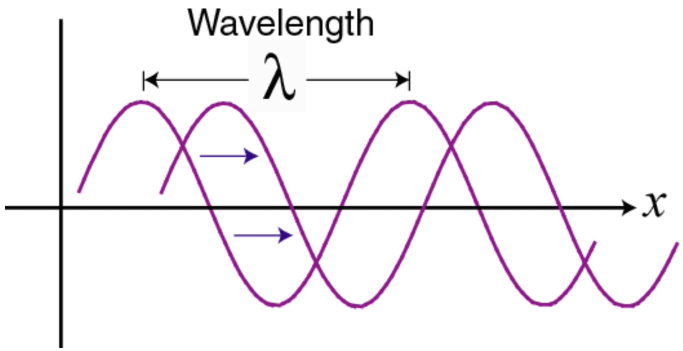
The Phase Velocity

How fast is the wave traveling?

Velocity is a reference distance divided by a reference time.



The phase velocity is the wavelength / period: $v = \lambda / \tau$

Since $f = 1/\tau$:

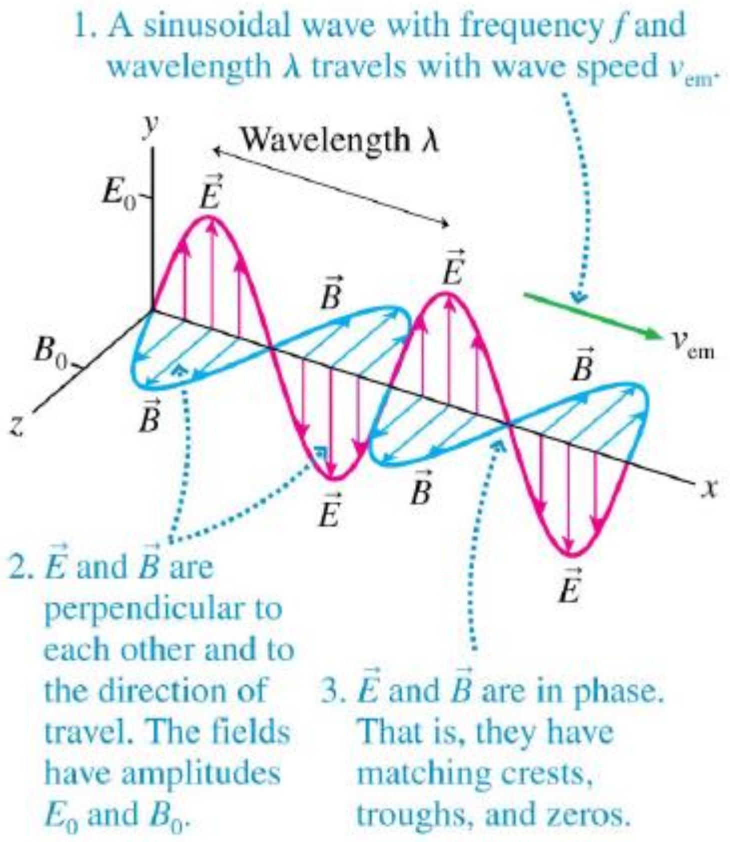
$$\boxed{v = \lambda f}$$

In terms of k, $k = 2\pi / \lambda$, and
the angular frequency, $\omega = 2\pi / \tau$, this is:

$$\boxed{v = \omega / k}$$

The propagation direction of a light wave

FIGURE 35.19 A sinusoidal electromagnetic wave.



$$\vec{v} = \vec{E} \times \vec{B}$$

Right-hand screw rule

The solution of electric filed wave equation is

$$E(z,t) = E_o \cos(\omega t - kz + \phi_0)$$

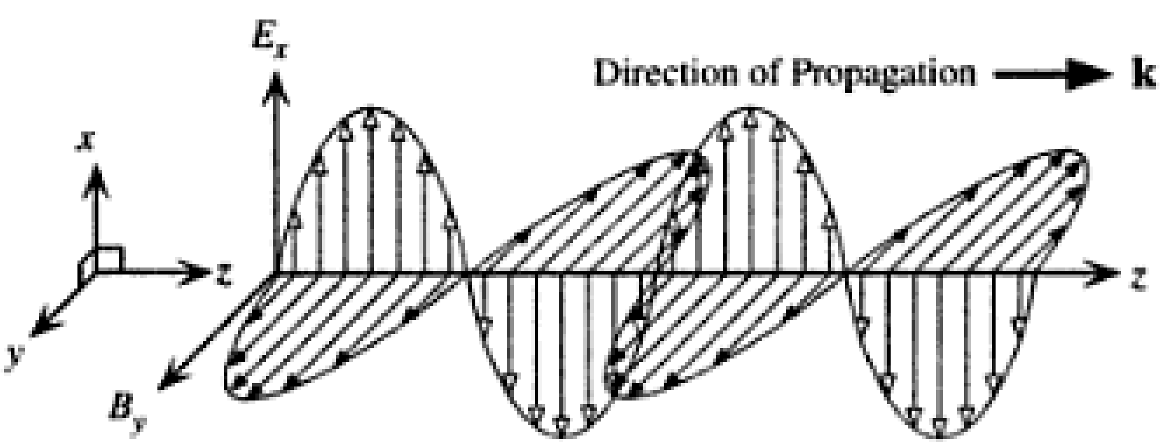
phase

Amplitud e

Angular frequency

Propagation constant or wave number

Phase constant

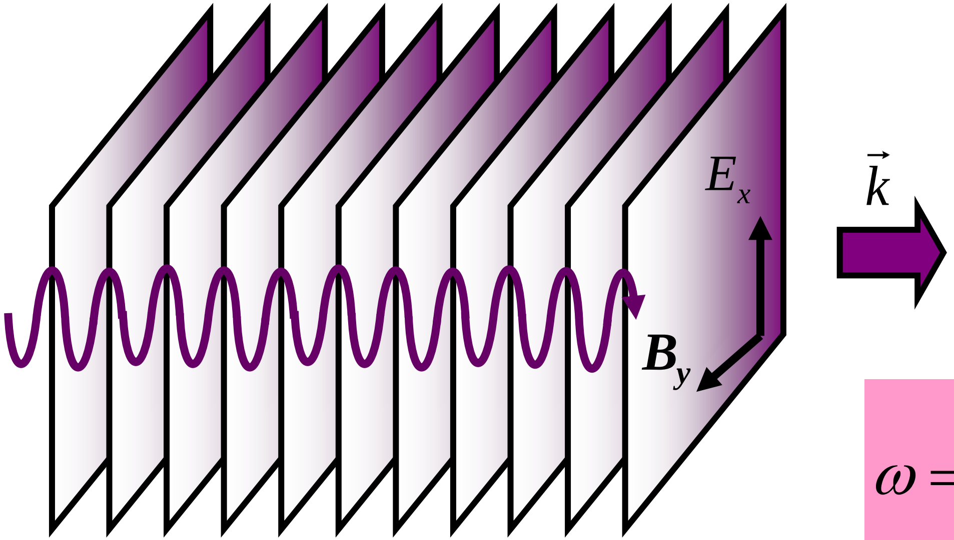


$$E(z,t) = E_o \cos(\omega t - kz + \phi_0)$$

It represents a monochromatic plane wave of infinite extent traveling in the positive z direction.

Wave front: any plane perpendicular to the direction of propagation (z axis), the phase of the wave is constant. A surface over which the phase of the wave is constant is known as wave front.

The velocity with which the constant phase moves is called phase velocity



We can get the expression for the phase velocity as follows

$$\omega dt - kdz = 0$$

Phase velocity,

$$V_p = \frac{dz}{dt} = \frac{\omega}{k}$$

$$\omega = \left| \frac{\partial \phi(z,t)}{\partial t} \right|$$

$$k = \left| \frac{\partial \phi(z,t)}{\partial z} \right|$$

Group velocity

Consider the superposition of two monochromatic waves with slightly different angular frequency

$$\omega_0 + \Delta\omega \text{ and } \omega_0 - \Delta\omega$$

Corresponding wave vectors

$$k_0 + \Delta k \text{ and } k_0 - \Delta k$$

Resulting signal

$$E_g = E_0 \cos\left[\left(\omega_0 + \Delta\omega\right)t - \left(k_0 + \Delta k\right)z\right] + E_0 \cos\left[\left(\omega_0 - \Delta\omega\right)t - \left(k_0 - \Delta k\right)z\right]$$

$$\cos[A] + \cos[B] = 2\cos\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right]$$

$$\cos\left[\left(\omega_0+\Delta\omega\right)t-\left(k_0+\Delta k\right)z\right]+\cos\left[\left(\omega_0-\Delta\omega\right)t-\left(k_0-\Delta k\right)z\right]$$

$$=2\cos\left[\frac{\left(\omega_0+\Delta\omega\right)t-\left(k_0+\Delta k\right)z+\left(\omega_0-\Delta\omega\right)t-\left(k_0-\Delta k\right)z}{2}\right]$$

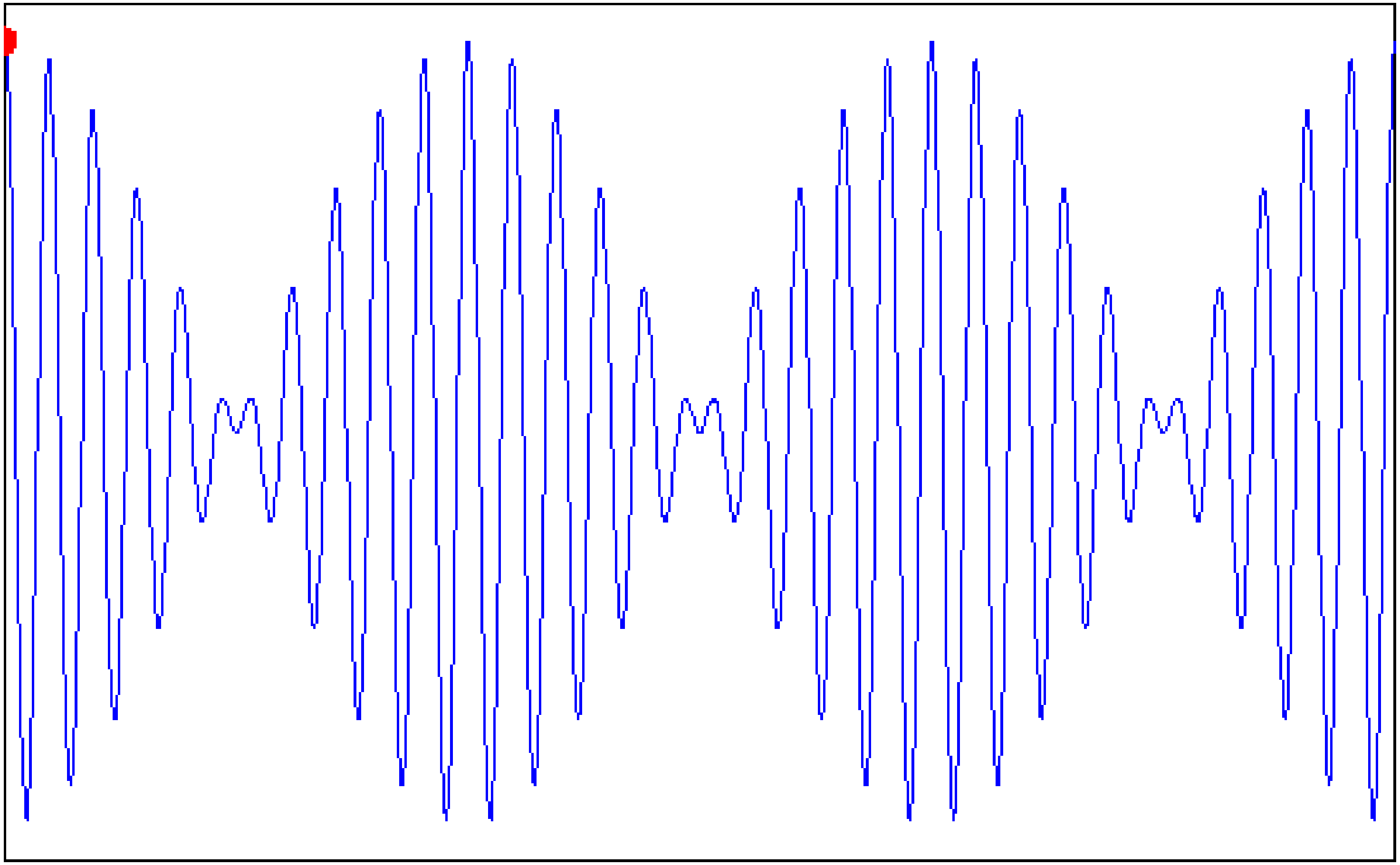
$$\times\cos\left[\frac{\left(\omega_0+\Delta\omega\right)t-\left(k_0+\Delta k\right)z-\left(\omega_0-\Delta\omega\right)t+\left(k_0-\Delta k\right)z}{2}\right]$$

$$=2\cos\left[\frac{2\left(\omega_0 t-k_0 z\right)}{2}\right]\times\cos\left[\frac{2\left(\Delta\omega t-\Delta k z\right)}{2}\right]$$

$$=2\cos\left[\omega_0 t-k_0 z\right]\ \cos\left[\Delta\omega t-\Delta k z\right]$$

It is considered as a wave of frequency ω_0. its amplitude is modulated by the function cos($\Delta\omega t$-Δkz) having angular frequency $\Delta\omega$ and phase constant Δk.

The Group Velocity

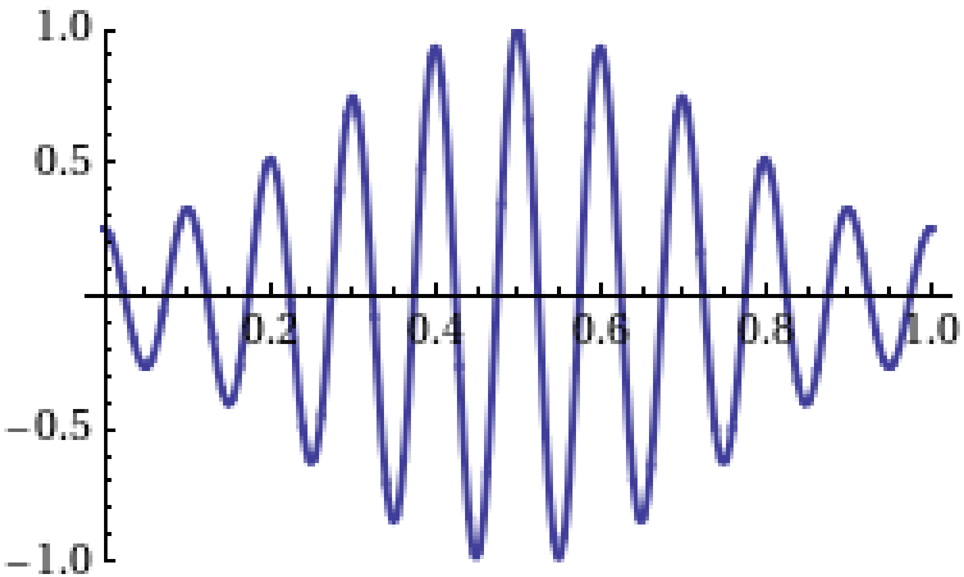


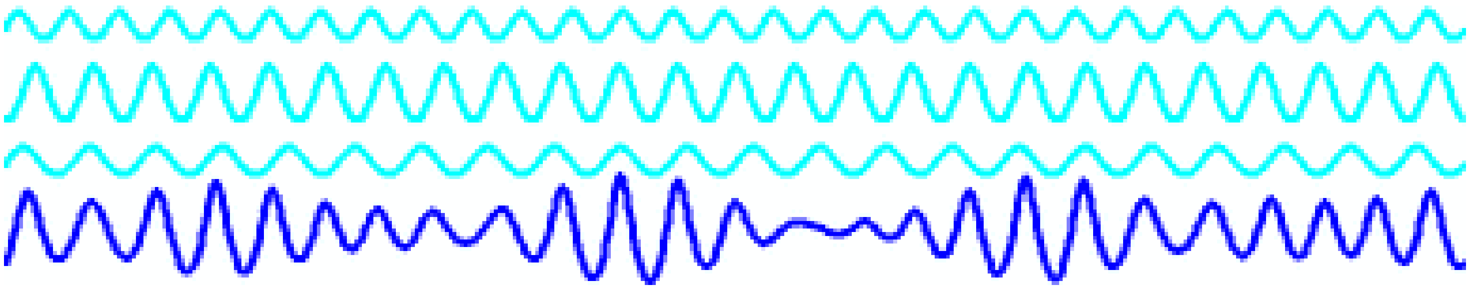
This is the velocity at which the overall shape of the wave's amplitudes, or the wave 'envelope', propagates. (= *signal velocity*)

Here, phase velocity = group velocity (the medium is *non-dispersive*)

Group velocity

Consider the superposition of two monochromatic wave with slightly different angular frequency

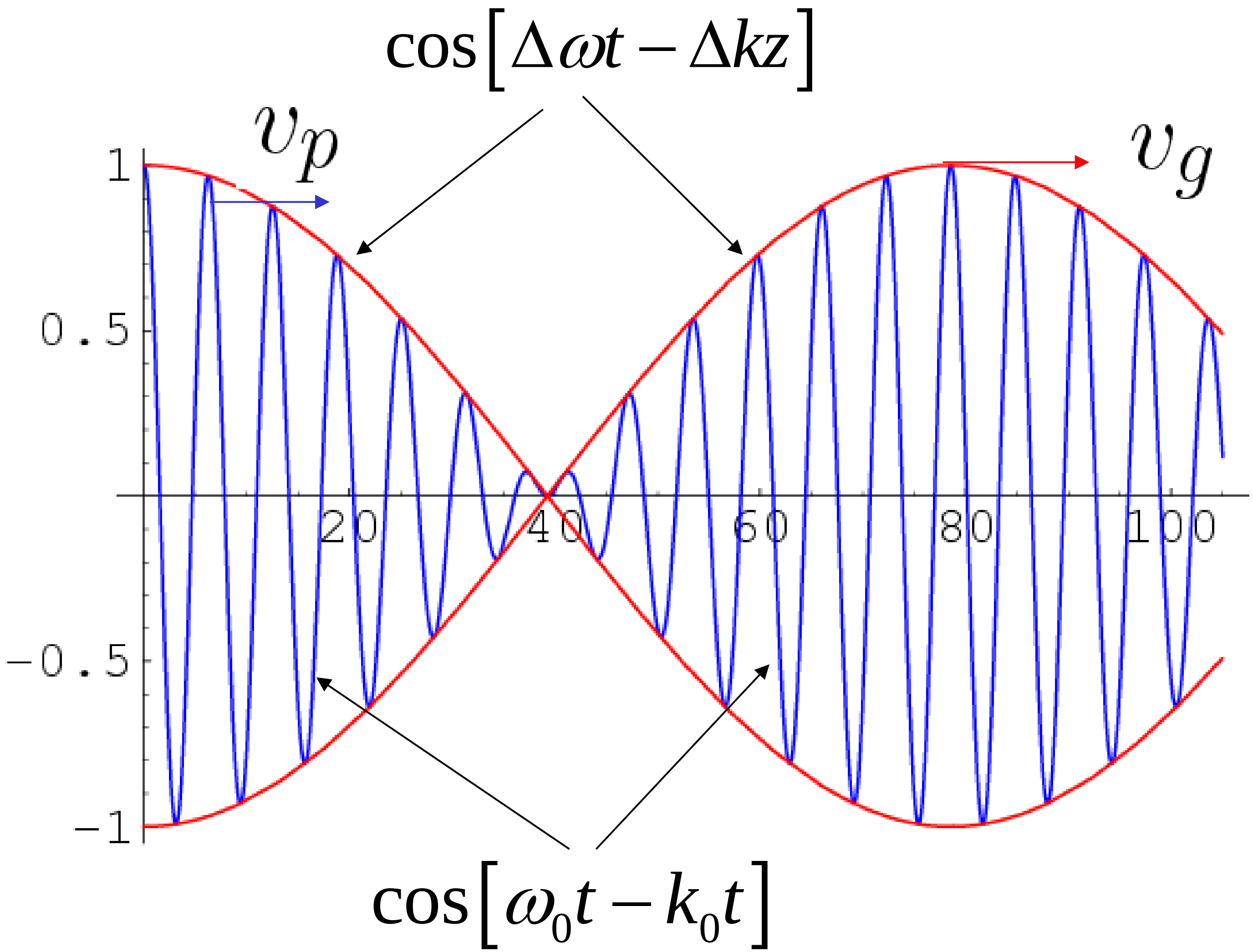




$$E_g = 2\cos\left[\omega_0 t - k_0 t\right]\ \cos\left[\Delta\omega t - \Delta k z\right]$$

The velocity with which the modulated function moves in space is called group velocity.

$$V_g = \frac{d\omega}{dk}$$



Group Index

Refractive index is defined as

$$n = \frac{Velocity\ of\ light_{vacuum}}{Velocity\ of\ light_{medium}}$$

$$n = \frac{c}{V_p} \qquad V_p = \frac{c}{n} = \frac{\omega}{k}$$

$$ck = n\omega$$

$$c\frac{d}{d\omega}(k) = \frac{d}{d\omega}(n\omega)$$

$$\frac{dk}{d\omega} = \frac{1}{c}\frac{d}{d\omega}(n\omega)$$

$$Group\ velocity, V_g = \frac{d\omega}{dk}$$

$$\frac{1}{V_g} = \frac{dk}{d\omega}$$

$$\frac{1}{V_g} = \frac{dk}{d\omega} = \frac{1}{c}\frac{d}{d\omega}(n\omega)$$

$$\frac{1}{V_g} = \frac{dk}{d\omega} = \frac{1}{c}\left[n\frac{d\omega}{d\omega} + \omega\frac{dn}{d\omega}\right]$$

$$\frac{1}{V_g} = \frac{dk}{d\omega} = \frac{1}{c}\left[n + \omega\frac{dn}{d\omega}\right]$$

$$\frac{1}{V_g} = \frac{dk}{d\omega} = \frac{1}{c}\left[n + \omega\frac{dn}{d\omega}\right]$$

$$\frac{1}{V_g} = \frac{dk}{d\omega} = \frac{\left[n + \omega\frac{dn}{d\omega}\right]}{c} = \frac{n_g}{c}$$

$$n_g = n + \omega\frac{dn}{d\omega}$$

Where *n_g* – Group Index

In terms of wavelength Group Index is written as

$$\omega\frac{dn}{d\omega} = \frac{2\pi c}{\lambda}\frac{dn}{d\left(\frac{2\pi c}{\lambda}\right)}$$

$$= \frac{\cancel{(2\pi c)}}{\lambda}\frac{dn}{\cancel{(2\pi c)}d\left(\lambda^{-1}\right)} = \frac{1}{\lambda}\frac{dn}{d\left(\lambda^{-1}\right)}$$

$$\omega\frac{dn}{d\omega} = \frac{1}{\lambda}\frac{dn}{(-)\lambda^{-2}d\lambda}$$

$$\omega\frac{dn}{d\omega} = -\lambda\frac{dn}{d\lambda}$$

$$n_g = n - \lambda\frac{dn}{d\lambda}$$