ME 6102: Design of Mechatronic Systems

Sampling, digitization, and signal processing

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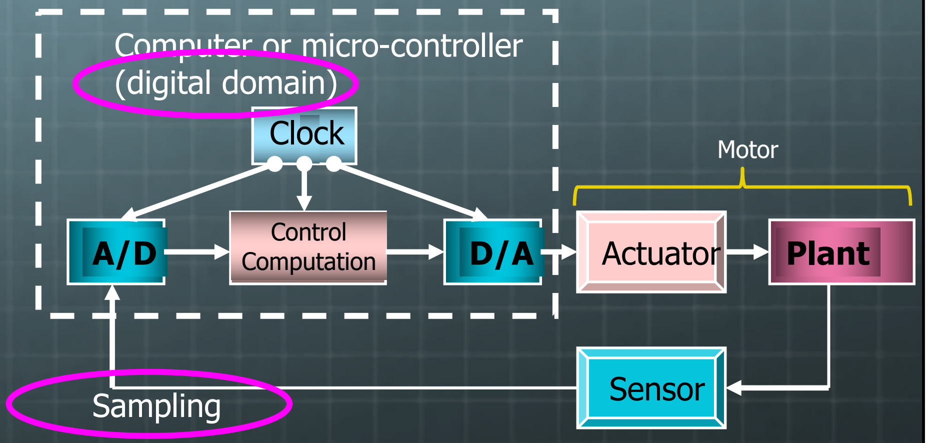
Outline

- Motivation
- Shannon sampling theorem & aliasing
- ZOH and digital domain system
- Effect of digitization on system
- Z transforms and their utility
- Basics of filters
- Implementation of digital control
  - Different ways

Digital control system configuration



Recall problem faced in Mechatronics lab

- Data overflow effect!
- Speed computed from logged data was much different from actual speed..

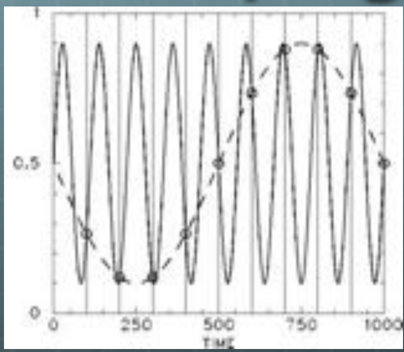
Sampling Fundes

- Basic idea: A continuous domain signal is represented as samples in digital domain.
- Q: are the samples true representation of original signal?
- Q: can it be possible to completely recover original signal from samples? What role sampling time or frequency has to play?
- PWM goes on motor as sampled signal, Q: What is effect of sampling on system dynamics?
- You have observed some discrepancy coming due to “insufficient” sampling (logging) frequency while processing steady state speed data.

Sampling Fundamentals

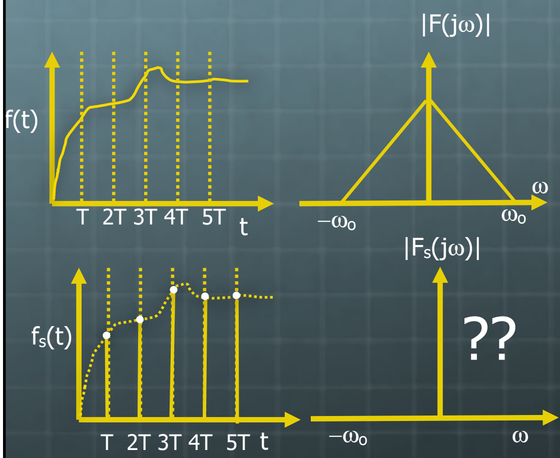
- **Example: movie with horse cart: observation about wheels**
  - **Cart is going forward**
  - **Wheels appear rotating in opposite direction → aliasing of signal**
- **Q: why is it so??**
- **Q: Can we quantify??**

Sampling Fundamentals



- **Example: Similar thing for sampling a sine wave with sampling time T**
  - **Original sine wave is lost**
  - **We see new sine wave at completely different frequency ➔ aliasing of signal**
- **Q: why is it so??**
- **Q: Can we quantify??**

Representation of signal and its sampled version



- General time domain signal = f(t) band limited to ω_o
- Q: how do you express its sampled version in mathematical expression?

$$f_s(t) = f(kT) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

- Q: What is frequency response of sampled signal? Using Fourier transform fundes:

$$\omega_s = \frac{2\pi}{T}$$

$$F_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F[j(\omega - k\omega_s)]$$

Shannon/Nyquist Sampling Theorem

- Two parts:
  - Sampling frequency should be more than 2 times the max frequency content in the signal (called Nyquist frequency): follows from

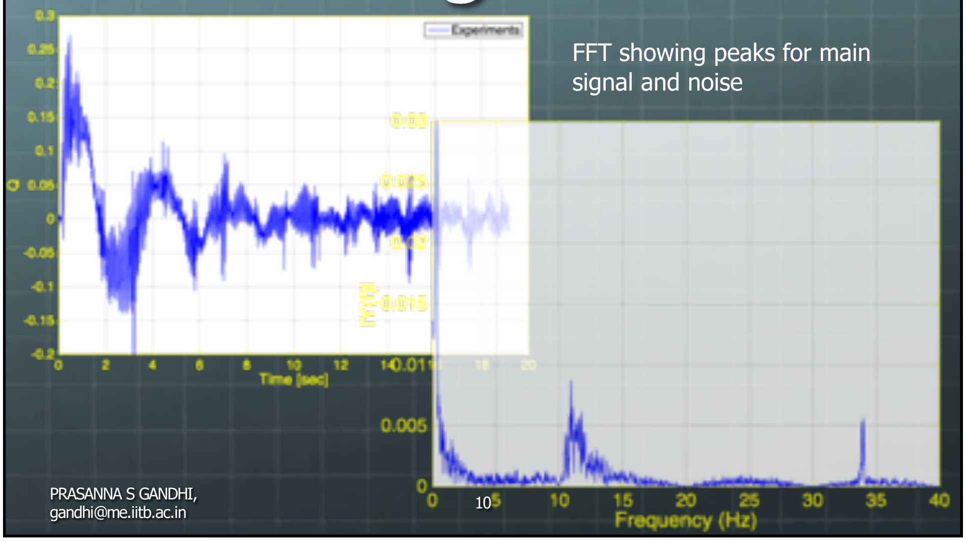
$$F_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F[j(\omega - k\omega_s)]$$

  - Shannon reconstruction
    - Built original signal using the samples by formula

$$\omega_s = \frac{2\pi}{T} \qquad f_r(t) = \sum_{k=-\infty}^{\infty} f_s(kT)\frac{\sin\frac{\omega_s(t-kT)}{2}}{\frac{\omega_s(t-kT)}{2}} = \sum_{k=-\infty}^{\infty} f_s(kT)\,\mathrm{sinc}\frac{\omega_s(t-kT)}{2}$$

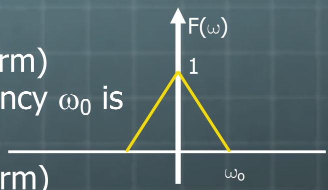
T = sampling time

Recorded Strain Guage Signal



FFT showing peaks for main signal and noise

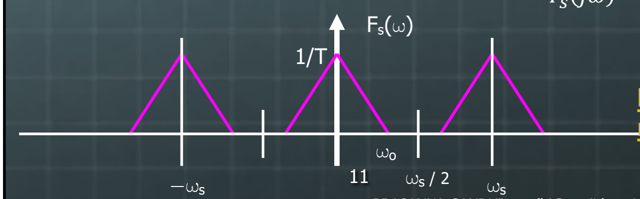
Sampling in Frequency Domain



If ***frequency response*** (fourier transform) original signal 'band limited' by Frequency ω_0 is represented as ➔

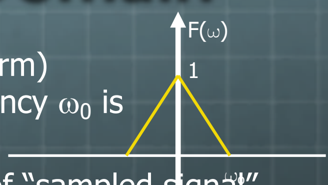
Then frequency response (Fourier transform) Of sampled signal when: frequency > $2w_o$ will be given based on Shannon sampling theorem as ➔

$$F_S(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F[j(\omega - k\omega_s)]$$



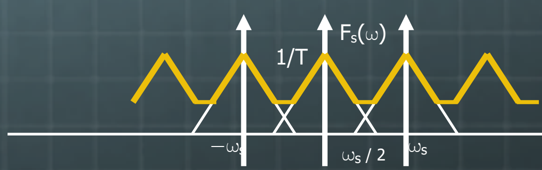
http://dspcan.homestead.com/files/Sdtft/dtftalia.htm

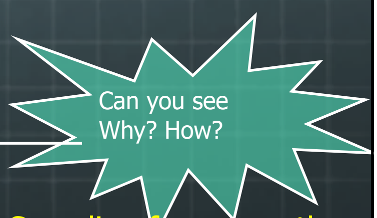
Aliasing in Frequency Domain



If ***frequency response*** (fourier transform) original signal 'band limited' by Frequency ω_0 is represented as ➔

Frequency response (Fourier transform) of "sampled signal" when: frequency < $2w_o$

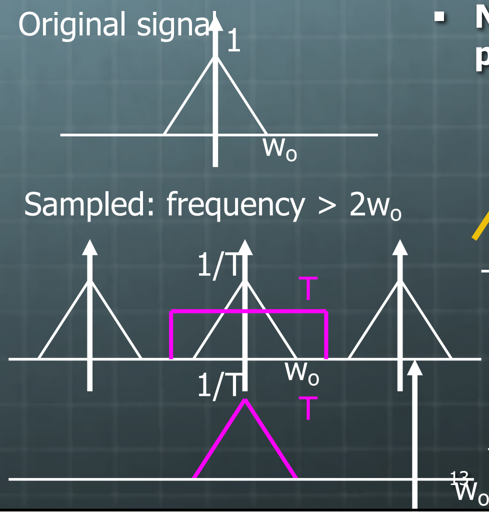


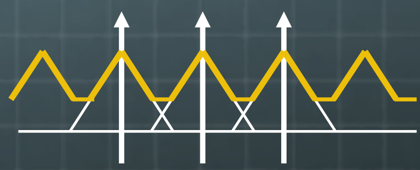


➔So what? What is implication of this? Sampling freq more than Twice the "bandwidth" of system

Shannon reconstruction

- **No reconstruction possible in this case**





Reconstruction formula:

$$f_r(t) = \sum_{k=-\infty}^{\infty} f(kT) \frac{\sin \frac{\pi(t-kT)}{T}}{\pi(t-kT)/T}$$

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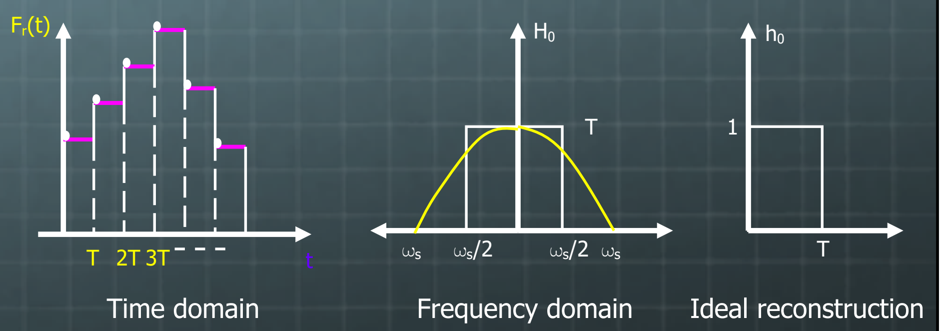
Other reconstructions

- **Zero order hold (ZOH) : most popular in mechatronic systems**



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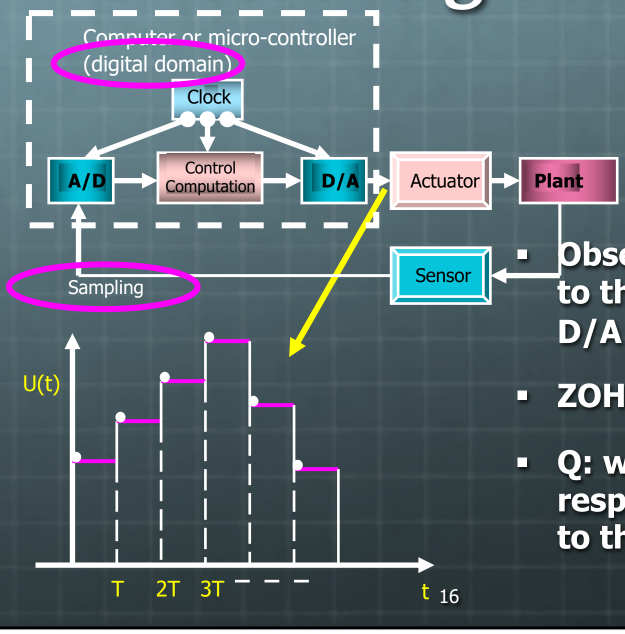
Applications: Sampling theorem

- Low pass analog filtering before sampling: To avoid aliasing effects coming after sampling
- Making sure sampling frequency is large enough to preserve signal contents
- If signal is to be processed further in mathematics with nonlinear operations then how do we do the sampling

Effect of digitization



- Observe input going to the plant through D/A or PWM
- ZOH signal
- Q: what is the response of system to the new input?

Effect of digitization

- Q: How do we analyze effect of such input U going on system? How system would respond to this input?
- We know for continuous input case that

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

- Apply it for discrete input from to see how system will evolve from x(k) to x(k+1)

$$x[(k+1)T] = e^{A[(k+1)T-kT]}x[kT] + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}Bu(\tau)d\tau$$

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Effect of digitization

- Simplify it further by change of variable in integration

$$s = (k+1)T - \tau; ds = -d\tau;$$

$$\tau = kT \rightarrow s = T; \tau = (k+1)T \rightarrow s = 0;$$

- This gives

$$x(k+1) = \boxed{e^{AT}}\, x(k) + \boxed{\int_0^T e^{As}Bds}\, u(k)$$

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Effect of digitization

- **Pulse response (conv in digital domain)**

Digital response for zero order hold is

$$x(k+1) = \phi(T)x(k) + \Gamma u(k)$$

$$y(k) = Cx(k)$$

$$\phi(T) = e^{AT}$$

$$\Gamma(T) = \int_0^T e^{As}\, ds\, B$$

$$x(k_0+1) = \phi(T)x(k_0) + \Gamma u(k_0)$$

$$x(k_0+2) = \phi(T)[\phi(T)x(k_0) + \Gamma u(k_0)] + \Gamma u(k_0+1)$$

$$\boxed{x(k) = \sum_{j=k0}^{k-1} \phi^{k-j-1}\Gamma u(j)}$$

← conv in digital domain

Effect of digitization

$$x(k) = \sum_{j=k0}^{k-1} \phi^{k-j-1}\Gamma u(j)$$

$$y(k) = \sum_{j=k0}^{k-1} C\phi^{k-j-1}\Gamma u(j)$$

$$h(l) = C\phi^{(l-1)}\Gamma$$

Pulse response function

$$h(k) = C\phi^{k-1}\Gamma$$

Effect of digitization

- **Pulse transfer function**

  q – Pulse transfer operator

  q x(k) = x(k+1)   ← We define

Zero order hold digital system response is:

$$qx(k) = x(k+1) = \phi(T)x(k) + \Gamma u(k)$$

$$x(k) = [qI - \phi(T)]^{-1}\Gamma u(k)$$
$$y(k) = C[qI - \phi(T)]^{-1}\Gamma u(k)$$

$$H(q) = C[qI - \phi(T)]^{-1}\Gamma$$ --Required transfer function

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Effect of digitization

- **Pulse transfer function**

  q – Pulse transfer operator

  q x(k) = x(k+1)   ← We define

$$H(q) = C[qI - \phi(T)]^{-1}\Gamma$$ --Required transfer function

- **This forms inspiration for definition or USE of the Z transforms**

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Z-transform Definition

- Z transform of pulse transfer operator results in algebraic multiplication as in s-domain.
- Definition:

$$X(z) = \mathbb{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Where Z is a complex variable

$$Z = re^{j\omega}$$

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Z-transform

- **Z transform properties:**
  - **Linearity**

$$Z[x_1(k)+x_2(k)]=X_1(z)+X2(z)$$

  - **Time shift**

$$Z[x(k-n)]=z^{-n}X(z)$$

  - **Convolution**

$$Z[x_1(k)*x_2(k)]=\sum_{k=0}^{\infty}X_1(k).X_2(k)$$

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Z-transform

- **Derivative**

$$Z[n.x(k)]=-z\frac{dX(z)}{dz}$$

- **Initial value theorem**

for

$$x(0)=\lim_{z\to\infty}X(z) \qquad x(k)=o \qquad k<0$$

- **Final value theorem**

$$\lim_{k\to\infty}x(k)=\lim_{z\to 1}(1-z^{-1})X(z)$$

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Z-transform: Use

- System representation in z domain similar to that in s domain
- Z transfer function: depends on
  - The sampling time
  - The kind of reconstruction

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Use of z-transform and z-inverse in finding solution to digital system

- Illustration simple example

  Ex. Find response of the following system in time domain

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

Solution:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

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Use of z-transform and z-inverse in finding solution to digital system

Using partial fraction expansion

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

Taking Z inverse transform

$$x(n) = 2\delta(n) - 9(\frac{1}{2})^n u(n) + 8u(n)$$

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Further directions

- **Control analysis and development tools similar to those in analog domain available**
- **Beyond the intended scope of this course**

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- **Fundamentals of filters**
- **Implementation of digital control**
  - **Different ways**

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Transforming system from one domain to other

- Bilinear transformation
- Domains:
  - Differential equations
  - State space
  - Transfer function
  - Discrete domain transfer function with shift operator q
  - Discrete domain transfer function with z transforms (using bilinear transforms)
  - Discrete time domain relation between yk and uk

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

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Why we Need Filters?

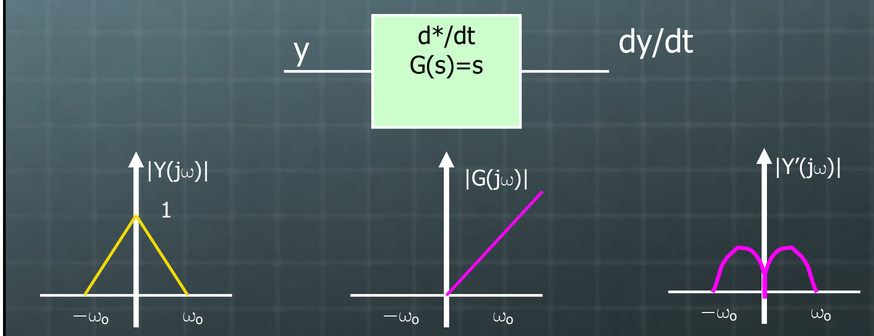
- Noise in the system
- Performance degradation because of noise in feedback
- Filters to remove usually high frequency noise
- Sources of noise??
  - Electromagnetic radiation
  - Differentiation

What happens with Differentiation in Frequency Domain?

- Differentiation operation on a signal in Laplace domain is multiplication by s so in frequency domain when $s = j\omega$ as a special case, it means amplitude of signal is multiplied by ω.
- Example, say signal is $y= a \sin\omega t$ amplitude is a. Then $dy/dt = a\omega \cos\omega t$.
- Graphical representation of general signal?

What happens with Differentiation in Frequency Domain?

- **General signal with band limited frequency content**



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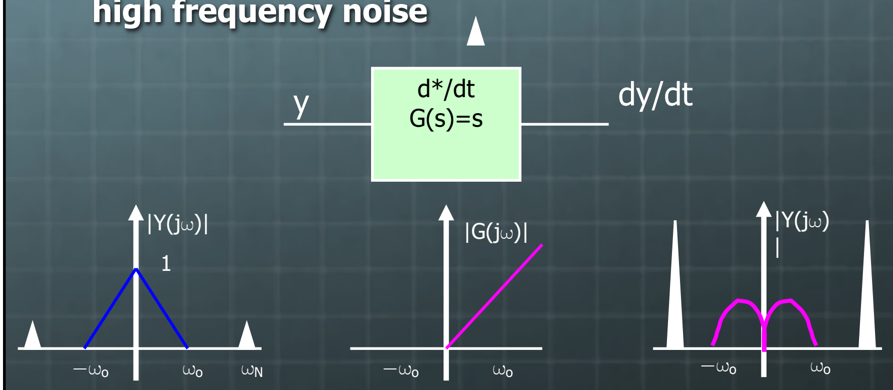
What happens with Differentiation in Frequency Domain?

- **Signal with band limited frequency content and high frequency noise**



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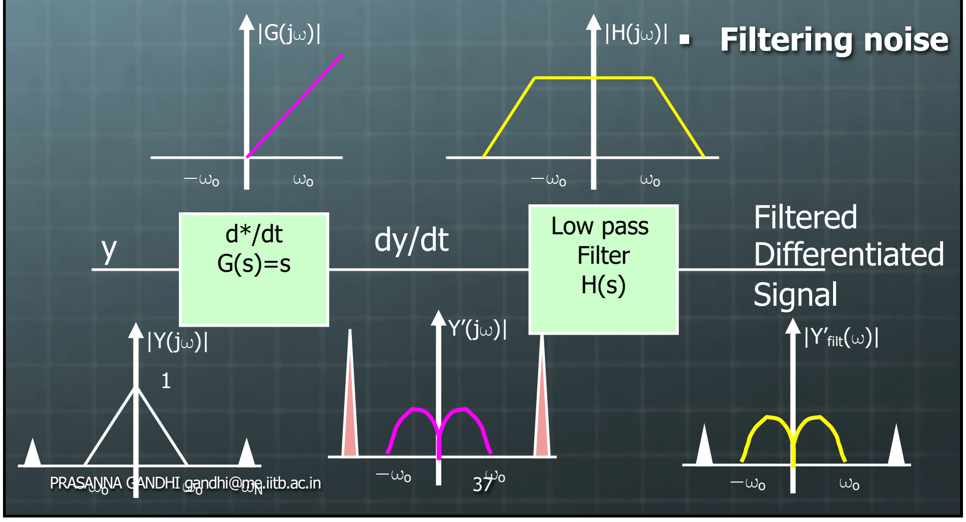
Q: how do we take care of this noise?

What happens with Differentiation in Frequency Domain?

- **Filtering noise**



Moving Average Filter and Implementation

- Moving average filter
- Implemented by averaging value at a given time instant based on past few values
- Need to update previous values sequentially at the end of ISR
  - x_1 = x
  - X_2 = x_1
  - …
- Open Q: What is frequency response of such filter? What are we doing in frequency domain?? Think

Moving Average Filter and Implementation

- Q: How do we know filter is working?
- Construct a case where noise effect is enhanced.
- Because of encoder we do not have much noise in position signal.
- There will be small noise in velocity signal because of approximate calculations. To enhance effect use kp = 0 and kd a very high value and FEEL the effect while moving motor.
- "FEEL" effect before and after filtering

Handling Friction

- Dither
- Compensation