Problem.

Continuous random variables.

The joint pdf of a two dimensional random variable (X, Y) is given by

$$f(x,y) = xy^2 + \frac{x^2}{8}, \quad 0 \le x \le 2, \; 0 \le y \le 1$$

Compute

i) $P(X>1)$

ii) $P(Y<1/2)$

iii) $P(X>1 / Y<1/2)$

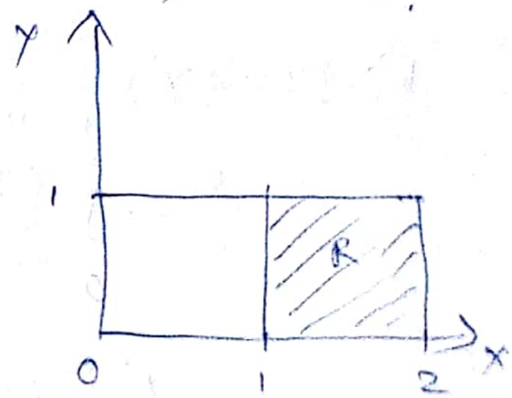
iv) $P(Y<1/2 / X>1)$

v) $P(X<Y)$

vi) $P(X+Y \le 1)$

Solution:- The given range is $0 \le x \le 2$, $0 \le y \le 1$.

i) $$P(X>1) = \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$



$$= \int_0^1 \left(y^2 \frac{x^2}{2} + \frac{x^3}{8 \times 3}\right) dy$$

$$= \int_0^1 \left(y^2 \cdot \frac{x^2}{2} + \frac{x^3}{24}\right)_1^2 dy$$

$P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\, dx\, dy$

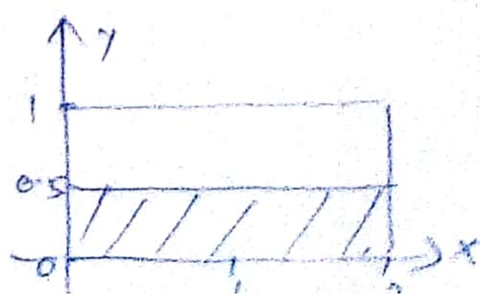
$$= \int_0^1 \left(\frac{y^2}{2}(4-1) + \frac{1}{24}(8-1)\right) dy$$

$$= \int_0^1 \left(\frac{3}{2} y^2 + \frac{7}{24}\right) dy = 3/2 \left(\frac{y^3}{3}\right)_0^1 + \frac{7}{24}(y)_0^1$$

$$= 3/2\,(1/3) + \frac{7}{24}(1) = 1/2 + \frac{7}{24} = \frac{12+7}{24} = \frac{19}{24}$$

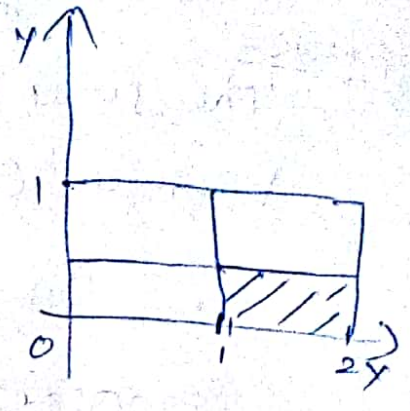
ii) $$P(Y<1/2) = \int_0^{1/2} \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^{1/2} \left(2y^2 + 1/3\right) dy = \frac{1}{4}$$



iii) $P(X>1, Y<\frac{1}{2}) = \int_0^{1/2}\int_1^2 \left(xy^2+\frac{x^2}{8}\right)dx\,dy$

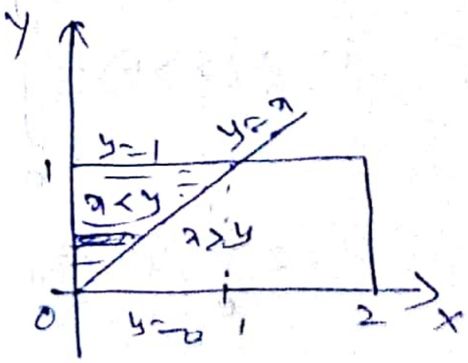
$= \frac{5}{24}.$



$P(X>1/Y<\frac{1}{2}) = \frac{P(X>1, Y<\frac{1}{2})}{P(Y<\frac{1}{2})} = \frac{5/24}{1/4} = \frac{5}{6}$ //

iv) $P(Y<\frac{1}{2}/X>1) = \frac{P(X>1, Y<\frac{1}{2})}{P(X>1)} = \frac{5/24}{19/24} = \frac{5}{19}$ //

v) $P(X<Y)$



$= \int_0^1\int_0^y \left(xy^2+\frac{x^2}{8}\right)dx\,dy$

$= \int_0^1 \left(y^2\left(\frac{x^2}{2}\right)_0^y + \frac{1}{8}\left(\frac{x^3}{3}\right)_0^y\right)dy$

$= \int_0^1 \left(\frac{y^4}{2} + \frac{1}{24}(y^3)\right)dy$

$= \frac{1}{2}\left(\frac{y^5}{5}\right)_0^1 + \frac{1}{24}\left(\frac{y^4}{4}\right)_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{53}{480}$

vi) $P(X+Y\leq 1) = \int_0^1\int_0^{1-y} \left(xy^2+\frac{x^2}{8}\right)dx\,dy$

$= \frac{13}{480}$