Lagrangian Multiplier

Thursday, 18 March 2021 10:01 AM

Recall: ① $f_{xx} > 0$ & $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$

② $f_{xx} < 0$ | $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$

① is minimum

② is maximum.

plot$(x, y, f(x,y))$

Saddle point

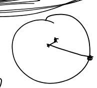
$$\boxed{f_{xx} \cdot f_{yy} - f_{xy}^2 < 0}$$

$$f_{xx} f_{yy} - f_{xy}^2 = 0$$

The test is inconclusive.

Lagrange Multiplier.

① Find closest point $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$



Closest - minimum

Farthest - maximum

Lagrange Multiplier Method - helps to find extrema values of constrained functions.

Def: Suppose $f(x,y) / f(x,y,z)$ and $g(x,y) / g(x,y,z)$ and $\nabla g \neq 0$ (partial derivative is not equal to zero) when $g(x,y) / g(x,y,z) = 0$

constraint

You may find x, y, z & λ by equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \; ; \; \frac{\partial f}{\partial y} = \lambda \cdot \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = \lambda \cdot \frac{\partial g}{\partial z}.$$

Problem:

Find greatest & smallest values of function $f(x,y) = xy$ takes on the ellipse

$$\boxed{\frac{x^2}{8} + \frac{y^2}{2} = 1} \; g(x,y).$$

Soln:

$$f(x,y) = xy \quad \& \quad g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$$

By Lagrange Multiplier.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \; ; \; \frac{\partial f}{\partial y} = \lambda \cdot \frac{\partial g}{\partial y}$$

$$\boxed{y = \lambda \cdot x/4} \; -① \qquad \boxed{x = \lambda \cdot y} \; -②$$

$$y = \lambda/4 \, (\lambda y) \quad (\text{From } 2)$$

$$y = \frac{\lambda^2 y}{4}$$

$y = 0$ (or)

$\lambda^2 \ldots = 0$

ellipse $x^2 \; y^2 \; 1$

$y = \frac{\lambda y}{4}$

$y - \frac{\lambda^2}{4} y = 0$

$a \cdot b = 0$ $\quad y\left(1 - \frac{\lambda^2}{4}\right) = 0$

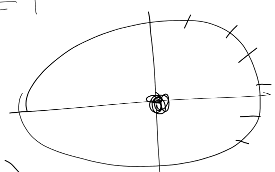
$y = 0$ (or) $1 - \frac{\lambda^2}{4} = 0$

ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$

Case(i) $y = 0$

$x = \lambda y$

$\boxed{x = 0}$ $\quad (0,0)$



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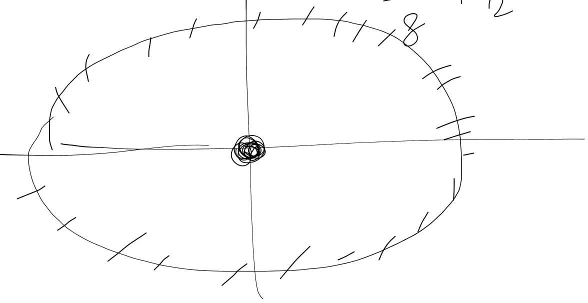

Case(ii) $1 - \frac{\lambda^2}{4} = 0$

$\frac{\lambda^2}{4} = 1$

$\lambda^2 = 4$

$\boxed{\lambda = \pm 2}$

$y \neq 0$ & $\lambda = \pm 2$ (By case(ii))

From 2 $\quad x = \lambda y \longrightarrow 2$

$\boxed{x = \pm 2y}$

$x = +2y$ & $x = -2y$

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$$

case i $\quad x = 2y$

$$\frac{(2y)^2}{8} + \frac{y^2}{2} = 1$$

$$\frac{\cancel{4}y^2}{\cancel{8}} + \frac{y^2}{2} = 1$$

$$2y^2 = 1$$

$$\frac{2y^2}{2} = 1$$

$$y^2 = 1$$

$$\boxed{y = \pm 1}$$

$$x = \lambda y$$

$$x = \pm 2y \quad (y = \pm 1)$$

$x = \pm 2(1)$ and $x = \pm 2(-1)$

$(-2,-1)$, $(-2,1)$ $(+2,-1)$ $(2,1)$

$$f(-2,-1) = xy = (-2)(-1) = +2$$

$$f(-2,1) = xy = (-2)(1) = -2$$

$$f(2,-1) = xy = (2)(-1) = -2$$

$$f(2,1) = xy = (2)(1) = 2$$

Smallest — -2

greatest — $+2$

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(3) Find max & min distance of point $(3, 4, 12)$ subject to

Sphere $x^2+y^2+z^2=4$

$g(x,y,z) = x^2+y^2+z^2-4$

$\boxed{f(x,y,z) = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}}$ $\Big\}$ -

$f(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2$ ✓

distan
$(x_1, \ldots$
$(x_2 \ldots$
$\sqrt{(x_2 \ldots}$

Lagrangian Multipleri

$\frac{\partial f}{\partial x} = \lambda \cdot \frac{\partial g}{\partial x}$ $\quad$ $\frac{\partial f}{\partial y} = \lambda \cdot \frac{\partial g}{\partial y}$, $\frac{\partial f}{\partial z} = \lambda \cdot \frac{\partial g}{\partial z}$

$2(x-3) = \lambda \cdot 2x$

$2x-6 = \lambda 2x$

$2x - \lambda 2x - 6 = 0$

$2x(1-\lambda) = 6$

$x(1-\lambda) = 3$

$2(y-4) = \lambda \cdot 2y$

$2y-8-\lambda 2y = 0$

$y(1-\lambda) = 4$

$2(z-12) = \lambda \ldots$

$z(1-\ldots$

$x = \frac{3}{1-\lambda}$ ✓ $\quad y = \frac{4}{1-\lambda}$ ✓ $\quad z = \frac{12}{1\ldots}$

Sub x, y, z in $g(x,y,z)$

$\ldots)^2 \quad (\frac{12}{\ldots})^2 \quad 1\ldots$

n ce

$y_1, z_1) \to$ Known

$\boxed{y_2, z_2)} \to$ Unknown

$-x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

(

2 2

$) = 12$

2 ✓

$$= \left(\frac{3}{(1-\lambda)}\right)^2 + \left(\frac{4}{1-\lambda}\right)^2 + \left(\frac{12}{1-\lambda}\right) - 4$$

$$\frac{9}{(1-\lambda)^2} + \frac{8}{(1-\lambda)^2} + \frac{144}{(1-\lambda)^2} = 4$$

$$\boxed{\lambda = -11/2 \quad \& \quad \lambda = +\frac{15}{2}}$$

Ex: Find the point on the plane $x + 2y + 3z = 13$ closeset to the point $(1, 1, 1)$

② Find the farthest distance from the point $(1, 1, 1)$ connected to the sphere $x^2 + y^2 + z^2 = 4$.

Refer 14.8 (Thomas calculus)