Jacobian Matrix

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Suppose: $u(x,y)$ & $v(x,y)$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = J(u,v) = J(u(x,y), v(x,y))$$

$u(x,y,z)$ & $v(x,y,z)$, $w(x,y,z)$

$$J(u,v,w) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Same: $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Properties:

$$J = \frac{\partial(u,v)}{\partial(x,y)} ; \quad J' = \frac{\partial(x,y)}{\partial(u,v)}$$

$$\boxed{J \cdot J' = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)}}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial u}{\partial u} = 1, \Rightarrow \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \boxed{\frac{\partial u}{\partial y}} \cdot \frac{\partial y}{\partial u}$$

$u(x,y)$

Property-2

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)}$$

$$u(x,y) = u(y,x)$$

$$u(x,y) = x + y \text{ (or) } \sin(x+y) \quad \alpha$$

$$\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{vmatrix} \quad \begin{matrix} u = (x,y) \\ v = (x,y) \\ u = (y,x) \\ v = (y,x) \end{matrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} ; \quad \frac{\partial(x,y)}{\partial(u,v)}$$

$$\Downarrow$$

$u(x,y)$ $\qquad x(u,v)$

Problem:

$x = r\cos\theta$; $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \qquad \frac{\partial y}{\partial r} =$$

$$\frac{\partial x}{\partial \theta} = \qquad \frac{\partial y}{\partial \theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial} & \end{vmatrix} \checkmark$$

Problem:

$x = r\cos\theta, \ y = r\sin\theta$.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta$$

$$= r$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

$$\boxed{\begin{array}{l} x = \rho\cos\phi, \ y = \rho\sin\phi \\ z = z \end{array}} \rightarrow \text{Given.}$$

$$\left.\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\right\} \text{To find}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} \rightarrow \text{Jacobian}$$

$$\boxed{\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho} \ \text{answer}$$

Problem:

Problem:

$$\boxed{\begin{array}{l} u = x^2 - y^2, \quad v = 2xy \\ x = r\cos\theta, \quad y = r\sin\theta \end{array}}$$

Given

To find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Formula by chain rule:

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3} \longrightarrow$$ what are the $\boxed{\text{independent}}$

Problem:

$$u = xyz, \quad v = x^2 + y^2 + z^2$$

$$w = x + y + z$$

$u - (xyz) = 0 \rightarrow f_1$

$v - (x^2+y^2+z^2) = 0$

$w - (x+y+z) = 0$

find $\frac{\partial(x,y,z)}{\partial(u,v,w)} \Rightarrow \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$

$$f(x,y,z,u,v,w) = 0 \qquad f(x,y) = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\ldots}$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= \frac{-1}{2(x-y)(y-z)(z-x)}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \cdot \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$