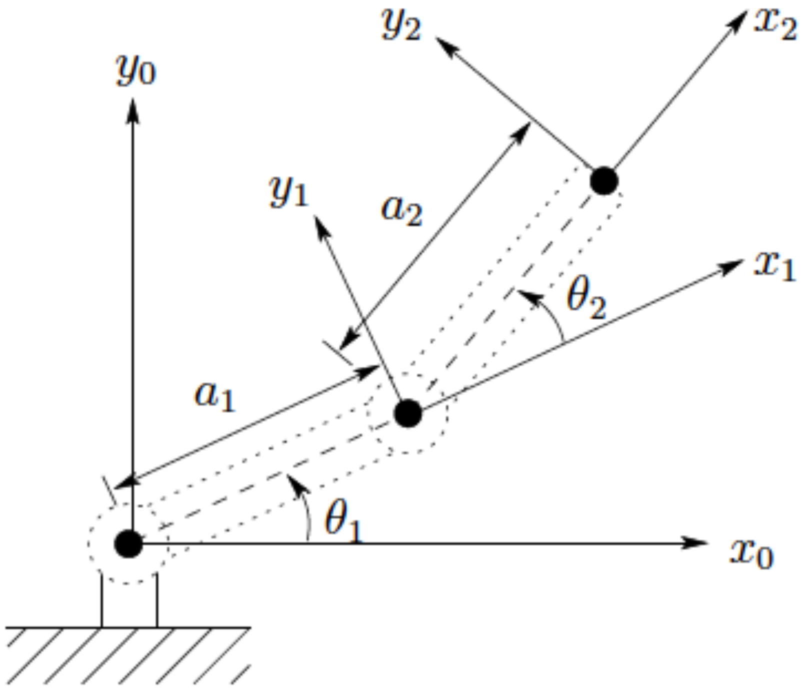
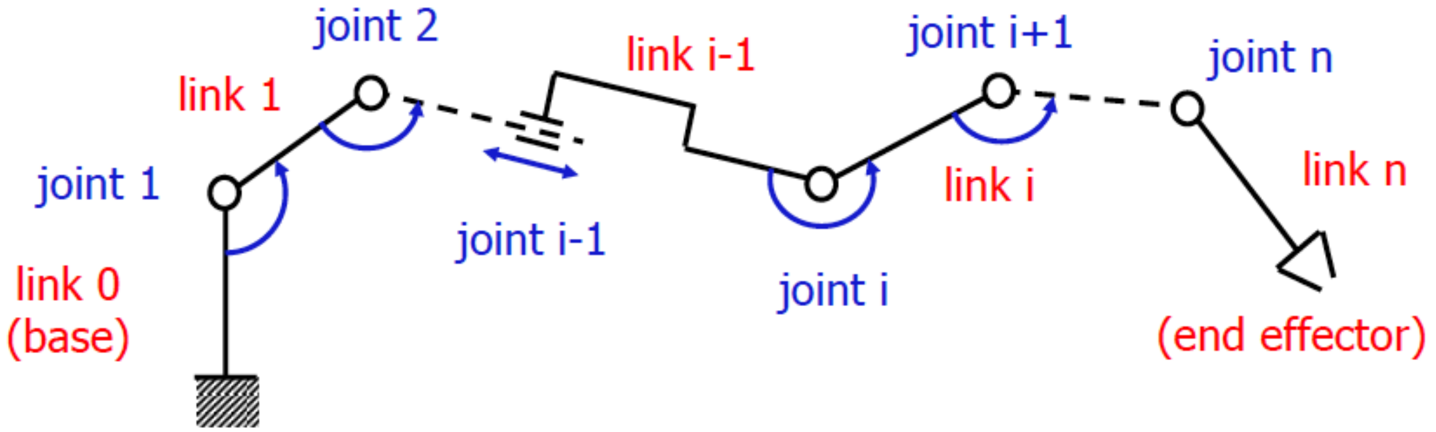
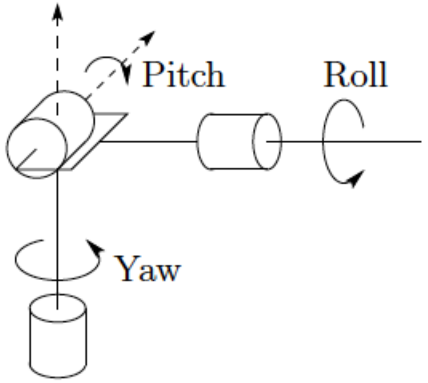
Recall : Two-link manipulator



$$
{}^{0}_{2}\mathrm{R} = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Numbering links and joints





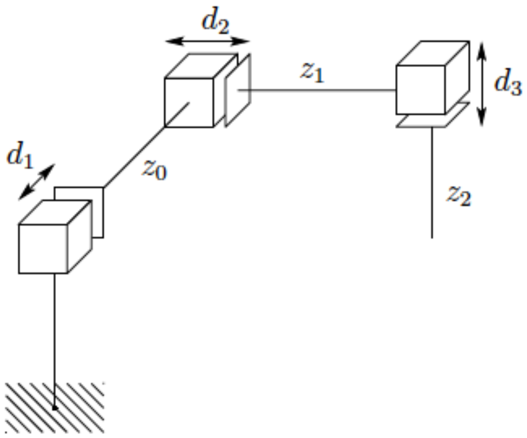
Links are numbered from 0 to n: Frame 0 is the base.

Joints are numbered from 1 to n: assign unit vector z_{i-1} along axis of joint I

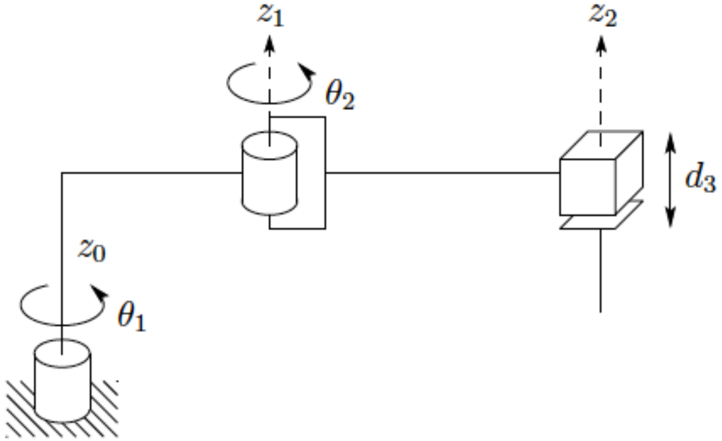
Frame i is fixed to link i: Joint i connects frame i to frame i-1

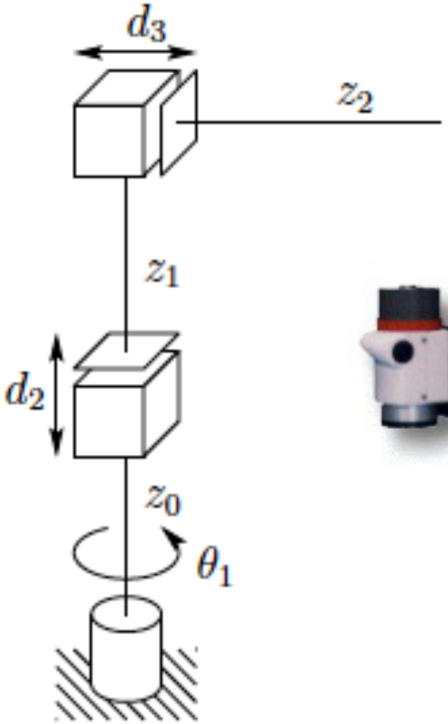
Associate a joint variable with i-th joint:

$$q_i = \begin{cases} \theta_i & \text{if the joint is revolute} \\ d_i & \text{if the joint is prismatic} \end{cases}$$

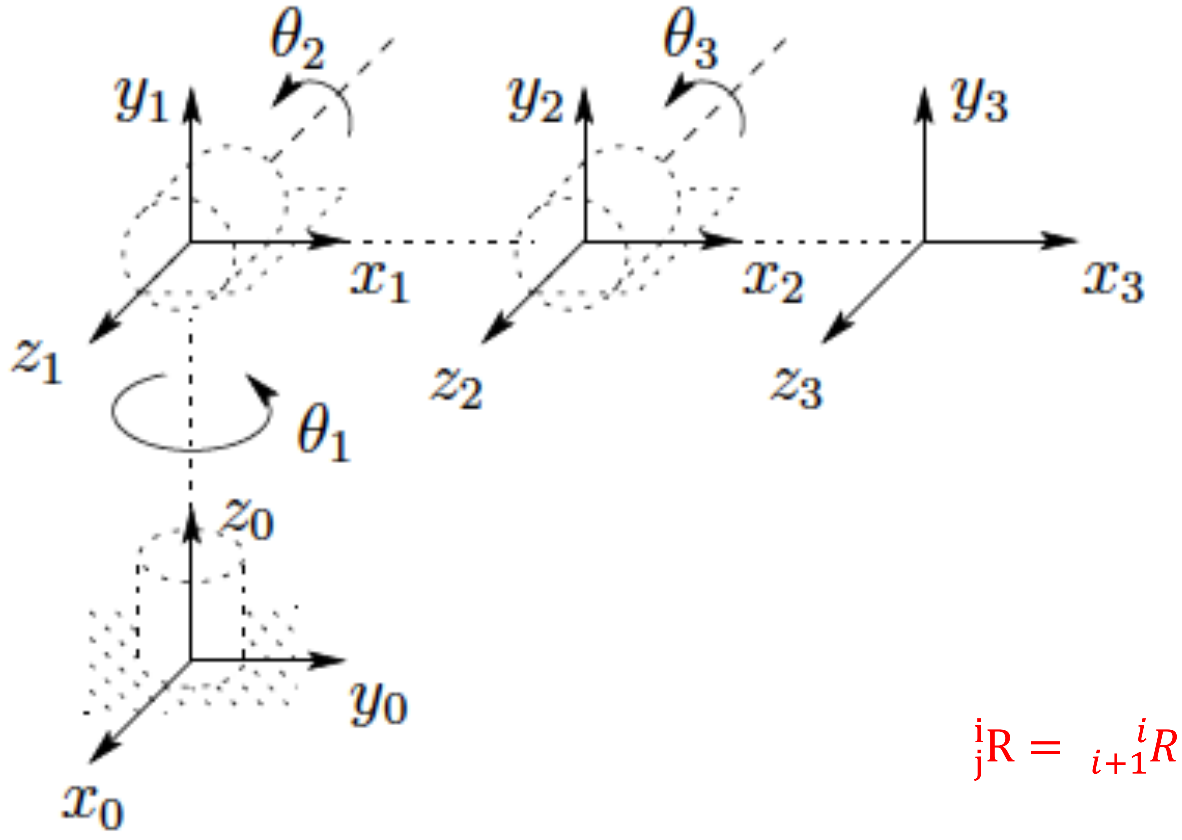


Numbering links and joints





Forward kinematics



Homogeneous transformation matrix $A_i = A_i(q_i)$ expresses the position of frame i with respect to frame i-1

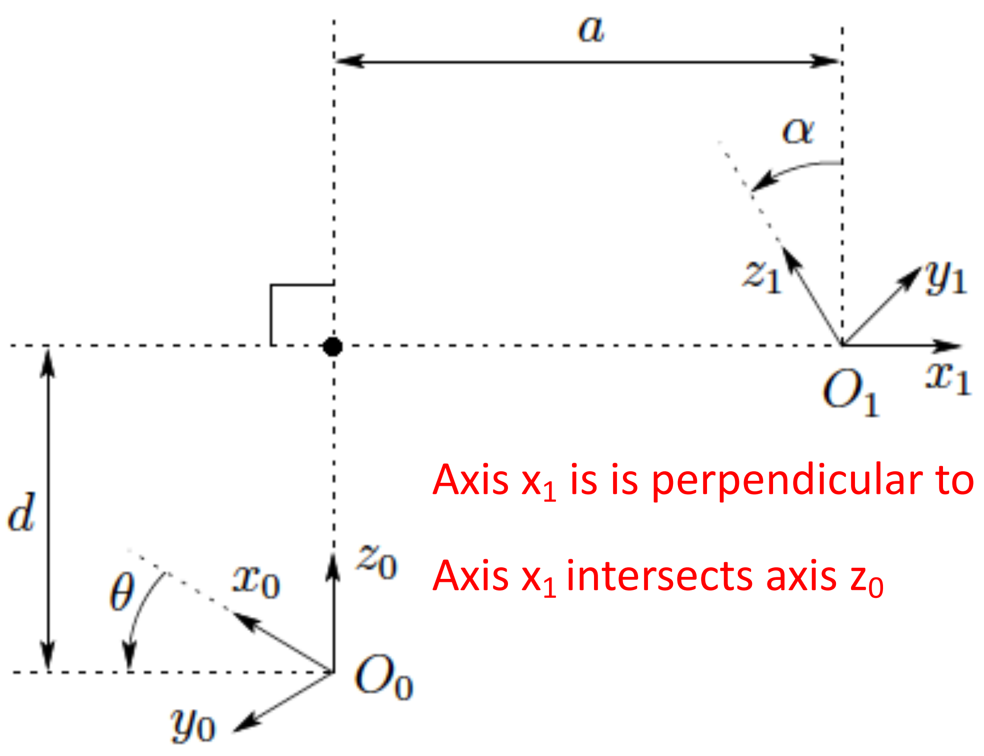
$${}^{i}_{j}T = \begin{cases} A_{i+1}A_{i+2}\ldots A_j & \text{if } j > i \\ I & \text{if } = j \\ {}^{j}_{i}T^{-1} & \text{if } i > j \end{cases}$$

$$A_i = \begin{bmatrix} {}^{i-1}_{\;\;i}R & {}^{i-1}O_i \\ 0 & 1 \end{bmatrix} \qquad {}^{j}_{i}T = \begin{bmatrix} {}^{i}_{j}R & {}^{i}O_j \\ 0 & 1 \end{bmatrix}$$

$${}^{i}_{j}R = {}^{\;\;i}_{i+1}R\,{}^{i+1}_{i+2}R \ldots {}^{j-1}_{\;\;j}R$$

$${}^{i}O_j = {}^{i}O_{j-1} + {}^{\;\;i}_{j-1}R\,{}^{j-1}O_j$$

Denavit-Hartenberg Convention



Axis x_1 is is perpendicular to axis z_0

Axis x_1 intersects axis z_0

$x_1 \cdot z_0 = 0 \Rightarrow r_{31} = 0$

$r_{11}^2 + r_{21}^2 = 1 \qquad (r_{11}, r_{21}) = (c\theta, s\theta)$

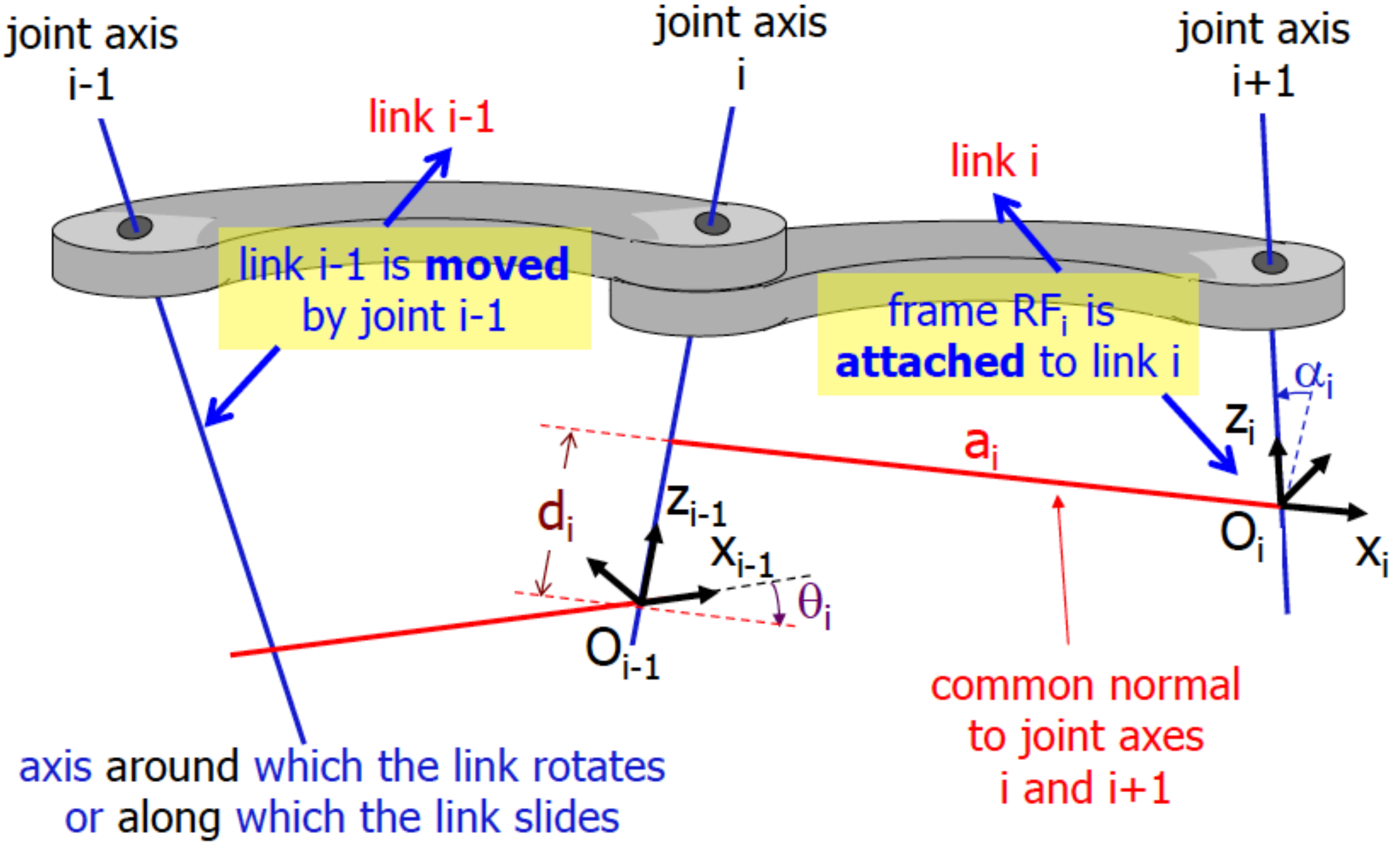
$r_{32}^2 + r_{33}^2 = 1 \qquad (r_{33}, r_{32}) = (c\alpha, s\alpha)$

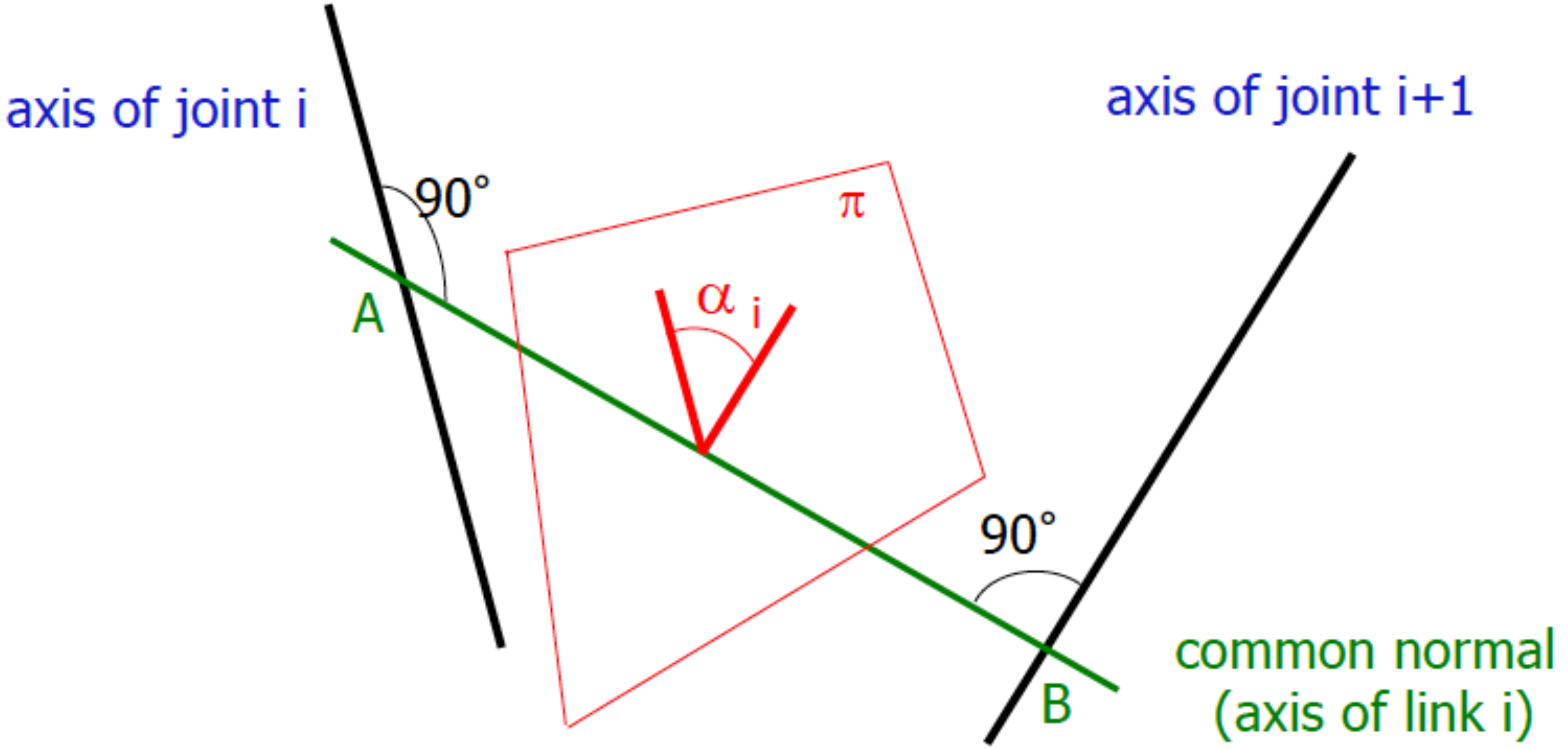
$$ {}^0_1R = R_z(\theta)R_x(\alpha) = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} $$

$$ {}^0O_1 = {}^0O_1 + d\,{}^0z_0 + a\,{}^0x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix} = \begin{bmatrix} ac_\theta \\ as_\theta \\ d \end{bmatrix} $$

Transformation matrix $A = R_z(\theta)T_z(\mathrm{d})T_x(\mathrm{a})R_x(\alpha) = \begin{bmatrix} {}^0_1R & {}^0O_1 \\ 0 & 1 \end{bmatrix}$

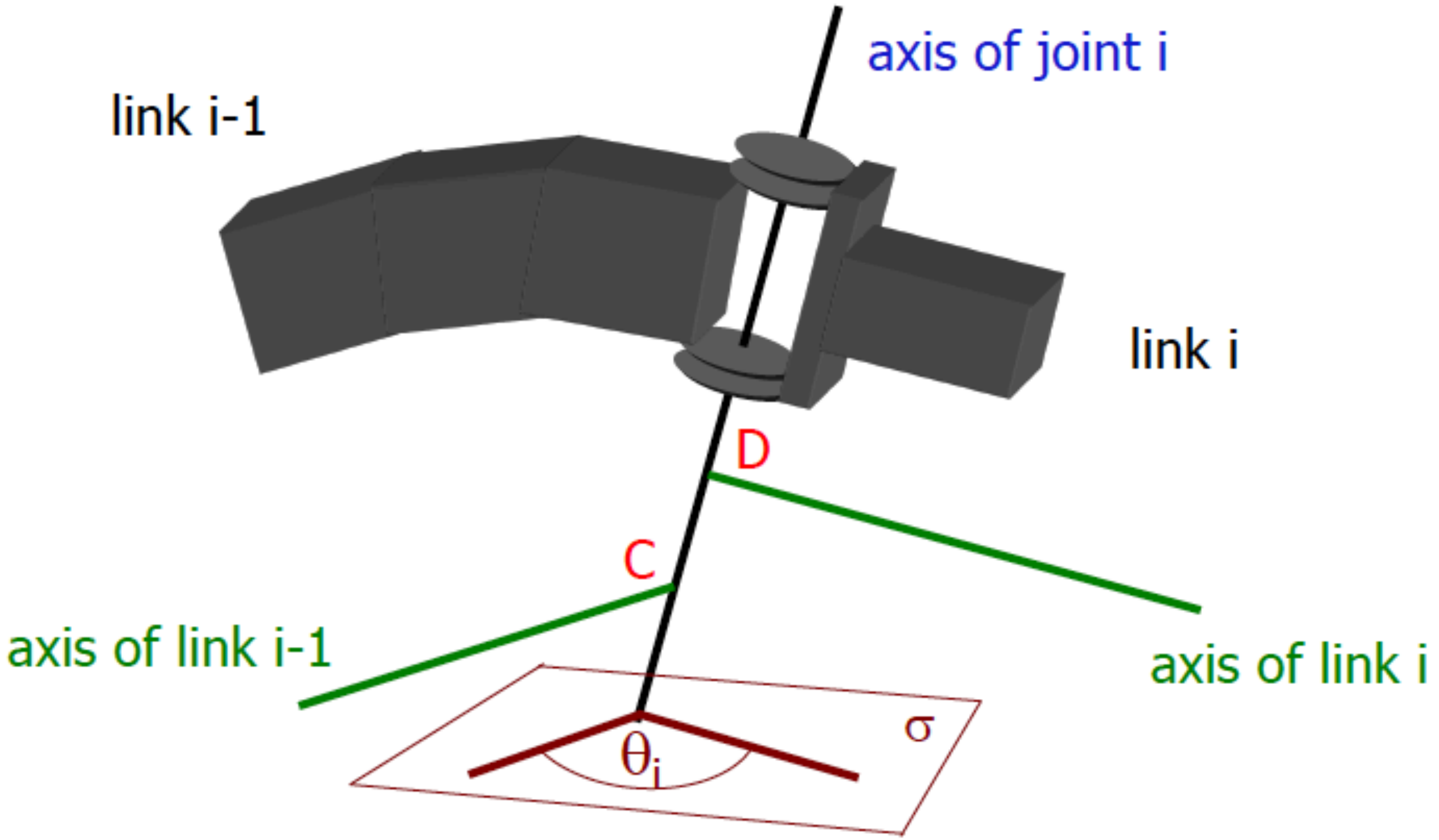
Denevit-Hartenberg Frame assignment





a_i = **displacement AB** between joint axes (always well defined)
α_i = **twist angle** between joint axes
— projected on a plane π orthogonal to the link axis

with sign (pos/neg)!

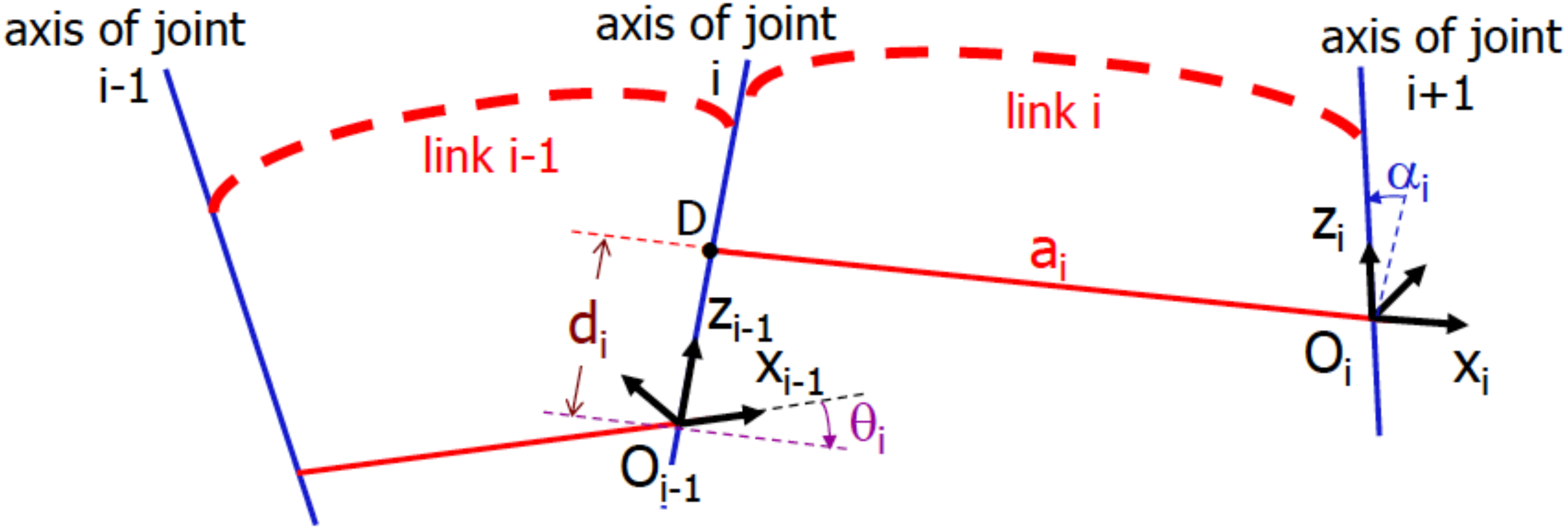


d_i = **displacement CD** (a variable if joint i is prismatic)

θ_i = **angle between link axes** (a variable if joint i is revolute)
— projected on a plane σ orthogonal to the joint axis

with sign (pos/neg)!

Denavit-Hartenberg parameters



- unit vector z_i along axis of joint i+1
- unit vector x_i along the common normal to joint i and i+1 axes (i $\rightarrow$ i+1)
- a_i = distance DO_i — positive if oriented as x_i (constant = "length" of link i)
- d_i = distance $O_{i-1}D$ — positive if oriented as z_{i-1} (variable if joint i is PRISMATIC)
- α_i = twist angle between z_{i-1} and z_i around x_i (constant)
- θ_i = angle between x_{i-1} and x_i around z_{i-i} (variable if joint i is REVOLUTE)

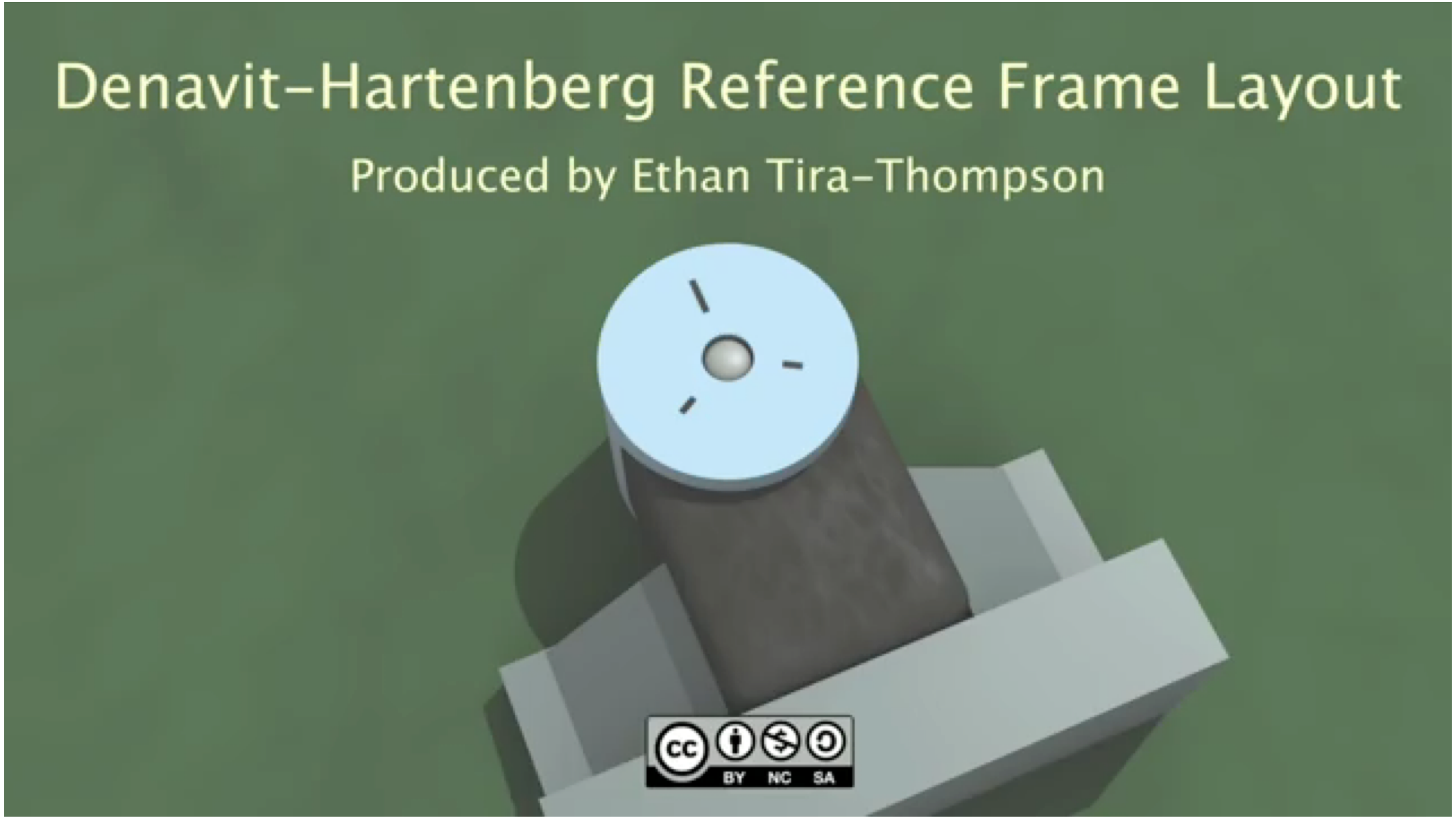
$$
\begin{aligned}
A_i &= Rot_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}Rot_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

DH frame assignment: special cases

- z_{i-1} and z_i are parallel
  - There are infinitely many common normal
  - Origin o_i can be anywhere along z_i
  - Choose o_i to simplify resulting equations
    - x_i can be directed towards z_{i-1} along the common normal or opposite of this
  - Example: Choose x_i along the common normal that passes through o_{i-1}
    - o_i is located at intersection of x_i with z_i
    - $d_i = 0$, $\alpha_i = 0$.
- z_{i-1} and z_i intersect
  - x_i is chosen normal to the plane formed by z_i and z_{i-1}
  - o_i can be anywhere along z_i
    - Often chosen as the point of intersection of z_i and z_{i-1}
    - $a_i = 0$

Ambiguities in defining DH frames

- *$frame_0$:* origin and x_0 axis are arbitrary
- *$frame_n$:* z_n axis is not specified (but x_n must be orthogonal to and intersect z_{n-1})
- *positive* direction of z_{i-1} (up/down on joint i) is arbitrary
  - choose one, and try to avoid "flipping over" to the next one
- *positive* direction of x_i (on axis of link i) is arbitrary
  - we often take $x_i = z_{i-1} \times z_i$ when successive joint axes are incident
  - when natural, we follow the direction "from base to tip"
- if z_{i-1} and z_i are *parallel:* common normal not uniquely defined
  - O_i is chosen arbitrarily along z_i , but try to "zero out" parameters
- if z_{i-1} and z_i are *coincident:* normal x_i axis may be chosen at will
  - again, we try to use "simple" constant angles (0, $\pi/2$)
  - this case may occur only if the two joints are of different kind (P & R)


Denavit-Hartenberg Reference Frame Layout
Produced by Ethan Tira-Thompson

Summary

Step 1: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-handed frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin o_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate o_i at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through o_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-handed frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = a$ along the direction z_{n-1}. Establish the origin o_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = s$ in the direction of the gripper closure and set $x_n = n$ as $s \times a$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-handed frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

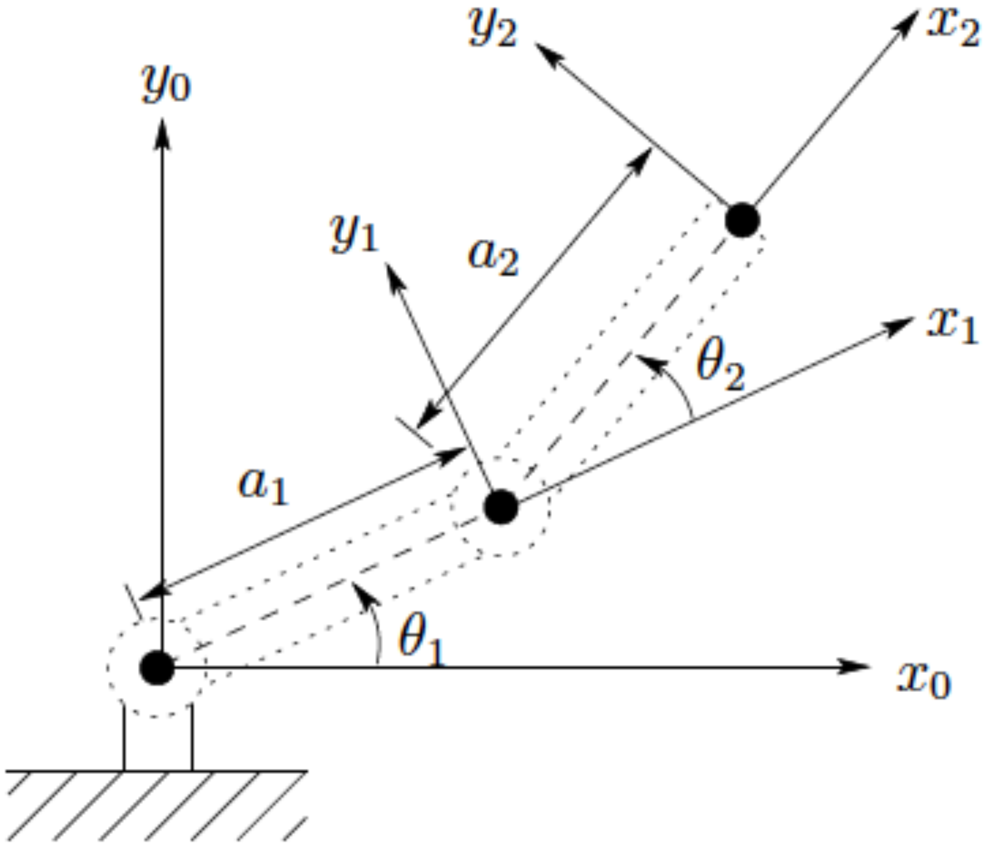
$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i.

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1}. θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

Step 9: Form ${}^{0}_{n}\mathrm{T} = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

Example: 2 link manipulator



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

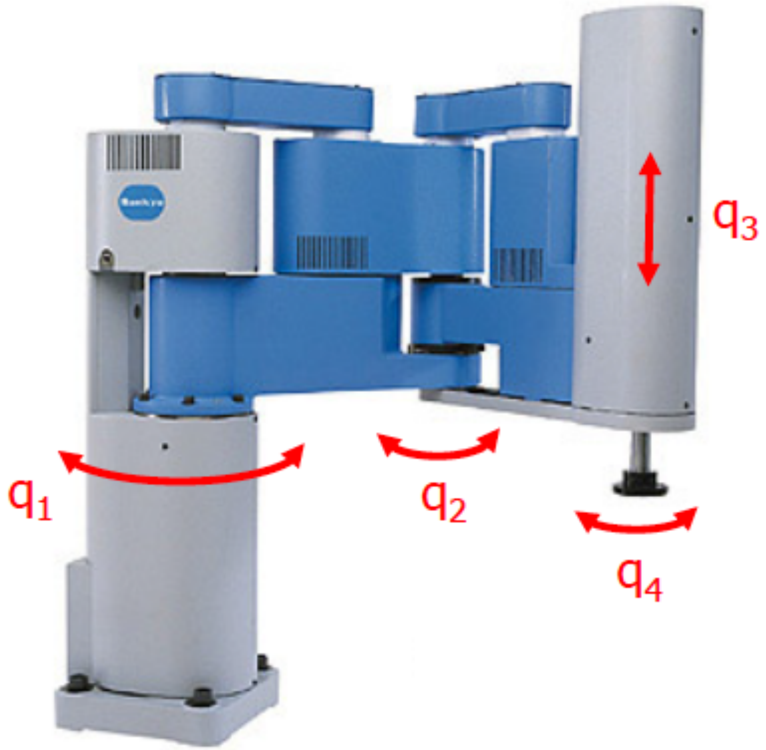
$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}_1^0T = A_1$$

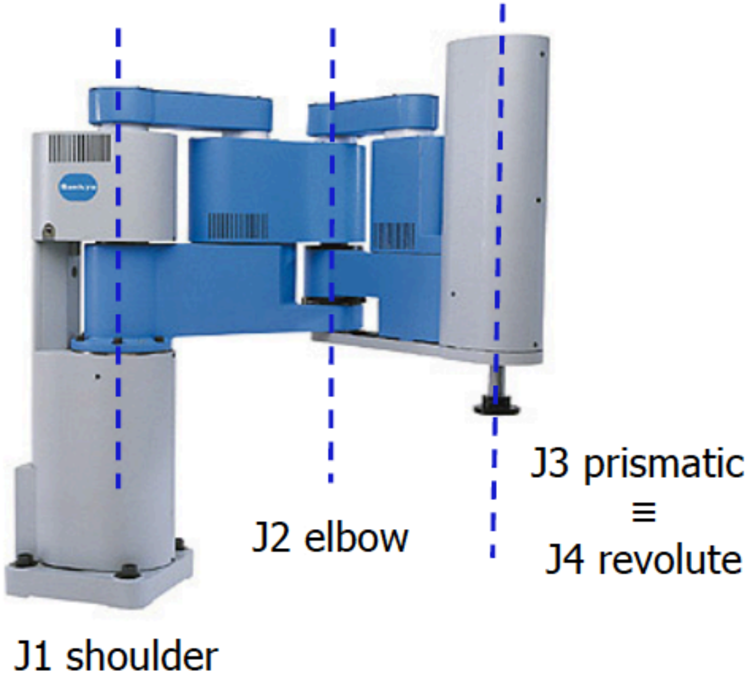
$${}_2^0T = A_1A_2$$

Example: Scara robot



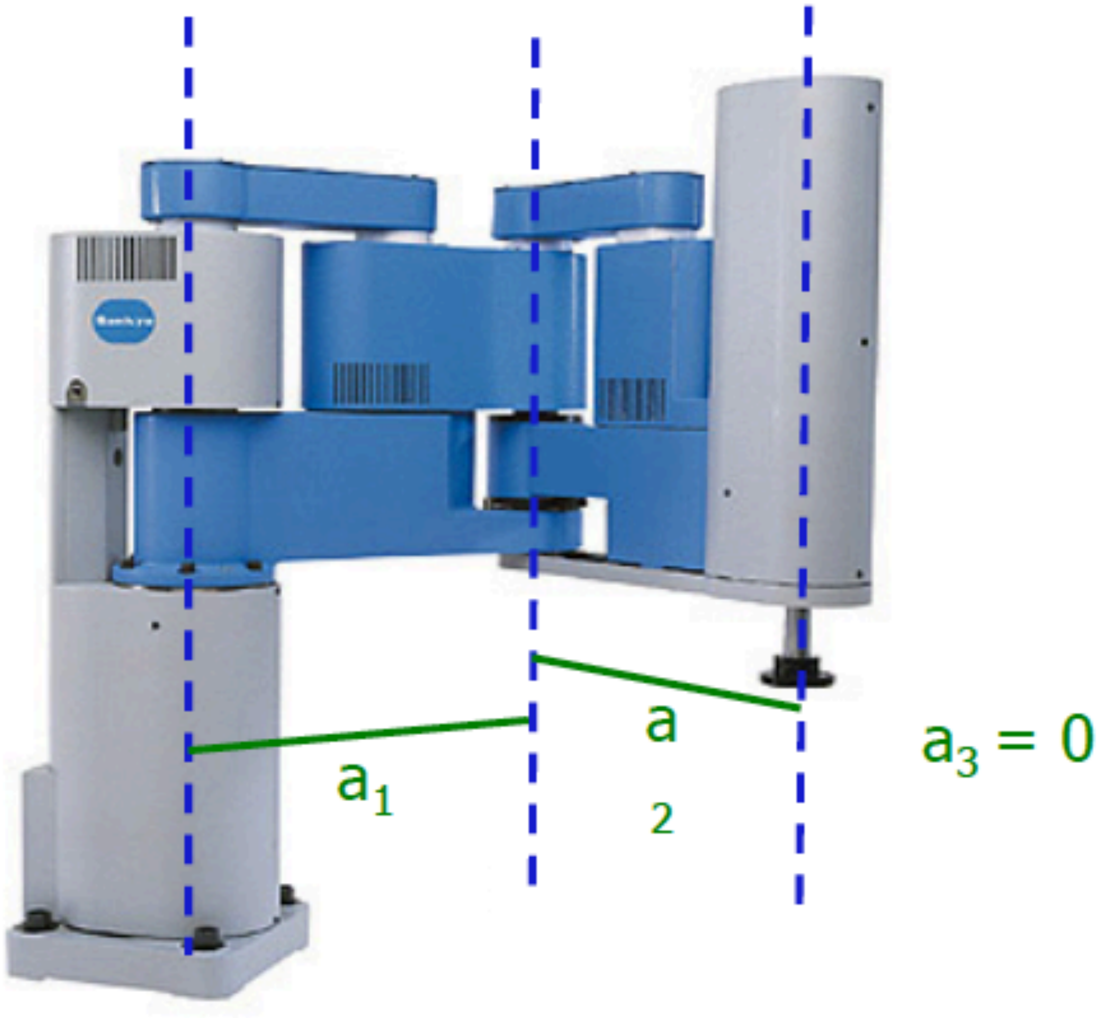
all parallel
(or coincident)

twists $\alpha_i = 0$
or π



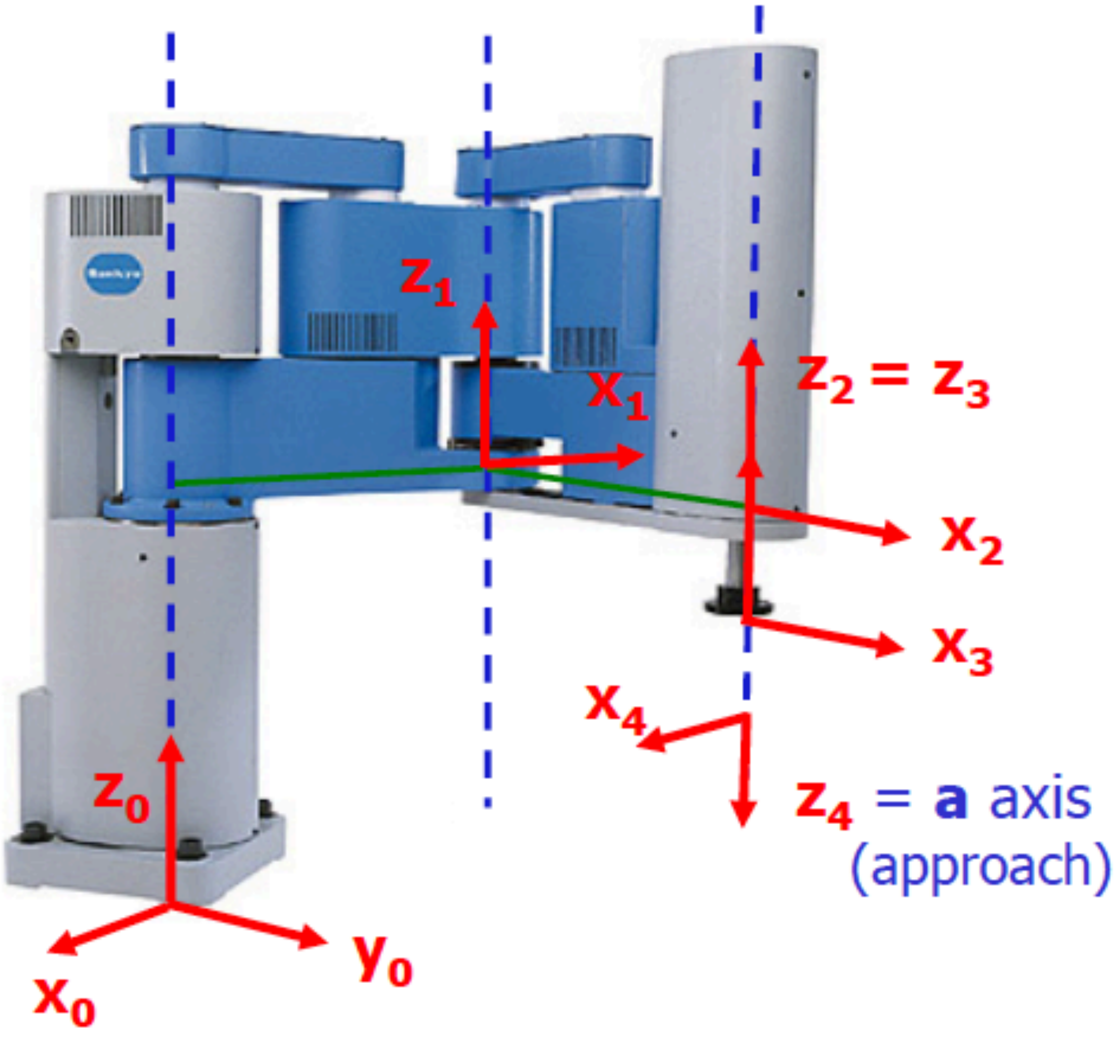
Step 2: Link axis

the vertical "heights"
of the link axes
are arbitrary
(for the time being)

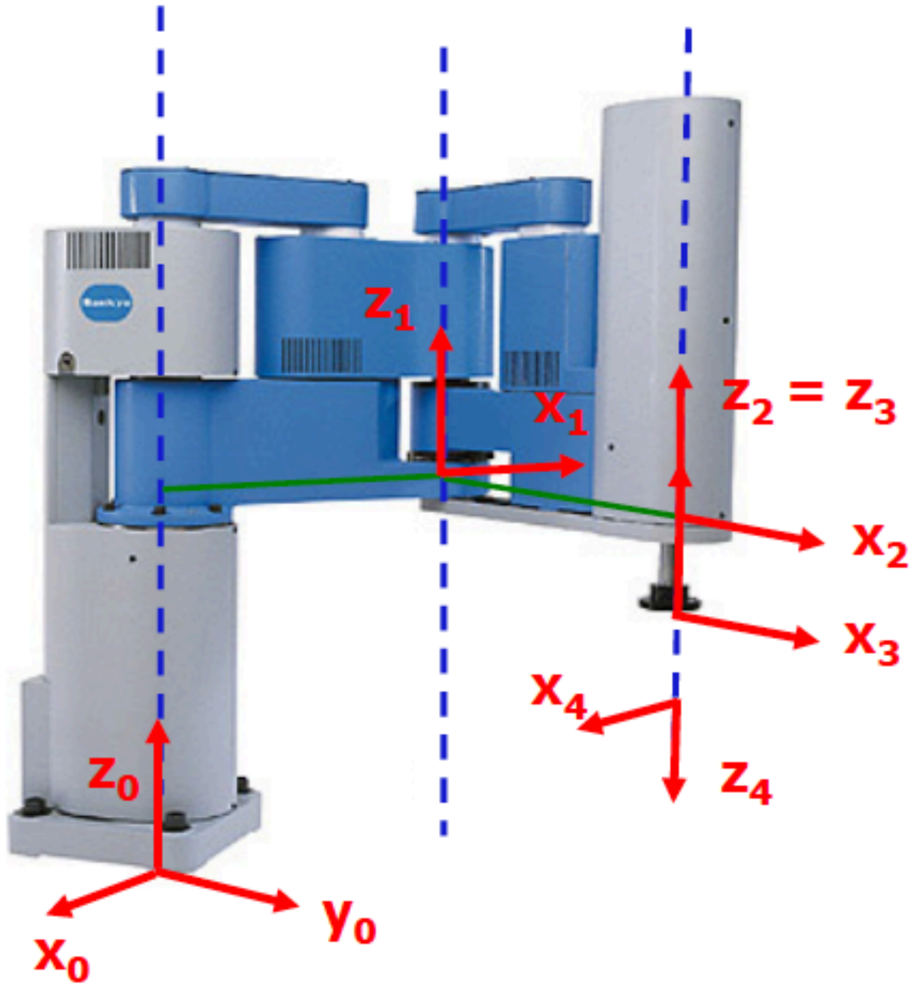


Step 3: Frames

axes $\mathbf{y_i}$ for i > 0
are not shown
(nor needed; they form
right-handed frames)



Step 4: Table of DH parameters



i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	q_1
2	0	a_2	0	q_2
3	0	0	q_3	0
4	π	0	d_4	q_4

note that:

- d_1 and d_4 could be set = 0
- here, it is $d_4 < 0$

Step 5: Transformation matrices

$$A_1(q_1) = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2(q_2) = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3(q_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} q &= (q_1, q_2, q_3, q_4) \\ &= (\theta_1, \theta_2, d_3, \theta_4) \end{aligned}$$

$$A_4(q_4) = \begin{bmatrix} c\theta_4 & s\theta_4 & 0 & 0 \\ s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & -1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 6: Forward kinematics

$$
{}^0_4T(q_1,q_2,q_3,q_4) = \begin{bmatrix} \boxed{\begin{matrix} c_{124} & s_{124} & 0 \\ s_{124} & -c_{124} & 0 \\ 0 & 0 & -1 \end{matrix}} & \boxed{\begin{matrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ d_1 + q_3 + d_4 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}
$$

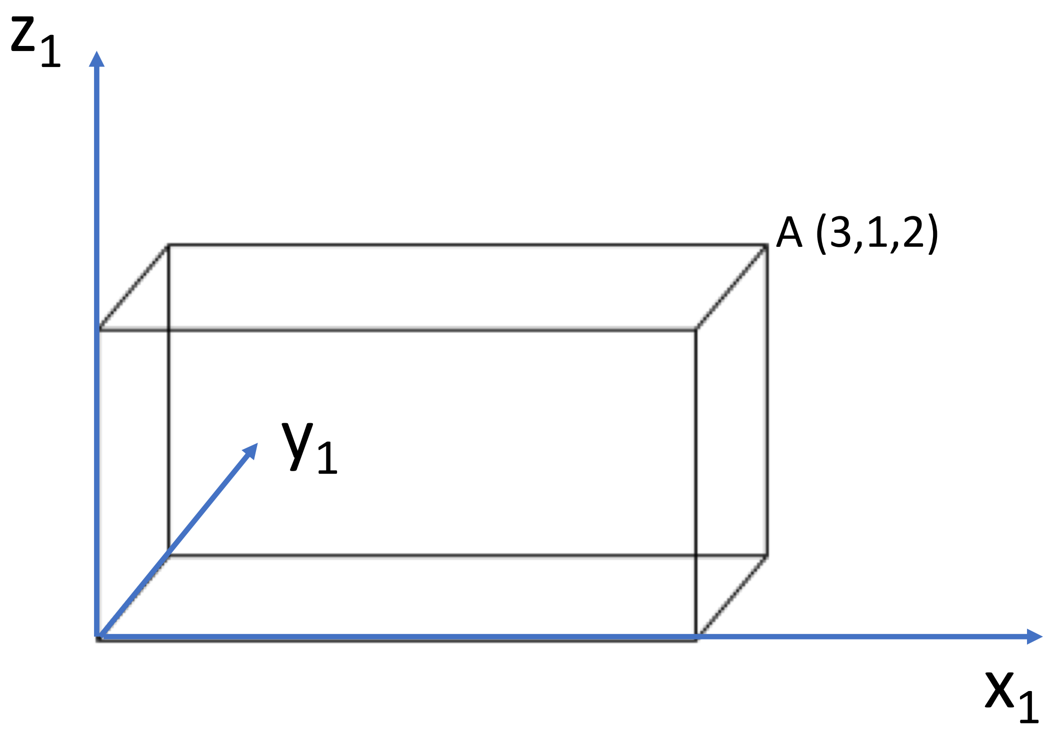
extract α_z from $R(q_1,q_2,q_4)$

take $p(q_1,q_2,q_3)$ as such

$$
r = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha_z \end{bmatrix} = f_r(q) = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ d_1 + q_3 + d_4 \\ q_1 + q_2 + q_4 \end{bmatrix} \in \mathbb{R}^4
$$

Task vector

Quiz



Consider the body shown with frame 1 ($x_1y_1z_1$) attached to the body. Initially, the body is placed such that frame 1 is perfectly aligned with a fixed frame XYZ with origin of frame one at the same location as origin of XYZ. Subsequently, the body is rotated in the following manner

1. Rotation by 45 degrees about axis x1
2. Rotation by 60 degrees about y1 (recall that y1 rotates with the body
3. Rotation by 90 degrees about the Z axis.

What is the new location of point A located on the body in the XYZ frame?

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_z(\theta) = \begin{bmatrix} c_\theta & -s_\phi & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$