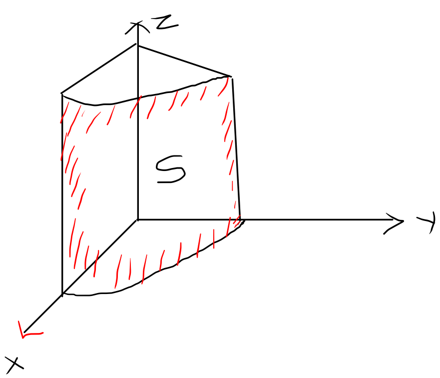
Pb: 1 Evaluate $\int_S \vec{F}\cdot\vec{N}\, ds$ where $\vec{F} = z\vec{i} + x\vec{j} + 3y^2 z\vec{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ to $z=5$

Soln:-



A parameterization for S is

$$X(s,t) = (\underbrace{4\cos s}_{x(s,t)}, \underbrace{4\sin s}_{y(s,t)}, \underbrace{t}_{z(s,t)}) \qquad 0 \le s \le \pi/2, \quad 0 \le t \le 5.$$

Normal vector

$$\vec{N}(s,t) = \frac{\partial(y,z)}{\partial(s,t)}\vec{i} - \frac{\partial(x,z)}{\partial(s,t)}\vec{j} + \frac{\partial(x,y)}{\partial(s,t)}\vec{k}$$

$$= \begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix}\vec{i} - \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}\vec{k}$$

$$= \begin{vmatrix} 4\cos s & 0 \\ 0 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} -4\sin s & 0 \\ 0 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} -4\sin s & 0 \\ 4\cos s & 0 \end{vmatrix}\vec{k}$$

$$= 4\cos s\,\vec{i} + 4\sin s\,\vec{j}.$$

$$\int_S \vec{F}\cdot\vec{N}\,dS = \int_0^1\int_0^{\pi/2} \vec{F}(x(s,t), y(s,t), z(s,t))\cdot\vec{N}(s,t)\,ds\,dt$$

$$= \int_0^1\int_0^{\pi/2} \vec{F}(4\cos s,\ 4\sin s,\ t)\cdot[4\cos s\ \vec{i} + 4\sin s\ \vec{j}]\,ds\,dt$$

$$= \int_0^1\int_0^{\pi/2} \left[\left[t\,\vec{i} + 4\cos s\,\vec{j} + 12\sin^2 s\, t\,\vec{k}\right]\right][4\cos s\,\vec{i} + 4\sin s\,\vec{j}]\,ds\,dt$$

$$= \int_0^1\int_0^{\pi/2} [4t\cos s + 16\cos s\sin s]\,ds\,dt$$

$$= \int_0^1\int_0^{\pi/2} [4t\cos s + 8\sin 2s]\,ds\,dt$$

$$= \int_0^1 [4t\sin s - 4\cos 2s]_0^{\pi/2}\,dt = \int_0^1 (4t+8)\,dt$$

$$= 90.$$

Greens Theorem:- If R is a closed region in XY plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

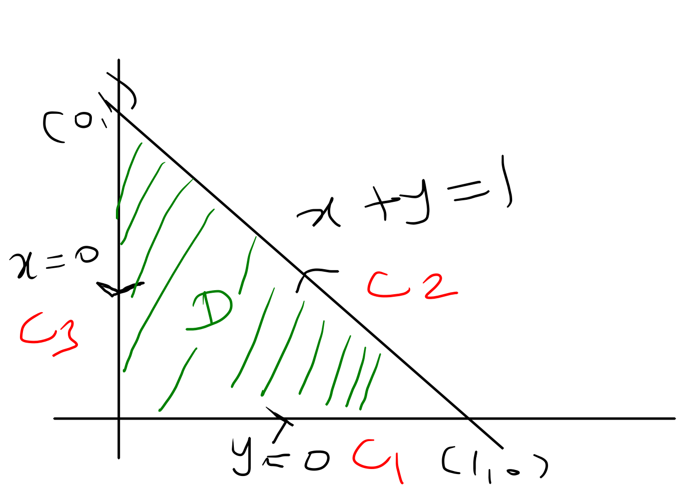
$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

where C is traversed in anticlockwise direction.

Pb:1 Verify Greens theorem for

$$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$ where C is the boundary of the region bounded by $x=0$, $y=0$; and $x+y=1$.

Soln:-



We can think of C is the union of C_1, C_2, C_3 where C_1 is the line segment joing $(0,0)$ to $(1,0)$

C_2 is the line segment joining $(1,0)$ to $(0,1)$

C_3 is the line segment joining $(0,1)$ to $(0,0)$.

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

We need to calculate $\int_C \underbrace{(3x^2-8y^2)}_{M}\,dx + \underbrace{(4y-6xy)}_{N}\,dy$

line integral along C_1 :- The parameterization of C_1 is

$$x = t \quad ; \quad y = 0 \qquad 0 \le t \le 1$$

$$\therefore dx = dt \qquad dy = 0$$

$$\therefore \int_{C_1} M\,dx + N\,dy = \int_0^1 3t^2\,dt = 3\left[t^3/3\right]_0^1 = 1.$$

line integral along C_2 :- The parameterization of C_2 is

$$x = t \;; \quad y = 1-t \qquad 1 \le t \le 0$$

$$\therefore dx = dt \;; \quad dy = -dt$$

$$\int_{C_2} M\,dx + N\,dy = \int_1^0 \left[3t^2 - 8(1-t)^2\right]dt + \left[4(1-t) - 6t(1-t)\right](-dt)$$

$$= \int_1^0 (-11t^2 + 26t - 12)\,dt = 8/3.$$

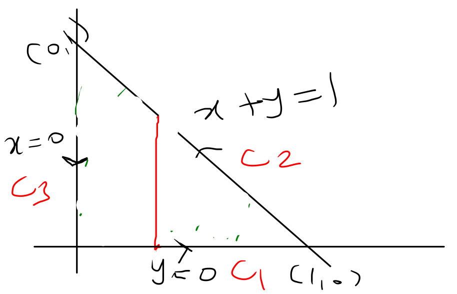
line integral along C_3 :-

$$y = t \;; \quad x = 0 \;; \qquad 1 \le t \le 0$$

$$dy = dt \qquad dx = 0$$

$$\therefore \int_{C_3} M\,dx + N\,dy = \int_1^0 4t\,dt = -2$$

$$\therefore \int_C M\,dx + N\,dy = 1 + 8/3 - 2 = 5/3.$$



$$\int_C \underbrace{(3x^2-8y^2)}_{M}\,dx + (\underbrace{4y-6xy}_{N})\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_D \left(-6y - (-16y)\right) dx\,dy = \iint_D 10y\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} 10y\,dx\,dy = \int_0^1 \left[5y^2\right]_0^{1-x} dx$$

$$= \int_0^1 5(1-x)^2\,dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= 5\left(0+\frac{1}{3}\right) = 5/3 \;.$$

$\therefore$ Green's theorem is verified.