Practice Question Bank

Calculus For Engineers

(MAT1011)

Department of Mathematics

VIT

Vellore Institute of Technology

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Contents

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Course Title: Calculus For Engineers

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Syllabus and Organization of Contents

Module 1 Application of Single Variable Calculus

Differentiation - Extrema on an Interval - Rolle's Theorem and the Mean Value Theorem - Increasing and Decreasing functions and First derivative test - Second derivative test - Maxima and Minima - Concavity. Integration - Average function value - Area between curves - Volumes of solids of revolution - Beta and Gamma functions – interrelation

(Chapter 1 of Practice Question Bank)

Module 2 Laplace Transform

Definition of Laplace transform - Properties - Laplace transform of periodic functions - Laplace transform of unit step function, Impulse function - Inverse Laplace transform - Convolution

(Chapter 2 of Practice Question Bank)

Module 3 Multi-variable Calculus

Functions of two variables - limits and continuity - partial derivatives - total differential - Jacobian and its properties

(Chapter 3 of Practice Question Bank)

Module 4 Application of Multi-variable Calculus

Taylor's expansion for two variables - maxima and minima – constrained maxima and minima - Lagrange's multiplier method

(Chapter 4 of Practice Question Bank)

Module 5 Multiple integrals

Evaluation of double integrals - change of order of integration - change of variables between Cartesian and polar coordinates - Evaluation of triple integrals - change of variables between Cartesian and cylindrical and spherical coordinates - Evaluation of multiple integrals using gamma and beta functions

(Chapter 5 of Practice Question Bank)

Module 6 Vector Differential Calculus

Scalar and vector valued functions – gradient, tangent plane - directional derivative - divergence and curl - scalar and vector potentials - Statement of vector identities - Simple problems

(Chapter 6 of Practice Question Bank)

Module 7 Vector Integral Calculus

Line, surface and volume integrals - Statements of Green's, Stoke's and Gauss divergence theorems - Verification and evaluation of vector integrals using them

(Chapter 7 of Practice Question Bank)

Chapter 1

Applications of Differentiation

1.1 Continuity and Differentiability

Definition 1.1.1 (Continuity)**.** Let $f : \mathscr{D} \to \mathbb{R}$, where $\mathscr{D} \subset \mathbb{R}$, and $c \in \mathscr{D}$. Then f is said to be continuous at c, if $\lim_{x\to c} f(x) = f(c)$.

Definition 1.1.2 (Differentiability)**.** Let $f : \mathscr{D} \to \mathbb{R}$, where $\mathscr{D} \subset \mathbb{R}$, and $c \in \mathscr{D}$. Then f is said to be differentiable at c, if $l = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and the limit value l is called the derivative $f'(c)$ of f at c.

Definition 1.1.3 (Critical Point)**.** An interior point c of the domain D of a function $f(x)$ is called a critical point of f, if $f'(c)$ is either undefined or is equal to zero.

Exercise 1.1.1. Find the critical point(s) of the following functions:

(a) $f(x) = x(4-x)^3$ for all $x \in \mathbb{R}$;

(b) $f(x) = x^2 - 32\sqrt{x}$ for all $x \in \mathbb{R}$

(c) $f(x) = x^2/(x-2)$ for all $x \in \mathbb{R}$.

1.2 Absolute Extrema on a Finite Closed Interval

Let $f : [a,b] \to \mathbb{R}$ and $c \in [a,b]$.

Definition 1.2.1. We say that $f(c)$ is the absolute maximum value on $[a,b]$, if $f(x) \leq f(c)$ for all $x \in [a,b]$, c is called a point of absolute maximum of f and $f(c)$ absolute maximum value of f. We say that $f(d)$ is the absolute minimum value on $[a,b]$, if $f(d) \leq f(x)$ for all $x \in [a,b]$, d is called a point of absolute minimum of f and $f(d)$ absolute minimum value of f.

Exercise 1.2.1. Find the absolute maxima and absolute minima of the following functions on the given intervals:

(a) $f(x) = (2x/3) - 5$ for all $x \in [-2,3]$,

(b) $f(x) = x^a(1-x)^b$ on $[0,1]$ where a and b are positive real numbers,

(c) $f(x) = \sqrt{5-x^2}$ on $[-2,1]$.

1.3 Mean Value Theorems

Theorem 1.3.1 (Rolle's Theorem)**.** Consider $f : [a,b] \to \mathbb{R}$. Suppose that

(a) f is continuous at every point of the closed interval $[a,b]$,

(b) f is differentiable on (a,b), and

(c) $f(a) = f(b)$.

Then there is *at least one* number c in (a, b) such that

$$f'(c) = 0. \tag{1.3.1}$$

Exercise 1.3.1. Explain why Rolle's theorem is not applicable to the following functions $f(x)$:

(a) $\dfrac{x^2 - 4x}{x - 2}$ on $[0, 4]$,

(b) $1 - (x - 1)^{2/3}$ on $[0, 2]$,

(c) $\tan x$ on $[0, \pi/4]$,

(d) $\sec x$ on $[0, 2\pi]$,

(e) $|x|$ on $[-1, 1]$.

Exercise 1.3.2. Verify whether Rolle's theorem and find an appropriate constant c of it for each of the following functions $f(x)$:

(a) $\sqrt{x} - x/3$ on $[0, 9]$;

(b) $(x - a)^m (x - b)^n$ on $[a, b]$;

(c) $(\sin x)/e^x$ on $[0, \pi]$,

(d) $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ on $[a, b]$, where $a > 0$,.

Exercise 1.3.3 (Self-check)**.** Explain the physical interpretation of Rolle's theorem.

Theorem 1.3.2 (The Mean Value Theorem)**.** Suppose that

(a) $y = f(x)$ is continuous on the closed interval $[a, b]$,

(b) f is differentiable on (a, b).

Then there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1.3.2}$$

Exercise 1.3.4. Explain why the mean value theorem is not applicable to the following functions:

(a) $f(x) = \begin{cases} \frac{\sin x}{x}, & (-\pi \le x < 0) \\ 0, & (x = 0) \end{cases}$

(b) $f(x) = \begin{cases} x^2 - x, & (-2 \le x \le -1) \\ 2x^2 - 3x - 3, & (-1 < x \le 0) \end{cases}$

(c) $f(x) = x^{2/3}$ in $[-1, 8]$.

Exercise 1.3.5. Verify the mean value theorem for each of the following functions $f(x)$ in the given interval and find an appropriate constant c in each case:

(a) $\log x$ on $x \in [1, e]$

(b) $f(x) = lx^2 + mx + n$ on $x \in [a, b]$

(c) $f(x) = x + 1/x$ on $[1/2, 1]$

(d) $f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \leq x \leq 2. \end{cases}$

(e) $f(x) = lx^2 + my^2 + n$ on $[a, b]$

Exercise 1.3.6. Interpret physically the mean-value theorem.

1.4 First Derivative Test for Local Extrema

Definition 1.4.1 (Monotonic Functions)**.** We say that $f : [a, b] \to \mathbb{R}$ is

- *increasing* on $[a, b]$, if $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$,
- *decreasing* on $[a, b]$, if $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$,
- *monotonic* on $[a, b]$, if it is either increasing or decreasing

Theorem 1.4.1 (Test of Monotonicity)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$,
- If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Exercise 1.4.1. Find the critical points of each of the following functions $f(x)$, and separate the intervals on which f is increasing and on which f is decreasing:

(a) $2x^3 - 18x$

(b) $x\sqrt{8 - x^2}$

(c) $\frac{x^2-3}{x-2}, x \neq 2$

(d) $x^2\sqrt{5 - x}$,

(e) $f(x) = x^4 - 8x^2 + 16 = (x^2 - 4)^2$

Theorem 1.4.2 (First Derivative Test for Local Maxima)**.** If $f'(x)$ changes its sign from positive to negative on passing through c from left to the right, that is

$$f'(x) \begin{cases} > 0 & \text{for } x < c \\ < 0 & \text{for } x > c \end{cases}$$

then f has a local maximum at c.

Theorem 1.4.3 (First Derivative Test for Local Minima). If $f'(x)$ changes its sign from negative to positive on passing through c from left to the right, that is

$$f'(x)\begin{cases} <0 & \text{for } x<c \\ >0 & \text{for } x>c \end{cases}$$

then f has a local minimum at c

Theorem 1.4.4 (First Derivative Test for Local Extrema). If $f'(x)$ positive or negative on both sides of c, then f does not have local extremum at c

Exercise 1.4.2. Using the first derivative test, identify the local extreme values of the following functions $f(x)$ in the given domain:

(a) $f(x)=\sqrt{25-x^2}$ in $[-5,5]$;

(b) $12x-x^3$ in $[-3,-\infty)$;

(c) x^3+3x^2+3x+1 for $x\leq 0$

(d) $f(x)=x^2-4x+4$ in $[1,\infty)$,

(e) $f(x)=x^2/(4-x^2)$ for $-2<x\leq 1$.

Exercise 1.4.3. Find the constants l and m so that $f(x)=lx^2+mx$ has an absolute maximum at the point $(1,2)$.

1.5 Second Derivative Test for Concavity

Definition 1.5.1 (Concavity). If the graph of a function $f(x)$ lies above the tangents at its points on an interval I, we say that C is *concave up* in I. While, If the graph of $f(x)$ lies below the tangents at its points on an interval I, we say that C is *concave down* in I.

Theorem 1.5.1 (Second Derivative Test). Let $y=f(x)$ be a plane curve C. Then the graph of $f(x)$ is concave up or down in I according as $f''(x)>0$ or $f''(x)<0$ for all $x\in I$ respectively.

Definition 1.5.2 (Point of Inflection). A point P on a curve $y=f(x)$ is called a *point of inflection*, if f is continuous at P and the concavity of the curve reverses on passing through P. Thus $P(c,f(c))$ is a point of inflection on the curve $y=f(x)$, if the sign of $f''(x)$ is different on either side of the ordinate $x=c$.

Exercise 1.5.1. Determine the critical points, points of local maxima and local minima of each of the following functions $f(x)$, and then identify the intervals on which f is concave up and concave down. Also, find the points of inflection in each case:

(a) $\frac{3}{4}(x^2-1)^{2/3}$

(b) $-2x^3+6x^2-3$

(c) $\sqrt[3]{x^3+1}$

(d) $f(x)=\frac{x^3}{3}-\frac{x^2}{2}-2x+\frac{1}{3}$

(e) $f(x)=\frac{x}{\sqrt{x^2+1}}$.

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 4.4-4.5, 4.8
2. Goldstein et al., *Calculus - Its Applications*, 13th Edition, Copyright © 2014 Pearson Edu., Sec. 2.3
3. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 3.3
4. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 4.1-4.4

1.6 Area of the Region bounded by two Plane Curves

The area A of a region $\mathscr{D}$ enclosed by the plane curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = a$ and $x = b$, where $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x)$ for $x \in [a, b]$, is given by the definite integral of $f(x) - g(x)$ from $x = a$ and $x = b$. That is

$$A = \int_{x=a}^{b} [f(x) - g(x)]\,\mathrm{d}x.$$

Exercise 1.6.1. Find the area of the region enclosed by

(a) the cubical parabola $y = x^3$ and the straight line $y = x$;

(b) the sinusoidal curves $y = \sin x$ and $y = \sin 2x$ between the ordinates $x = 0$ and $x = \pi$;

(c) the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ in the first quadrant;

(d) the graphs of $y = 2x$ and $y = x^2 - 8$

(e) the graphs of $y = |x|$ and $y = 1 - |x|$

(f) the curves $y = \sin x$ and $y = \cos x$ between the ordinates $x = 0$ and $x = \pi/2$

(g) the curves $y = \sin x$ and $y = \cos x$ between the ordinates $x = 0$ and $x = \pi$

(h) the parabolas $y = 20 + x - x^2$ and $y = x^2 - 5x$

(i) the astroid $\sqrt{x} + \sqrt{y} = 1$ and the line $x + y = 1$ in the first quadrant.

1.7 Average Value of a Function

Let $y = f(x)$ be continuous on $[a, b]$. The average value of f on the interval is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_{x=a}^{b} f(x)\,\mathrm{d}x. \tag{1.7.1}$$

Exercise 1.7.1. Find the average value f_{ave} of each of the following functions on the indicated interval:

(a) $2x^3 - 3x^2 + 4x - 1$, $[-1, 1]$

(b) $\sqrt{5x+1}$, $[0, 3]$

(c) $2/(x+1)^2$, $[3, 5]$

(d) $\cos 2x$, $[3, \pi/4]$

(e) $x^{2/3} - x^{-2/3}$, $[1, 4]$

(f) $f(x) = x\sqrt{x^2+16}$, $[0, 3]$

(g) $f(x) = |x| - 1$ on $[-1, 3]$

1.8 Volumes of Solids of Revolution

Disk Method: Let $f(x)$ be a continuous function on $[a, b]$. The volume of solid of revolution obtained by revolving the arc of the plane curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis, is

$$V = \int_a^b \pi y^2 \,\mathrm{d}x = \int_a^b \pi[f(x)]^2 \,\mathrm{d}x. \tag{1.8.1}$$

Let $g(y)$ be a continuous function on $[c, d]$. The volume of solid of revolution obtained by revolving the arc of the plane curve $x = g(y)$ from $y = c$ to $y = d$ about the y-axis, is

$$V = \int_c^d \pi x^2 \,\mathrm{d}y = \int_c^d \pi[g(y)]^2 \,\mathrm{d}y. \tag{1.8.2}$$

Exercise 1.8.1. Find the volume of the solid of revolution of

(a) the semi-circular arc $x^2 + y^2 = a^2$ from $x = -a$ to $x = a$ about the x-axis;

(b) the arc of the curve $y = x^3$ from $y = 0$ to $y = 8$ about the y-axis;

(c) the hyperbola $y^2 - x^2 = 1$ from $x = -a$ to $x = a$ about the x-axis

(d) the hyperbola $xy = 2$ about the y-axis, between the limits $y = 1$ to $y = 8$

(e) the arc of the parabola $y = \sqrt{x}$ from $x = 0$ to $x = 1$ about the x-axis.

Exercise 1.8.2. Regarding a cone of height h and radius a, as a solid of revolution of the straight line segment joining the vertex $(0, 0)$ to the point (a, h) from $y = 0$ to $y = h$ about the y-axis, find its volume.

Exercise 1.8.3. Regarding a cylinder of height h and radius a, as a solid of revolution of the rectangle with edges $x = 0$, $x = a$, $y = 0$ and $y = h$ about the y-axis, find its volume.

Washer Method: The volume of solid of revolution of the region enclosed by the plane curves $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$ from $x = a$ to $x = b$ about the x-axis, is

$$V = \int_a^b \pi[f(x)^2 - g(x)^2] \,\mathrm{d}x. \tag{1.8.3}$$

The volume of solid of revolution of the region enclosed by the plane curves $x = f(y)$ and $x = g(y)$ with $f(y) \geq g(y)$ from $y = c$ to $y = d$ about the y-axis, is

$$V = \int_c^d \pi[f(y)^2 - g(y)^2]\, \mathrm{d}y. \tag{1.8.4}$$

Exercise 1.8.4 (Self-check). Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the x-axis (between the given limits):

(a) $y = x^3$ and $y = x^2$

(b) $y^2 = 4(x-1)$ and $y = x - 1$

(c) $y = x^2 + 2$ and $y = 10 - x^2$

(d) $y = 1/x$ and $2y = 5 - 2x$

(e) by the parabola $y = x^2$ and the line $y = x$.

Exercise 1.8.5 (Self-check). Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the y-axis:

(a) $y = x^{1/3}$ and $x = 4y$, $x, y \geq 0$

(b) $x^2 - 2x$ and $y = x$

(c) $y = 16 - x$ and $y = 3x + 2$

(d) $y = x^3$ and $y = x^{1/3}$

Other Axes of Revolution

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. We integrate an appropriate cross-sectional area to find the volume.

Exercise 1.8.6 (Self-check). Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the given axis:

(a) $y = x^{1/2}$, $y = 0$ and $x = 9$, about $x = 9$

(b) $y = x^{1/2}$, $y = 0$ and $x = 9$, about $y = 3$

(c) $x = y^2$ and $x = y$, about $y = -1$

(d) $x = y^2$ and $x = y$, about $x = -1$

(e) $y = x^2$ aand $y = x^3$, about $x = 1$.

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition, Copyright © 2013 John Wiley & Sons, Sec. 6.1-6.2
2. Briggs et al, *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © 2013 Pearson Education, Inc., Sec. 6.2-6.3
3. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 5.1-5.2
4. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 5.2, 5.6, 6.1

1.9 Gamma and Beta Functions

Definition 1.9.1 (Gamma Function)**.**

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1}\,\mathrm{d}x, \quad p > 0. \tag{1.9.1}$$

Theorem 1.9.1 (Recurrence Relation)**.**

$$\Gamma(1) = 1 \quad \text{and} \quad \Gamma(p) = (p-1)\Gamma(p-1) \quad \text{for} \quad p > 1 \tag{1.9.2}$$

Theorem 1.9.2 (Gamma for Positive Integers)**.**

$$\Gamma(n) = (n-1)! \text{ for } n = 1, 2, 3, \ldots$$

Exercise 1.9.1. Evaliate the following:

(a) $\Gamma\left(\frac{3}{2}\right)$

(b) $\Gamma\left(\frac{5}{2}\right)$

(c) $\Gamma\left(\frac{7}{2}\right)$

(d) $\Gamma\left(\frac{9}{2}\right)$

(e) $\frac{\Gamma(6)}{2\Gamma(3)}$

Exercise 1.9.2. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Theorem 1.9.3 (Gamma for Negative Real Numbers)**.** For $p < 0, p \neq 0, -1, -2, \ldots$, we have

$$\Gamma(p) = \frac{1}{p} \cdot \Gamma(p+1) = \frac{1}{p} \cdot \frac{1}{p+1} \cdots \frac{1}{p+k} \cdot \Gamma(p+k+1),$$

where k is the least positive integer such that $p + k + 1 > 0$.

Exercise 1.9.3. Evaliate the following:

(a) $\Gamma\left(-\frac{1}{2}\right)$

(b) $\Gamma\left(-\frac{3}{2}\right)$

(c) $\Gamma\left(-\frac{5}{2}\right)$

(d) $\Gamma\left(-\frac{7}{2}\right)$

(e) $\dfrac{\Gamma(6)}{2\Gamma(3)}$

Definition 1.9.2 (Beta Function)**.**

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x, \quad p > 0,\ q > 0 \tag{1.9.3}$$

We have

(a) *Beta in terms of Trigonometric Sine and Cosine*

$$B(p, q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,\mathrm{d}\theta, \quad p > 0,\ q > 0. \tag{1.9.4}$$

(b) *Trigonometric integral in terms of Beta*

$$\int_0^{\frac{\pi}{2}} \sin^r\theta\cos^s\theta\,\mathrm{d}\theta = \frac{1}{2}\cdot B\left(\frac{r+1}{2}, \frac{s+1}{2}\right) \text{ for } r > 0,\ s > 0. \tag{1.9.5}$$

(c) *Beta in terms of Gamma*: If $p,\ q$ and r are positive real numbers, then $B(p, q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$

(d) $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$ for $p > 0$.

Exercise 1.9.4. Evaluate the following:

(a) $B(2, 5)$

(b) $B(4, 7)$

(c) $B(2, 3/2)$

(d) $B(9/2, 7/2)$

1.10 Evaluation of Integrals Using the Gamma

Exercise 1.10.1. Show that

(a) $\displaystyle\int_0^{\pi/2} \cos^9\theta\,\mathrm{d}\theta = \frac{128}{315}$

(b) $\displaystyle\int_0^{\pi/2} \sin^6\theta\cos^7\theta\,\mathrm{d}\theta = \frac{16}{3003}$

(c) $\displaystyle\int_0^{\pi/2} \sqrt{\cot\theta}\,\mathrm{d}\theta = \pi/2\sqrt{2}$

(d) $\left\{\int_0^{\pi/2} \sqrt{\sin\theta}\,\mathrm{d}\theta\right\}\left\{\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\,\mathrm{d}\theta\right\} = \pi$

(e) $\int_0^{\pi/2} \sqrt{\tan\theta}\,\mathrm{d}\theta = \pi/2\sqrt{2}$

(c) $\displaystyle\int_0^{\pi/2} \sqrt{\cot\theta}\,\mathrm{d}\theta = \pi/2\sqrt{2}.$

Exercise 1.10.2. Find $\displaystyle\int_0^1 x^{3/2}(1-x^2)^{5/2}\,\mathrm{d}x$, with the substitution $x = \sin\theta$.

Exercise 1.10.3. Evaluate $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^5}}\,\mathrm{d}x$, using the substitution $x = (\sin\theta)^{2/5}$.

Text and Reference books

1. Grewal, B. S., *Higher Engineering Mathematics*, 42nd Edition (2012), Khanna Publishers, Sec. 7.14-7.16

Chapter 2

The Laplace Transform

2.1 Laplace Transforms of Elementary Functions

Definition 2.1.1. Let $f : [0, \infty) \to \mathbb{R}$. The Laplace transform of $f(t)$ is given by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s), \tag{2.1.1}$$

where s is a *parameter* (real or complex). For the improper integral (2.1.1) to have finite value, s should be positive whenever s is real, and the real part of s should be positive if s is complex.

Theorem 2.1.1 (Dirichlet's conditions). The Laplace transform $\mathscr{L}\{f(t)\}$ of f exists, provided $f(t)$ is piecewise continuous over every finite interval, and is of exponential order as $t \to \infty$, that is $\lim_{t\to\infty} f(t)e^{-kt} = 0$ for some $k > 0$

The Laplace Transform of Real Powers of *t*

Theorem 2.1.2. Let p be a real number with $p > -1$, and $s > 0$. Then

$$\mathscr{L}\{t^p\} = \int_0^\infty t^p.e^{-st}\,\mathrm{d}t = \frac{\Gamma(p+1)}{s^{p+1}}, \tag{2.1.2}$$

where $\Gamma(r) = \displaystyle\int_0^\infty x^{r-1}\,e^{-x}\,\mathrm{d}x$ for $r > 0$.

Exercise 2.1.1. Using $\Gamma(1/2) = \sqrt{\pi}$ and recurrence formula $\Gamma(p+1) = p\Gamma(p)$, find

(a) $\mathscr{L}\left\{t^{-1/2}\right\}$

(b) $\mathscr{L}\left\{t^{5/2}\right\}$

(d) $\mathscr{L}\left\{t^{2/3}\right\}$.

The Laplace Transform of Positive Integer Powers of *t*

Taking p as a positive integer n, and using $\Gamma(n+1) = n!$ in (2.1.2), we get

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ for } n = 1, 2, 3, \ldots$$

Laplace Transform of Trigonometric Sine and Cosine

The Laplace transforms of $\sin at$ and $\cos at$ are

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2},\ \mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}.$$

Laplace Transform of the Exponential Signal

Theorem 2.1.3. The Laplace transform of the exponential signal e^{at} is given by

$$\mathscr{L}\left\{e^{at}\right\} = \int_0^\infty e^{at}\, e^{-st}\, \mathrm{d}t = \frac{1}{s-a},$$

provided $s > a$.

Exercise 2.1.2. Does $f(t) = e^{t^2}$ have the exponential order?

Linearity or Superposition of $\mathscr{L}$

Theorem 2.1.4. Let $\mathscr{L}\{f(t)\} = F(S)$, $\mathscr{L}\{g(t)\} = G(S)$. If a and b are real numbers, not both zero, then

$$\mathscr{L}\{af(t) + bg(t)\} = aF(s) + bG(s). \tag{2.1.3}$$

That is, the Laplace transform of linear combination of $f(t)$ and $g(t)$ equals the linear combination of their transforms $F(s)$ and $G(s)$.

Laplace Transform of Hyperbolic sine and Cosine

The Laplace transforms of $\sinh at$ and $\cosh at$ are given by

$$\mathscr{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \mathscr{L}\{\cosh at\} \frac{s}{s^2-a^2},$$

Exercise 2.1.3. Find the Laplace transform of each of the following functions $f(t)$ using the linearity:

(a) $\frac{e^{at}-e^{bt}}{a-b}$

(b) $\frac{ae^{at}-be^{bt}}{a-b}$

(c) $2\sin^2\left(\frac{at}{2}\right) = 1-\cos at$

(d) $at - \sin at$

(e) $\sin^3 at = \frac{3\sin at - \sin 3at}{4}$

(f) $\cos^3 at = \frac{3\cos at + \cos 3at}{4}$

(g) $\frac{\cos at - \cos bt}{b^2-a^2}$

(h) $\sinh at - \sin at$

(i) $\cosh at - \cos at$

Exercise 2.1.4. Find the Laplace transform of each of the following functions $f(t)$ using the linearity:

(a) $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

(b) $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$

(c) $3e^{3t} + 5t^4 - 4\cos 3t + 3\sin 4t$

(d) $e^{-3t} + 5e^t + 6\sin 2t - 5\cos 2t$

(e) $7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2$

(f) $4e^{-3t} - 2\sin 5t + 3\cos 2t - 2t^3 + 3t^4$

(g) $3t^2 + 6t + 4 + (5e^{2t})^2$

(h) $(t^2+1)^2 + 3\cosh 5t - 4\sinh t$

(i) $2e^{5t} + e^{-3t} + 5e^t + 5t - 2, t^2 - 5t - \sin 2t + e^{3t}$

(j) $t + \sin at, t - \cos at$

(k) $\sin at \sin bt, \sin at \cos bt, \cos at \cos bt$

(l) $(\sin at + \cos at)^2$

(m) $\sin(at+b), \cos(at+b)$

(n) $\sin\sqrt{t}$

2.2 Multiplication and Division by *t* and its Applications

Theorem 2.2.1 (Laplace Transform of, $f(t)$ Multiplied by t)**.** Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\{t\, f(t)\} = -\frac{\mathrm{d}F}{\mathrm{d}s}. \tag{2.2.1}$$

The Laplace transform of t times $f(t)$, is the negative of the derivative of Laplace transform.

In general,

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F}{\mathrm{d}s^n},\ n \geq 1. \tag{2.2.2}$$

Exercise 2.2.1. Find $\mathscr{L}\left\{te^{at}\right\}, \mathscr{L}\left\{t\sin at\right\}$ and $\mathscr{L}\left\{t\cos at\right\}$.

Exercise 2.2.2. Using the Laplace transform, evaluate the following integrals:

(a) $\displaystyle\int_0^\infty te^{-bt}\sin at\ \mathrm{d}t$, where $b > 0, a > 0$

(b) $\displaystyle\int_0^\infty t^2 e^{-t}\sin 3t\ \mathrm{d}t$

(c) $\displaystyle\int_0^\infty t^3 e^{-t}\sin t\ \mathrm{d}t$

Exercise 2.2.3. Find $\mathscr{L}\left\{\sin at + at\cos at\right\}$ and $\mathscr{L}\left\{\sin at - at\cos at\right\}$.

Theorem 2.2.2 (Laplace Transform of $f(t)$ divided by t)**.** Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_{u\,=\,s}^{\infty} F(u)\ \mathrm{d}u. \tag{2.2.3}$$

From (2.2.3), we see that

$$\int_0^{\infty}\left\{\frac{f(t)}{t}\right\} e^{-st}\,\mathrm{d}t = \int_{u\,=\,s}^{\infty} F(u)\ \mathrm{d}u = G(s).$$

Employing the limit as $s \to 0$, this gives

$$\int_0^{\infty}\left\{\frac{f(t)}{t}\right\}\mathrm{d}t = \lim_{s\to 0} G(s), \tag{2.2.4}$$

provided the limit exists.

Exercise 2.2.4. Find the Laplace transform of $f(t) = \frac{e^{-at}-e^{-bt}}{t}$, where a and b are positive with $a \neq 0$. Hence evaluate $\displaystyle\int_0^{\infty}\left(\frac{e^{-at}-e^{-bt}}{t}\right)\mathrm{d}t$.

Exercise 2.2.5. Establish that

(a) $\displaystyle\int_0^{\infty}\left\{\frac{\sin at}{t}\right\}\mathrm{d}t = \frac{\pi}{2}$

(b) $\displaystyle\int_0^{\infty}\left\{\frac{\sin^2 3t}{t}\right\} e^{-t}\,\mathrm{d}t = \frac{1}{2}\log\sqrt{37}$

Exercise 2.2.6. Prove the following:

(a) $\displaystyle\int_0^{\infty}\left\{\frac{1-e^{-t}}{t}\right\}\mathrm{d}t = \log\left(\frac{s+1}{s}\right)$

(b) $\displaystyle\int_0^{\infty}\left\{\frac{1-\cosh 2t}{t}\right\}\mathrm{d}t = \log\left(\frac{\sqrt{s^2-4}}{s}\right)$

(c) $\displaystyle\int_0^{\infty}\left\{\frac{e^{-t}}{\sqrt{t}}\right\}\mathrm{d}t = \sqrt{\pi}$

(d) $\displaystyle\int_0^{\infty}\left\{\frac{1-e^{-t}}{t}\right\} e^{-2t}\,\mathrm{d}t = \log(3/2)$

(e) $\displaystyle\int_0^{\infty}\left\{\frac{\cos at - \cos bt}{t}\right\}\mathrm{d}t = \log(b/a)$

2.3 Frequency and Time Shiftings

Theorem 2.3.1 (Frequency Shifting)**.** Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a),\ s > a. \tag{2.3.1}$$

Exercise 2.3.1. Find the Laplace transforms of $\mathscr{L}\{e^{at} \cdot \sqrt{t}\}$ and $\mathscr{L}\{e^{at}/\sqrt{t}\}$.

Exercise 2.3.2. Find the Laplace transforms of $\mathscr{L}\{e^{at} \cdot t\}$, $\mathscr{L}\{e^{at} \cdot t^2\}$ and $\mathscr{L}\{e^{at} \cdot t^3\}$.

Exercise 2.3.3. Find

(a) $\mathscr{L}\{e^{at} \cdot \cos bt\}$

(b) $\mathscr{L}\{e^{at} \cdot \sin bt\}$

(c) $\mathscr{L}\{e^{at} \cdot \cosh bt\}$

(d) $\mathscr{L}\{e^{at} \cdot \sinh bt\}$

2.4 Laplace Transform of Discontinuous Functions

Definition 2.4.1 (Heaviside Unit Step Function)**.** The Heaviside Unit Step Function with parameter $a > 0$, is defined by

$$H(t-a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a. \end{cases}$$

Theorem 2.4.1. The Laplace transform of the Heaviside's function is given by

$$\mathscr{L}\left\{H(t-a)\right\} = \int_0^\infty H(t-a)e^{-st}\,\mathrm{d}t = \int\limits_a^\infty e^{-st}\,\mathrm{d}t = \left|-\frac{e^{-st}}{s}\right|_a^\infty = \frac{e^{-as}}{s} \tag{2.4.1}$$

for $s > 0$.

Theorem 2.4.2. Let

$$g(t) = 1 - H(t-a) = \begin{cases} 1, & \text{if } t < a \\ 0, & \text{if } t \geq a. \end{cases}$$

Then $\mathscr{L}\left\{g(t)\right\} = \dfrac{1-e^{-as}}{s}$.

Theorem 2.4.3 (Laplace transform of Rectangular Pulse)**.** Let

$$R(t;a,b) = H(t-a) - H(t-b) = \begin{cases} 1, & \text{if } a < t < b \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$\mathscr{L}\left\{R(t;a,b)\right\} = \mathscr{L}\left\{H(t-a) - H(t-b)\right\} = \frac{e^{-as} - e^{-bs}}{s}.$$

Exercise 2.4.1. Find the Laplace transform of the following functions $f(t)$:

(a) $\begin{cases} 2, & \text{if } 0 \leq t < 3 \\ 0, & \text{elsewhere.} \end{cases}$

(b) $\begin{cases} 3, & 1 \le t < 4 \\ 0, & t < 1 \text{ or } t \ge 4. \end{cases}$

(c) $\begin{cases} -1/2, & 0 \le t < 1 \\ 3, & 1 \le t < 3/2 \\ 1/5, & t \ge 3/2. \end{cases}$

(d) $\begin{cases} 0, & \text{if } 0 \le t < 1 \\ 1/2, & \text{if } 1 \le t < 2 \\ 2/3, & \text{if } 2 \le t < 5 \\ 3/4, & \text{elsewhere.} \end{cases}$

Theorem 2.4.4 (Laplace Transform of Shifted Signal)**.** Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\{H(t-a)f(t-a)\} = e^{-as}F(s),\ a \ge 0. \tag{2.4.2}$$

Exercise 2.4.2.

(a) Given that $\mathscr{L}\{t^2\} = 2/s^3$, find $\mathscr{L}\{H(t-a)(t-a)^2\}$

(b) Given that $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$, what is $\mathscr{L}\{H(t-\pi)\sin(t-\pi)\}$?

Theorem 2.4.5 (Laplace Transform of Truncated Signal)**.** Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\{H(t-a)f(t)\} = e^{-as}\mathscr{L}\{f(t+a)\},\ a \ge 0. \tag{2.4.3}$$

Exercise 2.4.3 (Self-check)**.** Find the Laplace transform of the following functions $f(t)$:

(a) $\begin{cases} \cos(\pi t/2), & \text{if } 3 \le t < 5 \\ 0, & \text{elsewhere.} \end{cases}$

(b) $\begin{cases} t^3, & \text{if } 1 \le t < 2 \\ 0, & t < 1 \text{ or } t \ge 2. \end{cases}$

(c) $\begin{cases} -t^2, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ 0, & \text{elsewhere.} \end{cases}$

(d) $\begin{cases} 2, & 0 \le t < 4 \\ t/2, & 4 \le t < 6 \\ 3, & \text{elsewhere.} \end{cases}$

(e) $\begin{cases} t, & \text{if } 0 \le t < 1 \\ 2-t, & \text{if } 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$

2.5 Dirac Delta Function

The delta-function is used in mechanical problems, where an object like a string is hit with a hammer or beat a drum with a stick, a rather large force acts for a short interval of time.

Let $a \geq 0$. Given $\epsilon > 0$, consider a large amount of force

$$I_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & \text{if } a \leq t < a+\epsilon \\ 0, & \text{elsewhere,} \end{cases}$$

applied in a short interval $[a, a+\epsilon)$ of length ϵ. This is written as

$$I_\epsilon(t) = \frac{H(t-a) - H(t-a-\epsilon)}{\epsilon}.$$

The limit of $I_\epsilon(t)$ as $\epsilon \to 0$ is called the *Dirac delta* function $\mathscr{D}(t-a)$. That is

$$\mathscr{D}(t-a) = \lim_{\epsilon \to 0} \frac{H(t-a) - H(t-a-\epsilon)}{\epsilon}. \tag{2.5.1}$$

Theorem 2.5.1 (Laplace Transform of Dirac Delta Function)**.**

$$\mathscr{L}\left\{\mathscr{D}(t-a)\right\} = e^{-as},\ a \geq 0.$$

Exercise 2.5.1. Find the Laplace transform of the following functions $f(t)$:

(a) $\mathscr{D}(t-1) - \mathscr{D}(t-3)$

(b) $(t-3)\,\mathscr{D}(t-3)$

(c) $(\cos \pi t)\,\mathscr{D}(t-1)$

2.6 Periodic Functions

Definition 2.6.1 (Periodic Function)**.** A real valued function $f(t)$ is said to be a periodic function, if there exists a positive real number τ such that

$$f(t+\tau) = f(t) \text{ for all } t. \tag{2.6.1}$$

The least τ is called the period of f. Let $f(t)$ be a periodic function with period $\tau > 0$. Then its graph is repeated in regular intervals of length τ.

Theorem 2.6.1 (Laplace Transform of Periodic Functions)**.** Let $f(t)$ be a periodic function with period τ. Then its Laplace transform is given by

$$\mathscr{L}\left\{f(t)\right\} = \frac{1}{1-e^{-s\tau}} \int_0^\tau e^{-st}\, f(t)\, \mathrm{d}t. \tag{2.6.2}$$

Exercise 2.6.1. Find the Laplace transform of each of the following periodic functions $f(t)$:

(a) $\begin{cases} 1, & 0 < t < \frac{a}{2} \\ -1, & \frac{a}{2} < t < a, \end{cases}$ period a

(b) $\begin{cases} \sin \omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}, \end{cases}$ period $2\pi/\omega$

(c) $\sin 2t$ with period π

(d) $t(l-t)$ with period l

(e) $\begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a, \end{cases}$ with period $2a$

(f) $\begin{cases} \cos t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi, \end{cases}$ with period 2π

(g) $\begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \\ -1, & 2 < t < 3 \\ 0, & 3 < t < 4 \end{cases}$ with period 4

2.7 Inverse Laplace Transform of Elementary Functions

Theorem 2.7.1. We have

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$ for $n = 1, 2, 3, \ldots$

(b) $\mathscr{L}^{-1}\left\{\frac{1}{s^p}\right\} = \frac{t^{p-1}}{\Gamma(p)}$ for real $p > 0$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(d) $\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \cdot \sin at$

(e) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

(f) $\mathscr{L}^{-1}\left\{\frac{e^{as}}{s}\right\} = H(t-a)$, where $a \geq 0$

Theorem 2.7.2 (Linearity of the Inverse Laplace Transform)**.** Let $f(t) = \mathscr{L}^{-1}\left\{\bar{F}(s)\right\}$, $g(t) = \mathscr{L}^{-1}\left\{\bar{G}(s)\right\}$. Then for any scalars a and b,

$$\mathscr{L}^{-1}\left\{a\bar{F}(s) + b\bar{G}(s)\right\} = af(t) + bg(t). \tag{2.7.1}$$

We have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a} \cdot \sinh at = \frac{e^{at}-e^{-at}}{2a},\ \mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at = \frac{e^{at}+e^{-at}}{2}.$$

Theorem 2.7.3 (Inverse - First Shifting). If $\mathscr{L}^{-1}\{F(s)\} = f(t)$, then $\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$. That is

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at} \cdot \mathscr{L}^{-1}\{F(s)\} \cdot \tag{2.7.2}$$

We have

(a) $\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = e^{at} \cdot \frac{t^{n-1}}{(n-1)!}, n = 1, 2, , ...$

(b) $\mathscr{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = e^{-at} \cdot \frac{t^{n-1}}{(n-1)!}, n = 1, 2, , ...$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^p}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^p}\right\} = e^{at} \cdot \frac{t^{p-1}}{\Gamma(p)}$ for real $p > 0$

(d) $\mathscr{L}^{-1}\left\{\frac{1}{(s+a)^p}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^p}\right\} = e^{-at} \cdot \frac{t^{p-1}}{\Gamma(p)}$ for real $p > 0$

(e) $\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} = e^{at} \cdot \frac{\sin bt}{b}$

(f) $\mathscr{L}^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} = e^{-at} \cdot \frac{\sin bt}{b}$

(g) $\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^2-b^2}\right\} = e^{at} \cdot \frac{\sinh bt}{b}$

(h) $\mathscr{L}^{-1}\left\{\frac{1}{(s+a)^2-b^2}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^2-b^2}\right\} = e^{-at} \cdot \frac{\sinh bt}{b}$

Exercise 2.7.1. Employing Theorem 2.7.3, find

(a) $\mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}$

(b) $\mathscr{L}^{-1}\left\{\frac{s}{(s-a)^2+b^2}\right\}$

(c) $\mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\}$

(d) $\mathscr{L}^{-1}\left\{\frac{s}{(s-a)^2-b^2}\right\}$

(e) $\mathscr{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\}$

(f) $\mathscr{L}^{-1}\left\{\frac{s}{(s+a)^2+b^2}\right\}$

(g) $\mathscr{L}^{-1}\left\{\frac{s+a}{(s+a)^2-b^2}\right\}$

(h) $\mathscr{L}^{-1}\left\{\frac{s}{(s+a)^2-b^2}\right\}$.

Theorem 2.7.4 (Second Shifting). If $\mathscr{L}^{-1}\left\{\bar{F}(s)\right\} = f(t)$, then

$$\mathscr{L}^{-1}\left\{e^{-as}\bar{F}(s)\right\} = f(t-a) \cdot H(t-a) \tag{2.7.3}$$

where $a \geq 0$.

As an application of Theorem 2.7.4, we have

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^n}\right\} &= \left|\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\}\right|_{t\to(t-a)} H(t-a) = \frac{(t-a)^{n-1}}{(n-1)!}\cdot H(t-a)\\
\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s-b}\right\} &= \left|\mathscr{L}^{-1}\left\{\frac{1}{s-b}\right\}\right|_{t\to(t-a)} H(t-a) = e^{b(t-a)}\cdot H(t-a)\\
\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^2+b^2}\right\} &= \left|\mathscr{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right|_{t\to(t-a)} H(t-a)\\
&= \left|\frac{\sin bt}{b}\right|_{t\to(t-a)} H(t-a) = \frac{\sin b(t-a)}{b}\cdot H(t-a)\\
\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^2-b^2}\right\} &= \left|\mathscr{L}^{-1}\left\{\frac{1}{s^2-b^2}\right\}\right|_{t\to(t-a)} H(t-a)\\
&= \left|\frac{\sinh bt}{b}\right|_{t\to(t-a)} H(t-a) = \frac{\sinh b(t-a)}{b}\cdot H(t-a)\\
\mathscr{L}^{-1}\left\{\frac{se^{-as}}{s^2+b^2}\right\} &= \left|\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right|_{t\to(t-a)} H(t-a)\\
&= H(t-a)\left|\cos bt\right|_{t\to(t-a)} = H(t-a)\cos b(t-a)\\
\mathscr{L}^{-1}\left\{\frac{se^{-as}}{s^2-b^2}\right\} &= \left|\mathscr{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\}\right|_{t\to(t-a)} H(t-a)\\
&= H(t-a)\left|\cosh bt\right|_{t\to(t-a)} = H(t-a)\cosh b(t-a).
\end{aligned}$$

Exercise 2.7.2. Using any one of the above techniques, find the inverse of each of the following Laplace transforms:

(a) $\frac{1}{s}+\frac{s}{s^2+4}$ **(b)** $\frac{s^2-1}{s^3}$ **(c)** $\frac{(2+s)^2}{s^5}$

(d) $\frac{s^2-3s+4}{s^3}$ **(e)** $\frac{s}{s^2+16}+\frac{2}{s-3}+\frac{s+1}{s^3}$ **(f)** $\frac{3(s^2-2)^2}{2s^5}$

(g) $\frac{3s-1}{(s+1)^4}$ **(h)** $\frac{1}{s+4}-\frac{6}{(s-4)^2}$ **(i)** $\frac{3}{s-7}+\frac{1}{s^3}$

(j) $\frac{2s-5}{4s^2+25}+\frac{4s-18}{9-s^2}$ **(k)** $\frac{s}{s^2+6s+13}$ **(l)** $\frac{3}{s^2+6s+18}$

(m) $\frac{s}{2s^2+2s+1}$ **(n)** $\frac{3s+7}{s^2-2s-3}+\frac{2s-3}{s^2+4s+13}$ **(o)** $\frac{s+1}{s^2+4s+16}$

(p) $\frac{3s-1}{s^2-6s+2}$ **(q)** $\frac{s-4}{s^2-8s+10}$

Inverse Laplace Transform by Partial Fractions

Exercise 2.7.3 (Rational Fractions with Linear Factors in the Denominator)**.** Find the inverse of each of the following Laplace transforms:

(a) $\frac{1}{(s-1)(s+2)(s-3)}$

(b) $\frac{s-1}{(s-2)(s-3)(s-4)}$

(c) $\frac{s^2+1}{s(s-1)(s+1)(s-2)}$

(d) $\frac{1}{(s-1)(s+2)(s-3)}$

(e) $\frac{1}{(s-1)(s+2)(s-3)}$

Exercise 2.7.4. Rational Fractions with Linear Factors in the Denominator] Find the inverse of each of the following Laplace transforms:

(a) $\dfrac{3s}{(s-1)(s^2-4)}$ (b) $\dfrac{1}{(s+3)(s+7)}$ (c) $\dfrac{s-1}{(s+1)(s-3)}$

(d) $\dfrac{s-8}{s^2-3s-4}$ (e) $\dfrac{s^2-7s+5}{(s-1)(s-2)(s-3)}$ (f) $\dfrac{s^2+s-2}{s(s-2)(s+3)}$

(g) $\dfrac{s+10}{s^2-s-2}$ (h) $\dfrac{s-4}{s^2-4}$ (i) $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$

(j) $\dfrac{1}{s^3-s}$ (k) $\dfrac{6}{(s+2)(s-4)}$ (l) $\dfrac{s^2+9s-9}{s^3-9s}$

(m) $\dfrac{s^2-6s+4}{s^3-3s^2+2s}$ (n) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

Exercise 2.7.5 (Rational Fractions with Quadratic Factors in the Denominator)**.** Find the inverse of each of the following Laplace transforms:

(a) $\dfrac{s^2+1}{s(s-1)(s+2)^2}$ (b) $\dfrac{4s}{(s-1)(s+1)^2}$ (c) $\dfrac{s+1}{s^2(s+1)}$

(d) $\dfrac{1}{s^3(s+1)}$ (e) $\dfrac{s}{(s-1)^2(s+2)^2}$ (f) $\dfrac{4s+5}{(s-1)^2(s+2)}$.

Exercise 2.7.6 (Rational Fractions with Simple Quadratic Factors in the Denominator)**.** Find the inverse of each of the following Laplace transforms:

(a) $\dfrac{s^2-8}{(s^2+5)(s^2-7)}$ (b) $\dfrac{s^3}{s^4-a^4}$ (c) $\dfrac{s}{s^4+s^2+1}$

(d) $\dfrac{1}{(s^2-a^2)^2}$ (e) $\dfrac{s}{s^4+1}$ (f) $\dfrac{s}{s^2(s^2+9)}$

(g) $\dfrac{1}{s^4-16}$ (h) $\dfrac{s}{(s^2-1)^2}$

Exercise 2.7.7 (Linear and Quadratic Factors in the Denominator)**.** Find the inverse of each of the following Laplace transforms:

(a) $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ (b) $\dfrac{s+1}{(s^2+1)(s^2+4)}$ (c) $\dfrac{5s^2-7s+17}{(s-1)(s^2+4)}$

(d) $\dfrac{s^2+2s-4}{(s^2+9)(s-5)}$ (e) $\dfrac{s}{(s^2+2s+5)(s-7)}$ (f) $\dfrac{5s-7}{(s+3)(s^2+2)}$

(g) $\dfrac{s}{(s-1)(s^2+2s+2)}$ (h) $\dfrac{36}{s(s^2+1)(s^2+9)}$ (i) $\dfrac{1}{(s+1)(s+2)(s^2+2s+2)}$

2.8 Inverse Laplace Transform by Convolution Theorem

Definition 2.8.1 (Convolution)**.** Let f and g be piece-wise continuous on $[0,\infty)$, then the special product $f*g$, defined by the integral

$$(f*g)(t)=\int_0^t f(v)g(t-v)\,\mathrm{d}v \tag{2.8.1}$$

is called the *convolution integral* or simply *convolution* of f and g.

Theorem 2.8.1 (Convolution Theorem). Let f and g be piece-wise continuous on the interval $[0, \infty)$, and have exponential orders, then the Laplace transform of the convolution of f and g equals the product of the Laplace transforms of f and g, that is

$$\mathscr{L}(f * g)(t) = \mathscr{L}\left\{\int_0^t f(v)g(t-v)\,\mathrm{d}v\right\} = \mathscr{L}\{f(t)\} \cdot \mathscr{L}\{g(t)\} \tag{2.8.2}$$

The inverse form of the convolution theorem states that

$$\mathscr{L}^{-1}\{F(s) \cdot G(s)\} = (f * g)(t) = \int_0^t f(v)g(t-v)\,\mathrm{d}v, \tag{2.8.3}$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and $g(t) = \mathscr{L}^{-1}\{G(s)\}$.

Remark 2.8.1 (Division by s). Write $G(s) = 1/s$ in the convolution theorem, we have

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_{v=0}^t f(v)\,\mathrm{d}v, \tag{2.8.4}$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$. In other words,

$$\mathscr{L}\left\{\int_{v=0}^t f(v)\,\mathrm{d}v\right\} = \frac{F(s)}{s}. \tag{2.8.5}$$

Exercise 2.8.1. Use convolution theorem to find the following:

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+a)s}\right\}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s+a)}\right\}$

(c) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)s}\right\}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$

(e) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+4)(s-1)}\right\}$.

Exercise 2.8.2. Which method is convenient to find

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)s^2}\right\}$,

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$?

Also, find the inverse transform in each case.

Exercise 2.8.3. Use the convolution theorem to solve the following integral equations:

(a) $f(t) + \displaystyle\int_0^t (t-\tau)f(\tau)\,\mathrm{d}\tau = t$

(b) $f(t) = 2t - \int_0^t \sin\tau\, f(t-\tau)\ \mathrm{d}\tau$

(c) $f(t) = t\,e^t - \int_0^t \tau\, f(t-\tau)\ \mathrm{d}\tau$

(d) $f(t) = \cos t + \int_0^t e^{-\tau}\, f(t-\tau)\ \mathrm{d}\tau$

Text and Reference books

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2. Erwin Kreyszig, *Advanced Engineering Mathematics*, 10th Edition, John Wiley & Sons (2011), Ch. 6

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4. Peter V. O'Neil, *Advanced Engineering Mathematics*, 7th Ed., Cengage (2011), Sec. 3.3, 3.5

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Chapter 3

Multi-variable Differential Calculus

3.1 Geometry of Multi-variable Functions

Definition 3.1.1 (Functions of Two Variables)**.** Let $\mathscr{D} \subset \mathbb{R}^2$. A function $f : \mathscr{D} \to \mathbb{R}$ is called a real-valued function of *two* variables. The elements of $\mathscr{D}$ are ordered pairs (x, y) of real numbers, where x and y are called *input variables* and the real number $w = f(x, y)$, an *output variable* of the function f.

Definition 3.1.2 (Surface and Trace)**.** Let $f(x, y)$ be a two-variable function defined on $\mathscr{D} \subset \mathbb{R}^2$. The the *graph* $\{(x, y, f(x, y)) : (x, y) \in \mathscr{D}\}$ of $f(x, y)$ defines a *surface* $z = f(x, y)$ in space. The curve of intersection of the surface $z = f(x, y)$ and the plane $z = c$ is known as a *trace* or *plane section* of the surface in the plane $z = c$. Other traces are similarly defined.

Definition 3.1.3 (Level Curves and Contours)**.** Let $f(x, y)$ be a two-variable function defined on $\mathscr{D} \subset \mathbb{R}^2$. Let c be a constant. The curve lying in the xy-plane, given by $f(x, y) = c$ is called a *level curve* of f. Thus f takes on a constant value on its level curve. A *contour* $f(x, y) = c$ is regarded as a curve of intersection of the surface $z = f(x, y)$ and the plane $z = c$.

For $c > 0$, it lies at a height of c units from the xy-plane, and hence is a line of *constant elevation*. Whereas, for $c < 0$, it lies at a depth of c units from the xy-plane, and hence is a line of *constant depression*.

Definition 3.1.4 (Function of Three Variables)**.** A three-variable function is a mapping from $\mathscr{D}$ into $\mathbb{R}$, where $\mathscr{D} \subset \mathbb{R}^3$.

Definition 3.1.5 (Graph of a Function of Three Variables)**.** Let $w = f(x, y, z)$ be a function of three variables defined on $\mathscr{D} \subset \mathbb{R}^3$. The set of points $\{(x, y, z, f(x, y, z)) : (x, y, z) \in \mathscr{D}\}$ defines the *graph* of $f(x, y, z)$.

Definition 3.1.6 (Level Surface)**.** Let $w = f(x, y, z)$ be a function of three variables. The surface defined by $f(x, y, z) = c$, where c is a constant, is a *level surface* of f.

Thus f takes on a constant value on its level surface, and for various c-values, we get corresponding level surfaces.

Example 3.1.1. The level surfaces of $\phi = \sqrt{x^2 + y^2 + z^2}$ are the concentric spheres $x^2+y^2+z^2 = c$ with common centre at the Origin.

Cylindrical Polar Coordinates:

In 1671, Newton introduced the polar coordinate system, did not publish his result. However, the credit for the discovery of polar coordinates is attributed to J. Bernoulli as he published a paper containing polar coordinates in 1691. The formal extension of plane polar coordinates to three dimensions is the *cylindrical polar coordinate system*.

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z. \tag{3.1.1}$$

The inverse of the transformation (3.1.1) is given by

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\ z = z,$$

provided $r \neq 0$.

Remark 3.1.1. Cylindrical coordinates are used, if the value of a function $f(x, y, z)$ depends on the distance of the point (x, y, z) from the axis of symmetry, usually the z-axis, (*cylindrical symmetry*) - commonly found in towers, columns, and domes.

Spherical Polar Coordinates:

In 1773, Joseph Louis Lagrange while working on the total gravitational attraction, introduced *spherical polar coordinates* to compute the volume of an ellipsoid of revolution. The *spherical polar coordinates* ρ, ϕ and θ are described by the transformation:

$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi. \tag{3.1.2}$$

The inverse of the transformation (3.1.2) is given by

$$\rho = \sqrt{x^2 + y^2 + z^2},\ \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right),\ \theta = \tan^{-1}\left(\frac{y}{x}\right),$$

provided $\rho^2 \sin\phi \neq 0$.

Remark 3.1.2. Spherical coordinates are convenient to used when the value of a function $f(x, y, z)$ depends only on the distance of the point (x, y, z) from the origin (*spherical symmetry*).

3.2 Limit and Continuity

Definition 3.2.1 (Limit)**.** A function $f(x, y)$ is said to be continuous at a point $P(x_0, y_0)$ in the domain D of f, if the double limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists and is equals $f(x_0, y_0)$. The function f is said to be continuous on the domain $\mathscr{D}$, if it is continuous at every point of $\mathscr{D}$.

Theorem 3.2.1 (Existence of Limit)**.** If the f has different limits along different paths in $\mathscr{D}$ as (x, y) approaches (x_0, y_0), then the double or simultaneous limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Definition 3.2.2 (Continuity)**.** A function $f(x, y)$ is said to be continuous at a point $P(x_0, y_0)$ in the domain D of f, if the double limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists and is equals $f(x_0, y_0)$. The function f is said to be continuous on the domain $\mathscr{D}$, if it is continuous at every point of $\mathscr{D}$.

Exercise 3.2.1. Find the domain of definition of each of the following functions:

(a) $f(x, y) = \log(x^2 + y^2)$

(b) $f(x, y) = \frac{y}{x^2+1}$, $g(x, y) = \frac{x+y}{2+\cos x}$

(c) $f(x, y) = \sin\left(\frac{1}{xy}\right)$

(d) $f(x, y) = \sin\left(\frac{x^2+y^2}{x^2-3x+2}\right)$

Exercise 3.2.2. Show that the following functions have no limit as $(x, y) \to (0, 0)$, and hence are discontinuous at the origin $(0, 0)$:

(a) $f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} \frac{x-y}{x+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(c) $f(x,y) = \begin{cases} \frac{xy}{|xy|}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(d) $f(x,y) = \begin{cases} \frac{x^2-y}{x-y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(e) $f(x,y) = \begin{cases} \frac{x^4}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Exercise 3.2.3. By an appropriate substitution, show that the limit of

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

varies from -1 to 1 along the line $y = mx$ as $(x, y) \to (0, 0)$, and hence does not exist at the origin.

Exercise 3.2.4. Examine the continuity of the $f(x, y)$ at the origin, where

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Exercise 3.2.5. Use the polar coordinates to show that $f(x, y)$ is not continuous at the origin, where

$$f(x,y) = \begin{cases} \tan^{-1}\left(\frac{|x|+|x|}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Exercise 3.2.6. Using the polar coordinates, examine the continuity of the following functions at the origin:

(a) $f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} \cos\left(\frac{x^3-xy^2}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

3.3 Partial Differentiation

Let $u = f(x, y)$ be defined on a domain $\mathscr{D} \subset \mathbb{R}^2$ and $(x_0, y_0) \in \mathscr{D}$. The first order partial derivatives of f with respect to x and y are given by

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k},$$

which are denoted bt f_x and f_y respectively. Second order partial derivatives of f at any point are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \text{ or } f_{xx} = (f_x)_x,$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{ or } f_{yy} = (f_y)_y,$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{ or } f_{yx} = (f_y)_x,$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \text{ or } f_{xy} = (f_x)_y.$$

Higher order partial derivatives are similarly defined. The rules of finding partial derivatives of sum, difference, product and quotient are similar to those of ordinary derivatives.

Chain Rule of Partial Differentiation

Theorem 3.3.1 (One Intermediate Variable and Two Independent Variables)**.** Let $u = f(r)$, where $r = g(x, y)$. Then u is a composite function of x and y through the intermediate variable r, and

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x} = f'(r)\frac{\partial r}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial y} f'(r)\frac{\partial r}{\partial y}.$$

Theorem 3.3.2 (Two Intermediate Variables and One Independent Variable)**.** Let $u = f(x, y)$, where $x = g_1(t),\ y = g_2(t)$. Then u is a composite function of t through the intermediate variables x and y. The derivative of w with respect to t, given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{3.3.1}$$

is called the *total derivative* of f with respect to t.

Exercise 3.3.1. Using the definition, find the total derivative $\frac{\mathrm{d}u}{\mathrm{d}t}$ at a given value of t, and verify your result by the direct differentiation:

(a) $u = \sin\left(\frac{x}{y}\right)$, where $x = e^t, y = t^2$

(b) $u = \sin^{-1}(x - y)$, where $x = 3t, y = 4t^3$

(c) $u = x^2 + y^2$, where $x = \cos t + \sin t, y = \cos t - \sin t$

(d) $u = \frac{x+y}{z}$, where $x = \cos^2 t, y = \sin^2 t, z = 1/t$ at $t = 3$

(e) $u = xy + yz + zx$, where $x = e^t, y = e^{-t}, z = 1/t$

Exercise 3.3.2. Suppose that the partial derivatives of a function $f(x, y, z)$ at points on the helix $x = \cos t, y = \sin t, z = t$ are $f_x = \cos t, f_y = \sin t, f_z = t^2 + t - 2$. At what points on the curve, if any, can f take on extreme values?

Exercise 3.3.3. Let $f = x^2 e^{2y} \cos 3z$. Compute the total derivative $\frac{\mathrm{d}f}{\mathrm{d}t}$ at the point $(l, \log 2, 0)$ on the curve $x = \cos t, y = \log(t + 2), z = t$.

Theorem 3.3.3 (Two Intermediate Variables and Two Independent Variables)**.** Let $u = f(r, s)$, where $r = g_1(x, y)$, $s = g_2(x, y)$. Then u is a composite function of x and y through the intermediate variables r and s, and

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x},$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y}.$$

Exercise 3.3.4. Find the first order partial derivatives of each of the following functions:

(a) $f(x, y) = \frac{x}{x^2+y^2}$

(b) $f(x, y) = e^{xy} \sin(x + y)$

(c) $f(x, y) = x^y$

(d) $f(x, y) = \log_y x = \frac{\log x}{\log y}$

(e) $g(u, v) = v^2 e^{2u/v}$

(f) $h(x, y) = \cos^2(3x - y)^2$

(g) $f(x, y) = x^2 - xy + y^2$

(h) $f(x, y) = \frac{x+y}{xy-1}$

(i) $f(x, y) = \displaystyle\int_x^y g(\xi)\, d\xi$

(j) $f(t, \alpha) = \cos(2\pi t - \alpha)$

(k) $f(x, y, z) = \sin^{-1}(xyz)$

(l) $f(x, y, z) = 1 + xy^2 - 2z^2$

(m) $f(x, y, z) = x - e^{-xyz}$

(n) $g(r, \theta, z) = r(1 - \cos\theta) - z$

(o) $f(r, l, T, \omega) = \frac{1}{2rl}\sqrt{\frac{T}{\pi\omega}}$

Exercise 3.3.5. Find the partial derivative of each of the following functions with respect to each independent variable:

(a) $f(x, y, z) = x - \sqrt{y^2 + z^2}$

(b) $f(x, y, z) = x - \log(ax + by + cz)$

(c) $f(x, y, z) = e^{x^2+y^2+z^2}$

(a) $f(x, y) = x^2y + \cos y + y \sin x$

(b) $f(x, y) = \tan^{-1}(y/x)$

(c) $f(x, y) = y + \frac{x}{y}$

(d) $f(x, y) = \frac{x-y}{x^2+y}$

(e) $f(x, y) = ye^{x^2-y}$

Exercise 3.3.6. Verify whether $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for each of the following functions:

(a) $f(x, y) = \log(2x + 3y)$

(b) $f(x, y) = x \sin y + y \sin x + xy$

(c) $f(x, y) = xy^2 + x^2y^3 + x^3y^4$

(d) $f(x, y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

(e) $f(x, y) = (\log x) \tan^{-1}\left(x^2 + y^2\right)$

Exercise 3.3.7 (Three-dimensional Steady State Heat Flow). The three-dimensional Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{3.3.2}$$

is satisfied by the steady-state temperature distributions $\theta = f(x, y, z)$ in space, by gravitational and electrostatic potentials. The solutions of (3.3.2) are known as *harmonic functions*. Verify if each of the following functions is harmonic:

(a) $f(x, y) = \tan^{-1}\left(\frac{x}{y}\right)$

(b) $f(x, y) = \log\sqrt{x^2 + y^2}$

(c) $f(x, y) = e^{3x+4y} \cos 5z$

(d) $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$

(e) $f(x, y, z) = ax + by + cz$

Exercise 3.3.8 (One-dimensional Wave Equation)**.** Verify whether the following functions are solutions of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.3.3}$$

(a) $u(x, y) = \sin(x + ct)$

(b) $u(x, y) = \log[2(x + ct)]$

(c) $u(x, y, z) = \sin(x + ct) + \cos(x - ct)$

(d) $u(x, y, z) = 5\cos 3(x + ct) + e^{x+ct}$

Exercise 3.3.9 (One dimensional heat equation). The equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} \tag{3.3.4}$$

refers to the heat conduction along a metal bar without radiation.

(a) If $u = Ae^{-gx}\sin(nt - gx)$, show that $g = \sqrt{n/2\mu}$

(b) Show that $u = \frac{1}{\sqrt{t}} \cdot e^{-x^2/4a^2t}$ is a solution of (3.3.4)

(c) Show that $u = \sin(ax)e^{-bt}$ is a solution of (3.3.4), where a and b are constants.

Definition 3.3.1 (Total Differential). Let $u = f(x, y)$ be defined on a domain $\mathscr{D} \subset \mathbb{R}^2$ and $(x_0, y_0) \in \mathscr{D}$. If we move from (x_0, y_0) to a point $(x_0 + \mathrm{d}x, y_0 + \mathrm{d}y)$ nearby, the resulting change in u is given by its total differential:

$$\mathrm{d}u \text{ or } \mathrm{d}f = \left(\frac{\partial u}{\partial x}\right)_{(x_0,y_0)} \mathrm{d}x + \left(\frac{\partial u}{\partial y}\right)_{(x_0,y_0)} \mathrm{d}y. \tag{3.3.5}$$

3.3 Implicit Differentiation

Consider an implicit relation

$$f(x, y) = c, \tag{3.3.6}$$

where y is a differentiable function of x. Using the chain rule of partial differentiation, we see that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \text{ so that } \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Finding the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ from (3.3.6) is known as implicit differentiation.

Exercise 3.3.10. Find the indicated derivative from each of the following implicit relations:

(a) $r^2 = \sin 2\theta, \mathrm{d}r/\mathrm{d}\theta$

(b) $xy^2 = e^x - e^y, \mathrm{d}y/\mathrm{d}x$

(c) $y^2 = \log xy, \mathrm{d}y/\mathrm{d}x$

(d) $x^2 + y^2 = \log(x + y)^2, \mathrm{d}y/\mathrm{d}x$

Exercise 3.3.11. Show that the curves, given below are orthogonal at the indicated point:

(a) $y^2 = x^3$ and $2x^2 + 3y^3 = 5$, $(1, 1)$

(b) $y^3 + 3x^2y = 13$ and $2x^2 - 2y^2 = 3x$, $(2, 1)$

3.4 Jacobians

Definition 3.4.1 (Second Order Jacobian)**.** Consider the transformation

$$u = f(x, y),\ v = g(x, y). \tag{3.4.1}$$

The Jacobian of u and v with respect to x and y is given by

$$J\left(\frac{u, v}{x, y}\right) \equiv \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Exercise 3.4.1. Find $J\left(\frac{u,v}{x,y}\right)$, where

(a) $u = x \sin y,\ v = y \sin x$

(b) $u = x^2 - 2y,\ v = x + y$

(c) $u = ax + by,\ v = cx + dy$

(c) $u = x^2 - y^2,\ v = 2xy.$

Theorem 3.4.1 (Inverse Property of Jacobians)**.** If $J\left(\frac{u,v}{x,y}\right) \neq 0$, then the Jacobian transformation (3.4.1) is invertible, and

$$J\left(\frac{u, v}{x, y}\right) \cdot J\left(\frac{x, y}{u, v}\right) = 1 \text{ or } J\left(\frac{x, y}{u, v}\right) = 1/J\left(\frac{u, v}{x, y}\right). \tag{3.4.2}$$

Exercise 3.4.2. Let $u = x(1 - y),\ v = xy$. Then $J\left(\frac{x,y}{u,v}\right)$.

$$= 1/J\left(\frac{u, v}{x, y}\right) = 1/\begin{vmatrix} 1 - y & -x \\ y & x \end{vmatrix} = \frac{1}{x} = \frac{1}{u + v}.$$

Exercise 3.4.3. Verify the property, given in (3.4.1) for plane polar transformation.

Theorem 3.4.2 (Chain Rule of Jacobians)**.** Consider the transformations:

$$u = f(r, s),\ v = g(r, s),$$

where

$$r = p(x, y),\ s = q(x, y).$$

Then

$$J\left(\frac{u, v}{x, y}\right) = J\left(\frac{u, v}{r, s}\right) J\left(\frac{r, s}{x, y}\right). \tag{3.4.3}$$

Exercise 3.4.4. If $x = ar \cos\theta,\ y = br \sin\theta$. Let $u = r\cos\theta,\ v = r \sin\theta$. Then $x = au,\ y = bv$, find $J\left(\frac{x,y}{r,\theta}\right)$.

Definition 3.4.2 (Third Order Jacobian)**.** Consider the transformation

$$u = f(x, y, z),\ v = g(x, y, z)\ w = h(x, y, z). \tag{3.4.4}$$

The Jacobian of u, v and w with respect to x, y and z is given by

$$J\left(\frac{u, v, w}{x, y, z}\right) \equiv \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Exercise 3.4.5. Find $J\left(\frac{u,v,w}{x,y,z}\right)$, where:

(a) $u = \frac{2x-y}{2},\ v = \frac{y}{2}, w = \frac{z}{3}$

(b) $u = \frac{yz}{x},\ v = \frac{zx}{y}, w = \frac{xy}{z}$

(c) $u = x^2 - 2y,\ v = x + y + z,\ w = x - 2y + 3z.$

Exercise 3.4.6. Find $J\left(\frac{x,y,z}{u,v,w}\right)$, where:

(a) $u = x + y + z,\ uv = y + z,\ uvw = z$

(b) $u = \frac{yz}{x},\ v = \frac{zx}{y},\ w = \frac{xy}{z}.$

Exercise 3.4.7. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$, where the *cylindrical polar coordinates* r, θ and z are described by the transformation (3.1.1). Also, verify that $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} \cdot \frac{\partial(r,\theta,z)}{\partial(x,y,z)} = 1.$

Exercise 3.4.8. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}$, where the *spherical polar coordinates* ρ, ϕ and θ are described by the transformation(3.1.2). Then verify that

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \cdot \frac{\partial(\rho, \phi, \theta)}{\partial(x, y, z)} = 1.$$

Exercise 3.4.9. If $x = a\rho \sin\phi \cos\theta, y = b\rho \sin\phi \sin\theta, z = c\rho \cos\phi$, find $J\left(\frac{x,y,z}{\rho,\phi,\theta}\right)$.

Theorem 3.4.3 (Jacobians and Functional Dependence)**.** Consider the transformation (3.4.1). Then u and v are functionally dependent of each other if and only if

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0. \tag{3.4.5}$$

Exercise 3.4.10. Let $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ and $v = \sin^{-1} x + \sin^{-1} y$. Verify Theorem 3.4.3 and hence find the functional relationship between u and v.

Exercise 3.4.11. Let $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$. Verify that u and v are functionally dependent and hence find the functional relationship between u and v.

Theorem 3.4.4 (Jacobians and Functional Dependence)**.** Consider the transformation (3.4.4). Then u, v and w are functionally dependent if and only if

$$J(\frac{u, v, w}{x, y, z}) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0. \tag{3.4.6}$$

Exercise 3.4.12. Let $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x^2 + 2xy - zx$. Verify that u, v and w are functionally dependent and hence find the functional relationship among them.

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 13.1-13.5

2. Briggs et al., *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © (2013) Pearson Education, Inc., Sec. 13.1-13.5

3. Grewal, B. S., *Higher Engineering Mathematics*, 42nd Edition (2012), Khanna Publishers, Sec. 5.7

4. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 14.1-14.5

5. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 12.1-12.5

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Chapter 4

Applications of Multi-variable Differential Calculus

4.1 Taylor's theorem for Functions of Two Variables

Let $n \geq 1$. Suppose that $f(x, y)$ and its partial derivatives through order $n+1$ are continuous in a small neighborhood $\mathscr{R}(a, b)$, centered at the point (a, b). Then

$$
\begin{aligned}
f(a+h, b+k) &= \left|\left\{ e^{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}} \right\} f\right|_{(a,b)} \\
&= \left\{ 1 + \tfrac{1}{1!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right) + \tfrac{1}{2!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right)^2 \right. \\
&\qquad \left. + \tfrac{1}{3!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right)^3 + \cdots + \tfrac{1}{n!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right)^n \right\} f + R_n \\
&= f + \tfrac{1}{1!}\left(h\tfrac{\partial f}{\partial x} + k\tfrac{\partial f}{\partial y}\right) + \tfrac{1}{2!}\left(h^2\tfrac{\partial^2 f}{\partial x^2} + 2hk\tfrac{\partial^2 f}{\partial x \partial y} + k^2\tfrac{\partial^2 f}{\partial y^2}\right) \\
&\qquad + \tfrac{1}{3!}\left(h^3\tfrac{\partial^3 f}{\partial x^3} + 3h^2k\tfrac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\tfrac{\partial^3 f}{\partial x \partial y^2} + k^3\tfrac{\partial^3 f}{\partial y^3}\right) \\
&\qquad\quad + \cdots + \tfrac{1}{n!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right)^n f + R_n, \qquad (4.1.1)
\end{aligned}
$$

where f and all its partial derivatives are evaluated at the point (a, b), and

$$R_n = \tfrac{1}{(n+1)!}\left(h\tfrac{\partial}{\partial x} + k\tfrac{\partial}{\partial y}\right)^{n+1} f(a + ch + b + ck),$$

for some $0 < c < 1$, and R_n is called the Taylor's remainder after n terms.

In terms of subscript notation, we have

$$
\begin{aligned}
f(a+h, b+k) &= f + \tfrac{1}{1!}(hf_x + kf_y) + \tfrac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right) \\
&\qquad + \tfrac{1}{3!}\left(h^3 f_{xxx} + 3h^2k f_{yxx} + 3hk^2 f_{yyx} + k^3 f_{yyy}\right) + \cdots + R_n \qquad (4.1.2)
\end{aligned}
$$

In view of the continuity of the partial derivatives of all orders, we see that $f_{yxx} = f_{xxy}$, $f_{yyx} = f_{xyy}$ etc. Therefore, (4.1.2) is also written as

$$
\begin{aligned}
f(a+h, b+k) &= f + \tfrac{1}{1!}(hf_x + kf_y) + \tfrac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right) \\
&\qquad + \tfrac{1}{3!}\left(h^3 f_{xxx} + 3h^2k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right) + \cdots + R_n \qquad (4.1.3)
\end{aligned}
$$

Replacing (a, b) with $(0, 0)$ and then h with x, k with y in (4.1.3), we get

$$
\begin{aligned}
f(x, y) &= f + \tfrac{1}{1!}(xf_x + yf_y) + \tfrac{1}{2!}\left(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right) \\
&\qquad + \tfrac{1}{3!}\left(x^3 f_{xxx} + 3x^2y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right) + \cdots + R_n, \qquad (4.1.4)
\end{aligned}
$$

where f and all its partial derivatives are evaluated at $(0, 0)$.

Approximations using Taylor's theorem

For $n=1$: the linear approximation of f about the origin $(0,0)$, is given by

$$f(x,y) \approx f(0,0) + \tfrac{1}{1!}\left(x\tfrac{\partial f(0,0)}{\partial x} + y\tfrac{\partial f(0,0)}{\partial y}\right), \tag{4.1.5}$$

and the error in the linear approximation is given by

$$\begin{aligned} E(x,y) &= \tfrac{1}{2!}\left\{x^2\tfrac{\partial^2 f}{\partial x^2} + 2xy\tfrac{\partial^2 f}{\partial x\partial y} + y^2\tfrac{\partial^2 f}{\partial y^2}\right\} \\ &= \tfrac{1}{2!}\left\{x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right\}, \end{aligned} \tag{4.1.6}$$

where the second order partial derivatives are evaluated at (cx, cy) for some $0 < c < 1$.

For $n=2$, the quadratic approximation of f is given by

$$\begin{aligned} f(x,y) \approx f(0,0) &+ \tfrac{1}{1!}\left(x\tfrac{\partial f(0,0)}{\partial x} + y\tfrac{\partial f(0,0)}{\partial y}\right) \\ &+ \tfrac{1}{2!}\left(x^2\tfrac{\partial^2 f(0,0)}{\partial x^2} + 2xy\tfrac{\partial^2 f(0,0)}{\partial x\partial y} + y^2\tfrac{\partial^2 f(0,0)}{\partial y^2}\right), \end{aligned} \tag{4.1.7}$$

and the error in the approximation is given by

$$\begin{aligned} E(x,y) &= \tfrac{1}{3!}\left(x^3\tfrac{\partial^3 f}{\partial x^3} + 3x^2y\tfrac{\partial^3 f}{\partial x^2\partial y} + 3xy^2\tfrac{\partial^3 f}{\partial x\partial y^2} + y^2\tfrac{\partial^3 f}{\partial y^3}\right) \\ &= \tfrac{1}{3!}\left\{x^3 f_{xxx} + 3x^2yf_{xxy} + 3xy^2 f_{yyx} + y^3 f_{yyy}\right\}, \end{aligned} \tag{4.1.8}$$

where the third order partial derivatives are evaluated at (cx, cy) for some $0 < c < 1$.

For $n=3$, the cubic approximation of f is given by

$$\begin{aligned} f(x,y) \approx f &+ \tfrac{1}{1!}\left(xf_x + yf_y\right) + \tfrac{1}{2!}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right) \\ &+ \tfrac{1}{3!}\left(x^3 f_{xxx} + 3x^2yf_{xxy} + 3xy^2 f_{yyx} + y^3 f_{yyy}\right), \end{aligned} \tag{4.1.9}$$

where f and all its partial derivatives are evaluated at $(0,0)$.

Exercise 4.1.1 (Self-check). Use Taylor's formula to find the quadratic and cubic approximations for each of the following functions $f(x,y)$ at the origin:

(a) $y \sin x$

(b) $\sin x \cos y$

(c) $e^x \cos y$

(d) $\sin(x^2+y^2)$

(e) $\log_e(2x+y+1)$

(f) $1/(1-x-y)$

(g) $1/(1-x-y+xy)$

4.2 Unconstrained Local Extrema for Functions of Two variables

Let $f(x, y)$ be defined on a region R containing the point (a, b). We say that

- f has a *local maximum* at (a, b), if $f(x, y) \leq f(a, b)$ for all points (x, y) in a small neighbor hood of (a, b); (a, b) is a *point of local maximum* of f, and the functional value $f(a, b)$ gives the corresponding *local maximum value*,
- f has a *local minimum* at (a, b), if $f(a, b) \leq f(x, y)$ for all points (x, y) in a small neighbor hood of (a, b); (a, b) is a *point of local minimum* of f, and the functional value $f(a, b)$gives the corresponding *local minimum value*.

Definition 4.2.1 (Critical Point)**.** An interior point (a, b) is a *critical point* of f, if

- either $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} 0$ or
- or at least one of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ does not exist at (a, b).

Theorem 4.2.1 (Necessary Conditions for Local Extremum)**.** Let $f(x, y)$ have a local extremum at an interior point (a, b) of its domain R, then

$$\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0 \text{ at } (a, b), \tag{4.2.1}$$

provided the first partial derivatives exist at (a, b).

Definition 4.2.2 (Saddle Point)**.** A critical point (a, b) is a *saddle point* of a differentiable function f, if every neighborhood of (a, b) has points (x, y) such that $f(a, b) < f(x, y)$ and $f(a, b) > f(x, y)$.

Notation: $r = \frac{\partial^2 f}{\partial x^2},\ t = \frac{\partial^2 f}{\partial y^2},\ s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},\ D = rt - s^2 = \begin{vmatrix} r & s \\ s & t \end{vmatrix}.$

The number D is called the *discriminant* or *Hesasian* of second order.

Theorem 4.2.2 (Sufficient Conditions for Local Extremum)**.** Let $P(a, b)$ be a critical point of $f(x, y)$, where (4.2.1) hold good. Suppose that

- $D > 0$: Then f has a local maximum or local minimum at P. Further,
  (a) if $r < 0$, f will have a local maximum at P
  (b) if $r > 0$, f will have a local minimum at P; .
- $D < 0$: Then P will be a saddle point of f. The surface $z = f(x, y)$ crosses the tangent plane at (a, b).

Exercise 4.2.1. Find all the local extrema and saddle points, if any for the following functions:

(a) $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$

(b) $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

(c) $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$

(d) $f(x, y) = x^3 + 3xy + y^3$

(e) $f(x, y) = \log(x + y) + x^2 - y$

Exercise 4.2.2. Examine the following functions for the local extrema:

(a) $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$

(b) $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$

(c) $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

(d) $f(x, y) = 4xy - x^4 - y^4$

(e) $f(x, y) = 2(\log x + \log y) - 4x - y.$

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 13.8

2. Briggs et al., *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © (2013) Pearson Education, Inc., Sec. 13.8

3. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 14.7

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5. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 14.7, 14.9

4.3 Absolute Maxima and Minima on Closed and Bounded Sets

Theorem 4.3.1 (Extreme-value Theorem)**.** If $f(x, y)$ is continuous on a closed and bounded set $\mathscr{R}$, then f has both an absolute maximum and an absolute minimum on it.

The absolute extrema can occur either on the boundary or in the interior of of $\mathscr{R}$.

Theorem 4.3.2. If $f(x, y)$ has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.

To find absolute extrema of a continuous function f on a closed and bounded set $\mathscr{R}$, we employ the following steps:

(a) Find the critical points of f in the interior of $\mathscr{R}$, and then evaluate f there at,

(b) Evaluate f at the boundary points,

(c) The largest of the computed values is the absolute maximum and the smallest is the absolute minimum for f on $\mathscr{R}$.

Exercise 4.3.1. Find the absolute extrema of each of the following functions $f(x, y)$ on the indicated closed and bounded regions $\mathscr{R}$:

(a) $3xy - 6x - 3y + 7$; $\mathscr{R}$ is enclosed by the triangle with vertices $(0, 0)$, $(3, 0)$ and $(0, 5)$

(b) $xy - 4x$; $\mathscr{R}$ is enclosed by the triangle with vertices $(0, 0)$, $(0, 4)$, and $(4, 0)$

(c) xy^2; $\mathscr{R} = \{(x, y) : x \geq 0, y \geq 0$ and $x^2 + y^2 \leq 1\}$.

Exercise 4.3.2.

(a) Maximize $g(x, y, z) = xy + yz + xz$, where $x + y + z = 6$

(b) Find the least distance of the the point $(-6, 4, 0)$ from the cone $z^2 = x^2 + y^2$, $z \geq 0$

(c) Find the points on the graph of $z = x^2 + y^2 + 10$ nearest to the plane $x + 2y - z = 0$

(d) Find the shortest distance from $(2, -1, 1)$ to the plane $x + y - z = 2$

(e) Find the point on the plane $3x + 2y + z = 6$ that is nearest to the origin

(f) Find three positive numbers whose sum is 9 and whose sum of the squares is a minimum

(g) Find three numbers whose sum is 3 and whose product is maximum.

Exercise 4.3.3. Among all closed rectangular boxes of volume 27 cubic centimeters, what is the smallest surface area?

Exercise 4.3.4. Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the plane $x + y + z = 129$.

4.4 Lagrange Multiplier Method

Two-variable Function with One Constraint: Suppose that x and y are not independent, and connected in fact, by a side condition:

$$g(x, y) = k, \tag{4.4.1}$$

where k is a given constant. To find the extrema for $u = f(x, y)$ subject to the condition (4.4.1), we solve the equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \; \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \tag{4.4.2}$$

subject to the constraint (4.4.1) to get the maximum or minimum of f, we need.

Exercise 4.4.1.

(a) Find two non-negative numbers whose sum is 9 and the product of one number and the square of the other number is a maximum.

(b) A rectangular box with square base, open at the top, is to be made from 48 square feet of material. What dimensions will result in a box with the largest possible volume?

(c) A right circular cylindrical container with open top has surface area 3π square feet. What height h and base radius r will maximize its volume?

(d) Consider all triangles formed by lines passing through the point $(8/9, 3)$ and both the x and y axes. Find the dimensions of the triangle with the shortest hypotenuse.

(e) Find a rectangle of largest area, that can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y = 8 - x^3$.

(f) Find the dimensions (radius r and height h) of the cone of maximum volume, which can be inscribed in a sphere of radius 2.

Exercise 4.4.2.

(a) Find the extreme values of $f(x, y) = xy$ on (i) the unit circle, and (ii) the ellipse $x^2+2y^2 = 1$.

(b) Find the points on the curve $xy^2 = 54$, nearest to the origin.

(c) A closed right circular cylinder has a volume of 1000 cubic feet. The top and the bottom of the cylinder are made of metal that costs 2 dollars per square foot. The lateral side is wrapped in metal costing 2.50 dollars per square foot. Find the minimum cost of construction.

(d) Find the dimensions of the closed right circular cylindrical can of the least surface area, which contains the volume 16π cubic centimeters.

(e) If a cylinder is formed by revolving a rectangle of perimeter 12 inches about one of its edges, what dimensions of the rectangle will result in the cylinder of maximum volume?

(f) Find the length of the shortest ladder that will reach over an 8 feet high fence to a large wall which is 3 feet behind the fence.

(g) Find the maximum area of a right angled triangle whose perimeter is 4 units.

(h) The sum of the length and girth of a container of square cross section is a inches. Find its maximum volume.

Three-variable Function with One Constraint

Suppose that x, y and z are not independent, but connected by a side condition:

$$g(x, y, z) = 0. \tag{4.4.3}$$

To find the extrema for $u = f(x, y, z)$ subject to the condition (4.4.3), we solve the Langrangean equations

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}, \ \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y}, \ \frac{\partial f}{\partial z} = \lambda\frac{\partial g}{\partial z} \tag{4.4.4}$$

through the condition (4.4.3).

Exercise 4.4.3.

(a) Find all the critical points of $f(x, y, z) = x^2 - y^2$ on the surface of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

(b) Find the maximum and minimum values of $f(x, y, z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$.

(c) Discuss the extrema of $f(x, y, z) = x^3 + y^3 + z^3$, when $x + 2y + 3z = 4$.

(d) Find the extrema of $f(x, y, z) = \sqrt[3]{xyz}$ on the plane $x + y + z = k$.

(e) A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at a point on the probe's surface is $\theta = 8x^2 + 4yz - 16z + 600$. Find the hottest point on the probe's surface.

(f) Suppose that the Celsius temperature at the point (x, y, z) on the unit sphere is $\theta = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.

(g) Find the dimensions of an open rectangular box with maximum volume, if its surface covers 75 square units. What would be the dimensions, if the box is closed?

(h) Find the points on the hyperbolic cylinder $x^2 - z^2 = 1$, which are closest to the origin.

(i) Find the point on the plane $2x - 2y - z = 5$, which is at the least distance from the origin.

(j) Find the points on the surface $z = xy + 1$ nearest the origin.

(k) Extremize the squared distance function $f = x^2 + y^2 + z^2$ such that $x^2 - z^2 = 1$.

(l) Find the largest product the positive numbers x, y and z can have, if $x + y + z^2 = 16$.

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 13.9

2. Briggs et al., *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © (2013) Pearson Education, Inc., Sec. 13.9

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5. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 14.7, 14.8

Chapter 5

Multiple Integrals

5.1 Double Integral over a Rectangle

Theorem 5.1.1 (Fubini's Theorem over Rectangles)**.** Let $f(x,y)$ be continuous on $\mathscr{R} = \{a \leq x \leq b, c \leq y \leq d\}$ in the xy-plane. The double integral of $f(x,y)$ over $\mathscr{R}$ is determined through the iterated integrals:

$$\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_{x=a}^b \left\{ \int_{y=c}^d f(x,y)\,\mathrm{d}y \right\} \mathrm{d}x, \tag{5.1.1}$$

$$\int_c^d \int_a^b f(x,y)\ \mathrm{d}x\,\mathrm{d}y = \int_{y=c}^d \left\{ \int_{x=a}^b f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y. \tag{5.1.2}$$

The *limits of integration are all constants* in (5.1.1) and (5.1.2)

Remark 5.1.1. If $f(x,y) = g(x)h(y)$, then the iterated integral (5.1.1) can be written as

$$I = \left\{ \int_a^b g(x)\,\mathrm{d}x \right\} \left\{ \int_c^d h(y)\,\mathrm{d}y \right\}.$$

Exercise 5.1.1. Evaluate the following double integrals:

(a) $\displaystyle\int_0^{\log 4} \int_0^{\log 3} e^{x+y}\,\mathrm{d}x\,\mathrm{d}y$

(b) $\displaystyle\int_{-1}^2 \int_0^{\pi/2} y \sin x\ \mathrm{d}x\,\mathrm{d}y$

(c) $\displaystyle\int_0^3 \int_{-2}^0 (x^2 y - 2xy)\ \mathrm{d}y\,\mathrm{d}x$

(d) $\displaystyle\int_0^{\pi} \int_{-1}^1 xy^2\,\mathrm{d}x\,\mathrm{d}y$

(e) $\displaystyle I = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\,\mathrm{d}y\,\mathrm{d}x$

(f) $\displaystyle I = \int_0^{\pi/2} \int_0^{\pi/2} \sin(x+2y)\,\mathrm{d}x\,\mathrm{d}y$

Exercise 5.1.2. Evaluate the double integrals of the following functions over the given rectangles:

(a) $f(x,y) = \sqrt{x}/y^2,\ \mathscr{R} : 0 \leq x \leq 4,\ 1 \leq y \leq 2$

(b) $f(x,y) = xye^{x^2+y^2},\ \mathscr{R} : 0 \leq x \leq 1,\ 0 \leq y \leq 1$

(c) $f(x,y) = x - y^2,\ \mathscr{R} : 2 \leq x \leq 3,\ 1 \leq y \leq 2$

(d) $f(x,y) = 1 - \frac{x^2+y^2}{2},\ \mathscr{R} : 0 \leq x \leq 1,\ 0 \leq y \leq 1$

(e) $f(x,y) = \frac{x}{y} + \frac{y}{x}$, $\mathscr{R}: 1 \leq x \leq 4,\ 1 \leq y \leq 2$

(f) $f(x,y) = y/(1+xy)$, $\mathscr{R}: 0 \leq x \leq 1,\ 0 \leq y \leq 1$

(g) $f(x,y) = xy\cos(x^2y)$, $\mathscr{R}: 0 \leq x \leq 2,\ 0 \leq y \leq 1$

(h) $f(x,y) = y^2\cos x + y$, $\mathscr{R}: 0 \leq x \leq \pi/2,\ -3 \leq y \leq 3$

(i) $f(x,y) = ye^{-xy}$, $\mathscr{R}: 0 \leq x \leq 2,\ 0 \leq y \leq 3$

Volumes of Regions bounded below by Rectangles

The iterated integral (5.1.1) or (5.1.2) gives the **volume** of the region, under the surface $z = f(x,y)$, bounded below by the rectangle $\mathscr{R}$.

Exercise 5.1.3. Find the volume of the region, bounded under the surface $z = f(x,y)$, and bounded below by the rectangle $\mathscr{R}$ in each of the cases:

(a) $f(x,y) = 2\sin x\cos y, \mathscr{R} = \{0 \leq x \leq \pi/2, 0 \leq y \leq \pi/4\}$

(b) $f(x,y) = 1/xy, \mathscr{R}: \{1 \leq x \leq 2, 1 \leq y \leq 2\}$

(c) $f(x,y) = 3x + 4y, \mathscr{R}: \{1 \leq x \leq 2, 0 \leq y \leq 3\}$

(d) $f(x,y) = x\sin(x+y), \mathscr{R}: \{0 \leq x \leq \pi/6, 0 \leq y \leq \pi/3\}$

(e) $f(x,y) = 16 - x^2 - y^2, \mathscr{R}: \{0 \leq x \leq 2, 0 \leq y \leq 2\}$

(f) $f(x,y) = x + y + 1, \mathscr{R} = \{-1 \leq x \leq 1, -1 \leq y \leq 0\}$

(g) $f(x,y) = x^2 + y^2, \mathscr{R} = \{-1 \leq x \leq 1, -1 \leq y \leq 1\}$.

5.2 Double Integral Over General Regions

Exercise 5.2.1 (Given the non-constant Inner Limits)**.** Sketch the region $\mathscr{D}$ of integration and find each of the following double integrals:

(a) $I = \displaystyle\int_0^{\pi}\int_0^{\sin x} y\,\mathrm{d}y\,\mathrm{d}x$

(b) $I = \displaystyle\int_1^{2}\int_y^{y^2} \mathrm{d}x\,\mathrm{d}y$

(c) $I = \displaystyle\int_0^{1}\int_0^{y^2} 3y^3e^{xy}\,\mathrm{d}x\,\mathrm{d}y$

(d) $I = \displaystyle\int_0^{4}\int_0^{\sqrt{4-x^2}} xy\,\mathrm{d}y\,\mathrm{d}x$

(e) $\displaystyle\int_0^{1}\int_0^{\sqrt{1-x^2}} 8y\,\mathrm{d}y\,\mathrm{d}x$

Exercise 5.2.2. Sketch the region $\mathscr{D}$ of integration and find each of the following double integrals:

(a) $\displaystyle\int_0^{\pi/2}\int_0^{\cos y} e^x \sin y \,\mathrm{d}x\,\mathrm{d}y$

(b) $\displaystyle\int_1^2\int_0^y \mathrm{d}x\,\mathrm{d}y$

(c) $\displaystyle I=\int_0^1\int_{x^2}^{x^3}\left(x^2+y^2\right)\mathrm{d}y\,\mathrm{d}x$

(d) $\displaystyle I=\int_0^1\int_{y^4}^{y^2}\sqrt{y/x}\,\mathrm{d}x\,\mathrm{d}y$

(e) $\displaystyle\int_{-\pi/3}^{\pi/3}\int_0^{\sec x} 3\cos x\,\mathrm{d}y\,\mathrm{d}x$

Theorem 5.2.1 (Fubini's Theorem). Let $f(x,y)$ be continuous on a region $\mathscr{D}$.

(a) If $\mathscr{D}$ is described by the inequalities: $g_1(x)\le y\le g_2(x)$, $a\le x\le b$, where g_1 and g_2 are continuous on $[a,b]$. Then

$$\iint_{\mathscr{D}} f(x,y)\,\mathrm{d}A=\int_{x=a}^{b}\left\{\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\right\}\mathrm{d}x \tag{5.2.1}$$

(b) If $\mathscr{D}$ is described by the inequalities: $h_1(y)\le x\le h_2(y)$, $c\le y\le d$, where h_1 and h_2 are continuous on $[c,d]$. Then

$$\iint_{\mathscr{D}} f(x,y)\,\mathrm{d}A=\int_{y=c}^{d}\left\{\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\right\}\mathrm{d}y \tag{5.2.2}$$

Finding the Limits of Integration by Vertical Cross-sections:

Suppose we wish to evaluate $\displaystyle\iint_{\mathscr{D}} f(x,y)\,\mathrm{d}A$ by reducing to the form (5.2.1), where the integration is first with respect to y and then with respect to x. We employ the following steps:

(a) **Sketch** the region $\mathscr{D}$ and label the bounding curves

(b) **Find y-limits (inner):** Imagine a very thin vertical strip L with negligible width in the direction of increasing y passing through $\mathscr{D}$. Locate the points A and B, where L enters and leaves $\mathscr{D}$ respectively. Express y in terms of x at these points, say $y=g_1(x)$ to $y=g_2(x)$

(c) **Find x-limits (Outer):** Choose the (constant) x-limits that include all such vertical strips through $\mathscr{D}$, say $x=a$ to $x=b$

Thus $\displaystyle\iint_{\mathscr{D}} f(x,y)\,\mathrm{d}\mathrm{d}A=\int_{x=a}^{b}\left\{\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\right\}\mathrm{d}x.$

Finding the Limits of Integration by Horizontal Cross-sections:

Suppose we wish to evaluate $\displaystyle\iint\limits_{\mathscr{D}} f(x, y)\, \mathrm{d}A$ by reducing to the form (5.2.1), where the integration is first with respect to x and then with respect to y. We employ the following steps:

(a) **Sketch** the region $\mathscr{D}$ and label the bounding curves

(b) **Find x-limits (inner):** Imagine a very thin horizontal strip L with negligible width in the direction of increasing x passing through $\mathscr{D}$. Locate the points P and Q, where L enters and leaves $\mathscr{D}$ respectively. Express x in terms of y at these points, say $x = h_1(y)$ to $x = h_2(y)$

(c) **Find y-limits (Outer):** Choose the (constant) y-limits that include all such horizontal strips through $\mathscr{D}$, say $y = c$ to $y = d$

Thus $\displaystyle\iint\limits_{\mathscr{D}} f(x, y)\, \mathrm{d}A = \int_{y=c}^{d} \left\{ \int_{x=h_1(y)}^{h_2(y)} f(x, y)\, \mathrm{d}x \right\} \mathrm{d}y.$

Exercise 5.2.3. Find iterated integrals for $\iint_{\mathscr{D}} f(x, y)\, \mathrm{d}A$ by vertical and horizontal cross-sections, where $\mathscr{D}$ is bounded by the curve(s):

(a) $y = \sqrt{x}$, $y = 0$ and $x = 9$

(b) $y = \tan x$, $x = 0$ and $y = 1$

(c) $y = 3 - 2x$, $y = x$ and $x = 0$

(d) $y = 3x$ and $y = x^2$

(e) $y = e^x$, $x = 2$ and $y = 1$

Exercise 5.2.4. Find iterated integrals for $\iint_{\mathscr{D}} f(x, y)\, \mathrm{d}A$ by vertical and horizontal cross-sections, where $\mathscr{D}$ is bounded by the curve(s):

(a) $y = \sqrt{x}$, $y = 0$ and $x = 9$

(b) $y = \tan x$, $x = 0$ and $y = 1$

(c) $y = 3 - 2x$, $y = x$ and $x = 0$

(d) $y = 3x$ and $y = x^2$

(e) $y = e^x$, $x = 2$ and $y = 1$

Exercise 5.2.5. Evaluate $I = \iint_{\mathscr{R}} f(x, y)\, \mathrm{d}A$, where $\mathscr{R}$ is given against $f(x, y)$:

(a) $f(x, y) = x$, $\mathscr{R}$ is bounded by the line $y = x$ and the parabola $y = x^2$

(b) $f(x, y) = \frac{xy}{\sqrt{a^2 - y^2}}$, $\mathscr{R}$ is enclosed by the circle $x^2 + y^2 = a^2$

(c) $f(x, y) = \frac{\sin y}{y}$, $\mathscr{R}$ is enclosed by the curve $y = \sqrt{x}$ and the line $y = x$

(d) $f(x, y) = \frac{1}{\sqrt{2y-y^2}}$, $\mathscr{R}$ is bounded by the curve $x^2 = 4 - 2y$ in the first quadrant

(e) $y = e^x$, $x = 2$ and $y = 1$

Exercise 5.2.6. Determine the limits, and evaluate the double integral of each of the following functions $f(x, y)$ over the given region $\mathscr{R}$, by vertical and horizontal cross-sections:

(a) $f(x, y) = xy$; $\mathscr{R}$ is enclosed by $x^2 = 4ay$, the x-axis, and the line $x = 2a$

(b) $f(x, y) = y$; $\mathscr{R}$ is enclosed by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$

(c) $f(x, y) = (x + y)^2$; $\mathscr{R}$ is enclosed by the triangle with vertices $(0, 0)$, $(a, 0)$ and $(0, a)$

(d) $f(x, y) = xy(x + y)$; $\mathscr{R}$ is enclosed by the parabola $y = x^2$ and the line $y = x$

(e) $f(x, y) = x - y$; $\mathscr{R}$ is the region in the first quadrant, enclosed by the parabolas $y^2 = 3x$ and $y^2 = 4 - x$

5.3 Finding Volumes

Exercise 5.3.1. Compute the volume of the solid bounded

(a) above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed with edges $y = x$, $x = 0$ and $x + y = 2$ in the xy-plane

(b) in the first octant by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

(c) in the first octant bounded by the planes $x = 2$ and $y = 4$ and the cylinder $z = y^2$.

(d) enclosed by the planes: $y = 0, z = 0, y = 3, z = x$ and $x + z = 4$

(e) enclosed by the tetrahedron with faces: $x = 0, y = 0, z = 0$ and the plane $x + y + z = 1$

(f) cut from the square colunm $|x| + |y| \leq 1$ by the planes $z = 0$ and $3x + z = 3$

(g) bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane

(h) enclosed by the tetrahedron with faces: $x = 0, y = 0, z = 0$ and the plane $2x + 3y + z = 6$

(i) bounded above by the plane $z = 3x + y + 6$, below by the xy-plane, and on the sides by $y = 0$ and $y = 4 - x^2$

(j) bounded by the front and back by the planes $x = 2$, $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$

5.4 Reversing the Order of Integration

Example 5.4.1. Reverse the order of integration in the following integrals and hence evaluate them:

(a) $\displaystyle\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx$

(b) $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy} \, \mathrm{d}x \, \mathrm{d}y$

(c) $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} \, \mathrm{d}y \, \mathrm{d}x$

(d) $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \tfrac{1}{y^4+1} \, \mathrm{d}y \, \mathrm{d}x$

(e) $\displaystyle\int_0^a \int_{\sqrt{ax}}^a \tfrac{y^2}{\sqrt{y^4-a^2x^2}} \, \mathrm{d}y \, \mathrm{d}x$

(f) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, \mathrm{d}y \, \mathrm{d}x$

(g) $\displaystyle\int_0^a \int_{x/a}^{2\sqrt{x/a}} \left(x^2 + y^2\right) \mathrm{d}y \, \mathrm{d}x$

(h) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} \, \mathrm{d}y \, \mathrm{d}x$

(i) $\displaystyle\int_0^a \int_{y^2/a}^y \sqrt{y} \, \mathrm{d}x \, \mathrm{d}y$

(j) $\displaystyle\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right) \mathrm{d}x \, \mathrm{d}y$

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 14.5-14.7

2. Briggs et al., *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © (2013) Pearson Education, Inc., Sec. 14.1-14.2

3. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 15.1-15.2

4. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 13.1

5. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 15.1-15.2

5.5 Change of Variables in a Double Integral

Finding a Double Integral in Polar Form

To evaluate $I = \displaystyle\iint\limits_{\mathscr{R}} f(r, \theta) \, \mathrm{d}r \, \mathrm{d}\theta$:

(a) Sketch the region $\mathscr{R}$ and label the bounding curves

(b) Imagine a radius vector L from the pole in the direction of increasing r, passing through $\mathscr{R}$

(c) Locate the points P and Q, where L enters and leaves $\mathscr{R}$ respectively. Then express r in terms of θ at these points, say $r = \phi_1(\theta)$ at P, $r = \phi_2(\theta)$ at Q

(d) Choose the (constant) θ-limits that include all such radii, say $\theta = \alpha$ to $\theta = \beta$

(e) Thus the limits of integration are $r = \phi_1(\theta)$ to $\phi_2(\theta)$, and $\theta = \alpha$ to β

$$I = \int_{\theta=\alpha}^{\beta} \left\{ \int_{y=\phi_1(\theta)}^{\phi_2(\theta)} f(r,\theta)\,\mathrm{d}r \right\} \mathrm{d}\theta$$

Change of Variable - Cartesian into Polar:

Conversion of a double integral from Cartesian form to polar form is governed by the following equation:

$$\iint_{\mathscr{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathscr{G}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta,$$

where $\mathscr{G}$ is the Cartesian region $\mathscr{R}$, expressed in terms of polar coordinates.

Exercise 5.5.1. Evaluate each of the following double integrals through the polar transformation: $x = r\cos\theta, y = r\sin\theta$:

(a) $I = \displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,\mathrm{d}y\,\mathrm{d}x$

(b) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$

(c) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} y\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$

(d) $\displaystyle\int_0^1 \int_x^{\sqrt{1-x^2}} \tfrac{x}{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$

(e) $\displaystyle\int_0^a \int_y^a \tfrac{x^2}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$

(f) $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \mathrm{d}y\,\mathrm{d}x$

(g) $\displaystyle\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y\,\mathrm{d}x$

(h) $\displaystyle\int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^{y} \mathrm{d}x\,\mathrm{d}y$

(i) $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{0} \tfrac{2}{(1+x^2+y^2)^2}\,\mathrm{d}y\,\mathrm{d}x$

(j) $\displaystyle\int_1^2 \int_0^{\sqrt{2x-x^2}} \mathrm{d}y\,\mathrm{d}x$

Change of Variable - Cartesian into Elliptical Polar: Conversion of Cartesian double integral into elliptical polar form is governed by the following equation:

$$\iint_{\mathscr{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathscr{G}} f(ar\cos\theta, br\sin\theta)\,abr\,\mathrm{d}r\,\mathrm{d}\theta,$$

where $\mathscr{G}$ is the Cartesian region $\mathscr{R}$ expressed in elliptical polar coordinates $x = ar\cos\theta$, $y = br\sin\theta$.

Other Forms

Exercise 5.5.2 (Self-check)**.** Use the elliptical polar transformation $x = ar\cos\theta, y = br\sin\theta$ to evaluate

(a) $\iint (x+y)^2\,\mathrm{d}x\,\mathrm{d}y$

(b) $\iint (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$

over the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Exercise 5.5.3. Evaluate the double integral of $f(x,y) = (x+y)$ over the positive quadrant of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Exercise 5.5.4 (Double Integral over Unbounded Regions)**.** Compute the double integrals given below:

(a) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,\mathrm{d}y\,\mathrm{d}x$

(b) $\int_0^\infty \int_y^\infty xe^{-x^2/y}\,\mathrm{d}x\,\mathrm{d}y$

(c) $\int_1^\infty \int_0^{1/x^2} 2y\,\mathrm{d}y\,\mathrm{d}x$

(d) $\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3y}\,\mathrm{d}y\,\mathrm{d}x = 1$

(e) $\int_0^\infty \left(\frac{e^{-ax}-e^{-bx}}{x}\right)\mathrm{d}x = \int_0^\infty \int_a^b e^{-xy}\,\mathrm{d}y\,\mathrm{d}x$

(h) $\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$

(i) $\int_0^{\frac{\pi}{2}} \int_0^\infty \frac{r}{(r^2+a^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta$

(j) $\int_0^{\frac{\pi}{2}} \int_0^\infty \frac{r}{(r^2+a^2)^4}\,\mathrm{d}r\,\mathrm{d}\theta$

5.6 Areas of Plane Regions by Double Integral

(a) Cartesian Form: $A = \iint\limits_R \mathrm{d}x\,\mathrm{d}y = \iint\limits_R \mathrm{d}y\,\mathrm{d}x$

(b) Polar Form: $A = \iint_R r\,\mathrm{d}r\,\mathrm{d}\theta$

Exercise 5.6.1. Evaluate the area of the region

(a) enclosed between the coordinate axes and the line $x + y = 2$

(b) enclosed by the parabola $y^2 = x$ and the straight line $y = x$

(c) bounded by the the straight lines $y = x,\ 3y = x$ and $y = 2$

(d) enclosed by the circle $x^2 + y^2 = a^2$ in the first quadrant using the polar coordinates

(e) enclosed by the arc of the unit circle $x^2 + y^2 = 1$, the y axis and the line $y = x$ in the first quadrant using the polar coordinates

Average value of $f(x, y)$ over a plane region R is given by

$$f_{\text{ave}} = \frac{1}{\text{area of } R} \iint\limits_R f(x, y)\ \mathrm{d}A.$$

Mass, Moments, Centre of Mass, Moments of Inertia, Centroid

Let $f(x, y)$ be the linear density of a thin metal sheet bounding a region R. Then

Mass of the plate: $M = \iint\limits_R f(x, y)\ \mathrm{d}A$

First Moments about the coordinate axes:

$$M_x = \iint\limits_R y f(x, y)\ \mathrm{d}A,\ M_y = \iint\limits_R x f(x, y)\ \mathrm{d}A$$

Centre of Mass (x_c, y_c), where $x_c = \frac{M_y}{M},\ y_c = \frac{M_x}{M}$

Moments of Inertia

(a) about the x-axis: $I_x = \iint\limits_R y^2 f(x, y)\ \mathrm{d}A$

(b) about the y-axis is $I_y = \iint\limits_R x^2 f(x, y)\ \mathrm{d}A$

(c) about a line L is $I_L = \iint\limits_R r^2(x, y) f(x, y)\ \mathrm{d}A$, where $r(x, y)$ is the distance from (x, y) to L

(d) about the origin (polar moment) is $I_0 = I_x + I_y$

Centrod: When the density of the metal plate is constant, $f(x, y)$ cancels out of the numerator and denominator of the formulas of x_c and y_c. As such, the centre of mass becomes a feature of the object's shape and not of the material of which it is made. In this case, the center of mass is referred to as the **centroid** of the shape (x_C, y_C), where

$$x_C = \frac{M_y}{M} = \frac{\iint\limits_R x \, \mathrm{d}A}{\iint\limits_R \mathrm{d}A} \quad \text{and} \quad y_C = \frac{M_x}{M} = \frac{\iint\limits_R y \, \mathrm{d}A}{\iint\limits_R \mathrm{d}A}.$$

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 14.1-14.2
2. Briggs et al., *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © (2013) Pearson Education, Inc., Sec. 14.3, 14.7
3. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 15.3-15.4
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5. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 15.3-15.4

5.7 Triple Integrals

Limits all constants: The general forms are

$$\int_e^f \int_c^d \int_a^b f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z, \tag{5.7.1}$$

$$\int_a^b \int_c^d \int_e^f f(x, y, z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x, \tag{5.7.2}$$

$$\int_a^b \int_e^f \int_c^d f(x, y, z) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y \tag{5.7.3}$$

and so on. The limits will define a rectangular box $R = \{a \le x \le b, c \le y \le d\}$.

Exercise 5.7.1. Evaluate

(a) $\displaystyle\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$

(b) $\displaystyle\int_0^1 \int_0^2 \int_0^2 x^2 yz \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(x + y + z) \, \mathrm{d}x \, \mathrm{d}yz$

Non-contant Inner Limits:

Exercise 5.7.2. Evaluate

(a) $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz \; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

(b) $\int_0^1 \int_{y^2}^1 \int_1^{1-x} x \; \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

(c) $\int_{-1}^1 \int_0^{z} \int_{x-z}^{x+z} (x+y+z) \; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

(d) $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \; \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

(e) $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \; \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

5.8 Triple Integrals over General Regions

Exercise 5.8.1. Evaluate the triple integral of each of the following functions $f(x, y, z)$ over the given volume V:

(a) $xy + yz + zx$; V is enclosed by the rectangular box with edges $x = 0, x = 1; y = 0, y = 1; z = 0, z = 1$

(b) xyz; V is enclosed by the coordinate planes and the plane $x + y + z = 1$

(c) 1; V is enclosed by the planes $x = 0, y = 0, z = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Exercise 5.8.2 (Triple Integral using Spherical Polar Coordinates)**.** Evaluate the following triple integrals through the spherical polar coordinate transformation $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$:

(a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}} \; \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$
The volume is enclosed by the right circular cone $z = \sqrt{a^2 - y^2}$ with height $z = 1$ in the first octant

1. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$
The volume is enclosed by the positive octant of the unit sphere

(b) The triple integral of $f(x, y, z) = x + y + z$ over the volume enclosed by the sphere $x^2 + y^2 + z^2 = a^2$

(c) The triple integral of $f(x, y, z) = x^2 + y^2 + z^2$ over the volume enclosed by the upper hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$

(d) The triple integral of $f(x, y, z) = \frac{1}{\sqrt{1-x^2-y^2-z^2}}$ over the volume enclosed by positive octant of the unit sphere

2. $\int_0^\infty \int_0^\infty \int_0^\infty e^{-(x^2+y^2+z^2)} \; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

Exercise 5.8.3 (Triple Integral using Cylindrical Polar Coordinates). Using the cylindrical polar coordinates $x = r\cos\theta, y = r\sin\theta, z = z$, evaluate:

(a) $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{x^2+y^2} z^2 \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

(b) $\iiint (x+y+z)\,dV$ over the volume bounded by the coordinate planes, tha plane $z = h$ and the cylinder $x^2 + y^2 = 1$

(c) $\iiint z(x^2+y^2)dV$ over the volume bounded by the planes $z = 2$, $z = 3$, and the cylinder $x^2 + y^2 = 1$

(d) $\iiint (x^2+y^2+z^2)dV$ over the volume bounded by the planes $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 1$

5.9 Volumes of Solids by Triple Integral

The volume of a closed, bounded region R in space is $V = \displaystyle\iiint_R \mathrm{d}V$

- Cartesian: $V = \displaystyle\iint_R \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
- Spherical polar: $V = \displaystyle\iint_R \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$
- Cylindrical polar: $V = \displaystyle\iint_R r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$
- General: $V = \displaystyle\iint_R J\left(\frac{x,y,z}{u,v,w}\right)\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$

Exercise 5.9.1. Compute the volume of

(a) the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = a$

(b) the sphere $x^2 + y^2 + z^2 = a^2$ using the spherical polar coordinates

(c) the cylinder $x^2 + y^2 = a^2$ bounded by the planes $z = 0$ and $z = h$ using the cylindrical polar coordinates

(d) the portion of the sphere $x^2 + y^2 + z^2 = 1$ lying inside the cylinder $x^2 + y^2 = y$ using the cylindrical polar coordinates

(e) the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
[Hint: Use $x = a\rho\sin\phi\cos\theta,\ y = b\rho\sin\phi\sin\theta,\ z = c\rho\cos\phi$]

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 13.5-13.7

2. Briggs et al., *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © (2013) Pearson Education, Inc., Sec. 14.4-14.7

3. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 15.6-15.9

4. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 13.5-13.8

5. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 15.5, 15.7

Chapter 6

Vector Differential Calculus

6.1 Scalar and Vector Fields

Definition 6.1.1 (Scalar Point Function)**.** Let $\mathscr{D} \subset \mathbb{R}^3$. A function f that assigns a scalar to each point $(x, y, z) \in \mathscr{D}$ is known as a *scalar point function* or *scalar field*.

Example 6.1.1.

- *The distance function:* $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ relative to the origin
- *Temperature distribution in solid bodies:* $U(x, y, z)$
- *Fluid-flow problems:* Mass and density of metal plate
- *Earth's atmosphere:* Pressure of the air in the earth's atmosphere
- *Diffusion in lakes:* Concentration of a pollutant that spreads by diffusion in lakes
- *Potentials:* Gravitational and Electrostatic

Since scalar point function is a multi-variable function,

- its limit and continuity are interpreted in terms of path dependence,
- its first partial derivatives are interpreted in terms of slopes the tangent plane along the x- and y- axes,
- the differentiability in terms of its linear approximation.

Definition 6.1.2 (Two-dimensional Vector Fields)**.** Let $\mathscr{D} \subset \mathbb{R}^2$. A function **F** that assigns a vector **F**(x, y) to each point $(x, y) \in \mathscr{D}$ is known as a two-dimensional *vector point function* or *vector field*.

Thus if **F** is a two-dimensional vector field, we write

$$\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j} \tag{6.1.1}$$

Definition 6.1.3 (Three-dimensional Vector Fields)**.** Let $\mathscr{D} \subset \mathbb{R}^3$. A function **F** that assigns a vector **F**(x, y, z) to each point $(x, y, z) \in \mathscr{D}$ is known as a *vector point function* or *vector field*. Thus if **F** is a vector field, we write

$$\mathbf{F}(x, y, z) \text{ or } \vec{f}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k} \tag{6.1.2}$$

Note that **F** is continuous if and only if its components F_1, F_2 and F_3 are continuous. The partial derivatives of **F** are obtained in terms of the partial derivatives of its components F_1, F_2 and F_3.

Example 6.1.2 (Gravitational Field)**.** The gravitational force per unit mass that would be exerted on a small point mass m located at the origin is given by

$$\mathbf{F}(x, y, z) = -\tfrac{Gm}{r^3}\mathbf{r},$$

where G is the universal gravitational constant,

$$\mathbf{r} = \mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and $r = |\mathbf{r}|$.

Example 6.1.3 (Electrostatic Field). The electrostatic field intensity (force per unit charge) is expressed in SI units by

$$\mathbf{F}(x, y, z) = -\frac{q}{4\pi\epsilon_0 r^3}\mathbf{r},$$

where ϵ_0 is the permittivity of free space,

$$\mathbf{r} = \mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and $r = |\mathbf{r}|$.

Example 6.1.4 (Velocity Fields). A velocity vector field models the motion of air particles in a breeze at a single moment in time. Individual vectors indicate direction of motion, and their lengths indicate speed.

Example 6.1.5 (Other Vector Fields).

(a) *Force fields:* The force applied on a body to displace and its acceleration due to the force

(b) *Gravitational Field:* The gravitational force of attraction

(c) *Electric field:* The electric force of attraction or repulsion between two charged particles

(d) *Rigid body motion:* The angular and linear velocities, angular momentum at any point

Definition 6.1.4 (Curves and Paths in $\mathbb{R}^3$). Let $x, y, z : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$. Then the parametric equations

$$x = x(t),\ y = y(t),\ z = z(t)$$

define an arc of a space curve or path, where t is called a *parameter*. The vector form of a path from $t = a$ to $t = b$ is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, t \in [a, b]. \tag{6.1.3}$$

The functions $x(t)$, $y(t)$ and $z(t)$ are called *coordinate functions*.

The motion of Objects like particles, celestial bodies, or quantities, which changes with time, is described by curves.

- *Strings or Knots*: are closed curves in space
- *Molecules*: Like RNA or proteins can be modeled as curves
- *Computer Graphics*: Surfaces are represented by mesh of curves
- *Typography*: Fonts represented by Bezier curves
- *Space Time*: A curve in space-time describes the motion of particles

- *Topological Examples*: space-filling curves, boundaries of surfaces or knots

Definition 6.1.5 (Velocity and Acceleration)**.** Consider a space curve C represented by the parametric equations
$$x = x(t),\ y = y(t),\ z = z(t),$$
t being a parameter, where the associated position vector field

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{6.1.4}$$

denotes the displacement vector at any point P of a particle. Then

(a) $\dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ gives the velocity, the magnitude $|\dot{\mathbf{r}}|$ is called the *speed*. The *direction of motion* is given by $\dot{\mathbf{r}}/\ |\dot{\mathbf{r}}|$ The velocity vector is tangent to the curve at every point.

(b) $\ddot{\mathbf{r}} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$ gives the acceleration at a point P to C

(c) $\dddot{\mathbf{r}} = \frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}$ gives the jerk.

Special Differential Operators

- **Operator Del:** $\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$
- **Operator Del Squared or Laplacian Operator:**
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

6.2 Gradient of a Scalar Field

Definition 6.2.1 (Gradient of a Scalar Field)**.** Let $\phi(x, y, z)$ be a scalar point function defining a scalar field. Its *gradient* at any point $P(x, y, z)$ is given by

$$\text{grad}\ \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = \nabla\phi \tag{6.2.1}$$

Sigma notation of the gradient: grad $\phi = \sum \frac{\partial \phi}{\partial x}$, where x is replaced by y and z respectively when **i** is replaced with **j** and **k**.

Gradients and Level Surfaces

Consider a level surface

$$f(x, y, z) = c \tag{6.2.2}$$

of a scalar point function. By the chain rule of partial differentiation, we have

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = 0,$$

which can be written as

$$\left\{\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right\}\cdot\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i}+\frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j}+\frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{k}\right\}=0$$

Since $\frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i}+\frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j}+\frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{k}=\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is a tangent line at a point on the surface, the above equation shows that the gradient vector $\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$ is normal to the level surface (6.2).

(a) The normal to the level surface (6.2) at a point P is along its gradient at that point, and hence the unit outer normal at P is $\mathbf{n}=\text{grad }\phi/\,|\text{grad }\phi|$.

(b) Angle between two level surfaces is the angle between their normals, and is obtained by the formula $\cos\theta=\mathbf{n}_1\cdot\mathbf{n}_2$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit outer normals of the level surfaces at P.

Example 6.2.1 (Gradient of $f(r)=r^2$). Let $f(x,y,z)$ be a non-constant, continuous function of $r=\sqrt{x^2+y^2+z^2}$. Then, by the chain rule of partial differentiation,

$$\text{grad}\,\{f(r)\}=\tfrac{\partial f(r)}{\partial x}\mathbf{i}+\tfrac{\partial f(r)}{\partial y}\mathbf{j}+\tfrac{\partial f(r)}{\partial z}\mathbf{k}=f'(r)\left[\tfrac{\partial r}{\partial x}\mathbf{i}+\tfrac{\partial r}{\partial y}\mathbf{j}+\tfrac{\partial r}{\partial z}\mathbf{k}\right]=\tfrac{f'(r)}{r}\mathbf{r},$$

where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is the *position vector field.*

6.3 Directional Derivative of a Scalar Field

Definition 6.3.1 (Directional Derivative). Let $\phi(x,y,z)$ be a scalar point function defined on a region D in space and $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ be a constant vector in space. The *directional derivative* at any point $P(x,y,z)$ of D along $\mathbf{a}$ is given by

$$D_{\mathbf{a}}[\phi(P)]=\text{grad }\phi\cdot\hat{\mathbf{a}},$$

where $\hat{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|}$ is the unit vector along $\mathbf{a}$.

(a) The directional derivative is a generalization of the partial derivatives. It is a scalar.

(b) The directional derivative is the magnitude of the gradient at a certain point in a specific direction.

(c) The directional derivatives of ϕ at P along the coordinate axes are respectively $\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}$ and $\frac{\partial\phi}{\partial z}$.

(d) The directional derivative of ϕ is zero in the direction perpendicular to grad ϕ.

(e) The directional derivative of ϕ is maximum along grad ϕ and its maximum value is $|\text{grad }\phi|$.

Remark 6.3.1 (The Directional Derivative Along a Curve). If $f(x,y,z)$ is the temperature in a room and $\mathbf{r}$ is a curve with velocity $\dot{\mathbf{r}}$, then $\nabla f(\mathbf{r}(t))\cdot\dot{\mathbf{r}}$ is the temperature change, one measures on the point moving on a curve $\mathbf{r}(t)$ experiences.

Exercise 6.3.1. Evaluate the directional derivative $D_{\mathbf{a}}[\phi(P)]$ of each of the following scalar point functions ϕ at the point P along the given direction $\mathbf{a}$. Also find the direction and magnitude of the maximum directional derivative in each case:

(a) $\phi = x^3 + y^3 - z^3 - 3xyz,\ P(-1, 1, -1),\ \mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

(b) $\phi = xye^{x^2+z^2-5}, P(1, 3, -2),\ \mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

(c) $\phi = \frac{xz}{x^2+y^2}, P(1, -1, 1),\ \mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

(d) $\phi = x + y + 3z, P(1, 0, 3),\ \mathbf{a} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

(e) $\phi = 2x^2yz + 4xy^2,\ P(1, 2, 3),\ \mathbf{a}$ is the line $\frac{x-5}{4} = \frac{y}{-2} = \frac{z-4}{1}$

(f) $\phi = 2e^{2x-y+z},\ P(1, 3, 1),\ \mathbf{a}$ is the line $\mathbf{PQ}$ where $Q = (2, 1, 3)$

Exercise 6.3.2. Evaluate $D_{\mathbf{a}}[\phi(P)]$ of each of the following scalar point functions ϕ at the point P along the given direction **a**. Also find the direction and magnitude of the maximum directional derivative in each case:

(a) $\phi = \log(x^2 + y^2 + z^2) + xyz,\ P(1, 1, 1),\ \mathbf{a}$ is the normal to the surface $xy + yz + zx = 1$

(b) $\phi = x^2yz + 4xz^2,\ P(1, -2, 1),\ \mathbf{a}$ is the normal to the surface $x \log z - y^2 + 4 = 0$

(c) $\phi = xyz,\ P(1, 1, 1),\ \mathbf{a}$ is the normal to the surface $x^2z + y^2x + z^2y = 3$

(d) $\phi = x^2 + yz^2, P(1, -1, 3),\ \mathbf{a} = \text{grad}\,\phi$

(e) $\phi = x^3 + y^3 + z^3 - 3xyz,\ P(1, 1, 1),\ \mathbf{a}$ is the tangent to the curve $x = t,\ y = t^2,\ z = t^3$

(f) $\phi = x^2y^2z^2,\ P(1, 1, -1),\ \mathbf{a}$ is the tangent to the curve $x = e^t,\ y = 1+2\sin t,\ z = t-\cos t$, where $-1 \leq t \leq 1$

6.4 Divergence of Vector Fields

Definition 6.4.1 (Divergence of a Vector Field)**.** Consider a vector field

$$\mathbf{f} = \mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}.$$

Then

$$\text{div}\,\mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \tag{6.4.1}$$

Note that

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) = \nabla \cdot \mathbf{f} \tag{6.4.2}$$

Sigma notation of the gradient:

$$\text{div}\,\mathbf{f} = \sum \left(\mathbf{i} \bullet \tfrac{\partial \mathbf{f}}{\partial x}\right),\ \text{curl}\,\mathbf{f} = \sum \left(\mathbf{i} \times \tfrac{\partial \mathbf{f}}{\partial x}\right),$$

x is replaced by y and z respectively when **i** is replaced with **j** and **k**. **Physical Meaning of Divergence:** Consider a vector field **f** that represents a fluid velocity. Then

(a) The divergence of **f** at a point in a fluid is a measure of the rate at which the fluid is flowing away from or towards that point

(b) Divergence means that either the fluid is expanding or that fluid is being supplied by a source external to the field

Definition 6.4.2 (Solenoidal Field)**.** A vector field with zero divergence is called a solenoidal field.

Example 6.4.1.

(a) The magnetic field **B** is solenoidal

(b) The velocity field **F** of an incompressible fluid flow is solenoidal, since there are no sources and sinks and div **F** $= 0$

(c) The electric field **E** in neutral regions is solenoidal.

6.5 Curl of Vector Fields

Definition 6.5.1 (Curl of a Vector Field)**.** Consider a vector field represented by a vector point function

$$\mathbf{f} = \mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}.$$

Then

$$\text{curl } \mathbf{f} = \left(\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{f}$$

Example 6.5.1 (Curl of Fluid Motion)**.**

(a) The curl of the fluid velocity at a point in a fluid is a measure of the rotation of the fluid.

(b) In a 2D-flow, the curl of the fluid velocity is perpendicular to the motion and represents the direction of axis of rotation

Example 6.5.2 (Curl in Rotatory Motion)**.** Suppose that a rigid body is rotating in the xy-plane about the z-axis. Then the angular velocity of a particle at a point $P(\mathbf{r})$ of it is given by $\vec{\omega} = \omega\mathbf{k}$, where ω is its angular speed. We know that its linear velocity is

$$\mathbf{v} = \vec{\omega} \times \mathbf{r} = \omega\mathbf{k} \times (x\mathbf{i} + y\mathbf{j}) = -\omega(y\mathbf{i} + x\mathbf{j}).$$

Then

$$\text{curl}\,\mathbf{v} = -\omega\,\text{curl}\,(y\mathbf{i} + x\mathbf{j}) = -\omega \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\vec{\omega}.$$

Thus the curl of linear velocity is two times its angular velocity at P.

Definition 6.5.2 (Irrotational or Lamellar Fields)**.** A vector field having with zero curl is an *irrotational* field.

Example 6.5.3 (Position Vector Field **r**)**.** The vector field defined by the vector point function $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is known as the *position vector field*. Its magnitude is denoted by r. Thus $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

(a) grad $r = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\text{div}\,\mathbf{r} = 3$

(c) $\text{curl}\,\mathbf{r} = 0$ and $\mathbf{r}$ is an irrotational field.

6.6 Vector Identities

Derive the following results using appropriate vector identities:

1. $\text{grad}\, f(r) = \nabla f(r) = \left[\dfrac{f'(r)}{r}\right] \mathbf{r}$

    (a) $\text{grad}\, r^n = nr^{n-2}\mathbf{r},\ n \geq 1$

    (b) $\text{grad}\, r^{-n} = -nr^{-n-2}\mathbf{r},\ n \geq 1$

2. $\text{div grad}\, f(r) = \nabla^2[f(r)] = f''(r) + \left\{\dfrac{2}{r}\right\} f'(r)$. Hence find n such that $\text{grad}\, r^n$ is solenoidal.

3. $\text{curl grad}\, f(r) = \nabla \times \{\nabla f(r)\} = 0$, that is $\text{grad}\, f(r)$ is irrotational.

4. $\text{div}\, f(r)\mathbf{r} = 3f(r) + rf'(r)$

5. $\text{curl}\, f(r)\mathbf{r} = 0$, that is $f(r)\mathbf{r}$ is irrotational

6. $\text{curl}\, r\mathbf{f} = r(\text{curl}) + \left[\dfrac{f'(r)}{r}\right] \mathbf{r} \times \mathbf{f}$

7. $\text{grad div}\, f(r)\mathbf{r} = r(\text{curl}) + \left[4\left\{\dfrac{f'(r)}{r}\right\} + f''(r)\right] \mathbf{r}$

8. $\text{div}\, [r\ \text{grad}\, f(r)] = r\ \left[2\left\{\dfrac{f'(r)}{r}\right\} + f''(r)\right]$

9. $\text{div}\, (\mathbf{r} \times \text{grad}\, \phi) = 0$

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 13.5, 15.1

2. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 14.6, 16.1

3. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 12.6, 14.1

Chapter 7

Vector Integral Calculus

7.1 Line Integral and Green's Theorem

Consider a smooth curve C represented by the parametric equations

$$x = x(t),\ y = y(t),\ z = z(t) \quad (t \in [a, b]) \tag{7.1.1}$$

with the corresponding position vector field

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}. \tag{7.1.2}$$

Suppose that

$$\mathbf{F} = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k} \tag{7.1.3}$$

is a vector point function with continuous components f_i defined along C. Then *tangential line integral* or simply *line integral* of **F** along the curve C from a point $A(t = a)$ to $B(t = b)$ is given by

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_{t=a}^{b} [f_1\{x'(t)\} + f_2\{y'(t)\} + f_3\{z'(t)\}]\,\mathrm{d}t, \tag{7.1.4}$$

where f_1, f_2 and f_3 are expressed in terms of the parameter t using the relations (7.1.1). The line integral of **F** around a closed curve C is denoted by $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Exercise 7.1.1. Evaluate $\int \mathbf{f} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{f} = (3x^2 - y)\mathbf{i} + 2x\mathbf{j}$ along the parabola C: $y = 1 - x^2$ from $(-1, 0)$ to $(1, 0)$.

Exercise 7.1.2. Find the line integral of $\mathbf{f} = (x + y)\mathbf{i} + xy\mathbf{j}$ along the straight line C from $(0, 0)$ to $(0, 2)$ and then to $(1, 2)$.

Exercise 7.1.3. Evaluate $\int \mathbf{f} \cdot \mathrm{d}\mathbf{r}$ along each of the following paths C:

(a) $\mathbf{f} = (5xy - 6x^2)\mathbf{i} + (2y - 4x)\mathbf{j}$; C is the arc of the curve $y = x^3$ from $(1, 1)$ to $(2, 8)$.

(b) $\mathbf{f} = xy\mathbf{i} - x^2\mathbf{j}$; from $(0, 1)$ to $(1, 0)$ along the quarter circle $x^2 + y^2 = 1$

(c) $\mathbf{f} = 3xy\mathbf{i} - y^2\mathbf{j}$; C is the arc of the curve $y = 2x^2$ from $(0, 0)$ to $(1, 2)$

(d) $\mathbf{f} = 6y\mathbf{i} + x\mathbf{j}$; C is the arc of the curve $x = t^2,\ y = t^3$ from $(1, 1)$ to $(4, 8)$.

Exercise 7.1.4. Find the line integral of $\mathbf{f} = xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$ along the curve C: $x^2 + y^2 = 1,\ z = t$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

Exercise 7.1.5. Evaluate $\int \mathbf{f} \cdot \mathrm{d}\mathbf{r}$ along each of the following paths C:

(a) $\mathbf{f} = (3x^2 + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$; from $(0, 0, 0)$ to $(1, 1, 1)$ along $C \equiv x = t,\ y = t^2, z = t^3$

(b) $\mathbf{f} = (x^2 - 27)\mathbf{i} - 6yz\mathbf{j} + 8xz^2\mathbf{k}$; from $(0, 0, 0)$ to $(1, 0, 0)$, $(1, 0, 0)$ to $(1, 1, 0)$ and then $(1, 1, 0)$ to$(1, 1, 1)$.

Nice Applications of Line Integral

(a) If $\mathbf{F}$ represents a force field, $\displaystyle\int_A^B \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ gives the *work done* in moving a particle from a point A to a point B along a curve C.

(b) The line integral $\displaystyle\int_A^B \mathbf{E}\cdot \mathrm{d}\mathbf{r}$ gives the electrostatic potential difference between the points A and B along a smooth space curve C in a electrostatic field $\mathbf{E}$.

(c) If $\mathbf{f}$ is the velocity of a fluid particle, then $\displaystyle\oint_C \mathbf{f}\bullet \mathrm{d}\mathbf{r}$ is gives the *circulation* of the fluid around a closed curve C (a tidal basin or the turbine chamber of a hydroelectric generator, for example).

Exercise 7.1.6. Find the work done in moving a particle from the point A to a point B in the force field $\mathbf{F}$:

(a) $\mathbf{f} = xy\mathbf{i} + (x+y)\mathbf{j}$; from $A(-1,1)$ to $B(2,4)$ along the parabola $y = x^2$

(b) $\mathbf{f} = xy\mathbf{i} + (y-x)\mathbf{j}$; around the triangle with vertices $A(0,0)$, $B(1,0)$, $C(0,1)$ counterclockwise

(c) $\mathbf{f}$ is the gradient of $\psi(x,y) = (x+y)^2$; around the circle $x^2+y^2 = 4$ counterclockwise

(d) $\mathbf{f} = 3x^2\mathbf{i} + (2xz - y)\mathbf{j} + z\mathbf{k}$; from $A(0,0,0)$ to $B(2,1,3)$ along the straight line AB and along the curve $x = 2y^2,\ z = 2y^3$

(e) $\mathbf{f} = (2y+3)\mathbf{i} + xz\mathbf{j} + (yz - x)\mathbf{k}$; from $A(0,0,0)$ to $B(2,1,1)$ along the straight line AB and along the curve $x^2 = 4y,\ 3x^3 = 8z$.

Exercise 7.1.7. Find the potential difference observed in the electrostatic field $\mathbf{f} = e^{yz}\mathbf{i} + (xze^{yz} + z\cos y)\mathbf{j} + xye^{yz} + \sin y)\mathbf{k}$, when a charged particle is moves from $(1,0,1)$ to $(1,\pi/2,0)$ along the following paths. You should be clever enough to give the correct inference!

(a) the line segment $x = 1,\ y = \pi t/2,\ z = 1-t$,

(b) the line segment from $(1,0,1)$ to the origin, followed by the line segment from the origin to $(1,\pi/2,0)$,

(c) the line segment from $(1,0,1)$ to $(1,0,0)$, followed by the x-axis from $(1,0,0)$ to the origin, then by the parabola $y = \pi x^2/2$, $z = 0$ from there to $(1,\pi/2,0)$.

Green's Theorem in the Plane

Theorem 7.1.1. Let Ω be a simple region with a piecewise-smooth boundary C. If P and Q are scalar fields continuously differentiable on an open set that contains Ω, then

$$\oint_C [P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y] = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y, \tag{7.1.5}$$

where the line integral on the left hand side is over C in the counterclockwise direction.

Exercise 7.1.8. Evaluate $\oint_C (x - y^3)\,\mathrm{d}x + x^3\,\mathrm{d}y$, where C is the circle $x^2 + y^2 = 1$,

(a) directly

(b) by using Green's theorem.

Exercise 7.1.9. Using Green's theorem, find $\oint_C \mathbf{f} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{f} = \sin y\,\mathbf{i} + x(1 + \cos y)\,\mathbf{j}$ and C is the circle $x^2 + y^2 = a^2$,

Exercise 7.1.10. Using Green's theorem in the plane, find $\oint_C \mathbf{f} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{f}$ and the closed curve C are given below:

(a) $\mathbf{f} = (\cos x \sin y - xy)\,\mathbf{i} + (\sin x \cos y)\,\mathbf{j}$; C is the circle $x^2 + y^2 = 1$.

(b) $\mathbf{f} = (x^2 + y^2)\,\mathbf{i} + 3xy^2\,\mathbf{j}$; C is the circle $x^2 + y^2 = 4$.

Exercise 7.1.11. Use Green's theorem to evaluate $\oint_C [x^2(1 + y)\,\mathrm{d}x + (x^3 + y^3)\,\mathrm{d}y]$, where C is the square with edges $x = \pm 1, y = \pm 1$.

Exercise 7.1.12. Use Green's theorem to evaluate the following line integrals

(a) $\oint_C [(2x^2 - y^2)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y]$, where C is upper semi circle $x^2 + y^2 = 1$ plus the x-axis in the anti-clockwise direction.

(b) $\oint_C [(x^2 - \cosh y)\,\mathrm{d}x + (y + \sin x)\,\mathrm{d}y]$, where C is rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0, 1)$.

Areas of plane regions by Green's theorem

Theorem 7.1.2. The area of the region $\mathscr{D}$ enclosed by a positively oriented simple closed curve γ is given by

$$A = \frac{1}{2} \oint_\gamma [x\,\mathrm{d}y - y\,\mathrm{d}x]. \tag{7.1.6}$$

Exercise 7.1.13. Using the formula (7.1.6), find the area enclosed by the astroid γ: $x^{2/3} + y^{2/3} = 1$, parametrized as $x = \cos^3 t$, $y = b \sin^3 t$.

Exercise 7.1.14. Using the formula (7.1.6), find the area enclosed by the following simple closed curves γ:

(a) Ellipse $x^2/a^2 + y^2/y^2 = 1$

Answer. $ab\pi =$ square units

(b) One arch of the cycloid $x = a(t - \sin t, y = a(1 - \cos t)$

(c) γ is the boundary of the region $\mathscr{R}$, enclosed by the straight line $x + y = 2$, the y-axis and the parabola $y = x^2$.

Exercise 7.1.15. Verify (7.1.1) in each of the following triads (P, Q, C), where C is counterclockwise:

(a) $P = 2xy - x^2$, $Q = x^2 + y^2$; C is the boundary of the region enclosed by the parabolas $y = x^2$ and $y^2 = x$

(b) $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$; C is the boundary of the region enclosed by the triangle with edges $x = 0 = y, x + y = 1$

(c) $P = x^2 - y^2$, $Q = 2xy$; C is the rectangle with sides $x = 0 = y, x = a, y = b$

(d) $P = -y^3$, $Q = x^3$; C is the circle $x^2 + y^2 = a^2$.

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 15.2, 15.4
2. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 16.2, 16.4
3. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 14.2, 14.4
4. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 16.1, 16.2, 16.4

7.2 Surface Integral and Stoke's Theorem

Parametric Form of Surface

Let S be a piecewise smooth surface represented by the parametric form

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \quad \text{for all} \quad (u, v) \in D, \tag{7.2.1}$$

where D is the projection of S over the uv-plane, and hence is known as the *parameter* domain.

Surface Integral of Scalar Point Function

Let S be a piecewise smooth surface with the parametric form (7.2.1). The surface integral of a scalar point function $f(x, y, z)$ over S is given by

$$\iint\limits_S f(x, y, z)\ \mathrm{d}S = \iint\limits_D f[x(u, v), y(u, v), z(u, v)] \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}A, \tag{7.2.2}$$

where

$$\mathrm{d}A = \ \mathrm{d}u\,\mathrm{d}v \text{ or } \ \mathrm{d}v\,\mathrm{d}u \tag{7.2.3}$$

is the area element in the uv-domain.

Case (a): Suppose that S is given by the equation $z = \phi(x, y)$, and D lies in the xy-plane. Then x and y are treated as parameters and

$$\iint\limits_S f(x,y,z)\ \mathrm{d}S = \iint\limits_D f[x,y,\phi(x,y)] \left\{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + 1}\right\} \mathrm{d}A, \tag{7.2.4}$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ or $\mathrm{d}y\,\mathrm{d}x$ is the area element in the xy-domain.

Case (b): Suppose that S is given by the equation $x = \psi(y, z)$, and D lies in the yz-plane. Then y and z are treated as parameters and

$$\iint\limits_S f(x,y,z)\,\mathrm{d}S = \iint\limits_D f[\psi(y,z),y,z] \left\{\sqrt{\left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 + 1}\right\} \mathrm{d}A, \tag{7.2.5}$$

where $A = \mathrm{d}y\,\mathrm{d}z$ or $\mathrm{d}z\,\mathrm{d}y$ is the area element in the yz-domain.

Case (c): Suppose that S is given by the equation $y = \xi(x, z)$, and D lies in the xz-plane. Then z and x are treated as parameters and

$$\iint\limits_S f(x,y,z)\ \mathrm{d}S = \iint\limits_D f[x,\xi(x,z),z] \left\{\sqrt{\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2 + 1}\right\} \mathrm{d}A, \tag{7.2.6}$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}z$ or $\mathrm{d}z\,\mathrm{d}x$ is the area element in the xz-domain.

Case (d): Suppose that S is represented by the parametric form

$$x = a\cos\theta\sin\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\phi, \tag{7.2.7}$$

Then $\left|\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| = a^2\sin\phi$, and

$$\iint\limits_S f(x,y,z)\ \mathrm{d}S = \iint\limits_D f[a\cos\theta\sin\phi, a\sin\theta\sin\phi, \cos\phi]\, a^2\sin\phi\ \mathrm{d}\phi\ \mathrm{d}\theta. \tag{7.2.8}$$

Case (e): Suppose that S is represented by the parametric form

$$x = a\cos\theta,\ y = a\sin\theta,\ z = z. \tag{7.2.9}$$

Then $\left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z}\right| = a$, and

$$\iint\limits_S f(x,y,z)\ \mathrm{d}S = \iint\limits_D f[a\cos\theta, a\sin\theta, z]\, a\,\mathrm{d}z\,\mathrm{d}\theta. \tag{7.2.10}$$

Surface Area: If $f(x, y, z) = 1$, then (7.2.2) gives the area of the surface S as $A_S = \iint\limits_S \mathrm{d}S.$

Exercise 7.2.1. Find the surface integral $\iint\limits_S y \, \mathrm{d}S$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

Exercise 7.2.2. Compute the surface integral of $f(x, y, z) = x^2$ over the sphere $x^2 + y^2 + z^2 = 1$.

Exercise 7.2.3. Compute the surface area of the sphere $S: \; x^2 + y^2 + z^2 = a^2$.

Exercise 7.2.4. Find $\iint\limits_S z \, \mathrm{d}S$, where the surface S is enclosed by the cylinder $S_1: \; x^2 + y^2 = 1$, bounded below by the xy-plane S_2, and bounded above by the plane $S_3: \; z = x + 1$.

Exercise 7.2.5. Find the area of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Surface Integral of Vector Point Function

Let S be a piecewise smooth surface represented by the parametric form (7.2.1) and $\mathbf{n}$, the unit outer normal at a point P on S. Then the *normal component* of the surface integral of

$$\mathbf{F} = f_1(u, v)\,\mathbf{i} + f_2(u, v)\,\mathbf{j} + f_3(u, v)\,\mathbf{k} \text{ for } (u, v) \in D, \tag{7.2.11}$$

over S is given by

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint\limits_D \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \mathrm{d}A = \iint\limits_D \left[\mathbf{F} \, \frac{\partial \mathbf{r}}{\partial u} \, \frac{\partial \mathbf{r}}{\partial v} \right] \mathrm{d}u \, \mathrm{d}v, \tag{7.2.12}$$

where $\left[\mathbf{F} \, \frac{\partial \mathbf{r}}{\partial u} \, \frac{\partial \mathbf{r}}{\partial v} \right]$ is the scalar triple product of $\mathbf{F}$, $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$.

Remark 7.2.1. If S completely lies in the xy-plane, then $\mathbf{n} = \mathbf{k}$ and the inward unit normal is $-\mathbf{n}$, and $\mathrm{d}A = \mathrm{d}x \, \mathrm{d}y$.

Case (a): Suppose S is represented by the parametric form

$$x = x, \; y = x, \; z = z(x, y) \text{ (with the parameters } x \text{ and } y). \tag{7.2.13}$$

Then

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \begin{cases} \iint\limits_D \mathbf{F}[x, y, z(x, y)] \cdot \left(\frac{\partial z}{\partial x} \mathbf{i} + \frac{\partial z}{\partial y} \mathbf{j} + \mathbf{k} \right) \mathrm{d}x \, \mathrm{d}y, & \text{if } \mathbf{n} \text{ is upward} \\ \iint\limits_D \mathbf{F}[x, y, z(x, y)] \cdot \left(\frac{\partial z}{\partial x} \mathbf{i} + \frac{\partial z}{\partial y} \mathbf{j} - \mathbf{k} \right) \mathrm{d}x \, \mathrm{d}y, & \text{if } \mathbf{n} \text{ is downward,} \end{cases}$$

where D lies in the xy-plane.

Case (b): Suppose S is represented by the parametric form

$$x = a \cos\theta \sin\phi, \; y = a \sin\theta \sin\phi, \; z = a \cos\phi \text{ (the parameters are } \theta \text{ and } \phi). \tag{7.2.14}$$

Then $\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint\limits_D \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right) \mathrm{d}\phi \, \mathrm{d}\theta = \iint\limits_D \left[\mathbf{F} \, \frac{\partial \mathbf{r}}{\partial \phi} \, \frac{\partial \mathbf{r}}{\partial \theta} \right] \mathrm{d}\phi \, \mathrm{d}\theta.$

Case (c): Suppose S is represented by the parametric form

$$x = a \cos\theta, \; y = a \sin\theta, \; z = z \text{ (the parameters are } \theta \text{ and z).} \tag{7.2.15}$$

Then $\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint\limits_D \mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial\theta}\times\frac{\partial\mathbf{r}}{\partial z}\right)\mathrm{d}z\,\mathrm{d}\theta = \iint\limits_D \left[\mathbf{F}\ \frac{\partial\mathbf{r}}{\partial\theta}\ \frac{\partial\mathbf{r}}{\partial z}\right]\mathrm{d}\theta\,\mathrm{d}z.$

Remark 7.2.2. Let S be a piecewise smooth surface and **n** the unit outer normal at a point P on it. The *normal component* of the surface integral **F** over S is given by

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\ \mathrm{d}S = \iint\limits_{R_1} \left\{\frac{\mathbf{F}\cdot\mathbf{n}}{|\mathbf{n}\bullet\mathbf{k}|}\right\}\mathrm{d}x\,\mathrm{d}y \tag{7.2.16}$$

where R_1 is the projection of S on the xy-plane and the integrand here is expressed in terms of x and y using the equation of S.

Stoke's Theorem

Theorem 7.2.1. Let S be a smooth positively oriented surface with a smooth bounding curve C. If $\mathbf{F}(x, y, z)$ is a continuously differentiable vector field on an open set that contains S, then

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint\limits_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\ \mathrm{d}S. \tag{7.2.17}$$

Here $\mathrm{d}S$ is the differential surface area element and **n** is the unit outer normal to S at any point.

Exercise 7.2.6. Verify Stoke's theorem for each of the following vector fields

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

(b) $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$,

(c) $\mathbf{F} = (2x - y)\mathbf{i} + yz^2\mathbf{j} - y^2z\mathbf{k}$,

over the upper hemisphere S of radius a centered at the origin, with boundary C on the xy-plane.

Exercise 7.2.7. Verify Stoke's theorem for $\mathbf{F} = y^3\mathbf{i} + x^3\mathbf{j}$ over the surface S of the cylinder $x^2 + y^2 = 1$ whose upper cross-section is roofed by the plane $z = 1$ and the boundary C lies in the xy-plane.

Exercise 7.2.8. Apply Stoke's theorem to find $\oint\limits_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ over the surface S, which completely lies in the plane $x + z = a$ bounded by the curve C of intersection of the upper hemisphere $x^2 + y^2 + z^2 = a^2$ with the plane.

Exercise 7.2.9. Apply Stoke's theorem to find $\oint\limits_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where $\mathbf{F} = y\mathbf{i} + xz^2\mathbf{j} - zy^3\mathbf{k}$ over the surface S enclosed by the circle $C:\ x^2 + y^2 = 4$ in the plane $z = 3/2$.

Exercise 7.2.10. Use (7.2.17) to evaluate the following line integrals $\oint_C \mathbf{f}\cdot\mathrm{d}\mathbf{r}$:

(a) $\mathbf{f} = \sin z\mathbf{i} - \cos x\mathbf{j} + \sin y\mathbf{k}$, C is the boundary of the region $0 \le x \le 2\pi$, $0 \le y \le 1$ in the plane $z = 3$

(b) $\mathbf{f} = (x+y)\mathbf{i} + (2x-z)\mathbf{j} + (y+z)\mathbf{k}$, C is the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,3)$

Exercise 7.2.11. Verify (7.2.17) for each of the following triads $(\mathbf{f}, S, C)$, where C is the boundary of the surface S, oriented counterclockwise:

(a) $\mathbf{f} = (x^2+y^2)\mathbf{i} - 2xy\mathbf{j}$, S lies in the xy-plane, enclosed by the rectangle C with sides $x = \pm a$, $y = 0, y = b$

(b) $\mathbf{f} = (3xz^2 + x)\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}$, S is the surface of the cube with faces $x = y = z = 0$, $x = y = z = 1$open in the xy-plane, and its boundary C is the square with sides $x = 0 = y$, $x = y = 1$

(c) $\mathbf{f} = x\mathbf{i} + z^2\mathbf{j} + y^2\mathbf{k}$; S is the plane lying in the first octant

(d) $\mathbf{f} = -y\mathbf{i} + x\mathbf{j} - xyz\mathbf{k}$; S is the surface of the cone $x^2 + y^2 = z^2$ with the circle $x^2 + y^2 = 9$ as its boundary.

Exercise 7.2.12. Prove Faraday's Law by applying Stoke's theorem.

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 15.5, 15.8
2. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 16.7, 16.8
3. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 14.6, 14.8
4. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 16.5, 16.7

7.3 Volume Integral and Gauss' Divergence Theorem

Volume Integral as Flux across a Closed Surface: We recall that div**F** gives the flux of a vector field per unit volume. Therefore, the total flux across a smooth closed surface S enclosing a volume V is given by the triple integral $\displaystyle\iiint_V \operatorname{div} \mathbf{F}\ \mathrm{d}V$.

Theorem 7.3.1 (Gauss' Divergence Theorem)**.** Let V be the volume of a solid bounded by a closed, piecewise smooth oriented surface S. If the vector field $\mathbf{f}(x, y, z)$ is continuously differentiable throughout V, then

$$\iint_S \mathbf{F} \bullet \mathbf{n}\ \mathrm{d}S = \iiint_V (\operatorname{div} \mathbf{F})\ \mathrm{d}V, \tag{7.3.1}$$

where **n** is the outer unit normal.

Exercise 7.3.1. Verify (7.3.1) for each of the following pairs $(\mathbf{f}, S)$:

(a) $\mathbf{f} = (x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}$, S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1$, $0 \le z \le 1$

(b) $\mathbf{f} = 3xz\mathbf{i} + (y + xz)\mathbf{j} + (xy + 1)z\mathbf{k}$, S is the surface of the tetrahedron bounded by the three coordinate planes and the plane $2x + 3y + 6z = 12$

(c) $\mathbf{f} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$, S is the convex portion of the cylinder $x^2 + y^2 = 4$ plus upper and lower cross-sections in the planes $z = 0$ and $z = 3$

(d) $\mathbf{f} = 2xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$; S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ plus its plane section in the xy-plane.

Exercise 7.3.2. Use (7.3.1) to evaluate the following surface integrals $\iint_S \mathbf{f} \cdot \mathrm{d}\mathbf{n}\,\mathrm{d}S$:

(a) $\mathbf{f} = yz\mathbf{i}zx\mathbf{j} + xy\mathbf{k}$, S is the unit sphere $x^2 + y^2 + z^2 = 1$

(b) $\mathbf{f} = 2xy\mathbf{i} + yz^2\mathbf{j} + zx\mathbf{k}$, S is the rectangular box $0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3$

(c) $\mathbf{f} = x^2\mathbf{i} + y^2\mathbf{j} + 2z(xy - x - y)\mathbf{k}$, S is the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$

(d) $\mathbf{f} = e^x\mathbf{i} - ye^x\mathbf{j} - 3z\mathbf{k}$, S is the surface of the cylinder $x^2 + y^2 = 4, 0 \le z \le h$

(e) $\mathbf{f} = lx^2\mathbf{i} + my^2\mathbf{j} + nz^2\mathbf{k}$, S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$

(f) $\mathbf{f} = 4xz\mathbf{i} + xyz^2\mathbf{j} + 3z\mathbf{k}$, S is the surface of the cone $x^2 + y^2 = z^2$, bounded above by the plane $z = 4$

(g) $\mathbf{f} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$, bounded above by the xy-plane

Text and Reference Books

1. Anton, Bivens & Davis, *Calculus - Early Transcendentals*, 10th Edition (2013) John Wiley & Sons, Sec. 15.7
2. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 16.9
3. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 14.7
4. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 16.8

7.4 Irrotational Vector Fields

7.5 Irrotational and Conservative Vector Fields

Consider a vector field defined by a vector point function

$$\mathbf{f} = \mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}. \tag{7.5.1}$$

Then

$$\text{curl } \mathbf{f} = \left(\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{f}.$$

The field (7.5.1) is said to be *irrotational*, if its curl is zero. For instance, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is irrotational, since curl $\mathbf{r} = 0$.

Definition 7.5.1. A vector field (7.5.1) is said to be *conservative*, if the line integral $\int \mathbf{f} \cdot \mathrm{d}\mathbf{r}$ of $\mathbf{f}$ between any two points A and B is path independent. That is, $\int \mathbf{f} \cdot \mathrm{d}\mathbf{r}$ from any point A to other point B is the same, irrespective of the path between them.

An Equivalence for Conservative Fields

Theorem 7.5.1 (Equivalence Theorem)**.** Consider a vector field

$$\mathbf{f}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k},$$

where f_1, f_2 and f_3 are continuous on some open connected region $\mathscr{D}$. Then the following statements are equivalent:

(a) The field **f** is conservative on $\mathscr{D}$

(b) For any piecewise smooth closed curve C in $\mathscr{D}$, $\oint_C \mathbf{f} \cdot \mathrm{d}\mathbf{r} = 0$ (*Closed Loop Property*)

(c) There exists a scalar point function $\psi(x, y, z)$ such that $\mathbf{f} = \operatorname{grad} \psi$ on $\mathscr{D}$, ψ is known as the **potential** of the field (*Gradient Vector Field*)

(d) For any two points A and B in $\mathscr{D}$, $\int_A^B \mathbf{f} \cdot \mathrm{d}\mathbf{r} = \psi(B) - \psi(A)$, that is the line integral equals the difference of scalar potential between the points A and B (*Path Independence*).

Theorem 7.5.2. Every conservative vector field is irrotational.

Converse of Theorem 7.5.2 is not true. That is, an irrotational field may not be conservative. For example, consider the vector field

$$\mathbf{f} = -\left\{\frac{y}{x^2 + y^2}\right\}\mathbf{i} + \left\{\frac{x}{x^2 + y^2}\right\}\mathbf{j} \text{ for } (x, y) \in \mathbb{R}^2 - \{(0, 0)\} \tag{7.5.2}$$

is known to be irrotational. But $\oint_{x^2+y^2=1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi \neq 0$. Therefore, by Theorem 7.5.1, **f** is not conservative on $\mathbb{R}^2 - \{(0, 0)\}$. However, the same vector field restricted to $\mathscr{D} = \{(x, y) : x > 0\}$ is conservative with a potential function $\psi(x, y) = \tan^{-1}(y/x)$ on $\mathscr{D}$.

Remark 7.5.1. An irrotational vector field is conservative only if the domain is simply connected. A simply connected domain simply means - a path can be drawn between any two points in the domain and every such path drawn between two points can be transformed continuously into any other, preserving the endpoints. Intuitively - a simply connected domain can be contracted to a point, and does not contain any holes.

Exercise 7.5.1. Show that each of the following vector fields is conservative and hence find its scalar potential:

(a) $(6xy + z^3)\mathbf{i} + (3x^2 - z)\mathbf{j} + (3xz^2 - y)\mathbf{k}$

(b) $(x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}$

(c) $(4xy - 3x^2z^2)\mathbf{i} + 2x^2\mathbf{j} + 2x^3z\mathbf{k}$

(d) $(x^2 + xy^2)\mathbf{i} + (y^2 + x^2y)\mathbf{j}$

(e) $(e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$

(f) $\mathbf{r}/r$

(g) $(yze^{xyz} - 4x)\mathbf{i} + (zxe^{xyz} + z + \cos y)\mathbf{j} + (xyze^{xyz} + y)\mathbf{k}$

(h) $(\sin y + z \cos x)\mathbf{i} + (\sin z + x \cos y)\mathbf{j} + (\sin x + y \cos z)\mathbf{k}$

Exercise 7.5.2. Determine the constants (specified against each case) such that each of the following vector fields is conservative and hence find the respective scalar potential:

(a) $(x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x - cy + 2z)\mathbf{k};\ a, b, c$

(b) $(2xy + 3yz)\mathbf{i} + (x^2 + axz - 4z^2)\mathbf{j} + (3xy + 2byz)\mathbf{k}; a, b$

(c) $(y^2 + 2czx)\mathbf{i} + y(bx + cz)\mathbf{j} + (y^2 + cx^2)\mathbf{k};\ b, c$

(d) $(axy - z^3)\mathbf{i} + (bx^2 + 3z)\mathbf{j} + (6xz^2 + cy)\mathbf{k};\ a, b, c$

Text and Reference Books

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3. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 14.3
4. Thomus, J. B., *Calculus*, 12th Edition, Copyright © 2010 Pearson Edu., Sec. 16.3