MULTIPLE INTEGRALS

Change of Variables in Multiple Integrals

In this section, we will learn about:

The change of variables in double and triple integrals.

CHANGE OF VARIABLES IN SINGLE INTEGRALS

In one-dimensional calculus. we often use a change of variable (a substitution) to simplify an integral.

By reversing the roles of *x* and *u*, we can write the Substitution Rule (Equation 6 in Section 5.5) as:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$.

Another way of writing Formula 1 is as follows:

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\frac{dx}{du}\,du$$

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

A change of variables can also be useful in double integrals.

- We have already seen one example of this: conversion to polar coordinates.

The new variables r and θ are related to the old variables x and y by:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

The change of variables formula (Formula 2 in Section 15.4) can be written as:

$$\iint_R f(x, y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

where *S* is the region in the *rθ*-plane that corresponds to the region *R* in the *xy*-plane.

More generally, we consider a change of variables that is given by a transformation T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by:

$$\boxed{x = g(u, v) \qquad y = h(u, v)}$$

- We sometimes write these as: $x = x(u, v)$, $y = y(u, v)$

C^1 TRANSFORMATION

We usually assume that T is a C^1 transformation.

- This means that g and h have continuous first-order partial derivatives.

IMAGE & ONE-TO-ONE TRANSFORMATION

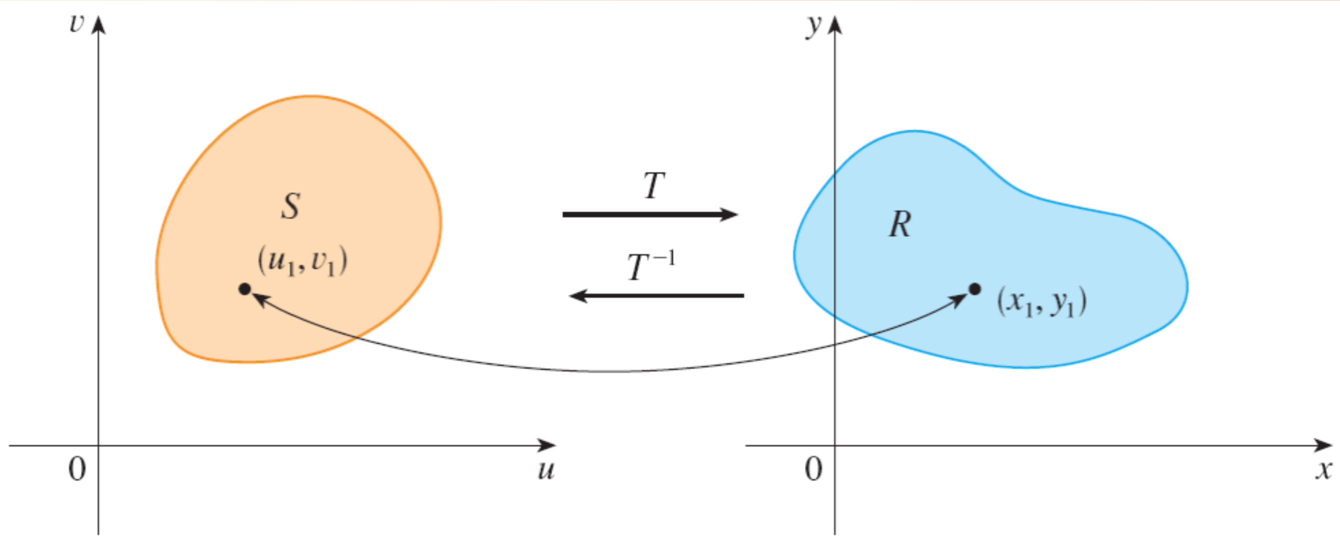
If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the image of the point (u_1, v_1).

If no two points have the same image, T is called one-to-one.

CHANGE OF VARIABLES

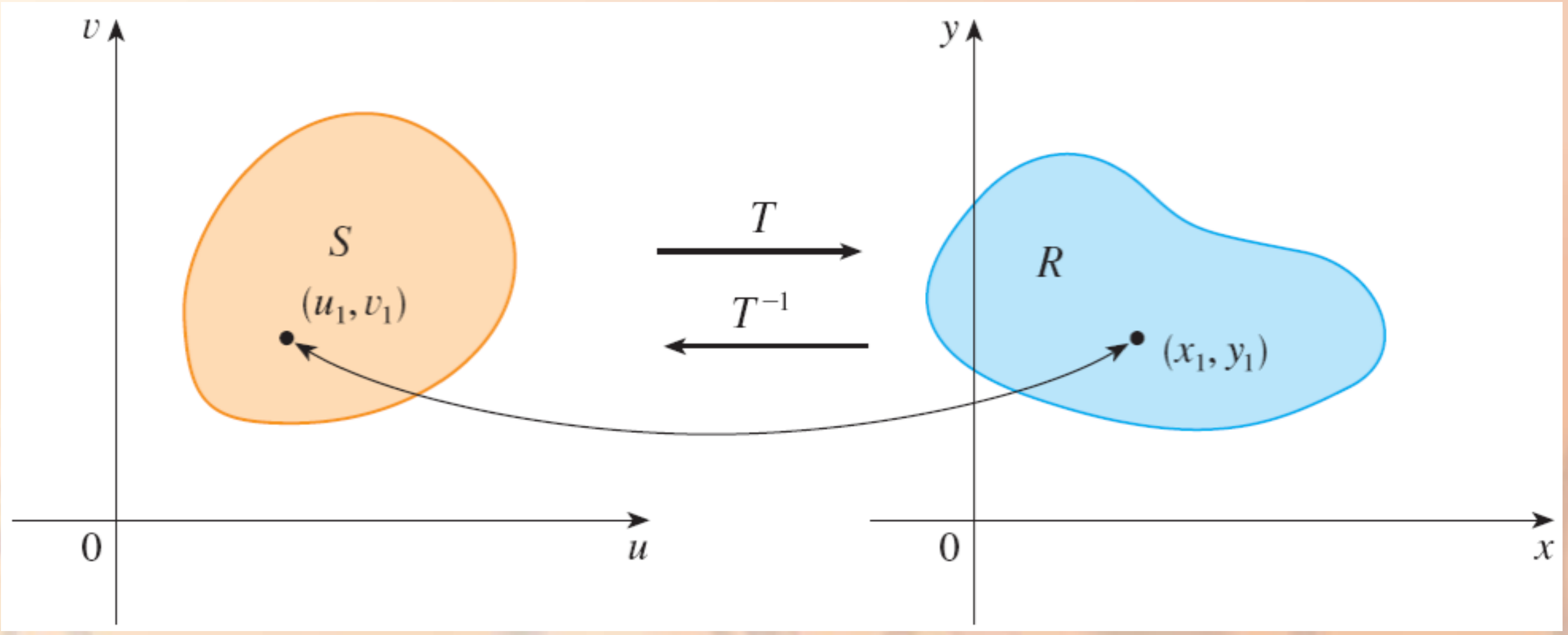
The figure shows the effect of a transformation *T* on a region *S* in the *uv*-plane.

- *T* transforms *S* into a region *R* in the *xy*-plane called the image of *S*, consisting of the images of all points in *S.*



INVERSE TRANSFORMATION

If T is a one-to-one transformation, it has an inverse transformation T^{-1} from the xy–plane to the uv-plane.



Then, it may be possible to solve Equations 3 for u and v in terms of x and y:

$$u = G(x, y)$$

$$v = H(x, y)$$

A transformation is defined by:

$$x = u^2 - v^2$$

$$y = 2uv$$

Find the image of the square

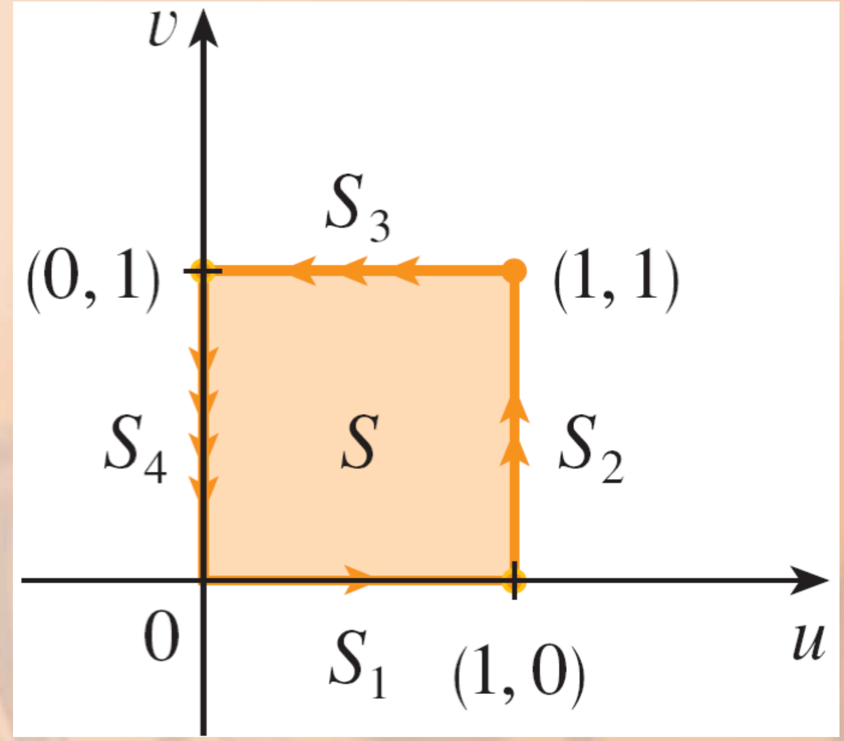
$$S = \{(u, v) \mid 0 \leq u \leq 1,\ 0 \leq v \leq 1\}$$

The transformation maps the boundary of *S* into the boundary of the image.

- So, we begin by finding the images of the sides of *S*.

The first side, S_1, is given by:

$$v = 0 \ (0 \leq u \leq 1)$$



From the given equations,

we have:

$$x = u^2,\ y = 0, \text{ and so } 0 \le x \le 1.$$

- Thus, S_1 is mapped into the line segment from (0, 0) to (1, 0) in the xy-plane.

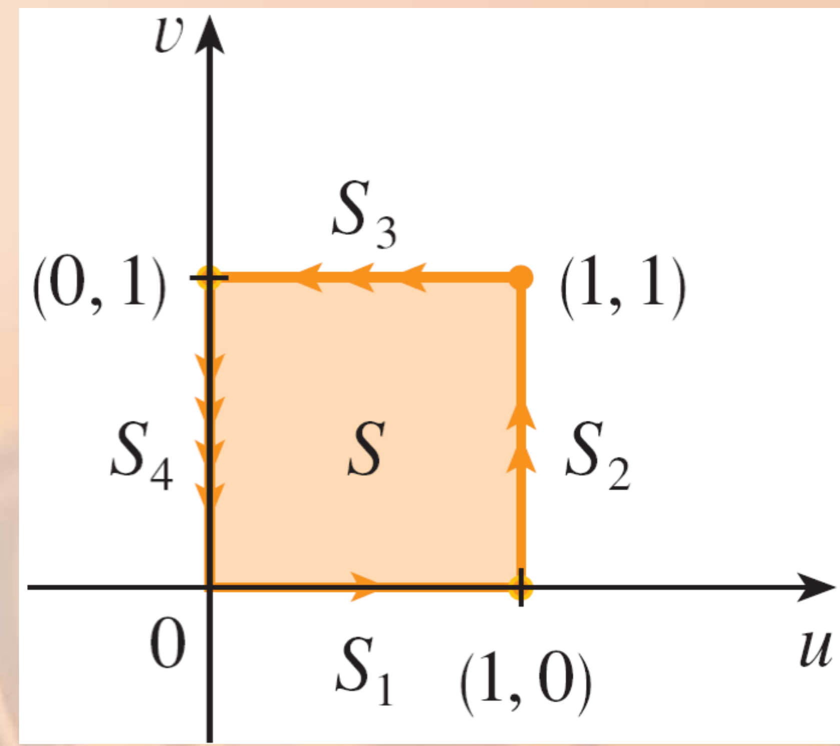
The second side, S_2, is:

$$u = 1 \ (0 \leq v \leq 1)$$

- Putting $u = 1$ in the given equations, we get:

$$x = 1 - v^2$$

$$y = 2v$$



Eliminating *v*, we obtain:

$$x = 1 - \frac{y^2}{4} \qquad 0 \le x \le 1$$

which is part of a parabola.

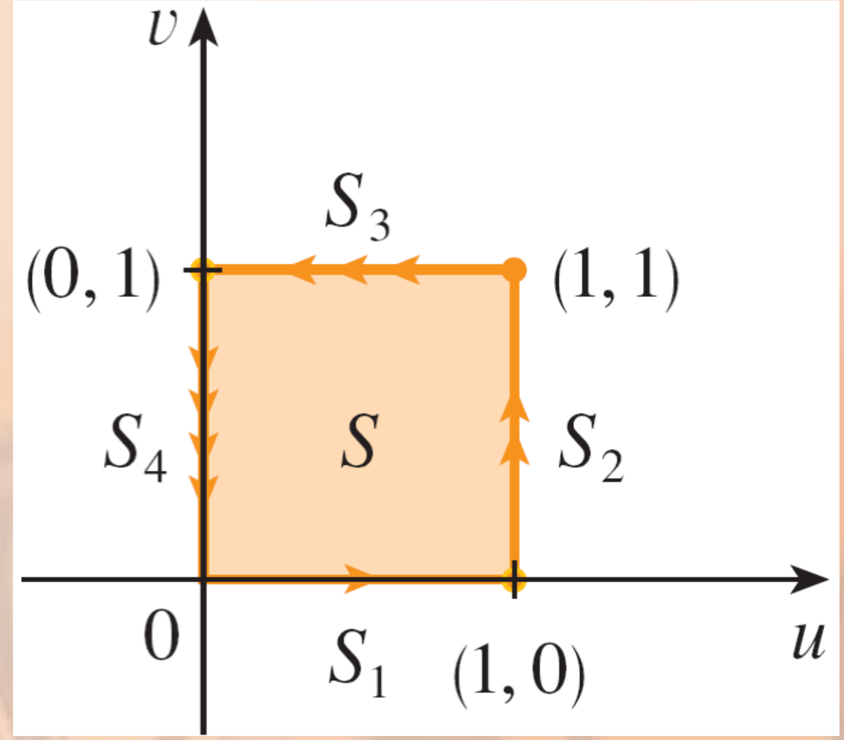
Similarly, S_3 is given by:

$$v = 1 \ (0 \leq u \leq 1)$$

Its image is

the parabolic arc

$$x = \frac{y^2}{4} - 1$$
$$(-1 \leq x \leq 0)$$



Finally, S_4 is given by:

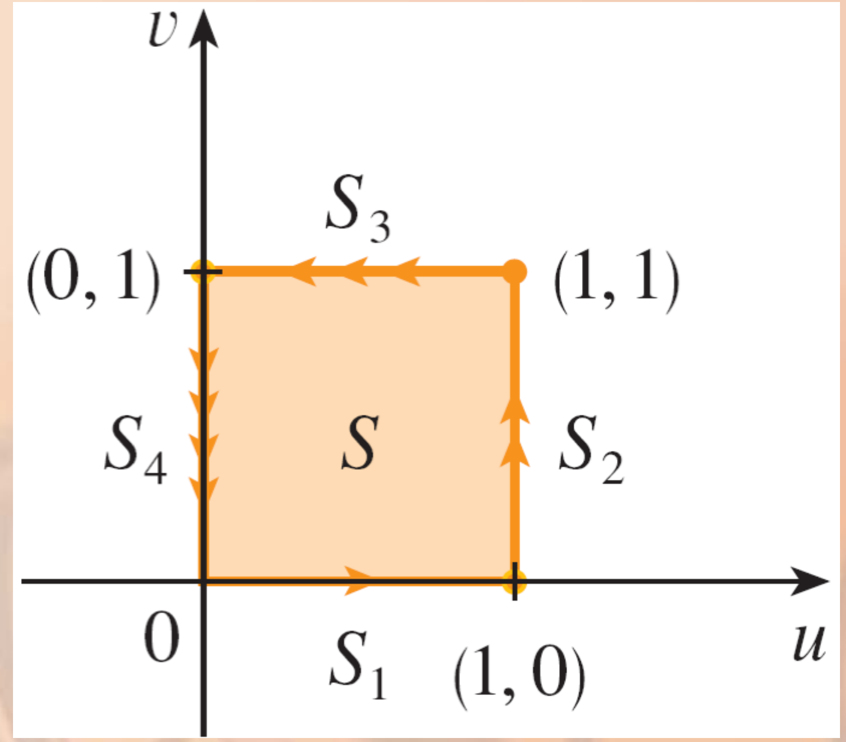
$$u = 0(0 \le v \le 1)$$

Its image is:

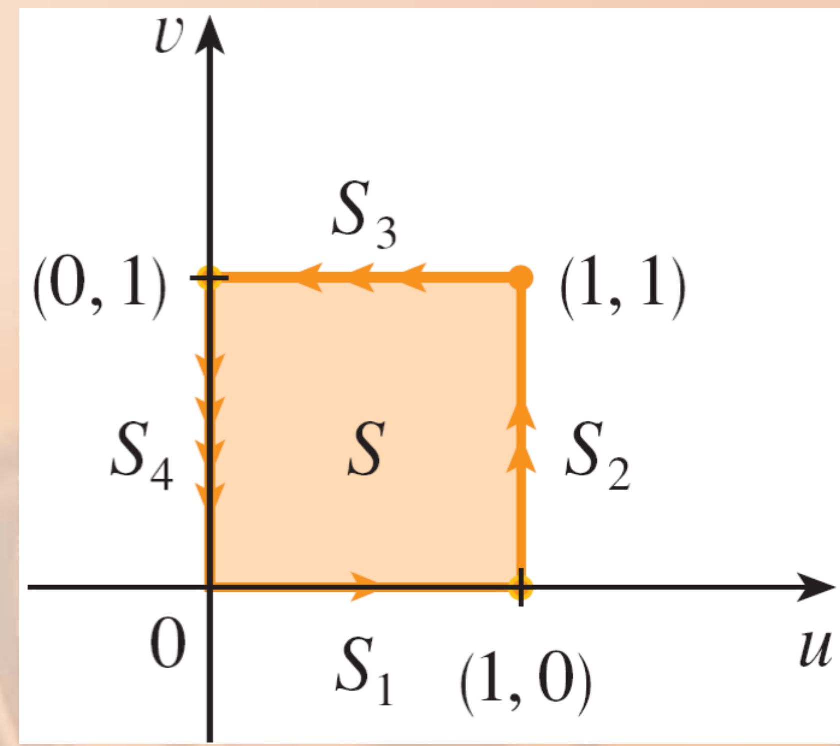
$$x = -v^2, \; y = 0$$

that is,

$$-1 \le x \le 0$$

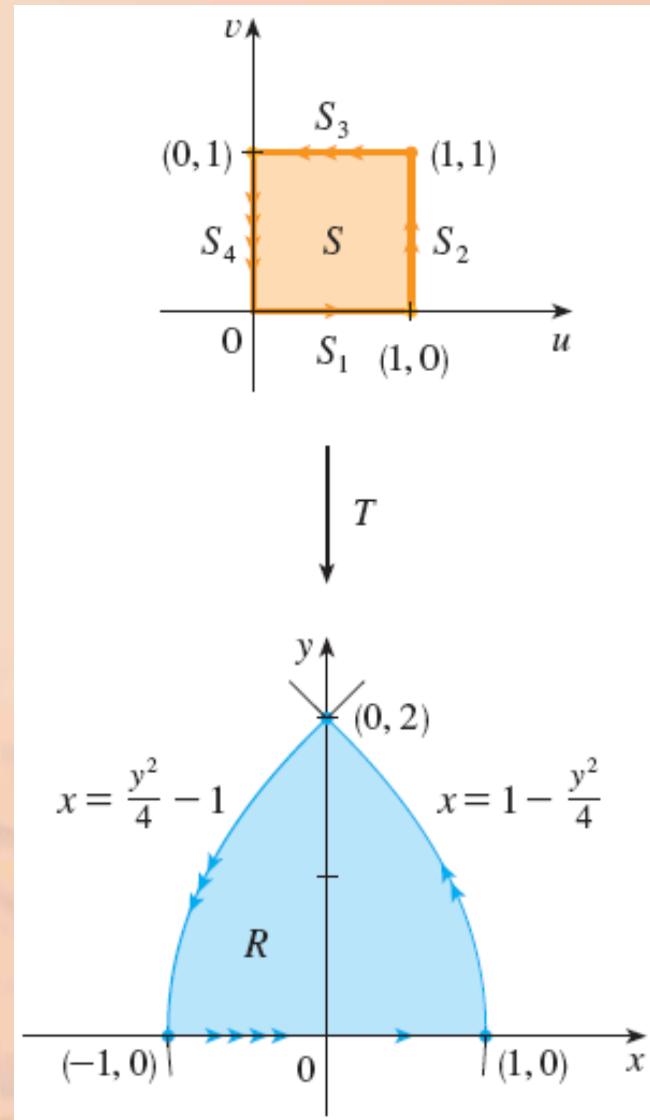


Notice that as, we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.



The image of S is the region R bounded by:

- The x-axis.
- The parabolas given by Equations 4 and 5.



Computing the cross product,

we obtain:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial u} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

JACOBIAN

The determinant that arises in this calculation is called the Jacobian of the transformation.

- It is given a special notation.

The Jacobian of the transformation *T* given by $x = g(u, v)$ and $y = h(u, v)$ is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Approximation 8

With this notation, we can use Equation 6 to give an approximation to the area ΔA of R:

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v$$

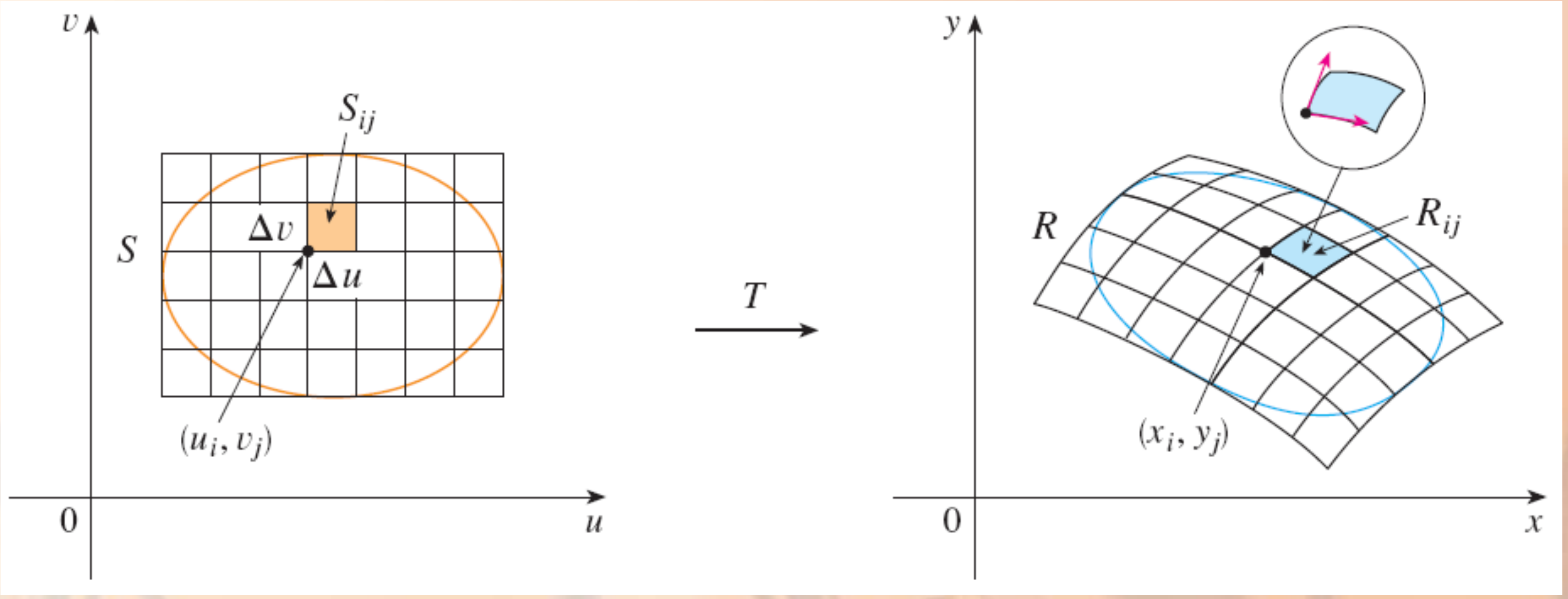
where the Jacobian is evaluated at (u_0, v_0).

JACOBIAN

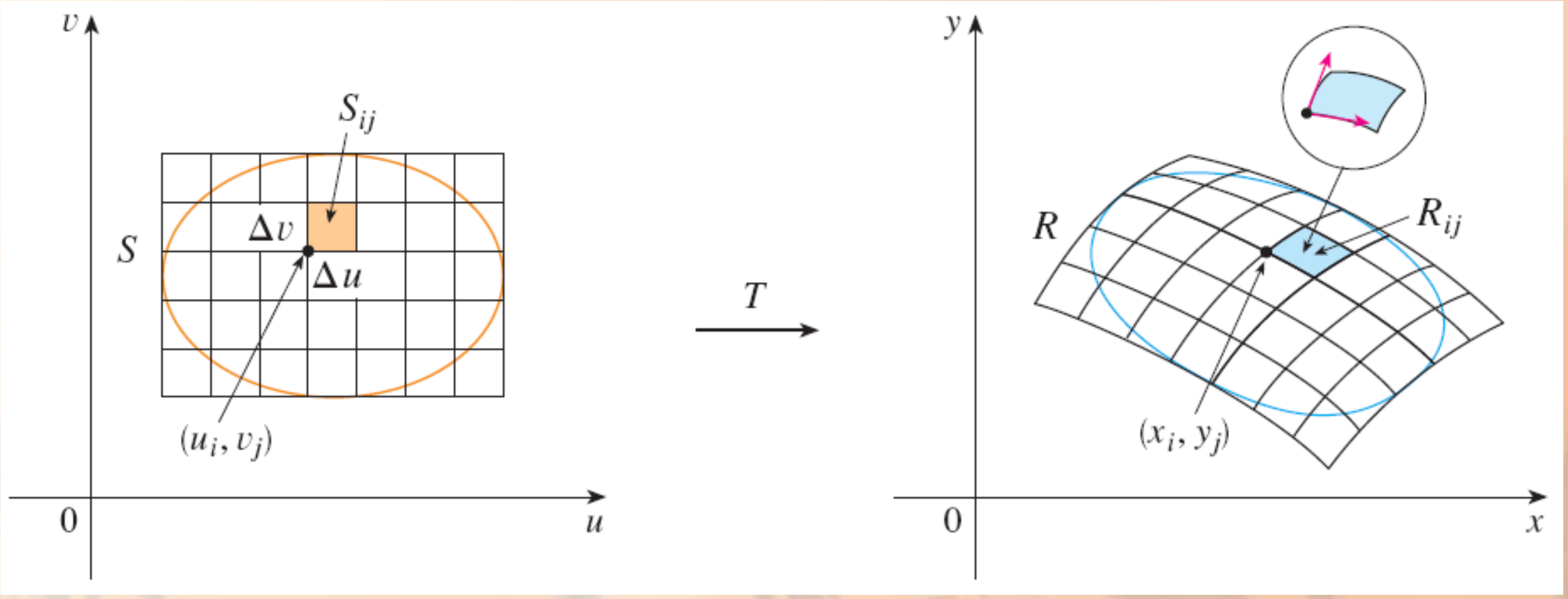
The Jacobian is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851).

- The French mathematician Cauchy first used these special determinants involving partial derivatives.
- Jacobi, though, developed them into a method for evaluating multiple integrals.

Next, we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}.



Applying Approximation 8 to each R_{ij} , we approximate the double integral of *f* over *R* as follows.



DOUBLE INTEGRALS

$$\iint_R f(x,y)\,dA$$

$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\,\Delta v$$

where the Jacobian is evaluated at (u_i, v_j).

Notice that this double sum is a Riemann sum for the integral

$$\iint_S f\left(g(u,v),h(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

The foregoing argument suggests that the following theorem is true.

- A full proof is given in books on advanced calculus.

Suppose:

- T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane.
- f is continuous on R and that R and S are type I or type II plane regions.
- T is one-to-one, except perhaps on the boundary of S.

Then,

$$\iint_R f(x,y)\,dA$$

$$= \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Theorem 9 says that we change from an integral in *x* and *y* to an integral in *u* and *v* by expressing *x* and *y* in terms of *u* and *v* and writing:

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2.

- Instead of the derivative dx/du, we have the absolute value of the Jacobian, that is,

$$|\partial(x, y)/\partial(u, v)|$$

CHG. OF VRBLS. (DOUBLE INTEG.)

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case.

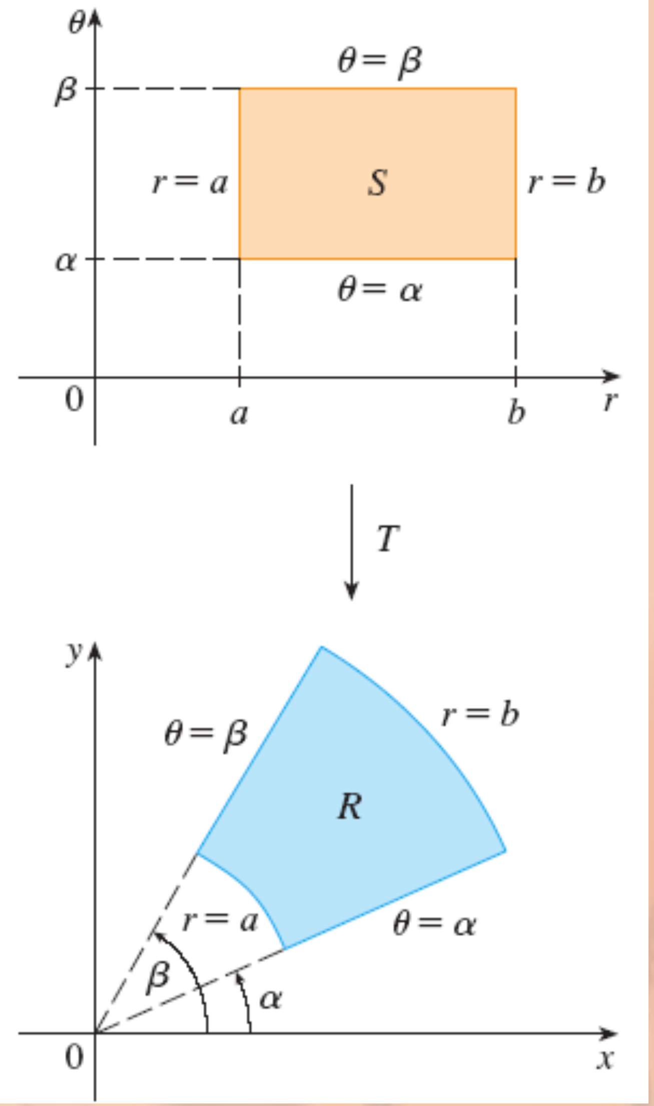
Here, the transformation *T* from the *rθ*-plane to the *xy*-plane is given by:

$$x = g(r, \theta) = r\cos\theta$$

$$y = h(r, \theta) = r\sin\theta$$

The geometry of the transformation is shown here.

- *T* maps an ordinary rectangle in the *rθ*-plane to a polar rectangle in the *xy*-plane.



The Jacobian of *T* is:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial x}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r > 0$$

So, Theorem 9 gives:

$$\iint_R f(x, y)\, dx\, dy$$

$$= \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dr\, d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

- This is the same as polar coordinate formula.

 Example 2

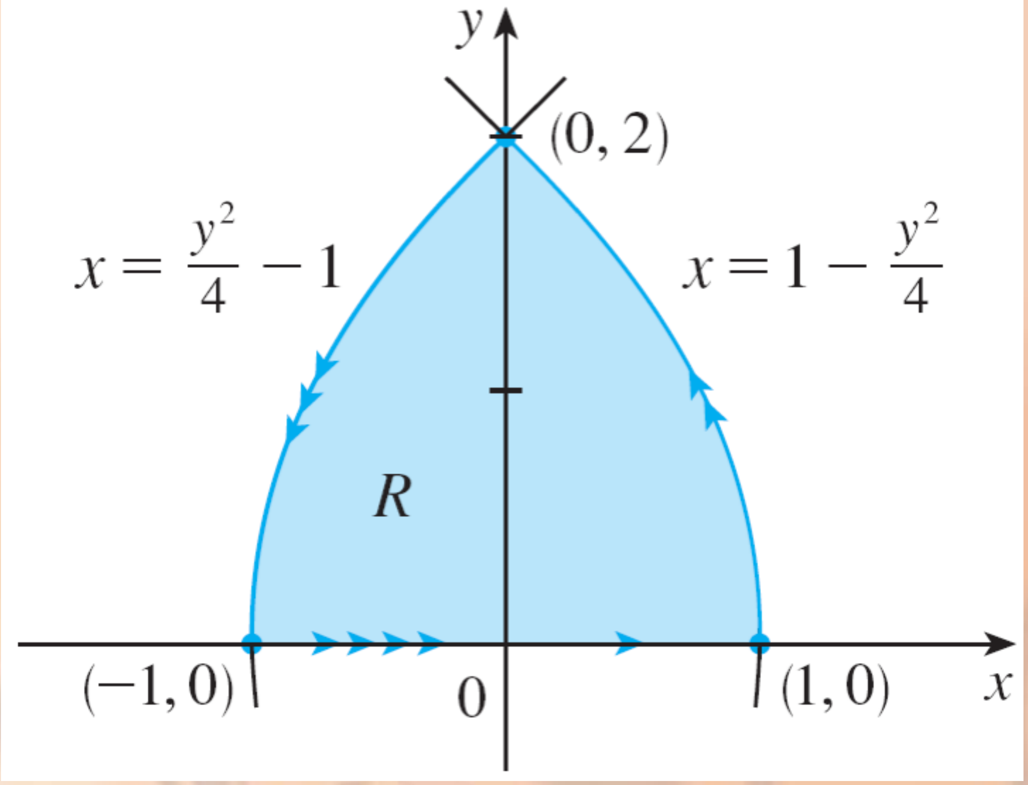
Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\iint_R y \, dA$$

where R is the region bounded by:

- The x-axis.
- The parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

The region R is pictured here.



 Example 2

In Example 1, we discovered that

$$T(S) = R$$

where S is the square [0, 1] x [0, 1].

- Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R.

First, we need to compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4u^2 + 4v^2 > 0$$

So, by Theorem 9,

$$\iint_R y\,dA = \iint_S 2uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

$$= \int_0^1\int_0^1 (2uv)4(u^2+v^2)\,du\,dv$$

$$= 8\int_0^1\int_0^1 (u^3v+uv^3)\,du\,dv$$

$$= 8\int_0^1 \left[\tfrac{1}{4}u^4v+\tfrac{1}{2}u^2v^3\right]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v+4v^3)\,dv = \left[v^2+v^4\right]_0^1 = 2$$

Note

Example 2 was not very difficult to solve as we were given a suitable change of variables.

If we are not supplied with a transformation, the first step is to think of an appropriate change of variables.

Note

If $f(x, y)$ is difficult to integrate,

- The form of $f(x, y)$ may suggest a transformation.

If the region of integration R is awkward,

- The transformation should be chosen so that the corresponding region S in the uv-plane has a convenient description.

 Example 3

Evaluate the integral

$$\iint_R e^{(x+y)/(x-y)}\,dA$$

where *R* is the trapezoidal region with vertices

(1, 0), (2, 0), (0, –2), (0,–1)

It isn't easy to integrate $e^{(x+y)/(x-y)}$.

So, we make a change of variables suggested by the form of this function:

$$u = x + y \qquad v = x - y$$

- These equations define a transformation T^{-1} from the *xy*-plane to the *uv*-plane.

 E. g. 3—Equation 11

Theorem 9 talks about a transformation *T* from the *uv*-plane to the *xy*-plane.

It is obtained by solving Equations 10 for *x* and *y*:

$$x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v)$$

The Jacobian of T is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

To find the region S in the uv-plane corresponding to R, we note that:

- The sides of R lie on the lines

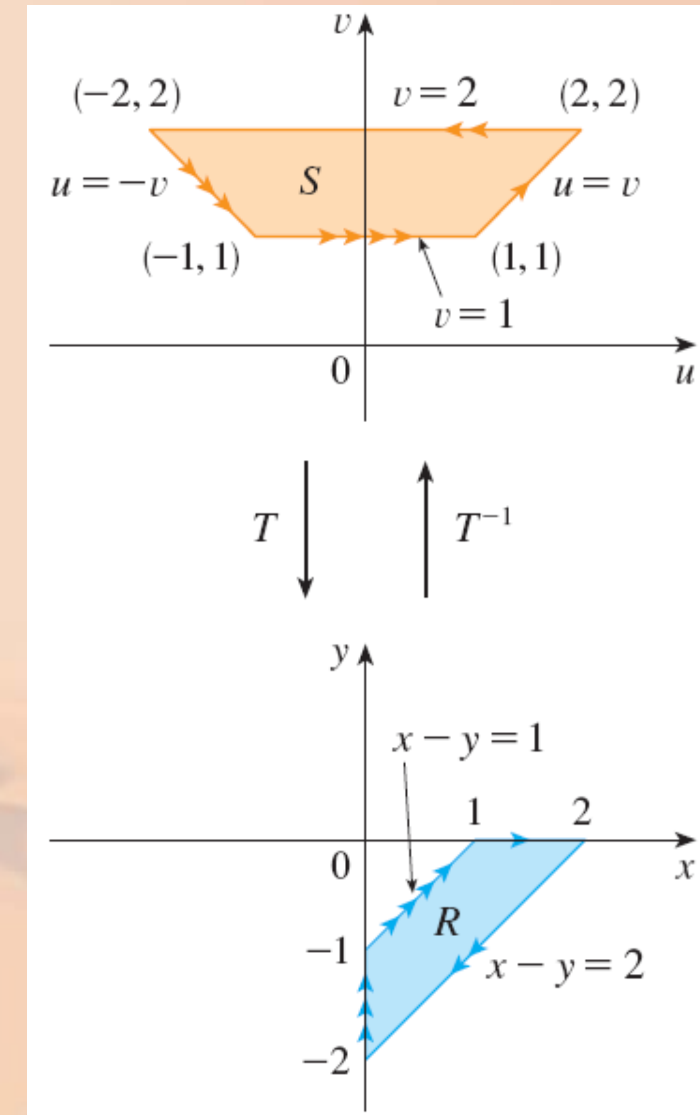
$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

- From either Equations 10 or Equations 11, the image lines in the uv-plane are:

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus, the region S is the trapezoidal region with vertices

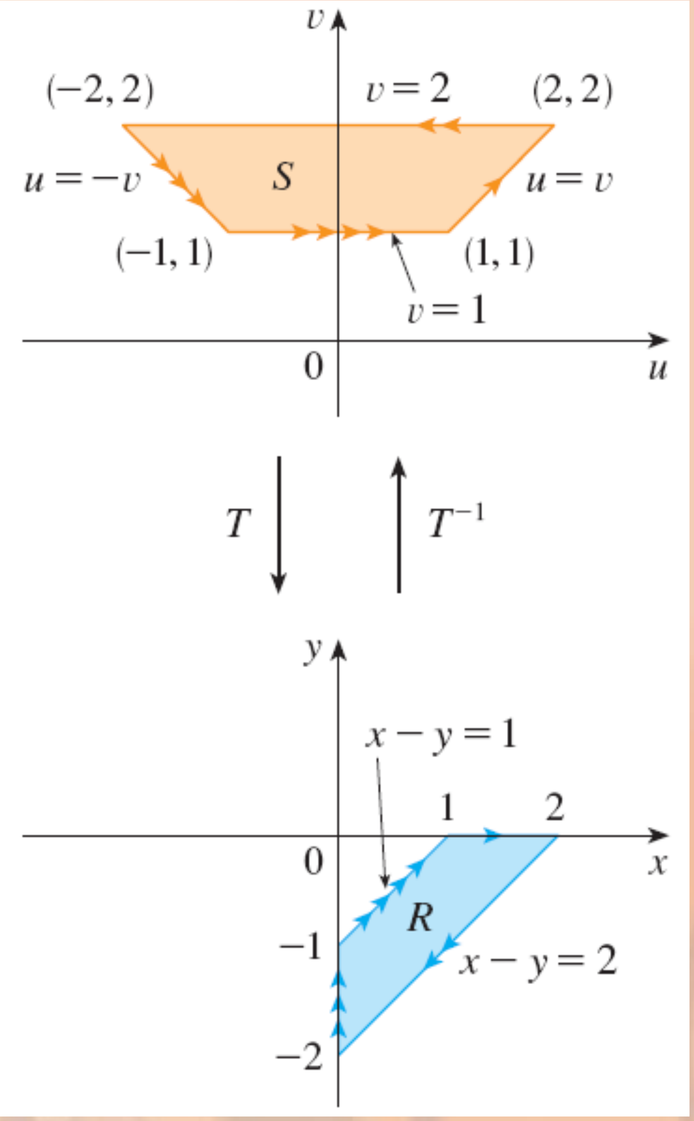
$$(1,\ 1),\ (2,\ 2),$$
$$(–2,\ 2),\ (–1\ ,1)$$



CHG. OF VRBLS. (DOUBLE INTEG.) **Example 3**

$S =$

$\{(u, v) \mid 1 \le v \le 2, -v \le u \le v\}$



So, Theorem 9 gives:

$$\iint_R e^{(x+y)/(x-y)}\,dA = \iint_S e^{u/v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right)du\,dv$$

$$= \tfrac{1}{2}\int_1^2 \left[ve^{u/v}\right]_{u=-v}^{u=v} dv$$

$$= \tfrac{1}{2}\int_1^2 (e - e^{-1})v\,dv = \tfrac{3}{4}(e - e^{-1})$$

There is a similar change of variables formula for triple integrals.

- Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

Equation 12

The Jacobian of *T* is this 3 x 3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

 Formula 13

Under hypotheses similar to those in Theorem 9, we have this formula for triple integrals:

$$\iiint_R f(x, y, z)\, dV$$

$$= \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Example 4

Use Formula 13 to derive the formula for triple integration in spherical coordinates.

- The change of variables is given by:

$$x = \rho \sin \Phi \cos \theta$$

$$y = \rho \sin \Phi \sin \theta$$

$$z = \rho \cos \Phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix}$$

$$- \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta)$$

$$- \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi$$

$$= -\rho^2\sin\phi$$

Since $0 \leq \phi \leq \pi$, we have $\sin \phi \geq 0$.

Therefore,

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \left| -\rho^2 \sin\phi \right|$$

$$= \rho^2 \sin\phi$$

Thus, Formula 13 gives:

$$\iiint_R f(x, y, z)\, dV$$

$$= \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \, \rho^2 \sin\phi \, d\rho \; d\theta \; d\phi$$

- This is equivalent to Formula 3 in Section15.8