(1)

$$u(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\, ds .$$

$$\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} . \qquad \operatorname{erf}(0) = 0 .$$

$$\lim_{x\to\infty} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2}\, ds = 1 .$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{2}{\sqrt{4\pi kt}} e^{-x^2/4kt}\right) = \frac{1}{\sqrt{\pi kt}} (-2x) e^{-x^2/4kt} \cdot \frac{1}{4kt}$$

$$= \frac{1}{k} \cdot \left(- \frac{1}{2t\sqrt{\pi kt}} x e^{-x^2/4kt} \right) .$$

$$u_t = \frac{2}{\sqrt{\pi}} e^{-x^2/4kt} \frac{x}{2\sqrt{k}} \left(-\frac{1}{2t^{3/2}}\right)$$

$$= -\frac{1}{2t\sqrt{\pi kt}} x e^{-x^2/4kt} \qquad \Rightarrow \quad u_{xx} = \frac{1}{k} u_t$$

$t>0$:

$$u(0,t) = \operatorname{erf}(0) = 0 .$$

$x>0$

$$u(x,0) = \lim_{t\to 0} \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) = \lim_{y\to\infty} \operatorname{erf}(y) = 1 .$$

② (a) $$w(x,t) = -2\,\frac{u_x(x,t)}{u(x,t)}$$

$$w_t = -2\cdot\left(\frac{u_{xt}}{u} - \frac{u_x\cdot u_t}{u^2}\right)$$

$$w_x = -2\left(\frac{u_{xx}}{u} - \frac{u_x^2}{u^2}\right)$$

$$w_{xx} = -2\left(\frac{u_{xxx}}{u} - \frac{u_{xx}\cdot u_x}{u^2} - \frac{2u_{xx}\cdot u_x}{u^2} + \frac{2u_x^3\cancel{u}}{u^{\cancel{4}3}}\right)$$

Aus $u_t - u_{xx} = 0$ folgt $u_{tx} - u_{xxx} = 0$.

Daher

$$w_t + w\cdot w_x = -2\left(\underbrace{\frac{u_{xt}}{u}}_{=\frac{u_{xxx}}{u}} - \frac{u_x u_t}{u^2} - \frac{2u_{xx}\cdot u_x}{u^2} + \frac{2u_x^3}{u^3}\right) = w_{xx}.$$

(b)

$$w(x,t) = -2\cdot\frac{-\frac{x}{4\pi t^2}\,e^{-x^2/4t}}{\frac{e^{-x^2/4t}}{2\pi t}} = \frac{x}{t}.$$

$$w_t = -\frac{x}{t^2} \qquad w_x = \frac{1}{t} \qquad w_{xx} = 0$$

$$w_t + w\cdot w_x = -\frac{x}{t^2} + \frac{x}{t}\cdot\frac{1}{t} = 0 = w_{xx}$$ ✓

③ (a) $v(x,t) = u(\alpha x, \alpha^2 t)$.

$t > 0$:

$$v(0,t) = u(0, \underbrace{\alpha^2 t}_{>0}) = 1.$$

$x \geq 0$:

$$v(x,0) = u(\underbrace{\alpha x}_{\geq 0}, 0) = 0.$$

$x \geq 0,\ t > 0$:

$$v_t - v_{xx} = \alpha^2 \cdot u_t(\alpha x, \alpha^2 t) - \underbrace{\alpha \frac{d}{dx} u_x(\alpha x, \alpha^2 t)}_{= \alpha^2 u_{xx}(\alpha x, \alpha^2 t)}$$

$$= 0.$$

Da die A-R-Aufgabe eindeutig lösbar ist, folgt

$$u(x,t) = u(\alpha x, \alpha^2 t).$$

Mit $\alpha = \frac{1}{2\sqrt{t}}$ folgt

$$u(x,t) = u\left(\frac{x}{2\sqrt{t}}, \frac{\cancel{t}}{4\cancel{t}}\right) = u\left(\frac{x}{2\sqrt{t}}, \frac{1}{4}\right).$$

(b)

$$\frac{\partial}{\partial t} u\left(\frac{x}{2\sqrt{t}}, \frac{1}{4}\right) = u_x\left(\frac{x}{2\sqrt{t}}, \frac{1}{4}\right) \cdot \left(-\frac{x}{4t^{3/2}}\right) = h'(\xi) \cdot \left(-2\xi^3\right) \cdot x^{-2}$$

$$\frac{\partial^2}{\partial x^2} u\left(\frac{x}{2\sqrt{t}}, \frac{1}{4}\right) = u_{xx}\left(\frac{x}{2\sqrt{t}}, \frac{1}{4}\right) \cdot \frac{1}{4t} = h''(\xi) \cdot \xi^2 \cdot x^{-2}$$

(3) Man erhält

$$2\xi h'(\xi) + h''(\xi) = 0 \quad , \text{ also}$$

$$h'(\xi) = c e^{-\xi^2} , \quad h(\xi) = c_1 \operatorname{erf}(\xi) + c_2 .$$

Es folgt mit den Randbedingungen

$$c_2 = c_1 \operatorname{erf}(0) + c_2 = h(0) = 1$$

$$c_1 + 1 = \lim_{\xi \to \infty} (c_1 \operatorname{erf}(\xi) + c_2) = \lim_{\xi \to \infty} h(\xi) = 0$$

Somit $h(\xi) = -\operatorname{erf}(\xi) + 1$. Es folgt

$$u(x,t) = -\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + 1 = 1 - \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} e^{-s^2}\, ds .$$

(4) $w(y,t) = v(y + z(t), t)$

$$w_t = v_x(y+z(t), t) \cdot \dot{z}(t) + v_t(y+z(t), t)$$

$$w_{yy} = v_{xx}(y + z(t), t)$$

Daher reicht es, dass $\dot{z}(t) = a(t)$, man nehme also etwa

$$z(t) = \int_0^t a(s)\, ds . \qquad \square.$$