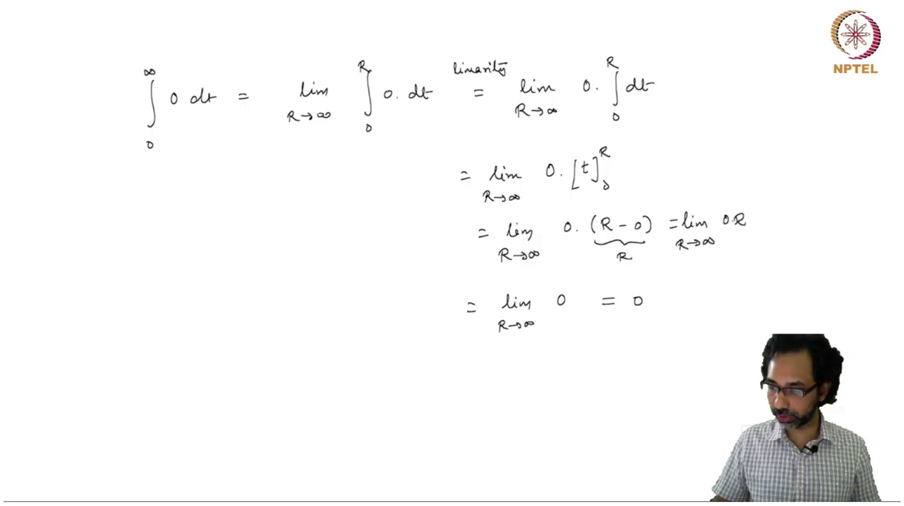
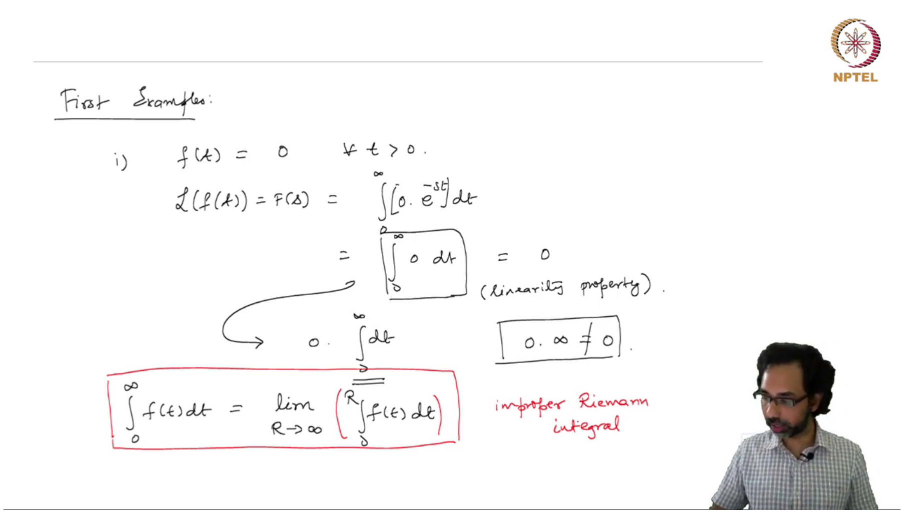


So, let us see how this is done so, we can write

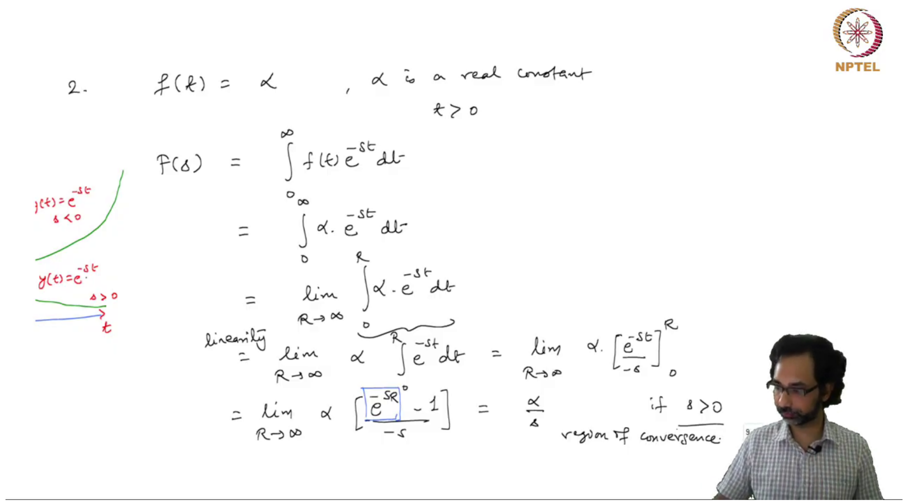
$$\int_0^\infty 0\ dt = \lim_{R\to\infty} 0 \cdot \int_0^R dt = \lim_{R\to\infty} 0 \cdot [t]_0^R$$
$$= \lim_{R\to\infty} 0 \cdot (R-0) = \lim_{R\to\infty} 0 \cdot R = \lim_{R\to\infty} 0 = 0.$$

So, this calculation can be formulized by using limits. (Refer Slide Time: 03:51)



So, one has to be careful when you have this kind of integral from 0 to ∞, it is always define by this formula using limits and this is what is called an improper Riemann integral because, the Riemann integral as you know can only be defined on intervals of finite length like, 0 to R. Here so, this constant in the red parenthesis is a Riemann integral.

But, then you have to take a limit as $R \to \infty$ and this limiting process may or may not exist and in when you have to take a limit on such unbounded domains this is called an improper Riemann integral. I will come back to this little bit later, but let us first compute some more functions. (Refer Slide Time: 05:03)



So, the second example is

$$2.\ f(t) = \alpha,\ \ \alpha \text{ is a real constant},\ \ t > 0.$$

So, again we have the Laplace Transform of this function

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \alpha e^{-st}dt.$$

Now, I am going to use again this limit formula so, this is

$$\lim_{R\to\infty} \int_0^R \alpha e^{-st}dt.$$

This limit inside the integral can be evaluated explicitly. First we use the linearity property. So, this a linearity property of integral by taking the constant α out of the integral so, we are left with

$$\lim_{R\to\infty} \alpha \int_0^R e^{-st}dt = \lim_{R\to\infty} \alpha \left[\frac{e^{-st}}{-s}\right]_0^R = \lim_{R\to\infty} \alpha \left[\frac{e^{-sR} - 1}{-s}\right] = \frac{\alpha}{s}.$$

So, if $s > 0$ then this term in the blue box this will go to 0 as $R \to \infty$ because, this is a graph of e^{-st} so, this is $y(t) = e^{-st}$, y is 1, when $t = 0$ and st goes to ∞, e^{-st} approaches 0.

So, it is helpful to keep these pictures in mind when you think about exponential function when $s > 0$ then it goes to 0 however, when $s < 0$, then you have a positive coefficient in front of t and then the graph looks like this. So, this is $y(t) = e^{-st}$, s negative and this is where s positive.

So, this is the exponential function when s is negative it goes to ∞ as $t \to \infty$ but, when s is positive it tends to 0 as $t \to \infty$. So, here in the end we will get after we evaluate this limit we will get $\frac{\alpha}{s}$. So, minus minus cancels so, we will we are just left with $\alpha.\frac{1}{s}$ or $\frac{\alpha}{s}$ provided that s is positive. So, this region of s where the Laplace Transform is well defined by well define means finite quantity is called the region of convergence of the Laplace Transform. So, this is the region of convergence.

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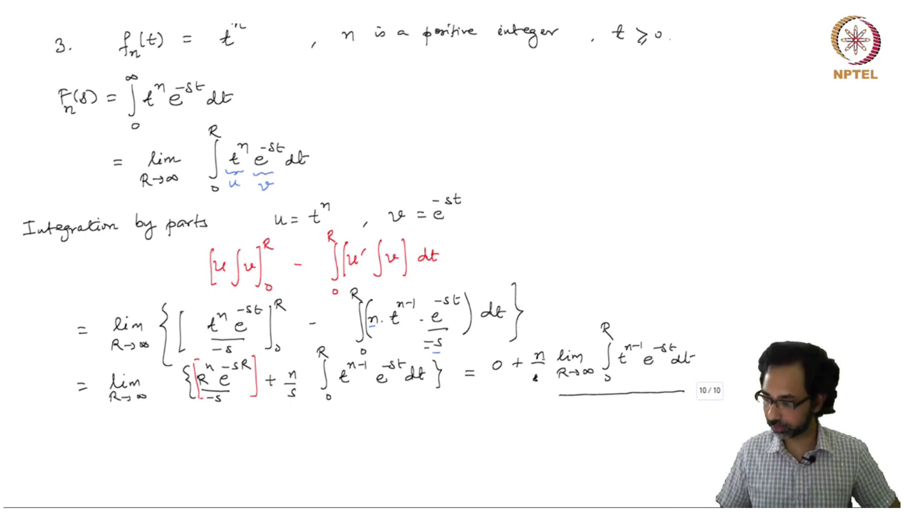

Now, let us look at another example $f(t) = t^n$ where n is a positive integer and domain of definition is $t \geq 0$. Now, let us try to compute this Laplace Transform. For notational purpose let $f_n(t) = t^n$

$$F_n(s) = \int_0^\infty t^n e^{-st} dt = \lim_{R\to\infty} \int_0^R t^n e^{-st} dt.$$

Here we can use integration by parts and you will see that integration by parts is a fundamental tool in computing the Laplace Transforms.

So, here we will have integration by parts by choosing our first function $u = t^n$ and the second function $v = e^{-st}$. So, this is our u and this is our v. Let me recall the integration by parts formula. So, this is given by

$$\int_0^R uv dt = \left[u \int_0^R v\right]_0^R - \int_0^R \left(u' \int v dt\right) dt$$

So, if you apply this formula what we will get

$$\begin{aligned}
&\lim_{R\to\infty}\left\{\left[\frac{t^n e^{-st}}{-s}\right]_0^R - \int_0^R \left(n\cdot t^{n-1}\cdot\frac{e^{-st}}{-s}\right)dt\right\}\\
&=\lim_{R\to\infty}\left\{\left[\frac{R^n e^{-sR}}{-s}\right] + \frac{n}{s}\int_0^R t^{n-1}\cdot\frac{e^{-st}}{-s}dt\right\}\\
&=0+\frac{n}{s}\lim_{R\to\infty}\int_0^R t^{n-1}\cdot\frac{e^{-st}}{-s}dt
\end{aligned}$$

Now, note at the first term $R^n e^{-st}$ when you take the limit inside limit is a linear operation so, you can take the limit term by term so, when you take this limit inside you will get 0 because, any power of n cannot go faster than exponential function again provided s is positive.

So, provided s is positive you will get 0 for this great graphic occur and now the next term so you get 0 in the first term and the next term is $\frac{n}{s}\lim\limits_{R\to\infty}\int_0^R t^{n-1}\cdot\frac{e^{-st}}{-s}dt$. Here note that this is again an improper Riemann integral but, here now we have a exponent which is less than the 1 which began with $n-1$ and we have factor of $\frac{n}{s}$.

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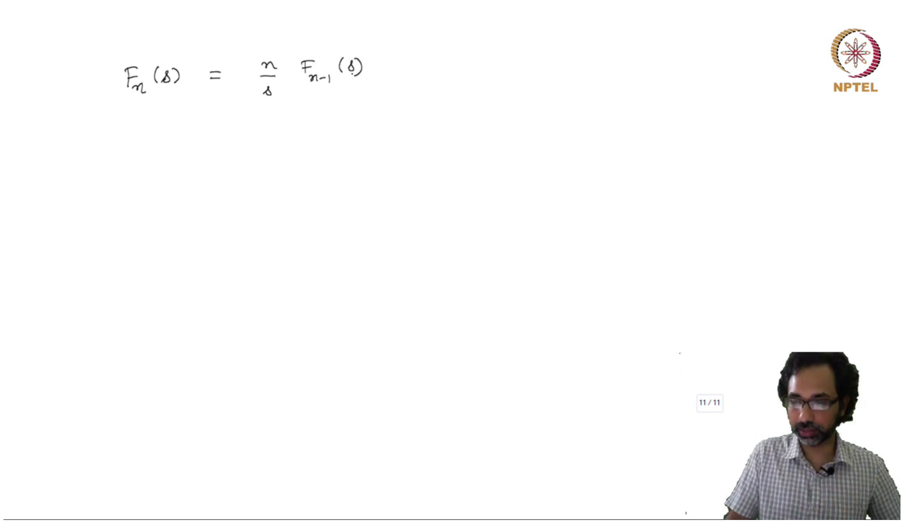

So, I am going to rewrite this as a recurrence relation. So, we started with $F_n(s) = \frac{n}{s}F_{n-1}(s)$.

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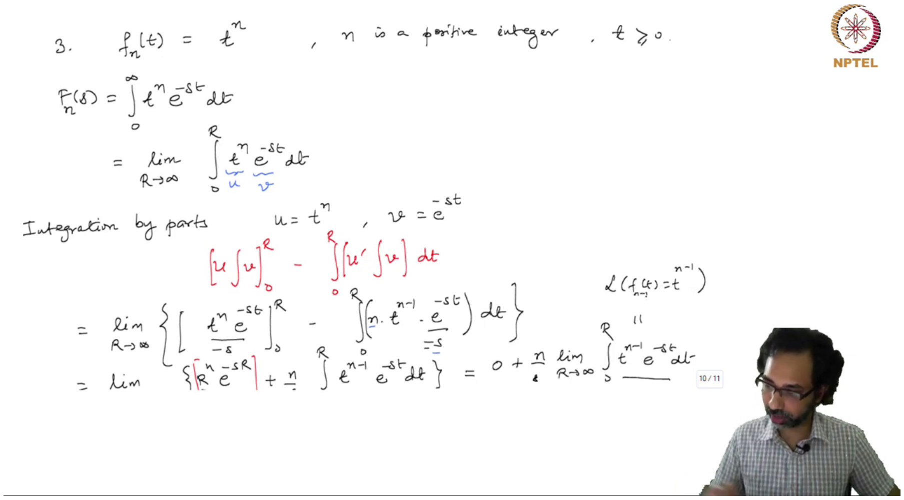

Because, note that this is the Laplace Transform of $f(t)$ which is t^{n-1} this is the Laplace Transform of F_{n-1}.

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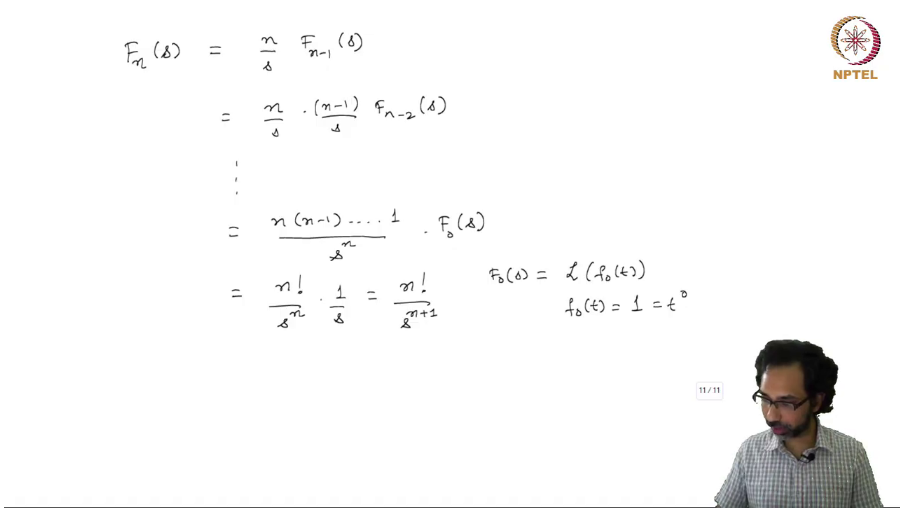

Now, we have the recurrence relation $F_n(s) = \frac{n}{s}F_{n-1}(s)$. So, we can keep substituting form various values of n and we get again and $\frac{n-1}{s}F_{n-2}(s)$. So, in this way we get $\frac{n(n-1)\cdots\cdots 1}{s^n}F_0(s)$. Now, note that numerator is just $n!F_0(s)$ and denominator is s^n where $F_0(s)$ is simply the Laplace Transform of $f_0(t)$ because, $f_0(t) = t^0 = 1$ and we have already computed the Laplace Transform of constant function. Now, this is a constant function equal to 1. So, the Laplace Transform of the constant function 1 is just $\frac{1}{s}$. So, this way we will get $\frac{n!}{s^{n+1}}$. Now, that we have seen how the limit is computed.

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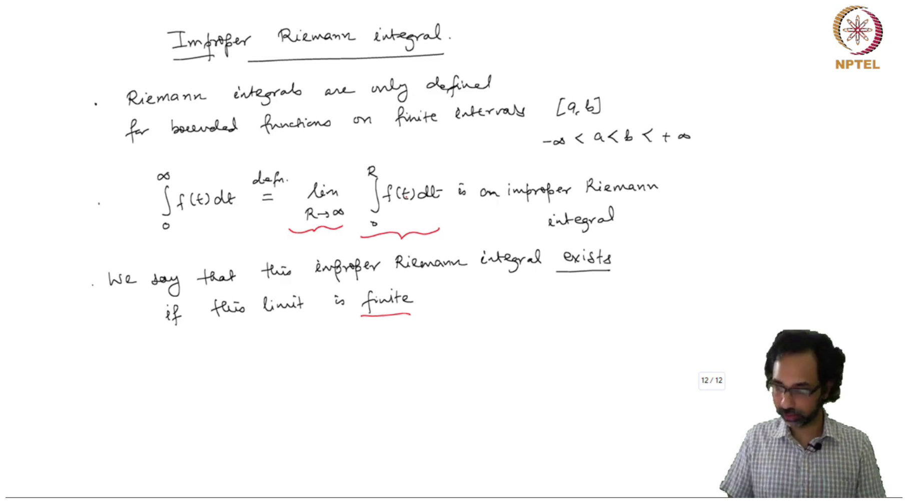

Let me go back to the notion of the improper Riemann integral so, first let me rewrite that the Riemann integrals are only defined for bounded functions on finite intervals of the form $[a, b]$ where $-\infty < a < b < +\infty$. So, a and b are finite real numbers.

Now, whenever you want to evaluate an integral of some function $f(t)$ over 0 to ∞. We define it so,

$$\int_0^\infty f(t)dt = \lim_{R\to\infty} \int_0^R f(t)dt.$$

So, first this is a improper Riemann integral and we say that this improper Riemann integral exists if this limit is finite meaning once you have evaluated this integral from 0 to R we get a function of R and then you take the limit as R goes to ∞ and after you pass to the limit you should get a finite value then we saying that the improper Riemann integral exists.

Now, it may happen that even though this integral exists for all R but, the limit may not exist. Here we implicitly assuming when we are trying to compute the limit that first of all this integral from 0 to R has to exist for this definition is concern.

So, in the next lecture we will see some examples of improper Riemann integrals and where this limit can fail to exists and we will give some sufficient conditions for this integral to exist meaning that will limit should be finite or those sufficient conditions on this functions F. So, we will see that in the next lecture. Thank you.