15

MULTIPLE INTEGRALS

POLAR COORDINATES

In plane geometry, the polar coordinate system is used to give a convenient description of certain curves and regions.

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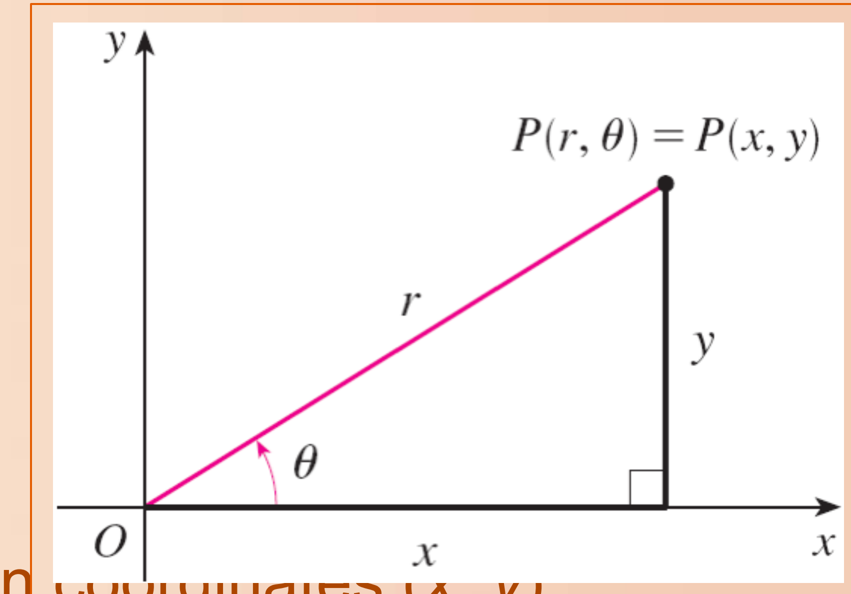
POLAR COORDINATES

The figure enables us to recall the connection between polar and Cartesian coordinates.



- If the point *P* has Cartesian coordinates (x, y) and polar coordinates (r, θ), then

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = y/x$$

CYLINDRICAL COORDINATES

In three dimensions there is a coordinate system, called cylindrical coordinates, that:

- Is similar to polar coordinates.
- Gives a convenient description of commonly occurring surfaces and solids.

Triple Integrals in Cylindrical Coordinates

In this section, we will learn about:

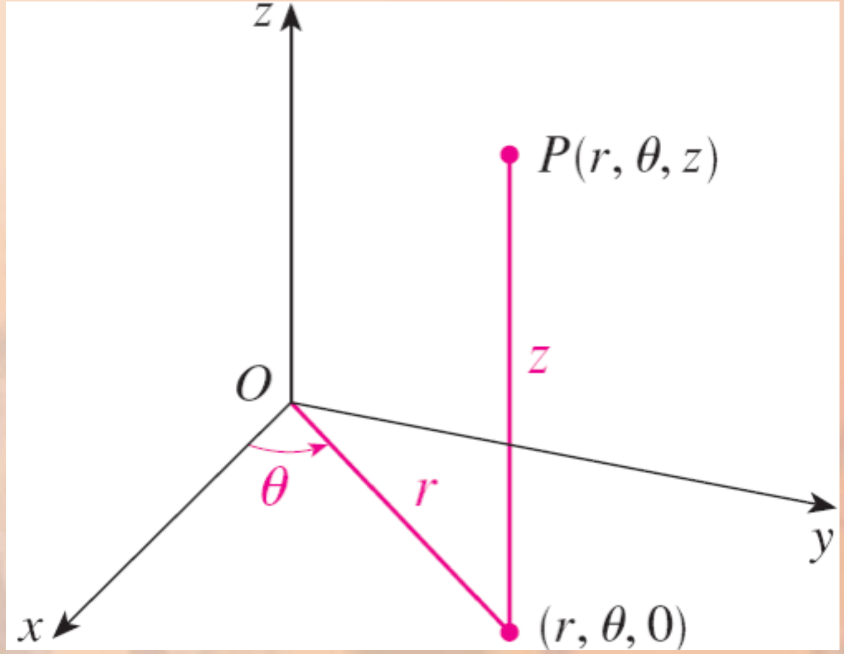
Cylindrical coordinates and

using them to solve triple integrals.

CYLINDRICAL COORDINATES

In the cylindrical coordinate system, a point *P* in three-dimensional (3-D) space is represented by the ordered triple (r, θ, z), where:

- r and θ are polar coordinates of the projection of *P* onto the *xy*–plane.
- z is the directed distance from the *xy*-plane to *P*.



To convert from cylindrical to rectangular coordinates, we use:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Equations 2

To convert from rectangular to cylindrical coordinates, we use:

$$r^2 = x^2 + y^2$$

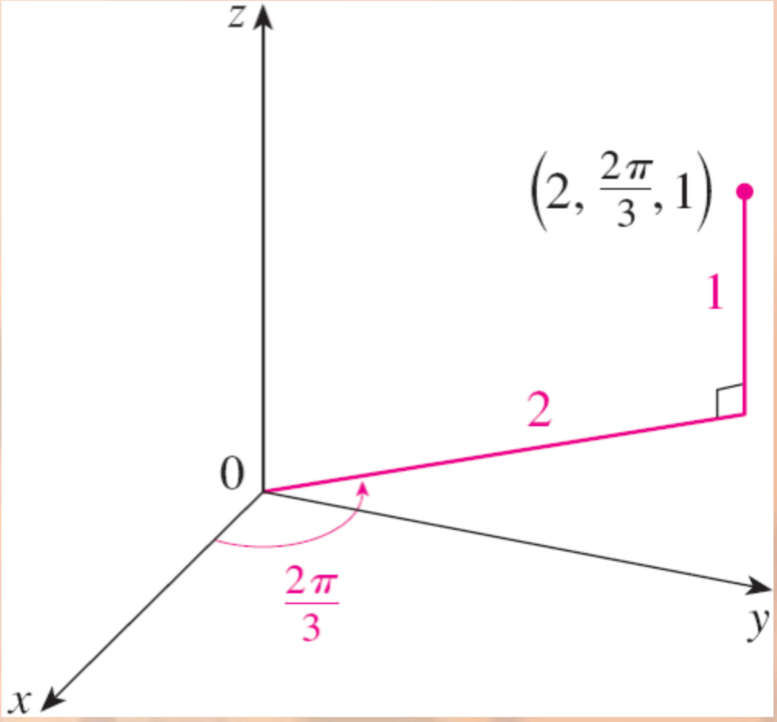
$$\tan \theta = y/x$$

$$z = z$$

a. Plot the point with cylindrical coordinates (2, $2\pi/3$, 1) and find its rectangular coordinates.

b. Find cylindrical coordinates of the point with rectangular coordinates (3, –3, –7).

The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted here.



From Equations 1, its rectangular coordinates are:

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

- The point is $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

From Equations 2, we have:

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1, \quad \text{so} \quad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore, one set of cylindrical coordinates is: $(3\sqrt{2}, -\pi / 4, -7)$

Another is: $(3\sqrt{2}, 7\pi / 4, -7)$

- As with polar coordinates, there are infinitely many choices.

CYLINDRICAL COORDINATES

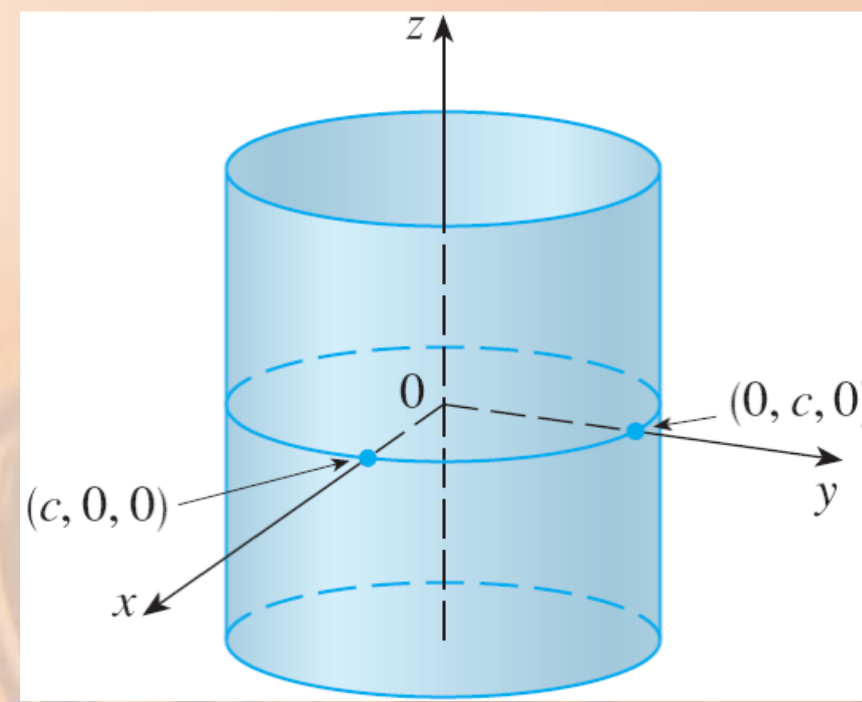
Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry.

- For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis.

CYLINDRICAL COORDINATES

- In cylindrical coordinates, this cylinder has the very simple equation $r = c$.

- This is the reason for the name "cylindrical" coordinates.



Describe the surface whose equation in cylindrical coordinates is $z = r$.

- The equation says that the z-value, or height, of each point on the surface is the same as r, the distance from the point to the z-axis.
- Since θ doesn't appear, it can vary.

So, any horizontal trace in the plane $z = k$ ($k > 0$) is a circle of radius k.

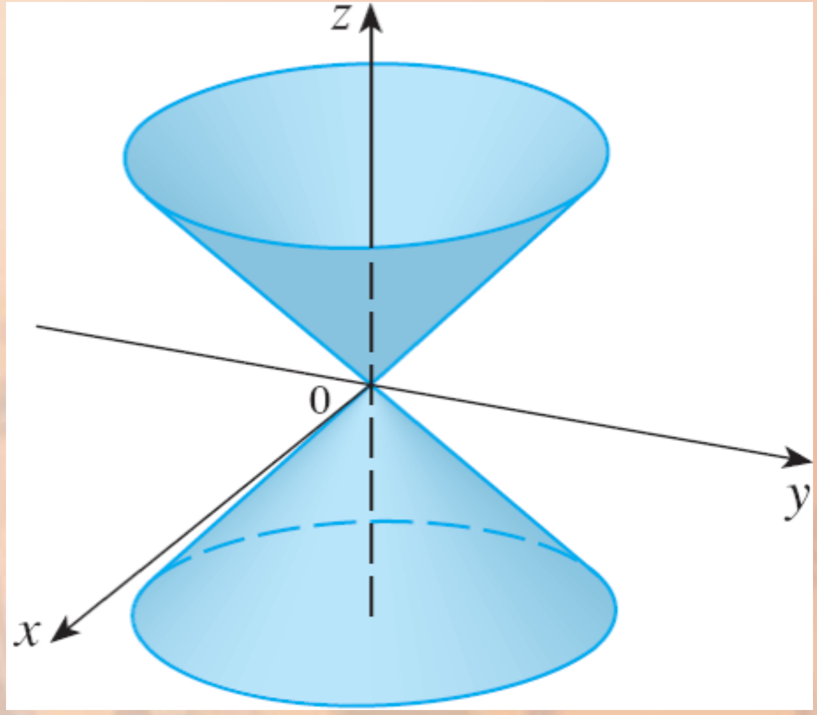
These traces suggest the surface is a cone.

- This prediction can be confirmed by converting the equation into rectangular coordinates.

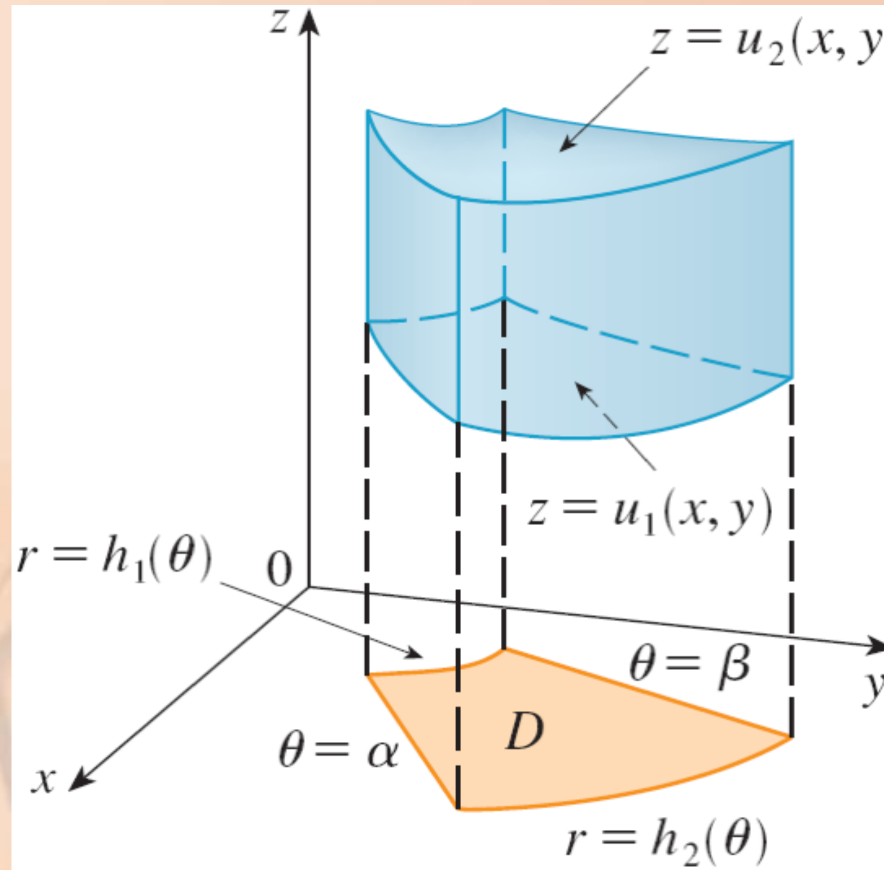
From the first equation in Equations 2, we have:

$$z^2 = r^2 = x^2 + y^2$$

We recognize the equation $z^2 = x^2 + y^2$ (by comparison with the table in Section 12.6) as being a circular cone whose axis is the z-axis.



Suppose that *E* is a type 1 region whose projection *D* on the *xy*-plane is conveniently described in polar coordinates.



EVALUATING TRIPLE INTEGRALS

In particular, suppose that *f* is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where *D* is given in polar coordinates by:

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

EVALUATING TRIPLE INTEGRALS **Equation 3**

We know from Equation 6 in Section 15.6 (Thomas Calculus)

that:

$$\iiint_E f(x, y, z)\, dV$$

$$= \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

However, we also know how to evaluate double integrals in polar coordinates.

In fact, combining Equation 3 with Equation 3 in Section 15.4 (Thomas Calculus), we obtain the following formula.

TRIPLE INTEGN. IN CYL. COORDS. **Formula 4**

$$\iiint_E f(x, y, z)\,dV$$

$$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

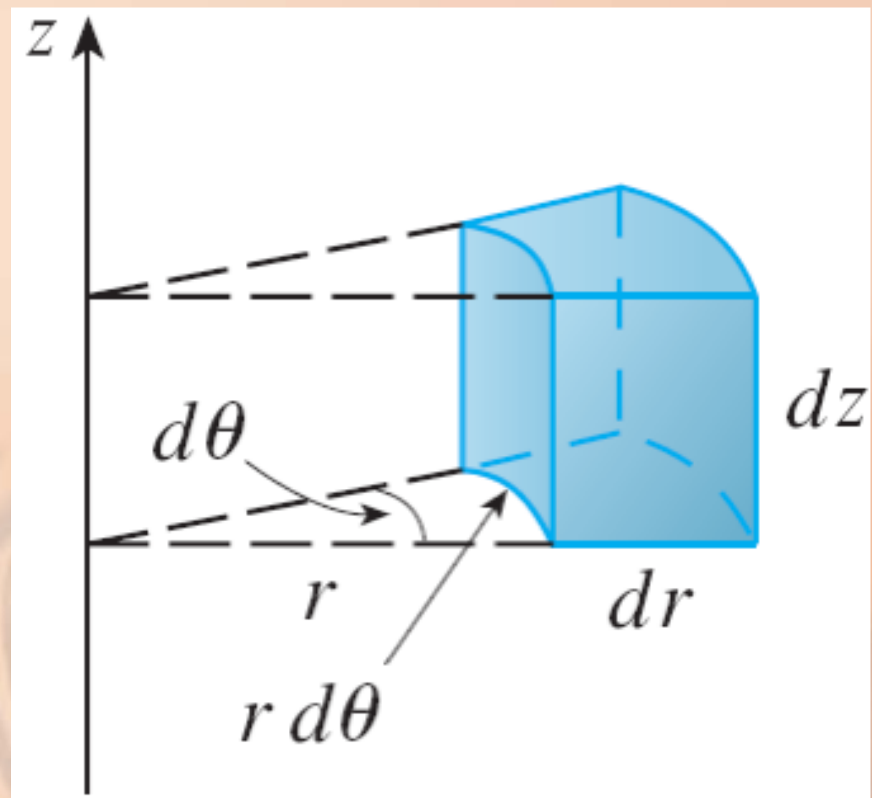
This is the formula for triple integration in cylindrical coordinates.

TRIPLE INTEGN. IN CYL. COORDS.

It says that we convert a triple integral from rectangular to cylindrical coordinates by:

- Writing $x = r \cos \theta$, $y = r \sin \theta$.
- Leaving z as it is.
- Using the appropriate limits of integration for z, r, and θ.
- Replacing dV by $r\, dz\, dr\, d\theta$.

The figure shows how to remember this.

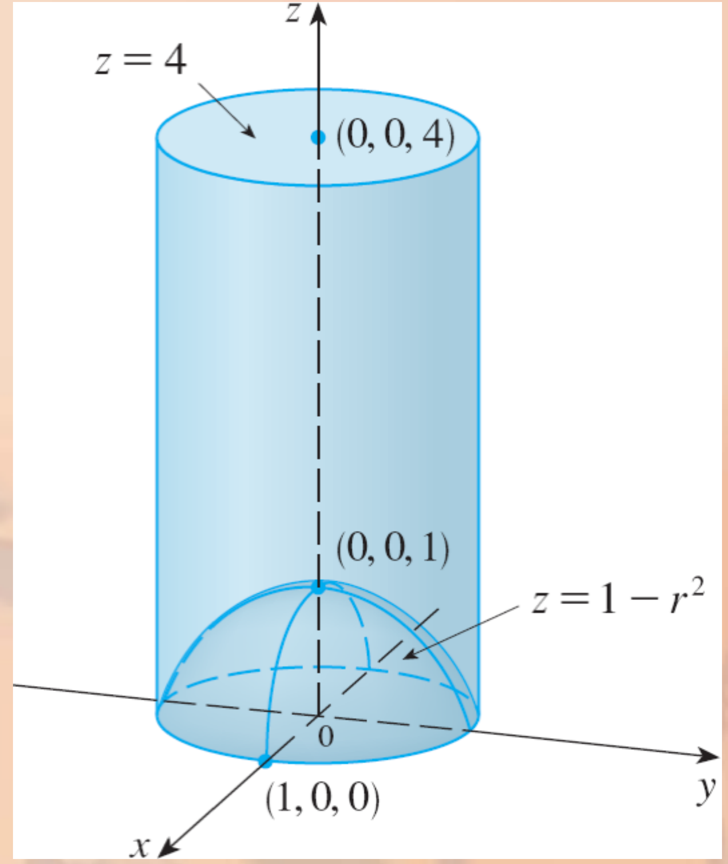


It is worthwhile to use this formula:

- When E is a solid region easily described in cylindrical coordinates.
- Especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

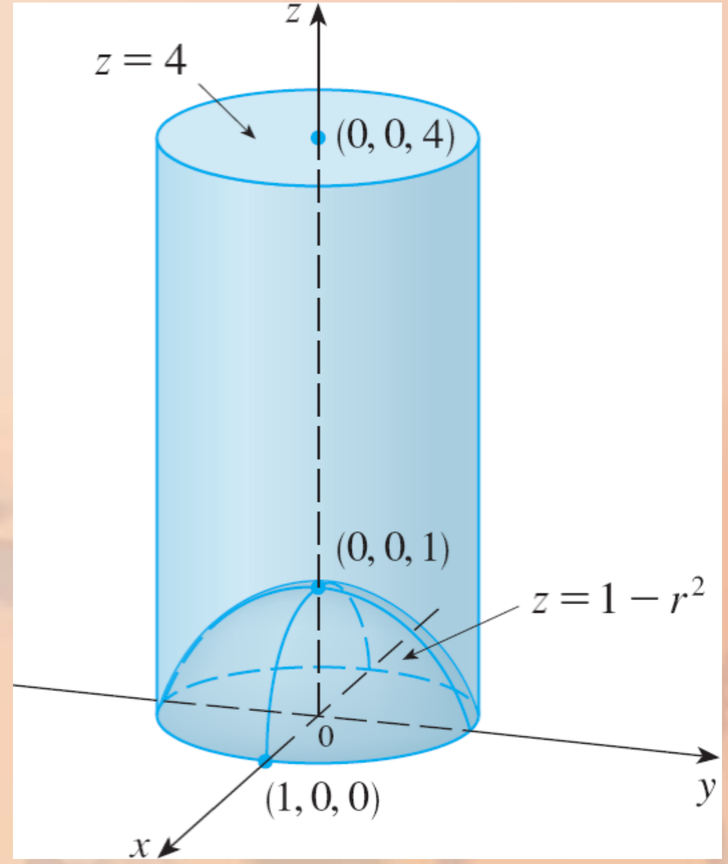
A solid lies within:

- The cylinder $x^2 + y^2 = 1$
- Below the plane $z = 4$
- Above the paraboloid $z = 1 - x^2 - y^2$



The density at any point is proportional to its distance from the axis of the cylinder.

Find the mass of *E*.



In cylindrical coordinates, the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$.

So, we can write:

$E =$

$\{(r, \theta, z) | \ 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4\}$

As the density at (x, y, z) is proportional to the distance from the z-axis, the density function is:

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant.

So, from Formula 13 in Section 15.6, the mass of *E* is:

$$m = \iiint_E K\sqrt{x^2+y^2}\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 Kr^2\left[4-\left(1-r^2\right)\right]dr\,d\theta$$

$$= K\int_0^{2\pi} d\theta \int_0^1 \left(3r^2+r^4\right)dr$$

$$= 2\pi K\left[r^3+\frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

 Example 4

Evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)dz\,dy\,dx$$

This iterated integral is a triple integral over the solid region

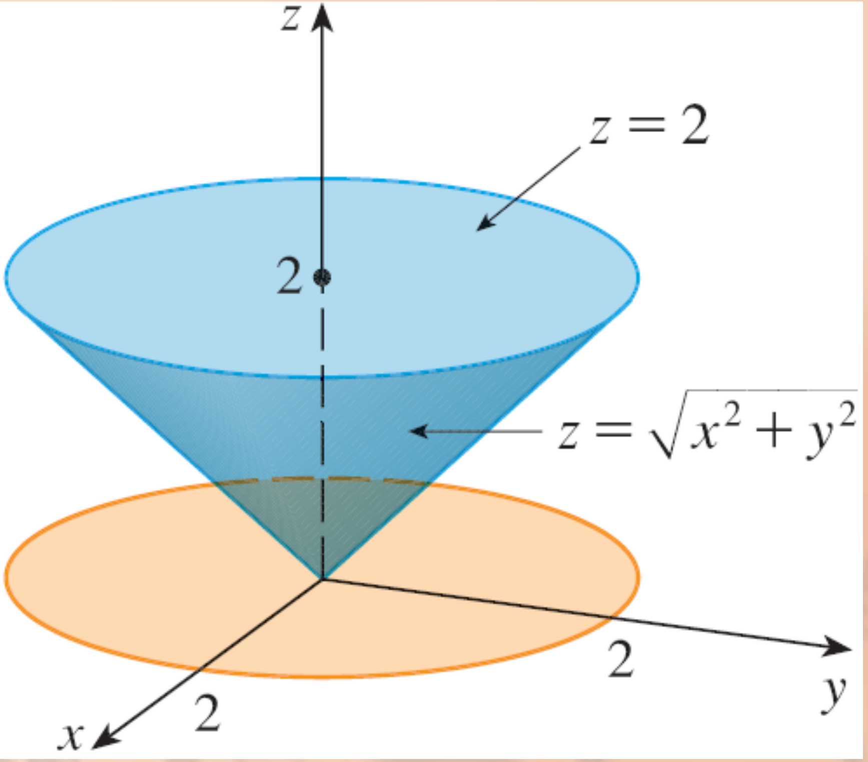
$$E = \{(x, y, z) \mid -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2\}$$

The projection of *E* onto the *xy*-plane is the disk $x^2 + y^2 \le 4$.

The lower surface of *E* is the cone

$$z = \sqrt{x^2 + y^2}$$

Its upper surface is the plane $z = 2$.



That region has a much simpler description in cylindrical coordinates:

$$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r \le z \le 2\}$$

- Thus, we have the following result.

EVALUATING TRIPLE INTEGRALS **Example 4**

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)dz\,dy\,dx$$

$$=\iiint_E\left(x^2+y^2\right)dV=\int_0^{2\pi}\int_0^2\int_r^2 r^2 r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^2 r^3(2-r)\,dr$$

$$=2\pi\left[\tfrac{1}{2}r^4-\tfrac{1}{5}r^5\right]_0^2$$

$$=\tfrac{16}{5}\pi$$