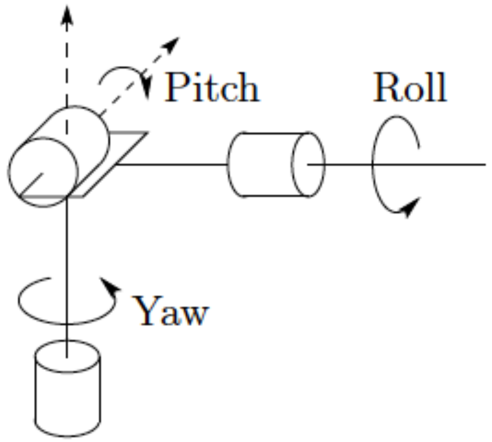
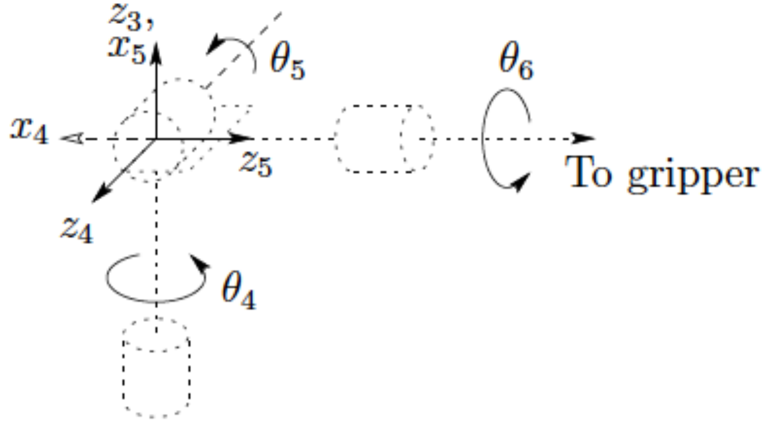
DH parameters example: Spherical wrist





Link	a_i	α_i	d_i	θ_i
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

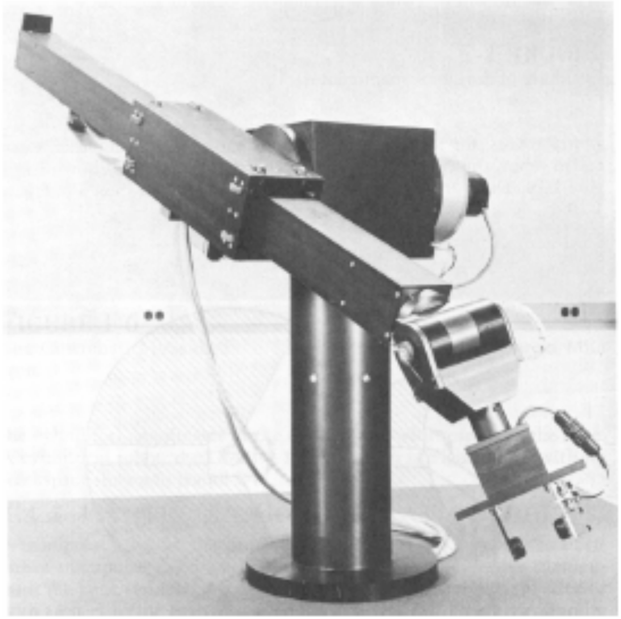
Spherical wrist: ZYZ Euler angles

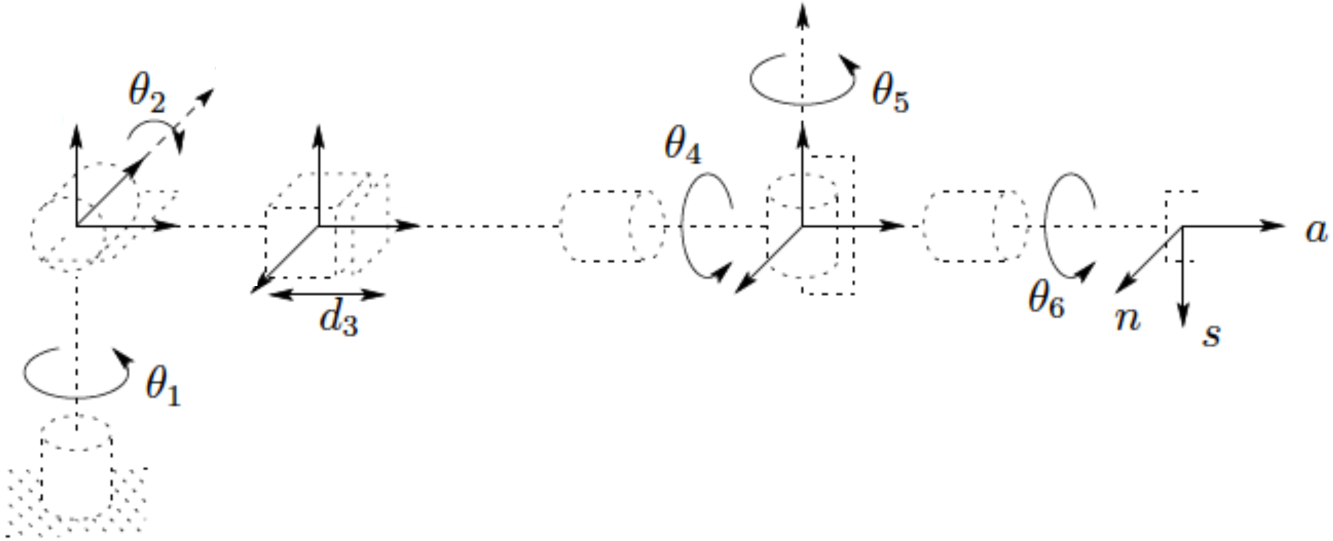
$$
\begin{aligned}
{}^0_4\mathrm{T} &= A_4A_5A_6 \\
&= \begin{bmatrix} {}^3_6R & p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

θ_4, θ_5 and θ_6 are the Euler angles ϕ, θ, and φ with respect to frame 3!

$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi}R_{y,\theta}R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

DH frames: Stanford Manipulator





DH parameters: Stanford Manipulator

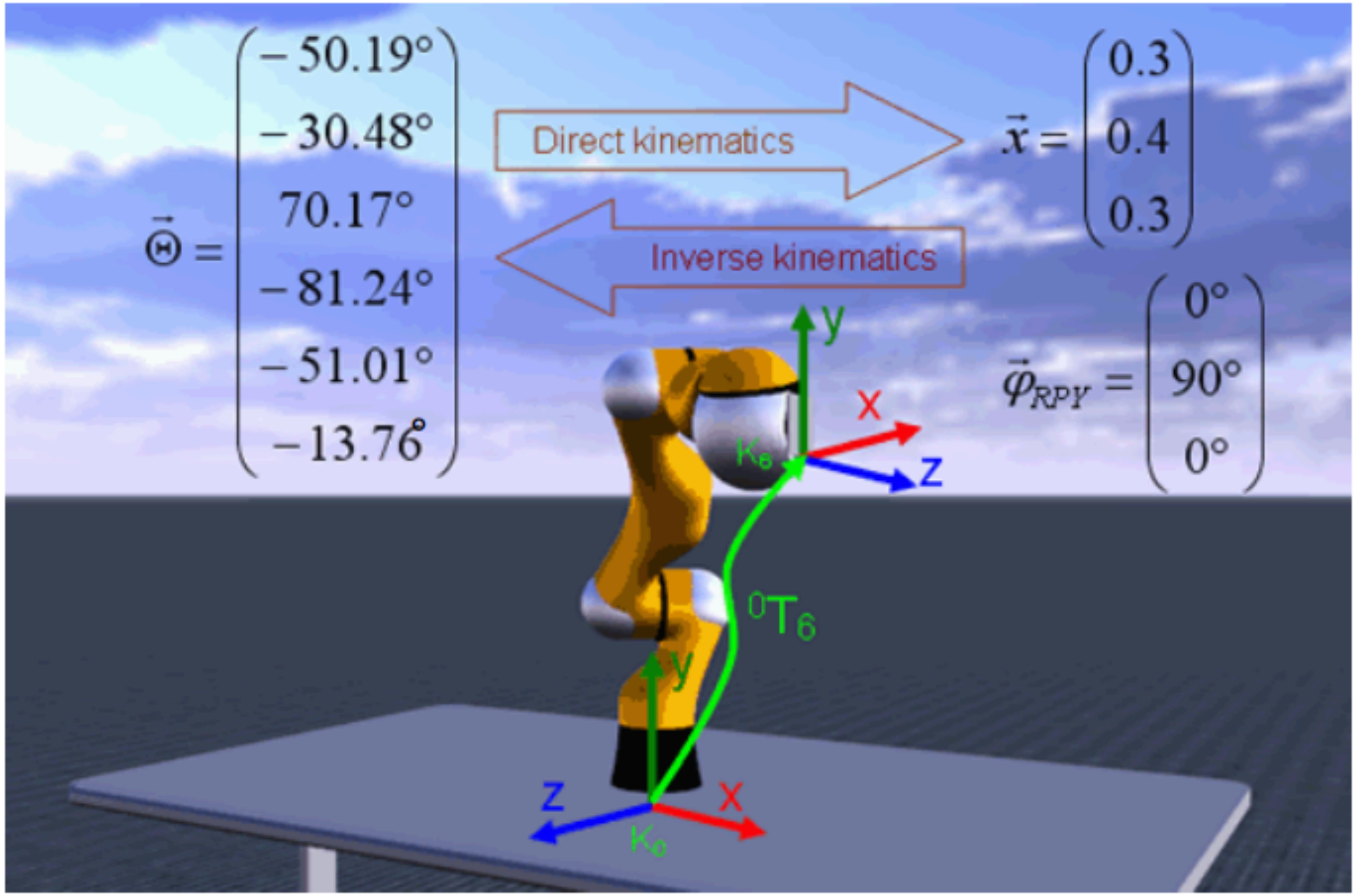
Link	d_i	a_i	α_i	θ_i
1	0	0	-90	$\theta^\star$
2	d_2	0	-90	$\theta^\star$
3	$d^\star$	0	0	0
4	0	0	-90	$\theta^\star$
5	0	0	$+90$	$\theta^\star$
6	d_6	0	0	$\theta^\star$

* joint variable

$${}^{0}_{6}\mathrm{T} = {}^{0}_{3}\mathrm{T}\, {}^{3}_{6}\mathrm{T}$$

$$\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6)\\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6)\\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6\\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6)\\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6)\\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6\\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5\\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5\\
r_{33} &= -s_2c_4s_5 + c_2c_5\\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5)\\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2)\\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}$$

Inverse Kinematics problem



Forward kinematics is always unique.

What about inverse kinematics?

Inverse kinematics problem

- Given a desired end-effector pose (position and orientation), find the values of joint variables that will realize it.
- Given

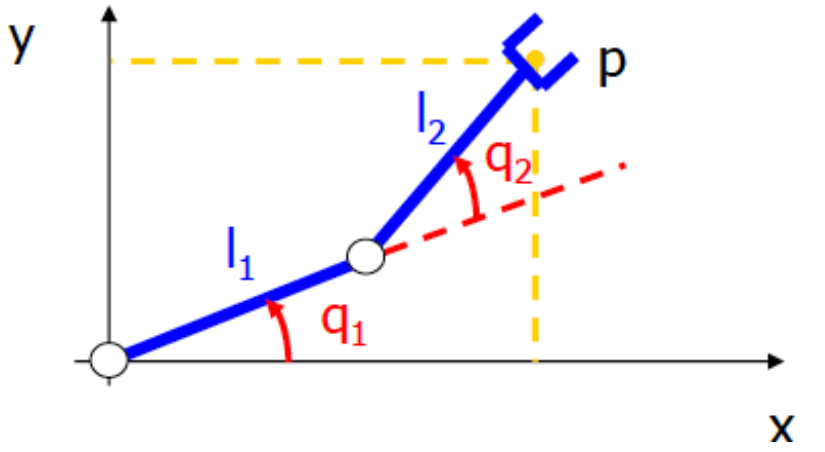
$$H = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

find (one or all) solutions of the equation

$$H = {}^{0}_{6}T(q_1, q_2, \dots, q_n)$$

- 12 nonlinear equations

Example: 2-link manipulator



$$p_x = l_1\, c_1 + l_2\, c_{12}$$

$$p_y = l_1\, s_1 + l_2\, s_{12}$$

$$p_x^2 + p_y^2 - (l_1^2 + l_2^2) = 2\, l_1\, l_2\, (c_1\, c_{12} + s_1\, s_{12}) = 2\, l_1\, l_2\, c_2$$

$$c_2 = (p_x^2 + p_y^2 - l_1^2 - l_2^2) / 2\, l_1\, l_2, \quad s_2 = \pm\sqrt{1 - c_2^2}$$

$$q_2 = \mathrm{atan2}(c_2, s_2)$$

What is the set of points that the end effector can reach?

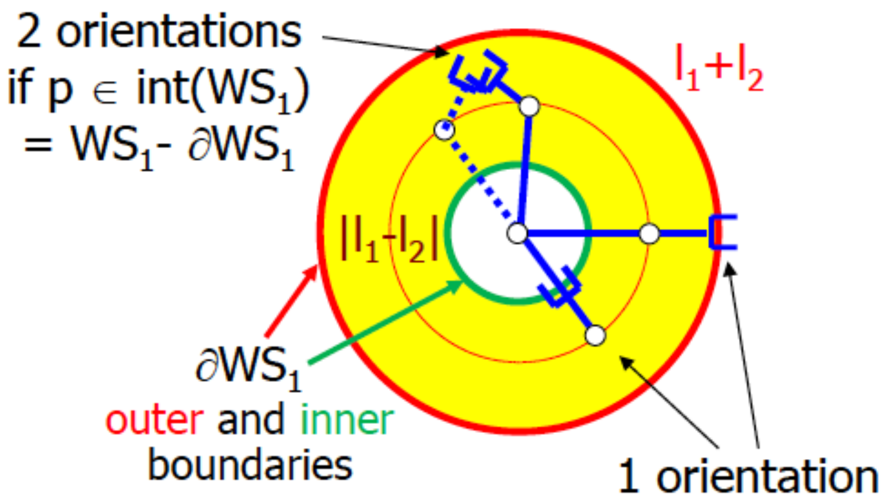
How many solutions are there if a point can be reached?

Can all positions be reached with any orientation?

Under what conditions can some positions be reached with any orientation?

Robot Workspace

- Primary workspace WS_1: set of all positions that can be reached with atleast one orientation
  - Out of WS_1 there is solution to the inverse kinematics problem
  - If $p \in WS_1$ there is at least one suitable orientation for which solution exists
- Secondary (or dexterous) workspace WS2: set of all positions that can be reached with any orientation
  - If $p \in WS_2$, a solution exists for any orientation



if $l_1 \neq l_2$

- $WS_1 = \{p \in \mathcal{R}^2 : |l_1-l_2| \leq \|p\| \leq l_1+l_2\} \subset \mathcal{R}^2$
- $WS_2 = \varnothing$

if $l_1 = l_2 = \ell$

- $WS_1 = \{p \in \mathcal{R}^2 : \|p\| \leq 2\ell\} \subset \mathcal{R}^2$
- $WS_2 = \{p = 0\}$

Solution methods

ANALYTICAL solution
(in closed form)

$\rightleftharpoons$

NUMERICAL solution
(in iterative form)

- preferred, if it can be found*
- use ad-hoc geometric inspection
- algebraic methods (solution of polynomial equations)
- systematic ways for generating a reduced set of equations to be solved

* sufficient conditions for 6-dof arms

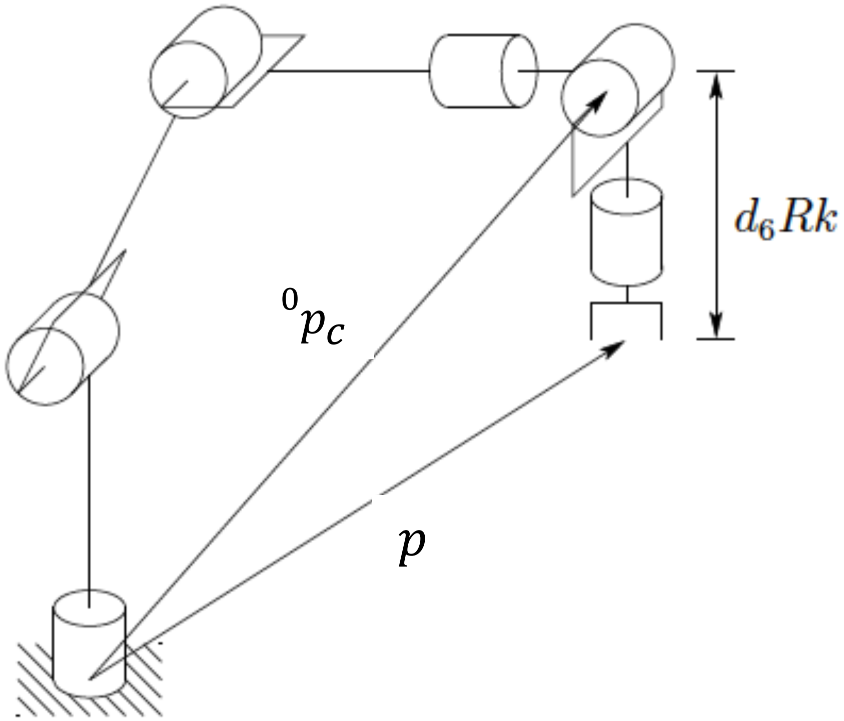
- 3 consecutive rotational joint axes are incident (e.g., spherical wrist), or
- 3 consecutive rotational joint axes are parallel

- certainly needed if $n>m$ (redundant case), or at/close to singularities
- slower, but easier to be set up
- in its basic form, it uses the (analytical) Jacobian matrix of the direct kinematics map

$$J_r(q) = \frac{\partial f_r(q)}{\partial q}$$

- Newton method, Gradient method, and so on...

Kinematic decoupling : 6DOF robot with spherical wrist



$$^0_6R(q_1, \dots, q_6) = R$$
$$^0_6p(q_1, \dots, q_6) = p$$

$$p = \ {}^0p_c + d_6\, R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

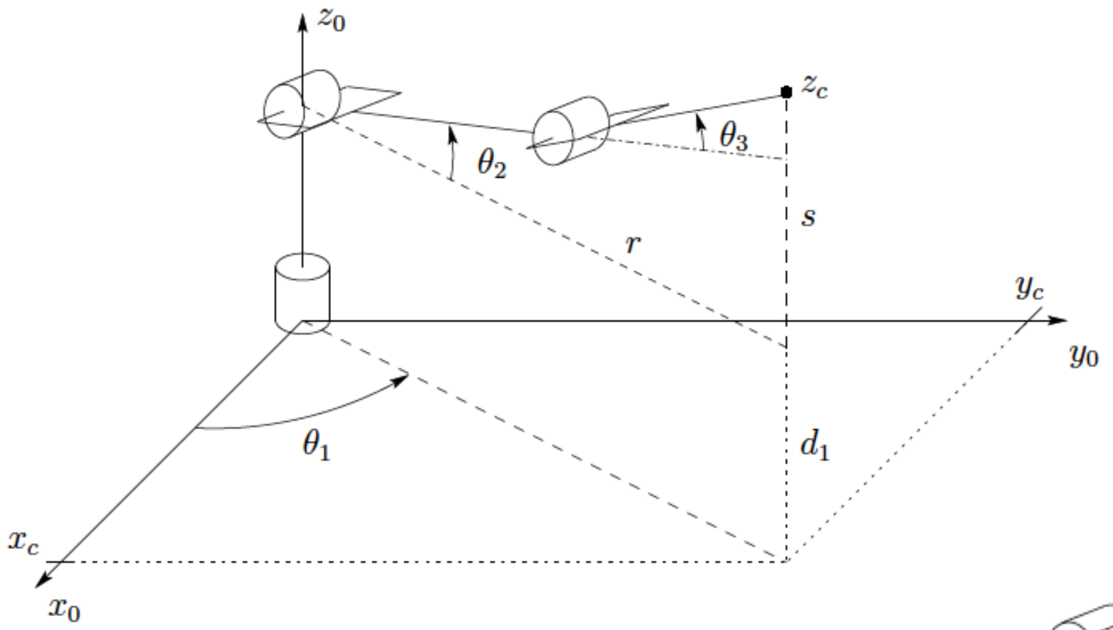
Solve for q1, q2, q3

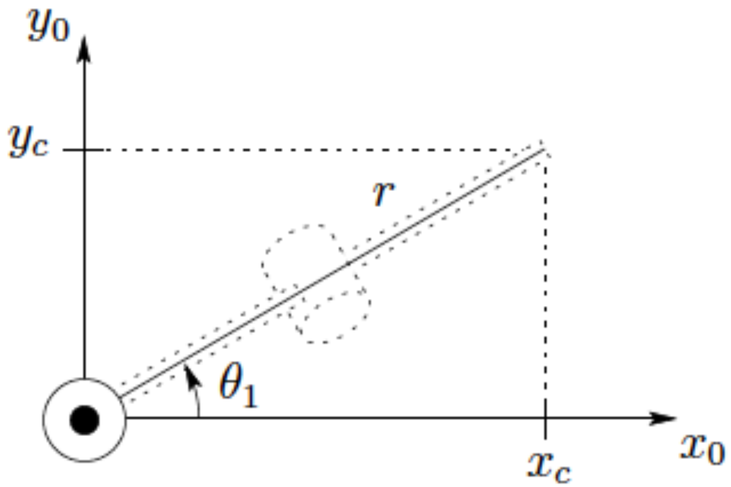
$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} p_x - d_6 r_{13} \\ p_y - d_6 r_{23} \\ p_z - d_6 r_{33} \end{bmatrix}$$

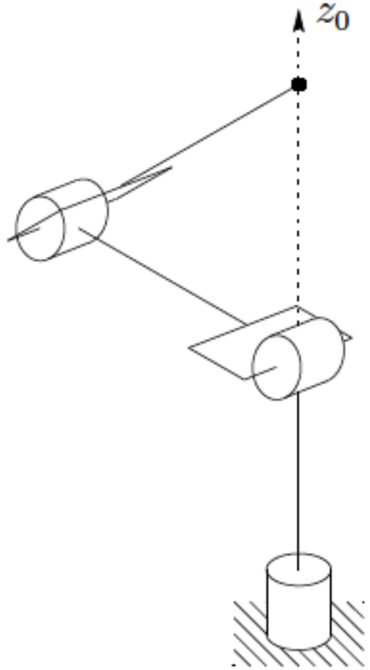
Solve for q4, q5, q6

$$R = {}^0_3R\, {}^3_6R \Rightarrow {}^3_6R = {}^0_3R^T R$$

Inverse kinematics: Elbow manipulator







Two solutions for θ_1 :

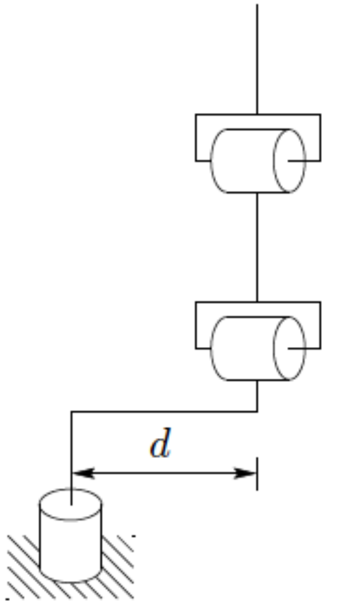
$$\theta_1 = \operatorname{atan2}(x_c, y_c)$$

$$\theta_1 = \pi + \operatorname{atan2}(x_c, y_c)$$

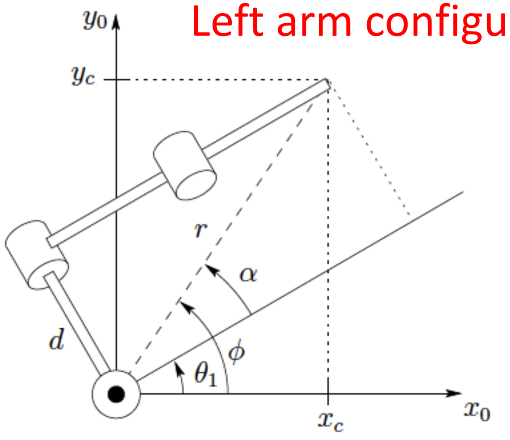
What if $x_c = y_c = 0$?

Infinite solutions for θ_1!

Elbow manipulator with shoulder offset



Left arm configuration

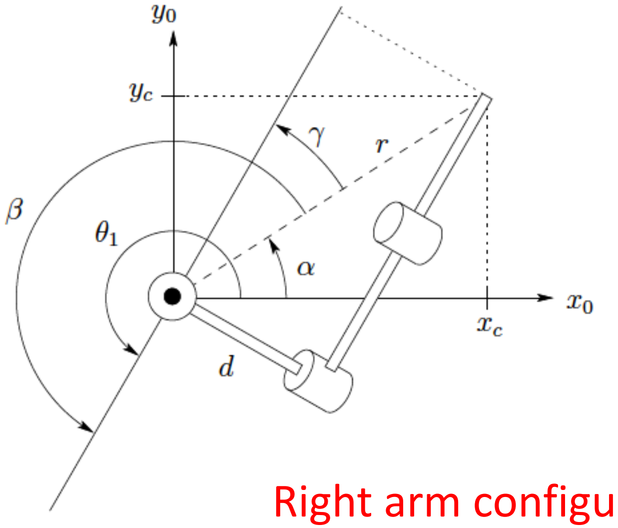


$$\phi = \mathrm{atan2}(x_c, y_c)$$

$$\alpha = \mathrm{atan2}\left(\sqrt{r^2 - d^2}, d\right)$$

$$= \mathrm{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right)$$

$$\theta_1 = \phi - \alpha$$



Right arm configuration

$$\alpha = \mathrm{atan2}(x_c, y_c)$$

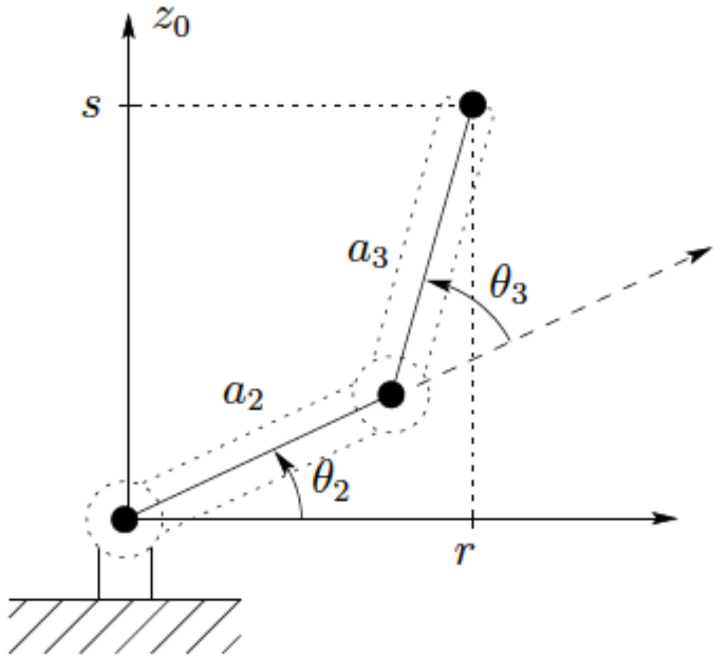
$$\beta = \gamma + \pi$$

$$\gamma = \mathrm{atan2}(\sqrt{r^2 - d^2}, d)$$

$$\beta = \mathrm{atan2}\left(-\sqrt{r^2 - d^2}, -d\right)$$

$$\theta_1 = \alpha + \beta$$

Inverse kinematics: Elbow manipulator

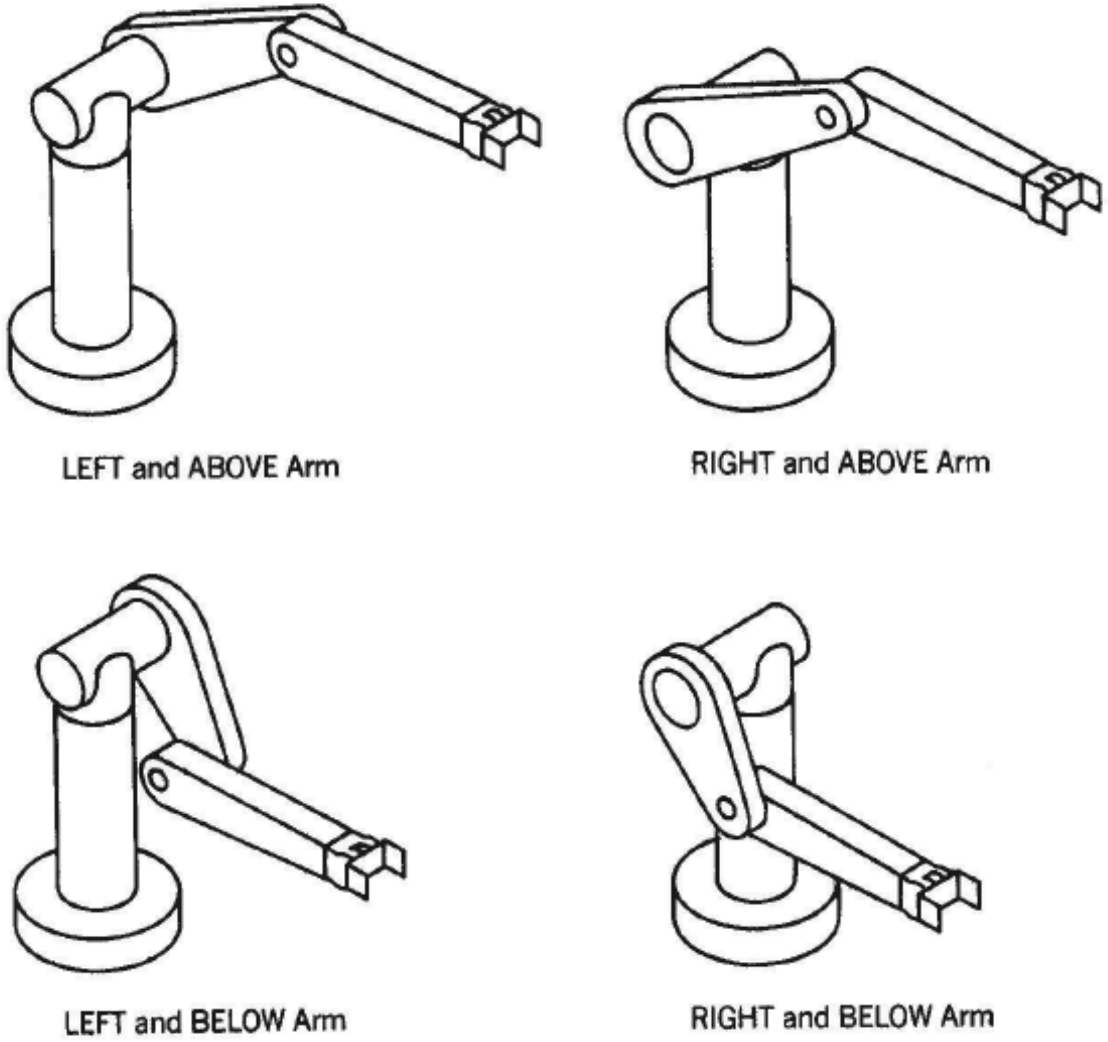


$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$
$$= \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} := D$$

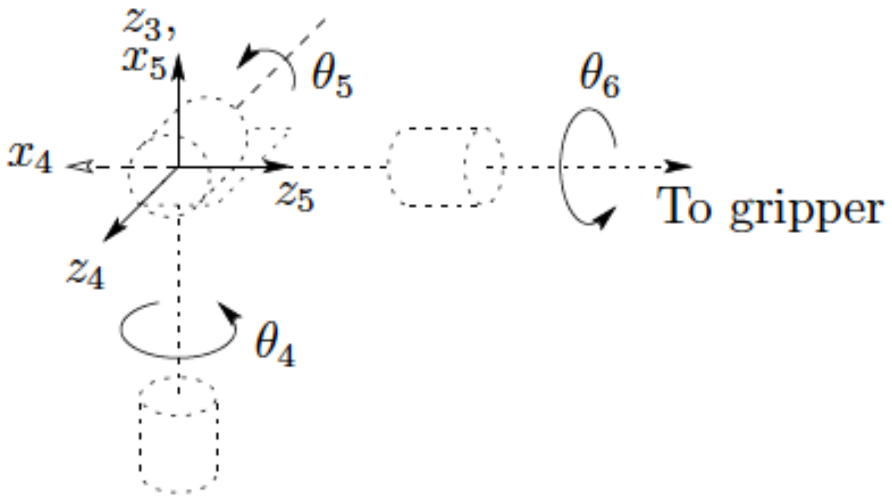
$$\theta_3 = \text{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$$

$$\theta_2 = \text{atan2}(r, s) - \text{atan2}(a_2 + a_3c_3, a_3s_3)$$
$$= \text{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \text{atan2}(a_2 + a_3c_3, a_3s_3)$$

Elbow manipulator: 4 solutions



Inverse orientation: spherical wrist



Recall: θ_4, θ_5 and θ_6 are the Euler angles ϕ, θ, and φ with respect to frame 3!

Inverse kinematics: elbow manipulator with wrist

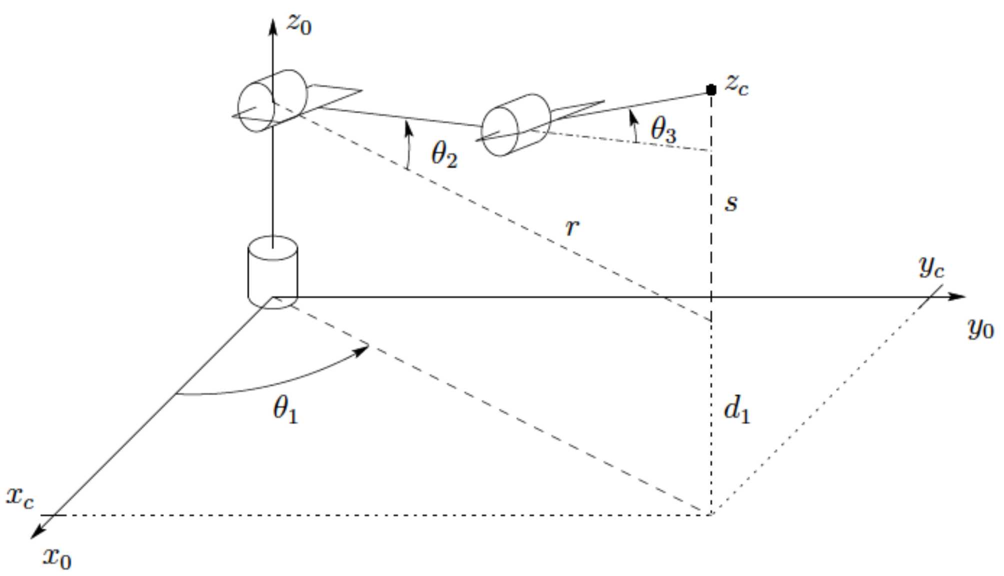


Fig. 3.13 Elbow manipulator

Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*

$$ {}^0_3R = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix} $$

$$ {}^3_6R = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} $$

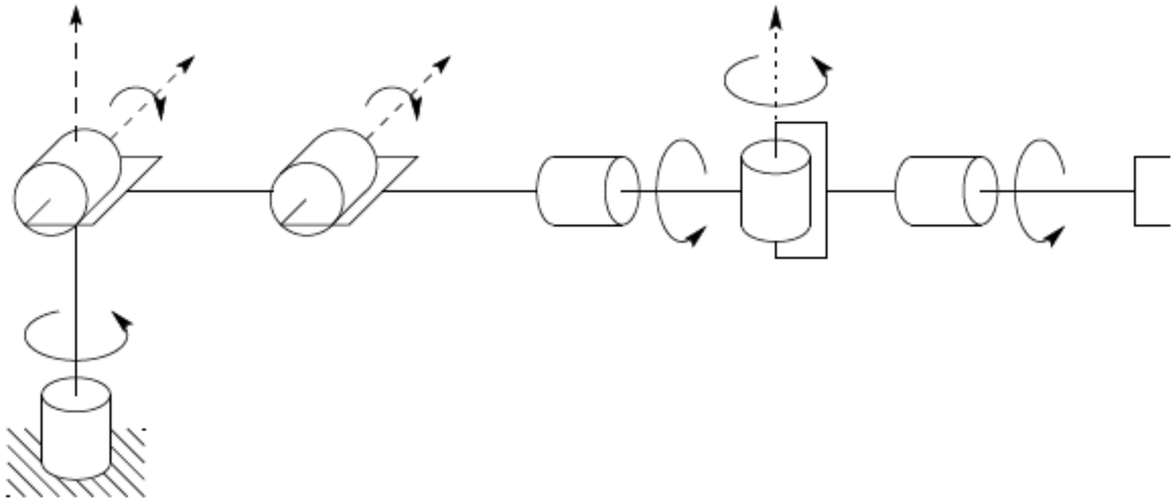
$$ {}^3_6R = {}^0_3R^T \; R $$

$$ \begin{aligned} \theta_4 &= \operatorname{atan2}(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33}, \\ &\qquad -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \\ \theta_6 &= \operatorname{atan2}(-s_1r_{11} + c_1r_{21}, s_1r_{12} - c_1r_{22}) \end{aligned} $$

$$ c_5 = s_1r_{13} - c_1r_{23} $$

$$ \theta_5 = \operatorname{atan2}\left(s_1r_{13} - c_1r_{23}, \pm\sqrt{1 - (s_1r_{13} - c_1r_{23})^2}\right) $$

Elbow manipulator: final solution



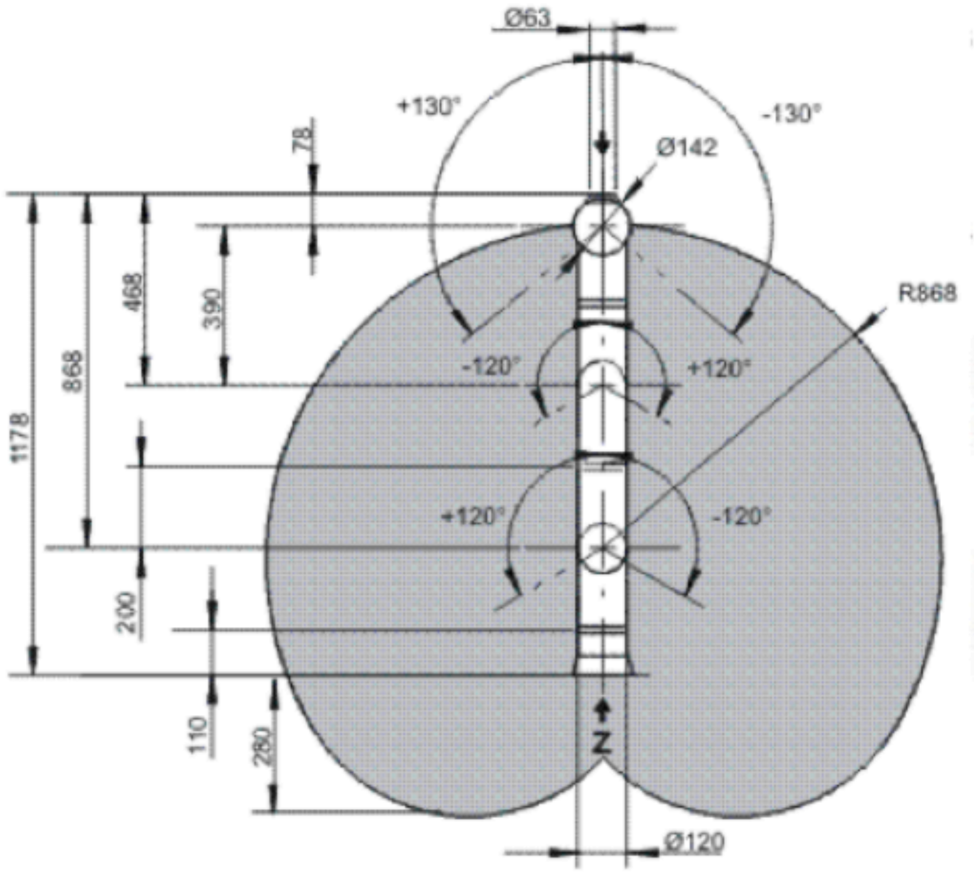
$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{aligned}
\theta_1 &= \operatorname{atan2}(x_c, y_c) \\
\theta_2 &= \operatorname{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \operatorname{atan2}(a_2 + a_3 c_3, a_3 s_3) \\
\theta_3 &= \operatorname{atan2}\left(D, \pm\sqrt{1 - D^2}\right), \\
&\qquad \textit{where } D = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2 a_3} \\
\theta_4 &= \operatorname{atan2}(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, \\
&\qquad -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}) \\
\theta_5 &= \operatorname{atan2}\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}\right) \\
\theta_6 &= \operatorname{atan2}(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22})
\end{aligned}$$

Problem

- Determine
  - Frames and table of DH parameters
  - Homogeneous transformation matrices
  - Forward kinematics



Side view (from the left)


Front view