ME 6102: Design of Mechatronic Systems

Mathematical Representation

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Outline

- Some properties of system
  - Linearity
  - Time invariance
- Different ways of representation
  - Ordinary differential equation form
  - Transfer function form
  - State space form
  - State form for nonlinear systems
- Solution of systems to get the response
  - ODE solution- analytical / numerical
  - Transfer function: Using Laplace and inverse Laplace
  - State space: Using eigenvalues

System Properties

$u_1 \longrightarrow y_1$

$u_2 \longrightarrow y_2$

$u_1+u_2 \longrightarrow y_1+y_2$

- **Linearity (VIMP)**
  - **Principle of superposition**
  - **Power ODE is one & linear terms**

  Example: Pendulum system
- **Time invariant**
  - **Start at t=0 or any other t, from same initial conditions → the response is same**
  - **Constant coef for TI systems**
  - **Time dependent coef: time varying system**

  Example: space vehicle

LTI system: linear time invariant

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Control System Representation

Different ways to represent

- **Differential equation form: equations governing system dynamics**
- **Transfer function**

  obtained by using Laplace transforms (linear systems)

$$G(s)=\frac{N(s)}{D(s)}=\frac{Y(s)}{U(s)}$$

- **State space form:**

$$\dot{x}=Ax+Bu$$
$$y=Cx+Du$$

  obtained by reducing diff. Equations by defining appropriate states x. Linear system only
- **State form for nonlinear systems** $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},u,t)$

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More on Laplace Transform

Review slide

Discussion on the following

- **Definition**
- **Standard tables of transforms**
- **Some important properties**
  - **Derivative**
  - **Final value theorem**
  - **Convolution integral**

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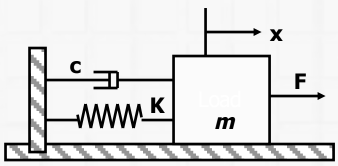
Example: spring mass system

Zero initial conditions



- **Differential equation form:**

  $m\ddot{x} + c\dot{x} + kx = u(t)$ **Newton's method**

- **Transfer function: input u(t), output x(t)**

  $G(s) = \frac{X(s)}{F(s)}$ $G(s) = \frac{1}{ms^2 + cs + k}$

Usually: zero initial conditions

- **State space form:**

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$x_1 = x$ $\dot{x}_1 = x_2$

$x_2 = \dot{x}$ $\dot{x}_2 = \frac{1}{m}(-cx_2 - kx_1)$

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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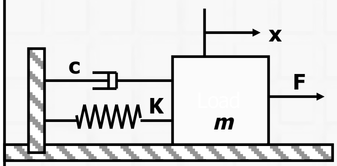
Example: spring mass system

Nonzero initial conditions



- **Differential equation form:**

  Newton's method

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

- **Transfer function: input u(t), output x(t)**

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$ms^2X(s) - ms\dot{x}(0) - mx(0) + csX(s) - cx(0) + kX(s) = F(s)$$

$$X(s) = \frac{F(s)}{ms^2 + cs + k} + \frac{m[s\dot{x}(0) + x(0)]}{ms^2 + cs + k} + \frac{cx(0)}{ms^2 + cs + k}$$

$$G(s) = \frac{X(s)}{F(s)}$$

➔ **Transfer function representation not possible for non-zero initial conditions**

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- **State space form: No problem**

Finding solutions to systems represented in the forms discussed

Solution 1: solving ODEs

- **Separation of variables and integration**
- **Solution = homogeneous part + complementary part**
- **Look for more details in "Advanced Engg. Mathematics" by Kreszig**

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Solution 2: Using Laplace transform and inverse

Idea: to express F(s) as

$$F(s) = \frac{P(s)}{Q(s)}$$

$$F(s) = \frac{a_1}{s+r_1} + \frac{a_2}{s+r_2} + \cdots$$

$$a_1 = \left(s+r_1\right)F(s)\Big|_{s\to -r_1}$$

Multiple poles

$$F(s) = \frac{a_1}{s+r} + \frac{a_2}{\left(s+r\right)^2} + \cdots + \frac{a_m}{\left(s+r\right)^m}$$

$$a_k = \frac{1}{(m-k)!}\frac{d^{m-k}}{ds^{m-k}}\left[F(s)\left(s+r\right)^m\right]_{s\to -r}$$

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Example

$$F(s)=\frac{s+1}{s^2\left(s^2+s+1\right)}$$

$$F(s)=\frac{a_1}{s}+\frac{a_2}{s^2}+\frac{a_3}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}+\frac{a_4}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}}$$

'roots' command in MATLAB

$$a_1=\left[\frac{d}{ds}\left(\frac{s+1}{s^2+s+1}\right)\right]_{s\to 0}=0$$

$$a_2=\left[\frac{s+1}{s^2+s+1}\right]_{s\to 0}=1$$

$$a_3=\left[\frac{\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)(s+1)}{s^2\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}\right]_{s\to -\frac{1}{2}+j\frac{\sqrt{3}}{2}}=\frac{j}{\sqrt{3}}$$

$$a_4=a_3^*=-\frac{j}{\sqrt{3}}$$

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Example

$$F(s)=\frac{s+1}{s^2\left(s^2+s+1\right)}$$

$$F(s)=\frac{1}{s^2}+\frac{j/\sqrt{3}}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}-\frac{j/\sqrt{3}}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}}$$

$$f(t)=t+\frac{j}{\sqrt{3}}e^{-\frac{1}{2}t+j\frac{\sqrt{3}}{2}t}-\frac{j}{\sqrt{3}}e^{-\frac{1}{2}t-j\frac{\sqrt{3}}{2}t}$$

$$f(t)=t-\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

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Concept of stability

- Very important characteristic of the transient performance of the system.
- An LTI system is stable if the following two notions of system stability are satisfied:

  (i) When the system is excited by a bounded input, the output is bounded (BIBO).

  →(ii) In the absence of input, output tends towards zero (the equilibrium state of the system) irrespective of initial conditions(this is also called as asymptotic stability).

More tangible conditions

i. If all the roots of the characteristic equation (poles) have negative real parts, the system is *stable*.

ii. If any ONE root of the characteristic equation has a positive real part or if there is a repeated root on the jw-axis, the system is *unstable*.

iii. If the condition (i) satisfied except for the presence of one or more non repeated roots on the jw-axis, the system is *marginally stable*.

Poles and zeros Definition

$$G(s) = \frac{N(s)}{D(s)} = \frac{Y(s)}{U(s)} = A\frac{\sum_{m=0}^{M}(s - s_{0m})}{\sum_{n=0}^{N}(s - s_{\infty n})}$$

← zeros

← poles

Pole zero plot and animation

http://www-es.fernuni-hagen.de/JAVA/PolZero/polzero.html

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Effect of Poles on system response

- The nature of impulse response g(t) depends on the poles of the transfer function G(s) which are the roots of the characteristic equation
- These roots may be both real and complex conjugate and may have multiplicity of various orders.
- The nature of responses contributed by all possible types of poles are shown in the following slides.
- HW: In each case find whether the system is stable or not with the help of BIBO condition (i.e check whether the area under the absolute-valued impulse curve is finite or not).

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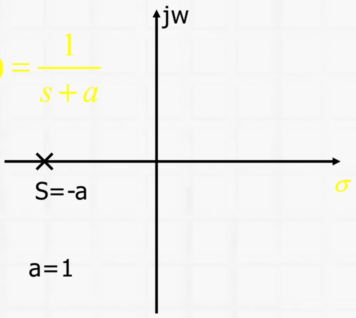
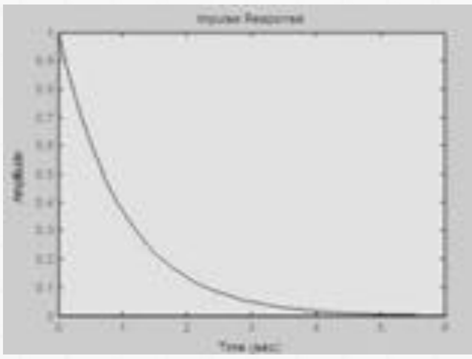
System with single real pole in left half s-plane.

$$G(s) = \frac{1}{s+a}$$





- **Impulse response is exponentially decaying (bounded output)**

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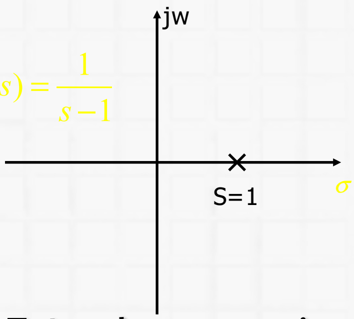
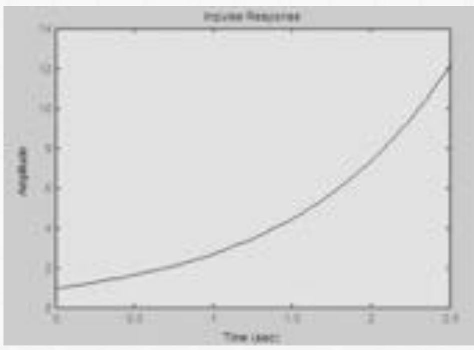
System with single real pole in right half s-plane.

$$G(s) = \frac{1}{s-1}$$





- **Impulse response is exponentially growing (unbounded output).**

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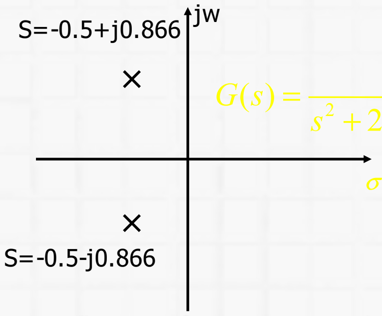
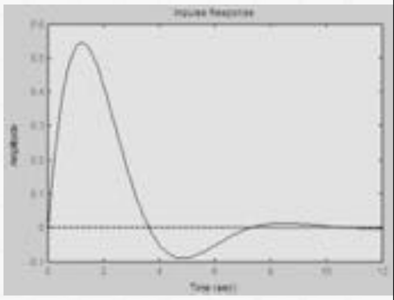
System with a pair of complex poles in left half s-plane.



$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n + \omega_n^2}$$



- Impulse response is exponentially decaying sinusoid (bounded output)

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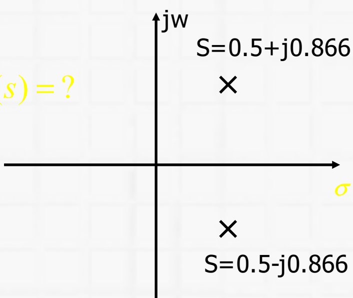
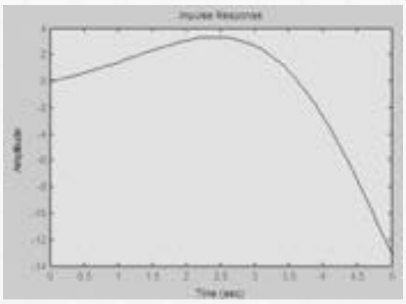
System with a pair of complex poles in right half s-plane.

$$G(s) = ?$$





- Impulse response is exponentially growing sinusoid (unbounded output)

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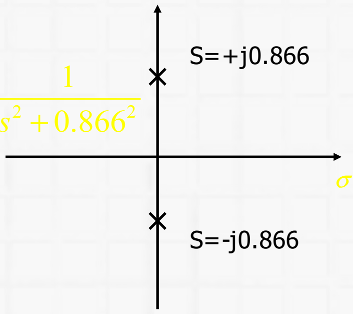
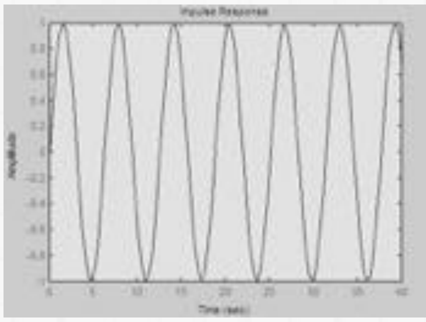
System with a pair of complex poles on jw axis.

$$G(s) = \frac{1}{s^2 + 0.866^2}$$





- Impulse response is sinusoidal (bounded output)

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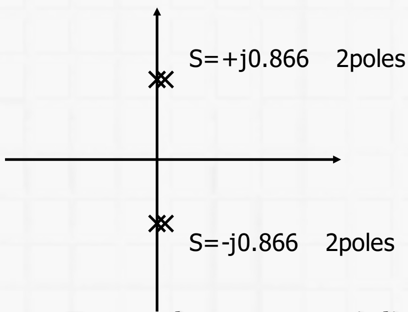
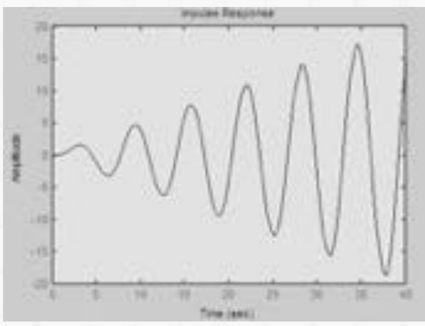
System with a double pair of complex poles on jw axis.





- Impulse response is linearly increasing sinusoid (unbounded output)

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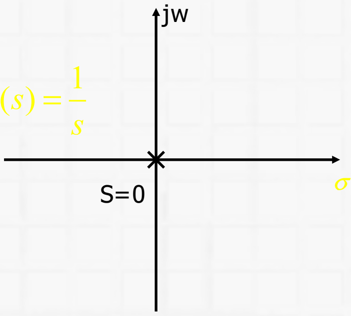
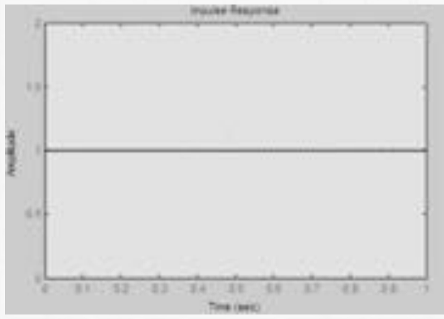
System with a pole at origin

$$G(s) = \frac{1}{s}$$





- Impulse response is constant (bounded output)
- Marginally stable system

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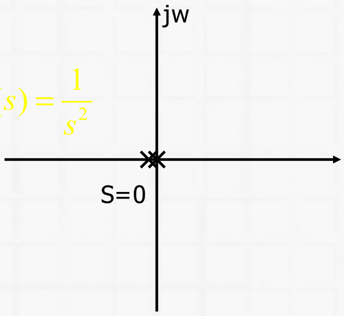
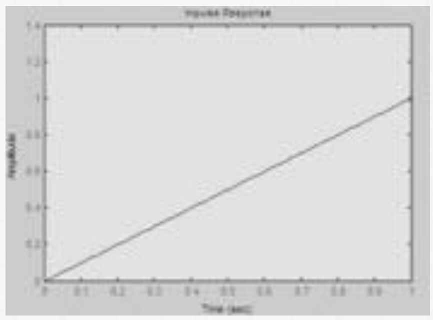
System with a double pole at origin

$$G(s) = \frac{1}{s^2}$$





- Impulse response is linearly increasing (unbounded output).

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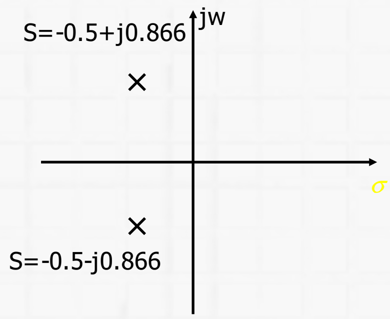
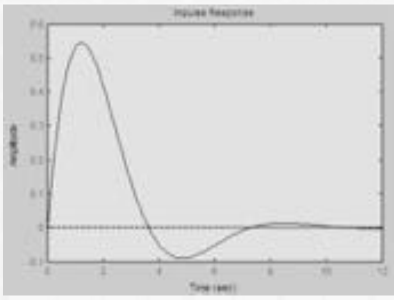
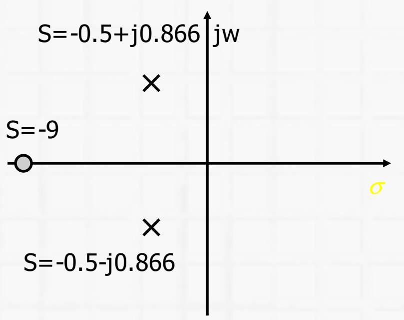
Effect of zeros on response

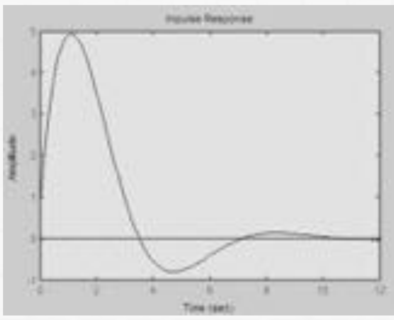
- Recall the impulse of the system with a pair of complex poles in left half s-plane.





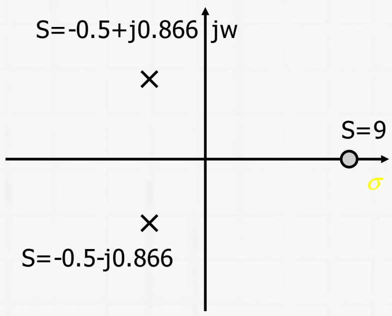
Add a left half s-plane zero to this system

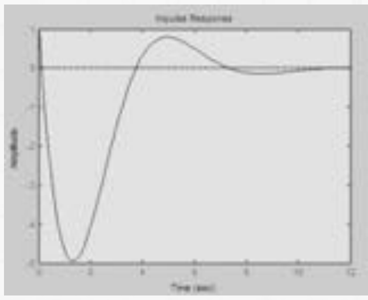




- We see no change in stability of the system except a change in the shape of the transient.

Add a right half s-plane zero to this system





- We see no change in stability of the system except a change in the shape of the transient either it is right or left half s-plane zero.

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Solution 3: solving SS

Similar to a first order system solution

- **SS model**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

- **Homogeneous part**

$$\mathbf{x}_h = e^{\mathbf{A}t}\mathbf{x}(0)$$

- **Particular solution**

$$\mathbf{x}_p = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau$$

- **Complete solution**

$$\mathbf{x}(t) = \mathbf{x}_h + \mathbf{x}_p \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau$$

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More in the lecture to follow later

Solution 3: solving SS

- **How to determine $e^{\mathbf{A}t}$??**
  - **Method 1: Using Laplace transform**

$$e^{\mathbf{A}t} = L^{-1}\{(s\mathbf{I}-\mathbf{A})^{-1}\}\mathbf{x}(0)$$

  - **Method 2: Using Caley-Hamilton theorem**

  See the example in notes

More discussion in lectures to follow

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Behavior of standard First and second order systems

Necessity

- **To get the basis for characterization of the response of other system**
- **Simplification in analysis of such systems**

Basis

- **Range of inputs like impulse, step, ramp are used for characterization**

➔Similar inputs used in the experimental characterization

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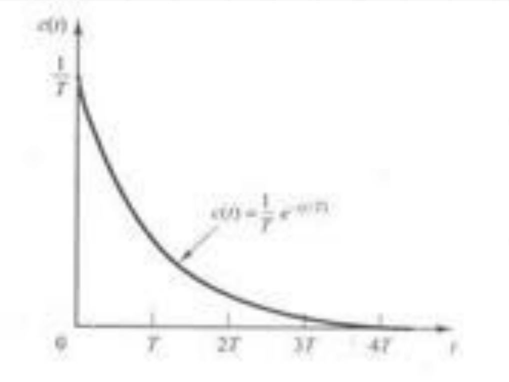
Standard Behavior

First order system: impulse



- **First order system**

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{Ts+1}$$

- **Unit impulse response**

$$U(s)=1$$

- **Using Laplace inverse**

$$y(t)=\frac{1}{T}e^{-t/T}$$

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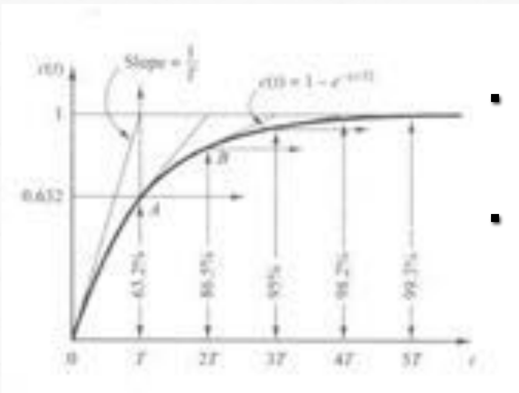
Standard Behavior

First order system: Step



- **First order system**

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{Ts+1}$$

- **Unit step response**

$$U(s)=\frac{1}{s}$$

- **Using Laplace inverse**

$$y(t)=1-e^{-t/T}$$

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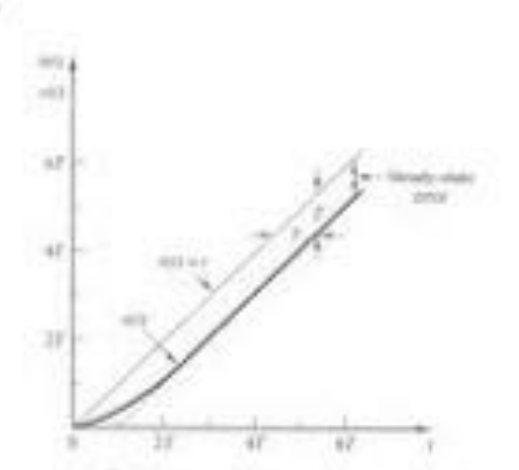
Standard Behavior

First order system: Ramp



- **First order system**

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{Ts+1}$$

- **Unit ramp response**

$$U(s)=\frac{1}{s^2}$$

- **Using Laplace inverse**

$$y(t)=t-[T-Te^{-t/T}]$$

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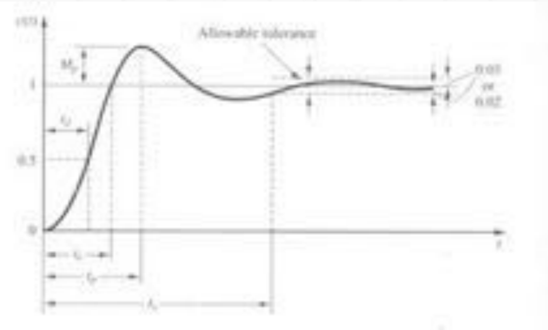
Standard Behavior

Second order system: Transient



- **Second order system**

$$G(s)=\frac{Y(s)}{U(s)}=\frac{K}{Js^2+Bs+K}$$

$$=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

- **Unit step response**

$$U(s)=\frac{1}{s}$$

- **Solution using Laplace inverse for different cases**

1. Underdamped
2. Critically damped
3. Overdamped

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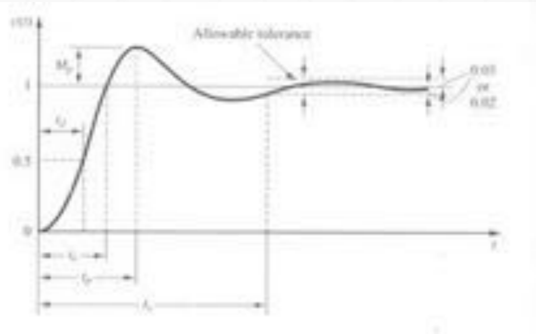
Standard Behavior

Second order system: Transient



Response terms

- Mp = Maximum overshoot
- ts = settling time
- tr = rise time
- Tp = peak time

These can be found by using the response

Standard formulae available in book

Use: for designing system Parameters/ control satisfying Certain constraints in response terms

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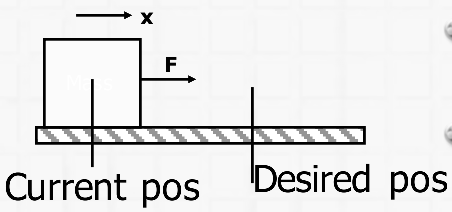
Concept of Feedback: Example



- **Goal: to place mass at a desired position**
- **Feedback???**

- Identify: quantity of interest (output) and quantity you can dictate (input)
- What is differential equation connecting input and output?

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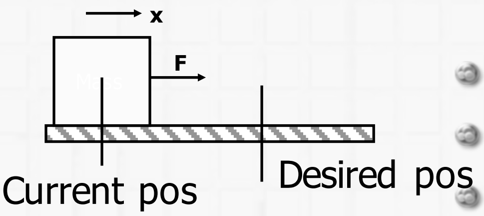
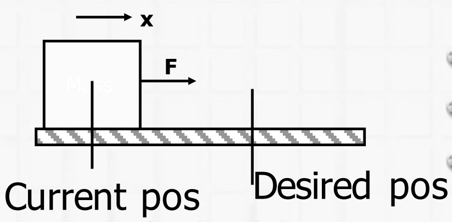
Concept of Feedback: Example



- **Goal: to place mass at a desired position**
- **Control input: Force F**
- **Output: displacement x**
- **Equation** $F = m\ddot{x}$

- Q: To take mass to desired position xd can you think what F should be applied?
  - One way is to define trajectory from current position x0 to xd and get corresponding $\ddot{x}(t)$ and apply $F = m\ddot{x}_d$ and hope that with this the desired position would be reached
  - Q: Can you see what is the problem with such solution

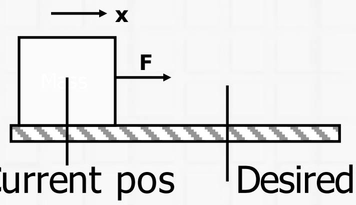
Concept of Feedback: Example



- **Goal: to place mass at a desired position**
- **Input: Force F**
- **Output: displacement x**
- **Equation** $F = m\ddot{x}$

- A: no feedback hence we would not know if we have really achieved the task, especially in the presence of external disturbances such as friction.
- Hence we need to sense the current position and make use of the same in developing F to be applied.
- Q: what function F=f(x) (notice NOT f(t)) should F be, so as to achieve the desired goal?? Can you think of?

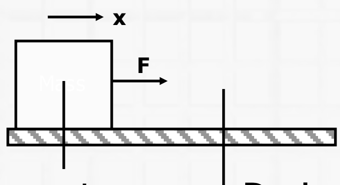
Concept of Feedback: Example



- **Goal: to place mass at a desired position**
- **Input: Force F**
- **Output: displacement x**
- **Equation** $F = m\ddot{x}$

- Q: How to come up with such function?
- One idea is to see error between current x and xd and make F to be some function of this error such that F = 0 when error is 0.
- Another idea is to consider virtual springs and dampers in the system such that equilibrium position of such spring mass damper system coincides with xd.
- To start with lets consider F to be $F = -k_p(x - x_d) = -k_p e$

Concept of Feedback: Example



- **Goal: to place mass at a desired position**
- **Input: Force F**
- **Output: displacement x**
- **Equation** $F = m\ddot{x}$

- Q: How do we know that such F would work towards achieving our goal?
- A: carry out analysis by substituting this F in equation of dynamics. Writing everything in e variable, thus

$$F = -k_p(x - x_d) = -k_p e = m\ddot{x} = m\ddot{e}$$

$$\therefore \quad m\ddot{e} + k_p e = 0$$

- This is equation of harmonic system without damping and for initial error e0 the final error will keep oscillating and never be zero as desired ☹ so how to take care of this!

Concept of Feedback: Example

- In effect what we have done is 'attached' a virtual spring to the mass whose equilibrium position is at xd which actually is leading to indefinite oscillations.
- A: Look at the final error equation and see what form we would desire for this equation. The form we would desire is

$$m\ddot{e}+k_d\dot{e}+k_pe=0$$

- In effect we would like to have additional virtual damper in the system which would reduce oscillations finally to zero meaning e = 0 which means x=xd ☺
- Q: can you see what expression for F would give the above equation of "closed loop dynamics" ??

Concept of Feedback: Example

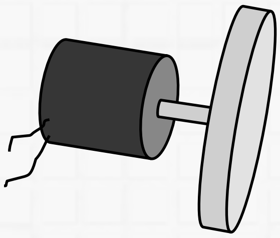
- Back substituting we would see that to get the desired equation of error dynamics $m\ddot{e}+k_d\dot{e}+k_pe=0$ we would need

$$F=-k_pe-k_d\dot{e}$$
$$=-k_p(x-x_d)-k_d\dot{x}$$

Notice both position and velocity are FED BACK

- Every time doing such analysis is difficult especially for more complex systems hence we would need additional tools to analyze and synthesize control input (F in this case) to achieve the desired goal in terms of desired output (x in this case)
- These tools we would learn are based on Laplace transforms hence need to revise Laplace transforms

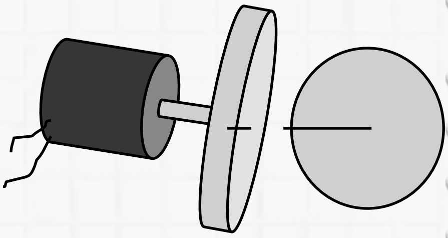
Concept of Feedback: Example



- Goal: to place disc at a desired angle
- Feedback??? Theta

- Identify: quantity of interest (output) : Theta and quantity you can dictate (input) : Torque
- Some may claim input can be voltage given to motor as well!! Possible but we then need motor dynamics to be considered in analysis as well
- The system is mathematically exactly same as previous system equation being $\tau = J\ddot{\theta}$
- HW: work out the control laws and analysis similar to the previous example!!

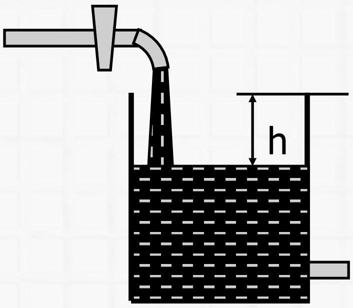
Feedback: Example



- Goal: to rotate disc with desired speed
- Feedback???
- Can the angle act as feedback?
- Desired output? Control input?
- What will be equation of dynamics in terms of speed ω?

- Can you develop control law that can achieve this goal?
- Q: Do you require kd term in this case?
- A: NO. we do not because since the system in the new variable is of first order there will be no overshoot and error response would be exponential!!

Feedback: Example



- Goal: to maintain height of liquid constant
- Feedback???

- Identify: quantity of interest (output) and quantity you can dictate (input)

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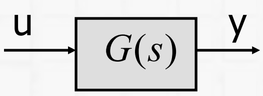
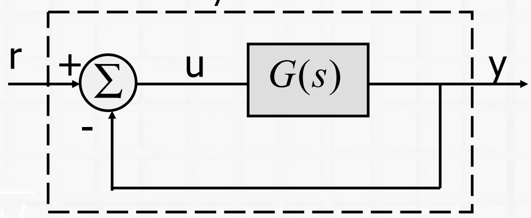
Open-loop vs closed-loop Transfer Function

LTI Systems with feedback

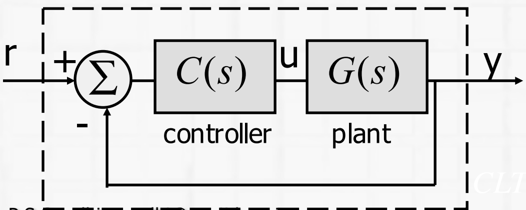


Unity feedback



Example $G(s) = \frac{1}{ms^2 + cs + k}$

$CLTF = \frac{G(s)}{1+G(s)}$



All manipulations algebraic in s domain

$CLTF = \frac{G(s)C(s)}{1+G(s)C(s)}$

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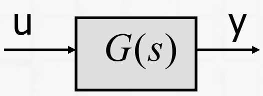
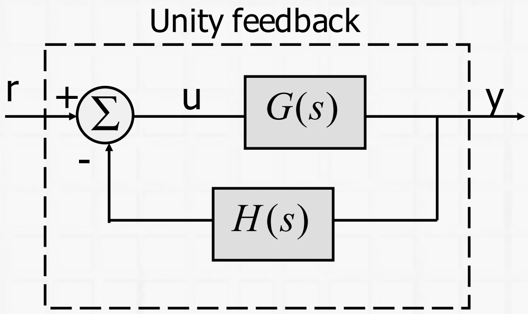
Open-loop vs closed-loop Transfer Function





Another configuration

$$CLTF = \frac{G(s)}{1 + H(s)G(s)}$$

All manipulations
algebraic in s domain

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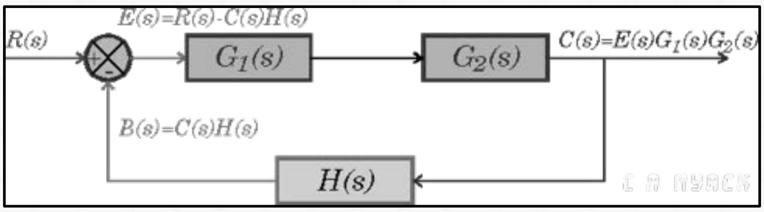
Another configuration



$$C(s) = R(s)\frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

$$C(s) = E(s)\frac{1}{1 + G_1(s)G_2(s)H(s)}$$

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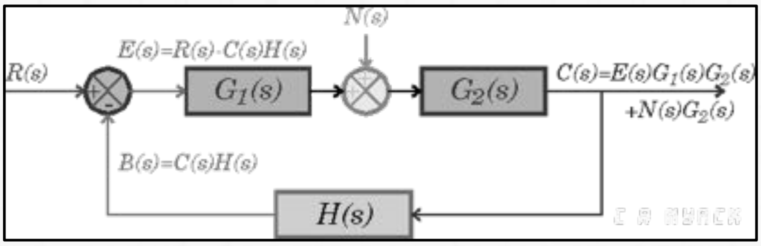
Disturbance



With $R(s) = 0$, the transfer function between $N(s)$ and $C(s)$

$$C(s) = N(s)\frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

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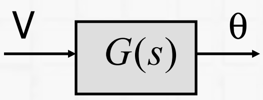
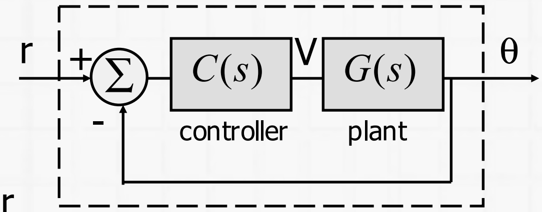
Motor with PD control

Motor position feedback





Open loop motor

$$G(s) = \frac{K_t\varphi}{J_m s^2 + B_m s + (KK_t\varphi^2 / R_a)s}$$

$$CLTF = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

$$C(s) = k_p + k_d s$$

Proportional part

Derivative part

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Important Concepts

- What is feedback?
- Open Loop Vs Closed Loop system

- How to process the desired feedback quantity?
- When goal is given, how to decide what should be the control algorithm? Is it unique?

We will answer these questions in this part of course

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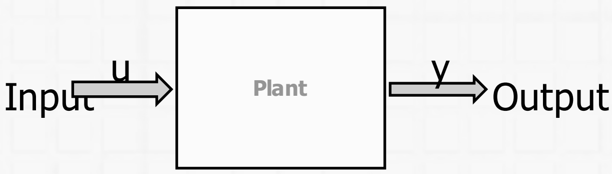
Input and output



- **Its important to know what are input and output variables given any system for control**
- **The relationship between input and output variables is obtained by applying standard physics law (Newtons, Kirchoff's etc.)**
- **From control perspective several other representations could be useful: one of them is transfer function representation**

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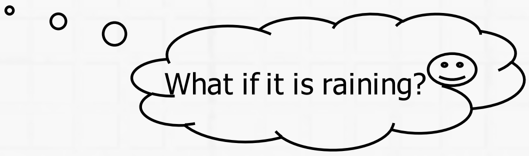
Open Loop Control

- No relation between input and output
- Useful for well-defined systems with low precision

Example:

- A sprinkler programmed to run at set times



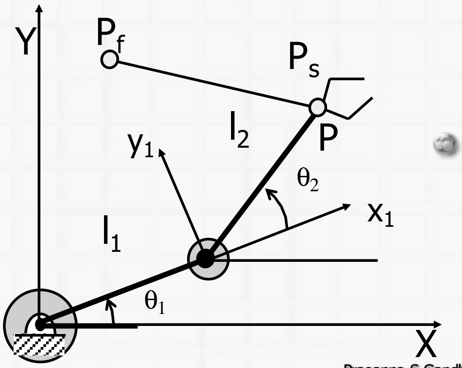
- Fan in your class / room
- Mass on surface system with F=f(t)

Closed Loop Control

- Monitor the output
- Compare the output with the reference input
- Take decision based on the difference

In case of sprinkler: soil moisture as a possible feedback

2-R Manipulator



Concept of Tracking Vs Regulation

- If we want to move from Ps to Pf without bothering about intermediate positions ➔ 'Regulation' problem
- If we want to draw straight line from Ps to Pf then kind of problem would result in maitaining θ1 as θ1d(t) and θ2 as θ2d(t). Such problems are termed as ➔ tracking problem in control context!

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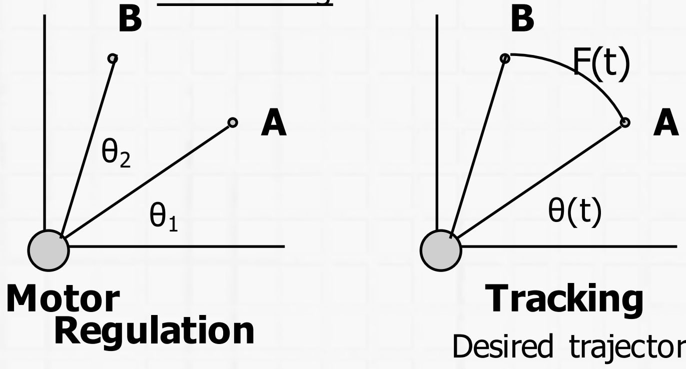
Tracking Vs Regulation

Eg: Robotic Arm What is meant by regulation and tracking in this case?



Desired trajectory is given function of time

Types of Systems

- Linear time invariant: represented by a linear ODE with constant coefficients
- Linear time variant: represented by a linear ODE with time-dependent coefficients
- Nonlinear: represented by nonlinear ODEs
- SISO – Single-input single-output
- MIMO – Multi-input multi-output
- LTI – Linear Time Invariant

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Solution of ODE

- Revise basics of solving an ODE
- Finding solution of homogeneous equation and finding complementary solution
- Example of simple second order system

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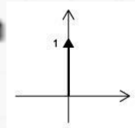
Transfer Function

- Impulse response: Output, $g(t)$, when input is a unit-impulse function $\delta(t)$, with $\boxed{\text{zero initial conditions}}$



- Transfer function: Laplace transform, $G(s)$, of the impulse response
- Also, $G(s)=X(s)/U(s)$. Why??

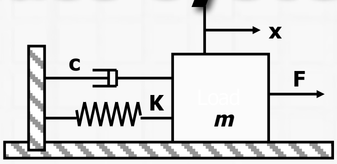
  where $X(s)$ and $U(s)$ are Laplace transforms of the output and

Poles and Zeros

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b_o s^m + b_1 s^{m-1} + \ldots.. + b_m}{a_o s^n + a_1 s^{n-1} + \ldots.. + a_n} ; m < n$$

- **Transfer function G(s)**
- **Poles: roots of denominator polynomial**
  - **More formal: a function is said to have a pole of order r at s=si if the** $\lim_{s \to s_i} [(s-s_i)^r G(s)]$ **has a finite nonzero value.**
- **Zeros: roots of numerator polynomial**
- **Characteristic equation is obtained by making denominator of the transfer function equal to zero**

Example: Spring mass system



What is the order of this system?

- Equation of motion:

$$\frac{d}{dt}\left(m\dot{x}\right)+kx=F$$

$$m\ddot{x}+kx=F$$

- $G(s)=X(s)/F(s)=???$

Meaning of Poles Example

- **Q: How poles are related to system response?**
- **Observe that the numerator of the TF is same as characteristics equation for solution of homogeneous part of differential equation**

Thank You

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Laplace Transform Definition

Review slide

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^{\infty} e^{-st} f(t)dt$$

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Laplace Transform Table

Review slide

f(t)	F(s)	
9b	$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
10 a	$re^{-at}\cos(bt+\theta)u(t)$	$\frac{[r\cos(\theta)]s+[ar\cos(\theta)-br\sin(\theta)]}{s^2+2as+(a^2+b^2)}$
10 b	$re^{-at}\cos(bt+\theta)u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$
10 c	$re^{-at}\cos(bt+\theta)u(t)$ $r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$ $\theta=\tan^{-1}\left[\frac{Aa-B}{A\sqrt{c-a^2}}\right]$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$
10 d	$e^{-at}\left[A\cos(bt)+\frac{B-Aa}{b}\sin(bt)\right]u(t)$ where: $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$

	f(t)	F(s)
1	$\delta(t)$	1
	$u(t)$	$\frac{1}{s}$
	$tu(t)$	$\frac{1}{s^2}$
	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
	$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
	$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
	$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$

Laplace Transform Properties

Review slide

	Operation	f(t)	F(s)
1	Addition	$f_1(t)+f_2(t)$	$F_1(s)+F_2(s)$
2	Scalar Multiplication	$kf(t)$	$kF(s)$
3a	Time Differentiation	$\frac{df}{dt}$	$sF(s)-f(0^-)$
3b		$\frac{d^2f}{dt^2}$	$s^2F(s)-sf(0^-)-\dot{f}(0^-)$
3c		$\frac{d^3f}{dt^3}$	$s^3F(s)-s^2f(0^-)-s\dot{f}(0^-)-\ddot{f}(0^-)$
4a	Time Integration	$\int_{0^-}^{t} f(\tau)d\tau$	$\frac{1}{s}F(s)$
4b		$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{s}F(s)+\frac{1}{s}\int_{-\infty}^{0^-} f(t)dt$
5	Time Shift	$f(t-t_0)u(t-t_0)$	$F(s)e^{-st_0}, t_0 \geq 0$
6	Frequency Shift	$f(t)e^{s_0t}$	$F(s-s_0)$
7	Frequency Differentiation	$-tf(t)$	$\frac{dF(s)}{ds}$

Laplace Transform Properties

Review slide

	Operation	f(t)	F(s)
8	Frequency Integration	$\frac{f(t)}{t}$	$\int_s^\infty F(z)dz$
9	Scaling	$f(at), a \geq 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
10	Time Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
11	Frequency Convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi j}F_1(s) * F_2(s)$
12	Initial Value	$f(0^+)$	$\lim_{s\to\infty} sF(s), n > m$
13	Final Value	$f(\infty)$	$\lim_{s\to\infty} sF(s)$, poles of $sF(s)$ in LHP

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