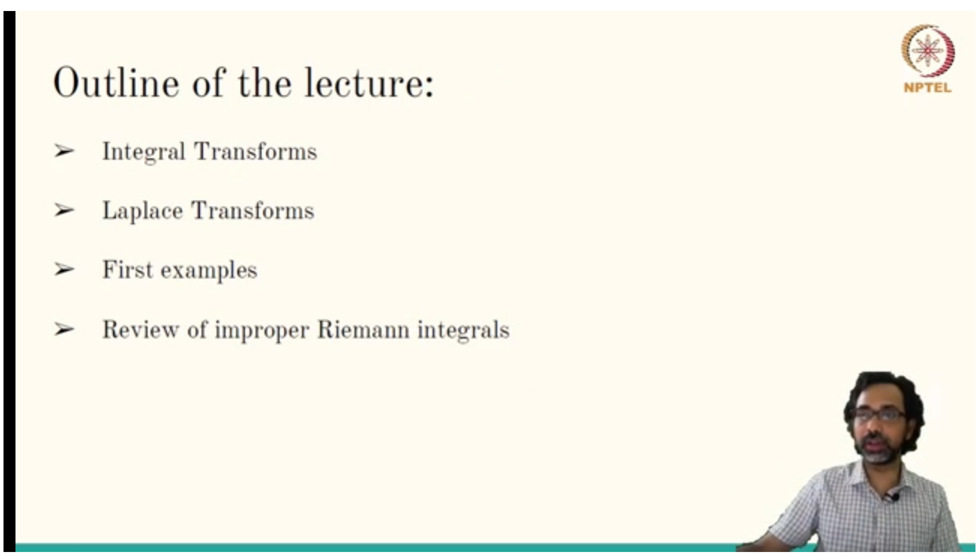
LAPLACE TRANSFORM

PROF. INDRAVA ROY

Lecture 1: Introduction and Motivation for Laplace Transforms – Part I

Welcome to the first lecture on Laplace Transforms on NPTEL. The origins of Laplace transform can be traced back to the work of Leonhard Euler, who used it to solve some differential equations but he did not develop the theory entirely. And it also appeared in the work of subsequent mathematicians like, Lagrange but it was Laplace who used really and put his teeth into the problem and develop the theory quite a lot.

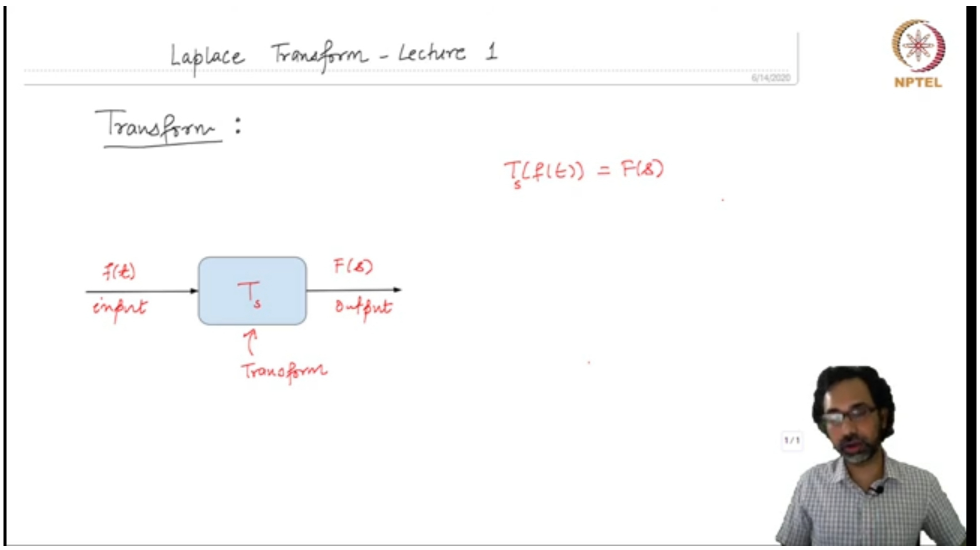
And as I said, it was used to not only solve differential equations but also other kinds of equations like, integral equations as well as difference or recurrence relations. So, let me give a brief outline of today's lecture. (Refer Slide Time: 01:06)



So today we will see the definition of integral transform, and next we will see that the Laplace transform is the special case of integral transforms, and then we will compute some first examples of Laplace transforms of some easy functions. And lastly, we will also review some

Department of Mathematics, The Institute of Mathematical Science, Chennai.

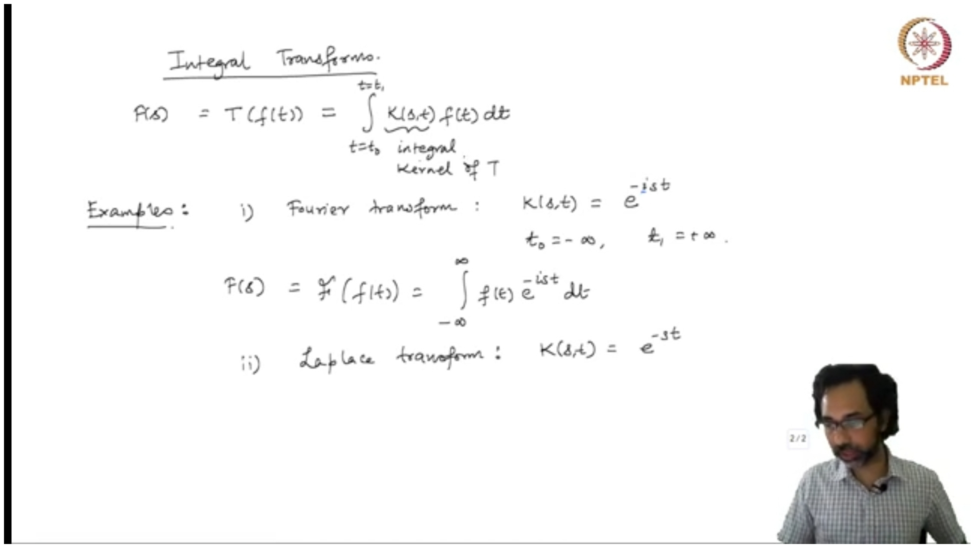
theoretical background on what are called improper Reimann integrals because the Laplace transform is defined as an improper Reimann integral, so it is useful to know what it is, and so we will see some basic facts about improper Reimann integrals. (Refer Slide Time: 01:49)



So, first let us see the definition of a transform before we talk about integral transform, let us see what are transforms. So, a mathematical transform can be described in pictorial form as a black box operation, well in this case it is a blue box. But let us suppose that on the left-hand side this is our input and the input go into this black box operation, whatever mathematical operation it describes, and you have some output, and the middle is the transform operation.

So, let me denote the transform by T and we suppose that in the input we get a function $f(t)$, so $f(t)$ is a function of the real parameter t and it this transform T takes $f(t)$ as an input. Now the output is another function which we shall denote by F with a different parameter s, so the transform changes the input or transforms the input $f(t)$ into this output $F(s)$. So, in mathematical formulation, we can write this as $T(f(t))$ and $T(f(t)) = F(s)$.

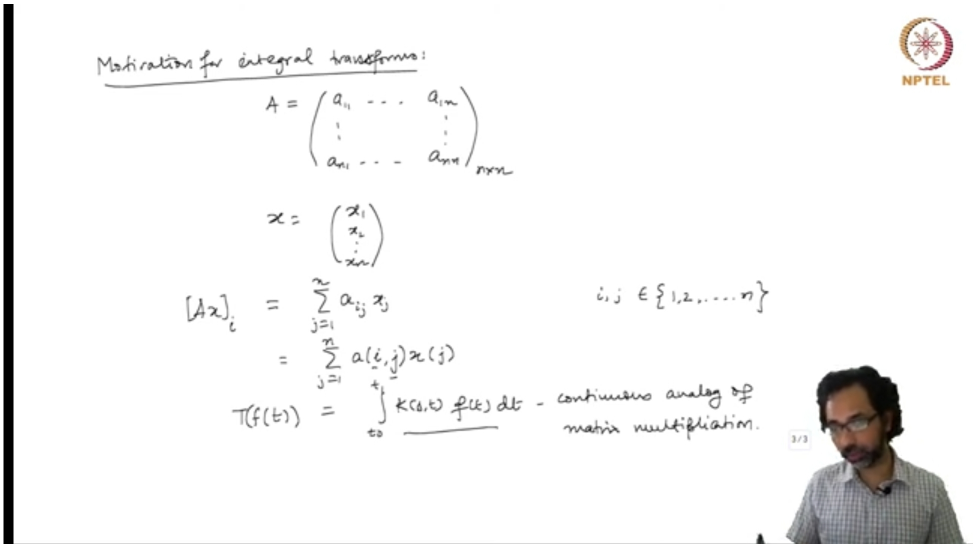
So, in mathematical formulation we can write this as an equation $T(f(t)) = F(s)$. So, it is obvious that this transform T should depend on this parameter s because it takes this function with variable t and outputs another function with the variable s. So, this is a layout for what a transform should do, replace a function and gives you another function. (Refer Slide Time: 04:38)



Now, what are integral transforms, so integral transforms are special case of transform. And in this case, $T(f(t))$ is given by an integral formula which is why it is called an integral transform and this is given by $T(f(t)) = \int_{t=t_0}^{t=t_1} K(s,t)f(t)dt$. So, this is an integral where this, this $K(s,t)$ is a function of two variables s and t and we are integrating against t, so what we are left with is a function of s. So, this is our $F(s)$ and it, this integration can be performed on the real line from the points t_0 to t_1. Now, you can allow t_1 or t_0 to be $+\infty$ or $-\infty$ respectively.

Now, let us see some examples of integral transforms. Just a terminology, this function $K(s,t)$ is known as an integral kernel of the integral transform of T. So, this has a special name, it is called a integral kernel of T. So, the first example can be given by the Fourier transform, this is a very well-known transform. In this case, the function $K(s,t) = e^{-ist}$ and $t_0 = -\infty$ and $t_1 = +\infty$. This is one of the transform. Here i is the complex number, $K(s,t) = e^{-ist}$ and the integration is from $-\infty$ to $+\infty$.

So, if we denote Fourier transform to be $\mathscr{F}$, then $F(s) = \mathscr{F}(f(t)) = \int_{-\infty}^{+\infty} f(t)e^{-ist}dt$. The second one is our Laplace transform and, in this case, our integral kernel $K(s,t) = e^{-st}$. So just note that, in the case of Fourier transform there is a complex number i but in the case of Laplace transform there is no i, that is the only difference. Now, let me say some words about this kind of formula, some motivation about how this formula can be thought of. (Refer Slide Time: 08:41)



Motivation for integral transforms: So, if we take an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & ...... & a_{1n} \\ . & .... & . \\ . & .... & . \\ . & .... & . \\ a_{n1} & ...... & a_{nn} \end{bmatrix} \tag{1}$$

And suppose that x is a column vector,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$$

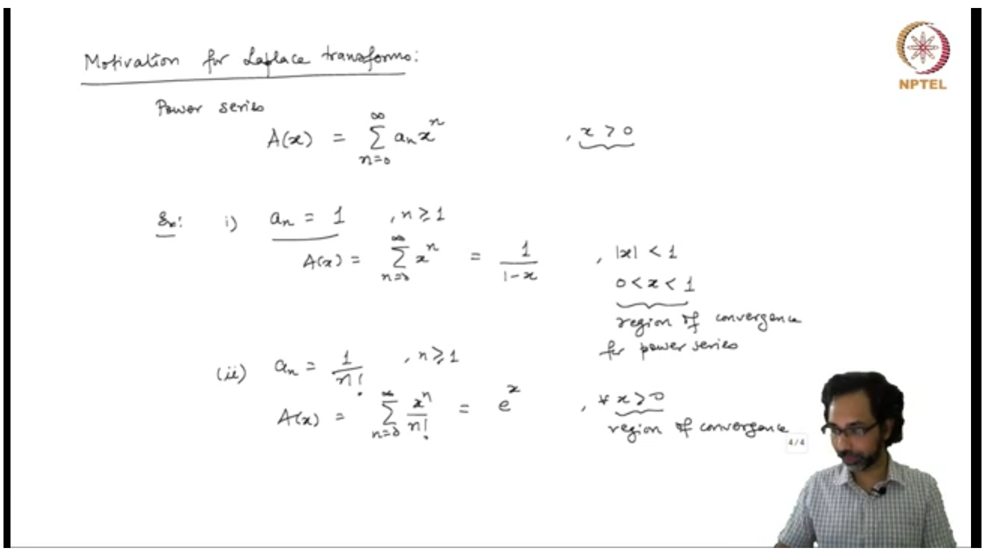
So, the what is the formula for Ax?, so Ax will be a column vector again, but I want to take the $i-$th component of this column vector. So, the formula is very easy, $[Ax]_i = \sum_{j=1}^{n} a_i j x_j$. So, we all know this formula is the matrix multiplication formula, but if you just rewrite this formula as follows

$$\begin{aligned} [Ax]_i &= \sum_{j=1}^{n} a_{ij} x_j, \ i, j \in \{1, 2, ..., n\} \\ &= \sum_{j=1}^{n} a(i, j) x(j) \end{aligned}$$

So, this is a function of two discrete variables and we can also think of this x as a function of this discrete variable j. So, now if you compare it with the formula for any integral transform $T(f(t)) = \int_{t_0}^{t_1} K(s,t)f(t)dt$. So we see very, these two formulas are quite similar.

The second one can be thought of as a continuous analog of the first one, meaning that this integral transform can be thought of as a continuous analog of matrix multiplication. This is just for an analogy and this kind of analogy will not help us in our computations, but it is just to relate this formula with something well known, which is the matrix multiplication formula.

(Refer Slide Time: 12:00)

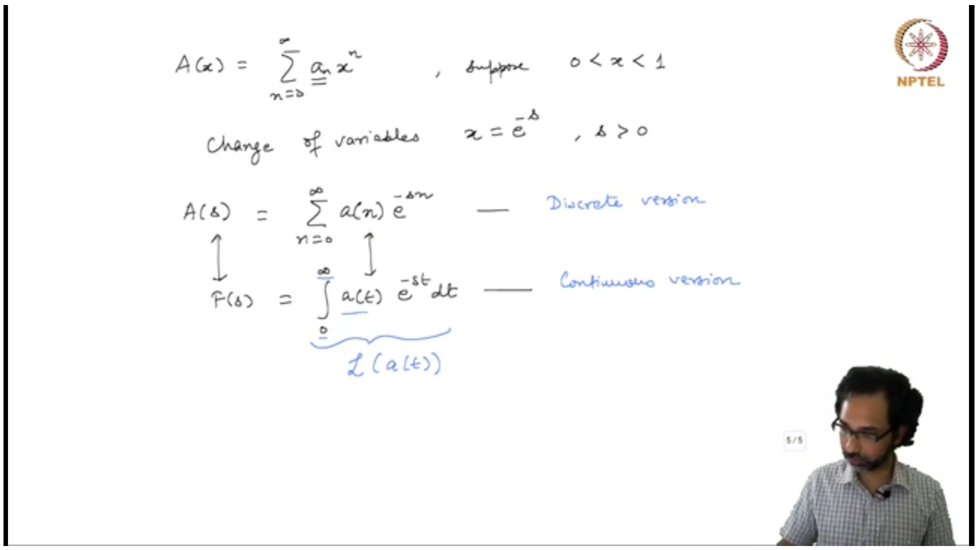


Now, let me talk about some motivation for the Laplace transform. Here, I am going to take what is called a power series and by a power series, I mean $\sum_{n=0}^{\infty} a_n x^n$, here x is a positive real number and a_n-s are arbitrary real numbers. So, this is a power series and usually when we have a such an infinite sum, we talk of the radius of convergence of the power series. For example, let us see an example of power series where all the a_n-s are 1. So, in this case, we have $Ax = \sum_{n=0}^{\infty} x^n$.

But now, this is just a geometric series and this converges to the function $\frac{1}{1-x}$ provided $|x| < 1$. So, since we have already taken x to be a positive number, so this means that x lies between 0 and 1 and so, this is our region of convergence for power series, given by $a_n = 1$ for all n. Now, we can also take $a_n = \frac{1}{n!}$. In this case, again we get something well known, $A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

And then this is are well-known exponential function, this is the Taylor series expansion of the exponential function and this is valid for all $x > 0$. So, in fact the entire positive real number is the region of convergence.

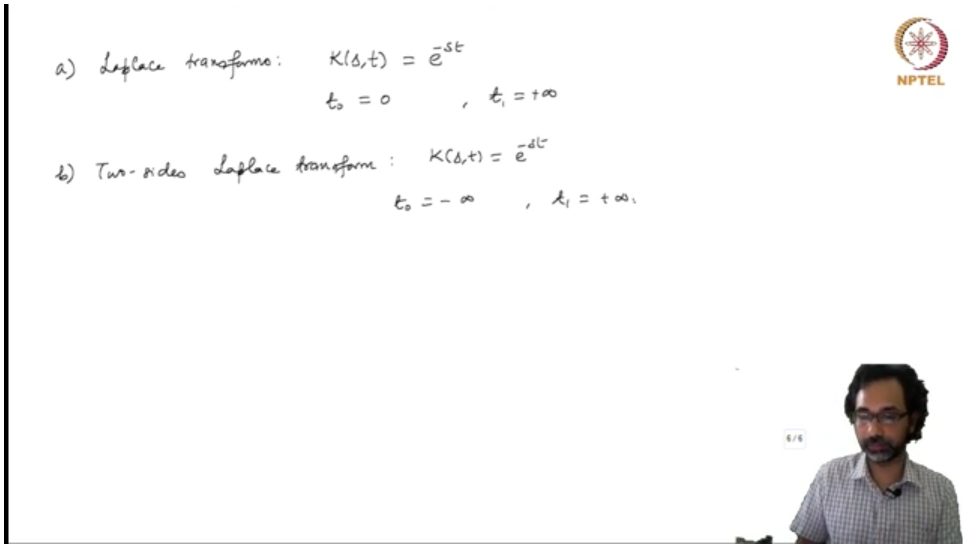
(Refer Slide Time: 14:55)



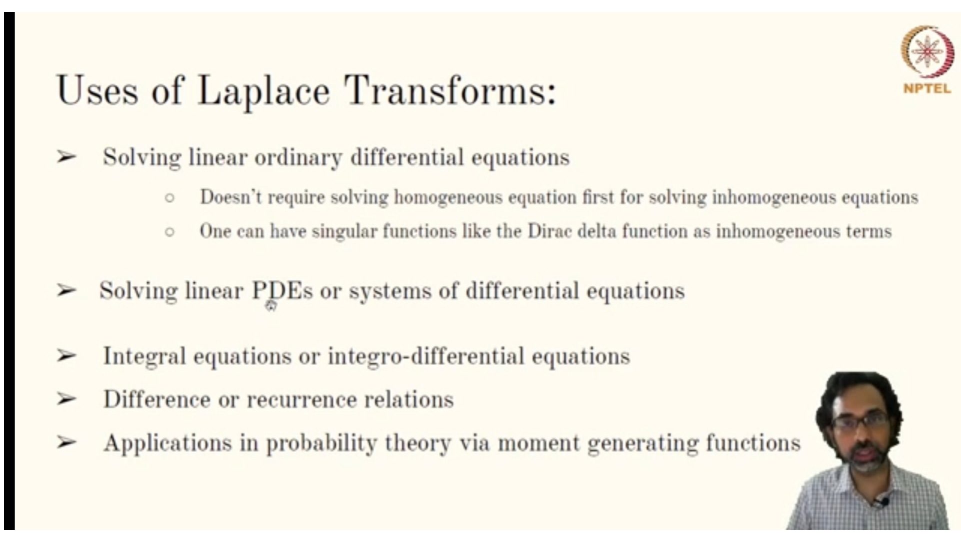
So, now that we know what a power series is and try to relate it to the definition of the Laplace transform. So, let me rewrite this formula for the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$, suppose, $0 < x < 1$. So, in many cases we have only a bounded radius of convergence, so I suppose that this region of convergence is between 0 and 1. Now in this region, I can make a change of variables by putting $x = e^{-s}$, where $s > 0$ and since $s > 0$, so e^{-s} is always lies between 0 and 1.

So, if we can write this power series now in terms of this change of variables, so we get $A(s) = \sum_{n=0}^{\infty} a(n)e^{-sn}$. Now, we are again in the familiar formula where this is a discrete version of the integral formula that we know before which was $F(s) = \int_0^{\infty} a(t)e^{-st}dt$. So here our f is just a, it's coefficient function and our A is, is just $F(s)$. So, pevious is the discrete version, and later is continuous version, and of course this is our Laplace transform of the function $a(t)$. So, sometimes we will also write it as $\mathscr{L}(a(t))$. So note that, here we have taken that units of integration from 0 to ∞, but we can also allow $-\infty$ to $+\infty$.

(Refer Slide Time: 18:06)



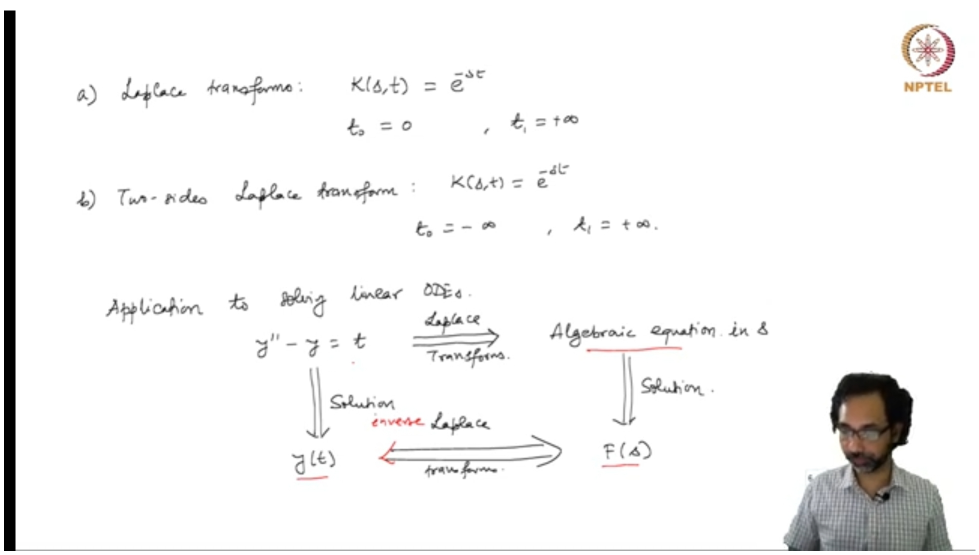
In this way, we get two versions of Laplace transforms; first one is the ordinary Laplace transform, where the integral kernel is given by e^{-st} and the limit of integration is taken from 0 to $+\infty$, and the second is so-called two-sided Laplace transform, where the integral kernel is again the same but now our limits of integration is from $-\infty$ to $+\infty$. (Refer Slide Time: 18:55)



Now, let me say some words about the uses of Laplace transforms. Laplace transforms can be used to solve ordinary, linear ordinary differential equations, and it has the advantage that you can directly use to solve such equations and it does not require solving homogeneous equation first, and then inhomogeneous equation, and so on. The other advantage is that one can have some singular functions like, the Dirac delta function which is not defined as a proper function as inhomogeneous terms meaning that the it can be taken as on the right-hand side of a ordinary, linear ordinary differential equation.

And one can still use Laplace transform techniques to solve such equations where other methods cannot or may not be used. So, in the same way it can also be used to solve linear partial differential equations or systems of differential equations. It can also be used to solve integral equations and integral differential equations, difference or recurrence relations, and it also has applications in probability theory via the so called moment generating functions.

Now except the last application, we will have the opportunity, we will have the opportunity to learn about all the rest of the applications. How to solve linear ordinary differential equations, partial differential equations, systems of differential equations as well as integral, integro differential or difference or recurrence relations as well. (Refer Slide Time: 20:37)



Now, let me say some words about this application to solving linear ODEs. In this case, if we have an ODE, let us say $y'' - y = t$, so this is a linear ordinary differential equation. Now, normally we will solve it using some method and we get the solution $y(t)$. Here, when we are using Laplace transforms first, we will transform this equation using Laplace transforms into an algebraic equation. So, we start with a differential equation and we get an algebraic equation.

So, this is of course much even easier to solve than, than a differential equation, so we can solve it. So we get, here an algebraic equation in the variable s and here you get some $F(s)$. The solution for this algebraic equation can be written as $F(s)$. And now this solution $y(t)$ and $F(s)$ are again related via Laplace transforms. So, if we know what kind of function gives you $F(s)$ as a Laplace transform, then we can invert this arrow, so this is called inverse Laplace transform, we can invert this arrow and from $F(s)$ we get back $y(t)$.

So, in this way where we start from a differential equation, get an algebraic equation, we solve the algebraic equation, and then get the so called inverse Laplace transform to get our solution to the original ordinary differential equation.