MULTIPLE INTEGRALS

MULTIPLE INTEGRALS

In this Module-5, we extend the idea of a definite integral to double and triple integrals of functions of two or three variables.

These ideas are then used to compute volumes, masses, and centroids of more general regions.

We also use double integrals to calculate probabilities when two random variables are involved.

We will see that polar coordinates are useful in computing double integrals over some types of regions.

Similarly, we will introduce two coordinate systems in three-dimensional space that greatly simplify computing of triple integrals over certain commonly occurring solid regions.

- Cylindrical coordinates
- Spherical coordinates

Double Integrals over Rectangles

In this section, we will learn about:

Double integrals and using them to find volumes and average values.

DOUBLE INTEGRALS OVER RECTANGLES

Just as our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid.

In the process, we arrive at the definition of a double integral.

DEFINITE INTEGRAL—REVIEW

First, let's recall the basic facts concerning definite integrals of functions of a single variable.

DEFINITE INTEGRAL—REVIEW

If $f(x)$ is defined for $a \leq x \leq b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$.

We choose sample points x_i^* in these subintervals.

 Equation 1

Then, we form the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

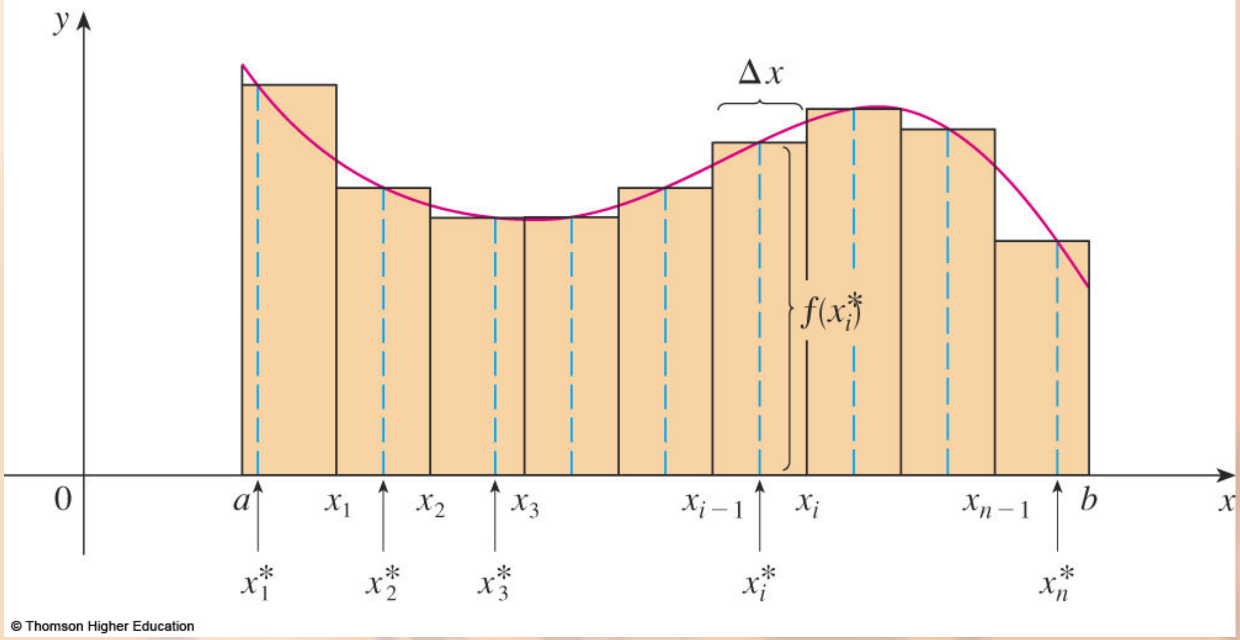
DEFINITE INTEGRAL—REVIEW **Equation 2**

Then, we take the limit of such sums as $n \to \infty$ to obtain the definite integral of *f* from *a* to *b*:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

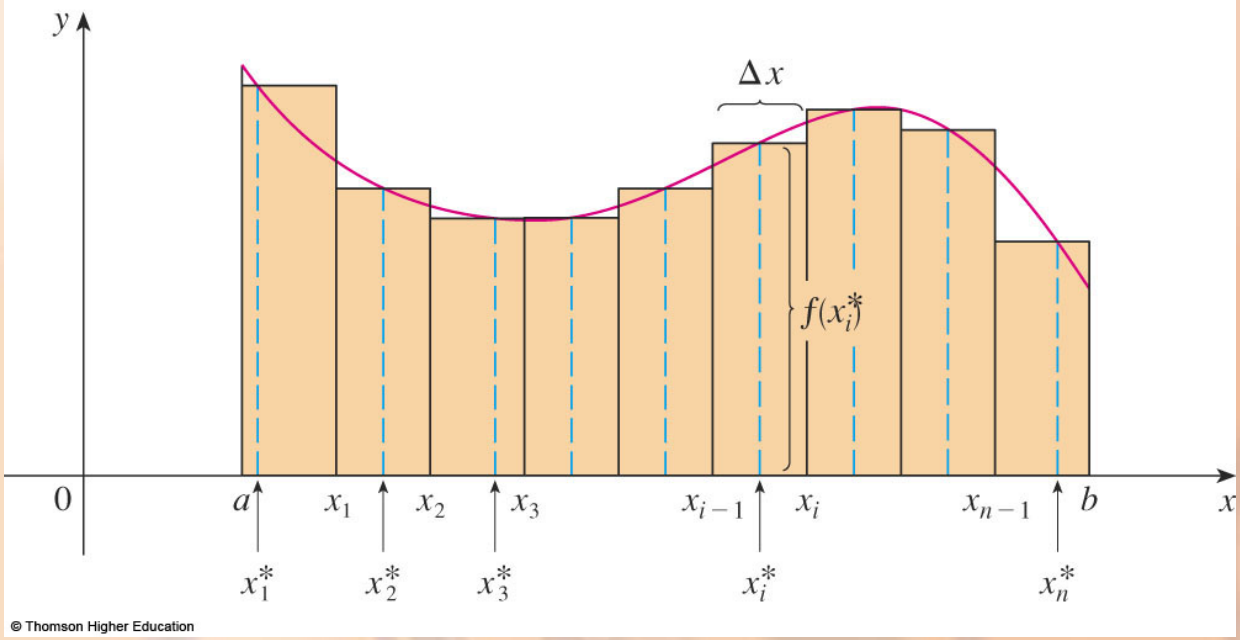
DEFINITE INTEGRAL—REVIEW

In the special case where $f(x) \geq 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles.



DEFINITE INTEGRAL—REVIEW

Then, $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b.



VOLUMES

In a similar manner, we consider a function *f* of two variables defined on a closed rectangle

$R = [a, b] \text{ x } [c, d]$

$= \{(x, y) \in \mathrm{R}^2 \mid a \leq x \leq b, c \leq y \leq d$

and we first suppose that $f(x, y) \geq 0$.

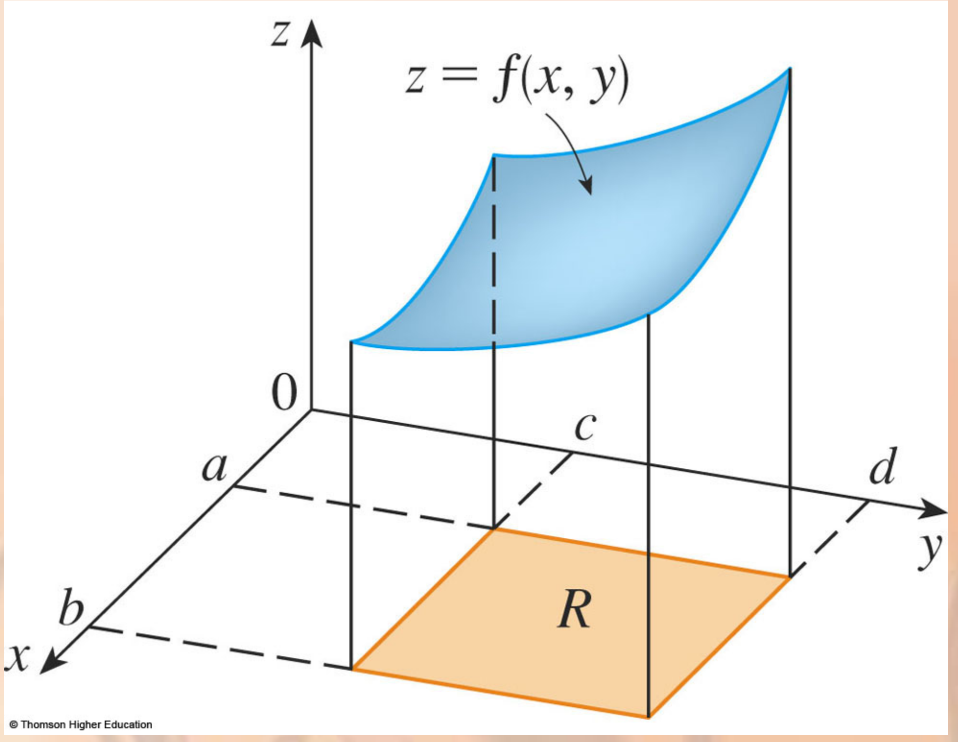
- The graph of *f* is a surface with equation $z = f(x, y)$.

VOLUMES

Let S be the solid that lies above R and under the graph of f, that is,

S = {(x, y, z) ä R³ | 0 ≤ z ≤ f(x, y), (x, y) ä R}

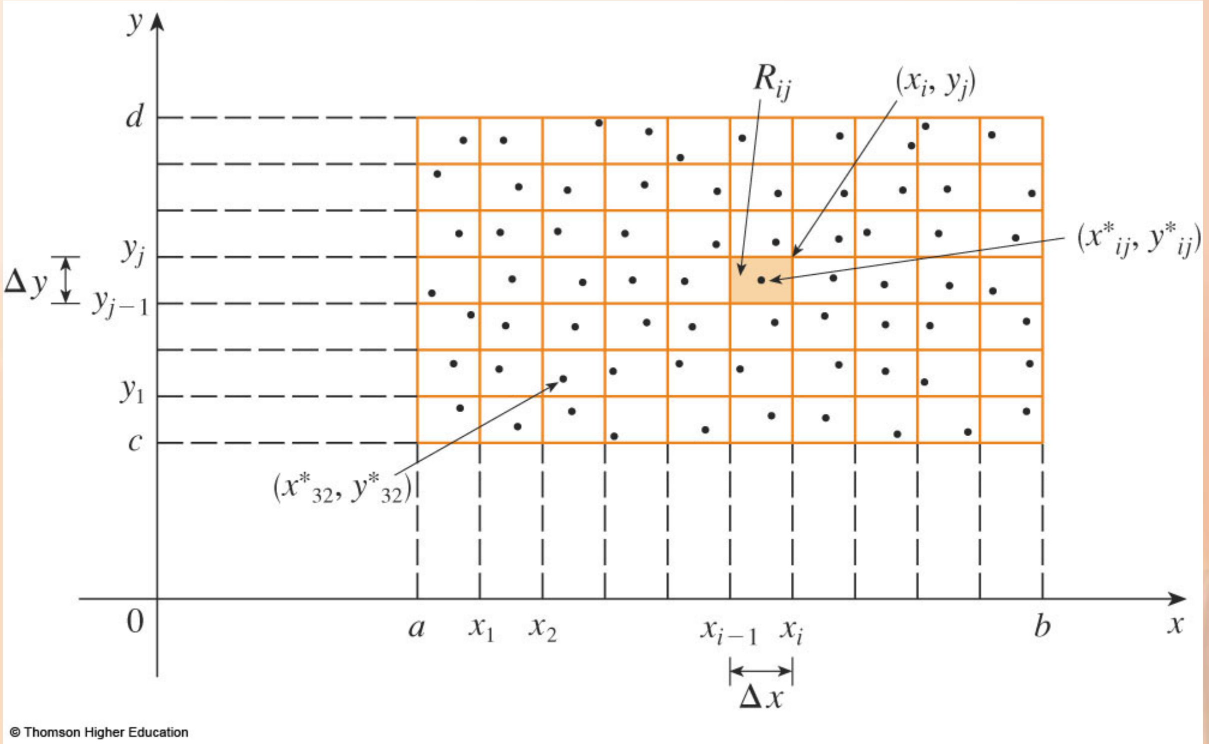
Our goal is to find the volume of S.



The first step is to divide the rectangle R into subrectangles.

- We divide the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$.
- Then, we divide $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$.

- Next, we draw lines parallel to the coordinate axes through the endpoints of these subintervals.

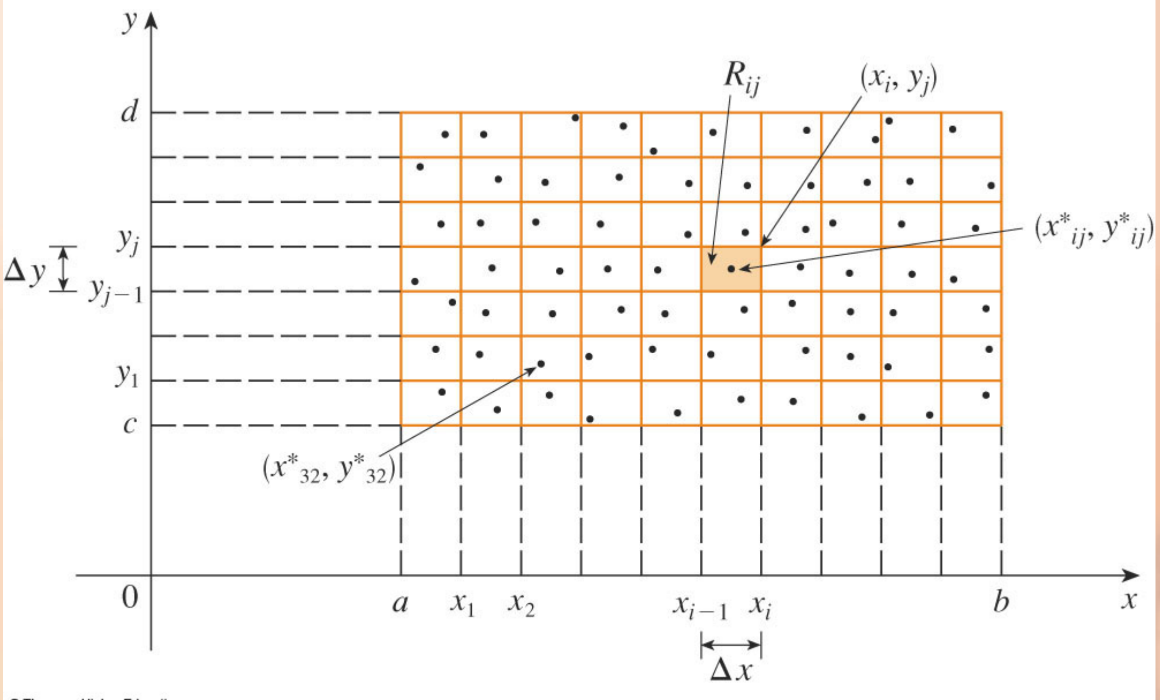


VOLUMES

- Thus, we form the subrectangles

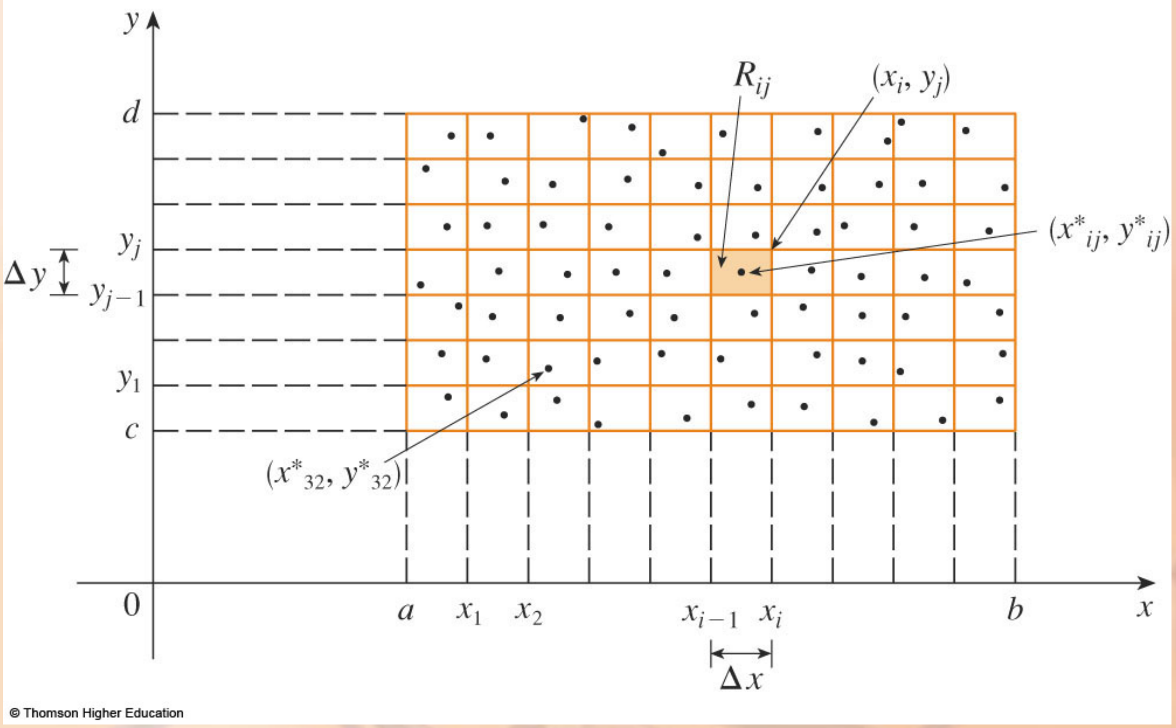
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$
$$= \{(x, y) \mid x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$



Let's choose a sample point $(x_{ij}{}^*, y_{ij}{}^*)$

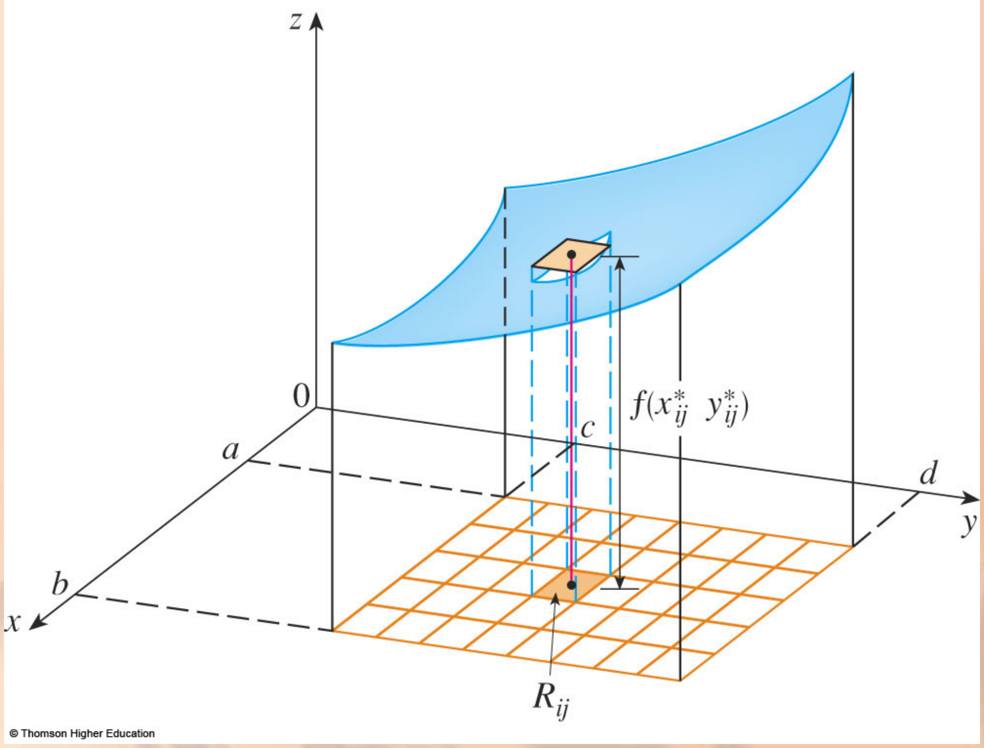
in each R_{ij}.



VOLUMES

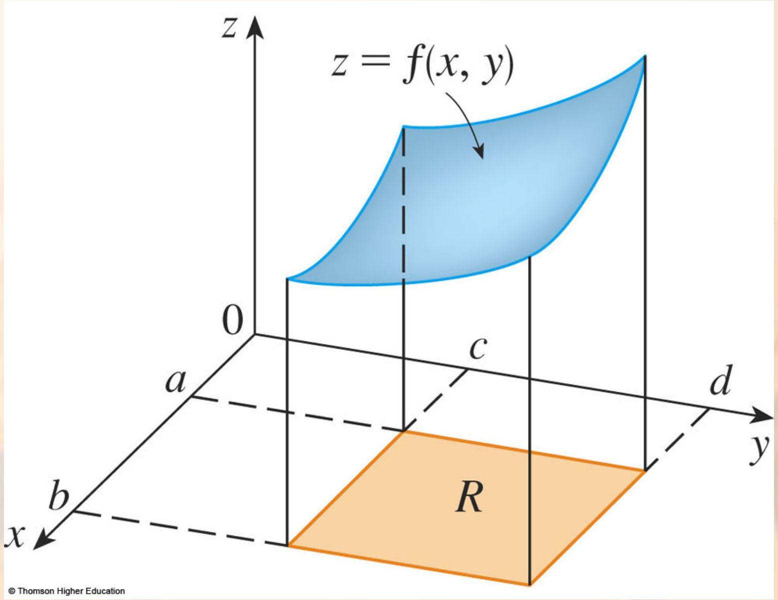
Then, we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with:

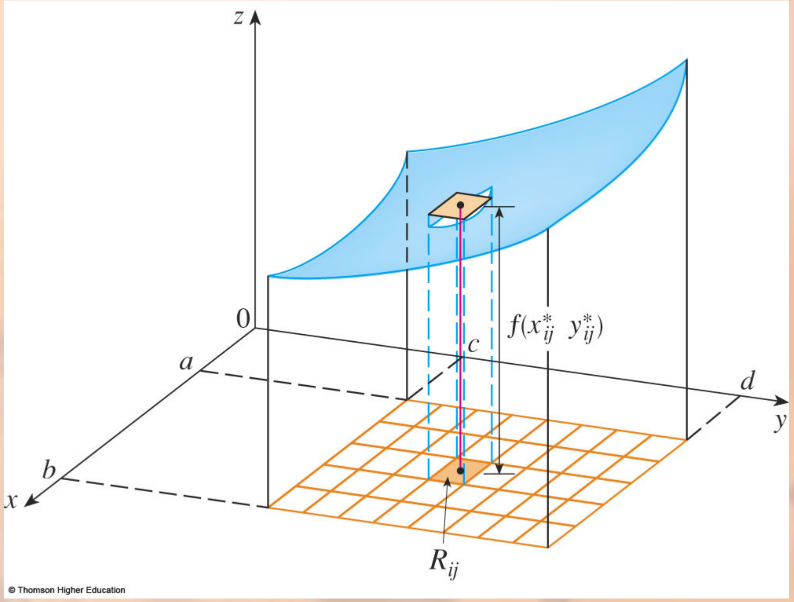
- Base R_{ij}
- Height $f(x_{ij}^*, y_{ij}^*)$



VOLUMES

Compare the figure with the earlier one.

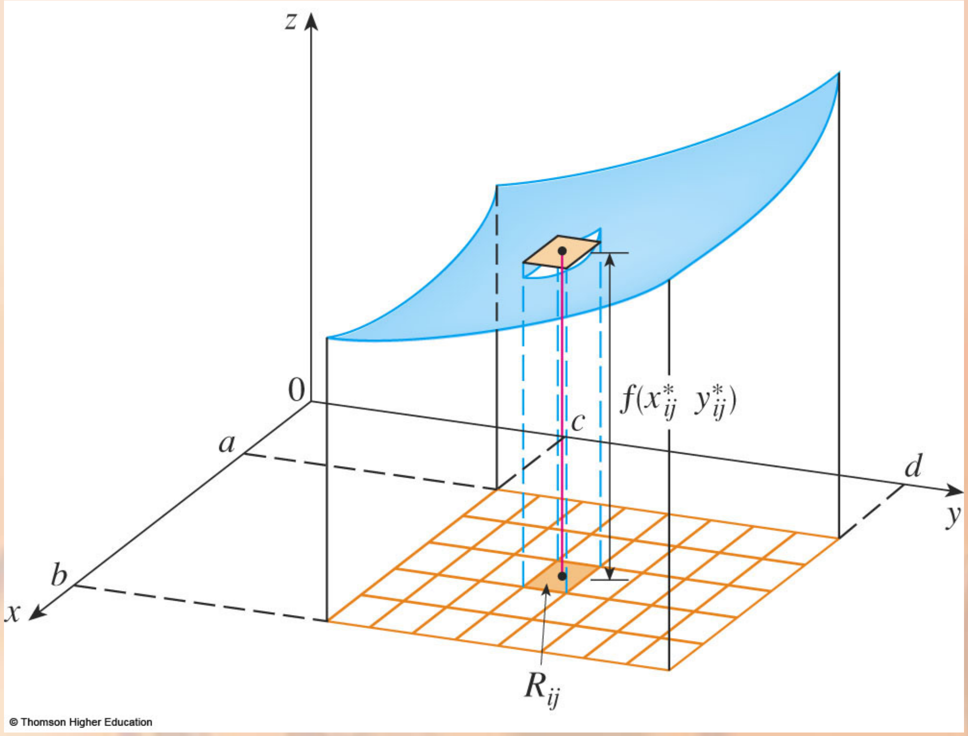




VOLUMES

The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\,\Delta A$$



VOLUMES

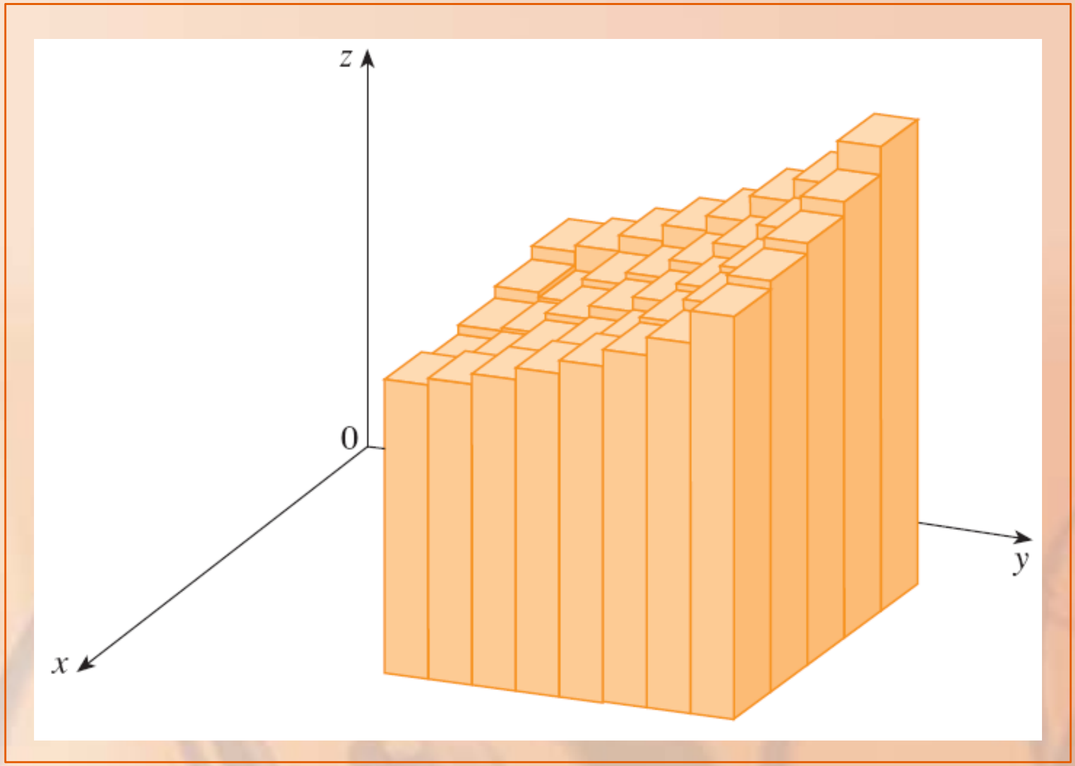
We follow this procedure for all the rectangles and add the volumes of the corresponding boxes.

VOLUMES

Equation 3

Thus, we get an approximation to the total volume of *S*:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$



This double sum means that:

- For each subrectangle, we evaluate *f* at the chosen point and multiply by the area of the subrectangle.

- Then, we add the results.

Equation 4

Our intuition tells us that the approximation given in Equation 3 becomes better as *m* and *n* become larger.

So, we would expect that:

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\,\Delta A$$

VOLUMES

Limits of the type that appear in Equation 4 occur frequently—not just in finding volumes but in a variety of other situations as well—even when *f* is not a positive function.

- So, we make the following definition.

Definition 5

The double integral of *f* over the rectangle *R* is:

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

DOUBLE INTEGRAL

The precise meaning of the limit in Definition 5 is that, for every number $\varepsilon > 0$, there is an integer N such that

$$\left| \iint_R f(x,y)\,dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \right| < \varepsilon$$

for:

- All integers m and n greater than N
- Any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij}^*

INTEGRABLE FUNCTION

A function *f* is called integrable if the limit in Definition 5 exists.

- It is shown in courses on advanced calculus that all continuous functions are integrable.
- In fact, the double integral of *f* exists provided that *f* is “not too discontinuous.”

INTEGRABLE FUNCTION

In particular,

if f is bounded [that is, there is a constant M such that $|f(x, y)| \leq$ for all (x, y) in R],

and

f is continuous there, except on a finite number of smooth curves,

then

f is integrable over R.

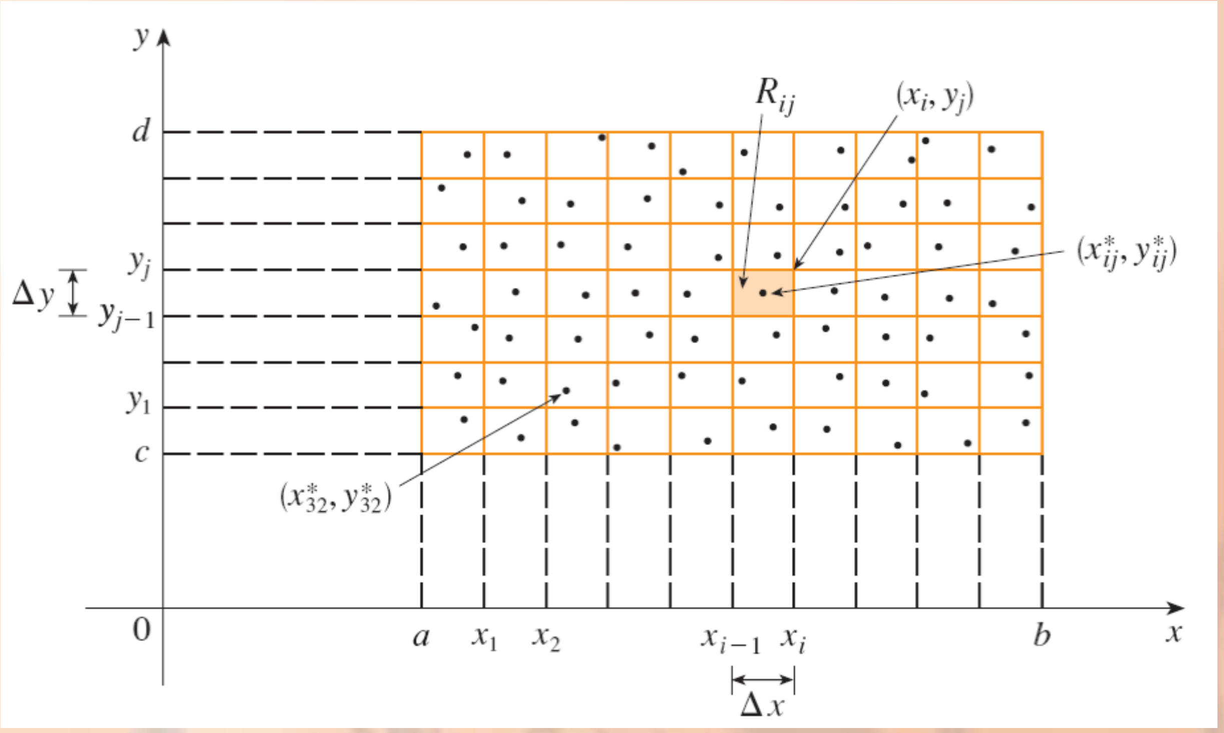
DOUBLE INTEGRAL

The sample point (x_{ij}^*, y_{ij}^*)

can be chosen to be any point

in the subrectangle R_{ij}^*.

However, suppose we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j)].



DOUBLE INTEGRAL

Equation 6

Then, the expression for the double integral looks simpler:

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i, y_i)\, \Delta A$$

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral, as follows.

DOUBLE INTEGRAL

If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is:

$$V = \iint_R f(x, y)\, dA$$

DOUBLE REIMANN SUM

The sum in Definition 5

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

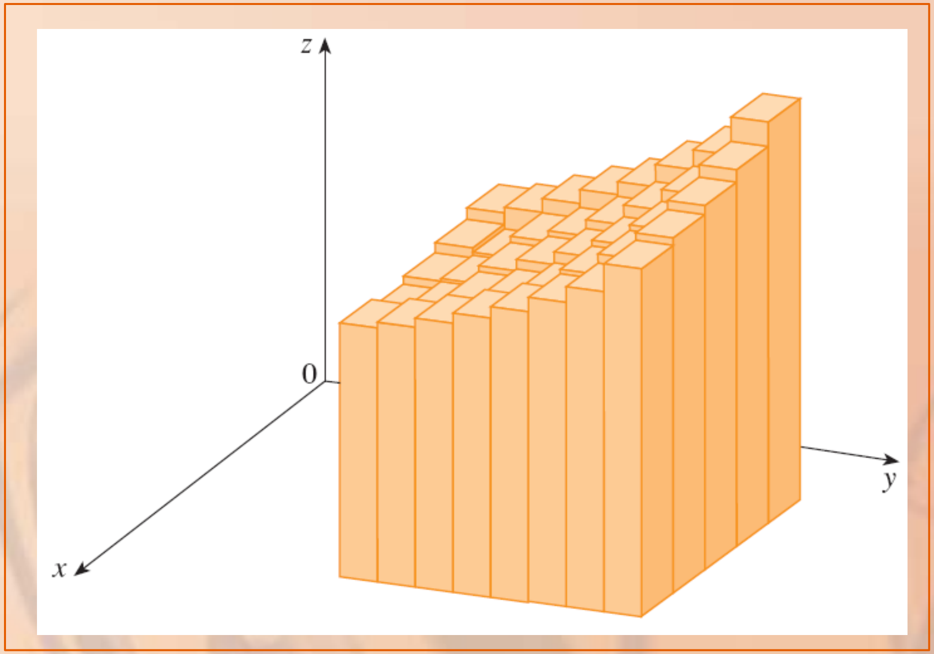
is called a double Riemann sum.

- It is used as an approximation to the value of the double integral.
- Notice how similar it is to the Riemann sum in Equation 1 for a function of a single variable.

DOUBLE REIMANN SUM

If f happens to be a positive function, the double Riemann sum:

- Represents the sum of volumes of columns, as shown.
- Is an approximation to the volume under the graph of f and above the rectangle R.



Example 1

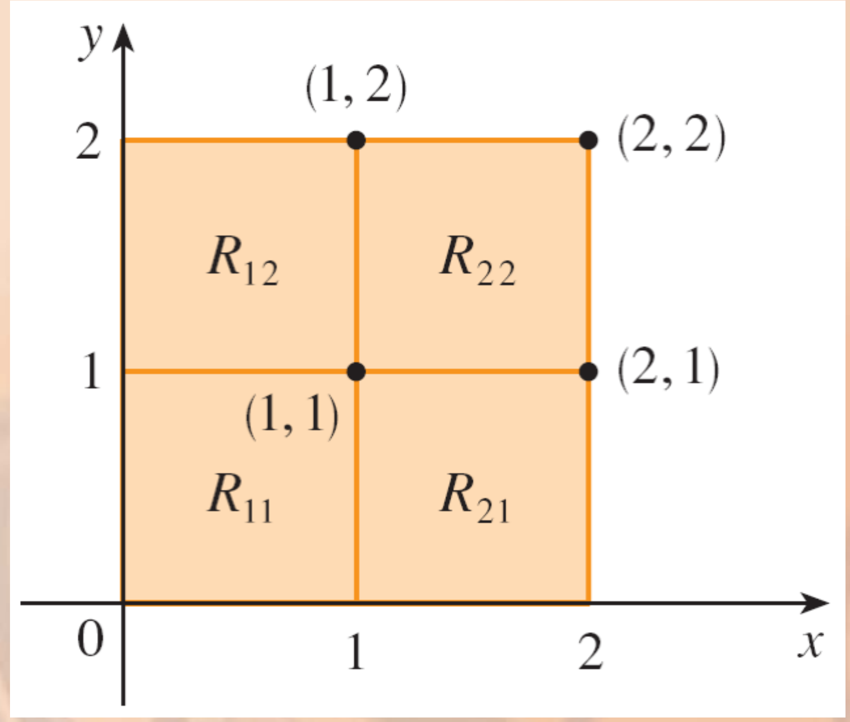
Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$.

- Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}.
- Sketch the solid and the approximating rectangular boxes.

Example 1

The squares are shown here.

- The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$
- The area of each square is 1.



Approximating the volume by the Riemann sum with $m = n = 2$, we have:

$$
\begin{aligned}
V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\,\Delta A \\
&= f(1,1)\,\Delta A + f(1,2)\,\Delta A + f(2,2)\,\Delta A \\
&= 13(1) + 7(1) + 10(1) + 4(1) \\
&= 34
\end{aligned}
$$

Example 1

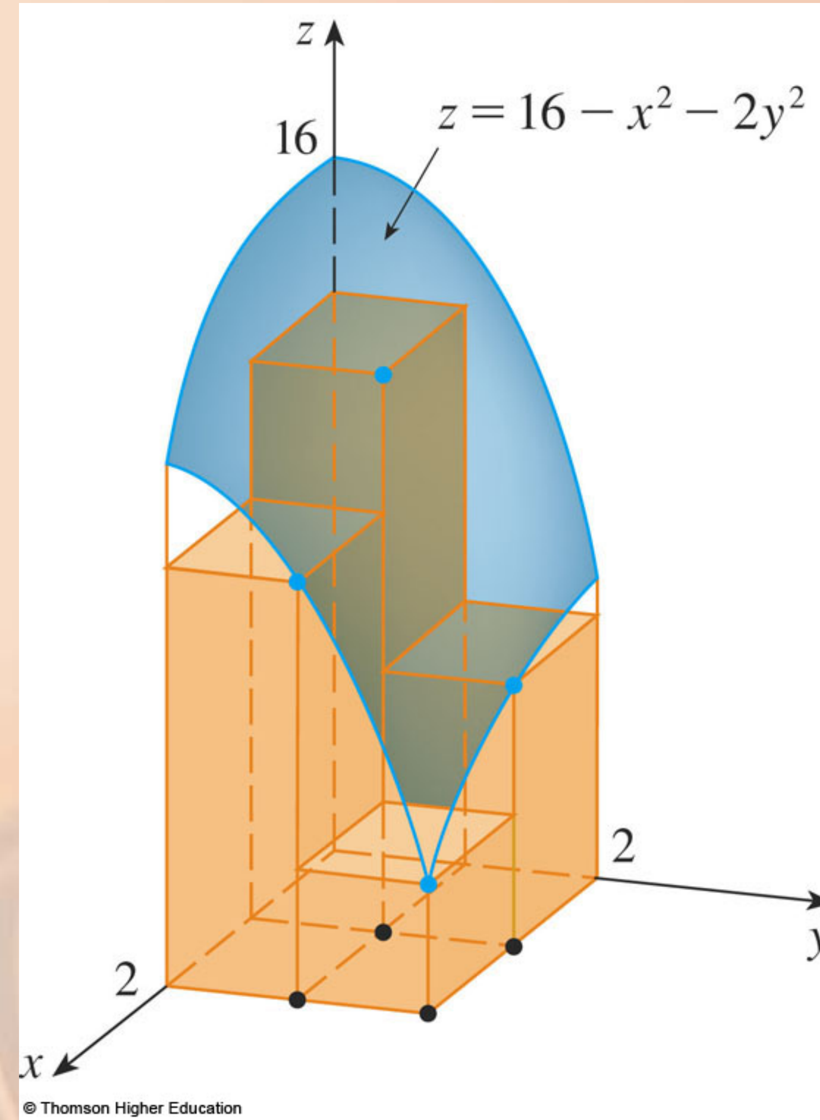
That is

the volume of

the approximating

rectangular boxes

shown here.

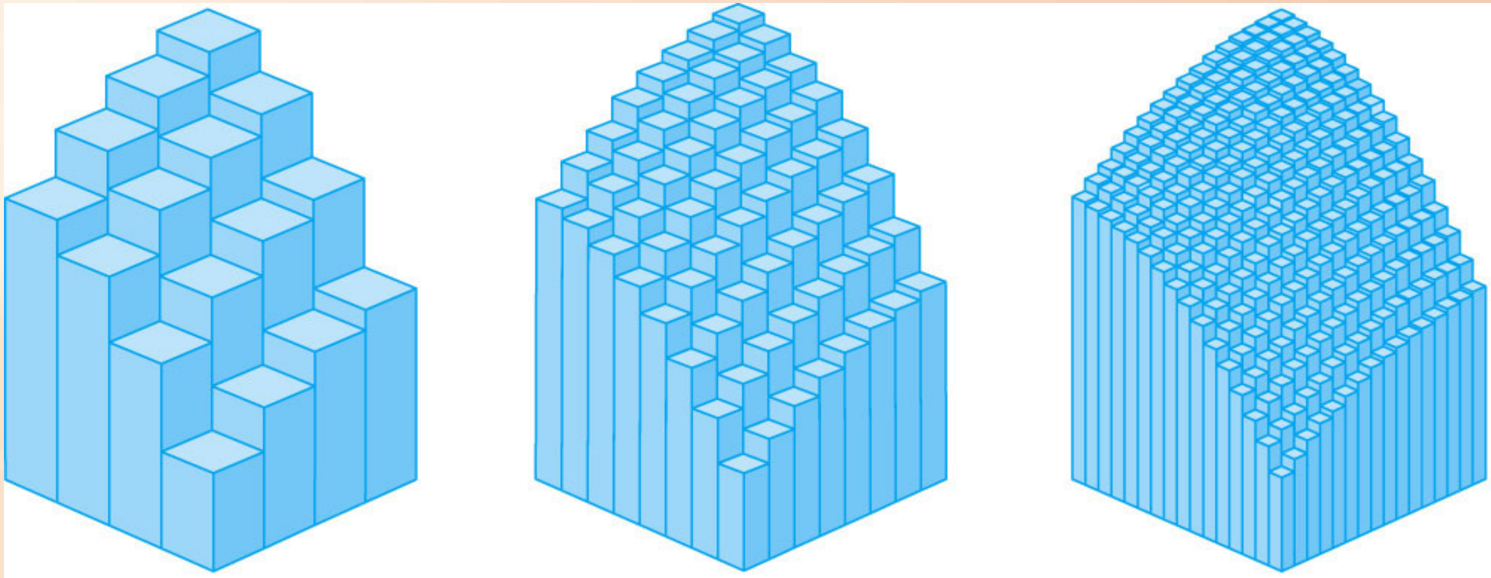


We get better approximations to the volume in Example 1 if we increase the number of squares.

DOUBLE INTEGRALS

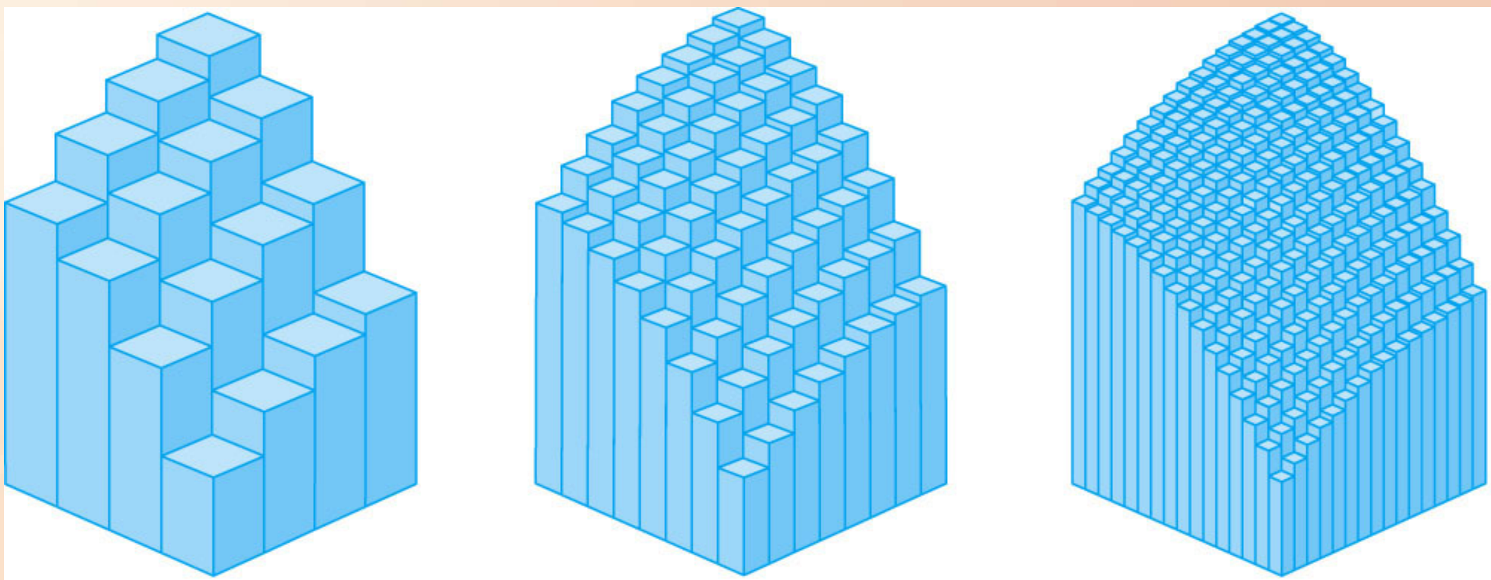
The figure shows how, when we use 16, 64, and 256 squares,

- The columns start to look more like the actual solid.
- The corresponding approximations get more accurate.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

In Section 15.2, we will be able to show that the exact volume is 48.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Example 2

If $R = \{(x, y) \mid -1 \le x \le 1, -2 \le y \le 2\}$, evaluate the integral

$$\iint_R \sqrt{1-x^2}\, dA$$

- It would be very difficult to evaluate this integral directly from Definition 5.
- However, since $\sqrt{1-x^2} \ge 0$, we can compute it by interpreting it as a volume.

Example 2

If $z = \sqrt{1 - x^2}$

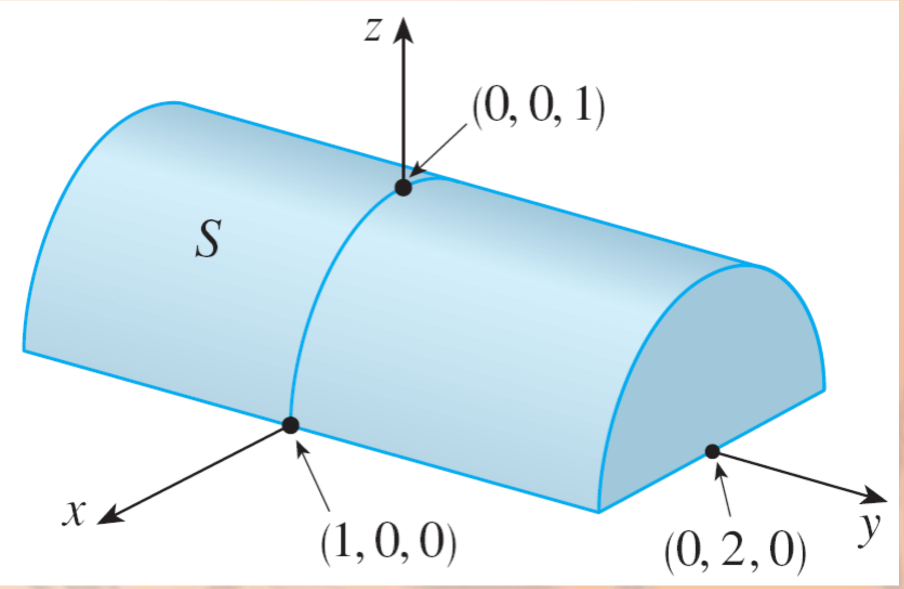
then

$$x^2 + z^2 = 1$$

$$z \geq 0$$

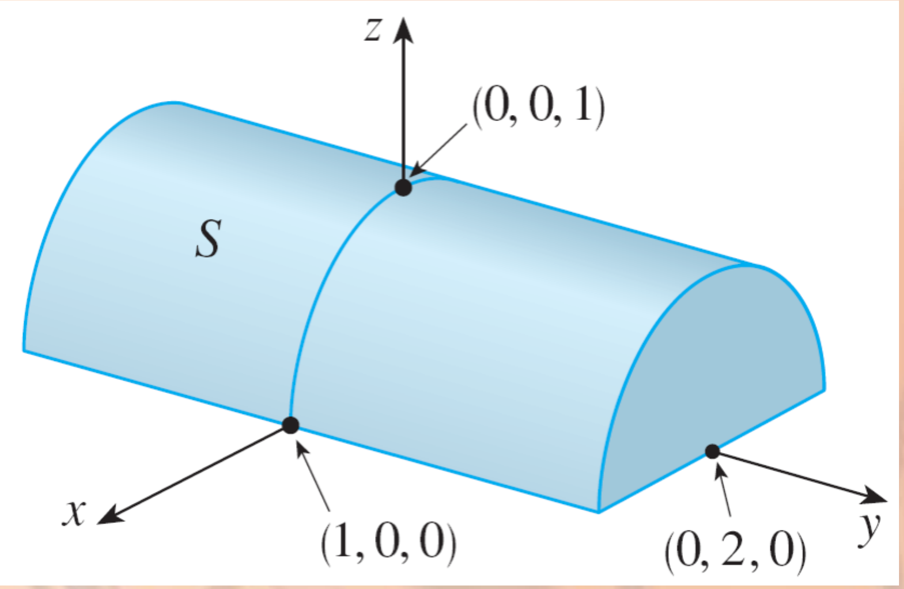
So, the given double integral represents the volume of the solid S that lies:

- Below the circular cylinder $x^2 + z^2 = 1$
- Above the rectangle R



The volume of *S* is the area of a semicircle with radius 1 times the length of the cylinder.

$$\iint_R \sqrt{1-x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$



NOTE

Remember that the interpretation of a double integral as a volume is valid only when the integrand *f* is a positive function.

NOTE

In the next class,

we will discuss how to interpret integrals

of functions that are not always positive

in terms of volumes.

AVERAGE VALUE

Recall from Section that the average value of a function *f* of one variable defined on an interval [*a*, *b*] is:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

AVERAGE VALUE

Similarly, we define the average value of a function *f* of two variables defined on a rectangle *R* to be:

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where *A*(*R*) is the area of *R*.

AVERAGE VALUE

If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{ave} = \iint_R f(x, y)\, dA$$

says that:

- The box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

PROPERTIES OF DOUBLE INTEGRALS

We now list three properties of double integrals.

- We assume that all the integrals exist.

PROPERTIES 7 & 8

These properties are referred to as the linearity of the integral.

$$\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

$$\iint_R cf(x,y)\, dA = c\iint_R f(x,y)\, dA$$

where *c* is a constant

PROPERTY 9

If $f(x, y) \geq g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$$