Module - 3

Multivariable Calculus

Functions of two variables

A symbol z which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write

$$\underbrace{z}_{\text{dependent}} = f\underbrace{(x,y)}_{\text{independent}} \text{ or } \phi(x,y).$$

Examples

1. Area of a rectangle is a fn of two variables (Length & Breadth)

2. The Volume of a right circular cylinder is $V = \pi r^2 h$ (r = radius & h height)

3. The total surface of a rectangular paraboloid is $2(xy + yz + zx)$ → Three variable fn.

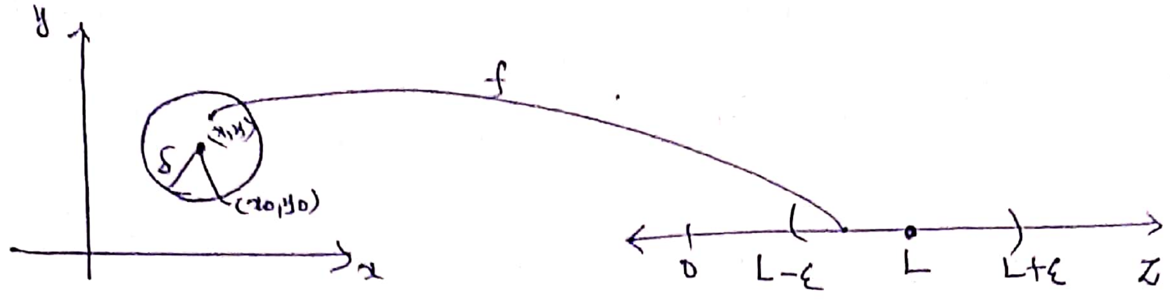
limits of the function of Two variables

We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\varepsilon > 0$, $\exists$ a corresponding number $\delta > 0$ s.t for all (x,y) in the domain of f,

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$



In the defn δ is the radius of a disc centered at (x_0, y_0). For all pts (x,y) within the disk, the function values $f(x,y)$ lie inside the corresponding interval $(L-\varepsilon, L+\varepsilon)$.

Example ① Determine if $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$ exists.

Soln:- Observe that the domain D of f is $\mathbb{R}^2 \setminus \{0,0\}$. And $f(0,y) = 0$ for $y \neq 0$ & $f(x,0) = 0$ for $x \neq 0$.

We guess that if the limit exists, it would be 0.

We start with any $\varepsilon > 0$. We want to choose a $\delta > 0$ s.t the following sentence becomes true.

If $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$ then $\left|\dfrac{4xy^2}{x^2+y^2}\right| < \varepsilon$.

Since $|y^2| = y^2 \leq x^2+y^2$ & $|x^2| = x^2 \leq x^2+y^2$, we have

$$\left|\frac{4xy^2}{x^2+y^2}\right| \leq 4|x| \leq 4\sqrt{x^2+y^2}$$

So, we choose $\delta = \dfrac{\varepsilon}{4}$. Assume that $0 < \sqrt{x^2+y^2} < \delta$. Then

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \leq 4\sqrt{x^2+y^2} < 4\delta = \varepsilon$$

Hence $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = 0$.

$$\left|\frac{4xy^2}{x^2+y^2}\right| \leq \left|4\frac{\cancel{(x^2+y^2)}x}{\cancel{x^2+y^2}}\right| = |4x| \leq 4\sqrt{x^2+y^2}$$

$\because y^2 \leq x^2+y^2$.

$|x^2| \leq x^2+y^2$

$|x| \leq \sqrt{x^2+y^2}$

Continuity of the function of two Variables

A function $f(x,y)$ is continuous at the point (x_0, y_0)

if (1) f is defined at (x_0, y_0)

(2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

* A function is continuous if it continuous at every point of its domain.

Example:- $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous in $\mathbb{R}^2$.

Soln:- At any pt other than origin, $f(x,y)$ is a rational number, therefore, it is continuous. To see that $f(x,y)$ is continuous at the origin, let $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{3}$, Assume that $\sqrt{x^2+y^2} < \delta = \varepsilon/3$. Then

$$\left| \frac{3x^2y}{x^2+y^2} - f(0,0) \right| \leq \left| \frac{3\,(\cancel{x^2+y^2})\,y}{\cancel{x^2+y^2}} \right| \leq 3|y| \leq 3\sqrt{x^2+y^2} < \varepsilon$$

Example:2 H.W $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous in $\mathbb{R}^2$. Why?

$$\boxed{\delta = \sqrt{\varepsilon}.}$$

$\sqrt{x^2+y^2} < \delta$ $\quad xy \leq x^2+y^2, \quad x^2-y^2 \leq x^2+y^2.$

$$|f(x,y) - 0| \leq \left| \frac{(\cancel{x^2+y^2})(x^2+y^2)}{\cancel{x^2+y^2}} \right| < \delta^2 = \varepsilon$$

$|f(x,y) - 0| < \varepsilon \Rightarrow$ ~~f is conti on zero~~ $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

Partial Derivatives

Let $z = f(x,y)$ be a function of two Variables x & y.

If we keep y as constant and vary x alone, then z is a fn of x only.

The derivative of z w.r.to x, treating y as constant, is called the partial derivative of z w.r.to x, and is at the pt (x_0, y_0) defined by

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

other notations, $\frac{\partial f}{\partial x}$, $f_x(x,y)$, $D_x f$.

|||ly

$$\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

Note: $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$.

Successive partial differentiation

$z = f(x,y)$ first derivative $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xy} \text{ or } f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

Problems

(1) $z = f(x,y) = x^3y - e^{xy}$

$$\frac{\partial f}{\partial x} = 3x^2y - ye^{xy}$$

$$\frac{\partial f}{\partial y} = x^3 - xe^{xy}$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy - y^2e^{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = -x^2e^{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^3 - xe^{xy})$$

$$= 3x^2 - (xye^{xy} + e^{xy})$$

$$= 3x^2 - e^{xy} - xye^{xy}$$

(2) $u = \log(x^3+y^3+z^3-3xyz)$ find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$?

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \qquad \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-xz)}}{\cancel{(x^2+y^2+z^2-xy-yz-xz)}(x+y+z)}$$

$$= \frac{3}{x+y+z}$$

(3) $u = (x-y)(y-z)(z-x)$ S.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

(4) $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ Ans 0

(5) $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ S.T $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$.

$$\frac{\partial u}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) + x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{y\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - 2y \tan^{-1}\left(\frac{x}{y}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{y^2}$$

$$= x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x \tan^{-1}\left(\frac{y}{x}\right) - y\right)$$

$$= 2x \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - 2y \tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= 1 - 2y \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \quad //$$

(6) If $u = x^y$, then S.T (i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = x^{y-1}[1 + y \log x]$.

(3,4)

Hint

$\boxed{\frac{d}{dx}(a^x) = e^{x \log a}}$

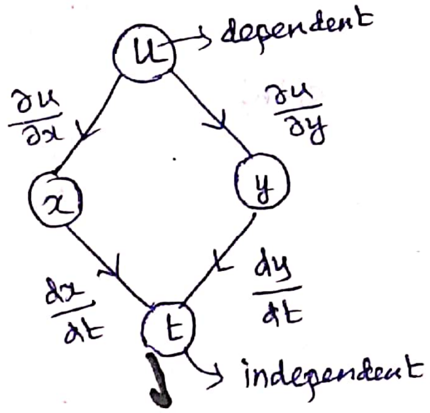
(7) $f(x,y) = \log\sqrt{x^2+y^2}$, S.T $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Total differential (or) Total derivatives

If $u = f(x,y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x,y)$. Thus, we can find the ordinary derivative $\frac{du}{dt}$ which is called the total derivative of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$. (which is different)

i.e.,

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$



Note: 1)

Composite function of one variable:

If $u = f(x,y,z)$ where x, y, z are all functions of a variable t, then we can similarly P.T

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

2. Differentiation of implicit functions:

If $f(x,y) = c$ be an implicit relation b/w x and y which defines as a differentiable function of x, then

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \left[\because \frac{\partial f}{\partial y} \neq 0\right]$$

3. Composite function of two variable:

If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$, then z is a function of u, v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

1. Find $\frac{dz}{dt}$ if $z = xy$ where $x = 2t^2$, $y = \sin t$

Soln:-

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= y \cdot 4t + x \cos t$$

$$\frac{dz}{dt} = 4t \sin t + 2t^2 \cos t.$$

2. Find $\frac{du}{dt}$ if $u = x^2+y^2+z^2$ where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$.

Soln:-

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2x \cdot e^t + 2y \cdot (e^t \cos t + \sin t\, e^t) + 2z\,(-e^t \sin t + \cos t\, e^t)$$

$$= 2e^{2t} + 2e^t \sin t\,(e^t \cos t + e^t \sin t) + 2e^t \cos t\,(-e^t \sin t + e^t \cos t)$$

$$= 2e^{2t} + \cancel{2e^{2t} \sin t \cos t} + 2e^{2t} \sin^2 t - \cancel{2e^{2t} \cos t \sin t} + 2e^{2t} \cos^2 t$$

$$= 2e^{2t}(2) = 4e^{2t}.$$

3. If $u = f(r,s,t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ S.T

$$\sum x \frac{\partial u}{\partial x} = 0.$$

Soln:- $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$= \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t} \quad \text{——} \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$= -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s} \quad \text{——} (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$= \frac{-y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t} \quad \text{——} (3)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}.$$

$$\therefore \sum x\frac{\partial u}{\partial x} = 0.$$

H.W

(4) If $u = x\log xy$, where $x^3+y^3+3xy=1$, find $\frac{du}{dx}$.

(5) If $z=(x,y)$ where $x=u^2-v^2$, $y=2uv$

P.T $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{4(u^2+v^2)}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

Soln:-

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}2u + \frac{\partial z}{\partial y}2v$$

$$= 2\left(u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right).$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial}{\partial u}\left(2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}\right)$$

$\frac{\partial}{\partial u} = 2\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)$

$\frac{\partial z}{\partial u} = 2\left(u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right)$

$$= 2\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}\right)$$

$$= 4\left(u^2\frac{\partial^2 z}{\partial x^2} + v^2\frac{\partial^2 z}{\partial y^2} + 2uv\frac{\partial^2 z}{\partial x\partial y}\right) \quad \text{——} (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}\,2u = 2\left(u\frac{\partial z}{\partial y} - v\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 2\left(u\frac{\partial}{\partial y} - v\frac{\partial}{\partial x}\right)\left(2u\frac{\partial z}{\partial y} - 2v\frac{\partial z}{\partial x}\right)$$

$$= 4\left(u^2\frac{\partial^2 z}{\partial y^2} + v^2\frac{\partial^2 z}{\partial x^2} - 2uv\frac{\partial^2 z}{\partial x\partial y}\right) \quad \text{— (2)}$$

Adding (1) & (2),

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \;//.$$

(6). If $u = f(x-y, y-z, z-x)$ S.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

$u = f(x-y, y-z, z-x)$

Let $\alpha = x-y$, $\beta = y-z$, $\gamma = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial x}$$

$$= \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(0) + \frac{\partial u}{\partial \gamma}(-1) = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \gamma} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta} \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial \beta} + \frac{\partial u}{\partial \gamma} \quad \text{— (3)}$$

Adding (1)+(2)+(3) $\Rightarrow$ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

H.W

(7) $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

S.T $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

$\alpha = \frac{1}{x} - \frac{1}{y}$, $\beta = \frac{1}{x} - \frac{1}{z}$

Jacobians and Properties

Definition

If $u = u(x,y)$ and $v = v(x,y)$ where x and y are independent, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is known as the Jacobian of u and v w.r.to x, y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J(u,v)$

IIIly, the Jacobian of three functions $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$ is defined as

$$J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties

① If u and v are functions of x and y, then

$$\underbrace{\frac{\partial(u,v)}{\partial(x,y)}}_{J} \times \underbrace{\frac{\partial(x,y)}{\partial(u,v)}}_{J'} = 1$$ [Inverse property of Jacobians].

② Chain rule :- If u, v are function of r, s and r, s are themselves functions of x, y i.e.,

$u = u(r,s)$, $v = v(r,s)$ and $r = r(x,y)$, $s = s(x,y)$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

③ If u, v, w are functionally dependent functions of three independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

Problems

1. $u = xy$, $v = x^2$ Find $\frac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} y & x \\ 2x & 0 \end{vmatrix} = -2x^2$$

2. $u = \frac{x^2}{y}$, $v = \frac{y^2}{x}$, find $J(x,y)$

$$J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

$$= 4 - 1 = 3$$

$$J(x,y) = \frac{1}{J(u,v)} = \frac{1}{3}.$$

3. $u = x(1-y)$, $v = xy(1-z)$, $w = xyz$.

P.T $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{x^2 y}$.

Soln:-

$$J(u,v,w) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1-y & -x & 0 \\ y(1-z) & x(1-z) & -xy \\ yz & xz & xy \end{vmatrix}$$

$$= x^2y \begin{vmatrix} 1-y & -1 & 0 \\ y(1-z) & 1-z & -1 \\ yz & z & 1 \end{vmatrix} = x^2y\left(1-y(1-z)+z\right) + 1\left(y(1-z)+yz\right)$$

$$= x^2y\left(1-y+1\left(y-\cancel{yz}+\cancel{yz}\right)\right) = x^2y$$

$$J(x,y,z) = \frac{1}{J(u,v,w)} = \frac{1}{x^2y}.$$

4. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$, $\frac{\partial(r,\theta)}{\partial(x,y)}$ and verify

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1.$$

Soln:-

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$r^2 = x^2+y^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\qquad 2r\frac{\partial r}{\partial x} = 2x$

$= \frac{x}{r}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{1}{r}.$$

$$\therefore JJ' = r \cdot \frac{1}{r} = 1.$$

5. Find the Jacobian of y_1, y_2, y_3 w.r.to x_1, x_2, x_3 if

$$y_1 = \frac{x_2x_3}{x_1}, \quad y_2 = \frac{x_1x_3}{x_2}, \quad y_3 = \frac{x_1x_2}{x_3}.$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix} = \frac{1}{x_1^2 x_2^2 x_3^2}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_1 x_3 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_2 x_1 \end{vmatrix} = \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 4.$$

6. If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

chain Rule, $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,\theta)}$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - 4x^2)(r)$$

$$= -4(x^2+y^2)(r) = -4r^2 \cdot r = -4r^3.$$

H.W

7. $u = x^2 + y^2$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$

8. If $x = uv$, $y = \frac{u}{v}$ S.T $JJ' = 1$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}.$$

$xy = u^2$ $\quad\quad$ $2u u_y = x$

$2u u_x = y$ $\quad\quad$ $u_y = \frac{x}{2u}$

$u_x = \frac{y}{2u}$

$v = \frac{x}{u}, \quad v = \frac{u}{y}$

$v^2 = \frac{x}{y}$

$2vv_x = \frac{1}{y} \qquad 2vv_y = -\frac{x}{y^2}$

$v_x = \frac{1}{2vy} \qquad v_y = \frac{-x}{2vy^2}$

$$J(u,v) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{y}{2u} & \frac{x}{2u} \\ \frac{1}{2vy} & \frac{-x}{2vy^2} \end{vmatrix}$$

$$= \frac{-xy}{4uvy^2} - \frac{x}{4uvy} = \frac{-x}{4uvy} - \frac{x}{4uvy}$$

$$= -\frac{x}{2uvy} = \frac{-v^2}{2uv} = \frac{-v}{2u}$$

$\therefore JJ' = 1.$

(5) 9. If we transform three dimensional cartesian coordinates (x,y,z) to spherical polar coordinates (r,θ,ϕ). S.T the Jacobian of x,y,z w.r.to r,θ,ϕ is $r^2 \sin\theta$.

Soln:-

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \left| \quad \right| = r^2\Big[\sin\theta\cos\phi(0 + \sin^2\theta\cos\phi) - \cos\theta\cos\phi(0 - \sin\theta\cos\theta\cos\phi) - \sin\theta\sin\phi(-\sin\theta\sin\phi - \cos^2\theta\sin\phi)\Big].$$

$$= r^2\left[\sin^3\theta\cos^2\phi + \sin\theta\cos^2\theta\cos^2\phi + \sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta\sin^2\phi\right].$$

$= r^2[\sin^3\theta + \sin\theta\cos^2\theta]$

$= r^2\sin\theta\,[1]$.

10. If $x = e^{v}\sec u$
H.W
$y = e^{v}\tan u$

Find J & J'.

11. $u = x+y+z$, $v = x^2+y^2+z^2$, $w = xy+yz+zx$

Find J.

12. $u = 3x+2y-z$, $v = x-2y+z$, $w = x(x+2y-z)$.

<u>Property</u> If u, v, w are functionally dependent functions of 3 independent variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

1. If $u = xy+yz+zx$, $v = x^2+y^2+z^2$, $w = x+y+z$ determine the functional relationship between u, v & w.

<u>Soln:-</u> $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad R_1 \to R_1 + R_2$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

$\therefore u, v$ & w are functionally dependent.

To find functional relationship just take

$w^2 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$

$\boxed{w^2 = v + 2u}$

2. $u = 3x+2y-z$, $v = x-2y+z$, $w = x^2+2xy-xz$

$\boxed{u^2 - v^2 = 8w}$

3. $x = ue^{v}\sin w$, $y = ue^{v}\cos w$, $z = u^2e^{2v}$ $\boxed{x^2+y^2 = z}$