① (a) $\phi \nabla \psi$ ist ein C^1-Vektorfeld. Es gilt mit Theorem 22.10

$$\int_A (\phi \Delta \psi + \langle \nabla \phi, \nabla \psi \rangle)\, dx \underset{\text{Kettenregel}}{=} \int_A \operatorname{div}(\phi \nabla \psi)\, dx$$

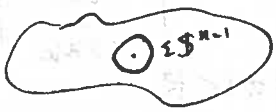
$$\underset{\text{Gauß}}{=} \int_{\partial A} \langle \phi \nabla \psi, \nu \rangle\, d\sigma = \int_{\partial A} \phi \cdot \underbrace{\langle \nabla \psi, \nu \rangle}_{= \frac{\partial \psi}{\partial \nu}}\, d\sigma .$$

(b) $$\int_A (\phi \Delta \psi - \Delta \phi \cdot \psi)\, dx = \int_A (\phi \Delta \psi + \langle \nabla \phi, \nabla \psi \rangle)\, dx$$

$$- \int_A (\psi \Delta \phi + \langle \nabla \psi, \nabla \phi \rangle)\, dx \underset{(a)}{=} \int_{\partial A} \phi \frac{\partial \psi}{\partial \nu}\, d\sigma - \int_{\partial A} \psi \frac{\partial \phi}{\partial \nu}\, d\sigma$$

$$= \int_{\partial A} \left(\phi \frac{\partial \psi}{\partial \nu} - \frac{\partial \phi}{\partial \nu} \cdot \psi \right) d\sigma .$$

② Es gilt $\cos \alpha(x) = \dfrac{\langle x, \nu(x) \rangle}{\|x\|}$. Ist $\varepsilon > 0$ so klein, dass $\overline{U_\varepsilon(0)} \subset A^\circ$, so ist A_ε ein Greenscher Bereich mit $\partial_{reg} A_\varepsilon = \partial_{reg} A \cup S^{n-1}(\varepsilon)$, wobei $\nu(x) = -x \cdot \frac{1}{\varepsilon}$ für $x \in S^{n-1}(\varepsilon)$. Mit Theorem 22.10



A_ε

folgt $\quad \left(F(x) = \frac{x}{\|x\|^n} \right)$

$$0 = \int_{A_\varepsilon} \operatorname{div} F(x) \underset{\text{Gauß}}{=} \int_{\partial A_\varepsilon} \langle F, \nu \rangle \, d\sigma =$$

$$= \int_{\partial A} \frac{\langle x, \nu(x) \rangle}{\|x\|^n} \, d\sigma(x) - \frac{1}{\varepsilon} \int_{S^{n-1}(\varepsilon)} \frac{\langle x, x \rangle}{\|x\|^n} \, d\sigma(x)$$

$$= \int_{\partial A} \frac{\cos\alpha(x)}{\|x\|^{n-1}} \, d\sigma(x) - \underbrace{\frac{1}{\varepsilon^{n-1}} A_{n-1}(S^{n-1}(\varepsilon))}_{\underset{\text{Satz 22.8}}{=} A_{n-1}(S^{n-1}) = \omega_n},$$

denn $\quad \operatorname{div} F(x) = \sum_{j=1}^{n} \left(\frac{1}{\|x\|^n} - \frac{x_k \cdot \frac{n}{2} \cdot (x_1^2 + \dots + x_n^2)^{\frac{n-2}{2}} \cdot 2x_k}{\|x\|^{2n}} \right.$

$$= \frac{1}{\|x\|^{n+2}} \sum_{j=1}^{n} \left(\|x\|^2 - n \cdot x_k^2 \right) = 0$$

für alle $\quad x \in \mathbb{R}^n, \ x \neq 0. \quad$ Somit

$$\int_{\partial A} \frac{\cos\alpha(x)}{\|x\|^{n-1}} \, d\sigma(x) = \omega_n .$$

③ $\quad$ Sei $\quad M_k = \sup_{x \in \mathbb{R}^n} \|x\|^{n+\delta} \left| \frac{\partial f_k(x)}{\partial x_k} \right| < \infty .$

Es gilt $\quad \forall R > 0$:

$$\int_{1 \leq \|x\| < R} |\operatorname{div} F(x)| \, dx \leq \sum_{k=1}^{n} \int_{1 \leq \|x\| < R} \left| \frac{\partial f_k(x)}{\partial x_k} \right| dx$$

$$(③) \quad \leq \left(\sum_{k=1}^{n} M_k\right) \cdot \underbrace{\int_{1 \leq \|x\| < R} \frac{1}{\|x\|^{n+\delta}}\, dx}$$

$$\underset{\text{Satz 22.11}}{=} \omega_n \cdot \int_{1}^{R} \frac{dr}{r^{1+\delta}}$$

Da $\lim_{R \to \infty} \int_1^R \frac{dr}{r^{1+\delta}}$ existiert $(< \infty)$, existiert nach dem Majorantenkriterium das Integral $\int_{\mathbb{R}^n} |\operatorname{div} F(x)|\, dx$, d.h. $\int_{\mathbb{R}^n} \operatorname{div} F(x)\, dx$ konvergiert absolut. Definiere $m_k(R) = \sup_{\|x\| = R} \|x\|^{n-1} |f_k(x)|$

Nun

$$\int_{\mathbb{R}^n} \operatorname{div} F(x)\, dx = \lim_{R \to \infty} \int_{\|x\| < R} \operatorname{div} F(x)\, dx =$$

$$\underset{\text{Gauß}}{=} \lim_{R \to \infty} \int_{\|x\| = R} \langle F, \nu(x) \rangle\, d\sigma(x) =: (*)$$

Es gilt für $x \in \partial B_R(0)$: $\nu(x) = \frac{x}{R}$, also

$$\left| \int_{\|x\| = R} \langle F(x), \nu(x) \rangle\, d\sigma(x) \right| \leq \frac{1}{R} \sum_{j=1}^{n} \int_{\|x\| = R} \overbrace{|x_k f_k(x)|}^{\leq \|x\| \cdot |f_k(x)|}\, d\sigma(x)$$

$$\leq \frac{1}{R} \cdot \left(\sum_{j=1}^{n} m_j(R)\right) \cdot \underbrace{\int_{\|x\| = R} \frac{1}{\|x\|^{n-2}}\, d\sigma(x)}$$

$$= \frac{1}{R^{n-2}} A_{n-1}(S^{n-1}(R))$$

$$\underset{\text{Satz 22.8}}{=} R \cdot \omega_n$$

$$\leq \quad \omega_n \cdot \sum_{j=1}^{n} m_j(R) \longrightarrow 0 \quad (R \to \infty) \text{ nach Voraus-}$$

setzung. Damit gilt $\int_{\mathbb{R}^n} \operatorname{div} F(x)\, dx = (*) = 0$.

(4) Es gilt $E(x) = f(r) \cdot \frac{x}{r} \qquad (r = \|x\|)$

$$Q(r) = \int_{B_r(0)} \rho(x)\, dx$$

$$= \int_{B_r(0)} \operatorname{div} E(x)\, dx = \frac{1}{r} \int_{S^2(r)} \langle E(x), x \rangle\, d\sigma(x)$$

$$= \int_{S^2(r)} f(\|x\|)\, d\sigma(x) \underset{\text{Satz 22.8}}{=} f(r) \cdot \omega_3 \cdot r^2$$

Da $\omega_3 = 4\pi$, folgt $f(r) = \dfrac{Q(r)}{4\pi r^2}$.

(a) Für $r \leq R - \varepsilon$ ist $\rho(x) = 0$ und $Q(r) \equiv 0$,

also $E(x) = f(r) \cdot \frac{x}{r} = 0$

(b) Für $r \geq R + \varepsilon$ ist $Q(r) \equiv Q$, also

$$E(x) = f(r) \frac{x}{r} = \frac{Q}{4\pi r^2} \cdot \frac{x}{r}.$$