MULTIPLE INTEGRALS

Triple Integrals in Spherical Coordinates

In this section, we will learn how to:

Convert rectangular coordinates to spherical ones and use this to evaluate triple integrals.

SPHERICAL COORDINATE SYSTEM

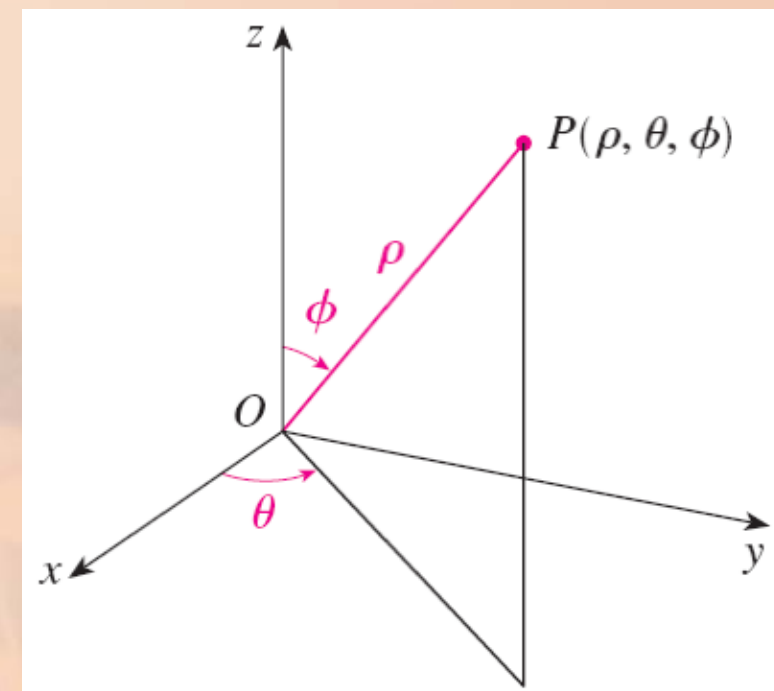
Another useful coordinate system in three dimensions is the spherical coordinate system.

- It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

SPHERICAL COORDINATES

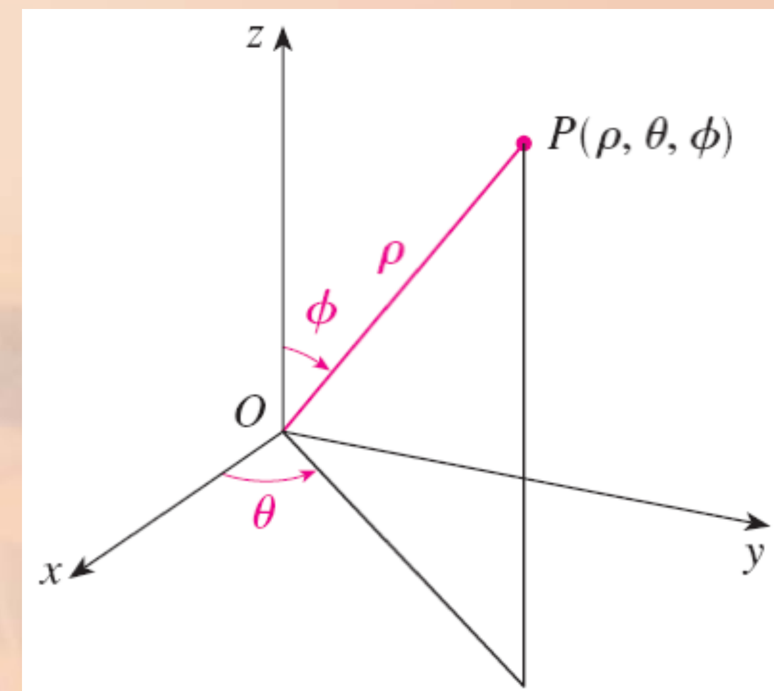
The spherical coordinates (ρ, θ, Φ) of a point P in space are shown.

- $\rho = |OP|$ is the distance from the origin to P.
- θ is the same angle as in cylindrical coordinates.
- Φ is the angle between the positive z-axis and the line segment OP.



Note that:

- $\rho \geq 0$
- $0 \leq \phi \leq \pi$



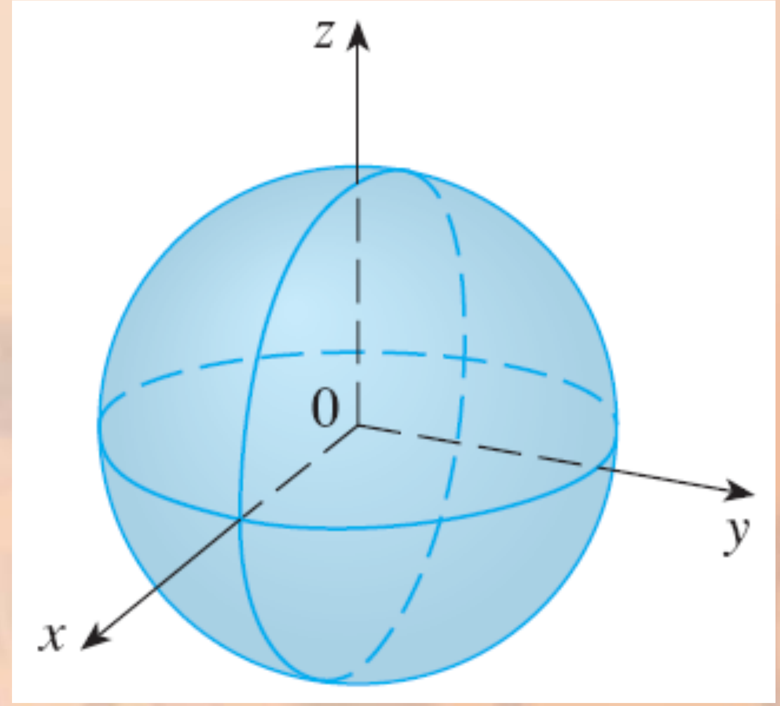
SPHERICAL COORDINATE SYSTEM

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

SPHERE

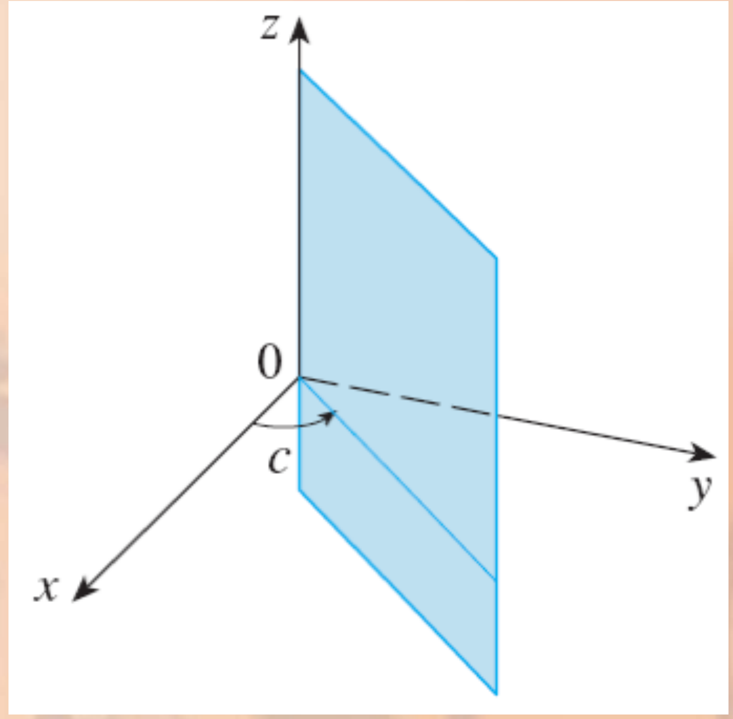
For example, the sphere with center the origin and radius *c* has the simple equation $\rho = c$.

- This is the reason for the name "spherical" coordinates.



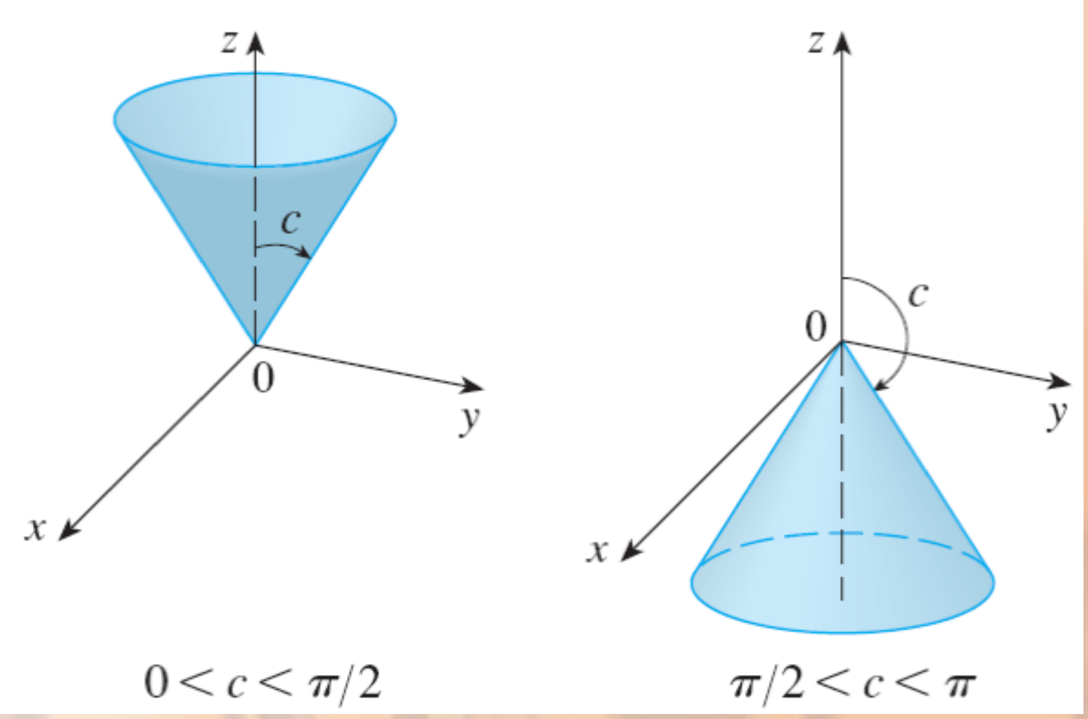
HALF-PLANE

The graph of the equation $\theta = c$ is a vertical half-plane.



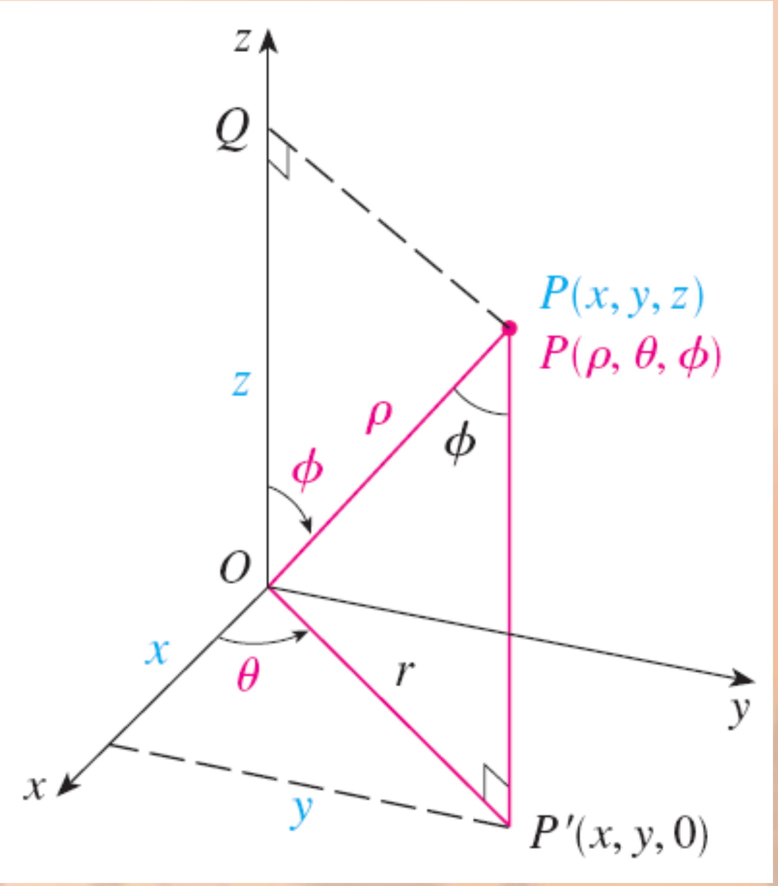
HALF-CONE

The equation $\Phi = c$ represents a half-cone with the z-axis as its axis.



SPHERICAL & RECTANGULAR COORDINATES

The relationship between rectangular and spherical coordinates can be seen from this figure.



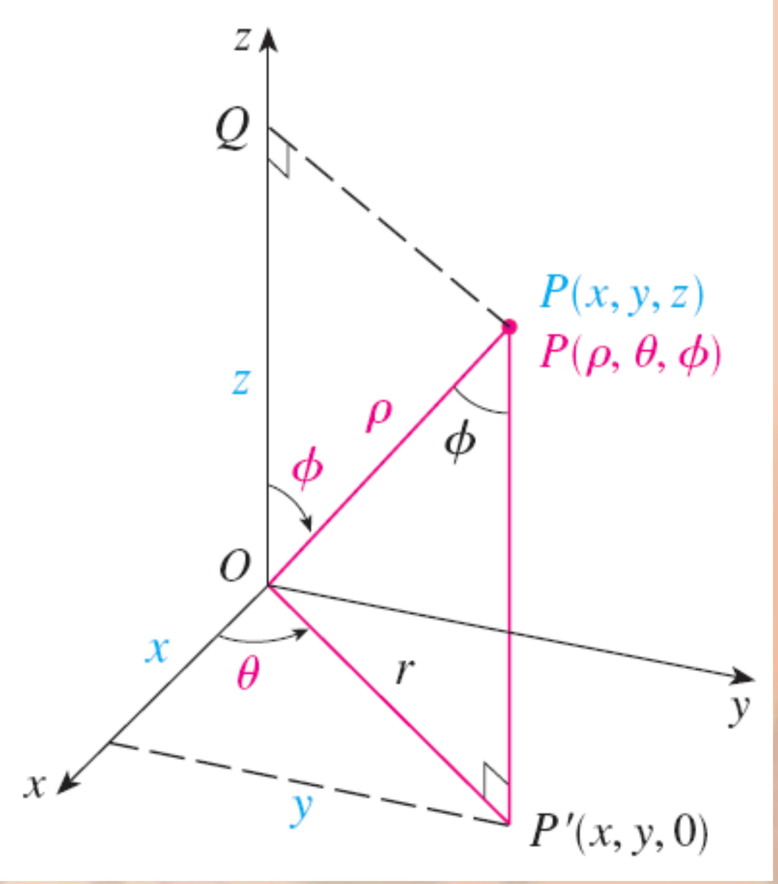
SPHERICAL & RECTANGULAR COORDINATES

From triangles *OPQ* and *OPP'*,

we have:

$$z = \rho \cos \Phi$$

$$r = \rho \sin \Phi$$



- However,
  $x = r \cos \theta$
  $y = r \sin \theta$

So, to convert from spherical to rectangular coordinates, we use the equations

$$x = \rho \sin \Phi \cos \theta$$

$$y = \rho \sin \Phi \sin \theta$$

$$z = \rho \cos \Phi$$

Also, the distance formula shows that:

$$\rho^2 = x^2 + y^2 + z^2$$

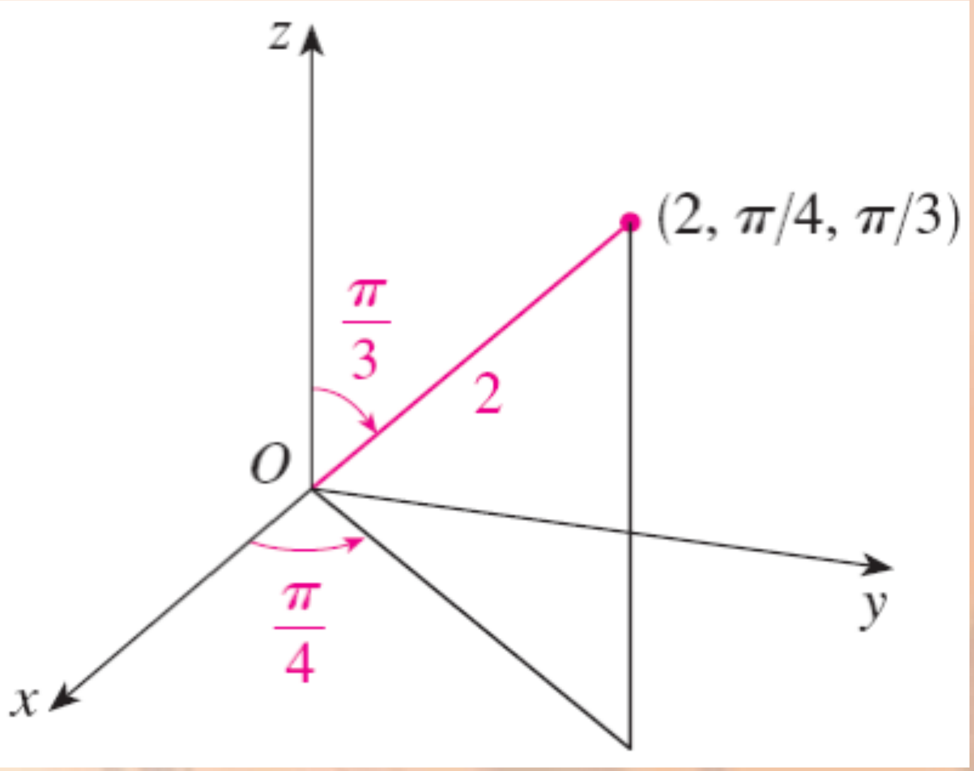
- We use this equation in converting from rectangular to spherical coordinates.

Example 1

The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates.

Plot the point and find its rectangular coordinates.

We plot the point as shown.



From Equations 1, we have:

$$x = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$x = \rho \sin\phi \sin\theta = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi = 2\cos\frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus, the point (2, π/4, π/3) is

$$\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$$

in rectangular coordinates.

The point $\left(0, 2\sqrt{3}, -2\right)$ is given in rectangular coordinates.

Find spherical coordinates for the point.

From Equation 2, we have:

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{0+12+4} \\ &= 4\end{aligned}$$

So, Equations 1 give:

$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = 0 \qquad \theta = \frac{\pi}{2}$$

- Note that $\theta \neq 3\pi/2$ because $y = 2\sqrt{3} > 0$

Therefore, spherical coordinates of the given point are:

$$(4, \pi/2 , 2\pi/3)$$

EVALUATING TRIPLE INTEGRALS WITH SPH. COORDS.

In the spherical coordinate system,

the counterpart of a rectangular box is

a spherical wedge

$$E = \{(\rho, \theta, \phi) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

where:

$a \geq 0$, $\beta - \alpha \leq 2\pi$, $d - c \leq \pi$

EVALUATING TRIPLE INTEGRALS

Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

EVALUATING TRIPLE INTEGRALS

Thus, we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\Phi = \Phi_k$.

However, that sum is a Riemann sum for the function

$$F(\rho, \theta, \phi)$$

$$= f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi$$

- So, we have arrived at the following formula for triple integration in spherical coordinates.

TRIPLE INTGN. IN SPH. COORDS. **Formula 3**

$$\iiint_E f(x, y, z)\,dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where *E* is a spherical wedge given by:

$$E = \left\{(\rho, \theta, \phi) \,\middle|\, a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\right\}$$

Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing:

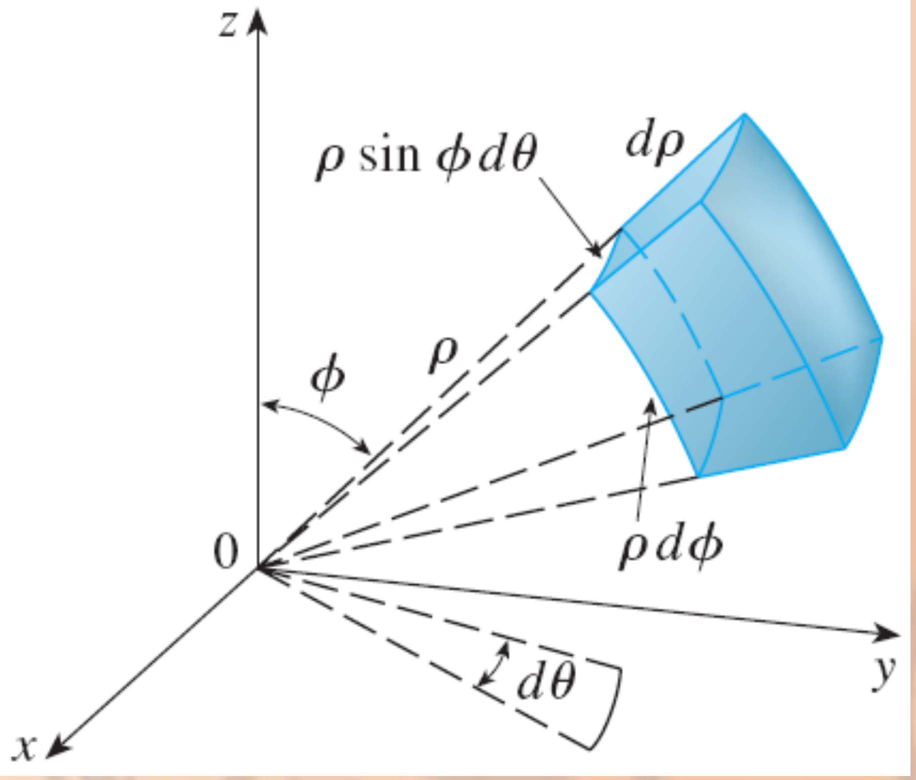
$$x = \rho \sin \Phi \cos \theta$$

$$y = \rho \sin \Phi \sin \theta$$

$$z = \rho \cos \Phi$$

That is done by:

- Using the appropriate limits of integration.
- Replacing *dV* by $\rho^2 \sin \Phi \, d\rho \, d\theta \, d\Phi$.



TRIPLE INTGN. IN SPH. COORDS.

The formula can be extended to include more general spherical regions such as:

$$E =$$

$$\left\{ (\rho, \theta, \phi) \,\middle|\, \alpha \le \theta \le \beta, c \le \phi \le d, g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi) \right\}$$

- The formula is the same as in Formula 3 except that the limits of integration for ρ are $g_1(\theta, \Phi)$ and $g_2(\theta, \Phi)$.

TRIPLE INTGN. IN SPH. COORDS.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Evaluate

$$\iiint_B e^{\left(x^2+y^2+z^2\right)^{3/2}}\,dV$$

where *B* is the unit ball:

$$B = \left\{(x, y, z)\,\middle|\,x^2 + y^2 + z^2 \le 1\right\}$$

As the boundary of *B* is a sphere, we use spherical coordinates:

$$B = \left\{ (\rho, \theta, \phi) \middle| 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi \right\}$$

- In addition, spherical coordinates are appropriate because:

$$x^2 + y^2 + z^2 = \rho^2$$

So, Formula 3 gives:

$$\iiint_B e^{\left(x^2+y^2+z^2\right)^{3/2}}\, dV$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{\left(\rho^2\right)^{3/2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^{1} \rho^2 e^{\rho^3} d\rho$$

$$= \Big[-\cos\phi\Big]_0^{\pi}\left(2\pi\right)\left[\tfrac{1}{3}e^{\rho^3}\right]_0^1 = \tfrac{4}{3}\pi\left(e-1\right)$$

It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates.

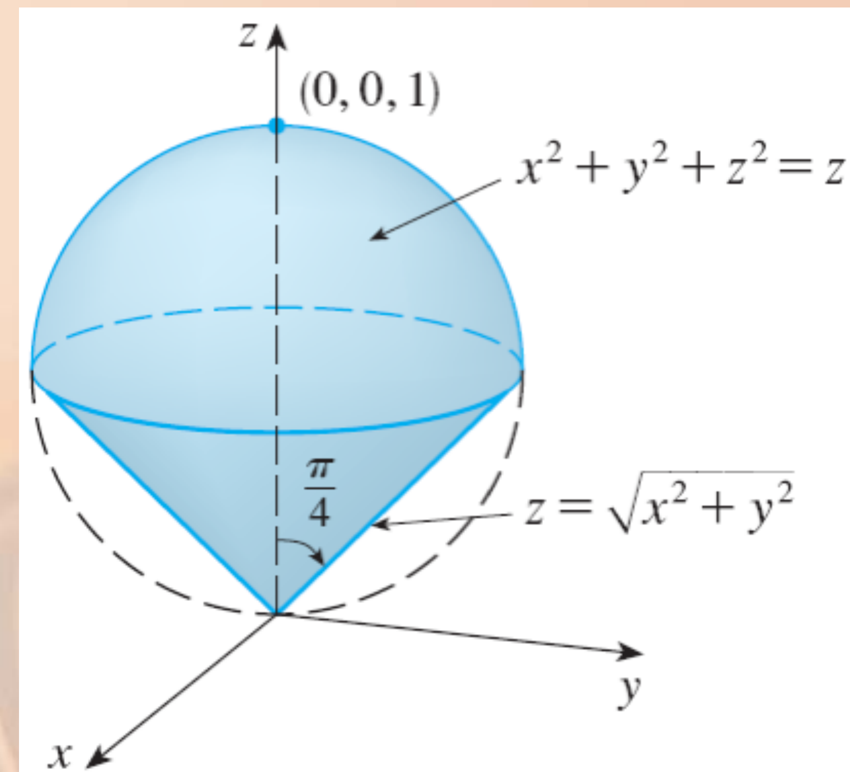
- In rectangular coordinates, the iterated integral would have been:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{\left(x^2+y^2+z^2\right)^{3/2}}\,dz\,dy\,dx$$

Use spherical coordinates to find the volume of the solid that lies:

- Above the cone

  $$z = \sqrt{x^2 + y^2}$$
- Below the sphere $x^2 + y^2 + z^2 = z$



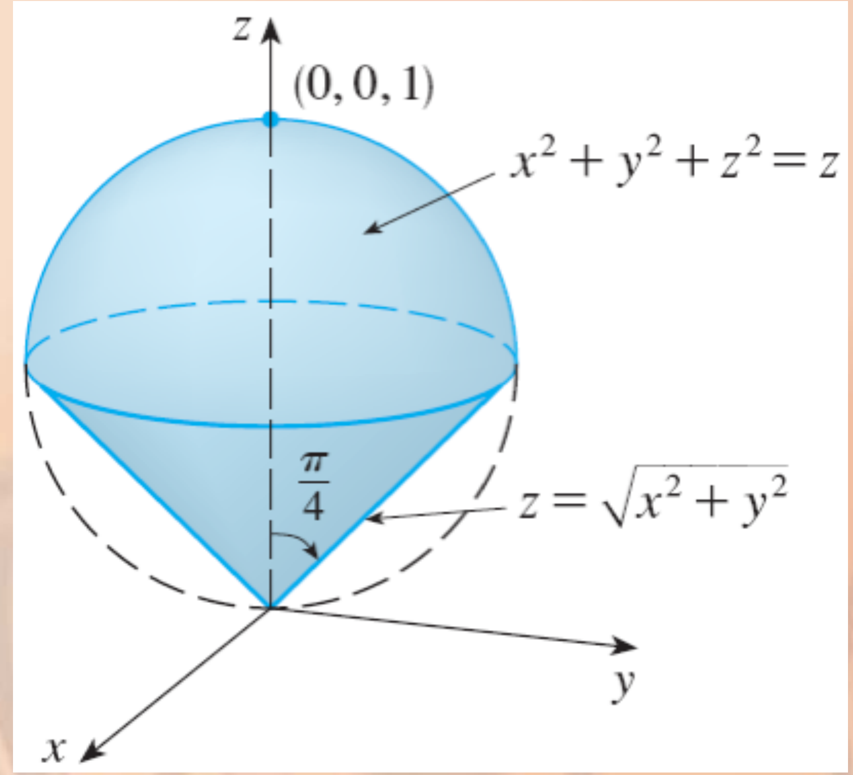
Notice that the sphere passes through the origin and has center (0, 0, ½).

- We write its equation in spherical coordinates as:

$$\rho^2 = \rho \cos \Phi$$

or

$$\rho = \cos \Phi$$



 Example 4

The equation of the cone can be written as:

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$$

$$= \rho\sin\phi$$

- This gives:

$$\sin\Phi = \cos\Phi$$

or

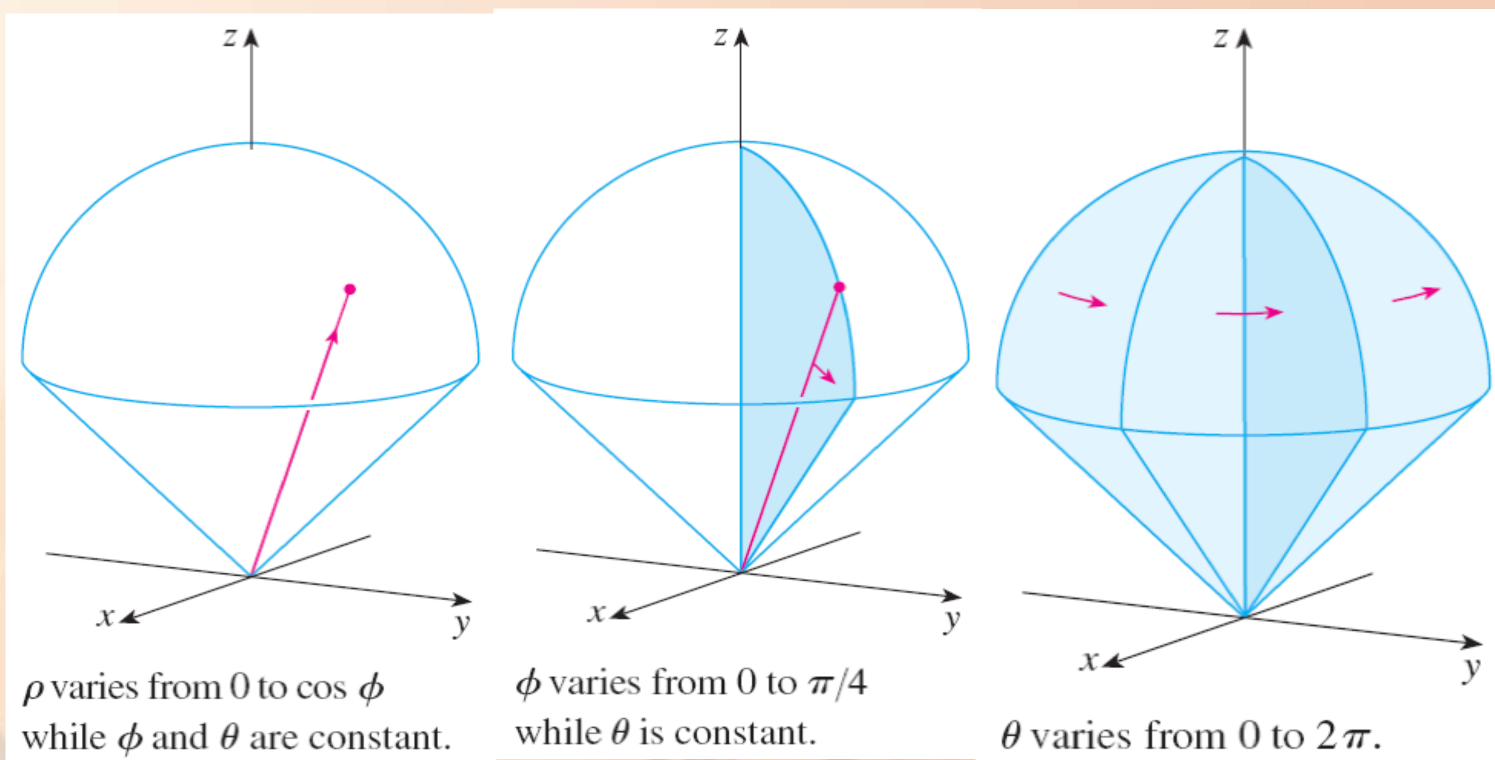
$$\Phi = \pi/4$$

Thus, the description of the solid *E* in spherical coordinates is:

$$E = \{(\rho, \theta, \phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/4, 0 \le \rho \le \cos\phi\}$$

 Example 4

The figure shows how E is swept out if we integrate first with respect to ρ, then Φ, and then θ.



The volume of *E* is:

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi$$

$$= \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$

This figure gives another look (this time drawn by Maple) at the solid of Example 4.