ME 6102: Design of Mechatronic Systems

Control Design: Linear SS

P.S. Gandhi
Mechanical Engineering
IIT Bombay

State Space Control Design

- Basic idea
  - Consider standard form
  - Check controllability and observability properties
  - Design observer to estimate states based on measured output
  - Design controller

Model

- Transform a given system in state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Check for controllability : see notes
- Check for observability and design Leunberger : See notes
- Choose pole locations for placing poles: See example

Nonlinear Domain Control

- Lyapunov theory : One of the important ANALYSIS techniques (no synthesis is possible)
- Literature explores Lypunov theory to present control design based on several techniques: sliding mode control, nested saturation .. For example.
- We gather here gist of Lyapunov theory from application perspective to nonlinear rigid body mechanical systems

Lyapunov Theory

- Enables analysis of nonlinear controllers
- Synthesis of nonlinear, adaptive, robust controllers possible for both regulation and trajectory tracking applications

- First we need to study some mathematical preliminaries
- Define various stability notions

Locally Positive Definite Function (lpdf)

A function $V : R_+ \times R^n \to R$ is said to be a locally positive definite function(lpdf) if

(i) It is continuous

(ii) $V(t, \mathbf{0}) = 0 \forall t \geq 0$

(iii) There exist a constant r>0 and a function α of class K such that

$$\alpha(\|\mathbf{x}\|) \leq V(t, \mathbf{x}), \forall t \geq 0, \forall \mathbf{x} \in B_r$$

Locally Positive Definite Function (lpdf)

$$\alpha(\|\mathbf{x}\|) \le V(t,\mathbf{x}), \forall t \ge 0, \forall \mathbf{x} \in B_r$$

Concept of norm ||.|| of x
Distance! Example:

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 \ldots + x_n^2}$$

Class K function
→ continuous
→ strictly increasing
→ positive for x other than zero
→ zero for x = 0

Positive Definite Function

A function $V : R_+ \times R^n \to R$ is said to be positive definite function (pdf) if

(i) It is continuous

(ii) $V(t,\mathbf{0}) = 0, \forall t \ge 0$

(iii) There exist a function α of class K such that

$$\alpha(\|\mathbf{x}\|) \le V(t,\mathbf{x}), \forall t \ge 0,$$

hold for all $X \in R^n$

Mathematical Preliminaries

Positive Definite Function (pdf)

(i) Continuous
(ii) V(t,0) = 0 for all t
(iii) a function condition on V(t,x) for nonzero x

Conditions valid for all

$$X \in R^n$$

Examples

$$V(t,\mathbf{x}) = (t+1)(x_1^2 + x_2^2) \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$V(t,\mathbf{x}) = x_1^4 + x_2^2 + x_3^2 \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$V(t,\mathbf{x}) = x_1^4 + x_2^2$$

Mathematical Preliminaries

Other Definitions

V is <u>decrescent</u> if there exist a contstant r>0 and a function β of class K such that

$$V(t,\mathbf{x}) \le \beta(\|\mathbf{x}\|), \forall t \ge 0, \forall \mathbf{x} \in B_r$$

V is <u>radially unbounded</u> if $\alpha(\|\mathbf{x}\|) \le V(t,\mathbf{x}), \forall t \ge 0,$

for all $X \in R^n$ **and for some continuous function** α **with additional property that**

$$\alpha(r) \to \infty \text{ as } r \to \infty$$

V is a <u>negative definite function</u> if –V is an lpdf and a negative definite function if –V is a pdf

Quadratic Functions

- Very popular function used in Lyapunov theory applied to robots
- Quadratic functions can be written in the form

$$V(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

$$\begin{aligned} V(\mathbf{x}) &= a_{11}x_1^2 + a_{22}x_2^2 + \ldots + a_{nn}x_n^2 \\ &+ 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \ldots + 2a_{1n}x_1x_n \\ &+ 2a_{23}x_2x_3 + \ldots + 2a_{2n}x_2x_n \\ &+ \ldots.. + 2a_{(n-1)n}x_{n-1}x_n \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{12} & a_{22} & & \\ . & . & . & a_{(n-1)n} \\ a_{1n} & & & a_{nn} \end{bmatrix}$$

Quadratic Functions

- Theorem: Quadratic functions are positive definite functions if determinants of successive minors of *A* are all positive

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{12} & a_{22} & & \\ . & . & . & a_{(n-1)n} \\ a_{1n} & & & a_{nn} \end{bmatrix}$$

Quadratic Functions

- Theorem: Quadratic functions are positive definite functions if eigenvalues of A are all positive

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{12} & a_{22} & & \\ . & . & . & a_{(n-1)n} \\ a_{1n} & & & a_{nn} \end{bmatrix} \qquad eig_i(A) > 0 \quad i = 1,2,..n$$

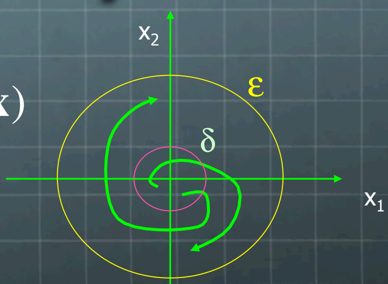
Lyapunov's Notions of Stability

- **Stability: Characteristic of equilibrium of**

$$\dot{\mathbf{x}} = f(t, \mathbf{x})$$

- **Equilibrium is obtained by solving for x in**

$$f(t, \mathbf{x}) = \mathbf{0}$$



- Equilibrium **0** is said to be stable if for each ε>0 and each $t_0 \in R_+$ there exists a δ=δ(ε,t_0) such that

$$\| \mathbf{x}_0 \| < \delta(\varepsilon, t_0) \Rightarrow \| \mathbf{s}(t, t_0, \mathbf{x}_0) \| < \varepsilon, \forall t \geq t_0$$

Lyapunov's Notions of Stability

$$\dot{\mathbf{x}} = f(t,\mathbf{x})$$

- Equilibrium 0 is attractive if for each $t_0 \in R_+$ there is an $\eta(t_0) > 0$ such that

$$\| \mathbf{x}_0 \| < \eta(t_0) \Rightarrow \| \mathbf{s}(t,t_0,\mathbf{x}_0) \| \to 0 \quad \text{as } t \to \infty$$

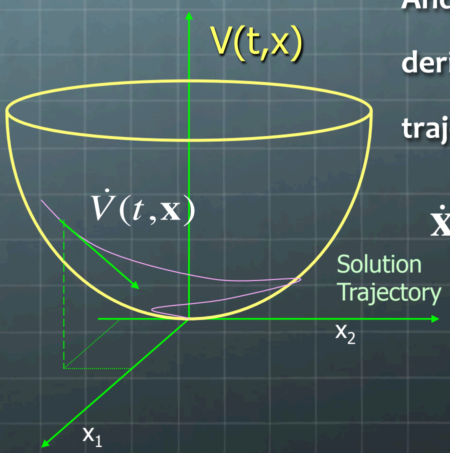
- Equilibrium is said to be asymptotically stable if it is stable and attractive

Derivative along Trajectories

Definition: Let $V : R_+ \times R^n \to R$ be continuously differentiable with respect to all of the arguments and let ∇V denote the gradient of V with respect to x (written as a row vector). Then the Function $\dot{V} : R_+ \times R^n \to R$ is defined by

$$\dot{V}(t,\mathbf{x}) = \frac{\partial V}{\partial t}(t,\mathbf{x}) + \nabla V(t,\mathbf{x})\, f(t,\mathbf{x}) \qquad \dot{\mathbf{x}} = f(t,\mathbf{x})$$

Derivative along Trajectories



And is called the derivative of V along the trajectories of

$$\dot{\mathbf{x}}(t) = f\left[t,\mathbf{x}(t)\right], \forall t \geq 0$$

Fundamentals of Lyapunov Theory

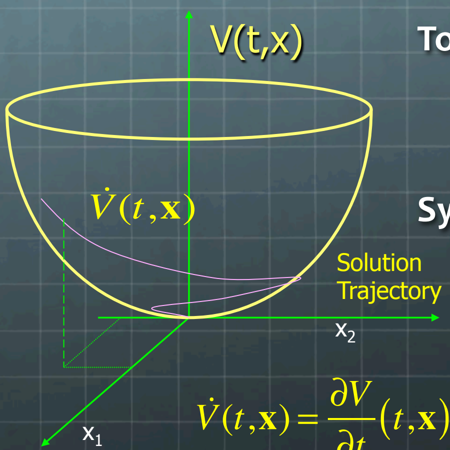
Dynamic system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)$$

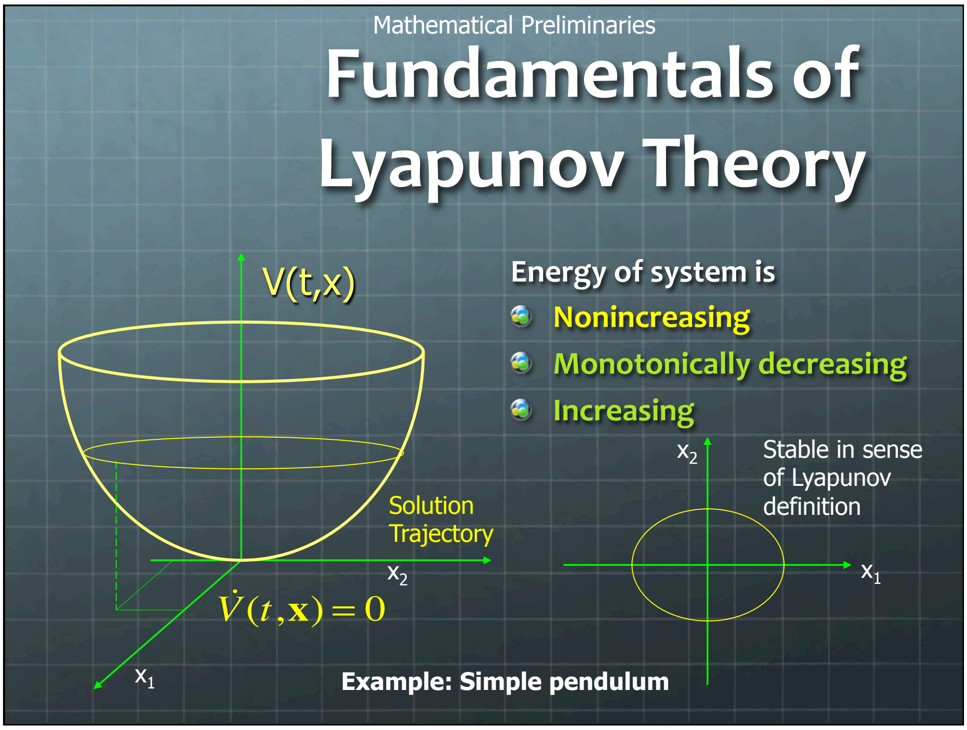
Suppose 0 is equilibrium

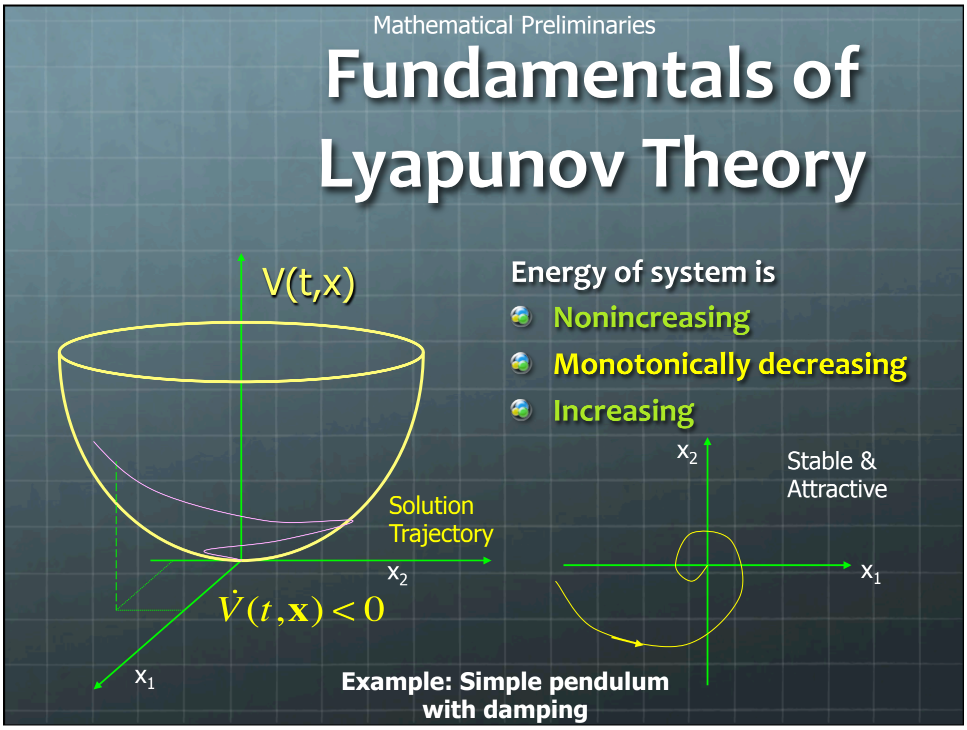
Total energy (E): zero at origin and positive otherwise. E is considered to be V here.

System is perturbed from origin: observe E
- Nonincreasing
- Monotonically decreasing
- Increasing



$$\dot{V}(t,\mathbf{x}) = \frac{\partial V}{\partial t}(t,\mathbf{x}) + \nabla V(t,\mathbf{x})\, f(t,\mathbf{x})$$


Mathematical Preliminaries
Fundamentals of Lyapunov Theory
V(t,x)
Energy of system is
Nonincreasing
Monotonically decreasing
Increasing
Solution Trajectory
x_2
$\dot{V}(t,\mathbf{x}) = 0$
x_1
x_2
Stable in sense of Lyapunov definition
x_1
Example: Simple pendulum


Mathematical Preliminaries
Fundamentals of Lyapunov Theory
V(t,x)
Energy of system is
Nonincreasing
Monotonically decreasing
Increasing
Solution Trajectory
x_2
$\dot{V}(t,\mathbf{x}) < 0$
x_1
x_2
Stable & Attractive
x_1
Example: Simple pendulum with damping

Mathematical Preliminaries

Fundamentals of Lyapunov Theory

- Based on previous analysis we can conclude:
  - When derivative of energy type PDF function is zero the equilibrium of the system is stable
  - If it is strictly less than zero then the equilibrium of the system is stable and attractive → asymptotically stable
  - There is definite relationship between stability and properties of V and $\dot{V}$
- Lyapunov generalized this relationship and came up with theorems on stability

Lyapunov's Theorm: Stability

Consider a nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$

The equilibrium x=0 of the system is

stable if there exist a lpdf C^1

$$V : R_+ \times R^n \to R$$

and a constant r>0 such that

$$\dot{V}(t, \mathbf{x}) \le 0, \forall t \ge t_0, \forall \mathbf{x} \in B_r$$

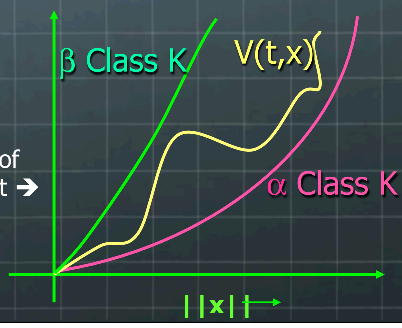
Where $\dot{V}$ is evaluated along the

trajectories of the system

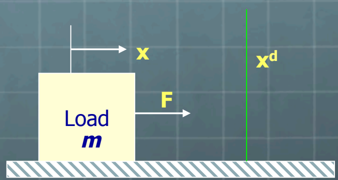
Lyapunov's Theorm: Asymptotic Stability

The equilibrium 0 of the system is uniformly asymptotically stable if there exist a C^1 decrescent lpdf V such that $-\dot{V}$ is an lpdf

Definition of Decrescent ➔



Lyapunov Stability: Example



Aim: Point to point control. Take mass to the final position x^d

PD control: $m\ddot{x} = F$

$$F = -K_p e - K_d \dot{x}$$

$$m\ddot{x} = -K_p e - K_d \dot{x}$$

$$m\ddot{e} + K_d \dot{e} + K_p e = 0$$

Lyapunov function candidate
Energy based

$$V = \frac{1}{2} m\dot{e}^2 + \frac{1}{2} K_p e^2$$

$$\dot{V} = m\dot{e}\ddot{e} + K_p e\dot{e}$$

$$= \dot{e}(-K_p e - K_d \dot{e}) + K_p e\dot{e}$$

$$= -K_d \dot{e}^2 \qquad \leq 0$$

Lyapunov Stability: PD Control of Mechanical System

Nonlinear dynamic equations of a fully actuated mechanical system

$$\sum_{j=1}^{n} d_{jk}(q)\ddot{q}_j + \sum_{i,j=1}^{n} C_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

← Equation obtained by Lagrange formulation

$$J_{m_k}\ddot{\theta}_{m_k} + \left(B_{m_k} + \frac{K_b K_m}{R}\right)\dot{\theta}_{m_k} = \frac{K_m}{Rv_k} - r_k\tau_k$$

$$\theta_m = \frac{1}{r_k} q_k$$

← Dynamics of motor actuator

$$\frac{1}{r_k^2} J_m \ddot{q}_k + \frac{1}{r_k^2} B\dot{q}_k = \frac{K_m}{r_k R} v_k - \tau_k$$

← Dynamics of motor Actuator in generalized coordinate

Lyapunov Stability: PD Control

$$\underbrace{\frac{1}{r_k^2} J_m}_{J} \ddot{q}_k + \sum_{j=1}^{n} d_{jk}\ddot{q}_j + \sum_{i,j=1}^{n} C_{ijk}\dot{q}_i\dot{q}_j + \frac{1}{r_k^2} B\dot{q}_k + g_k = \underbrace{\frac{K_m}{r_k R} v_k}_{u_k}$$

Where $B = B_{m_k} + \frac{K_b Lm}{R}$

Dynamics coupled with actuator

In matrix form these equations of motions can be written as

$$\big(D(q) + J\big)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u$$

Same Dynamics

Approximation considering nonlinearities as disturbance and equations in motor variable

$$J_{eff}\ddot{\theta}_{mk} + B_{eff}\dot{\theta}_{mk} = KV_k - d_k r_k$$

We can have Linear control Using this

PD Control

An independent joint PD-control scheme can be written in vector from as

$$u = K_p\tilde{q} - K_D\dot{q}$$

Where $\tilde{q} = q^d - q$ is the error between the desired joint displacement q^d and the actual joint displacement q, and

$K_{p,}K_D$ are diagonal matrices of proportional and derivative gain.

PD Control

To show that the above control law achieves zero steady state error consider an energy based Lyapunov function candidate

$$V = \frac{1}{2}\dot{q}^T\left(D(q)+J\right)\dot{q} + \frac{1}{2}\tilde{q}^T K_p\tilde{q}$$

The first term is the kinetic energy of the robot and the second term accounts for the proportional feedback $K_p\tilde{q}$ (spring elastic energy)

PD Control

Time derivative of V is given by

$$\dot{V} = \frac{1}{2}\dot{q}^T\left(D(q)+J\right)\ddot{q} + \frac{1}{2}\dot{q}^T\dot{D}(q)\dot{q} - \dot{q}^T K_p\tilde{q}$$

Solving for $\left(D(q)+J\right)\ddot{q}$ with g(q)=0 and substituting the expression into the

Above gives

$$\dot{V} = \dot{q}^T\left(U - C(q,\dot{q}) + B\dot{q}\right) + \frac{1}{2}\dot{q}^T\dot{D}(q)\dot{q} - \dot{q}^T K_p\tilde{q}$$

$$= \dot{q}^T\left(U - B\dot{q} - K_p\tilde{q}\right) + \underbrace{\frac{1}{2}\dot{q}^T\left(\dot{D}(q) - 2C(q,\dot{q})\right)\dot{q}}_{=0}$$

$$= \dot{q}^T\left(U - B\dot{q} - K_p\tilde{q}\right)$$

Recall

$$u = K_p\tilde{q} - K_D\dot{q}$$

PD Control

$$= \dot{q}^T\left(U - B\dot{q} - K_p\tilde{q}\right)$$

Substituting PD control law $u = K_p\tilde{q} - K_D\dot{q}$

$$\dot{V} = \dot{q}^T\left(K_D + B\right)\dot{q} \le 0$$

Notice that there is no term corresponding to q

Above analysis shows that V is decreasing as long as $\dot{q}$ is not zero. This sufficient to prove that manipulator can reach a position where $\dot{q} = 0$

but $q \neq q^d$ is not proved yet!

→ Stability in sense of Lyapunov

→ No asymptotic stability conclusion

→ We need additional tool

LaSalle's theorem

LaSalle's Theorem

Suppose the system is autonomous $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

Suppose there exist a C^1 function

$$V : R_+ \times R^n \to R$$

such that

(1) V is a pdf and a radially unbounded

(2) $\dot{V}(t,0) \le 0, \forall t \ge 0, \forall x \in R^n$

(3) Define

$$R = \left\{ x \in R^n : \exists t \ge 0 \text{ such that } \dot{V}(t,x) = 0 \right\}$$

LaSalle Theorem

$$R = \left\{ x \in R^n : \exists t \ge 0 \text{ such that } \dot{V}(t,x) = 0 \right\}$$

and suppose R does not contain any trajectories of the system other than the trivial trajectory x=0. Then the equilibrium 0 is globally uniform asymptotically stable

PD Control

We now apply this theorem to our PD control problem

Suppose $\dot{V}=0$, then it implies that $\ddot{q}=0$ and $\dot{q}=0$ since

$$\dot{V}=\dot{q}^T\left(K_D+B\right)\dot{q}\leq 0$$

From the equation of motion for closed loop PD control $\left(D+J\right)\ddot{q}+C\left(q,\dot{q}\right)\dot{q}=-K_p\tilde{q}-K_D\dot{q}$ we must then have $0=-K_p\tilde{q}$ which implies that $\tilde{q}=0$

PD Control

LaSalle's theorem then implies that system is asymptotically stable.

In case there is gravitational term present equation must be modified to read

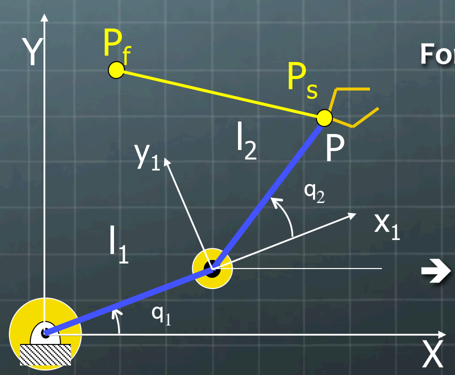
$$\dot{V}=\dot{q}^T\left(U-g(q)-B\dot{q}-K_p\tilde{q}\right)$$

The presence of the gravitational term means that PD control alone cannot guarantee asymptotic stability for point-to-point movement

Overall Conclusion: PD control can asymptotically stabilize general rigid body mechanical system

→Trajectory tracking

Tracking Problem



Task: q need to go along a smooth (Continuous and differentiable) trajectory

For example, for 2R manipulator shown if we would like to go from Ps to Pf along straight line in 3 sec, how would you plan joint motions?

➔ Motion or path planning problem

Tracking Problem

- Analysis of computed torque controllers can be carried out
- It usually leads to cases in which situation is similar to what we saw for PD control
- LaSalle's theorems could not be applied in this case and more advanced tools are necessary to establish stability proofs

High Performance Tracking Controller

Let $\tau = Da + Cv + Bv - K_d r$

Where

$$a = \dot{v}$$

$$e = q - q^d$$

$$v = \dot{q}^d - \wedge e$$

$$r = \dot{q} - v = \dot{e} + \wedge e$$

$K_d, \wedge$ are +ve definite matrices

Called Li-Slotine Controller

Analysis: next class

Summary

- Lyapunov theory: stability theorems
- Application to robots: PD control analysis
- Tracking controller