Laplace Transform
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Lecture 4
Properties of Laplace Transforms – I - Part 1

In the last lectures, we have computed some basic examples of Laplace Transforms for some easy to compute functions. Now, to compute more Laplace transform of more functions we will need to establish some theoretical properties of the Laplace transform which will help us in the computations.

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So in this lecture, we will do this. And today we will see some properties like, linearity property of the Laplace transform, change of scale property, and the first and second shifting properties.

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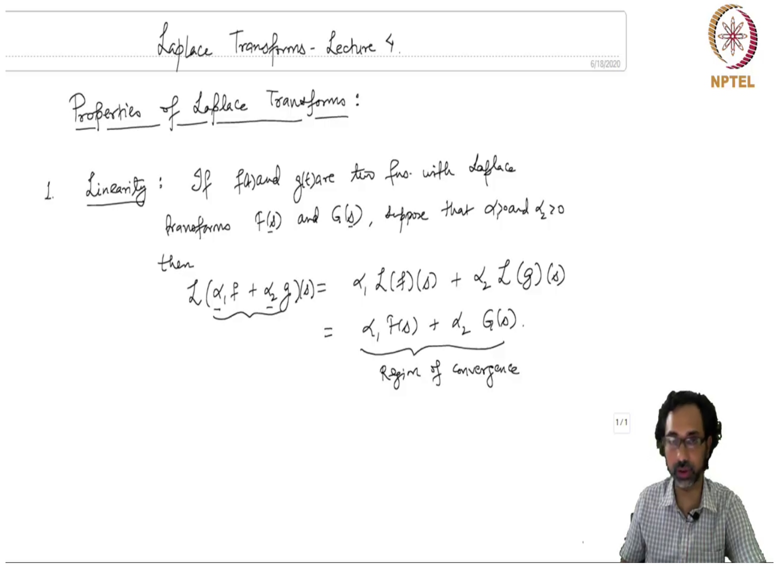

Properties of Laplace Transforms:

Let us begin with the linearity property of the Laplace transform.

1.Linearity property: Suppose we have two functions f and g with variables in t with Laplace transforms $F(s)$ and $G(s)$ respectively, that is, the Laplace transforms are variables in s and α_1 and α_2 are positive constants. Then the Laplace transform of the function $\left(\alpha_1 f + \alpha_2 g\right)$, that is

$$\mathscr{L}\left(\alpha_1 f + \alpha_2 g\right)(s) = \alpha_1 \mathscr{L}(f)(s) + \alpha_2 \mathscr{L}(g)(s)$$

$$= \alpha_1 F(s) + \alpha_2 G(s).$$

In other words, when you multiply your functions with constants, those constants can be taken out of the Laplace transform, and when you add two functions, the Laplace transform of the added function is the sum of the Laplace transforms of the individual functions. This is the linearity property of the Laplace transform.

Here, we should mention what is the region of convergence for the Laplace transform of this function, that is what is the region of convergence of $\alpha_1 F(s) + \alpha_2 G(s)$.

Remark: Whenever we compute a Laplace transform, we have to specify the region of convergence as well, otherwise the answer is deemed incomplete.

So, let us try to understand what is the region of convergence of this function $\alpha_1 F(s) + \alpha_2 G(s)$.

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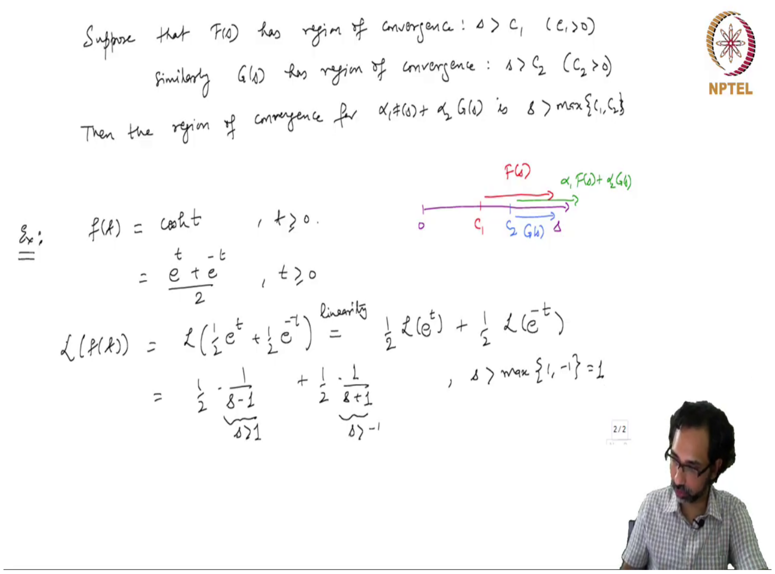

Suppose $F(s)$ has region of convergence $s > c_1$ for some constant $c_1 > 0$. Similarly suppose, $G(s)$ has the region of convergence given by $s > c_2$ for some constant $c_2 > 0$. Then, the region of convergence for $\alpha_1 F(s) + \alpha_2 G(s)$ is given by the region $s > \max\{c_1, c_2\}$.

So, as in the above picture, we have the s line and the constant c_1 on it. The region of the red arrow describes the region of convergence of $F(s)$. Similarly, we have the constant c_2 and the region of the blue arrow describes the region of convergence of $G(s)$. Then the maximum of $\{c_1, c_2\}$ will be the region of convergence of $\alpha_1 F(s) + \alpha_2 G(s)$ and in this case the region of the green arrow describes this region.

So in general, whenever you have two regions of convergence for functions, f and g, you can take the intersection of this two and you will get the region of convergence for the sum. So, let us try to compute an example using this property.

So let us suppose that, $f(t)$ is equal to the hyperbolic cosine function, $cosht$. Recall that,

$cosht = \frac{e^t + e^{-t}}{2}$, for $t \geq 0$. So, the Laplace transform of this function $f(t)$ will be,

$\mathscr{L}(f(t)) = \mathscr{L}\{\frac{1}{2}e^t + \frac{1}{2}e^{-t}\}$.

Using linearity property, we get,

$$\mathscr{L}(f(t)) = \frac{1}{2}\mathscr{L}\{e^{t}\} + \frac{1}{2}\mathscr{L}\{e^{-t}\}$$

$$= \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}.$$

Now, $\frac{1}{s-1}$ has the region of convergence $s > 1$ and $\frac{1}{s+1}$ has the region of convergence $s > -1$. So the sum is defined in the region $s > \max\{1, -1\}$ that is $s > 1$.

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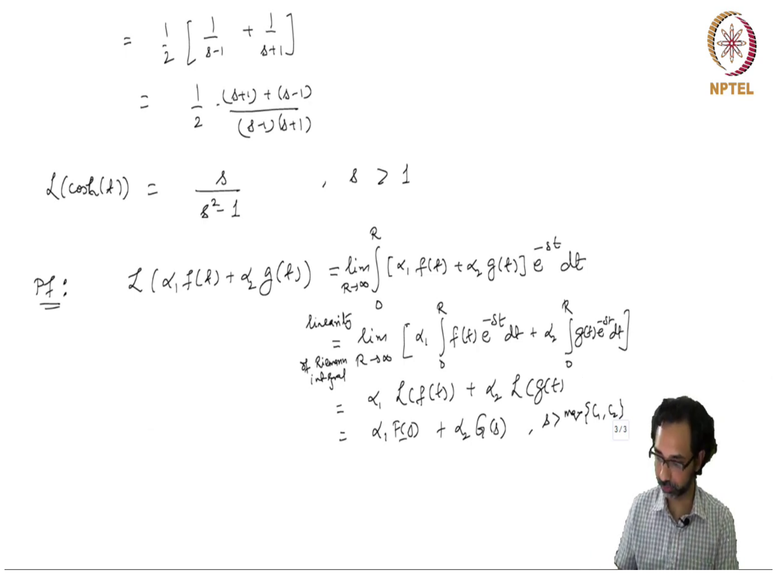

So, let us try to compute this expression.

$$\mathscr{L}(cosht) = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) = \frac{s}{s^2-1},$$ and this is valid for $s > 1$.

So we see that, this linearity property helps us compute the Laplace transform of more functions. So let us try to give a very easy proof for the linearity property.

Proof: So we try to compute the Laplace transform of $\left(\alpha_1 f(t) + \alpha_2 g(t)\right)$, and this is by definition. So , using the definition of the improper Riemann integral we have,

$$\mathscr{L}\left(\alpha_1 f(t) + \alpha_2 g(t)\right) = \lim_{R\to\infty}\int_0^R \left[\alpha_1 f(t) + \alpha_2 g(t)\right] e^{-st}\,dt$$

$$= \lim_{R \to \infty} \left[\alpha_1 \int_0^R f(t) e^{-st} dt + \alpha_2 \int_0^R g(t) e^{-st} dt \right]$$

$$= \alpha_1 \mathcal{L}(f(t)) + \alpha_2 \mathcal{L}(g(t))$$

$$= \alpha_1 F(s) + \alpha_2 G(s), \text{ for } s > \max\{c_1, c_2\}.$$

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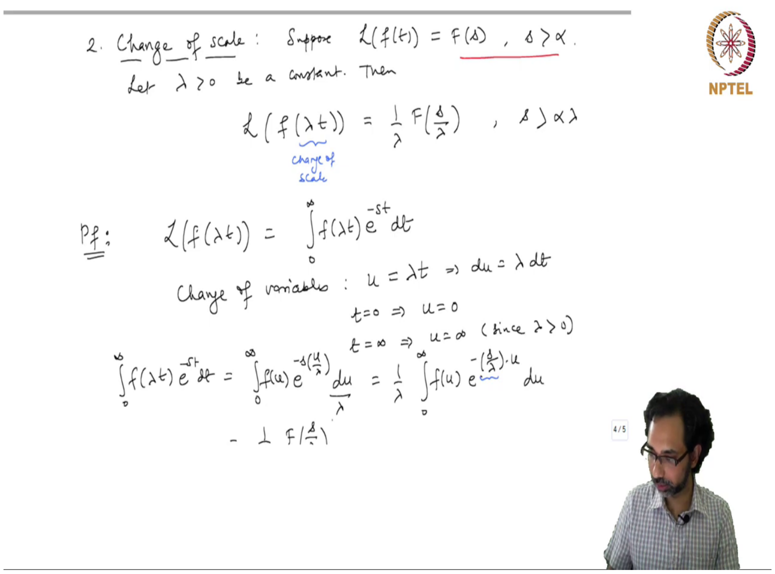

Now, let us go to the second property, which is the change of scale.

2.Change of Scale: The change of scale property says that, given a function $f(t)$ with the Laplace transform $F(s)$ for $s > \alpha$, and given a constant $\lambda > 0$, the Laplace transform of

$$\mathcal{L}\left(f(\lambda t)\right) = \frac{1}{\lambda} F\left(\frac{s}{\lambda}\right), \text{ such that } s > \alpha\lambda.$$

So this is like a scaling, that is why it is called change of scale, and we are changing the scale of the domain.

So let us try to give a proof for this result.

Proof: So, by definition of the Laplace transform of $f(t)$ we have given that,

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t) e^{-st} dt.$$

One can use improper Riemann integrals and write the limits, etc. Since now it is clear what it should be, I am just writing 0 infinity for shorthand notation.

So to compute the Laplace transform of $f(\lambda t)$, we have,

$$\mathscr{L}\left(f(\lambda t)\right) = \int_0^\infty f(\lambda t)e^{-st}dt.$$

So we can make a change of variables, and we put $u = \lambda t$. This means, $du = \lambda dt$. So now let us see what are the limits of the integration. When t equal to 0, this means that u equal to 0 because λ is a positive number, and when t equals infinity, this also implies u equals plus infinity. So again, the limits of integration are from 0 to infinity.

So we get, $$\mathscr{L}\left(f(\lambda t)\right) = \frac{1}{\lambda}\int_0^\infty f(u)e^{\frac{-su}{\lambda}}du = \frac{1}{\lambda}F\left(\frac{s}{\lambda}\right).$$

Now, the only thing that remains is to show that the region of convergence is $s > \alpha\lambda$, but this is clear. Note that, original function $F(s)$ was defined in the region $s > \alpha$. So, since we are changing the scale, here we should put $\frac{s}{\lambda} > \alpha$. And this is equivalent of saying that, $s > \alpha\lambda$. So, we have shown the change of scale property.