MULTIPLE INTEGRALS

Double Integrals over General Regions

In this section, we will learn:

How to use double integrals to find the areas of regions of different shapes.

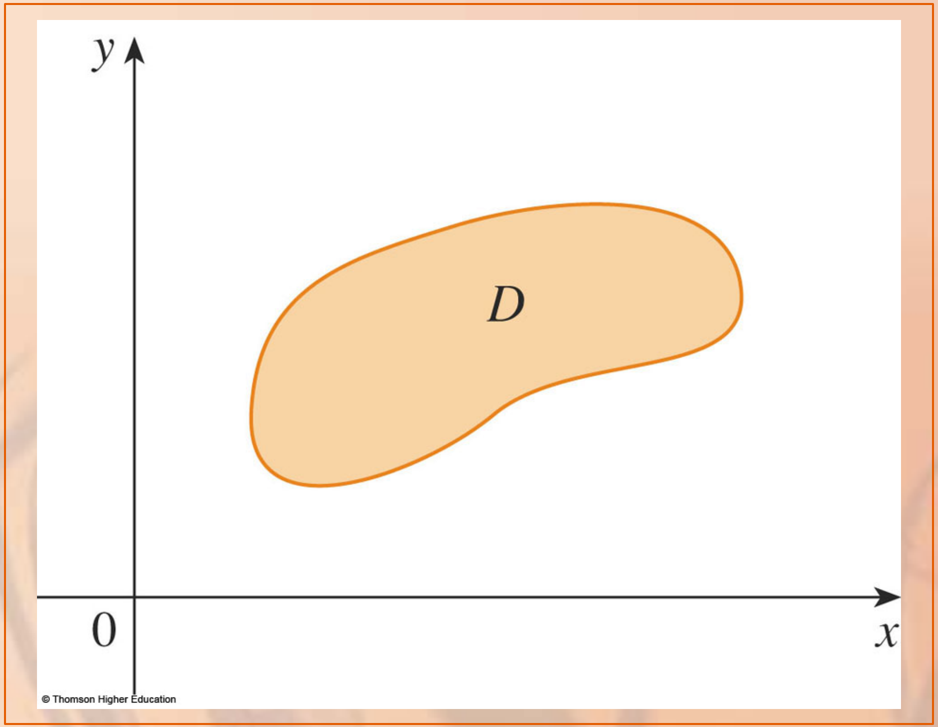
SINGLE INTEGRALS

For single integrals, the region over which we integrate is always an interval.

DOUBLE INTEGRALS

For double integrals, we want to be able to integrate a function *f* not just over rectangles but also over regions *D* of more general shape.

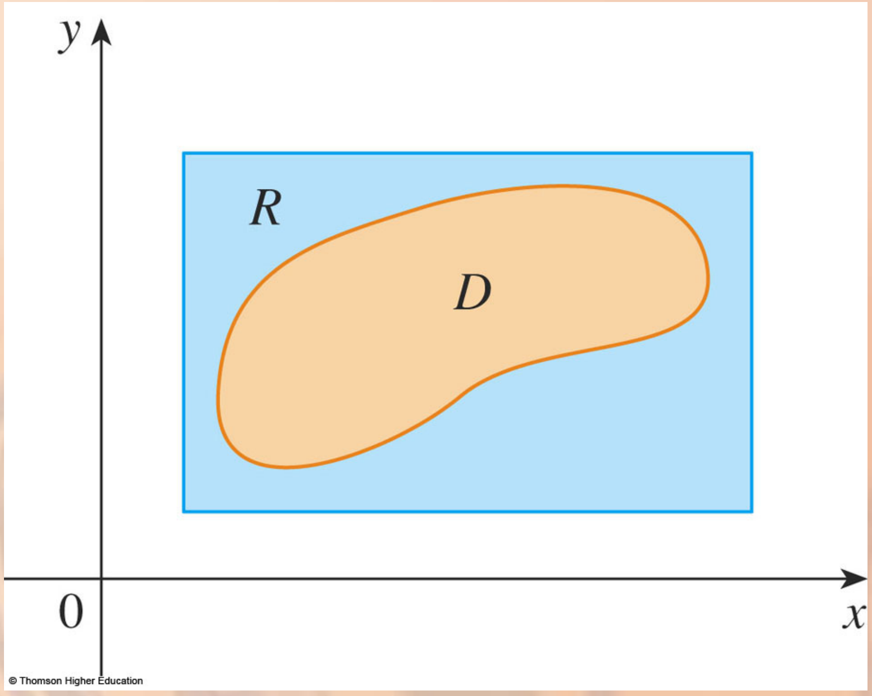
- One such shape is illustrated.



DOUBLE INTEGRALS

We suppose that *D* is a bounded region.

- This means that *D* can be enclosed in a rectangular region *R* as shown.



Equation 1

Then, we define a new function *F* with domain *R* by:

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Definition 2

If *F* is integrable over *R*, then we define the double integral of *f* over *D* by:

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

where *F* is given by Equation 1.

Definition 2 makes sense because *R* is a rectangle and so $\iint_R F(x, y)\, dA$

has been previously defined.

The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D—and so they contribute nothing to the integral.

- This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geq 0$,

we can still interpret

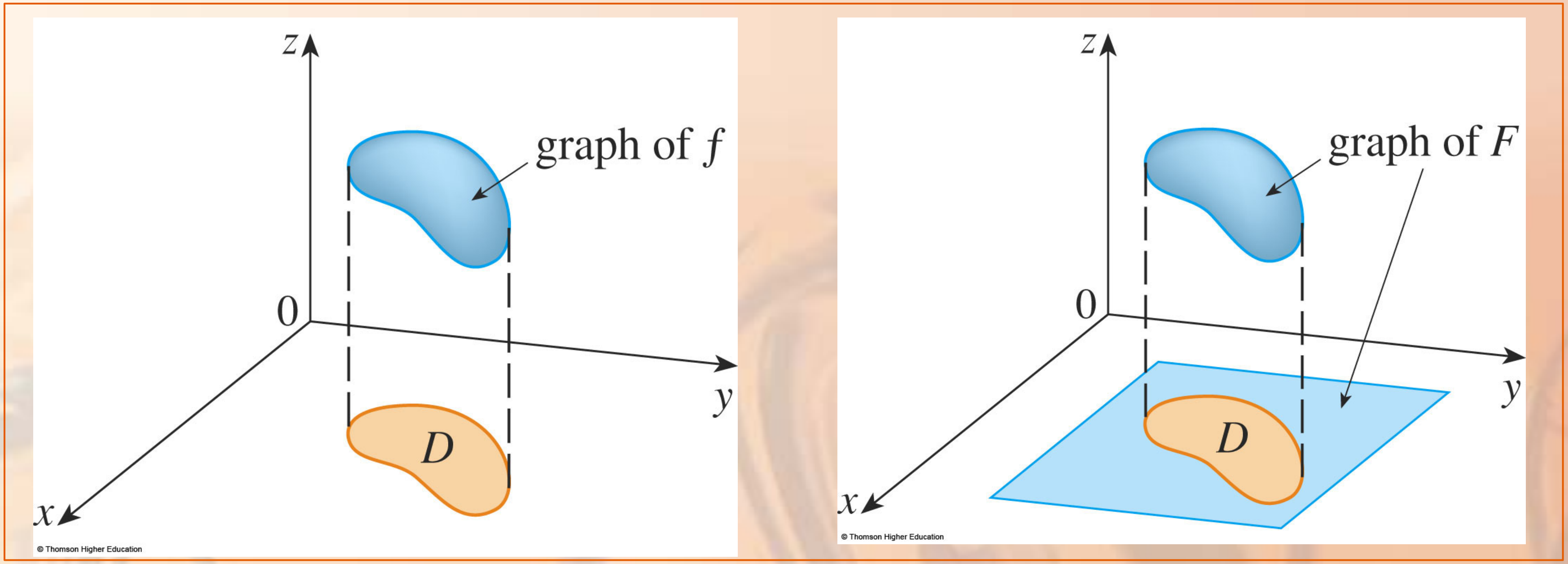
$$\iint_D f(x, y)\, dA$$

as the volume of the solid that lies above D

and under the surface $z = f(x, y)$ (graph of f).

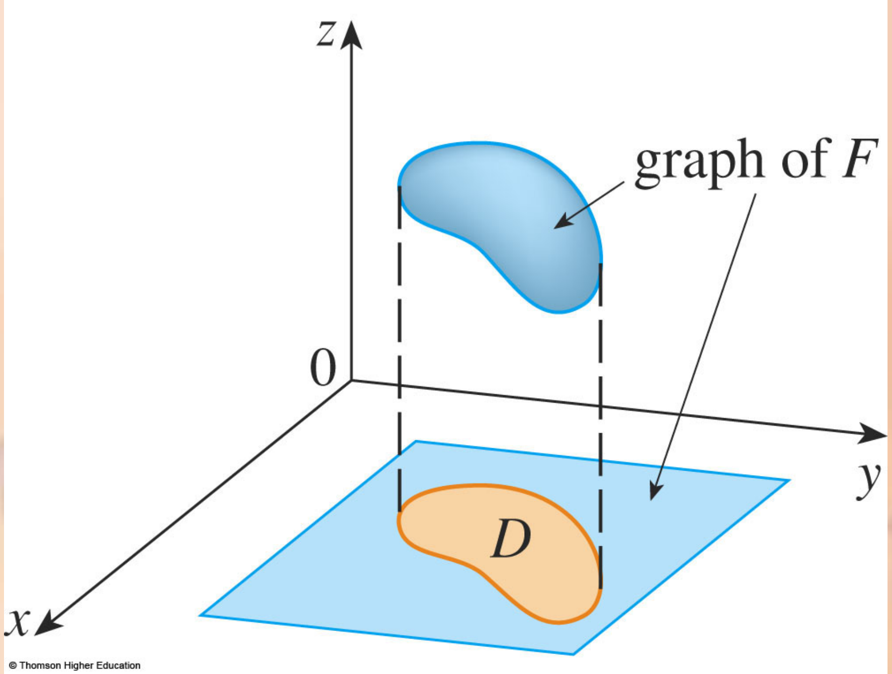
You can see that this is reasonable by:

- Comparing the graphs of *f* and *F* here.
- Remembering $\iint_R F(x,y)\,dA$ is the volume under the graph of *F*.



DOUBLE INTEGRALS

This figure also shows that *F* is likely to have discontinuities at the boundary points of *D*.



DOUBLE INTEGRALS

Nonetheless, if *f* is continuous on *D* and the boundary curve of *D* is "well behaved" then it can be shown that $\iint_R F(x, y)\, dA$ exists and so $\iint_D f(x, y)\, dA$ exists.

- In particular, this is the case for the following types of regions.

In particular, this is the case for the following types of regions.

TYPE I REGION

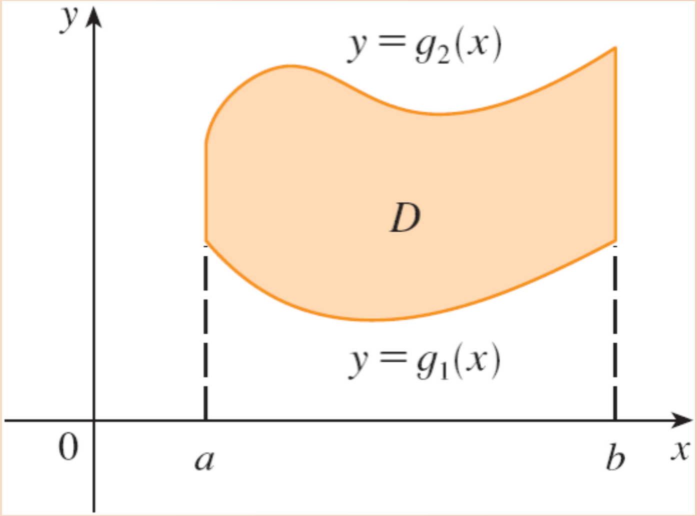
A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x, that is,

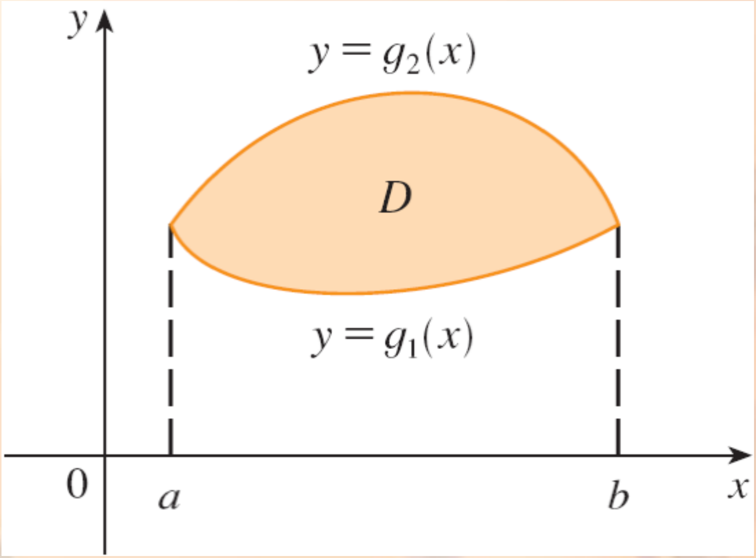
$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

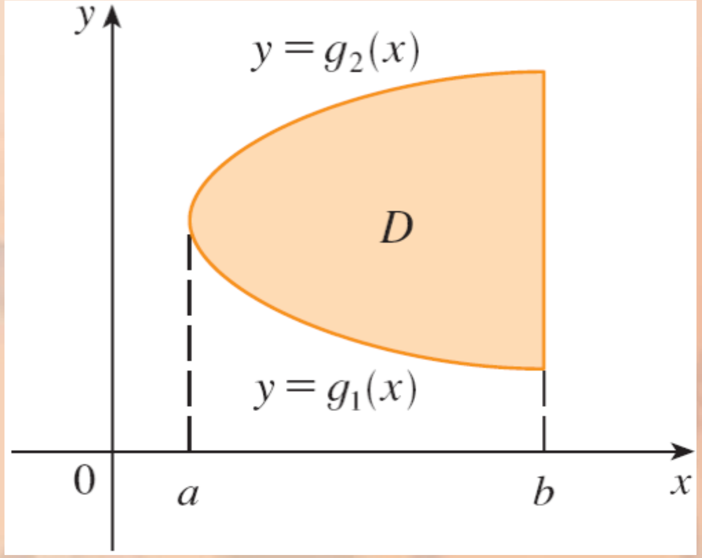
where g_1 and g_2 are continuous on $[a, b]$.

TYPE I REGIONS

Some examples of type I regions are shown.

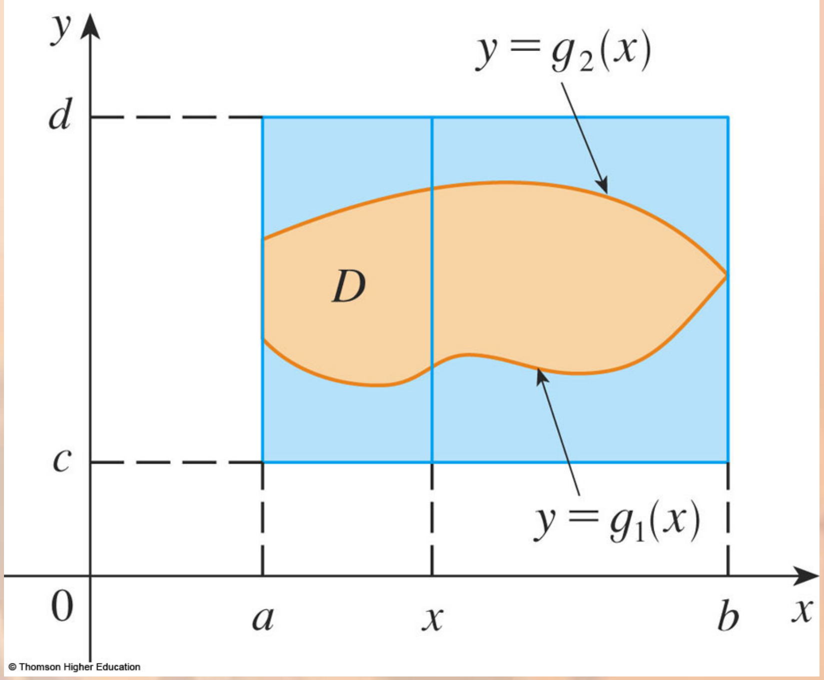






TYPE I REGIONS

To evaluate $\iint_D f(x, y)\,dA$ when *D* is a region of type I, we choose a rectangle *R* = [*a*, *b*] x [*c*, *d*] that contains *D.*



TYPE I REGIONS

Then, we let F be the function given by Equation 1.

- That is, F agrees with f on D and F is 0 outside D.

TYPE I REGIONS

Then, by Fubini's Theorem,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

$$= \int_a^b \int_c^d F(x,y)\,dy\,dx$$

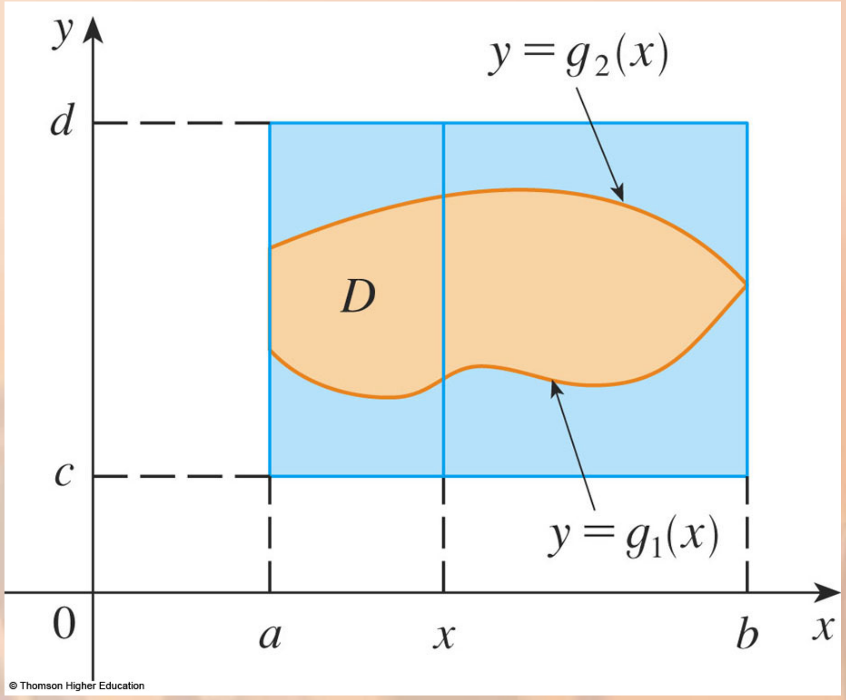
TYPE I REGIONS

Observe that $F(x, y) = 0$

if $y < g_1(x)$ or

$y > g_2(x)$ because (x, y)

then lies outside D.



TYPE I REGIONS

Therefore,

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy$$

$$= \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \leq y \leq g_2(x)$.

TYPE I REGIONS

Thus, we have the following formula that enables us to evaluate the double integral as an iterated integral.

TYPE I REGIONS

Equation 3

If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

TYPE I REGIONS

The integral on the right side of Equation 3 is an iterated integral.

- The exception is that, in the inner integral, we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

TYPE II REGIONS

Equation 4

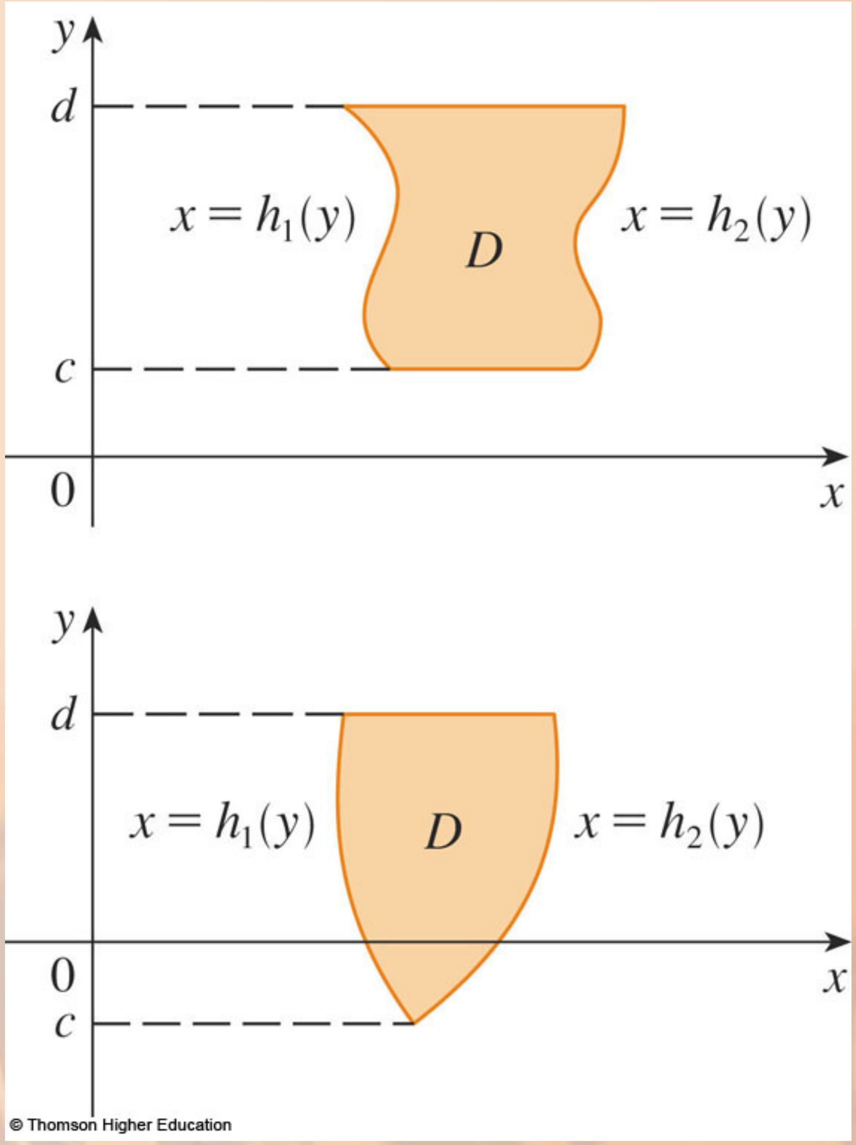
We also consider plane regions of type II, which can be expressed as:

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous.

TYPE II REGIONS

Two such regions are illustrated.



TYPE II REGIONS

Equation 5

Using the same methods that were used in establishing Equation 3, we can show that:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

where *D* is a type II region given by Equation 4.

Example 1

Evaluate $$\iint_D (x + 2y)\, dA$$

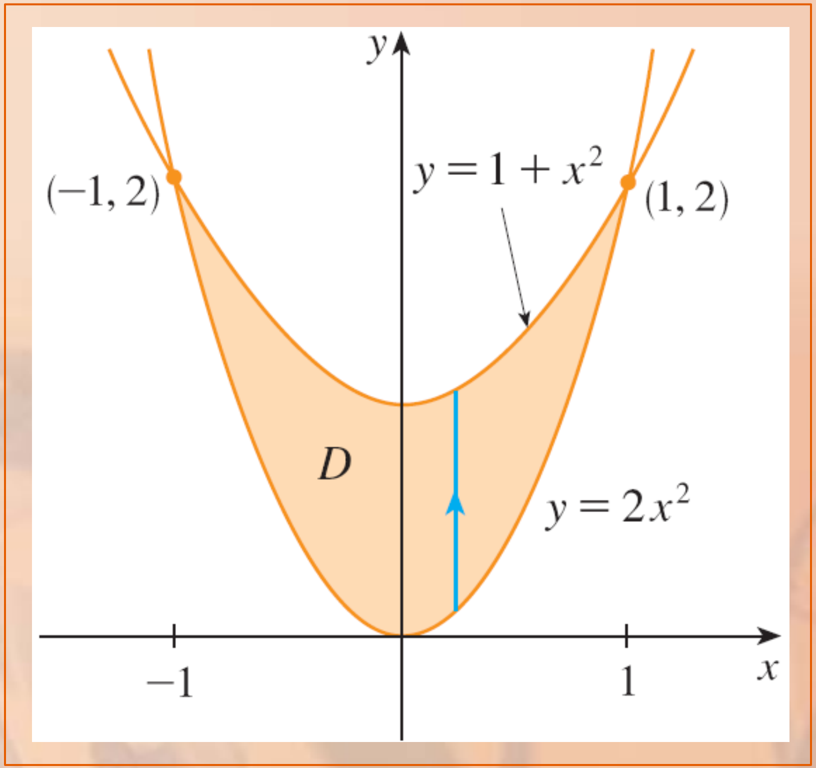
where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

TYPE II REGIONS

Example 1

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$.

- Thus, $x = \pm 1$.

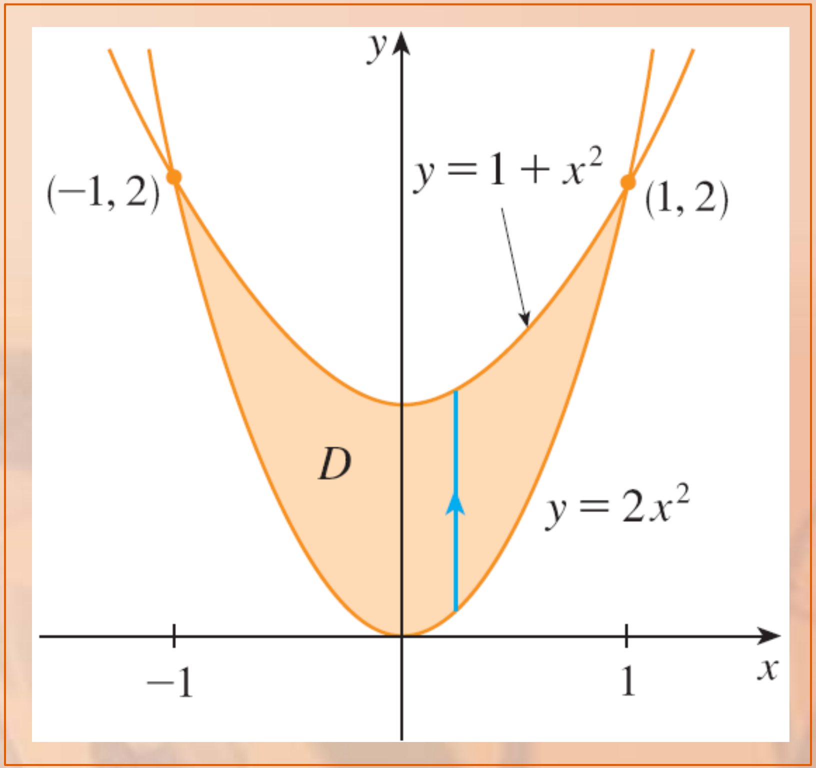


Example 1

We note that the region D is a type I region but not a type II region.

- So, we can write:

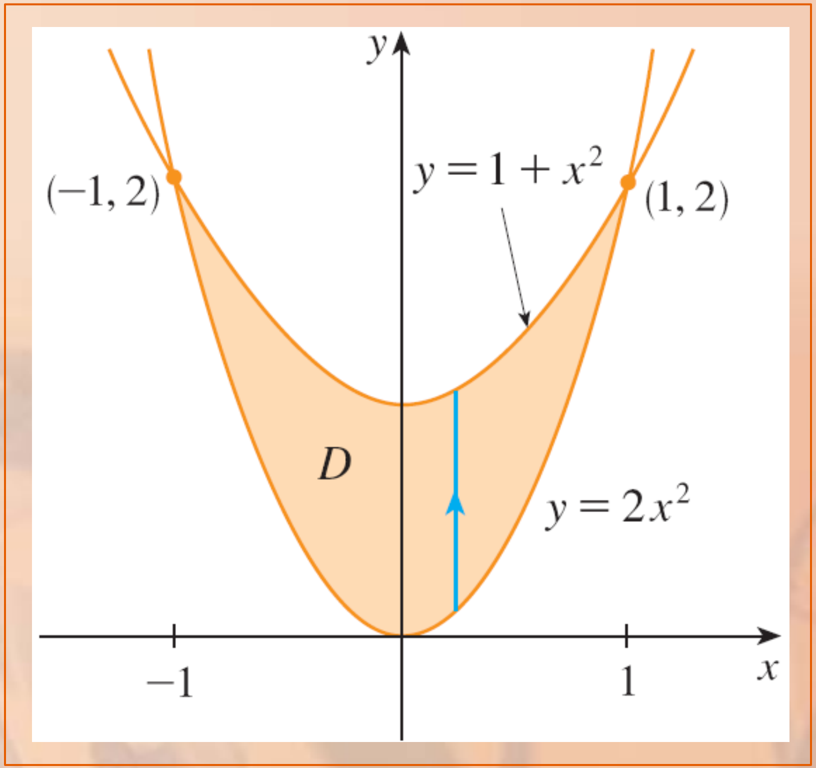
  $D = \{(x, y) \mid -1 \le x \le 1,$
  $\quad 2x^2 \le y \le 1 + x^2\}$



Example 1

The lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$.

- So, Equation 3 gives the following result.



TYPE II REGIONS

Example 1

$$\iint_D (x+2y)\,dA$$

$$= \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{-1}^{1} [xy + y^2]_{y=2x^2}^{y=1+x^2}\,dx$$

$$= \int_{-1}^{1} [x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2]\,dx$$

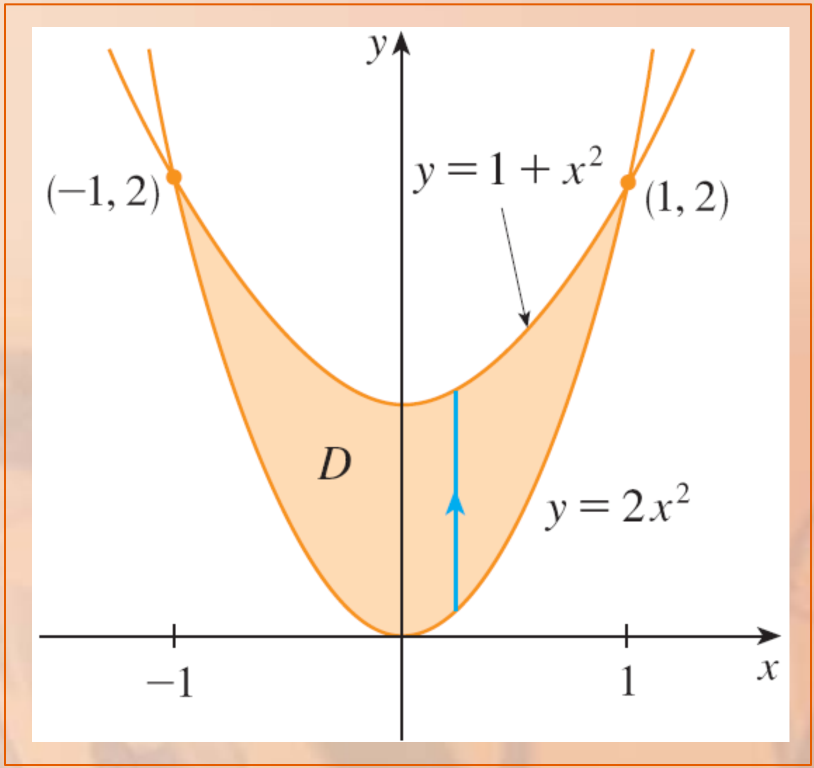
$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big]_{-1}^{1} = \frac{32}{15}$$

NOTE

When we set up a double integral as in Example 1, it is essential to draw a diagram.

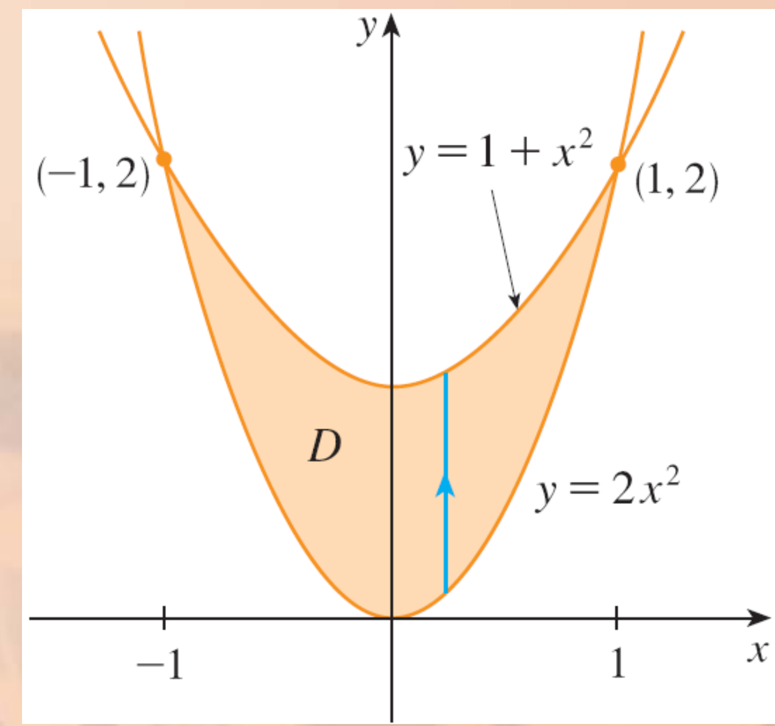
- Often, it is helpful to draw a vertical arrow as shown.



NOTE

Then, the limits of integration for the inner integral can be read from the diagram:

- The arrow starts at the lower boundary $y = g_1(x)$, which gives the lower limit in the integral.
- The arrow ends at the upper boundary $y = g_2(x)$, which gives the upper limit of integration.



NOTE

For a type II region, the arrow is drawn horizontally from the left boundary to the right boundary.

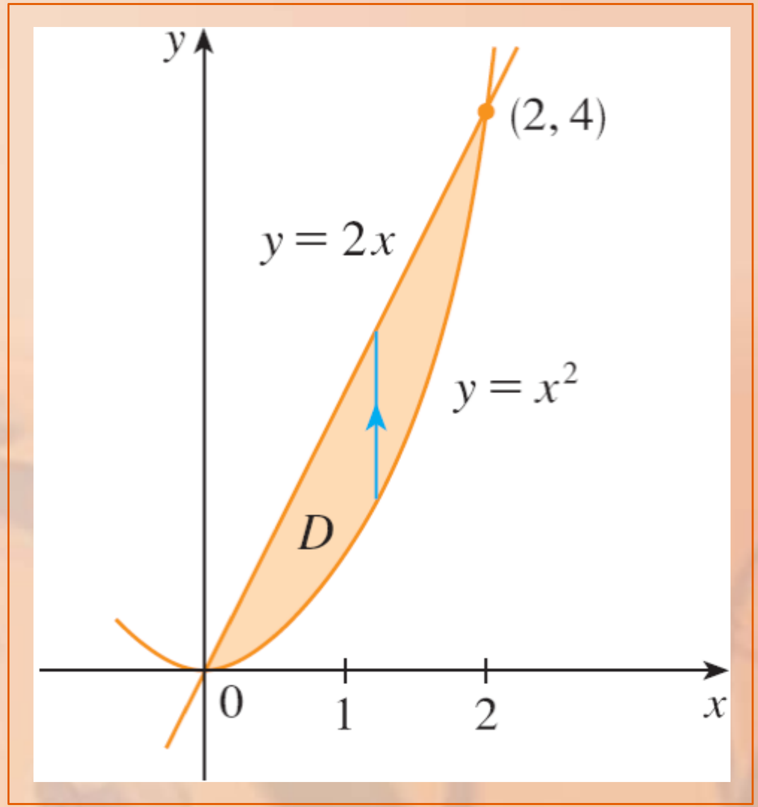
Example 2

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy–plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

From the figure, we see that D is a type I region and

$$D = \{(x, y) \mid 0 \leq x \leq 2,\ x^2 \leq y \leq 2x\}$$

- So, the volume under $z = x^2 + y^2$ and above D is calculated as follows.



E. g. 2—Solution 1

$$V$$

$$= \iint_D (x^2 + y^2)\, dA$$

$$= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3} \right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx$$

$$= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \Bigg]_0^2$$

$$= \frac{216}{35}$$

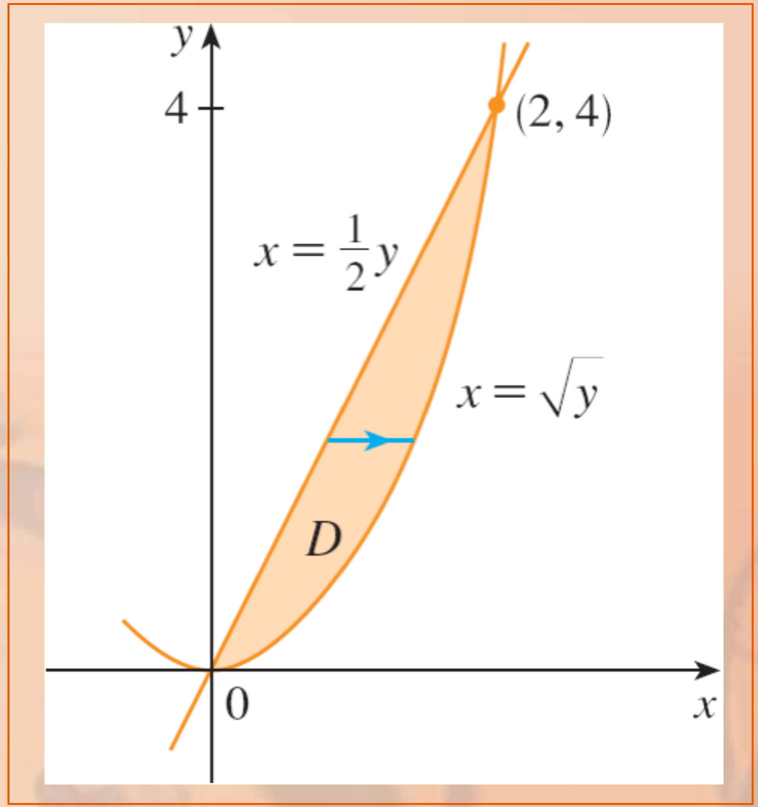
TYPE II REGIONS

E. g. 2—Solution 2

From this figure, we see that D can also be written as a type II region:

$$D = \{(x, y) \mid 0 \le y \le 4, \tfrac{1}{2}y \le x \le \sqrt{y}\}$$

- So, another expression for V is as follows.



E. g. 2—Solution 2

$$V = \iint_D (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{1}{2}}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy$$

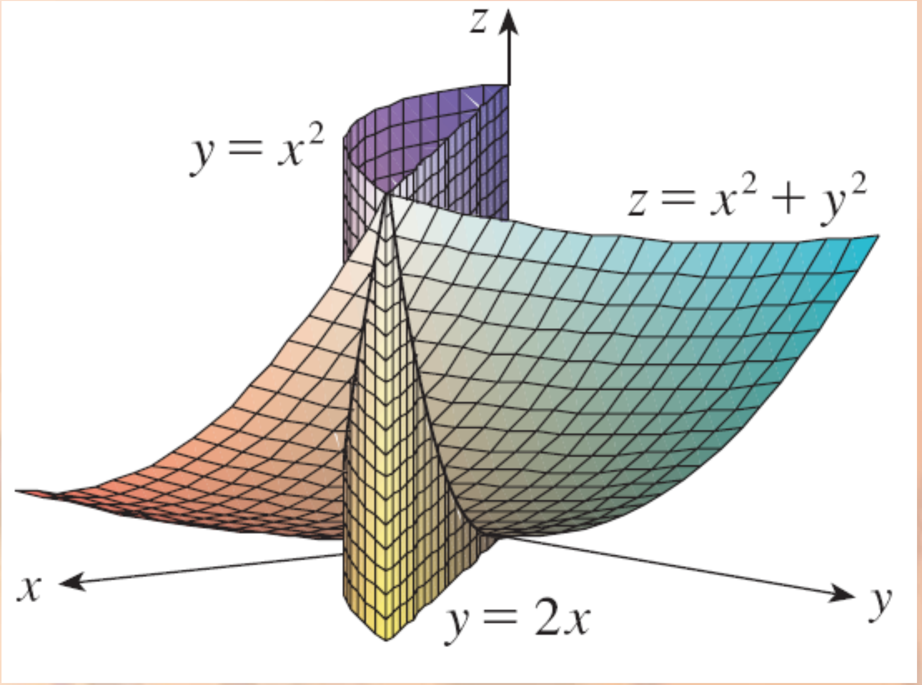
$$= \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \right) dy$$

$$= \tfrac{2}{15} y^{5/2} + \tfrac{2}{7} y^{7/2} - \tfrac{13}{96} y^4 \Big]_0^4 = \tfrac{216}{35}$$

The figure shows the solid whose volume is calculated in Example 2.

It lies:

- Above the *xy*-plane.
- Below the paraboloid $z = x^2 + y^2$.
- Between the plane $y = 2x$ and the parabolic cylinder $y = x^2$.



Example 3

Evaluate

$$\iint_D xy\,dA$$

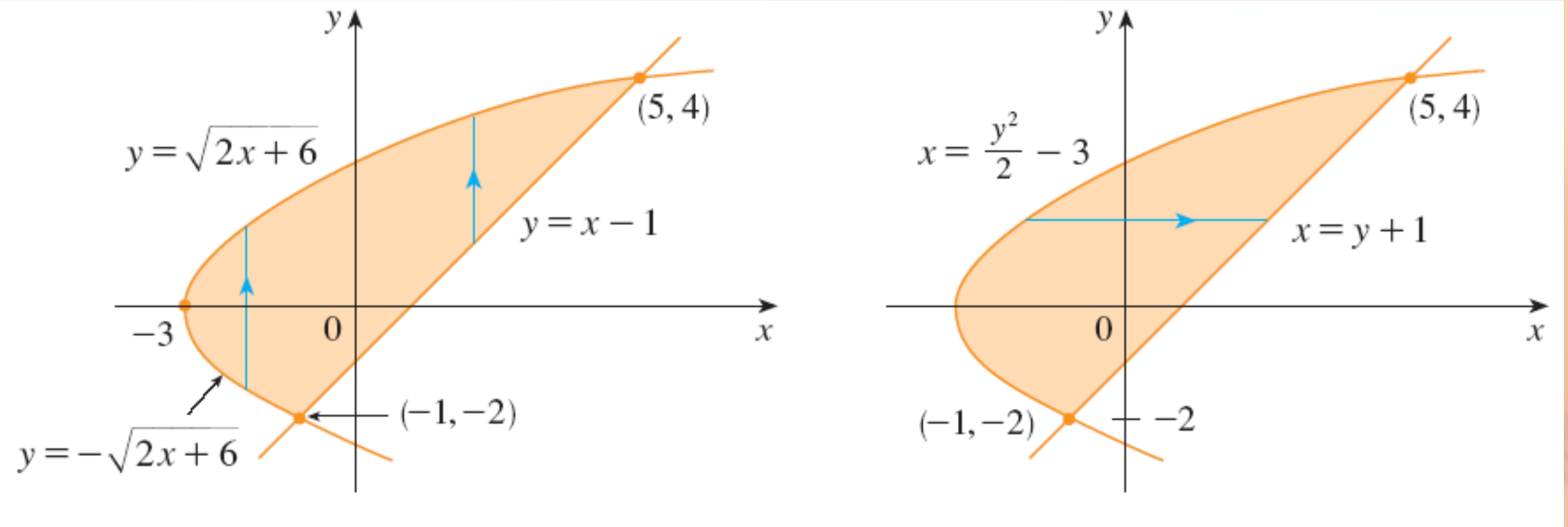
where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$

TYPE I & II REGIONS

Example 3

The region D is shown.

- Again, D is both type I and type II.



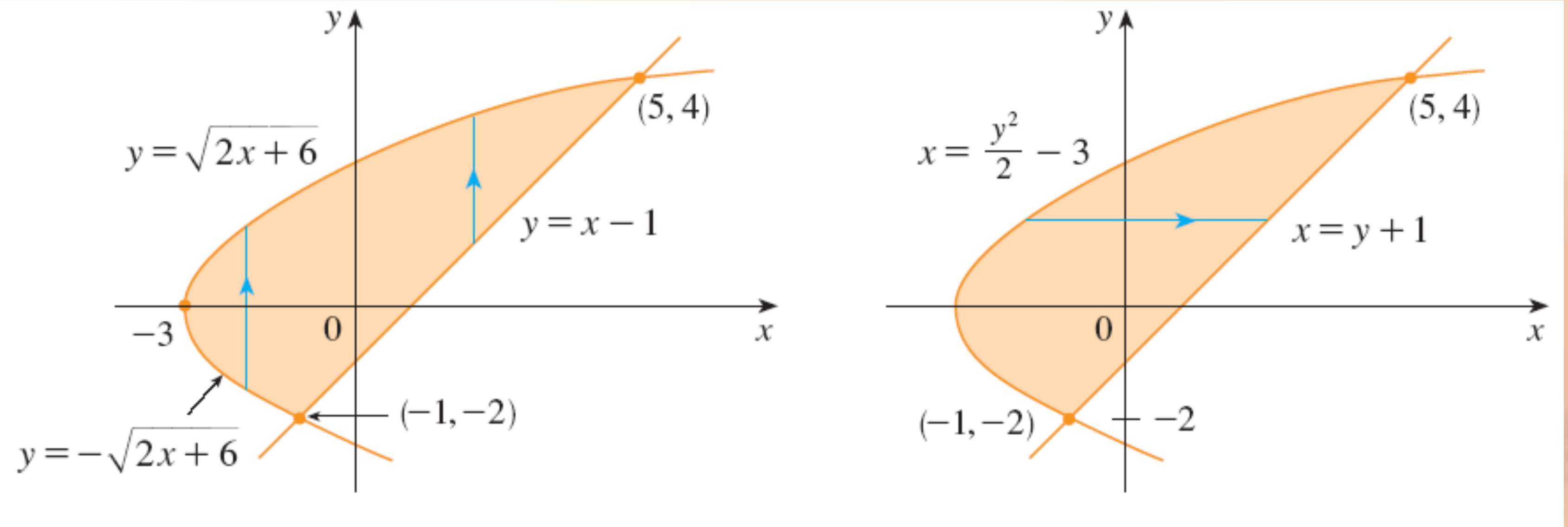
(a) D as a type I region

(b) D as a type II region

TYPE I & II REGIONS

Example 3

However, the description of *D* as a type I region is more complicated because the lower boundary consists of two parts.



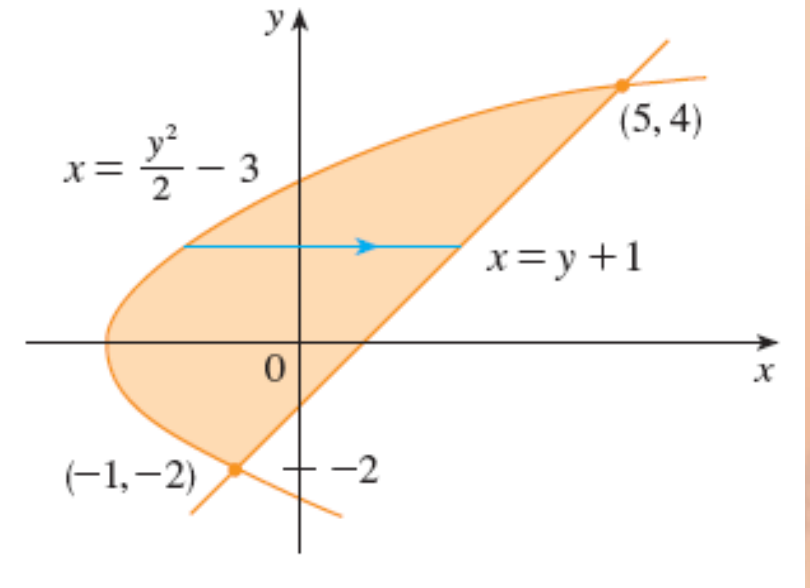
(a) *D* as a type I region

(b) *D* as a type II region

Hence, we prefer to express D as a type II region:

$$D = \{(x, y) \mid -2 \le y \le 4,\ 1/2y^2 - 3 \le x \le y + 1\}$$

- Thus, Equation 5 gives the following result.



D as a type II region

Example 3

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy$$

$$= \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - (\tfrac{1}{2}y^2-3)^2\right]dy$$

$$= \tfrac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy$$

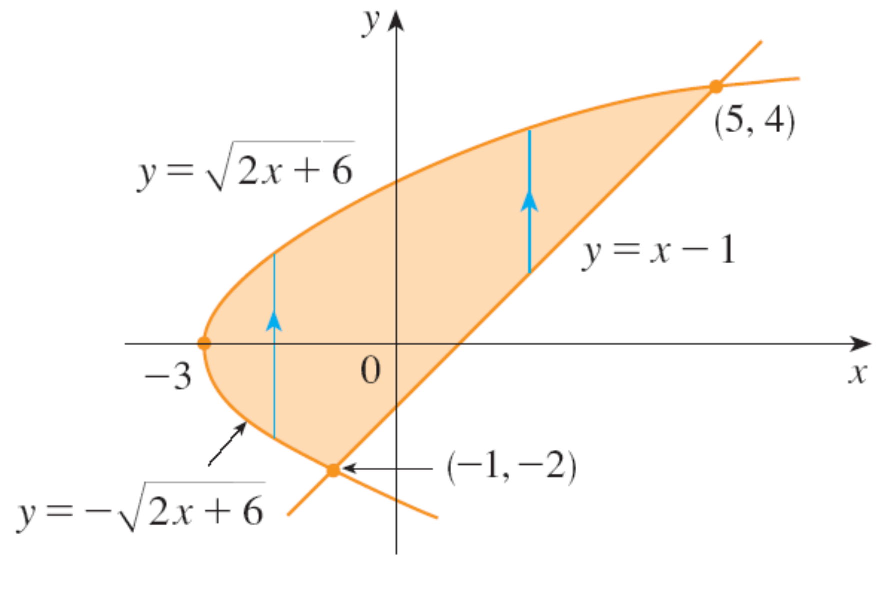
$$= \tfrac{1}{2}\left[-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\right]_{-2}^{4} = 36$$

Example 3

If we had expressed D as a type I region, we would have obtained:

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

- However, this would have involved more work than the other method.



D as a type I region

Example 4

Find the volume of the tetrahedron bounded by the planes

$$x + 2y + z = 2$$

$$x = 2y$$

$$x = 0$$

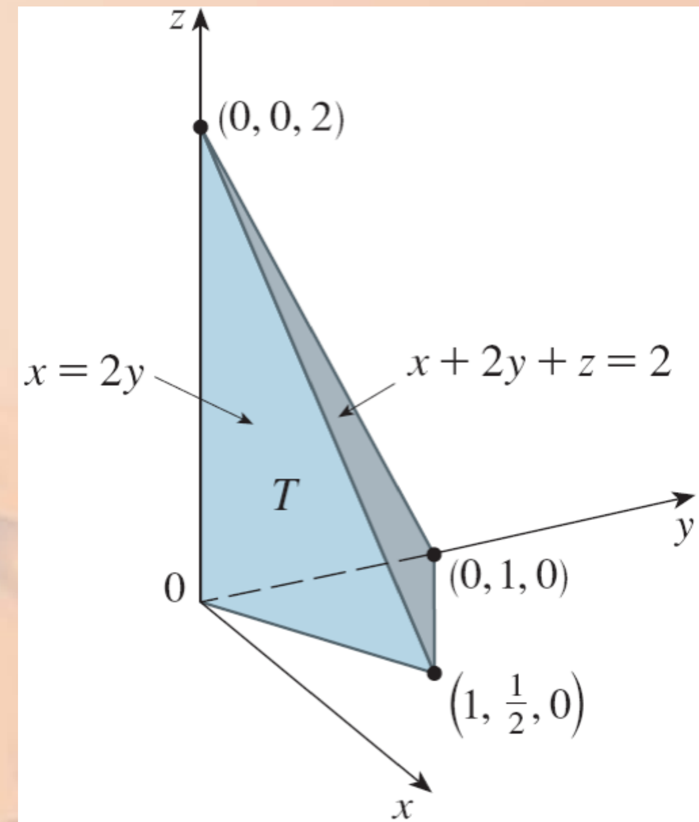
$$z = 0$$

In a question such as this, it's wise to draw two diagrams:

- One of the three-dimensional solid
- One of the plane region *D* over which it lies

Example 4

The figure shows the tetrahedron T bounded by the coordinate planes $x = 0$, $z = 0$, the vertical plane $x = 2y$, and the plane $x + 2y + z = 2$.

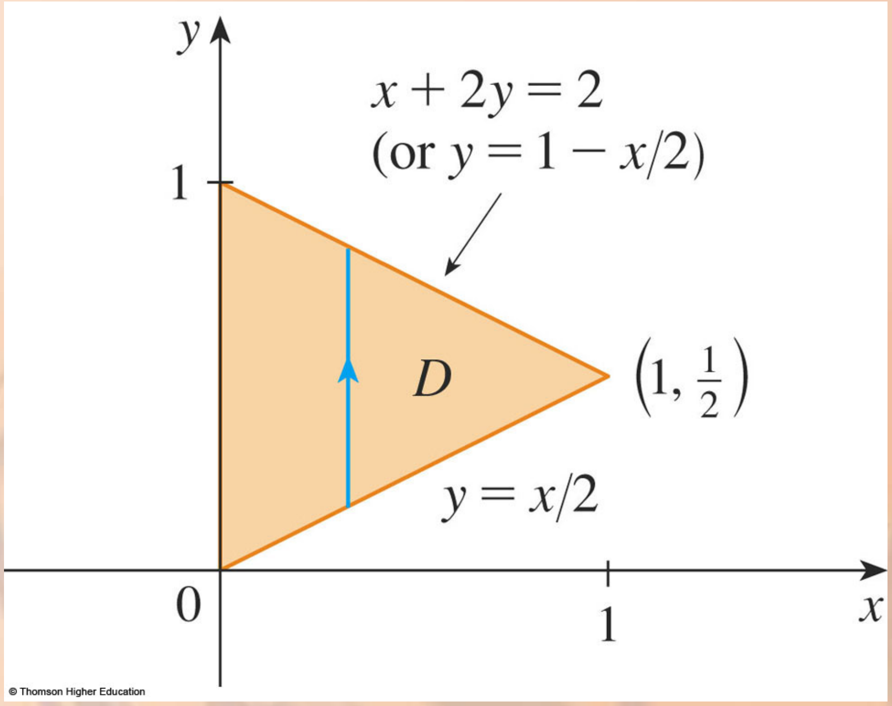


DOUBLE INTEGRALS

Example 4

As the plane $x + 2y + z = 0$ intersects the xy-plane (whose equation is $z = 0$) in the line $x + 2y = 2$, we see that:

- T lies above the triangular region D in the xy-plane within the lines
  $x = 2y$
  $x + 2y = 2$
  $x = 0$



Example 4

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$.

So, the required volume lies under the graph of the function $z = 2 - x - 2y$ and above

$$D = \{(x, y) \mid 0 \le x \le 1, x/2 \le y \le 1 - x/2\}$$

Example 4

Therefore,

$$
\begin{aligned}
V &= \iint_D (2 - x - y)\, dA \\
&= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx \\
&= \int_0^1 \left[2y - xy - y^2 \right]_{y=x/2}^{y=1-x/2} dx
\end{aligned}
$$

Example 4

$$= \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4} \right] dx$$

$$= \int_0^1 \left(x^2 - 2x + 1\right) dx$$

$$= \frac{x^3}{3} - x^2 + x \Big]_0^1$$

$$= \frac{1}{3}$$

Example 5

Evaluate the iterated integral

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dy\, dx$$

- If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin\left(y^2\right) dy$
- However, it's impossible to do so in finite terms since $\int \sin\left(y^2\right) dy$ is not an elementary function. (See the end of Section 7.5)

Hence, we must change the order of integration.

- This is accomplished by first expressing the given iterated integral as a double integral.

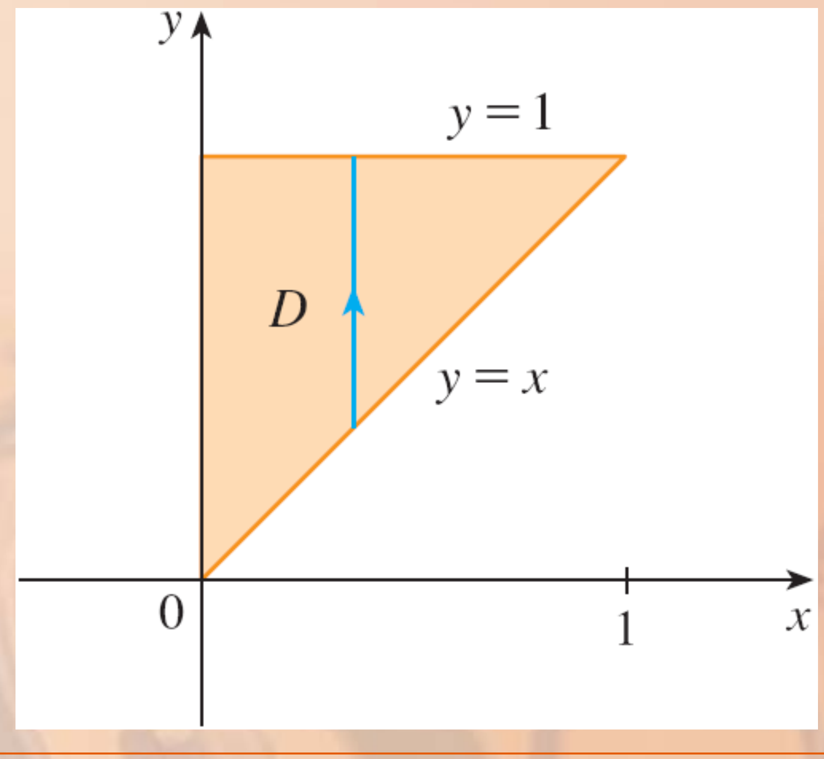
- Using Equation 3 backward, we have:

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dy\, dx = \iint_D \sin\left(y^2\right) dA$$

where $D = \{(x, y) \mid 0 \le x \le 1, x \le y \le 1\}$

 Example 5

We sketch that region *D* here.

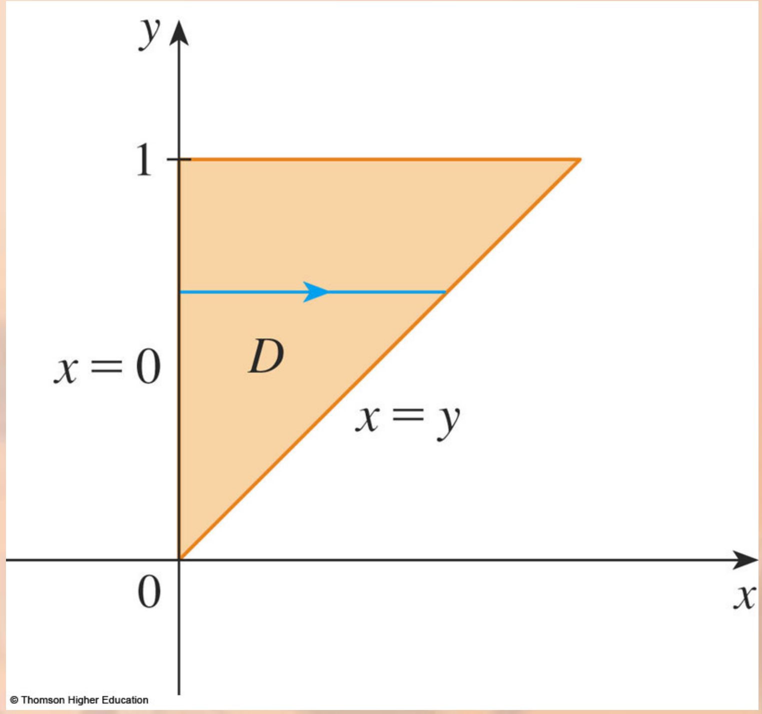


Example 5

Then, from this figure, we see that
an alternative description of D is:

$$D = \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$$

- This enables us to use Equation 5 to express the double integral as an iterated integral in the reverse order, as follows.



Example 5

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dy\, dx = \iint_D \sin\left(y^2\right) dA$$

$$= \int_0^1 \int_0^y \sin\left(y^2\right) dx\, dy$$

$$= \int_0^1 \left[x \sin\left(y^2\right)\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin\left(y^2\right) dy$$

$$= -\tfrac{1}{2} \cos\left(y^2\right)\Big]_0^1$$

$$= \tfrac{1}{2}(1 - \cos 1)$$

PROPERTIES OF DOUBLE INTEGRALS

We assume that all the following integrals exist.

- The first three properties of double integrals over a region *D* follow immediately from Definition 2 and Properties 7, 8, and 9 in Section 15.1

PROPERTIES 6 AND 7

$$\iint_D \left[f(x,y) + g(x,y)\right] dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$

PROPERTY 8

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

$$\iint_D f(x, y)\, dA \geq \iint_D g(x, y)\, dA$$

PROPERTIES

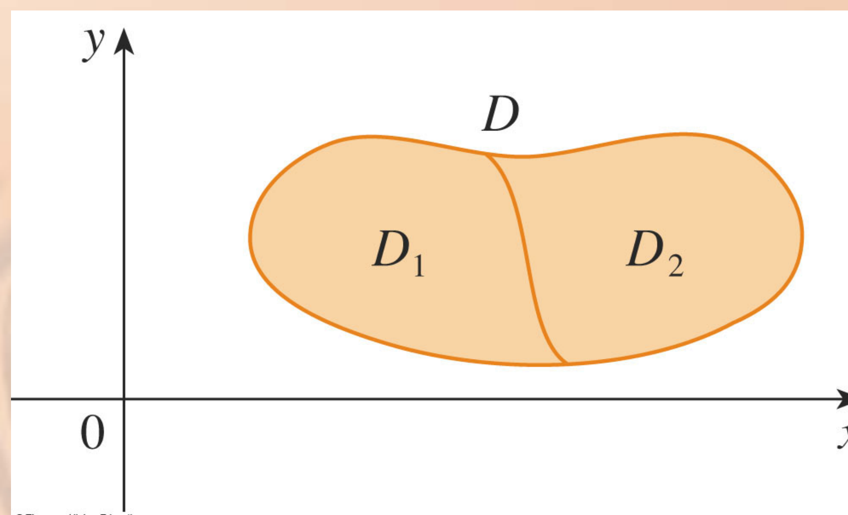
The next property of double integrals is similar to the property of single integrals given by the equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

PROPERTY 9

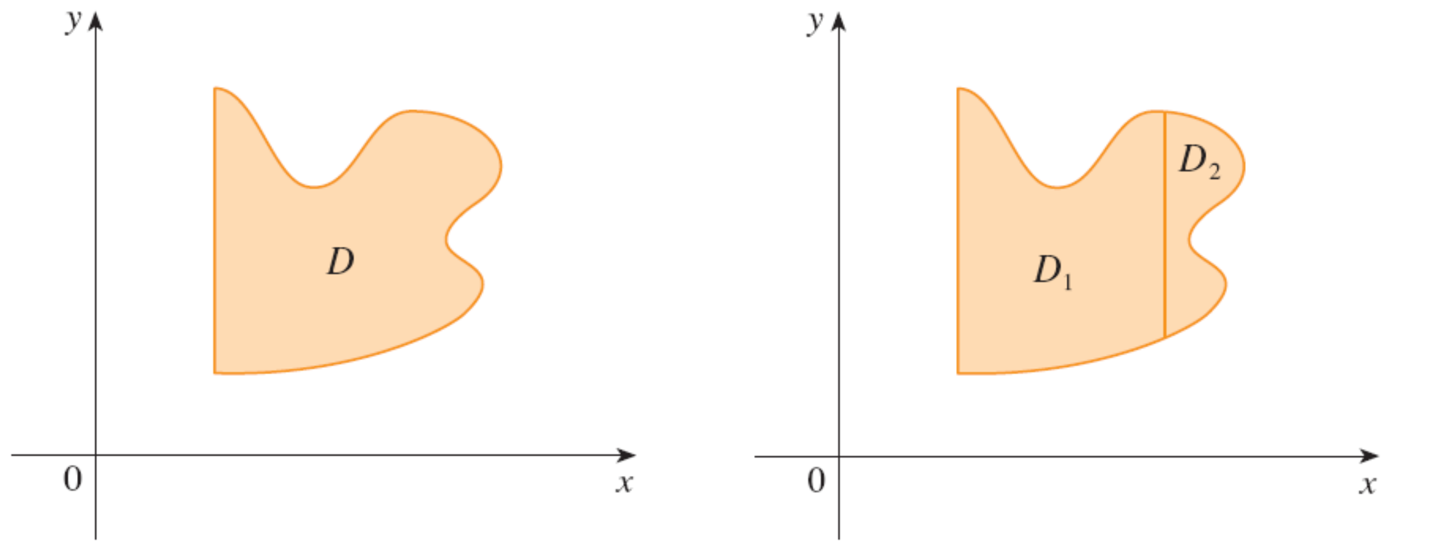
If $D = D_1$ È D_2, where D_1 and D_2 don’t overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



PROPERTY 9

Property 9 can be used to evaluate double integrals over regions *D* that are neither type I nor type II but can be expressed as a union of regions of type I or type II.



(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

PROPERTY 10

Equation 10

The next property of integrals says that, if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

$$\iint_D 1\, dA = A(D)$$

PROPERTY 10

The figure illustrates why Equation 10 is true.

- A solid cylinder whose base is *D* and whose height is 1 has volume $A(D) \cdot 1 = A(D)$.
- However, we know that we can also write its volume as

$$\iint_D 1\, dA$$