Page No.
Date : / /

Q) Find maximum and minimum distance of (3, 4, 12) subjected to sphere $x^2+y^2+z^2=4$.

Given, sphere $x^2+y^2+z^2=4$ with centre (0,0,0)

and distance from point $P(x,y,z)$ to $A(3,4,12)$ is

$$d^2 = (x-3)^2+(y-4)^2+(z-12)^2.$$

So objective function (ie to be ~~extremized~~ found maxima/minima is)

$$f(x,y) = d^2 = (x-3)^2+(y-4)^2+(z-12)^2$$

and the bounding function is

$$g(x,y,z) = x^2+y^2+z^2-4.$$

According to Lagrange's Multiplier method, extreme points lie when

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

i.e $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \lambda \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle$

$$\Rightarrow \langle 2(x-3), 2(y-4), 2(z-12) \rangle$$

$$= \lambda \langle 2x, 2y, 2z \rangle \quad — ①$$

Solving for corresponding values.

$$2(x-3) = 2\lambda x \quad — ②$$
$$2(y-4) = 2\lambda y \quad — ③$$
$$2(z-12) = 2\lambda z \quad — ④$$

and constraint $x^2 + y^2 + z^2 = 4$ — ⑤

solving ②,

$$x - 3 = \lambda x$$
$$x(1-\lambda) = 3$$
$$\therefore x = \frac{3}{1-\lambda} \quad — ⑥$$

Ily, $y - 4 = \lambda y$

$$y = \frac{4}{1-\lambda} \quad — ⑦$$

and $z = \dfrac{12}{1-\lambda}$ — ⑧

Page No.
Date : / /

put ⑥, ⑦, ⑧ in ⑤,

$$\frac{9}{(1-\lambda)^2} + \frac{16}{(1-\lambda)^2} + \frac{144}{(1-\lambda)^2} = 4$$

$$169 = 4(1-2\lambda+\lambda^2)$$

$$4\lambda^2 - 8\lambda + 4 - 169 = 0$$

$$4\lambda^2 - 8\lambda - 165 = 0 \quad - ⑨$$

Solving ⑨, $\lambda = \frac{15}{2}, -\frac{11}{2}$.

put $\lambda = 15/2$.

$$x = \frac{3}{1-\frac{15}{2}} = -\frac{6}{13}$$

$$y = \frac{4}{1-\frac{15}{2}} = -\frac{8}{13}.$$

$$z = \frac{12}{1-\frac{15}{2}} = -\frac{24}{13}.$$

Page No.
Date : / /

put $\lambda = -\frac{11}{2}$,

$$x = \frac{3}{1+11/2} = \frac{6}{13}$$

$$y = \frac{4}{1+11/2} = \frac{8}{13}$$

$$z = \frac{12}{1+11/2} = \frac{24}{13}.$$

Now, evaluating function at

$$f\left(-\frac{6}{13}, -\frac{8}{13}, -\frac{24}{13}\right) = \left(-\frac{6}{13}-3\right)^2 + \left(-\frac{8}{13}-4\right)^2 + \left(-\frac{24}{13}-12\right)^2$$

$$= 225.$$

$$f\left(\frac{6}{13}, \frac{8}{13}, \frac{24}{13}\right) = \left(\frac{6}{13}-3\right)^2 + \left(\frac{8}{13}-4\right)^2 + \left(\frac{24}{13}-12\right)^2$$

$$= 121$$

$\therefore$ Maximum distance $= 225$ at $\left(-\frac{6}{13}, -\frac{8}{13}, -\frac{24}{13}\right)$.

$\therefore$ Minimum distance $= 121$ at $\left(\frac{6}{13}, \frac{8}{13}, \frac{24}{13}\right)$.

Q) Find the point on the plane $x + 2y + 3z = 13$ closest to point $(1,1,1)$.

Given, ~~bouum~~ boundary condition $x + 2y + 3z = 13$.

given function is distance from $P(x,y,z)$ to $A(1,1,1)$

i.e $f(x,y,z) = d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2$

and boundary function $g(x,y,z) = x + 2y + 3z - 13$

Using Lagranges multiplier method

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

$$\Rightarrow \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \lambda \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle$$

~~So~~

$$\Rightarrow \langle 2(x-1), 2(y-1), 2(z-1) \rangle = \lambda \langle 1, 2, 3 \rangle$$

So,

$$2(x-1) = \lambda$$
$$2(y-1) = 2\lambda$$
$$2(z-1) = 3\lambda$$

Solving we get.

Page No.
Date : / /

$$\therefore x = 1 + \frac{\lambda}{2}$$

$$y = 1 + \lambda$$

$$z = 1 + \frac{3\lambda}{2}$$

Given boundary condition

$$g(x,y,z) = x + 2y + 3z - 13 = 0 \text{ (let)}$$

$$\Rightarrow \quad x + 2y + 3z = 13$$

$$\left(1 + \frac{\lambda}{2}\right) + 2(1+\lambda) + 3\left(1 + \frac{3\lambda}{2}\right) = 13$$

$$\frac{2+\lambda}{2} + 2 + 2\lambda + 3 + \frac{9\lambda}{2} = 13$$

$$\frac{2 + \lambda + 9\lambda + 10 + 4\lambda}{2} = 13$$

$$\cancel{12 + 11\lambda = 26}$$

$$\cancel{11\lambda = 14}$$

$$\cancel{\lambda = \frac{14}{11}}$$

$$12 + 14\lambda = 26$$

$$14\lambda = 14$$

$$\therefore \lambda = 1$$

Page No.
Date : / /

$\therefore x = 1 + \frac{\lambda}{2} = \frac{3}{2}$

Ily $y = 1 + \lambda = 2$

$z = 1 + \frac{3\lambda}{2} = \frac{5}{2}$.

$\therefore$ point $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ lying on plane is closest to $(1,1,1)$.

Page No.
Date : / /

Q) Find the farthest distance from point $(1,1,1)$ connected to sphere $x^2+y^2+z^2=4$

Given, boundary condn $x^2+y^2+z^2=4$

and boundary function $g(x,y,z) = x^2+y^2+z^2-4$.

We have to maximise distance, so function is

$f(x,y,z) = d^2 = (x-1)^2+(y-1)^2+(z-1)^2$

Using Lagranges multiplier

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \lambda \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle$$

$$\Rightarrow \left\langle 2(x-1), 2(y-1), 2(z-1) \right\rangle = \lambda \left\langle 2x, 2y, 2z \right\rangle$$

Solving, we get

$\cancel{2}(x-1) = \cancel{2}\lambda x$

$\Rightarrow$ ~~$x = 1+\lambda$~~ $\quad x - \lambda x = 1$

$\therefore x = \dfrac{1}{1-\lambda}$.

~~Illy, $2(y-1) = 2\lambda$~~

~~$\therefore y = 1+\lambda$.~~

Illy, $y-1 = y\lambda \Rightarrow y(1-\lambda) = 1$

$\therefore y = \dfrac{1}{1-\lambda}$.

Date : / /

and $(z-1) = \lambda z$
$\Rightarrow z - \lambda z = 1$
$\therefore z = \dfrac{1}{1-\lambda}$

put x, y, z in terms of λ in $g(x, y, z) = 0$

i.e. $x^2 + y^2 + z^2 - 4 = 0$

$$\frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 4$$

$\therefore 1 = 4(1-\lambda)^2$
$\Rightarrow (1-\lambda)^2 = \dfrac{1}{4}$

$1 - \lambda = \dfrac{1}{2}, -\dfrac{1}{2}$

$\therefore \lambda = \dfrac{1}{2}, \dfrac{3}{2}.$

if $\lambda = 1/2$,

$$(x, y, z) = \left(\frac{1}{1-1/2}, \frac{1}{1-1/2}, \frac{1}{1-1/2} \right)$$
$$= (2, 2, 2).$$

ii) $\lambda = 3/2$

$$(x, y, z) = \left(\frac{1}{1-3/2}, \frac{1}{1-3/2}, \frac{1}{1-3/2}\right)$$

$$= (-2, -2, -2).$$

Now

$$f(2,2,2) = (2-1)^2 + (2-1)^2 + (2-1)^2$$
$$= 3$$

$$f(-2,-2,-2) = (-2-1)^2 + (-2-1)^2 + (-2-1)^2$$
$$= 3 \times 9$$
$$= 27$$

$\therefore$ $(-2,-2,-2)$ is the farthest point from $(1,1,1)$ connected to $x^2 + y^2 + z^2 = 4$.