LAPLACE TRANSFORM

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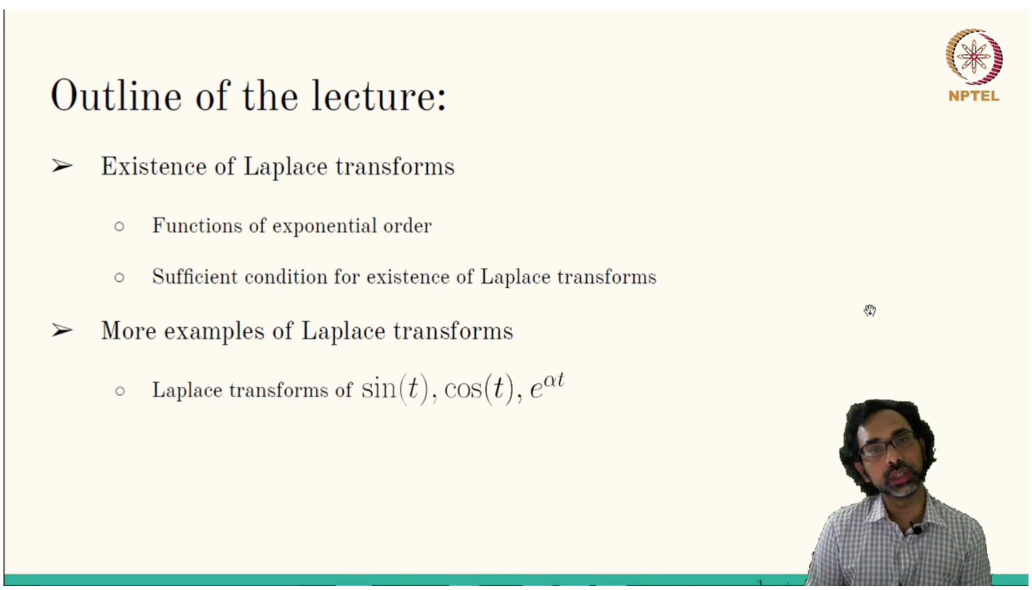
Lecture 5: Existence of Laplace transforms and Examples

Now, we have seen that improper Riemann integrals may or may not exist and that we have given sufficient conditions for the existence of improper Riemann integrals. Now we will ask the same question about the existence of Laplace transforms.



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In this lecture we will see definition of functions of exponential order. We will give sufficient conditions for the existence of Laplace transforms, just like we did for existence of improper Riemann integrals. We will continue our computations of more examples of Laplace transforms and in this lecture we will see Laplace transforms of Sin function, Cos function as well as the exponential function.

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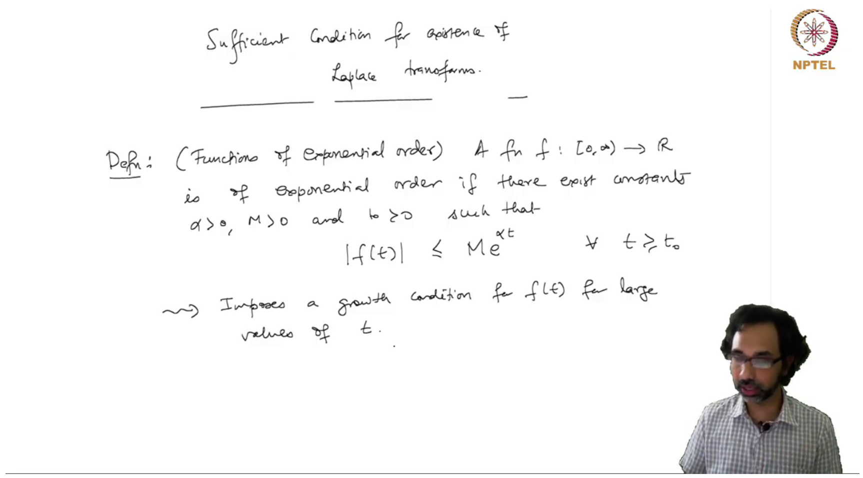
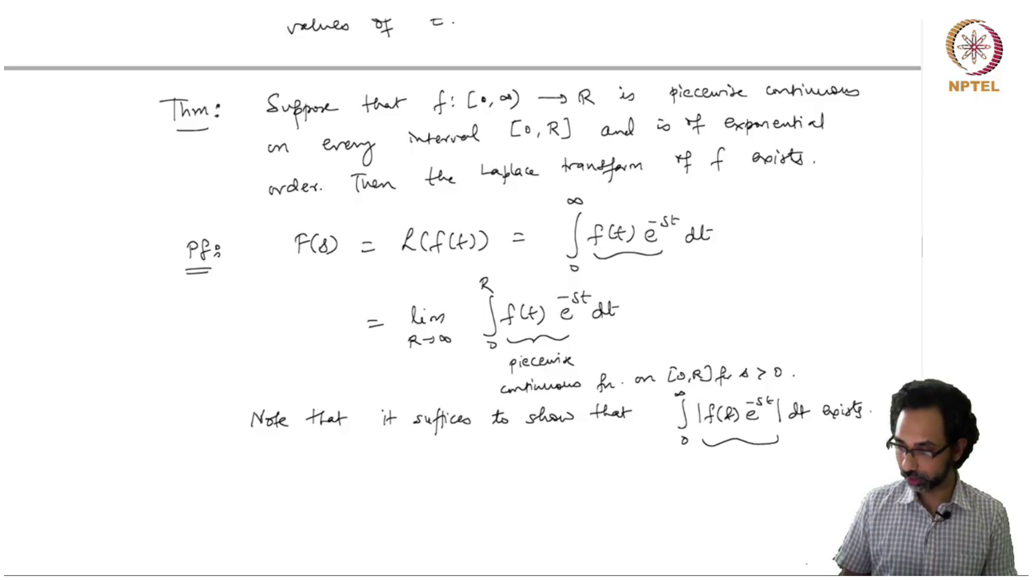


(Refer Slide Time: 1:08) Now with this theory in mind, we can state the sufficient condition for existence of Laplace transforms. For this we make a definition and this is about functions of exponential order.

Definition: A function $f : [0, \infty) \longrightarrow \mathbf{R}$ is of exponential order, if there exist constants $\alpha > 0$, $M > 0$, and $t > 0$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t \geq 0$.

This definition imposes a growth condition for the function $f(t)$ for large value of t. It says that it cannot be grow faster than some multiple of an exponential function. So, we will see that once we have this kind of once we impose this kind of growth condition on the function the Laplace transforms will exist.



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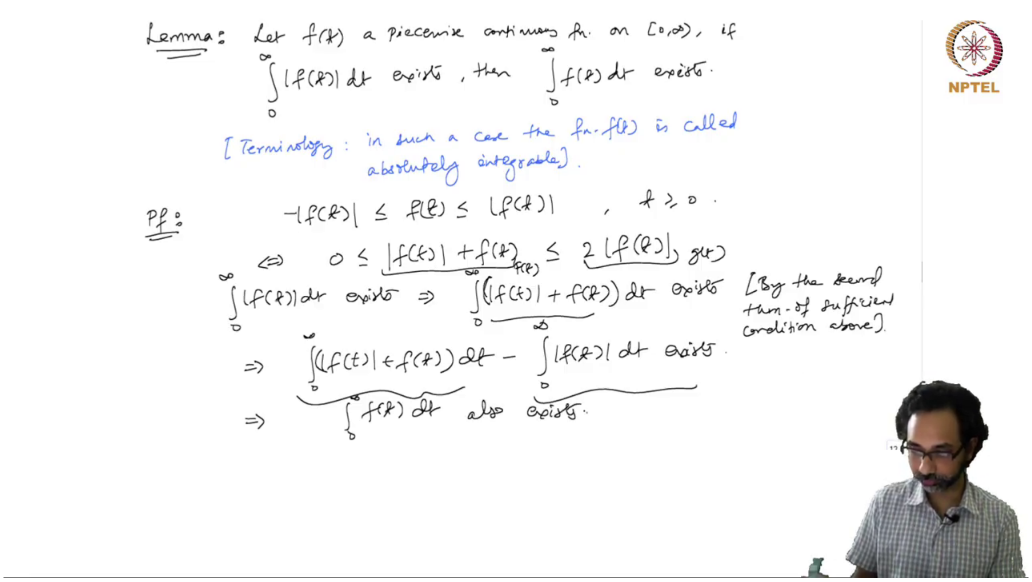
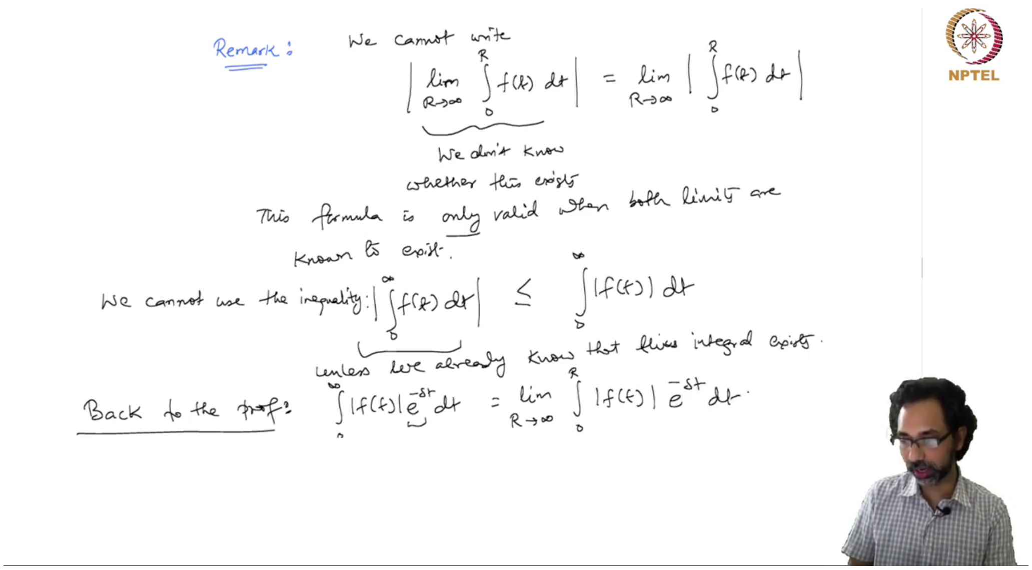
So, let us state this as a theorem. Suppose that $f : [0, \infty) \longrightarrow \mathbf{R}$ is piece wise continues on every interval $[0, R)$ and is of exponential order, then the Laplace transform of f exists, okay? So, this proof is quiet easy. We have to show the Laplace transform of this function $f(t)$ which I denote by $F(s)$ which is $\int_0^\infty f(t)e^{-st}dt$ By definition this is again the $\lim_{R\to\infty} \int_0^R f(t)e^{-st}dt$

Now, because f is piece wise continues on every interval this function the multiplication of $f(t)e^{-st}$ will be piece wise continues on every interval $[0, R]$. First note that it suffices to show that the integral $\int_0^R |f(t)e^{-st}|dt$ exists. So, we are claiming that if you just show this, as a consequence we can infer that the original integral, also converges. So, this is because of the following Lemma.



(Refer Slide Time: 6:09) It says that for f a piecewise, continuous function, piecewise continuous function on the whole interval 0 to ∞ if $\int_0^\infty |f(t)|dt$ exists, then $\int_0^\infty f(t)dt$ exists. So, let us see the quite easy proof for this statement. So, we begin by noting that we have the inequality $-f(t)$ less than or equal to minus modulus of $f(t)$ is less than or equal to mod of $f(t)$ for any t greater than or equal to 0. So, then this implies is the same thing as writing 0 less than or equal to mod $f(t)$ plus $f(t)$ is less than or equal to two times mod $f(t)$. Just adding modulus of $f(t)$ on the on both the inequalities we get this inequality and now we know that integral $\int_0^\infty |f(t)|dt$ exists. So, this implies that $\int_0^\infty (|f(t)| + f(t))dt$ by the second theorem of sufficient condition above.

So, we have seen that in this case, we have this is our dt and this is our $f(t)$, well I am not going to use the same notation here, but let us call it $F(t)$. so, $F(t)$ is positive and it is bounded above by an integrated function dt. So therefore, capital $F(t)$ is also integrable. So, this we wrote down as a theorem for a sufficient condition for the existence of improper Riemann integral. And now I am going to subtract. So, this implies that integral 0 to infinity modulus of $f(t)$ plus $f(t)$ dt minus integral $\int_0^\infty |f(t)|dt$ this also exist because now it is a difference of two finite things, therefore it must also be finite. And now I can use the linearity property. To get that, the modulus of $f(t)$ dt also exists.



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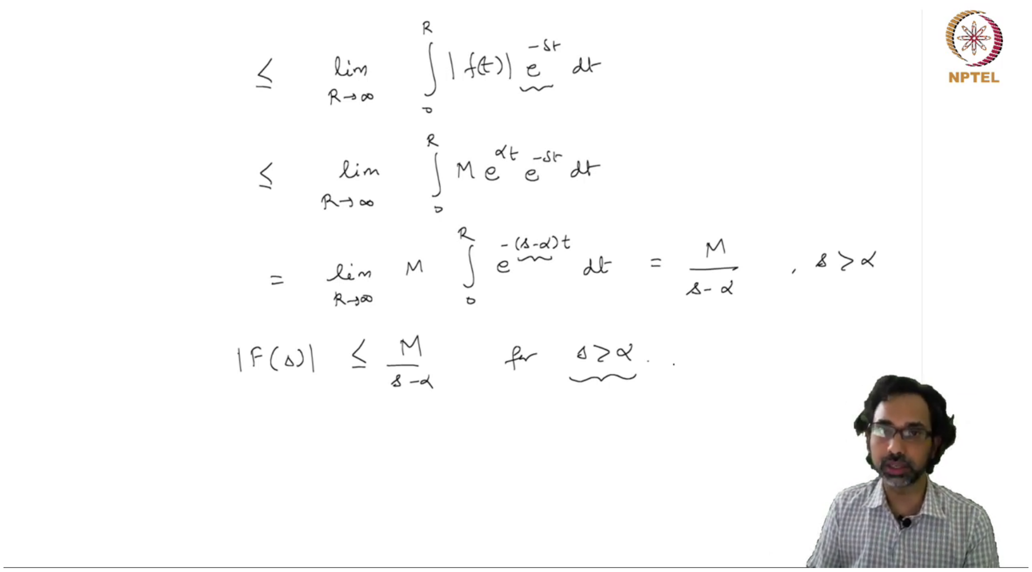
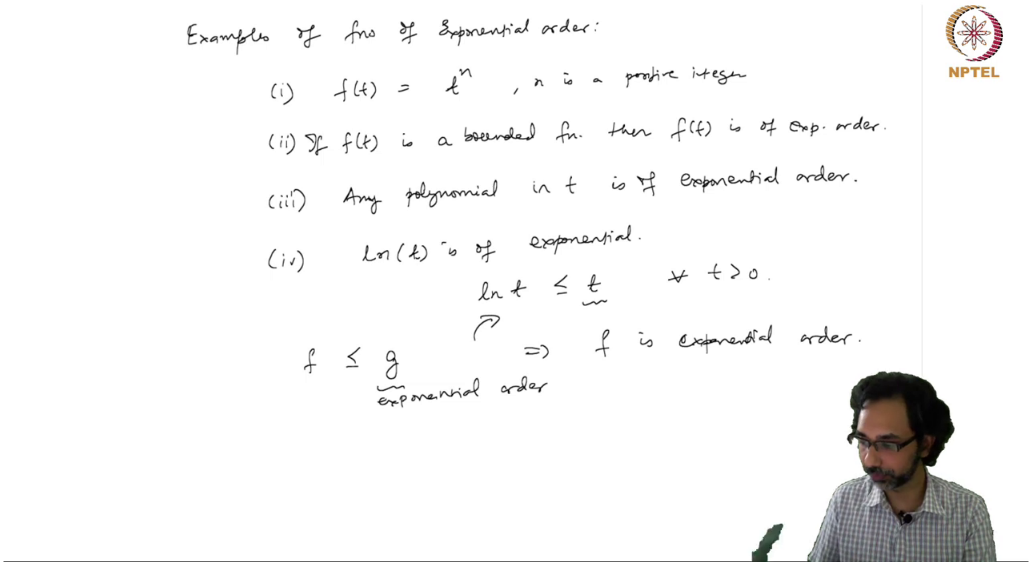
So, as a remark, notice that we cannot write. So, why we did all this trickery with the modulas of $f(t)$ plus $f(t)$ which is this fact here, which says that we cannot write that $|\lim_{R\to\infty} \int_0^R f(t)dt| = \lim_{R\to\infty} |\int_0^R f(t)dt|$.

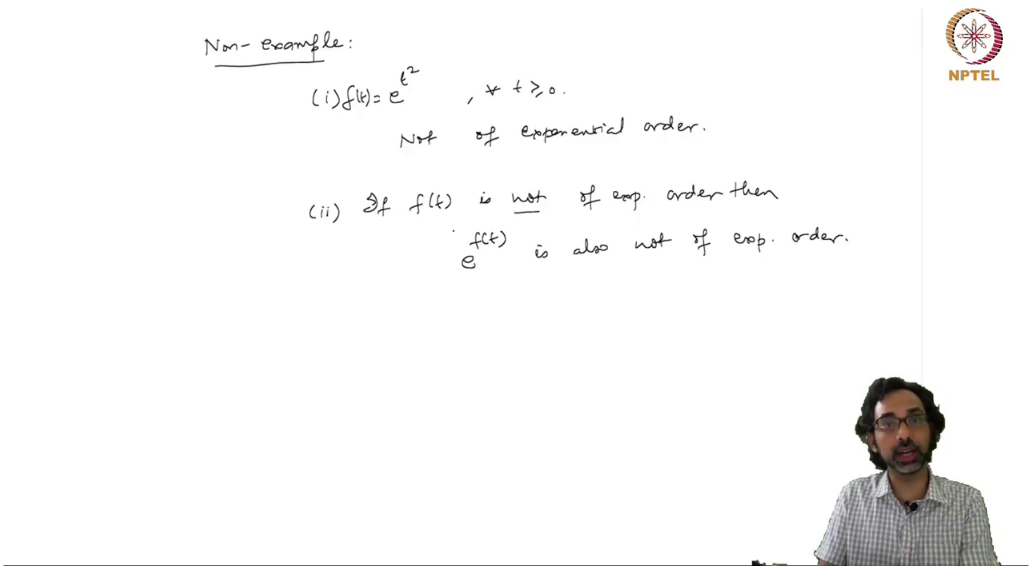
We cannot write this because we do not know yet whether this exists or not, so we are trying to prove that this exists. And we cannot use this interchange of modulas and limits unless we know that both limits exist. So, this formula only valid when both limits are known to exist. Therefore we had to do this trickery with inequalities and using the comparison theorem for convergence of for the existence of improper Riemann integrals for positive functions. So, here also another warning is that we cannot use $\int_0^\infty f(t)dt \leq \int_0^\infty |f(t)|dt$ unless we know this exists. I saw in some books that this has been written and it is unfortunately incorrect because we do not know whether this limit this integral inside the modulus exists yet. So, we cannot write this inequality.



So, now we go back to the proof. Proof for the theorem, and so we only want to show that the integral $\int_0^\infty |f(t)|dt$ is finite. We just have to prove that integral zero to infinity modulus of $\int_0^\infty f(t)e^{-st}dt$ I can take $e^{st}dt$ to the modulus, because this is always positive and but note that this is $lim_{R\to\infty} \int_0^R |f(t)e^{-st}dt|$ by definition, is less than or equal to limit R goes to infinity integral 0 to R modulus $f(t)e^{-st}$ since, this is always positive the modulus will not affect it. And now we use the fact that $f(t)$ is of exponential order. So, this is $\int_0^R Me^{\alpha(t)}e^{-st}$okay? And this is limit R as goes to infinity you can take M outside the integral and then you get e to the power minus S minus α times t dt. But this limit can easily be evaluated and you just get $M/s - \alpha$. So, provided as s is greater than α so you want this to be positive this coefficient s minus α to be positive this is equals to $M/s - \alpha$. Therefore, we get the modulus of $F(s)$ is less than or equal to$M/s - \alpha$ for s greater than α. Therefore, $F(s)$ is finite for $s > \alpha$. So, the Laplace transform exists in the region $s > \alpha$. Now, whenever we write the Laplace Transform of function, we should always mention the region where it is valid. Here the region of convergence is $s > \alpha$. But we should always mention what is the region in which the Laplace transform is valid.

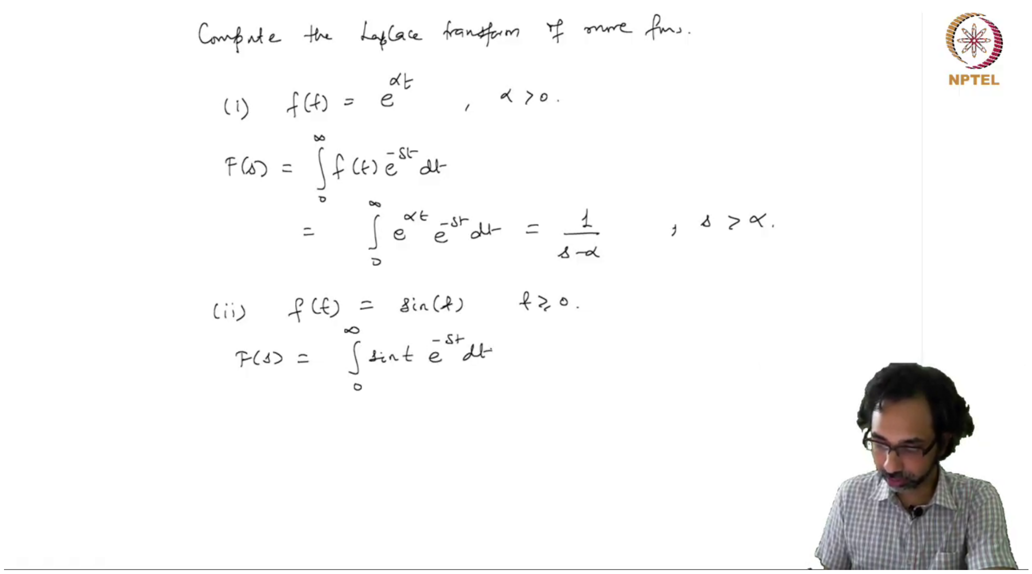


So, what are the examples of functions of exponential order? So, first is $f(t) = t^n$, where n is a positive integer. This is of exponential order. Secondly, if $f(t)$ is a bounded function, then $f(t)$ is of exponential order. Third is any polynomial in t is of exponential order. So, by the way, in the second example you have all the trigonometry functions like sin and Cos, because they are all bounded functions. But from the first one, since all powers of t are of exponential order, then any polynomial of t is also of exponential order. And here we are using that if two functions are of exponential order. Then there some also of exponential order. Another example is $ln(t)$, log to the base e, this is of exponential order. Simply because $ln(t) \leq t$ for all positive t and since t is of exponential order then $ln(t)$ is also of exponential order, which means that if f and g are two functions, $f \leq g$ and g this is of exponential order. This implies that f is of exponential order.



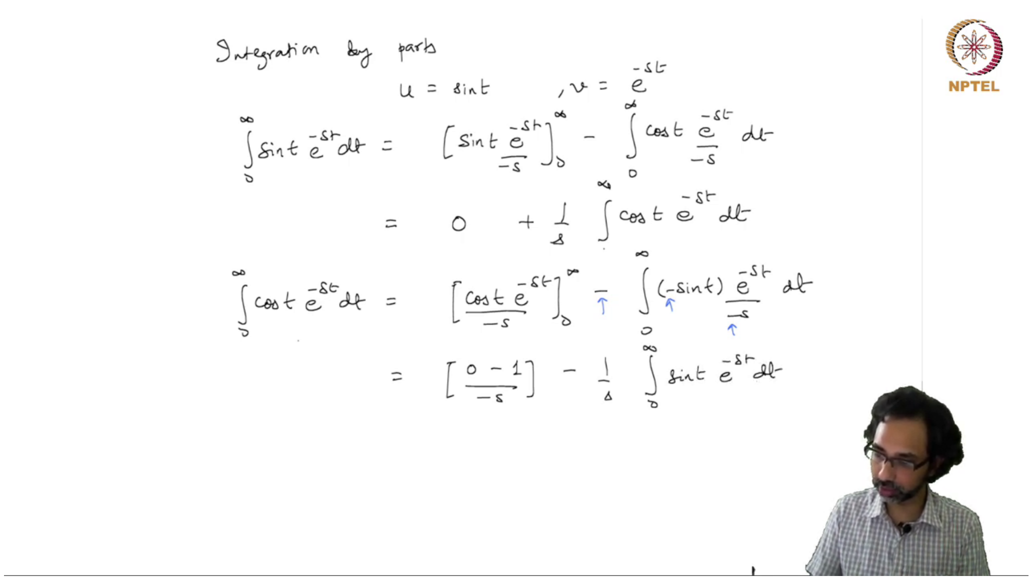
Now, let us see non example where a function is not of exponential order. First is that is say $f(t) = e^{t^2}$ for all t positive. Now, it can be checked that by taking logarithms, that this is not of exponential order. We can produce new functions from this function $f(t)$, which is not of exponential order. For example if $f(t)$ is not of exponential order then $e^{f(t)}$ is also not of exponential order. So, one can have many examples of functions which are not of exponential order, but note that our condition is only a sufficient condition for the existence of Laplace transforms, and there exist functions of which are not of exponential order, which also help. For which there Laplace transform exists. So, it is only a sufficient condition, but not a necessary condition.

So, now that we have some theoretical background for the existence of Laplace transforms an existence of improper Riemann integral integrals, let us go back to the , computation of the Laplace transforms of some more functions. So, let us compute the Laplace transform of more functions.

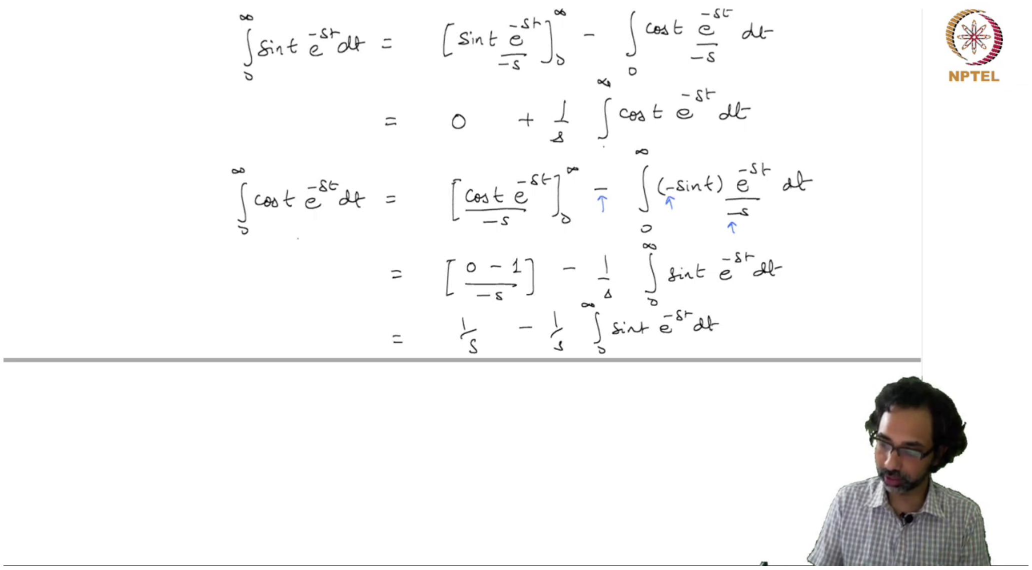


So, the first one is $f(t) = e^{\alpha t}$, α is positive. Note that this is now a function of exponential order. So, it's Laplace Transform will exist. Now, let us compute 0 to infinity $f(t)e^{-st}dt$, this is 0 to infinity, exponential αt minus st dt and this we have computed before. This is $1/(s-\alpha)$ for $s > \alpha$. Okay? So, the Laplace Transform of $e^{\alpha t} = 1/(s-\alpha)$, for $s > \alpha$.

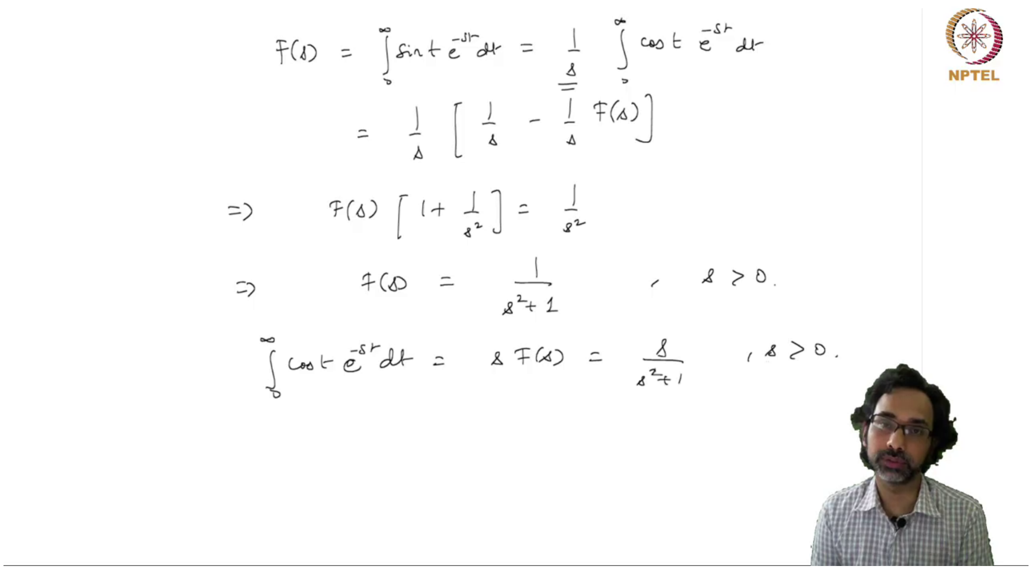
Now, let us consider $f(t) = Sin(t)$ for $t > 0$. So, $F(s) = \int_0^\infty Sin(t)e^{-st}dt$.



Now we will again imply integration by parts. So, I take u to be $Sin(t)$ and v to be exponential minus st, then you get this 0 to infinity $Sin(t)$ exponential minus $stdt$ this is equal to, so let us evaluate this by integration by parts. So, the first term is $Sin(t)$ exponential minus st by minus s 0 to infinity, minus integral 0 to infinity $Cos(t)$ exponential minus st by minus st dt.



So, let us see what the first term is the at infinity this goes to 0 because $Sin(t)$ is bounded above by one and exponential minus st will go to 0. So, you get 0 and at 0 you again get 0 because $Sin(t)$ will be 0. So, you get a 0 here in the first term and then you get a plus 1 over s integral 0 to infinity $Cos(t)e^{-st}dt$. Now, I am going to apply integration by parts one more time, for Laplace transforms of the Cos function so 0 to infinity $Cos(t)e^{-st}dt$. Now let us evaluate the Laplace transform of $Cos(t)$, then we can again imply integration by parts. So, you will get $e^{-st}/-s$ 0 to ∞. So, you get a minus sign outside one over s integral 0 to infinity $Sin(t)e^{-st}dt$. So, in the end we get $1/s$ minus $1/s \int_0^\infty Sin(t)e^{-st}dt$. (Refer Slide Time: 24:32)



Now, I am going to plug this in the equation that we found before. So, $F(s) = \int_0^\infty Sin(t)e^{-st}$. This is equal to $1/s \int_0^\infty Sin(t)e^{-st}dt$. Now we can plug the value of this Laplace transform of $Cos(t)$, which is one over s minus 1 over s and the Laplace transform of $Sin(t)$ which is again $F(s)$.

So, now we have this equation for $F(s)$ on both sides, so we can solve for $F(s)$ so you will get $F(s)$ 1 plus 1 over s^2 equals 1 over s^2 . So, you get $F(s) = 1/s^2 + 1$. So, this is the Laplace transform of the Sin function. But now that we have compute it for the Sin function we know how the Laplace transform of Sin and Cos are related it is just 1 over s times the Laplace transform of the Cos function. Therefore, the Laplace transform of the Cos function is just s time the Laplace transform of the Sin function $sF(s)$ which is $s/s^2 + 1$.

So, we have compute it the Laplace transforms of some more functions and we have also looked at the existence criteria for the Laplace transform. In the next lectures we will see some more property is of Laplace transforms more theoretical properties which will help us to compute the Laplace transforms of even more variety of functions.