Electromagnetic wave equations

Electromagnetic wave Equations

Faraday's law (Maxwell's 3^{rd} Equation)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

$$\vec{B} = \mu \vec{H}.$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}.$$

Take Curl on both sides

$$\nabla \times \nabla \times \vec{E} = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t}. \quad (1)$$

Maxwell's fourth Equation

$$\nabla \times H = J + \frac{\partial D}{\partial t}.$$

$$\nabla \times H = \sigma E + \varepsilon \frac{\partial E}{\partial t}.$$

where $D = \varepsilon E$
$J = \sigma E$, (Ohm's law).

Differentiating

$$\nabla \times \frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(\nabla \times H)$$

$$= \frac{\partial}{\partial t}\left(\sigma E + \varepsilon \frac{\partial E}{\partial t}\right).$$

$$\nabla \times \frac{\partial H}{\partial t} = \sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2}. \quad \text{--- (2)}$$

Substitute eqn ② into eqn ①

$$\nabla \times \nabla \times E = -\mu \left[\sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2} \right]$$

$$= -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \quad (3)$$

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E \quad (4)$$

$$\nabla \cdot E = \frac{1}{\varepsilon} \nabla \cdot D$$

Since there is no net charge within the conductor, the charge density $\rho = 0$.

$$\nabla \cdot D = 0$$

$$\nabla \cdot E = 0.$$

$$\nabla \times \nabla \times E = -\nabla^2 E \quad (5)$$

Comparing equations 3 and 5

$$\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} + \mu\varepsilon \frac{\partial^2 E}{\partial t^2}.$$

$$\boxed{\nabla^2 E - \mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0} \quad (6)$$

This is the wave equation for electric field.

Wave equation for magnetic field

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$= \sigma E + \varepsilon \frac{\partial E}{\partial t}.$$

Take curl on both sides

$$\nabla \times \nabla \times H = \sigma \nabla \times E + \varepsilon \nabla \times \frac{\partial E}{\partial t}. \quad \text{--- (7)}$$

from Maxwell's third equation (Faraday's law)

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad \text{--- (8)}$$

After Differentiating

$$\nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial^2 H}{\partial t^2}. \quad \text{--- (9)}$$

Use equations (8) & (9) in eqn (7).

$$\nabla \times \nabla \times H = -\mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2}.$$

$$\nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H.$$

$$= -\nabla^2 H.$$

$$[\nabla \cdot B = \nabla \cdot H = 0]$$

$$\nabla^2 H = \mu\sigma \frac{\partial H}{\partial t} + \mu\varepsilon \frac{\partial^2 H}{\partial t^2}.$$

$$\boxed{\nabla^2 H - \mu\sigma \frac{\partial H}{\partial t} - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0.}$$

This is the wave equation for magnetic field.

Wave Equations for free space

For free space, the conductivity of the medium is zero.

$$\sigma = 0$$

and there is no charge containing in it; $\rho = 0$.

The wave equations become

$$\nabla^2 E - \mu \varepsilon \frac{\partial^2 E}{\partial t^2} = 0.$$

$$\nabla^2 H - \mu \varepsilon \frac{\partial^2 H}{\partial t^2} = 0.$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} ; \quad \mu_r = \frac{\mu}{\mu_0}$$

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}.$$

When $\mu_r = 1$ and $\varepsilon_r = 1$.

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{1}{\sqrt{\varepsilon \mu}}.$$

$$\Rightarrow \quad \varepsilon \mu = \frac{1}{c^2}.$$

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

$$\nabla^2 H - \frac{1}{c^2} \frac{\partial^2 H}{\partial t^2} = 0.$$

Verify that the following equations satisfy the one-dimensional wave equations:

$$E_y(x,t) = E_0 \cos(kx - \omega t);$$

$$B_z(x,t) = B_0 \cos(kx - \omega t);$$

assume $\omega = kc$.

Solution:

$$E_y(x,t) = E_0 \cos(kx-\omega t) \quad (1)$$

$$B_z(x,t) = B_0 \cos(kx-\omega t) \quad (2).$$

One dimensional wave equations

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y(x,t)}{\partial t^2} \quad (3).$$

$$\frac{\partial^2 B_z(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_z(x,t)}{\partial t^2} \quad (4).$$

Differentiating eqn (1) w.r.to x

$$\frac{\partial E_y(x,t)}{\partial x} = -k E_0 \sin(kx-\omega t)$$

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = -k^2 E_0 \cos(kx-\omega t). \quad (5)$$

Differentiating eqn (1) w.r.to 't'.

$$\frac{\partial E_y(x,t)}{\partial t} = \omega E_0 \sin(kx - \omega t)$$

$$\frac{\partial^2 E_y(x,t)}{\partial t^2} = -\omega^2 E_0 \cos(kx - \omega t) \quad \text{— (6)}.$$

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y(x,t)}{\partial t^2}$$

$$= \left(-k^2 + \frac{\omega^2}{c^2}\right) E_0 \cos(kx - \omega t) \quad \text{— (7)}$$

Substitute $\omega = kc$.

eqn (7) becomes

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y(x,t)}{\partial t^2} = 0$$

Do the samething for eqn (2).

Both equations satisfy the wave equations.