Problem 5·32. *The joint probability distribution of two random variables X and Y is given by : $P(X = 0, Y = 1) = \frac{1}{3}$, $P(X = 1, Y = -1)\ \frac{1}{3}$, and $P(X = 1, Y = 1) = \frac{1}{3}$.*

Find (i) Marginal distributions of X and Y, and (ii) the conditional probability distribution of X given Y = 1.

Solution. $P(X = -1)$

$= \sum_y P(X = -1, Y = y)$

$= P(X = -1, Y = -1)$

$+ P(X = -1, Y = 0)$

$+ P(X = -1, Y = 1) = 0$

Similarly $P(X = 0) = \frac{1}{3}$

and $P(X = 1) = \frac{2}{3}$

Y \ X	−1	0	1	Marginal Y
−1	0	0	$\frac{1}{3}$	$\frac{1}{3}$
0	0	0	0	0
1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Marginal (X)	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Thus

Marginal distribution of X is :

Values of X, x :	−1	0	1
P (X = x) :	0	$\frac{1}{3}$	$\frac{2}{3}$

Marginal distribution of Y is :

Values of Y, y :	−1	0	1
P(Y = y) :	$\frac{1}{3}$	0	$\frac{2}{3}$

(*ii*) The conditional probability distribution of X given Y is :

$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$. Now

$P(X = -1 \mid Y = 1) = \frac{P(X = -1, Y = 1)}{P(Y = 1)} = 0$, $P(X = 0 \mid Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{1/3}{2/3} = \frac{1}{2}$

$P(X = 1 \mid Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{1/3}{2/3} = \frac{1}{2}$

Thus the conditional distribution of X given Y = 1 is :

Values of X = x	−1	0	1
P(X = x ∣ Y = 1)	0	$\frac{1}{2}$	$\frac{1}{2}$

Example 5·33. *For the adjoining bivariate probability distribution of X and Y, find :*

(i) $P(X \le 1, Y = 2)$,

(ii) $P(X \le 1)$,

(iii) $P(Y \le 3)$, *and*

(iv) $P(X < 3, Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below :

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(i) $P(X \le 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) = 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(iii) $P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(iv) $P(X < 3, Y \le 4) = P(X = 0, Y \le 4) + P(X = 1, Y \le 4) + P(X = 2, Y \le 4)$

$$= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right) = \frac{9}{16}.$$

Example 5·34. *For the joint probability distribution of two random variables X and Y given below :*

X \ Y	1	2	3	4	Total
1	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{9}{36}$
3	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{8}{36}$
4	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{9}{36}$
Total	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	1

Find (i) the marginal distributions of X and Y, and

(ii) conditional distribution of X given the value of Y = 1 and that of Y given the value of X = 2.

Solution. The marginal distribution of X is defined as :

$$P(X = x) = \sum_y P(X = x, Y = y)$$

$$\therefore \quad P(X=1) = \sum_y P(X=1, Y=y)$$
$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$$
$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}.$$

Similarly $P(X=2) = \sum_y P(X=2, Y=y) = \frac{9}{36}$; $P(X=3) = \sum_y P(X=3, Y=y) = \frac{8}{36}$

and $P(X=4) = \sum_y P(X=4, Y=y) = \frac{9}{36}$.

Similarly, we can obtain the marginal distribution of Y.

MARGINAL DISTRIBUTION OF X

Values of X, x	1	2	3	4
$P(X=x)$	$\frac{10}{36}$	$\frac{9}{36}$	$\frac{8}{36}$	$\frac{9}{36}$

MARGINAL DISTRIBUTION OF Y

Values of Y, y	1	2	3	4
$P(Y=y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$

(*ii*) The conditional probability function of X given Y is defined as follows :

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}. \text{ Therefore}$$

$$\therefore \quad P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{4/36}{11/36} = \frac{4}{11}.$$

$$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

$$P(X=3 \mid Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)} = \frac{5/36}{11/36} = \frac{5}{11}$$

$$P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Hence the conditional distribution of X given $Y = 1$ is :

x :	1	2	3	4
$P(X=x \mid Y=1)$:	$\frac{4}{11}$	$\frac{1}{11}$	$\frac{5}{11}$	$\frac{1}{11}$

Similarly, we can obtain the conditional distribution of Y for $X = 2$ as given below :

y :	1	2	3	4
$P(Y=y \mid X=2)$:	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{9}$

Example 5·35. *A two-dimensional r.v. (X, Y) have a bivariate distribution given by :*

$$P(X=x, Y=y) = \frac{x^2+y}{32}, \text{ for } x = 0, 1, 2, 3 \text{ and } y = 0, 1.$$

Find the marginal distributions of X and Y.

(b) A two-dimensional r.v. (X, Y) have a joint probability mass function :

$$p(x, y) = \frac{1}{27}(2x+y), \text{ where } x \text{ and } y \text{ can assume only the integer values 0, 1 and 2.}$$

Find the conditional distribution of Y for X = x.

Solution. (*a*) We have

Y \ X	0	1	2	3	*Marginal distribution of Y*, $P(Y = y)$
0	0	$\frac{1}{32}$	$\frac{4}{32}$	$\frac{9}{32}$	$\frac{14}{32}$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{18}{32}$
Marginal distribution of X, $p(X = x)$	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{19}{32}$	1

The marginal probability distribution of X is given by :

$$p(X = x) = \sum_y P(X = x, Y = y) \text{ and is}$$

tabulated in last row of above table.

The marginal probability distribution of Y is given by :

$$P(Y = y) = \sum_x P(X = x, Y = y) \text{ and is}$$

tabulated in last column of above table.

(*b*) The joint probability function :

$$p_{XY}(x, y) = \frac{1}{27}(2x + y) \; ; x = 0, 1, 2; \; y = 0, 1, 2$$

gives the following table of joint probability distribution of X and Y.

JOINT PROBABILITY DISTRIBUTION $p(x, y)$ OF X AND Y

X \ Y	0	1	2	$f_X(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$

For example, $p(0, 0) = \frac{1}{27}(0 + 2 \times 0) = 0$, $\quad p(1, 0) = \frac{1}{27}(0 + 2 \times 1) = \frac{2}{27}$;

$p(2, 0) = \frac{1}{27}(0 + 2 \times 2) = \frac{4}{27}$; and so on,

The conditional distribution of Y for $X = x$ is given by :

$$P_{Y|X}(Y = y \mid X = x) = \frac{p_{XY}(x,y)}{p_X(x)} \text{ and is}$$

obtained in the adjoining table .

CONDITIONAL DISTRIBUTION OF Y FOR $X = x$

Y \ X	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$
2	$\frac{4}{15}$	$\frac{5}{15}$	$\frac{6}{15}$

Example 5·36. *Two discrete random variables X and Y have the joint probability*

Example 5·37. *If X and Y are two random variables having joint density function :*

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y) \,;\, 0 \le x < 2,\ 2 \le y < 4 \\ 0 \,,\ otherwise \end{cases}$$

Find (i) $P(X < 1 \cap Y < 3)$, *(ii)* $P(X + Y < 3)$, *and (iii)* $P(X < 1 \mid Y < 3)$.

Solution. We have

(*i*) $P\ (X < 1 \cap Y < 3) \quad = \int_{-\infty}^{1} \int_{-\infty}^{3} f(x, y)\, dx\, dy = \int_{0}^{1} \int_{2}^{3} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{3}{8}$

(*ii*) $P(X + Y < 3) \quad = \int_{0}^{1} \int_{2}^{3-x} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{5}{24}$

(*iii*) $P(X < 1 \mid Y < 3) \quad = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)} = \frac{3/8}{5/8} = \frac{3}{5}$

$$\left[\text{From part } (i) \text{ and } P(Y < 3) = \int_{0}^{2} \int_{2}^{3} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{5}{8}\right]$$

Example 5·38. *Suppose that two-dimensional continuous random variable (X, Y) has joint p.d.f. given by :*

$$f(x, y) = \begin{cases} 6x^2 y,\ 0 < x < 1,\ 0 < y < 1 \\ 0,\ elsewhere \end{cases}$$