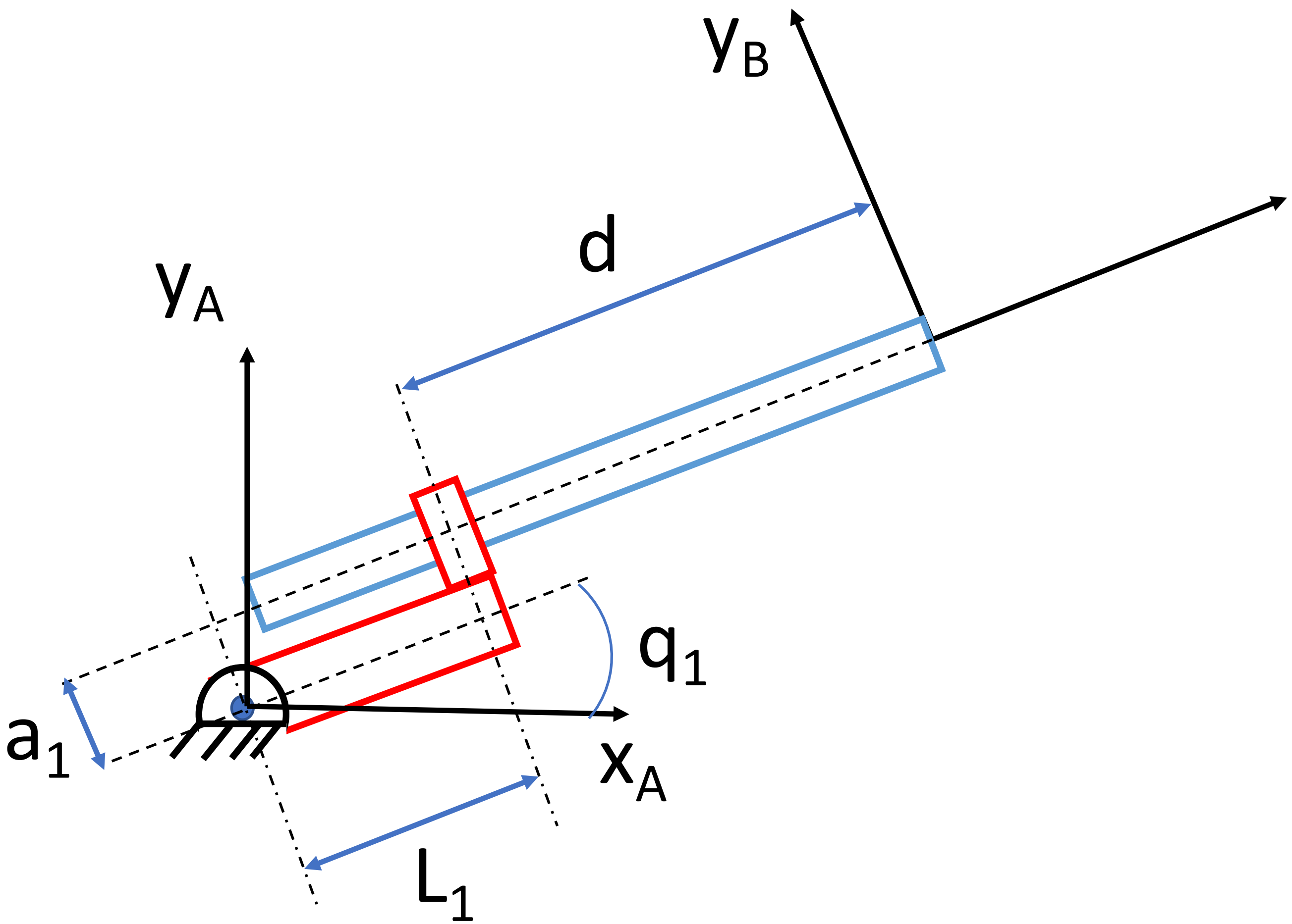
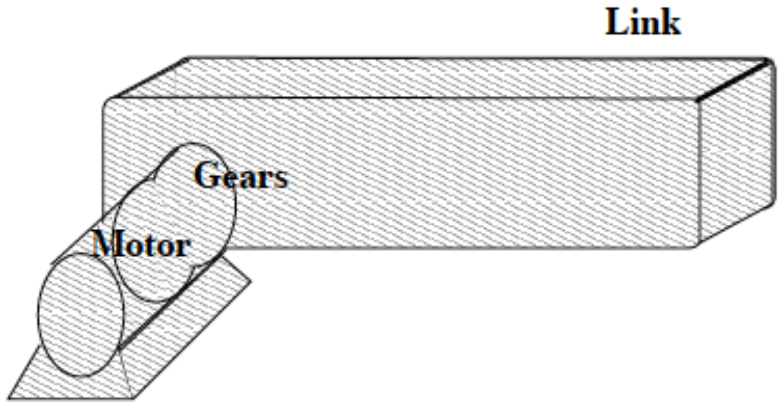
Determine the joint torques needed to support a load of 1 N, acting along positive y_A axis, at the tip of this manipulator when $q_1 = 30^o$ and $d_1 = 1$. Assume $l_1 = 1$.



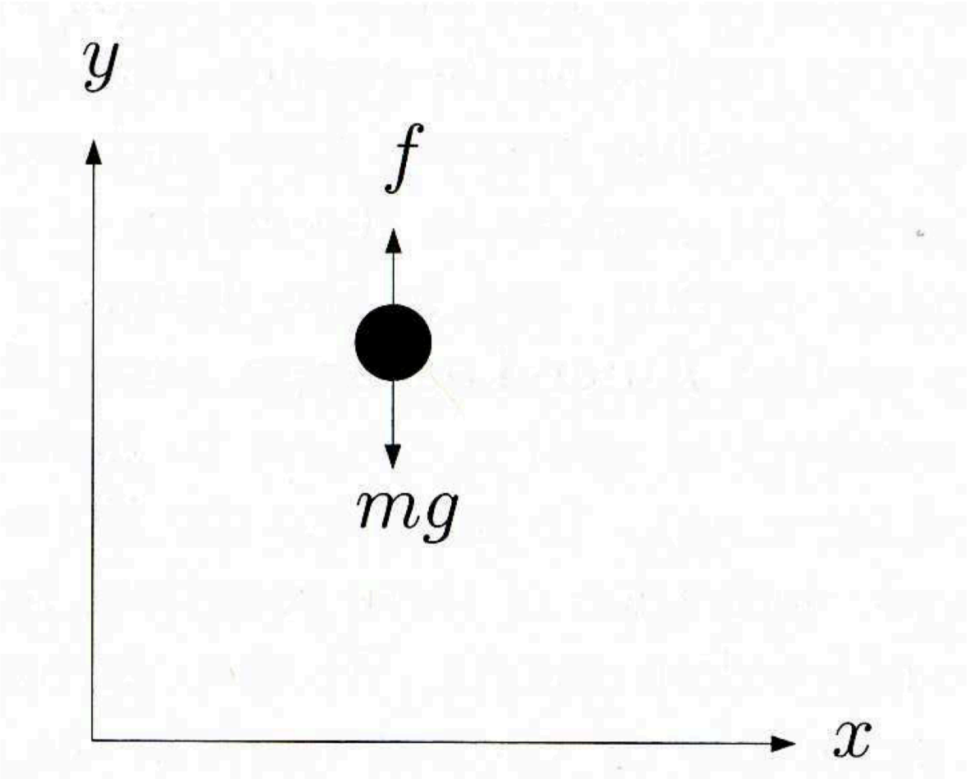
Dynamics: Euler-Lagrange Equations

One DOF system

Single link robot arm



Example: One-DOF system



- f is the external force
- mg is the force acting on the particle due to gravity

Equation of motion as per Newton's second law

$$m\,\ddot{y} = \Sigma F_i = f\ - mg$$

Example: One-DOF system

The equation of motion of the particle

$$m\,\ddot{y} = \Sigma F_i = f\ - mg$$

can be rewritten in a different way!

$$m\ddot{y} = \frac{d}{dt}\left(m\frac{dy}{dt}\right) = \frac{d}{dt}(m\ \frac{\partial}{\partial \dot{y}}[\frac{1}{2}\dot{y}^2] = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{y}}\right)$$

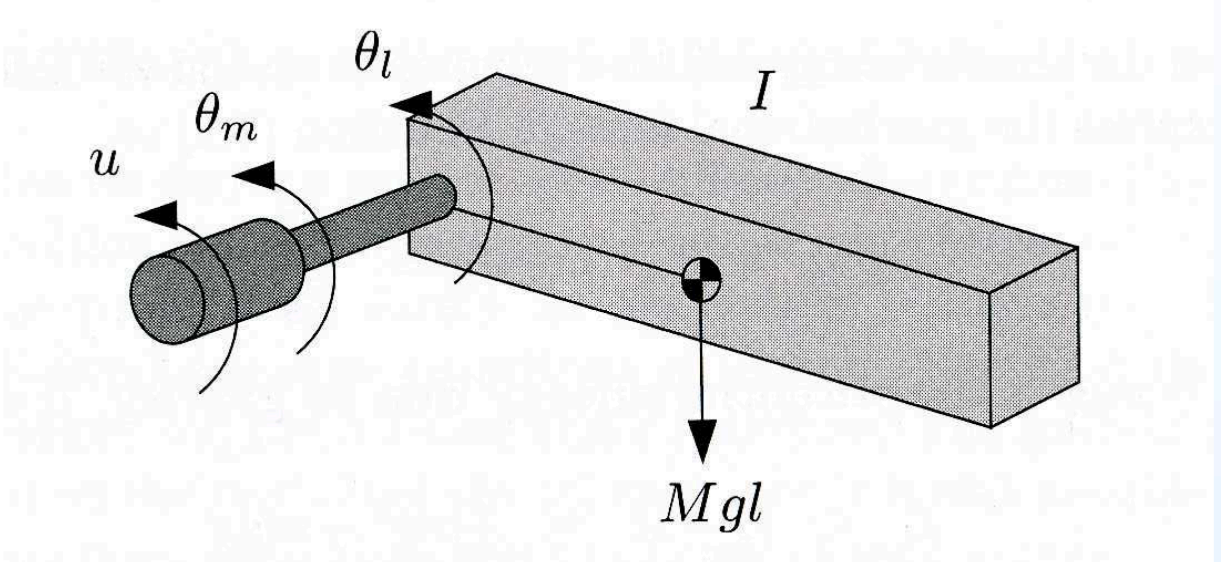
$$mg = \frac{\partial}{\partial y}[mgy] = \frac{\partial}{\partial y}P$$

with $K = \frac{1}{2}m\dot{y}^2$ and $P = mgy$ as the kinetic and potential energy.

Newton's second law can be rewritten as

$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\mathcal{L}\right) - \frac{\partial}{\partial y}\mathcal{L} = f$ with the Lagrangian, $\mathcal{L}(y, \dot{y}) = K\ - P$.

Example: Single-link Arm



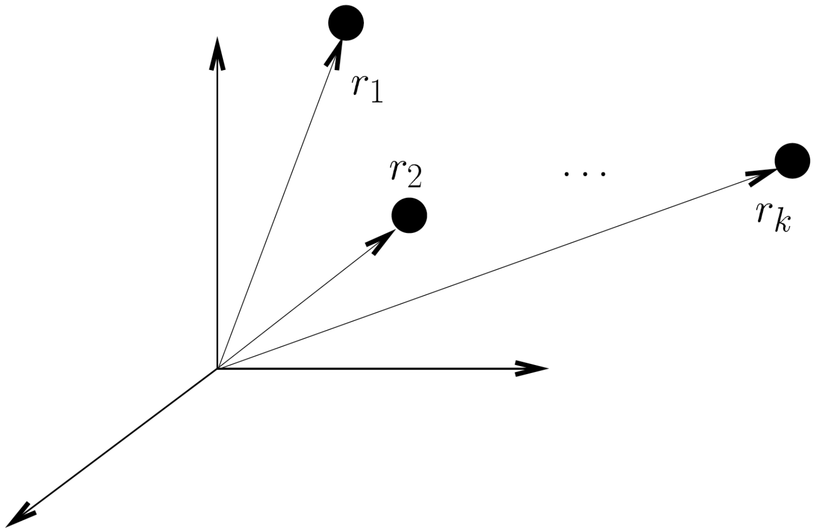
A rigid link (θ_l) coupled to a DC motor (θ_m), through a gear box.

$$\theta_m = r\theta_l$$

- Kinetic energy: $K = \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_l\dot{\theta}_l^2 = \frac{1}{2}(r^2J_m + J_m)\dot{\theta}_l^2$
- Potential energy: $P = Mgl(1 - \cos\theta_l)$
- The Lagrangian is $\mathcal{L} = K - P$, and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}\mathcal{L}\right) - \frac{\partial}{\partial\theta}\mathcal{L} = (r^2J_m + J_l)\ddot{\theta} + Mgl\sin\theta_l = ru$$

Holonomic constraints



- Unconstrained system of k particles
- Degrees of freedom are $3k$
- The number of DoFs is less if the system is constrained.

A constrained imposed on k particles (with coordinates $r_1, r_2, \dots, r_k \in R^3$) is called holonomic, if it is an equality constraint of the form

$$g_i(r_1, r_2, \dots, r_k) = 0 \qquad i = 0, 1, 2, \dots, l$$

and non-holonomic otherwise.

Presence of constraint implies presence of a constraint force, that forces this constraint to hold.

Holonomic constraints

Example: Two particles joined by a massless rigid wire of length l.

$$r_1, r_2 \in R^3 : \left|\left|r_1 - r_2\right|\right|^2 = (r_1 - r_2)^T (r_1 - r_2) = l^2$$

In general,

$$g_i(r_1, r_2, \dots, r_k) = 0 \qquad i = 0, 1, 2, \dots, l$$

Differentiating,

$$\frac{d}{dt} g_i(r_1, r_2, \dots, r_k) = \frac{\partial g_i}{\partial r_1}\frac{dr_1}{dt} + \frac{\partial g_i}{\partial r_2}\frac{dr_2}{dt} + \cdots + \frac{\partial g_i}{\partial r_k}\frac{dr_k}{dt} = 0$$

or,

$$\frac{\partial g_i}{\partial r_1} dr_1 + \frac{\partial g_i}{\partial r_2} dr_2 + \cdots + \frac{\partial g_i}{\partial r_k} dr_k = 0$$

Generalized coordinates

If the system is subject to holonomic constraints then

- If a system consists of k particles, it may be possible to express their coordinates as a functions of fewer than $3k$ variables

$$r_1 = r_1(q_1, \dots, q_n), r_2 = r_2(q_1, \dots, q_n), \dots, r_k = r_k(q_1, \dots, q_n)$$

- The smallest set of variables is called generalized coordinates

- The smallest number n is called the number of degrees of freedom

- If the system consists of an infinite number of particles, it might have finite number of degrees of freedom

Virtual displacements

Given a set of k particles and a holonomic constraints

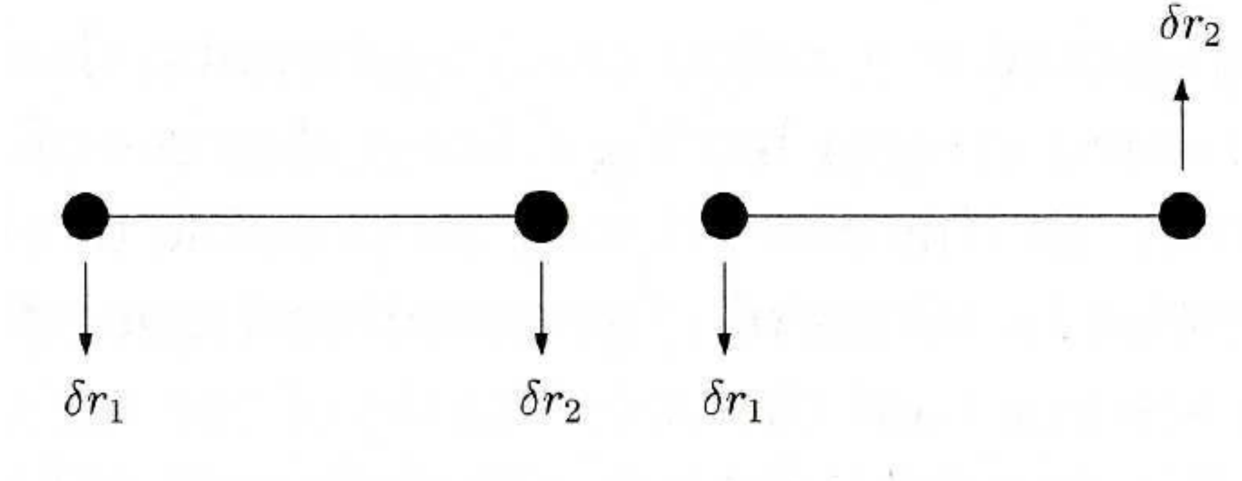
$$g_i(r_1, \dots, r_k) = 0, \qquad i = 1, 2, \dots, l$$

a set of infinitesimal displacements $\delta r_1, \delta r_2, \dots, \delta r_k$ that are consistent with the constraint, i.e.

$$\frac{\partial g_i}{\partial r_1}\delta r_1 + \frac{\partial g_i}{\partial r_2}\delta r_2 + \dots + \frac{\partial g_i}{\partial r_k}\delta r_k = 0, \qquad i = 1, 2, \dots, l$$

are called virtual displacements.

Virtual displacements of a rigid bar



These infinitesimal motions do not destroy the constraint

$$(r_1 - r_2)^T (r_1 - r_2) = l^2$$

If r_1 and r_2 are perturbed

$$r_1 \rightarrow r_1 + \delta r_1 \qquad r_2 \rightarrow r_2 + \delta r_2$$

that is

$$(r_1 + \delta r_1 - r_2 - \delta r_2)^T (r_1 + \delta r_1 - r_2 - \delta r_2) = l^2$$

or,

$$(r_1 - r_2)^T (\delta r_1 - \delta r_2) = 0$$

D'Alembert's principle

Consider a system of k particles, suppose that

- The system has holonomic constraints, that is some of the particles are exposed to constraint forces f_i^c.
- There are externally applied forces f_i^e on the particles.
- The system is moving

Then the work done by all forces applied to the i^{th} particle along each set of virtual displacements is zero, if we add the inertia forces

$$\sum_i \left(f_i^e - \frac{d}{dt}[m\dot{r}_i]\right)^T \delta r_i = 0$$

D'Alembert's principle

Virtual displacements are computed as

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j, \qquad\qquad i = 1, 2, \dots, k$$

Then

$$\sum_{i=1}^{k} f_i^{e^T} \delta r_i = \sum_{i=1}^{k} f_i^{e^T} \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} f_i^{e^T} \frac{\partial r_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{j=1}^{n} \psi_j \delta q_j$$

Functions ψ_j are called generalized forces.

D'Alembert's principle

The second term can be rewritten as

$$\sum_{i=1}^{k} \frac{d}{dt} m_i \dot{r}_i{}^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i^T \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j$$

Now,

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i}^{k} \left\{ \frac{d}{dt} \left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_j} \right] \right\}$$

D'Alembert's principle

Hence,

$$\sum_{i=1}^{k}\sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j}\right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]\right\}\right]\delta q_j$$

Now,

$$v_i = \dot{r}_i = \sum_{j=1}^{n}\frac{\partial r_i}{\partial q_j}\delta \dot{q}_j \;\Rightarrow\; \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] = \sum_{l=1}^{n}\frac{\partial^2 r_i}{\partial q_j \partial q_l}\backslash\dot{\mathrm{q}}_i = \frac{\partial}{\partial q_j}\left[\sum_{l=1}^{n}\frac{\partial r_i}{\partial q_l}\delta \dot{q}_l\right] = \frac{\partial v_i}{\partial q_j}$$

D'Alembert's principle

Hence,

$$\sum_{i=1}^{k}\sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j}\right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]\right\}\right]\delta q_j$$

$$= \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j}\right] - m_i v_i^T \frac{\partial v_i}{\partial q_j}\right\}\right]\delta q_j$$

$$= \sum_{j=1}^{n}\left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}\right]\delta q_j$$

where,

$$K = \sum_{i=1}^{n}\frac{1}{2} m_i |v_i|^2$$

D'Alembert's principle

To summarize,

$$\sum_i \left(f_i^e - \frac{d}{dt}[m\dot{r}_i]\right)^T \delta r_i = 0$$

with

$$\sum_{i=1}^{k} \frac{d}{dt} m_i \dot{r}_i^T \delta r_i = \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}\right] \delta q_j , \qquad \sum_{i=1}^{k} f_i^{eT} \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

is

$$\sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j\right] \delta q_j = 0$$

If δq_j are independent

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j = 0, \qquad i = 1, \dots, n$$

D’Alembert’s principle

If ψ_j are functions are of particular form, then the equations are

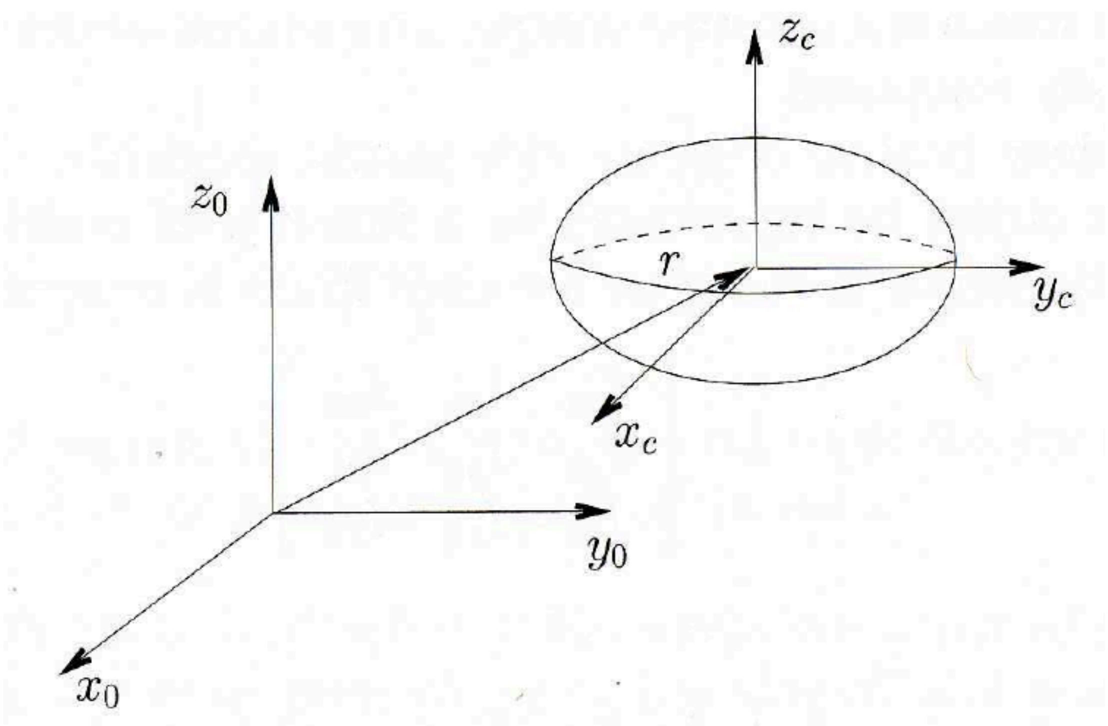
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j,$$

$$\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j,$$

$$\mathcal{L} = K - P,$$

$$i = 1, \dots, n$$

Computing kinetic energy



Kinetic energy of a rigid body comprises of kinetic energy of translation and kinetic energy of rotation

$$K = \frac{1}{2}m|v_c|^2 + \frac{1}{2}\,\omega^T I_i \omega$$

Computing kinetic energy

Computing angular velocity

$$S(\omega) = \dot{R}(t)R^T(t) \quad \rightarrow \omega$$

Matrix I_i is the inertia tensor.

In the body frame, it is constant $I_c = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

To compute inertia tensor in the inertial frame, we can use the formula

$$I_i = R(t)I_cR^T(t)$$

Kinetic energy for n-link manipulator

Kinetic energy of link i

$$K = \frac{1}{2} m \left| v_{c_i} \right|^2 + \frac{1}{2}\, \omega_i^T I_i \omega_i$$

The generalized coordinates q are usually the the joint angles (for revolute joints) and positions (for prismatic joints)

We need to express

- $v_{c_i} = \dot{r}_{c_i}$ as a function of generalized coordinates q and velocities $\dot{q}$ where, r_{c_i} is the position of the center of mass of link i

- ω_i as a function of generalized coordinates q and velocities $\dot{q}$

$$v_{c_i} = J_{v_{c_i}}(q)\dot{q}, \qquad \omega_i = J_{\omega_i}(q)\dot{q}$$

Total energy for n-link manipulator

Total kinetic energy

$$K = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} m_i J_{v\,c_i}^T(q) J_{v\,c_i}(q) \; + J_{\omega\,i}^T(q) R_i(q) I_i R_i^T(q) J_{\omega\,i}(q)\right]\dot{q}$$

$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} \quad = \sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j$$

Total potential energy

$$P = \sum_{i}^{n} P_i = \sum_{i}^{n} m_i g r_{c\,i}$$

Computing Jacobians $J_{v_{c_i}}$ and J_{ω_i}

Follow the same approach that was used to determine end-effector velocities

Using D-H frames,

$$J_{v_{c_i}}^{(k)} = \begin{cases} {}^0z_{k-1} & \text{for prismatic joint, k < i} \\ {}^0z_{k-1} \times \left[r_{c_i} - o_{k-1}\right], & \text{for revolute joint, } k < i \\ 0 & k > i \end{cases}$$

$$J_{\omega_i}^{(k)} = \begin{cases} 0 & \text{for prismatic joint, k < i} \\ {}^0z_{k-1} & \text{for revolute joint, } k < i \\ 0 & k > i \end{cases}$$

Equations of motion

$$\psi_i = -\frac{\partial P}{\partial q_i} + \tau_i$$

where τ_i is the joint torque applied at the i^{th} joint.

Hence, if we define $\mathcal{L} = K - P$

Equations of motion are given by

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial (K-P)}{\partial q_i}, \qquad \mathrm{i} = 1, \ldots, \mathrm{n}$$

Equations of motion

$$\frac{\partial K}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q}\right] = \sum_{j=1}^{n} d_{ij}\dot{q}_j$$

$$\Rightarrow \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} = \frac{d}{dt}\left[\sum_{j=1}^{n} d_{ij}\dot{q}_j\right] = \sum_{j=1}^{n} d_{ij}\ddot{q}_j + \sum_{j=1}^{n}\frac{d}{dt}\left[d_{ij}(q)\right]\dot{q}_j$$

$$= \sum_{j=1}^{n} d_{ij}\ddot{q}_j + \sum_{j=1}^{n}\left(\sum_{k=1}^{n}\frac{\partial d_{ij}(q)}{\partial q_k}\dot{q}_k\right)\dot{q}_j$$

$$= \sum_{j=1}^{n} d_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial d_{ij}(q)}{\partial q_k} + \frac{\partial d_{ik}(q)}{\partial q_j}\right)\dot{q}_k\dot{q}_j$$

Equations of motion

$$\frac{\partial(K-P)}{\partial q_i} = \frac{\partial}{\partial q_i}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} - \frac{\partial}{\partial q_i}P\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_i}D(q)\right]\dot{q} - \frac{\partial}{\partial q_i}P$$

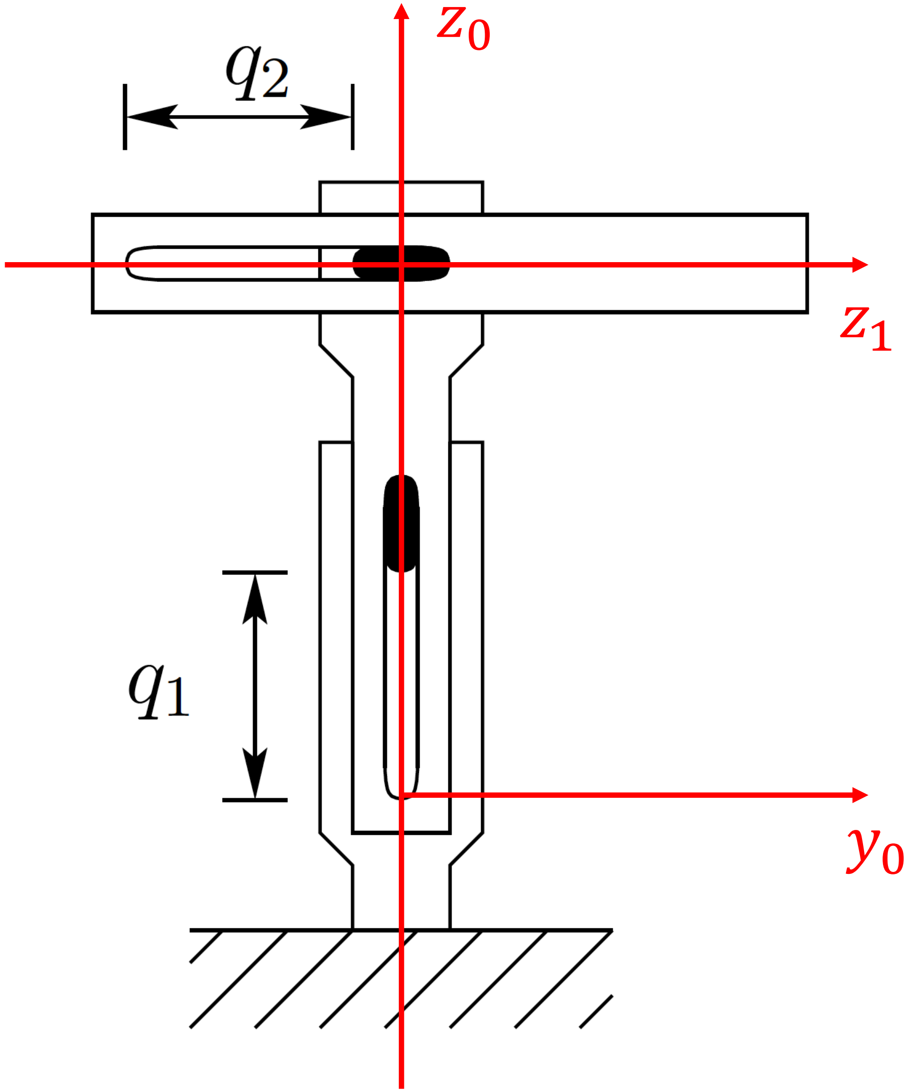
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial d_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k - \frac{\partial}{\partial q_i}P$$

Subtracting, the equations of motion are

$$\sum_{j=1}^{n} d_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j}^{n}\sum_{k}^{n}\underbrace{\boxed{\left(\frac{\partial d_{ik}}{\partial q_j} + \frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{jk}}{\partial q_i}\right)}}_{c_{ijk}}\dot{q}_j\dot{q}_k + \underbrace{\boxed{\frac{\partial}{\partial q_i}P}}_{g(q)} = \tau_i$$

$$i = 1, 2, \ldots, n$$

Example: Cartesian manipulator



	θ	d	α	a
1	0	q_1	$-\frac{\pi}{2}$	0
2	0	q_2	0	0

DH parameters

Only prismatic joints: $J_\omega = 0$

$$J_{v_{c_1}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad J_{v_{c_2}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Cartesian manipulator: equations of motion

$$v_{c_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{q}, \qquad v_{c_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \dot{q}$$

Hence, the kinetic and potential energy are

$$K = \frac{1}{2} \dot{q}^T \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \dot{q}, \qquad P = g(m_1 + m_2)q_1 + Const$$

The equations of motion are:

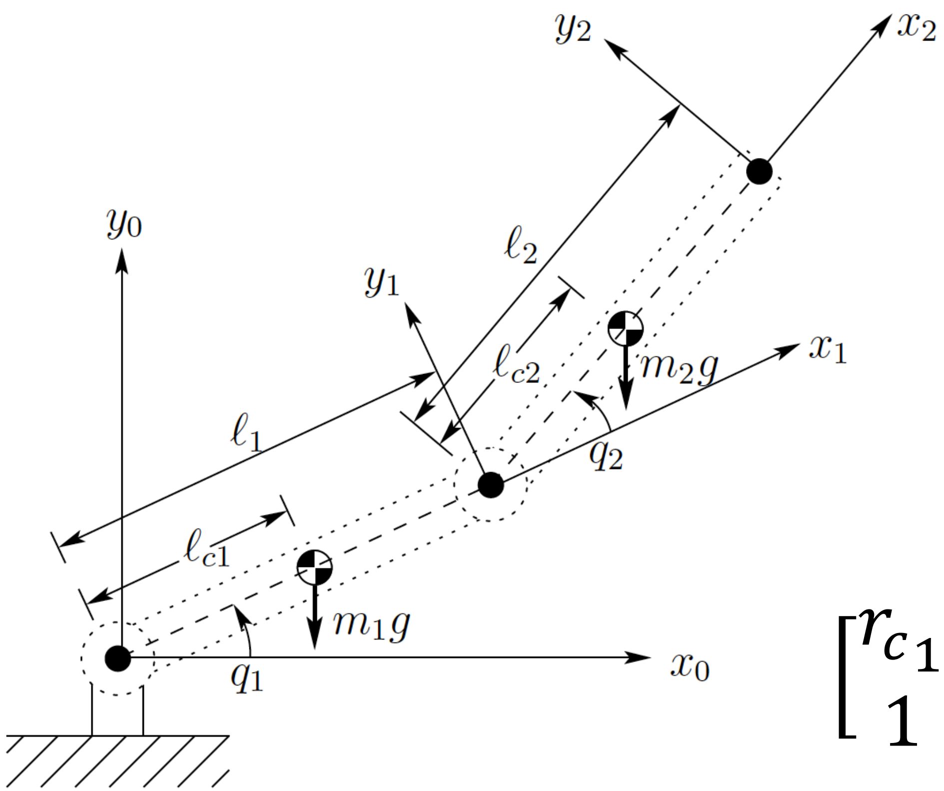
$$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_i} - \frac{\partial (K - P)}{\partial q_i} = \tau_i$$

or,

$$(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) = \tau_1$$

$$m_2 \ddot{q}_2 = \tau_2$$

Example: Two-link manipulator



	$\boldsymbol{\theta}$	d	$\boldsymbol{\alpha}$	$\boldsymbol{a}$
1	q_1	0	0	l_1
2	q_2	0	0	l_2

DH parameters

$$\begin{bmatrix} r_{c_1} \\ 1 \end{bmatrix} = {}^0_1T \begin{bmatrix} l_1 - l_{c_1} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} r_{c_2} \\ 1 \end{bmatrix} = {}^0_2T \begin{bmatrix} l_2 - l_{c_2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Jacobians: Two-link manipulator

$$J_{v\,c_1}^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c_1} c_1 \\ l_{c_1} s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_{c_1} s_1 \\ l_{c_1} c_1 \\ 0 \end{bmatrix}, \qquad J_{v\,c_1}^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad J_{v\,c_1} = [J_{v\,c_1}^1 \quad J_{v\,c_1}^2]$$

$$J_{v\,c_2}^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_1 c_1 + l_{c_2} c_{12} \\ l_1 s_1 + l_{c_2} s_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_{c_2} s_{12} \\ l_1 c_1 + l_{c_2} c_{12} \\ 0 \end{bmatrix},$$

$$J_{v\,c_2}^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c_2} c_{12} \\ l_{c_2} s_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} -l_{c_2} s_{12} \\ l_{c_2} c_{12} \\ 0 \end{bmatrix}, \qquad J_{v\,c_2} = [J_{v\,c_2}^1 \quad J_{v\,c_2}^2]$$

$$J_{\omega_1} = [z_0 \quad 0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad J_{\omega_2} = [z_0 \quad z_1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Kinetic Energy: Two-link manipulator

Linear and angular velocities

$$v_{c_1} = J_{v_{c_1}}\dot{q}, \quad \omega_1 = J_{\omega_1}\dot{q}, \qquad v_{c_2} = J_{v_{c_2}}\dot{q}, \quad \omega_2 = J_{\omega_2}\dot{q}$$

Translational kinetic energy

$$K_{trans} = \frac{1}{2}m_1 v_{c_1}^T v_{c_1} + \frac{1}{2}m_2 v_{c_2}^T v_{c_2} = \frac{1}{2}\dot{q}^T[m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}}]\dot{q}$$

Rotational kinetic energy

$$K_{rot} = \frac{1}{2}\dot{q}^T\left[J_{\omega_1}^T(q)R_1(q)I_1R_1^T(q)J_{\omega_1}(q) + J_{\omega_2}^T(q)R_2(q)I_2R_2^T(q)J_{\omega_1}(q)\right]\dot{q}$$

$$= \frac{1}{2}\dot{q}^T\left\{(I_{33})_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + (I_{33})_2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\}\dot{q}$$

Hence,

$$D(q) = m_1 J_{v_{c_1}}^T J_{v_{c_1}} + m_2 J_{v_{c_2}}^T J_{v_{c_2}} + \begin{bmatrix}(I_{33})_2 + (I_{33})_1 & (I_{33})_2\\ (I_{33})_2 & (I_{33})_2\end{bmatrix}$$

Equations of motion: Two-link manipulator

$$d_{11} = m_1 l_{c\,1}^2 + m_2\left(l_1^2 + l_{c\,2}^2 + 2 l_1 l_{c_2} c_2\right) + (I_{33})_1 + (I_{33})_2$$

$$d_{12} = d_{21} = m_2\left(l_{c_2}^{\;2} + l_1 l_{c_2} c_2\right) + (I_{33})_2, \qquad d_{22} = m_2 l_{c\,2}^2 + (I_{33})_2$$

$$c_{ijk} = \left(\frac{\partial d_{ik}}{\partial q_j} + \frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{jk}}{\partial q_i}\right)$$

$$c_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0,$$

$$c_{112} = c_{121} = \frac{1}{2}\left(\frac{\partial d_{12}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_1}\right) = -m_2 l_1 l_{c_2} s_2 =: h,$$

$$c_{122} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h, \qquad c_{211} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h,$$

$$c_{212} = c_{221} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0, \qquad c_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = -h$$

Equations of motion: Two-link manipulator

Potential energy

$$\mathrm{P}_1 = m_1 g l_{c_1} s_1, \qquad \mathrm{P}_2 = m_2 g\left(l_1 s_1 + l_{c_2} s_{12}\right), \qquad P = P_1 + P_2$$

Hence,

$$\phi_1 = \frac{\partial P}{\partial q_1} = \left(m_1 l_{c_1} + m_2 l_1\right) g c_1 + m_2 l_{c_2} g c_{12}, \qquad \phi_2 = \frac{\partial P}{\partial q_2} = m_2 g l_{c_2} c_{12}$$

Equations of motion

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{111}\dot{q}_1^2 + c_{112}\dot{q}_1\dot{q}_2 + c_{121}\dot{q}_2\dot{q}_1 + c_{122}\dot{q}_2^2 + \phi_1 = \tau_1,$$
$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{211}\dot{q}_1^2 + c_{212}\dot{q}_1\dot{q}_2 + c_{221}\dot{q}_2\dot{q}_1 + c_{222}\dot{q}_2^2 + \phi_1 = \tau_1$$

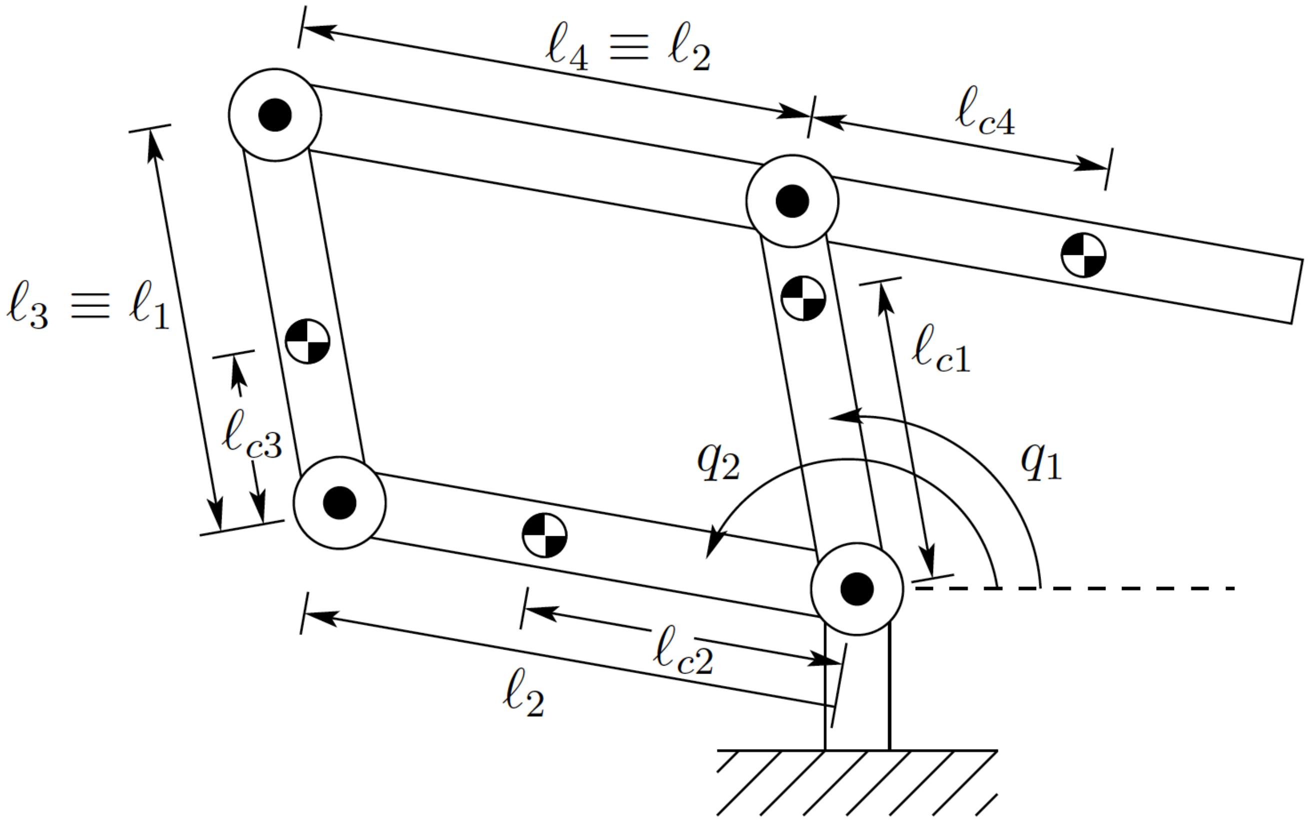
In matrix form,

$$D(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

where,

$$C(q,\dot{q}) = \begin{bmatrix} h\dot{q}_2 & h\dot{q}_2 + h\dot{q}_1 \\ -h\dot{q}_1 & 0 \end{bmatrix}; \qquad G(q) = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Example: Five bar linkage



$$\begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = \begin{bmatrix} l_{c1} c_1 \\ l_{c1} s_1 \end{bmatrix},$$

$$\begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = \begin{bmatrix} l_{c2} c_2 \\ l_{c2} s_2 \end{bmatrix},$$

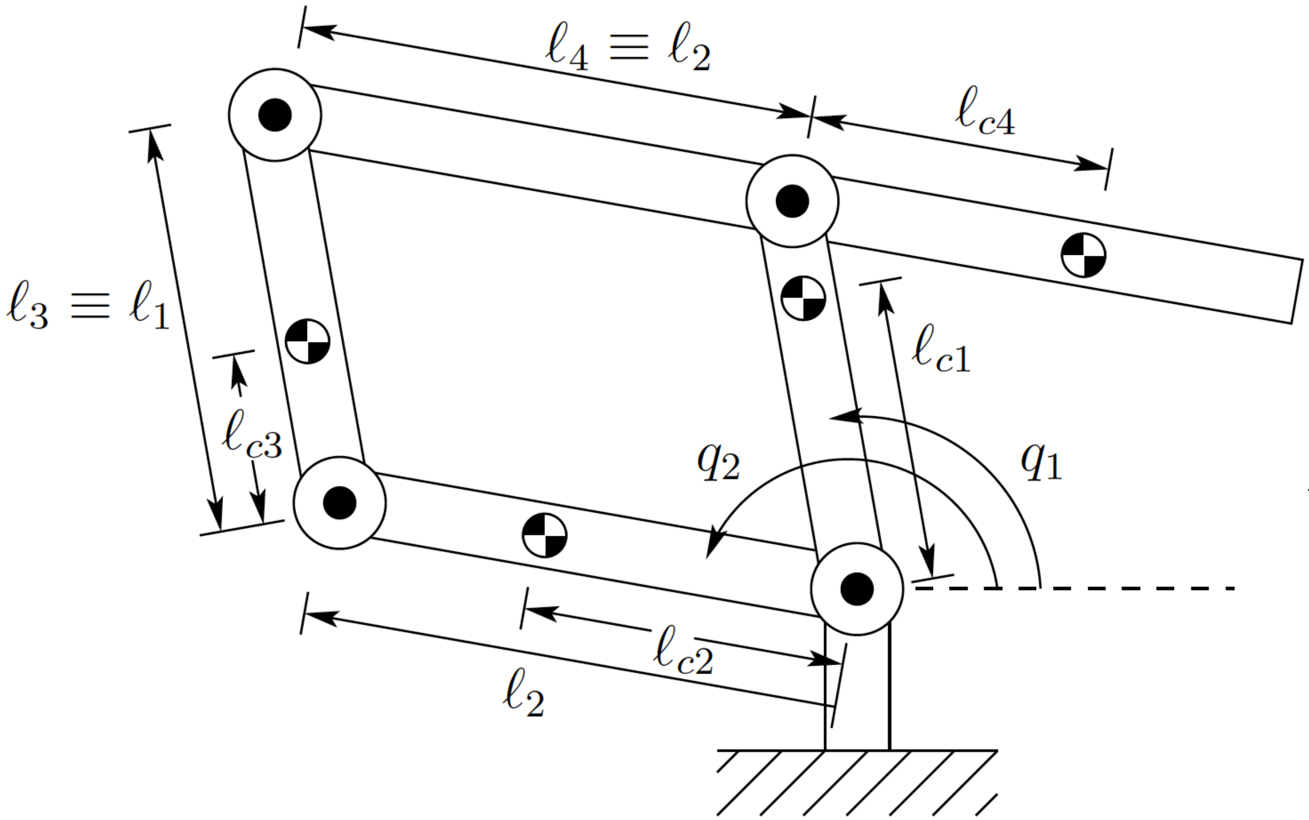
$$\begin{bmatrix} x_{c3} \\ y_{c3} \end{bmatrix} = \begin{bmatrix} l_2 c_2 \\ l_2 s_2 \end{bmatrix} + \begin{bmatrix} l_{c3} c_1 \\ l_{c3} s_1 \end{bmatrix},$$

$$\begin{bmatrix} x_{c4} \\ y_{c4} \end{bmatrix} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \end{bmatrix} - \begin{bmatrix} l_{c4} c_2 \\ l_{c4} s_2 \end{bmatrix}$$

$$v_{c1} = \begin{bmatrix} -l_{c1} s_1 & 0 \\ l_{c1} c_1 & 0 \end{bmatrix} \dot{q}, \qquad v_{c2} = \begin{bmatrix} 0 & -l_{c2} s_2 \\ 0 & l_{c2} c_2 \end{bmatrix} \dot{q},$$

$$v_{c3} = \begin{bmatrix} -l_{c3} s_1 & -l_2 s_2 \\ l_{c3} c_1 & l_2 c_2 \end{bmatrix} \dot{q}, \qquad v_{c4} = \begin{bmatrix} -l_1 s_1 & l_{c4} s_2 \\ l_1 c_1 & -l_{c4} c_2 \end{bmatrix} \dot{q}$$

Example: Five bar linkage



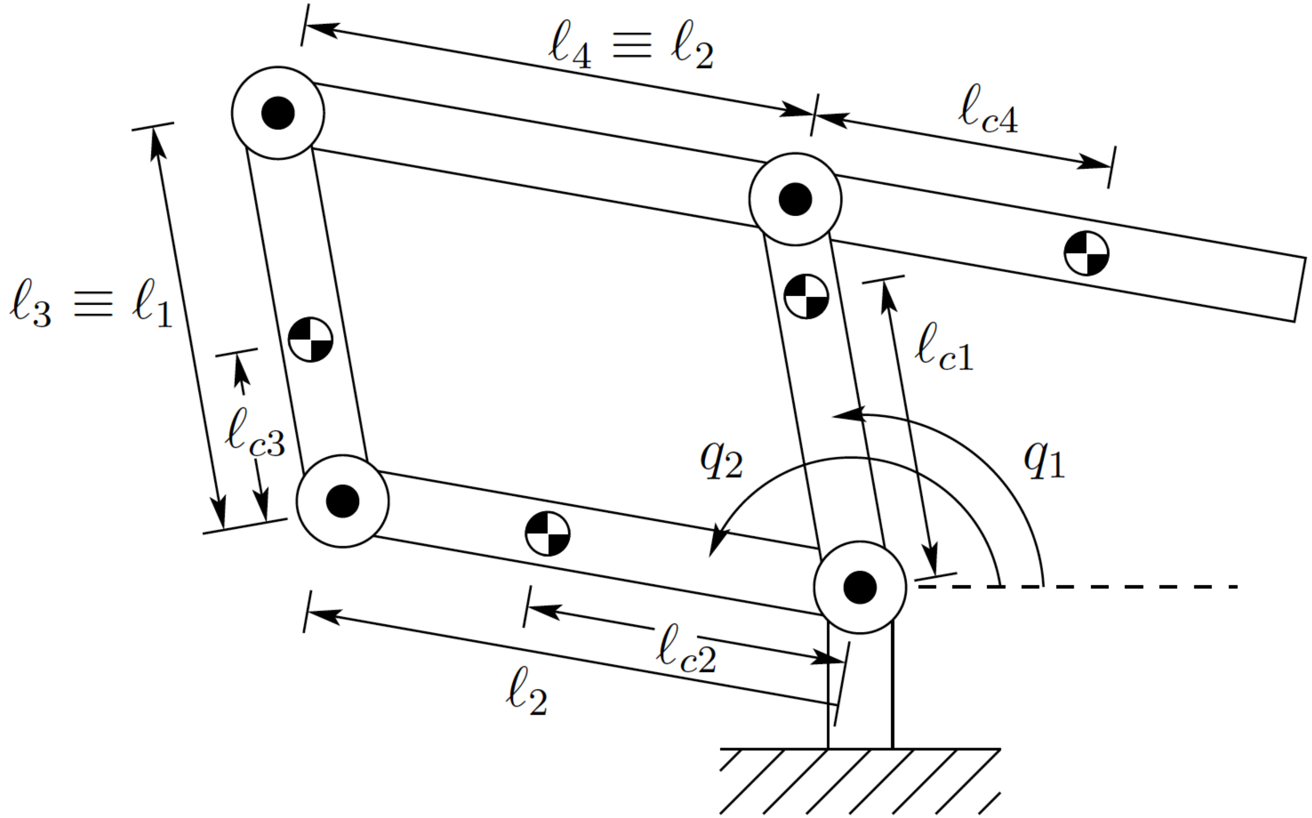
$$\omega_1 = \omega_3 = \dot{q}_1 \hat{k}, \qquad \omega_2 = \omega_4 = \dot{q}_2$$

$$D(q) = \sum_{i=1}^{4} m_i J_{vci}^T J_{vci} + \begin{bmatrix} I_1 + I_3 & 0 \\ 0 & I_2 + I_4 \end{bmatrix}$$

0 if $m_2 l_2 l_{c3} = m_4 l_1 l_{c4}$

$$d_{11}(q) = m_1 l_{c1}^2 + m_3 l_{c3}^2 + m_4 l_4^2 + I_1 + I_3,$$
$$d_{21}(q) = d_{12}(q) = (m_2 l_2 l_{c3} - m_4 l_1 l_{c4}) \cos(q_2 - q_1),$$
$$d_{22}(q) = m_2 l_{c2}^2 + m_3 l_2^2 + m_4 l_{c4}^2 + I_2 + I_4$$

Example: Five bar linkage



Constant diagonal inertia matrix

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no Coriolis/Centrifugal terms!

Potential energy

$$P = \sum_{i=1}^{4} y_{ci} = g(m_1 l_{c1} + m_3 l_{c3} + m_4 l_1)s_1 + g(m_2 l_{c2} - m_4 l_{c4} + m_3 l_2)s_2$$

Hence,

$$\phi_1 = g(m_1 l_{c1} + m_3 l_{c3} + m_4 l_1)c_1$$
$$\phi_2 = g(m_2 l_{c2} - m_4 l_{c4} + m_3 l_2)c_2$$

Equations of motion:

$d_{11}\ddot{q}_1 + \phi_1 = \tau_1,$ $\qquad$ $d_{22}\ddot{q}_2 + \phi_2 = \tau_2$

Properties: Skew-symmetry and passivity

- Matrix $N(q,\dot{q}) = \dot{D}(q) - 2C(q,\dot{q})$ is skew-symmetric

$$\dot{d}_{kj} = \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \,,$$

$$n_{kj} = \dot{d}_{kj} - 2c_{kj} = \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} - \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \right] \dot{q}_i = \sum_{i=1}^{n} \left[\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j} \right] \dot{q}_i$$

$$= -n_{jk}$$

- Passivity property: There exists a constant $\beta \geq 0$, such that

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta \geq -\beta, \qquad \forall T \geq 0$$

Amount of energy produced by the system has a lower bound given by $-\beta$.

Properties: Passivity and total energy

- Total energy in the system $H = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$
- Along the system trajectory,

$$\dot{H} = \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T \frac{\partial P}{\partial q}$$

$$= \dot{q}^T\big(\tau - C(q,\dot{q}) - G(q)\big) + \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T \frac{\partial P}{\partial q}$$

$$= \dot{q}^T \tau + \frac{1}{2}\dot{q}^T \left(\dot{D}(q) - 2C(q,\dot{q})\right)\dot{q} = \dot{q}^T \tau$$

- Integrating,

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta = H(T) - H(0) \geq -H(0)$$