① In allen Fällen ist Grad $p = 0$ und Grad $q \geq 2$.

(a) $x^6+1 = (x^3+i)(x^3-i)$

$= (x-i)(x-e^{i\pi 7/6})(x-e^{i\pi 11/6})$

$\cdot (x+i)(x+e^{i\pi 7/6})(x+e^{i\pi 11/6})$

Singularitäten in der oberen HE:

$i \quad , \quad -e^{i\pi 7/6} = e^{i\pi/6} \quad , \quad -e^{i\pi 11/6} = e^{i\pi 5/6}$

$$\operatorname*{Res}_{z=i} \frac{1}{x^6+1} = \left.\frac{1}{6x^5}\right|_{x=i} = \frac{1}{6i}$$

$$\operatorname*{Res}_{z=e^{i\pi/6}} \frac{1}{z^6+1} = \frac{1}{6e^{i\pi 5/6}}$$

$$\operatorname*{Res}_{z=e^{i\pi 5/6}} \frac{1}{z^6+1} = \frac{1}{6e^{i\pi 25/6}} = \frac{1}{6e^{i\pi/6}}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} \underset{i = e^{i\pi/2}}{=} \frac{\pi}{3}\left(1 + e^{i\pi\left(\frac{1}{2}-\frac{5}{6}\right)} + e^{i\pi\left(\frac{1}{2}-\frac{1}{6}\right)}\right)$$

$$= \frac{\pi}{3}\left(1 + e^{-i\pi/3} + e^{i\pi/3}\right)$$

$$= \frac{\pi}{3} + \frac{2\pi}{3}\cos\frac{\pi}{3} \underset{\cos\frac{\pi}{3} = \frac{1}{2}}{=} \frac{2\pi}{3}.$$

(b) $z^4+1 = (z^2+i)(z^2-i)$

$$= \left(z+\frac{1-i}{\sqrt{2}}\right)\left(z-\frac{1-i}{\sqrt{2}}\right)\left(z+\frac{1+i}{\sqrt{2}}\right)\left(z-\frac{1+i}{\sqrt{2}}\right)$$

Nullstellen in der oberen HE: $\pm \frac{1\pm i}{\sqrt{2}}$

$$\operatorname*{Res}_{z=\pm\frac{1\pm i}{\sqrt{2}}} \frac{1}{z^4+1} = \frac{1}{4z^3}\Big|_{z=\pm\frac{1\pm i}{\sqrt{2}}} = \frac{\sqrt{2}}{4i(1\pm i)}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}\left(\frac{1}{1+i}+\frac{1}{1-i}\right)$$

$$= \frac{\pi}{2\sqrt{2}}(1-i+1+i) = \frac{\pi}{\sqrt{2}} = \frac{\pi\sqrt{2}}{2}.$$

(c) $(z^2+1)(z^2+4)^2 = (z+i)(z-i)(z-2i)^2(z+2i)^2$

Nullstellen in der oberen HE: $i, 2i$

$2i$ ist eine <u>doppelte</u> Nullstelle!

$$\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)(z^2+4)^2} = \frac{1}{(z+i)(z^2+4)^2}\Big|_{z=i} = \frac{1}{2i\cdot 3^2} = \frac{1}{18i}$$

$$\operatorname*{Res}_{z=2i} \frac{1}{(z^2+1)(z^2+4)^2} = \frac{\partial}{\partial z}\left(\frac{1}{(z^2+1)(z+2i)^2}\right)\Big|_{z=2i}$$

$$= -\frac{2z\cdot(z+2i)^2 + 2(z+2i)(z^2+1)}{(z^2+1)^2(z+2i)^4}\Big|_{z=2i}$$

$$= \left[-\frac{2z}{(z^2+1)^2(z+2i)^2} - \frac{2}{(z^2+1)(z+2i)^3}\right]\Big|_{z=2i}$$

((1) (c))

$$= -\frac{4i}{(-3)^2(4i)^2} - \frac{2}{-3(4i)^3} = -\frac{1}{2^2 3^2 i} - \frac{1}{2^5 3 i}$$

$$= -\frac{3+8}{2^5 3^2 i} = -\frac{11}{2^5 3^2 i}$$

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2} \underset{\substack{\text{Integrand} \\ \text{gerade}}}{=} \frac{1}{2}\int_{-\infty}^{\infty} \cdots = \pi i \left(\frac{1}{2\cdot 3^2 \cdot i} - \frac{11}{2^5 3^2 i} \right)$$

$$= \pi \cdot \frac{16-11}{2^5 3^2} = \frac{5\pi}{288}.$$

(2)

$$|e^{iz}| = |e^{ix-y}| = e^{-y} \leq 1 \quad \text{für alle}$$

$z = x+iy$ mit $\operatorname{Im} z = y \geq 0$.

(a) $$g(z) = \frac{e^{iz}}{1+z^2}, \quad h(z) = \frac{1}{1+z^2}$$

$$\lim_{\operatorname{Im} z \geq 0,\, z \to \infty} \frac{1}{1+z^2} = 0$$

Mit Satz 18.12 folgt

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z=z_j}\left(h(z)\cdot e^{iz}\right)$$

wobei $z_1, \dots, z_k$ die Singularitäten von h in der obere HE sind.

$$\frac{1}{(1+z^2)} = \frac{1}{(z-i)(z+i)}$$ Einzige Singularität in der oberen HE: i

$$\operatorname{Res}_{z=i} \frac{e^{iz}}{1+z^2} = \lim_{z\to i} \frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}.$$ Es folgt

$$\int_0^{\infty} \frac{\cos x}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{2}} \cdot \frac{e^{-1}}{2\cancel{i}} = \frac{\pi}{2e}.$$

(b) $$\lim_{\operatorname{Im} z \geq 0,\, z \to \infty} \frac{z}{z^2+a} = \lim_{\operatorname{Im} z \geq 0,\, z \to \infty} \frac{1}{z + \frac{a}{z}} = 0$$

$$z^2 + a = (z - i\sqrt{a})(z + i\sqrt{a})$$

$a > 0 \Rightarrow$ Einzige Singularität in der oberen HE ist

$+i\sqrt{a}$.

$$\operatorname*{Res}_{z = i\sqrt{a}} \frac{z e^{i\omega z}}{z^2+a} = \lim_{z \to i\sqrt{a}} \frac{z e^{i\omega z}}{z + i\sqrt{a}} = \frac{\cancel{i\sqrt{a}}\, e^{-\omega\sqrt{a}}}{2\cancel{i\sqrt{a}}}$$

Mit Satz 18.13

$$\int_{-\infty}^{\infty} \frac{x e^{i\omega x}}{x^2+a}\, dx = \cancel{2}\pi i \cdot \frac{e^{-\omega\sqrt{a}}}{\cancel{2}} = \pi i\, e^{-\omega\sqrt{a}}$$

Weiter

$$\int_0^{\infty} \frac{x \sin(\omega x)}{1+x^2}\, dx = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{x(\cos(\omega x) + i\sin(\omega x))}{1+x^2}\, dx$$

$$= \frac{1}{2i} \int_{-\infty}^{\infty} \frac{x e^{i\omega x}}{1+x^2}\, dx = \frac{\pi}{2} \cdot e^{-\omega\sqrt{a}}.$$

③ $\left|\frac{z}{1+z}\right| \leq$ const, da

$$\lim_{|z|\to\infty} \left|\frac{z}{1+z}\right| = \lim_{|z|\to\infty} \left|\frac{1}{1/z+1}\right| = 1 .$$

$$\operatorname*{Res}_{z=-1} \frac{1}{z^\alpha(1+z)} = \lim_{z\to -1} \frac{1}{z^\alpha} = \frac{1}{e^{+i\pi\alpha}}$$

Mit Satz 18.14 gilt

$$\int_0^\infty \frac{dx}{x^\alpha(1+x)} = \frac{2\pi i}{(1-e^{-2\pi i\alpha})e^{i\pi\alpha}} = \pi\cdot\frac{2i}{e^{i\pi\alpha}-e^{-i\pi\alpha}}$$

$$= \frac{\pi}{\sin(\pi\alpha)} .$$

(4) In allen Fällen kann man die entsprechenden DGL-Systeme lösen und so $e^A = e^{tA}|_{t=1}$ berechnen. Wir wollen aber die Reihen ausrechnen.

(a)
$$A^2 = \begin{pmatrix} \alpha^2-\beta^2 & -2\alpha\beta \\ 2\alpha\beta & \alpha^2-\beta^2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} \alpha^3-3\alpha\beta^2 & -3\alpha^2\beta+\beta \\ 3\alpha^2\beta-\beta^3 & \alpha^3-3\alpha\beta^2 \end{pmatrix}$$

Sei $z = \alpha + i\beta$.

<u>Behauptung</u>: Für alle $m \in \mathbb{N}$ gilt

$$A^m = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \text{ mit } z^m = u+iv.$$

<u>Beweis</u>: $A^0 = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $z^0 = 1 + i\cdot 0$ ✓

<u>$m \rightsquigarrow m+1$</u> Sei $z^m = u+iv$, $A^m = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$.

Dann gilt
$$A^{m+1} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \cdot \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$$
$$= \begin{pmatrix} \alpha u - \beta v & -(\beta u + \alpha v) \\ \beta u + \alpha v & \alpha u - \beta v \end{pmatrix}$$

und $z^{m+1} = (\alpha + i\beta)(u+iv) = (\alpha u - \beta v) + i(\beta u + \alpha v)$.

Somit ist die Beh. gezeigt und

$$e^A = \sum_{m=0}^{\infty} \frac{1}{m!} A^m = \begin{pmatrix} \operatorname{Re} e^z & -\operatorname{Im} e^z \\ \operatorname{Im} e^z & \operatorname{Re} e^z \end{pmatrix} = \begin{pmatrix} \cos\beta\cdot e^\alpha & -\sin\beta\, e^\alpha \\ \sin\beta\, e^\alpha & \cos\beta\, e^\alpha \end{pmatrix}$$

(b) Sei $x = \alpha + \beta$, $y = \alpha - \beta$.

$$A^2 = \begin{pmatrix} \alpha^2+\beta^2 & 2\alpha\beta \\ 2\alpha\beta & \alpha^2+\beta^2 \end{pmatrix} \quad A^3 = \begin{pmatrix} \alpha^3+3\alpha\beta^2 & 3\alpha^2\beta+\beta^3 \\ 3\alpha^2\beta+\beta^3 & 3\alpha\beta^2+\alpha^3 \end{pmatrix}$$

<u>Behauptung</u>: Für alle $m \in \mathbb{N}$ gilt

$$A^m = \begin{pmatrix} u & v \\ v & u \end{pmatrix} \quad \text{mit } x^m = u+v \text{ und } y^m = u-v$$

<u>Beweis</u>: $m=0$: $A^0 = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad x^0 = 1+0$, $y^0 = 1-0$ ✓

$m \rightsquigarrow m+1$ Sei $A^m = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$ mit $x^m = u+v$, $y^m = u-v$

$$A^{m+1} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} u & v \\ v & u \end{pmatrix} = \begin{pmatrix} \alpha u + \beta v & \beta u + \alpha v \\ \beta u + \alpha v & \alpha u + \beta v \end{pmatrix}$$

$$(\alpha u + \beta v) \pm (\beta u + \alpha v) = (\alpha \pm \beta)(u \pm v) = \begin{cases} x \cdot x^m = x^{m+1} & + \\ y \cdot y^m = y^{m+1} & - \end{cases}$$ ✓

Daher $$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \begin{pmatrix} \frac{1}{2}(e^{\alpha+\beta} + e^{\alpha-\beta}) & \frac{1}{2}(e^{\alpha+\beta} - e^{\alpha-\beta}) \\ \frac{1}{2}(e^{\alpha+\beta} - e^{\alpha-\beta}) & \frac{1}{2}(e^{\alpha+\beta} + e^{\alpha-\beta}) \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\beta \cdot e^\alpha & \sinh\beta \cdot e^\alpha \\ \sinh\beta \cdot e^\alpha & \cosh\beta \cdot e^\alpha \end{pmatrix}.$$

(4) (c) $[\lambda E, B] = \lambda B - \lambda B = 0$

Also $e^A = e^{\lambda E + B} = \underbrace{e^{\lambda E}}_{= e^\lambda \cdot E} \cdot e^B = e^\lambda \cdot e^B$.

(d) $A^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & & \ddots & 1 \\ & & \ddots & & 0 \\ 0 & & & & 0 \end{pmatrix}$ $\quad A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

$$A^m = \begin{pmatrix} \overbrace{0 \cdots 0}^{m+1} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & \ddots & 1 \\ & & & & 0 \\ 0 & & & & 0 \end{pmatrix} \qquad A^m = 0 \quad m > n$$

$$e^A = \begin{pmatrix} 0 & 1 & 1/2 & \cdots & 1/(n-1)! \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1/2 \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}.$$