①

(a) $L = \lim_{n \to \infty} \sqrt[n]{(1/3)^n} = 1/3$, $R = \frac{1}{L} = 3$

(b) $\sqrt[n]{3^n/n} = 3/\sqrt[n]{n} \longrightarrow 3/1 = 3 =: L$, $R = \frac{1}{L} = 1/3$

(c) $$\frac{6^n}{2^{n+1} + 5^n} \cdot \frac{2^{n+2} + 5^{n+1}}{6^{n+1}} = \frac{1}{6} \cdot \frac{4 \cdot 2^n + 5 \cdot 5^n}{2 \cdot 2^n + 5^n} = \frac{1}{6} \cdot \frac{4 \cdot (\frac{2}{5})^n + 5}{2 \cdot (\frac{2}{5})^n + 1}$$

$$\longrightarrow \frac{1}{6} \cdot \frac{0+5}{0+1} = \frac{5}{6} =: L, \quad R = \frac{1}{L} = 6/5$$

(d) $$\sum_{n=0}^{\infty} 4^n z^{2n} = \sum_{n=0}^{\infty} 2^{2n} z^{2n}$$

$$\sqrt[2n]{2^{2n}} = 2 =: L \quad , R = \frac{1}{L} = 1/2 .$$

$$\sqrt[2n+1]{0} = 0$$ ↖ größten HP

(e) $$\underbrace{\frac{(n+1)^{n+1}}{(n+1)!}}_{=(n+1)(n+1)^n} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} \cdot \frac{\cancel{n!}}{\cancel{n!}}$$

$$= \left(1 + \frac{1}{n}\right)^n \longrightarrow e =: L, \quad R = \frac{1}{L} = \frac{1}{e}$$

(f) $$\frac{(2n+1)^{2n+1}}{2^{2n+2}(2n+2)!} \cdot \frac{2^{2n}(2n)!}{(2n-1)^{2n-1}} = \frac{(2\cancel{n+1})(2n) \cdot (\cancel{2n})!}{4(2n+2)(2\cancel{n+1}) \cdot (\cancel{2n})!}$$

$$\cdot \left(\frac{2n+1}{2n-1}\right)^{2n} = \frac{1}{4} \cdot \underbrace{\left(\frac{n}{n+1}\right)}_{\to 1} \cdot \underbrace{\left(\frac{2n+1}{2n-1}\right)}_{\to 1} \cdot \underbrace{\left(1 + \frac{2}{2n-1}\right)^{2n-1}}_{\to e^2}$$

$$\longrightarrow \frac{e^2}{4} =: L, \quad R = \frac{1}{L} = \frac{4}{e^2}$$

(g) $$\sqrt[n]{\left(\frac{n+1}{n}\right)^{n^2}} = \left(1 + \frac{1}{n}\right)^n \longrightarrow e =: L, \quad R = \frac{1}{\cancel{L}} = \frac{1}{e}$$

2 (a) $$\sum_{n=1}^{\infty} n z^n = z \sum_{n=1}^{\infty} n z^{n-1} = z \frac{\partial}{\partial z} \frac{1}{1-z} = \frac{z}{(1-z)^2}$$

(b) $$\sum_{n=1}^{\infty} n^2 z^n = \sum_{n=1}^{\infty} n(n+1) z^n - \sum_{n=1}^{\infty} n z^n = z \sum_{n=0}^{\infty} (n+1)(n+2) z^n - \frac{z}{(1-z)^2}$$

$$= z \left(\frac{\partial}{\partial z}\right)^2 \frac{z^2}{1-z} - \frac{z}{(1-z)^2} = + \frac{2z}{(1-z)^3} - \frac{z}{(1-z)^2}$$

$$= \frac{z(1+z)}{(1-z)^3}$$

(c) $$\sum_{n=1}^{\infty} n^3 z^n = \sum_{n=1}^{\infty} n(n+1)(n+2) z^n - 3 \sum_{n=1}^{\infty} n^2 z^n - 2 \sum_{n=1}^{\infty} n z^n$$

$$= z \left(\frac{\partial}{\partial z}\right)^3 \frac{z^3}{1-z} - 3 \frac{z(z+1)}{(1-z)^3} - 2 \frac{z}{(1-z)^2}$$

$$= \frac{z\left(6 + 3(z+1)(1-z) - 2(1-z)^2\right)}{(1-z)^4}$$

$$= \frac{z(1+4z+z^2)}{(1-z)^4}.$$

Alle Reihen konvergieren absolut und lok.-glm auf $U_1(0)$, denn $\sqrt[n]{n} \to 1$, $\sqrt[n]{n^2} \to 1$, $\sqrt[n]{n^3} \to 1$.

③

<u>Direkte Lösung</u>:

$$\frac{y'}{y} = \frac{x}{1-x} = \frac{1}{1-x} - 1$$

$$y = e^{-\log(1-x) - x} + C = \frac{1}{1-x} e^{-x} + C$$

$$y(0) = e^{-0-0} + C = 1 + C \overset{!}{=} 1 \Rightarrow C = 0.$$

<u>Potenzreihenansatz</u> $\quad y(x) = \sum_{k=0}^{\infty} y_k x^k \;, \; y(0) = y_0 = 1.$

DGL: $$0 = y' - \frac{x}{1-x} y = \sum_{k=0}^{\infty} (k+1) y_{k+1} x^k - x \sum_{k=0}^{\infty} x^k \cdot \sum_{k=0}^{\infty} y_k x^k$$

$$= \sum_{k=0}^{\infty} (k+1) y_{k+1} x^k - x \cdot \sum_{k=0}^{\infty} \Big(\sum_{\ell=0}^{k} y_\ell \Big) x^k$$

$$= \sum_{k=0}^{\infty} \Big((k+1) y_{k+1} - \sum_{\ell=0}^{k-1} y_\ell \Big) x^k$$

$$\Rightarrow \quad y_{k+1} = \frac{1}{k+1} \sum_{\ell=0}^{k-1} y_\ell \qquad (\forall k \geq 0)$$

Lösung entwickeln: $$\frac{1}{1-x} e^{-x} = \sum_{k=0}^{\infty} x^k \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = \sum_{k=0}^{\infty} \Big(\sum_{\ell=0}^{k} \frac{(-1)^\ell}{\ell!} \Big) x^k$$