PARTIAL DERIVATIVES

3.3
Partial Derivatives

In this section, we will learn about:

Various aspects of partial derivatives.

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

- Then, we are really considering a function of a single variable x:

$$g(x) = f(x, b)$$

PARTIAL DERIVATIVE

If g has a derivative at a, we call it the partial derivative of f with respect to x at (a, b).

We denote it by:

$$f_x(a, b)$$

 Equation 1

Thus,

$$f_x(a, b) = g'(a)$$

where $g(x) = f(a, b)$

By the definition of a derivative, we have:

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

Equation 2

So, Equation 1 becomes:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

PARTIAL DERIVATIVE

Similarly, the partial derivative of f with respect to y at (a, b), denoted by $f_y(a, b)$, is obtained by:

- Keeping x fixed ($x = a$)
- Finding the ordinary derivative at b of the function $G(y) = f(a, y)$

Equation 3

Thus,

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

Equations 4

If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

NOTATIONS

There are many alternative notations for partial derivatives.

- For instance, instead of f_x, we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the first variable) or $\partial f/\partial x$.
- However, here, $\partial f/\partial x$ can't be interpreted as a ratio of differentials.

NOTATIONS FOR PARTIAL DERIVATIVES

If $z = f(x, y)$, we write:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x}$$

$$= f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}$$

$$= f_2 = D_2 f = D_y f$$

To compute partial derivatives, all we have to do is:

- Remember from Equation 1 that the partial derivative with respect to x is just the ordinary derivative of the function g of a single variable that we get by keeping y fixed.

RULE TO FIND PARTIAL DERIVATIVES OF $z = f(x, y)$

Thus, we have this rule.

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 1

If

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

find

$$f_x(2, 1) \text{ and } f_y(2, 1)$$

Holding y constant and differentiating with respect to x, we get:

$$f_x(x, y) = 3x^2 + 2xy^3$$

- Thus,

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3$$
$$= 16$$

Holding x constant and differentiating with respect to y, we get:

$$f_y(x, y) = 3x^2y^2 - 4y$$

- Thus,

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1$$
$$= 8$$

Example 2

If

$$f(x, y) = 4 - x^2 - 2y^2$$

find $f_x(1, 1)$ and $f_y(1, 1)$ and

interpret these numbers as slopes.

We have:

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

If

$$f(x, y) = \sin\left(\frac{x}{1+y}\right)$$

calculate

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

Example 3

Using the Chain Rule for functions of one variable, we have:

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

Example 4

Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Example 4

To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

- Solving for $\partial z/\partial x$, we obtain:

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to *y* gives:

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Partial derivatives can also be defined for functions of three or more variables.

- For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

It is found by:

- Regarding y and z as constants.
- Differentiating $f(x, y, z)$ with respect to x.

FUNCTIONS OF MORE THAN TWO VARIABLES

If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed.

- However, we can't interpret it geometrically since the graph of f lies in four-dimensional space.

FUNCTIONS OF MORE THAN TWO VARIABLES

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the i th variable x_i is:

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

Then, we also write:

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$

- Holding y and z constant and differentiating with respect to x, we have:

$$f_x = ye^{xy} \ln z$$

- Similarly,

$$f_y = xe^{xy} \ln z \qquad f_z = e^{xy}/z$$

HIGHER DERIVATIVES

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.

SECOND PARTIAL DERIVATIVES

So, we can consider their partial derivatives

$$(f_x)_x\,,\ (f_x)_y\,,\ (f_y)_x\,,\ (f_y)_y$$

These are called the second partial derivatives of f.

NOTATION

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

SECOND PARTIAL DERIVATIVES

Thus, the notation f_{xy} (or $\partial^2 f/\partial y \partial x$) means that we first differentiate with respect to x and then with respect to y.

In computing f_{yx}, the order is reversed.

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

- In Example 1, we found that:

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

- Hence,

$$f_{xx} = \frac{\partial}{\partial x}\left(3x^2 + 2xy^3\right) = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y}\left(3x^2 + 2xy^3\right) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}\left(3x^2y^2 - 4y\right) = 6x^2y - 4$$

Notice that $f_{xy} = f_{yx}$ in Example 6.

- This is not just a coincidence.
- It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

-

CLAIRAUT'S THEOREM

Suppose f is defined on a disk D that contains the point (a, b).

If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Partial derivatives of order 3 or higher can also be defined.

- For instance, $$f_{xyy} = \left(f_{xy}\right)_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x}$$
- Using Clairaut's Theorem, it can be shown that

$$f_{xyy} = f_{yxy} = f_{yyx}$$

if these functions are continuous.

Example 7

Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$

- $f_x = 3\cos(3x + yz)$
- $f_{xx} = -9\sin(3x + yz)$
- $f_{xxy} = -9z\cos(3x + yz)$
- $f_{xxyz} = -9\cos(3x + yz) + 9yz\sin(3x + yz)$

PARTIAL DIFFERENTIAL EQUATIONS

Partial derivatives occur in partial differential equations that express certain physical laws.

LAPLACE'S EQUATION

For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called Laplace's equation after

Pierre Laplace (1749–1827).

Solutions of this equation are called harmonic functions.

- They play a role in problems of heat conduction, fluid flow, and electric potential.

 Example 8

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

- $u_x = e^x \sin y$
- $u_y = e^x \cos y$
- $u_{xx} = e^x \sin y$
- $u_{yy} = -e^x \sin y$
- $u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$

- Thus, u satisfies Laplace's equation.

WAVE EQUATION

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform.

- This could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

- $u_x = \cos(x - at)$
- $u_{xx} = -\sin(x - at)$
- $u_t = -a \cos(x - at)$
- $u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}$

- So, u satisfies the wave equation.