ME 6102: Design of Mechatronic Systems

Mathematical Modeling

P.S. Gandhi
Mechanical Engineering
IIT Bombay

Outline

- Necessity of modeling (mechatronic view)
  - Assess system performance off line
  - Develop controllers
- Criterion: Simplicity vs. accuracy
- Basis of modeling
  - Physics of the system
    Newton's law, Kirchoff's law, other phenomena
  - Empirical methods
    Input/output observation of system
  - Simplified models
    Linearised model, Assumed modes method
- Examples:
  - motor model
  - hysteresis in harmonic drives

Need for modeling

- Assess system performance
  - Towards design and synthesis of systems sensitivity to parameters (Manufacturing tolerance perspective)

off-line Simulations to know the system
  - Behavior
  - Response to disturbances
  - Response to various inputs
- Develop controllers
  - Test performance of a controller
  - Get control input history
- Substantial saving in time and cost of control system design

Criteria for modeling

- Simple model amenable to development of control
- Accuracy of representation of dynamics of interest

Example

Friction model at: Low speeds vs. high speeds

Empirical models vs. models based on physics

Basis of Modeling

- Based on physics of phenomenon

Example: Newton's Law: for dynamical systems

Kirchoff's Law: for electrical circuits

Laws of optics: for optical systems

- Based on input output experimental observations

Hysteresis models

Some chemical processes

Some fluid dynamics models

Steps in Modeling

- Identify the most important physical phenomenon
- Assess their complexity to be represented using equations governing physics

→ Take decision regarding I/O model vs. physics

- Apply corresponding laws to get the final equations
- Verify models by matching model simulation and experimental results

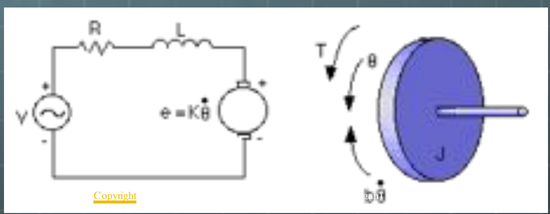
Example 1

Modeling based on physics of problem

DC Motors: Modeling

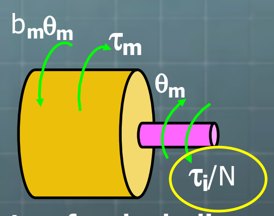
Based on Physics

- **Simple model of DC motor**

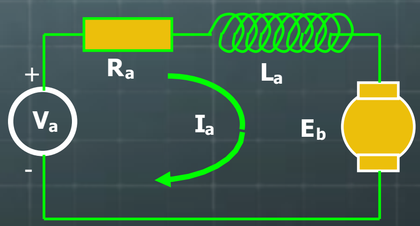


- **Applications: plenty of areas: computers, printers, robots, automobile, automatic CNC machines, and so on..**

Motor Dynamics



Armature free body diagram



Electrical circuit diagram

Mechanical

$$J_m\ddot{\theta}_m + B_m\dot{\theta}_m + \frac{\tau_i}{N} = \tau_m$$

$$\tau_m = K_t\varphi I_a$$

Electrical

$$V_a = I_aR_a + E_b + L\frac{dI_a}{dt}$$

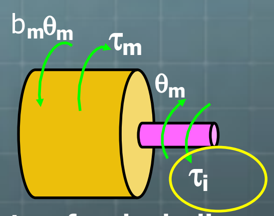
$$E_b \propto \varphi\omega$$

$$E_b = K\varphi\omega$$

Identification of motor model parameters

- Need to develop realistic simulation and test control development before implementation
- Inherently nonlinear system from identification perspective because of friction in the system
- Friction characteristics and identification!

Motor Dynamics + friction



Armature free body diagram

Mechanical

$$J_m\ddot{\theta}_m + B_m\dot{\theta}_m + \tau_f + \tau_l = \tau_m$$

$$\tau_m = K_t\varphi I_a$$

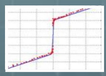
How to model friction?

- Coulomb friction/ viscous friction
- Transfer function

Friction models

- Coulomb friction model
- Coulomb and Viscous friction model
- Stribeck model for viscous friction

Friction model



Total friction

$$\tau_{fr} = F_f +$$

LuGre part

$$F_f = \sigma_0 \chi + \sigma_1 \dot{\chi} + \sigma_2 \dot{\theta}_m,$$

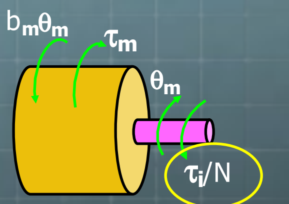
$$\dot{\chi} = \dot{\theta}_m - \frac{|\dot{\theta}_m|}{g(\dot{\theta}_m)} \chi,$$

$$\sigma_0 g(\dot{\theta}_m) = F_c + (F_s - F_c) e^{-(\dot{\theta}_m / v_s)^2},$$

$$\tau_{frp} =$$

σ_0 = Spring stiffness (pre-sliding)
σ_1 = Damping (pre-sliding)
σ_2 = Damping (steady state)
F_c = Coulomb friction
F_s = Static friction
χ = Friction state
v_s = Stribeck velocity

Motor Dynamics + load

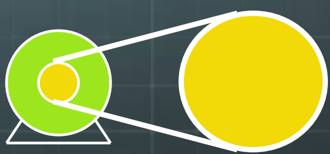


Armature free body diagram

Mechanical

$$J_m \ddot{\theta}_m + B_m \dot{\theta}_m + \frac{\tau_i}{N} = \tau_m$$

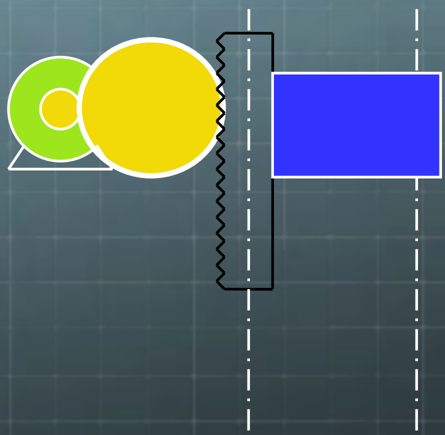
$$\tau_m = K_t \varphi I_a$$



How to account for load inertia?

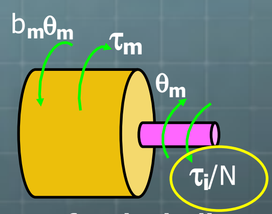
Concept of equivalent inertia: discuss

CD ROM Gross Tracking



- Identify the most important physical phenomenon
- Discussion on equivalent inertia of the system as seen on motor side
- Energy based method to find equivalent inertia of the system

Motor Dynamics + load



Armature free body diagram

More complicated loads

Slider crank system: for constant torque what would be time evolution of position

Ways to deal with these and more complex problems

Need for lagrange formulation

Example of CD ROM Fine Tracking and Focussing Systems Modeling

Basic Mechanical Plant

- Q:whats the DOF?
- How to model? Spr mass model
- Any other thing possible?
- Whats the criteria??

Lagrange formulation to model dynamics

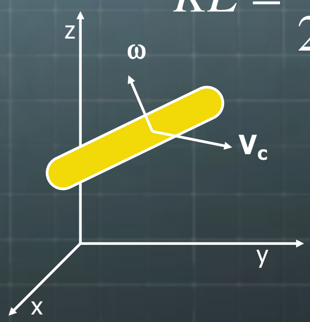
Euler-Lagrange Equation

- Method based on energy
- Equations are obtained without considering the internal reaction forces
- Ideal for more complex robotic manipulator configurations. Ex. Complex 3D robot, flexible link robot
- Better than Newton's method for robotic applications

Kinetic Energy

- Total kinetic energy of one robot link is

$$KE = \frac{1}{2} m \mathbf{v}_c^T \mathbf{v}_c + \frac{1}{2} \omega^T I \omega$$



$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

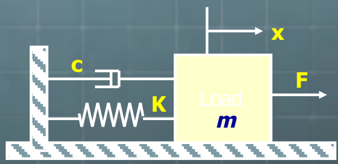
Euler-Lagrange Equation

- Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$

- Where τ_i is the external force in the direction of q_i
- L = KE-PE
- KE is the kinetic energy and PE is the potential energy

Spring Mass System



Kinetic energy of the system KE= $\frac{1}{2}m\dot{x}^2$

Potential energy of the system PE= $\frac{1}{2}kx^2$

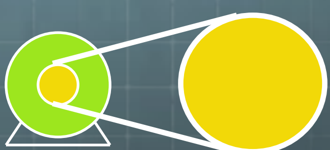
L=KE-PE= $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

From Lagrange equation

$$\frac{d}{dt}\left(m\dot{x}\right) + kx = F$$

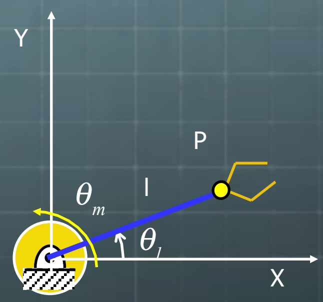
$$m\ddot{x} + kx = F$$

Example: Belt drive with and without flexible belt



- Identify the most important physical phenomenon
- Derivation in class: Seeing inertia transferred to motor shaft
- Increase in number of degrees of freedom when belt is considered flexible
- Derivation of energies and equation of motion

Example: Single Link Manipulator with gear box



Gear ratio between motor and manipulator is n:1

Kinetic energy of the system

$$K = \tfrac{1}{2} J_m \dot{\theta}_m^2 + \tfrac{1}{2} J_l \dot{\theta}_l^2$$

$$= \tfrac{1}{2} (J_m + J_l / n^2) \dot{\theta}_m^2$$

Potential energy of the system

$$V = Mgl\left(1 - \cos(\theta_l)\right)$$

Example: Single Link Manipulator with gear box

Lagrangian

$$L = \frac{1}{2}(J_m + J_l / n^2)\dot{\theta}_m^2 - Mgl\left(1 - \cos(\theta_l)\right)$$

From Lagrange equation

$$\left(J_m + \frac{J_l}{n^2}\right)\ddot{\theta}_m + \frac{Mgl}{n}\sin\frac{\theta_m}{n} = \tau$$

τ **Consists of motor torque input u and damping torque**

Example: Single Link Manipulator with gear box

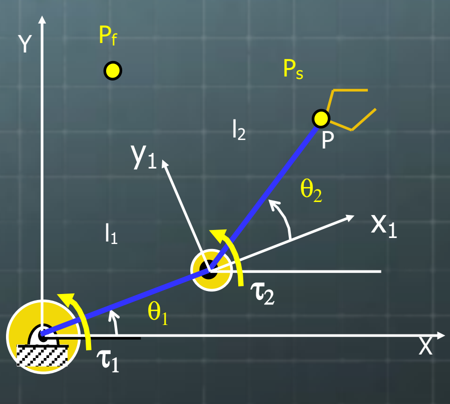
$$\tau = u - \left(B_m + B_l / n^2\right)\dot{\theta}_m$$

Thus equation governing system dynamics is

$$J\ddot{\theta}_m + B\dot{\theta}_m + C\sin(\theta_m / n) = u$$

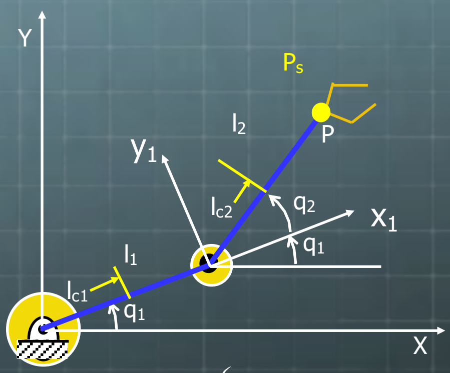
Where
$$J = (J_m + J_l / n^2)$$
$$B = \left(B_m + B_l / n^2\right)$$
$$C = \frac{Mgl}{n}$$

Dynamics



- **Q: Given forces and torques can we find the joint angles history??**
- **Differential Equations of motion could be obtained by Lagrange formulation!**
- **We consider a general case first and come back to this system**

Lagrangian Formulation for 2-R Manipulator



Kinetic energy is computed
Using jacobian expression

$$v_{c1} = J_{v_{c1}} \dot{q}$$

$$J_{v_{c1}} = \begin{pmatrix} -l_{c1}\sin q_1 & 0 \\ l_{c1}\cos q_1 & 0 \end{pmatrix}$$

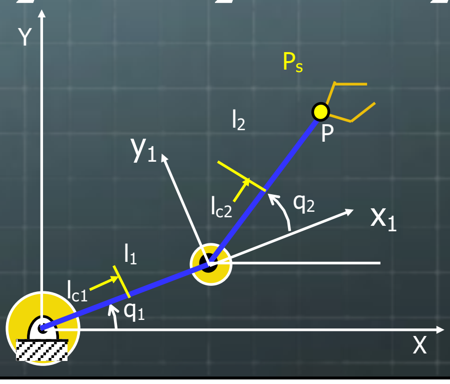
$$v_{c2} = J_{v_{c2}} \dot{q}$$

$$J_{v_{c2}} = \begin{pmatrix} -l_1 \sin q_1 - l_{c2}\sin(q_1+q_2) & -l_{c2}\sin(q_1+q_2) \\ l_1\cos q_1 + l_{c2}\cos(q_1+q_2) & l_{c2}\cos(q_1+q_2) \end{pmatrix}$$

Lagrangian Formulation for 2-R Manipulator

Translational part of kinetic energy

$$\frac{1}{2} m_1 v_{c1}^T v_{c1} + \frac{1}{2} m_2 v_{c2}^T v_{c2} = \frac{1}{2}\dot{q}\left\{ m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} \right\}\dot{q}$$



Rotational part??

Lagrangian Formulation for 2-R Manipulator

Angular velocities of link 1 and 2 are

$$\omega_1 = \dot{q}_1$$

$$\omega_2 = \dot{q}_1 + \dot{q}_2$$

Rotational Kinetic energy $= \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2$

$$\frac{1}{2}\dot{q}^T \left\{ I_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + I_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \dot{q}$$

Potential energy of system

$$V = \left(m_1 l_{c1} + m_2 l_1\right) g \sin q_1 + m_2 l_{c2} g \sin\left(q_1 + q_2\right)$$

2-R Manipulator

Using Lagrange equation and arranging the terms

$$d_{11} = m_1 l_{c1}^2 + m_2\left(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos q_2\right) + I_1 + I_2$$

$$d_{12} = d_{21} = m_2\left(l_{c2}^2 + l_1 l_{c2} \cos q_2\right) + I_2$$

$$d_{22} = m_2 l_{c2}^2 + I_2$$

Element of matrix C

$$C_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$

$$C_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h$$

$$C_{121} = C_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 l_1 l_{c2} \sin q_2 = h$$

$$C_{122} = C_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0$$

$$C_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h$$

$$C_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0$$

2-R Manipulator

$$\frac{\partial V}{\partial q_1} = \left(m_1 l_{c1} + m_2 l_1\right) g\cos q_1 + m_2 l_{c2} g\cos\left(q_1 + q_2\right)$$

$$\frac{\partial V}{\partial q_2} = m_2 l_{c2} \cos\left(q_1 + q_2\right)$$

Final dynamical equations of the system

$$\boxed{\begin{array}{l} d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_1\dot{q}_2 + c_{221}\dot{q}_2^2 + \dfrac{\partial V}{\partial q_1} = \tau_1 \\ d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + \dfrac{\partial V}{\partial q_2} = \tau_2 \end{array}}$$

Recap: Euler-Lagrange Equation

Case of General mechanical rigid body systems

- **Equation**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$

- **Where τ_i is the external force in the direction of q_i**
- **L = KE-PE**
- **KE is the kinetic energy and PE is the potential energy**
- **Next general case of rigid robot link**

Kinetic Energy

- **Total kinetic energy of one robot link is**

$$KE_i = \frac{1}{2} m_i \mathbf{v}_{ci}^T \mathbf{v}_{ci} + \frac{1}{2} \omega_i^{\mathrm{T}} I_i \omega_i$$

Expressing V_{ci} and ω_i in terms of selected generalized coordinates q_i

$$KE = \frac{1}{2} \dot{\mathbf{q}}^T D(\mathbf{q}) \dot{\mathbf{q}} = \frac{1}{2} \sum_{i,j}^{n} d_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j$$

- **Hence Lagrangian L is**

$$L = KE - PE$$

$$= \frac{1}{2} \sum_{i,j}^{n} d_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j - PE(\mathbf{q})$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_{j} d_{kj}(\mathbf{q}) \dot{q}_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}(\mathbf{q})\dot{q}_j$$

$$= \sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial PE}{\partial q_k}$$

Euler-Lagrange Equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$

$$\sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial PE}{\partial q_k} = \tau_k$$

$k = 1,2,\ldots,n$

Interchanging order of summation and using symmetry

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j$$

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \sum_{i,j}\frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j$$

Hence finally Euler-Lagrange equation becomes

$$\sum_j d_{kj}(\mathbf{q})\ddot{q}_j + \sum_{i,j} c_{ijk}(\mathbf{q})\dot{q}_i\dot{q}_j + \frac{\partial PE}{\partial q_k} = \tau_k$$

$$k = 1,2,\ldots,n$$

This can further be written as

$$D(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$

General form of Equations "Mechanical Systems"

$$D(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$

$$c_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$

Important mathematical properties:

- **Matrix D(q) is symmetric positive definite**
- **Matrix $\dot{D}(\mathbf{q}) - 2C(\mathbf{q},\dot{\mathbf{q}})$ is a skew symmetric matrix**

These properties hold for mechanical systems with rigid bodies and time invariant mass

Energies in Mech Electrical domain

- Kinetic energy

$$T = \frac{1}{2} L i^2$$ Inductance

- Potential energy

$$V = \frac{1}{2} C V^2$$ Capasitor

$$C = \frac{\varepsilon A}{g}$$

- Resistive loss

- Kinetic energy

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$ Motion of mass

- Potential Energy

$$V = \frac{1}{2} k x^2$$ Spring

$$V = mgh$$ Gravity

- Damping loss

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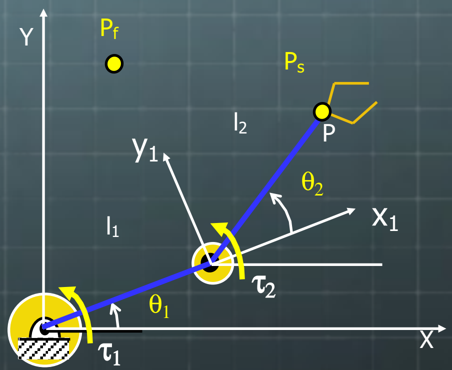
Newton's Method

Application to robot

- First carry out kinematic analysis of the robot
- Find out accelerations
- Draw free body diagrams of robot links
- Apply Newton's law
- For complex 3D problems one can write down the vector equations of force summation for each link and use them recursively to eliminate the internal coupling forces

Newton's Method

- **Application to 2R manipulator**
  - **First carry out kinematic analysis of the robot**
  - **Find out accelerations of link CGs**
  - **Draw free body diagrams of robot links**
  - **Apply Newton's laws**

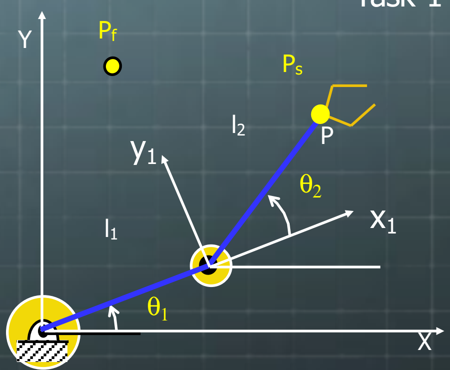


Carry out exercise and verify with Lagrange formulation!!

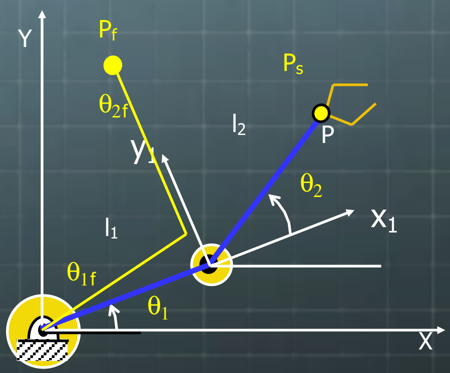
Regulation problem

Task 1→

- **Lets say "we want the manipulator to move P from P_s to P_f", Not bothered about path**
- **Think what steps you will take to achieve this point-to-point movement**



Regulation problem



- **Steps**

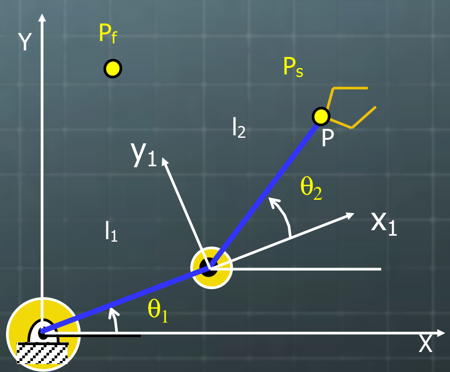
1. **Find out <u>desired</u> joint angles corresponding to P_s and P_f by inverse kinematics**

$$\theta_s^d = \theta_{1s}^d, \theta_{2s}^d$$

$$\theta_f^d = \theta_{1f}^d, \theta_{2f}^d$$

2. **<u>Command</u> your motors to move joints simultaneously to go from starting point to the final point**

Regulation problem

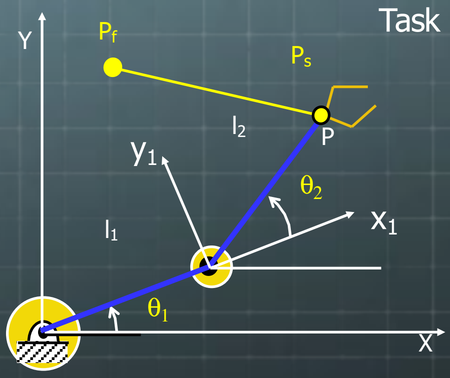


Questions

- **What is meant by <u>command</u>?**
- **How much should be the motor torque? What does it depend on?**
- **What if we want robot to take P to the final position faster?**
- **How do we make sure that after reaching final position links stay there?**

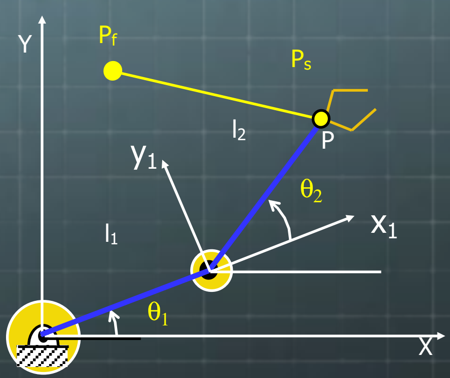
Answers proper design and understanding control

Tracking Problem



- **Lets say "we want the manipulator to move P from P_s to P_f in a straight line (desired trajectory)"** ← Task 2→
- **Think what steps you will follow to achieve this *trajectory tracking***

Tracking problem



- **Steps**

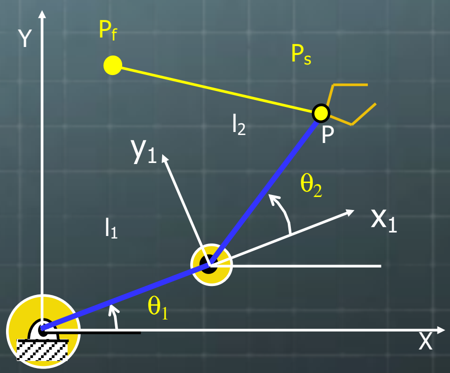
1. **Find out desired joint angle *trajectories* by inverse kinematics**

$$\theta_i^d = \theta_i^d(t)$$

2. **Command your motors to move joints simultaneously to follow the desired position**

Q: What would be the best controller to Do this job?? PID, Feed forward etc.

Tracking Problem



Questions

- What is meant by <u>command</u> in this case?
- How much should be the motor torque? What does it depend on?
- What if we want robot to take P to the final position faster?
- How much deviation from the straight line will occur?

Answers proper design and understanding control