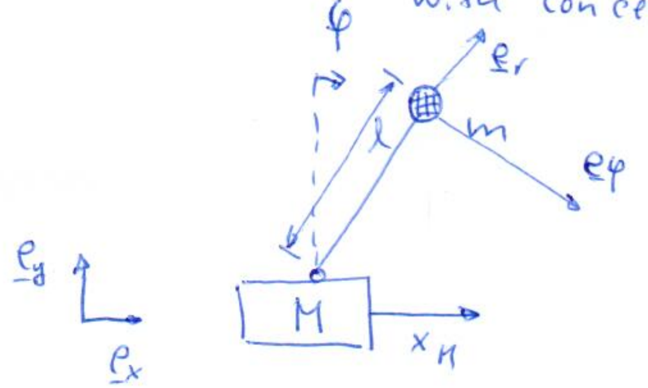
Inverted Pendelum

□ Mathmatical modeling using Newton-Euler balance equation

- Introduction of the body coordinates and their derivation

two bodies: Cart with mass M, Pendelum with concentrated mass m



- Coordinate of the Cart

$$\underline{x}_M = x_M \, \underline{e}_x \tag{1}$$

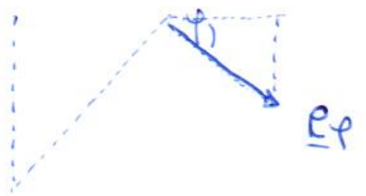
- Coordinate of the Pendelum (mass point)

$$\underline{x}_m = x_M \, \underline{e}_x + l \, \underline{e}_r \tag{2}$$

- conversion of $\underline{e}_r$ into $\underline{e}_x$ and $\underline{e}_y$ coordinates

$$\underline{e}_r = \sin\varphi \; \underline{e}_x + \cos\varphi \; \underline{e}_y \qquad (3)$$

- conversion of $\underline{e}_\varphi$ into $\underline{e}_x$ and $\underline{e}_y$ coordinates



$$\underline{e}_\varphi = \cos\varphi \; \underline{e}_x - \sin\varphi \; \underline{e}_y \qquad (4)$$

- matrix notation

$$\begin{pmatrix} \underline{e}_r \\ \underline{e}_\varphi \end{pmatrix} = \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix} \begin{pmatrix} \underline{e}_x \\ \underline{e}_y \end{pmatrix}$$

- inversion results in the calculation of $\underline{e}_x, \underline{e}_y$ from $\underline{e}_r, \underline{e}_\varphi$

$$\begin{pmatrix} \underline{e}_x \\ \underline{e}_y \end{pmatrix} = \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}^{-1} \begin{pmatrix} \underline{e}_r \\ \underline{e}_\varphi \end{pmatrix}$$

$$\begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}^{-1} = \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}$$

calcution with $A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$

Which results in

$$\underline{e}_x = \underline{e}_r \sin\varphi + \overset{\underline{e}_\varphi}{\cancel{\cos\varphi}} \cos\varphi \qquad (5)$$

$$\underline{e}_y = \underline{e}_r \cos\varphi - \underline{e}_\varphi \sin\varphi \qquad (6)$$

- derivation of the coordinates (preparation for further calculation)

$$\dot{\underline{x}}_m = \frac{d}{dt}\left(x_H \underline{e}_x + l \underline{e}_r \right) = \dot{x}_H \underline{e}_x + l \dot{\underline{e}}_r$$

with

$$\dot{e}_r = \frac{d}{dt}\left(\sin\varphi \, \underline{e}_x + \cos\varphi \, \underline{e}_y \right)$$

$$\dot{e}_r = \dot{\varphi} \underbrace{\left(\cos\varphi \, \underline{e}_x - \sin\varphi \, \underline{e}_y \right)}_{\underline{e}_\varphi \text{, see (4)}} = \dot{\varphi} \, \underline{e}_\varphi$$

we obtain

$$\dot{\underline{x}}_m = \dot{x}_H \underline{e}_x + l \dot{\varphi} \underline{e}_\varphi$$

The second derivation of x_m is given by

$$\ddot{\underline{x}}_m = \frac{d}{dt}\left(\dot{\underline{x}}_m \right) = \frac{d}{dt}\left(\dot{x}_H \underline{e}_x + l \dot{\varphi} \underline{e}_\varphi \right)$$

$$= \ddot{x}_H \underline{e}_x + l \ddot{\varphi} \underline{e}_\varphi + l \dot{\varphi} \dot{\underline{e}}_\varphi$$

with $\dot{\underline{e}}_\varphi$ given as

$$\dot{\underline{e}}_\varphi = \frac{d}{dt}\left(\cos\varphi \, \underline{e}_x - \sin\varphi \, \underline{e}_y \right)$$

$$= \dot{\varphi}\left(-\sin\varphi \, e_x - \cos\varphi \, \underline{e}_y \right)$$

$$\dot{\underline{e}}_\varphi = -\dot{\varphi} \underbrace{\left(\sin\varphi \, e_x + \cos\varphi \, \underline{e}_y \right)}_{\underline{e}_r \text{, see (3)}} = -\dot{\varphi} \, \underline{e}_r$$

we obtain

$$\ddot{\underline{x}}_m = \ddot{x}_H \underline{e}_x + l\ddot{\varphi}\,\underline{e}_\varphi - l\dot{\varphi}^2\,\underline{e}_\varphi$$

$\ddot{\underline{x}}_m$ in r, φ coordinates:

$$\ddot{\underline{x}}_m = \ddot{x}_H(\underline{e}_r \sin\varphi + \underline{e}_\varphi \cos\varphi) + l\ddot{\varphi}\,\underline{e}_\varphi - l\dot{\varphi}^2\,\underline{e}_r$$

$$\ddot{\underline{x}}_m = \ddot{x}_H \sin\varphi\,\underline{e}_r + \ddot{x}_H \cos\varphi\,\underline{e}_\varphi + l\ddot{\varphi}\,\underline{e}_\varphi - l\dot{\varphi}^2\,\underline{e}_r$$

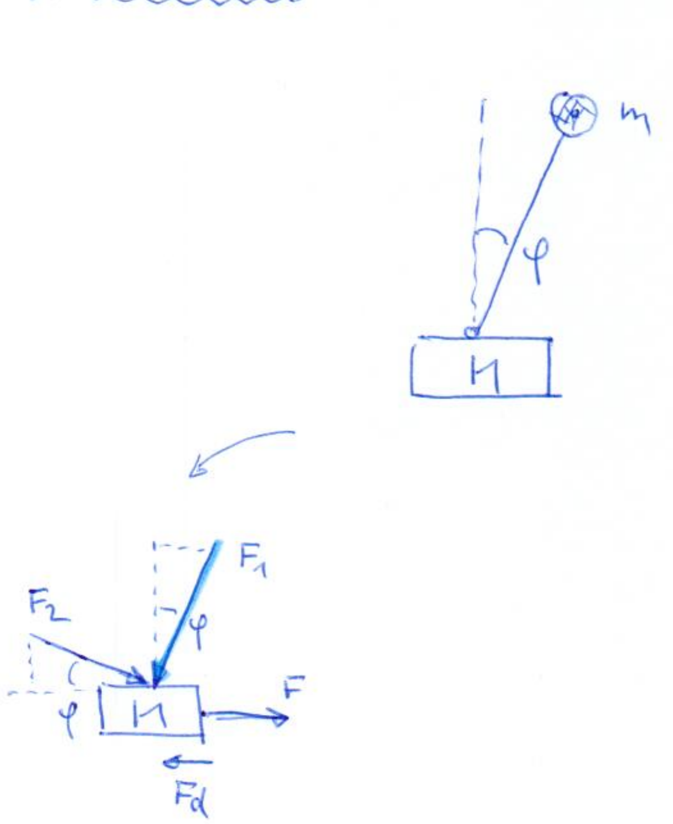
$$\ddot{\underline{x}}_m = (\ddot{x}_H \sin\varphi - l\dot{\varphi}^2)\,\underline{e}_r + (\ddot{x}_H \cos\varphi + l\ddot{\varphi})\,\underline{e}_\varphi \quad (7)$$

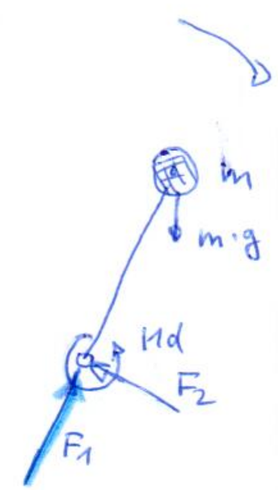
$\ddot{x}_m$ in x, y coordinates

$$\ddot{\underline{x}}_m = \ddot{x}_H\,\underline{e}_x + l\ddot{\varphi}(\cos\varphi\, e_x - \sin\varphi\, \underline{e}_y) - l\dot{\varphi}^2(\sin\varphi\,\underline{e}_x + \cos\varphi\,\underline{e}_y)$$

$$\ddot{\underline{x}}_m = (\ddot{x}_H + l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2\sin\varphi)\underline{e}_x + (-l\ddot{\varphi}\sin\varphi - l\dot{\varphi}^2\cos\varphi)\,\underline{e}_y \quad (8)$$

- decomposition of the inverted pendulum on the cart





- cart

$\underline{e}_x$: $\ddot{x}_M M = F - F_d - F_1 \sin\varphi + F_2 \cos\varphi$ (just in x-coordinate)

with $F_d = d\,\dot{x}_M$

we obtain

$$\ddot{x}_M M = F - d\,\dot{x}_M - F_1 \sin\varphi + F_2 \cos\varphi \qquad (9)$$

- pendulum

$$\ddot{\underline{x}}_m m = F_1 \underline{e}_r - F_2 \underline{e}_\varphi - mg\,\underline{e}_y$$ (mix of three coordinates)

$$\ddot{\underline{x}}_m = \frac{d^2}{dt^2}\left(x_M \underline{e}_x + l\,\underline{e}_r\right) = \begin{cases} (7) & \text{in } \underline{e}_r, \underline{e}_\varphi \text{ coordinates} \\ (8) & \text{in } \underline{e}_x, \underline{e}_y \text{ coordinates} \end{cases}$$

- Pendelum in $\underline{e}_\varphi$, $\underline{e}_r$ coordinates :

$$\underbrace{\underline{\ddot{x}}_m}_{(7)} \; m = F_1 \, \underline{e}_r - F_2 \, \underline{e}_\varphi - m g \underbrace{\underline{e}_y}_{(6)}$$

$$\left[(\ddot{x}_M \sin\varphi - l \dot{\varphi}^2) \, \underline{e}_r + (\ddot{x}_M \cos\varphi + l \ddot{\varphi}) \, \underline{e}_\varphi \right] m$$

$$= F_1 \underline{e}_r - F_2 \, \underline{e}_\varphi - m g \, (\underline{e}_r \cos\varphi - \underline{e}_\varphi \sin\varphi)$$

$$\underline{e}_r : \; m (\ddot{x}_M \sin\varphi - l \dot{\varphi}^2) = F_1 - mg \cos\varphi \qquad (10)$$

$$\underline{e}_\varphi : \; m (\ddot{x}_M \cos\varphi + l \ddot{\varphi}) = -F_2 + m g \sin\varphi \qquad (11)$$

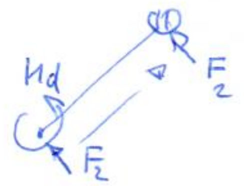
- Pendelum in $\underline{e}_x$, $\underline{e}_y$ coordinates

$$\underbrace{\underline{\ddot{x}}_m}_{(8)} \cdot m = F_1 \underbrace{\underline{e}_r}_{(3)} - F_2 \underbrace{\underline{e}_\varphi}_{(4)} - m g \underbrace{\underline{e}_y}_{\text{remain as it is}}$$

$$\underline{e}_x : \; m (\ddot{x}_M + l \ddot{\varphi} \cos\varphi - l \dot{\varphi}^2 \sin\varphi) = F_1 \sin\varphi - F_2 \cos\varphi \qquad (12)$$

$$\underline{e}_y : \; m (-l \ddot{\varphi} \sin\varphi - l \dot{\varphi}^2 \cos\varphi) = F_1 \cos\varphi + F_2 \sin\varphi - mg \qquad (13)$$

$\sum M|_{\text{around } m} = 0$



$$M_d - F_2 \, l = 0$$

$$F_2 = \frac{M}{l} = \frac{d_{Mt} \cdot \dot{\varphi}}{l} \qquad (14)$$

(12) in (9)

$$-F_1 \sin\varphi + F_2 \cos\varphi = -m(\ddot{x}_M + l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2 \sin\varphi)$$

$$\ddot{x}_M M = F - d\dot{x}_M \underbrace{- F_1 \sin\varphi + F_2 \cos\varphi}$$

$$\ddot{x}_M M = F - d\dot{x}_M - m\ddot{x}_M - ml\ddot{\varphi}\cos\varphi + l\dot{\varphi}^2 \sin\varphi \, m$$

$$\boxed{\ddot{x}_M (M+m) + m(l\ddot{\varphi}\cos\varphi - l\dot{\varphi}^2 \sin\varphi) + d\dot{x}_M = F \qquad (15)}$$

(14) in (11)

$$\boxed{m(\ddot{x}_M \cos\varphi + l\ddot{\varphi}) = -\frac{d_{Mt}\,\dot{\varphi}}{l} + mg\sin\varphi \qquad (16)}$$

(15), (16) are the final motion eq. of the inverted pendulum on a cart.

- Mathmatical modelling using the energy method according to Lagrange

Lagrange equation (general form)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = F_i \qquad (17)$$

where D denotes the dissipative function with the damping caused by friction

$$D_t = \frac{1}{2}\, d\, \dot{x}_M^2 \qquad \text{translation} \qquad (18)$$

$$D_r = \frac{1}{2}\, d_{MF}\, \dot{\varphi}^2 \qquad \text{rotation} \qquad (19)$$

$$D = \Sigma D_j$$

and F_i : generalized forces

q_i : generalized coordinates

L : Lagrange function

$$L = E_{kin} - E_{pot} \qquad (20)$$

$$E_{kin} = \sum_j E_{kin_j} - \sum_j E_{pot_j}$$

- generalized coordinates $\quad q_1 = \varphi \quad , \quad q_2 = x_M \qquad (21)$

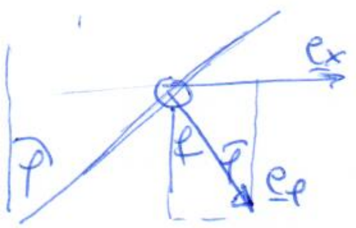
- generalized forces $\quad F_1 = 0 \quad , \quad F_2 = F \qquad (22)$

- Kinetic energies in the system

$$E_{kin} = \frac{1}{2} M \dot{x}_M^2 + \frac{1}{2} m \dot{x}_m^2$$

$$\dot{x}_M^2 = |\dot{\underline{x}}_M|^2 \quad , \quad \dot{x}_m^2 = |\dot{\underline{x}}_m|^2 \quad , \quad \dot{\underline{x}}_m = \dot{x}_M \underline{e}_x + l \dot{\varphi} \underline{e}_\varphi$$

$$|\dot{\underline{x}}_m|^2 = \dot{x}_M^2 + (l\dot{\varphi})^2 + 2 \dot{x}_M l \dot{\varphi} \cos\varphi$$



$$\underline{e}_x \cdot \underline{e}_\varphi = \cos\varphi$$

$$\leadsto E_{kin} = \frac{1}{2} M \dot{x}_M^2 + \frac{1}{2} m \left(\dot{x}_M^2 + (l\dot{\varphi})^2 + 2 \dot{x}_M l \dot{\varphi} \cos\varphi \right)$$

- Potential energy

$$E_{pot} = m g l \cos\varphi$$

- Lagrange function $\quad L = E_{kin} - E_{pot}$

$$L = \frac{1}{2} M \dot{x}_M^2 + \frac{1}{2} m\left(\dot{x}_M^2 + (l\dot{\varphi})^2 + 2\dot{x}_M l \dot{\varphi} \cos\varphi\right) - m g l \cos\varphi$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{2} m\left(2 l^2 \dot{\varphi} + 2 \dot{x}_M l \cos\varphi\right) = m l\left(l \dot{\varphi} + \dot{x}_M \cos\varphi\right)$$

$$\frac{\partial L}{\partial \varphi} = \frac{1}{2} m\left(-2 \dot{x}_M l \dot{\varphi} \sin\varphi\right) + m g l \sin\varphi$$

$$\frac{\partial L}{\partial \varphi} = m l \sin\varphi\left(g - \dot{x}_M \dot{\varphi}\right)$$

$$\frac{\partial L}{\partial \dot{x}_M} = \frac{1}{2} M \, 2 \dot{x}_M + \frac{1}{2} m \, 2 \dot{x}_M + \frac{1}{2} m \, 2 l \dot{\varphi} \cos\varphi$$

$$= (M + m) \dot{x}_M + m l \dot{\varphi} \cos\varphi$$

$$\frac{\partial L}{\partial x_M} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = m l\left(l \ddot{\varphi} + \ddot{x}_M \cos\varphi - \dot{x}_M \sin\varphi \cdot \dot{\varphi}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_M}\right) = (M+m) \ddot{x}_M + m l \ddot{\varphi} \cos\varphi - m l \dot{\varphi} \sin\varphi \cdot \dot{\varphi}$$

$$= (M+m) \ddot{x}_M + m l\left(\ddot{\varphi} \cos\varphi - \dot{\varphi}^2 \sin\varphi\right)$$

$$D = \frac{1}{2} d \dot{x}_H^2 + \frac{1}{2} d_{Hf} \dot{\varphi}^2$$

$$\frac{\partial D}{\partial \dot{\varphi}} = d_{Hf} \dot{\varphi}$$

$$\frac{\partial D}{\partial \dot{x}_H} = d \dot{x}_H$$

All previous terms result in using (17), (21), and (22)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} + \frac{\partial D}{\partial \dot{\varphi}} = 0$$

$$ml(l\ddot{\varphi} + \ddot{x}_H \cos\varphi - \dot{x}_H \sin\varphi \dot{\varphi}) - ml \sin\varphi (g - \dot{x}_H \dot{\varphi})$$

$$+ d_{Hf} \dot{\varphi} = 0$$

$$\rightsquigarrow ml(l\ddot{\varphi} + \ddot{x}_H \cos\varphi - \sin\varphi\, g) + d_{Hf} \dot{\varphi} = 0$$

$$\boxed{ml(\ddot{x}_H \cos\varphi + l\ddot{\varphi}) - mgl \sin\varphi + d_{Hf} \dot{\varphi} = 0} \quad (23)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_M}\right) - \frac{\partial L}{\partial x_M} + \frac{\partial D}{\partial \dot{x}_M} = F$$

$$\boxed{(M+m)\ddot{x}_M + ml(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) - 0 + d\,\dot{x}_M = F} \quad (24)$$

Note that (23) $\hat{=}$ (16)

(24) $\hat{=}$ (15)

$\rightsquigarrow$ both methods the Newton Euler balance equation (page 1-5) and the Lagrange energy methods give the same results!