Find the value of $\frac{\partial x}{\partial z}$ at $(1,-1,-3)$ if $xz+y\ln x-x^2+4=0$ defines 'x' as function of two independent variables y and z and the partial derivatives exist.

Sol^n: Since 'x' is function of 'y' and 'z' and partial derivatives.

$$xz-y\ln x-x^2+4=0.$$

Diff. wrt. z,

$$\Rightarrow \frac{\partial(xz)}{\partial z}-\frac{\partial y\ln x}{\partial z}-\frac{\partial(x^2+4)}{\partial z}=0$$

$$\Rightarrow x+z\frac{\partial x}{\partial z}-\frac{y}{x}\frac{\partial x}{\partial z}-2x.\frac{\partial x}{\partial z}=0.$$

$$\Rightarrow \frac{\partial x}{\partial z}=\frac{-x}{z-\frac{y}{x}-2x}$$

$$=\frac{-x^2}{xz-y-2x^2}$$

At $(x,y,z)=(1,-1,3)$,

$$\frac{\partial x}{\partial z}=\frac{-1}{1\times3-(-1)-2\times1^2}=\frac{-1}{3+1+2}=-1/6$$

$$\boxed{\therefore \frac{\partial x}{\partial z}=-\frac{1}{6}}$$

Suppose that we substitute polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ in a differentiable fxⁿ $f(x,y) = \omega$.

1). Show that $\frac{\partial w}{\partial r} = f_x\cos\theta + f_y\sin\theta$.

2). $\frac{1}{r}\frac{dw}{d\theta} = -f_x\sin\theta + f_y\cos\theta$.

Solⁿ

$$f(x,y) = \omega \quad \text{———} \quad (1)$$

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Differentiating (1) wrt. r,

$$\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial \omega}{\partial r}$$

$$\Rightarrow f_x\cdot\frac{\partial(r\cos\theta)}{\partial r} + f_y\frac{\partial r\sin\theta}{\partial r} = \frac{\partial\omega}{\partial r}$$

$$\Rightarrow \boxed{f_x\cos\theta + f_y\cdot\sin\theta = \frac{\partial\omega}{\partial r}}$$ proved.

Now, Differentiating eqⁿ (1) wrt. θ,

$$\Rightarrow \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial\theta} = \frac{\partial\omega}{\partial\theta}$$

$$\Rightarrow f_x\cdot\frac{\partial r\cos\theta}{\partial\theta} + f_y\cdot\frac{\partial r\sin\theta}{\partial\theta} = \frac{\partial\omega}{\partial\theta}$$

$$\Rightarrow f_x\cdot(-r\cdot\sin\theta) + f_y(r\cdot\cos\theta) = \frac{\partial\omega}{\partial\theta}$$

$$\Rightarrow \boxed{-f_x\cdot\sin\theta + f_y\cdot\cos\theta = \frac{1}{r}\frac{\partial\omega}{\partial\theta}}$$ proved.

Find the derivative of function at P_0 in direction of u.

1. $f(x,y) = 2xy - 3y^2$, $P_0(5,5)$, $\vec{u} = 4i+3j$

sol^n Given that :- $\vec{u} = 4\hat{i} + 3\hat{j}$, $|\vec{u}| = \sqrt{u^2} = \sqrt{4^2+3^2} = 5$

Then, Unit vector in direction of u is:-

$$\hat{u} = \frac{4}{5}i + \frac{3}{5}j$$

For fn, $f(x,y) = 2xy - 3y^2$, gradient, $\nabla f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.

$$= \langle 2y, 2x - 6y \rangle.$$

At $P_0(5,5)$,

$$\nabla f(x,y) = \langle 2\times5, 2\times5 - 6\times5 \rangle$$
$$= \langle 10, -20 \rangle.$$

Directional derivative in direction of $\vec{u}$ at point $P_0(5,5)$ is:-

$$D_u \cdot f(5,5) = \nabla \cdot f(5,5) \cdot \hat{u}$$

$$= \langle 10, -20 \rangle \cdot \frac{1}{5} \cdot \langle 4,3 \rangle.$$

$$= \frac{1}{5} \langle 40 - 60 \rangle$$

$$= -4$$

$\therefore$ The directional derivative of $f(x,y)$ at P_0 is 4.

4). Find all the local maxima, Local minima and saddle points of the function.

1). $f(x,y) = 2x^2+3xy+4y^2-5x+2y$.

↳ Sol:-

First we need to find the critical points of given function.

To find critical points, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0$.

i.e. $\nabla f(x,y) = 0$

$\Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial y}{\partial y}\right) = (0,0)$.

$\Rightarrow (4x+3y-5,\ 3x+8y+2) = (0,0)$.

Comparing corresponding values, $4x+3y-5=0$

$\Rightarrow 4x+3y=5$ —— ①

$3x+8y+2=0$

$\Rightarrow 3x+8y=-2$ —— ②

Performing $3①-4②$, we get,

$-23y = 23$

$\Rightarrow y = -1$

From ①, $4x+3y=5$

$\Rightarrow x = \frac{5+3}{4} = 2$ $\therefore (x,y) = (2,-1)$.

So, there's only one maximum critical point at $(2,-1)$.

Then, $f_x = 4x+3y-5$ $f_{xy} = 3$ $f_{xx} = 4$

$f_y = 3x+8y+2$ $f_{yy} = 8$ $f_{yx} = 3$

Then, the determinant of Hessian Matrix, $\begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$ is:-

$D = f_{xx} \cdot f_{yy} - f_{xy}^2 = 4\times 8 - 3^2$

$\therefore D = 23$

At $(2,-1)$,

$f_{xx}(2,-1) = 4 > 0$.   Since, f_{xy} and $D > 0$ at $(2,-1)$,

$D(2,-1) = 23 > 0$. $f(x,y)$ is minimum at $(2,-1)$.

Min. Value is:-

$f(2,-2) = 2\times 2^2 + 3\times 2\times(-1) + 4\times(-1)^2 - 5\times 2 + 2\times(-1)$

$= 8-6+4-10-2$

$= -6$

$\therefore$ Min value is -6 at $(2,-1)$.

Use the method of langrange's multiplier to find.

1. Minimum on hyperbola: The minimum value of $x+y$ subject to constraints $xy=16$, $x>0$, $y>0$.
2. Maximum on a line: The maximum value of $x \cdot y$, subject to $x+y=16$. Comment on geometry of each solution.

↳ sol^n →

$f(x,y) = x+y$ to be minimized subject to $xy=16$, $x>0$, $y>0$.

Suppose, $t(x,y) = xy - 16 = 0$.

By langrange's multiplier's method:

$\nabla f(x,y) = \lambda \nabla t(x,y)$

$\Rightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \lambda \cdot \left(\frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}\right)$

$\Rightarrow (1,1) = \lambda (y,x)$

Here, we get, $x = \frac{1}{\lambda}$, $y = \frac{1}{\lambda}$.

Substituting x & y from above in boundary function,

$xy - 16 = 0$

$\Rightarrow \frac{1}{\lambda} \cdot \frac{1}{\lambda} = 16$

$\Rightarrow \lambda = \pm \frac{1}{4}$ (-ve is discarded because, $x,y>0$).

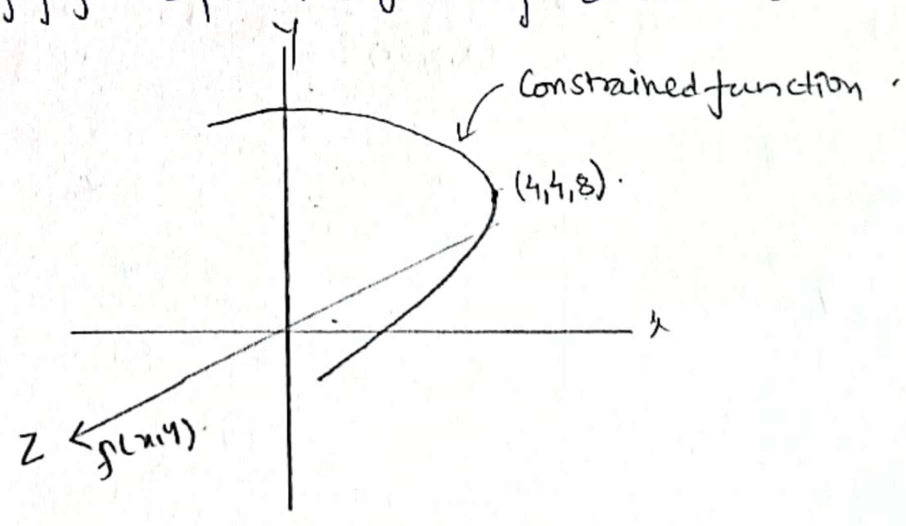
Hence, taking $\lambda = \frac{1}{4}$; $x = \frac{1}{\lambda} = 4$

$y = \frac{1}{\lambda} = 4$. $\quad \therefore (x,y) = (4,4)$.

At $(4,4)$, $f(4,4) = x+y = 4+4 = 8$

$\therefore$ Min value is 8 at $(4,4)$.

Following figure depicts the geometry at minima.



Scanned with CamScanner

2). Solⁿ:-

Given function:-
$f(x,y) = xy$ ——— ①

constraints $x+y=16$

Suppose $t(x,y) = x+y-16$

From Langrange's Multiplier,

$\nabla f(x,y) = \lambda \cdot \Delta t(x,y)$.

$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \lambda\left(\frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}\right)$

$= (y,x) = \lambda(1,1) = (\lambda,\lambda)$.

comparing the corresponding values, we get,

$y=\lambda, \quad x=\lambda$

Substituting 'x' and 'y' in constraint equation.

$\Rightarrow \cancel{t(x,y)} = x+y=16$

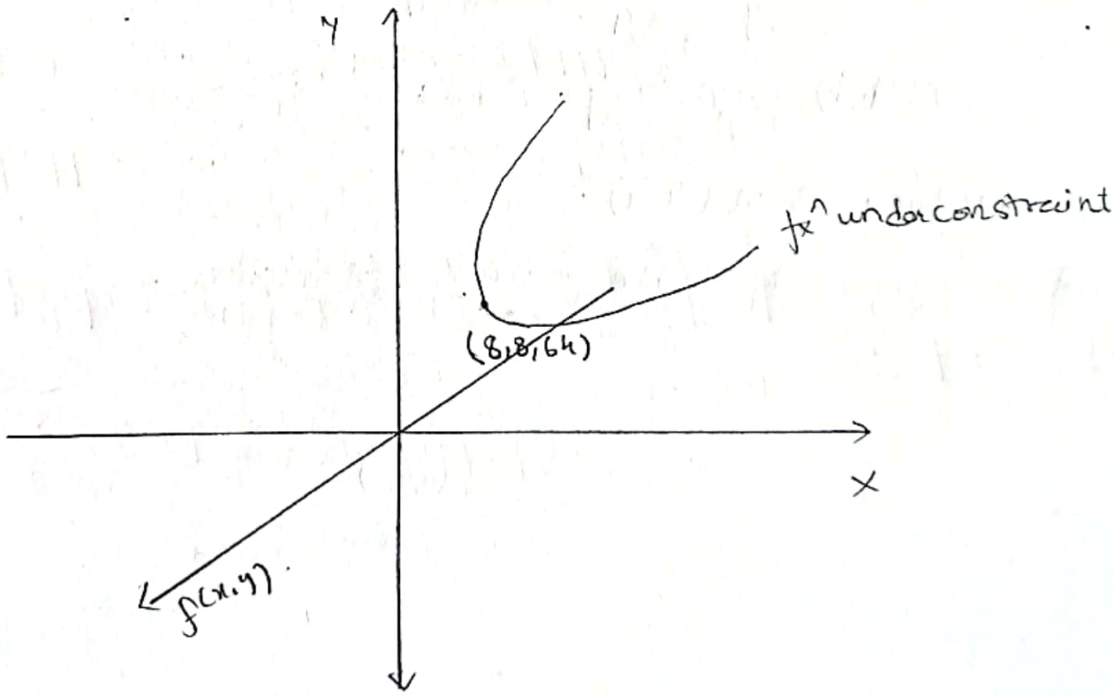
$\Rightarrow 2\lambda = 16 \Rightarrow \lambda = 8$

$\therefore x=8$ & $y=8$

Hence, the value of $f(x,y) = xy$ is max at $(8,8)$.

$\therefore f(x,y) = xy$
$= 8 \cdot 8$
$= 64$ (max value).

Following figure depicts it's geometry at $(8,8)$.



6) Use Taylor's formula for $f(x,y) = \ln(2x+y+1)$ at origin to find quadratic and cubic approximations of 'f' near origin.

Sol^n.

$f(x,y)$ to be expanded at $(0,0)$.

At $(0,0)$, $f(x,y) = \ln(1) = 0$.

Now, the partial derivatives are:-

$f_x = \dfrac{2}{(2x+y+1)}$. At $(0,0)$, $f_x = 2$

$f_y = \dfrac{1}{(2x+y+1)}$. At $(0,0)$. $f_y = 1$

$f_{xy} = \dfrac{-2 \cdot 1}{(2x+y+1)^2}$. At $(0,0)$, $f_{xy} = \dfrac{-2}{1} = -2$.

$f_{xx} = \dfrac{-2 \cdot 2}{(2x+y+1)^2}$. At $(0,0)$, $f_{xx} = -4$

$f_{yy} = \dfrac{-1}{(2x+y+1)^2}$, At $(0,0)$. $f_{yy} = -1$.

$f_{xxx} = \dfrac{-4(-2)}{(2x+y+1)^3}$, At $(0,0)$. $f_{xxx} = 8$

$f_{yyy} = \dfrac{-1\cdot(-2)}{(2x+y+1)^3}$, At $(0,0)$, $f_{yyy} = 2$

$f_{xyy} = \dfrac{-2\cdot(-2)}{(2x+y+1)^3}$, At $(0,0)$. $f_{xyy} = 4$

$f_{xxy} = \dfrac{-4\cdot 2\cdot(-2)}{(2x+y+1)^3}$. At $(0,0)$ $f_{xxy} = 16$.

Quadratic approximation at $(0,0)$ is:-

$$f(x,y) \approx f(0,0) + \frac{1}{1!}\left[f_x(0,0)(x-0) + f_y(0,0)(y-0)\right] + \frac{1}{2!}\left[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right].$$

$$\Rightarrow f(x,y) \approx 0 + 2x + y - \frac{1}{2}(4x^2 + 4xy + y^2).$$

$$= 2x + y - 2x^2 - 2xy - \frac{y^2}{2}.$$

which is the quadratic approximation of $f(x,y) = \ln(2x+y+1)$.

For cubical approximation,

$$f(x,y) \approx f(0,0) + \left(f_x(0,0)\cdot(x-0) + f_y(0,0)(y-0)\right) + \frac{1}{2!}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) + \frac{1}{3!}\left(f_{xxx}(0,0)x^3 + f_{yyy}(0,0)y^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}xy^2\right).$$

$$\Rightarrow f(x,y) = \text{second-order approximation} + \frac{1}{3!}\left(f_{xxx}x^3 + f_{yyy}y^3 + 3f_{xxy}x^2y + 3f_{xyy}xy^2\right).$$

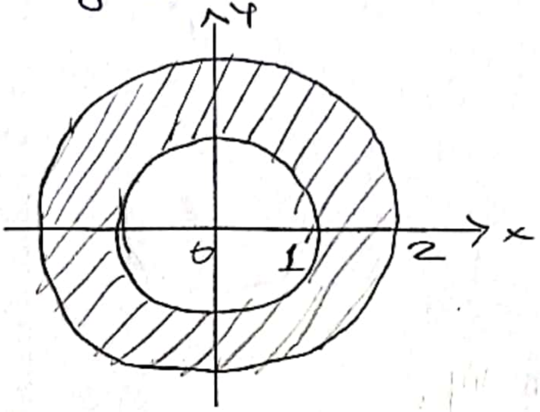
$$= 2x + y - 2x^2 - 2xy + \frac{y^2}{2} + \frac{1}{6}\left(16x^3 + 2y^3 + 24x^2y + 12xy^2\right).$$

$$= 2x + y - 2xy - \frac{y^2}{2} + \frac{16}{6}x^3 + \frac{y^3}{8} + 4x^2y + 2xy^2$$

which is the cubic approximation of $f(x,y) = \ln(2x+y+1)$ at origin.

Find the area of circular washer of outer and inner radii 2 and 1 units respectively, using (a) Fubini's Theorem (b) Simple geometry.

(a)



For outer radii, eq^n of circle is:- $x^2+y^2=4$ ——— (i)

For inner radii, eq^n of circle is:- $x^2+y^2=1$ ——— (ii)

Using the polar coordinates in above equations, substitute $x=r\cos\theta$, $y=r\sin\theta$

Then, In eq^n (i),

$x^2+y^2=4$

$\Rightarrow r^2\cos^2\theta+r^2\sin^2\theta=4$

$\Rightarrow r^2=4$

$\Rightarrow r=2$

In eq^n (ii),

$x^2+y^2=1$

$\Rightarrow r^2\cos^2\theta+r^2\sin^2\theta=1$

$\Rightarrow r^2=1$

$\Rightarrow r=1$

From Fubini's Theorem,

$$A=\iint_R f(x,y)\,d = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\cdot r\,dr\cdot d\theta$$

Taking $\theta_1=0°$, $\theta_2=2\pi$, $r_1=1$, $r_2=2$,

$$\therefore A = \int_{\theta_1}^{\theta_2}\int_{1}^{2} r\,dr.\,d\theta$$

$$= \int_{\theta_1}^{\theta_2}\left[\frac{r^2}{2}\right]_1^2 d\theta$$

$$= \int_{\theta_1}^{\theta_2}\frac{3}{2}\,d\theta$$

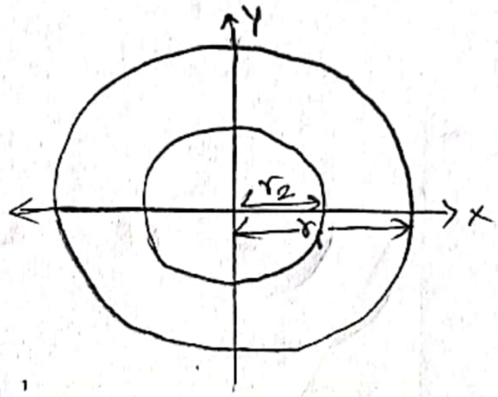
$$= \frac{3}{2}\left[\theta\right]_0^{2\pi}$$

$= 3\pi$ square units

is the area by fubini's theorem.

Scanned with CamScanner

(5).



By simple geometry,

Area of bigger circle $(A_1) = \pi r_1^2$

Area of smaller circle $(A_2) = \pi r_2^2$

Area of washer $(\Delta A) = A_1 - A_2$

$= \pi r_1^2 - \pi r_2^2$

Since $r_1 = 2$ units, $r_2 = 1$ units.

$\Delta A = \pi \cdot 2^2 - \pi \cdot 1^2$

$= 3\pi$ square units.

which is the area by simple geometry.

Hence, the area calculated by both methods is same and hence both are valid methods.

8). Calculate the area enclosed by leaf of rose given by $\boxed{r = 12\cos 3\theta}$

Sol.

$r = 12\cos 3\theta$ is eqⁿ of rose.

Then, for the area of rose, we convert it to polar form.

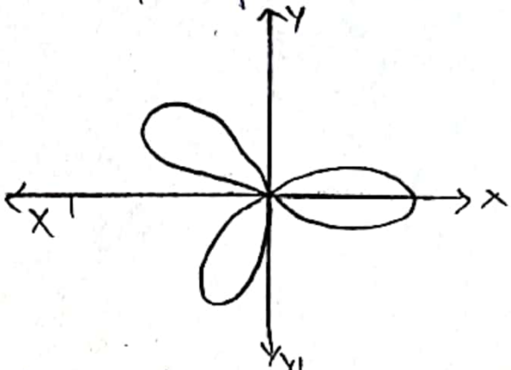
Let, $r=0$, for upper limit of θ.

Then, $r = 12\cos 3\theta = 0$

$\Rightarrow \cos 3\theta = 0$

$\Rightarrow \theta = \left(\frac{\pi}{2}\right)\cdot\frac{1}{3}$

$= \frac{\pi}{6}$



So, θ goes from 0 to $\pi/6$, and r goes from 0 to $12\cos 3\theta$.

Symmetrically, there are 3 leaves of the rose given by equation as shown above.

The area of one leaf is:

$$A = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/6} \left[\frac{r^2}{2}\right]_0^{12\cos 3\theta} d\theta$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^{\pi/6} 144\cdot\cos^2 3\theta\, d\theta$$

$$= 144\int_0^{\pi/6} \cos^2 3\theta\, d\theta$$

$$= 144\int_0^{\pi/6} \frac{(1+\cos 6\theta)}{2} d\theta$$

$$= 72\int_0^{\pi/6} (1+\cos 6\theta)\, d\theta$$

$$= 72\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/6}$$

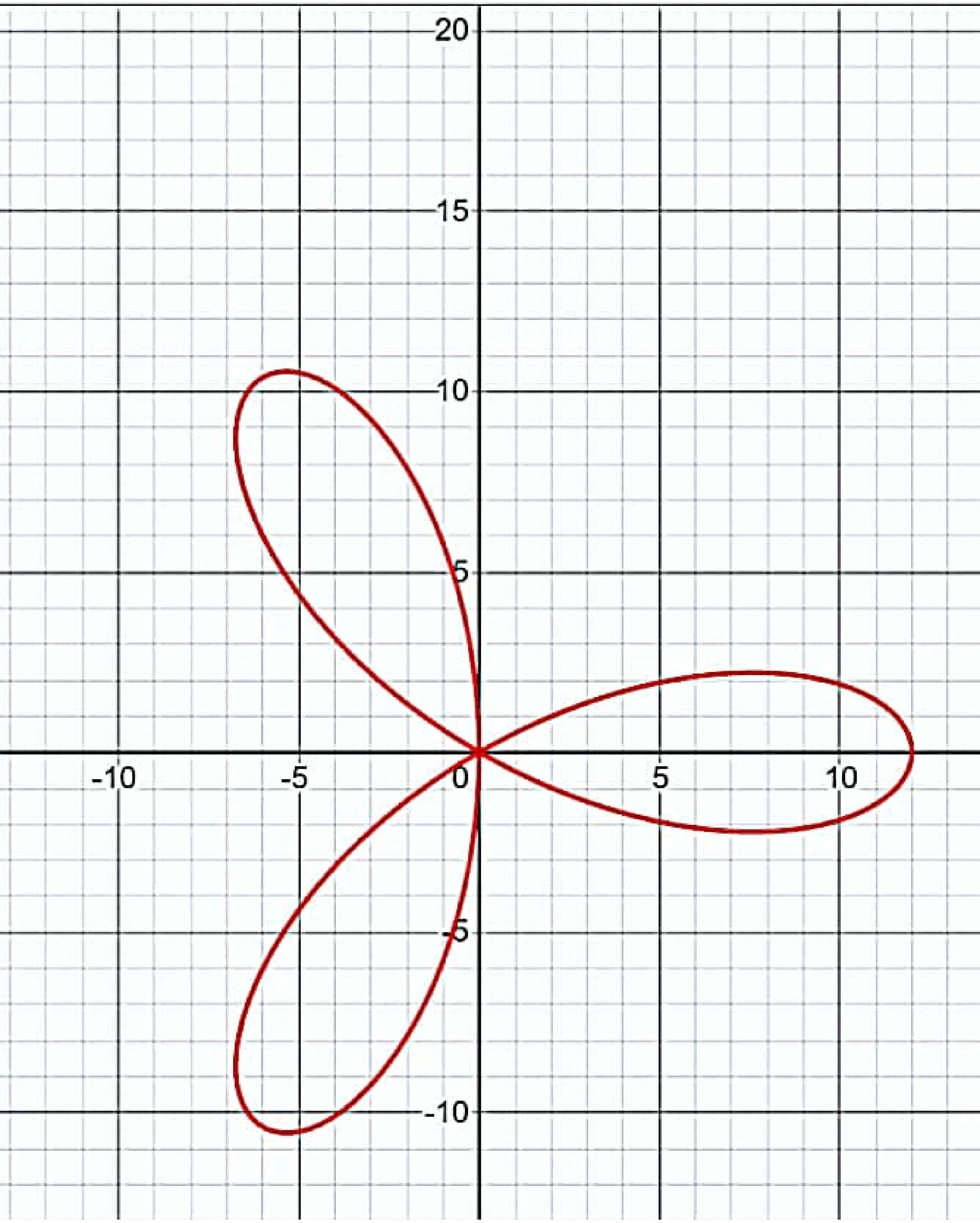
$$= 72\cdot\left(\frac{\pi}{6} + \frac{\sin\pi}{6} - 0 - \frac{\sin 0}{6}\right)$$

$$= 72\cdot\left(\frac{\pi}{6} + 0\right)$$

$= 12\pi$ square units.

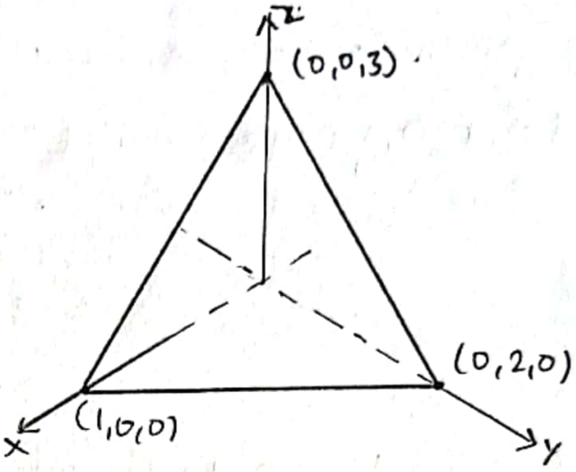
Hence, the area enclosed by one leaf is 12π square units.

By all three leaves, the area enclosed is 36π square units.


20
15
10
5
0
-5
-10
-10
-5
5
10

20BDS0405

Q) Find the volume of the region enclosed by tetrahedron in first octant bounded by plane passing through (1,0,0), (0,2,0) and (0,0,3) and coordinate planes.



In tetrahedron as shown above, the intercepts on the axes are 1,2 and 3 respectively. Hence, the eq^n of plane is

$$\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1 \quad \text{(In ~~....~~ intercept form).}$$

$$\Rightarrow 6x+3y+2z-6=0$$

$$\Rightarrow z=\frac{6-3y-6x}{2}$$

In first octant, ~~minimum~~ max limit of z is $\frac{6-3y-6x}{2}$ and min limit is 0.

Then considering xy plane, take z=0, the above eq^n transforms to:-

$$x+\frac{y}{2}=1$$

$$\Rightarrow 2x+y-2=0$$

$$\Rightarrow y=2-2x$$

Again, here the limit of y is from 0 to 2-2x.

For limit of x, the question has given 0 to 1.

Hence, Volume of tetrahedron inscribed by coordinate planes is:-

$$V=\int_0^1\int_0^{2-2x}\int_0^{\frac{6-3y-6x}{2}} dz\,dy\,dx$$

$$=\int_0^1\int_0^{2-2x}[z]_0^{\frac{6-3y-6x}{2}}\,dy\,dx$$

$$=\frac{1}{2}\int_0^1\int_0^{2-2x}(6-3y-6x)\,dy\,dx$$

$$=\frac{1}{2}\int_0^1\left(6y-\frac{3}{2}y^2-6xy\right)_0^{2-2x}dx$$

$$=\frac{1}{2}\int_0^1\left(6(2-2x)-\frac{3}{2}\cdot 4(1-x)-6x(2-2x)\right)dx$$

$$=\frac{1}{2}\int_0^1\left(12-12x-6+6x-12x+12x^2\right)dx$$

$$=\frac{1}{2}\int_0^1(6x^2+6-12x)\,dx$$

$$=\frac{1}{2}\left[2x^3+6x-6x^2\right]_0^1 dx \quad = \frac{1}{2}\times(2\times1+6\times1-6\times1) = 1 \text{ cubic unit.}$$

Hence, the total enclosed volume is 1 cubic unit.