Conical Singularities and Signature Change in the Kerr-(Anti)-de Sitter Spacetime

Physics 590 Undergraduate Thesis

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Abstract.

The current cosmological paradigm of a ΛCDM model suggests that the correct field equations describing the structure of spacetime are the modified Einstein field equations. Within this framework, the spacetime describing an isolated rotating black hole in a universe with a background energy density of Λ is the Kerr-(Anti)-de Sitter solution. In this report, I examine certain fundamental features of the spacetime, such as the possible existence of conical singularities on surfaces of constant r and t, and the existence of regions of opposite signature. The analysis of conical singularities is accomplished using two methods. First, I compare the Kerr-(Anti)-de Sitter metric to a metric for a 2-surface that is known to result in cone points. Second, I perform a calculation of the total curvature of the Kerr-(Anti)-de Sitter manifold. Using both of these methods, I found that the existence of cone points was contingent on whether or not one chooses to include a factor of $1 + \frac{\Lambda}{3}a^2$ in the metric. In particular, if this factor is included, there are no cone points on surfaces of constant r and t in Kerr-(Anti)-de Sitter. The examination of cone points indirectly lead to an analysis of the signature of the spacetime. It was found that for certain values of the cosmological constant and the black hole's spin, there exist regions of opposite signature. However, though these regions exist, they cannot be connected physically, i.e. by null or timelike geodesics, and so remain distinct.

Contents

1. Introduction

It is well known that the universe is thought to be described by the ΛCDM model, which consists of cold dark matter and dark energy in addition to the baryonic matter with which we are familiar [1]. Dark energy is commonly ascribed to the cosmological constant, Λ, that appears in Einstein's modified field equations. According to modern astrophysical observations, the cosmological constant accounts for approximately 73% of the universe's total energy distribution [2]. This fact makes it an important term to consider in any discussion of the large-scale structure of the universe.

The structure of space and time in the universe is described to a remarkable degree of accuracy by Einstein's general theory of relativity. Within this framework, the standard description of empty space, and that used in the special theory of relativity, is given by the Minkowski spacetime. In geometrized units ($\mathrm{G} = \mathrm{c} = 1$) and spherical polar coordinates, the so-called line element corresponding to this spacetime is given as

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2. \tag{1}$$

This spacetime geometry is the *massless vacuum* solution to Einstein's equations without Λ. However, in light of astrophysical observations providing strong evidence for the ΛCDM cosmological model, we are forced to consider the inclusion of the cosmological constant in the field equations. This necessarily leads to a redefinition of what is considered to be vacuum. The Minkowski spacetime no longer holds as the vacuum solution when Λ is included. Rather, the massless vacuum solution to Einstein's equations with Λ is given by the de Sitter spacetime [3]:

$$ds^2 = (1 - \frac{\Lambda}{3} r^2) dt^2 - \frac{dr^2}{1 - \frac{\Lambda}{3} r^2} - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2. \tag{2}$$

This geometry describes an empty universe with the addition of the cosmological constant. In modern cosmological models, it is considered to be the asymptotic end state of universes that began with a big bang and that do not recollapse.

Perhaps one of the most striking theoretical predictions of general relativity is the existence of black holes. Once thought to be likely unphysical, black holes are now considered to be pervasive throughout the universe. The most physically relevant description of a black hole in the general theory of relativity is given by the Kerr solution to the field equations. This solution describes an isolated rotating black hole in an otherwise empty universe [3]. Though it is not directly applicable to rotating black holes in the centres of galaxies, it provides a model for what might be expected to remain after all matter has been absorbed into the black hole. It can be verified that if one removes the mass and spin in the Kerr solution, this spacetime reduces to Minkowski, as expected. However, as with the Minkowski spacetime, this is a solution to Einstein's equations without Λ and consequently does not fit into the current cosmological paradigm. The corresponding spacetime for Einstein's equations with the cosmological constant is the Kerr-(Anti)-de Sitter spacetime [3], which describes a rotating black hole embedded in a universe with a background energy density of Λ. The denomination "anti", allows for the possibility that $\Lambda < 0$. In coordinates similar to those normally used in the original

Kerr spacetime, the geometry is as follows:

$$\begin{aligned} ds^2 = &- \frac{\rho^2}{\Delta_r}dr^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 - \frac{\Delta_\theta \sin^2(\theta)}{\rho^2}\left[a\frac{dt}{\Xi} - (r^2+a^2)\frac{d\phi}{\Xi}\right]^2 \\ &+ \frac{\Delta_r}{\rho^2}\left[\frac{dt}{\Xi} - a\sin^2(\theta)\frac{d\phi}{\Xi}\right]^2 \end{aligned} \tag{3}$$

where

$$\Delta_r = (r^2+a^2)\left(1 - \frac{\Lambda}{3}r^2\right) - 2mr \tag{4a}$$

$$\Delta_\theta = 1 + \frac{\Lambda}{3}a^2\cos^2(\theta) \tag{4b}$$

$$\rho^2 = r^2 + a^2\cos^2(\theta) \tag{4c}$$

$$\Xi = 1 + \frac{\Lambda}{3}a^2. \tag{4d}$$

Analogous to the original Kerr solution, this spacetime reduces to regular de Sitter space if $a = m = 0$ (where a is the spin and m is the mass). This is explored in more depth in Section 7. This spacetime is the focus of my analysis. Though it is such an important spacetime to study in the context of modern astrophysics, it is by no means perfectly understood. In particular, the spacetime might exhibit interesting topological features such as cone points. I begin my analysis by examining a claim made by Akcay and Matzner [4], which states that the term Ξ in equation (3) eliminates conical singularities in the geometry near $\theta = 0$ and $\theta = \pi$. The examination of this problem subsequently lead to an analysis of a more fundamental feature of the spacetime. This feature is that the Kerr-(Anti)-de Sitter spacetime exhibits regions with opposite signature, which is unheard of in physical solutions of general relativity. I introduce the concept of signature in depth in Section 5, before examining the signature of Kerr-(Anti)-de Sitter. However, before I tackle this analysis, I begin with a review of the central concepts in general relativity.

2. Concepts in General Relativity

In the general theory of relativity, gravitation is described by the *geometry* of a 4-dimensional spacetime or manifold. The geometry of a particular spacetime is characterized by the *metric tensor*, $g_{\alpha\beta}(x^\gamma)$, which is a function of the coordinates on the spacetime, x^γ. Given a general coordinate 4-vector, $dx^\alpha = (dx^1, dx^2, dx^3, dx^4)$, the metric tensor is related to the spacetime analogue of the line element in the following way:

$$ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta \tag{5}$$

where repeated indices denote an implicit sum, as per the Einstein summation convention. The line element ds^2 and the metric tensor provide equivalent representations of the spacetime. Within the framework of general relativity, particles being acted upon uniquely by gravity move on the manifold along curves known as *affinely parameterized geodesics*. These are curves $x^\alpha = x^\alpha(\lambda)$, where λ is an affine parameter. Such curves obey the geodesic equation

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{6}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ is a Christoffel symbol of the second kind and is built out of the metric tensor and its first partial derivatives. Further, if one defines the tangent 4-vector to be

$$u^{\alpha} = \dot{x}^{\alpha} = \frac{dx^{\alpha}}{d\lambda} \tag{7}$$

then the quantity $\mathcal{L}$ defined by the inner product

$$\mathcal{L} = \frac{1}{2}u^{\alpha}u_{\alpha} = \frac{1}{2}g_{\alpha\beta}u^{\alpha}u^{\beta} \tag{8}$$

is *constant* along curves that obey equation (6). In fact, the quantity $\mathcal{L}$ is the standard Lagrangian and it can be shown that the geodesic equation is entirely equivalent to the Euler-Lagrange equations of motion:

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^{\alpha}}\right) - \frac{\partial\mathcal{L}}{\partial x^{\alpha}} = 0. \tag{9}$$

The geometry of a given spacetime is theoretically obtained by means of *Einstein's field equations*. In geometrized units, the original equations (i.e. without Λ) are given as

$$G^{\alpha}_{\beta} = 8\pi T^{\alpha}_{\beta} \tag{10}$$

where G^{α}_{β} is the *Einstein tensor* and T^{α}_{β} is the *energy-momentum tensor*. The Einstein tensor is the only symmetric, divergenceless ($\nabla_{\beta}G^{\beta}_{\alpha} = 0$, where ∇_{β} is the covariant derivative) tensor constructed out of the metric tensor and its first two partial derivatives. The energy-momentum tensor describes the composition of the background that forms the spacetime. For instance, vacuum is described by $T^{\alpha}_{\beta} = 0$. In addition to this, all conservation laws of the background are obtained from this tensor via $\nabla_{\beta}T^{\beta}_{\alpha} = 0$. It follows that Einstein's equations and the divergence free character of G^{α}_{β} guarantee the conservation laws. Given a particular form of the energy-momentum tensor, one solves the field equations in order to obtain the corresponding metric tensor, which describes the geometry of the manifold, and consequently, all particle motion on the manifold.

The more relevant equations in contemporary analysis are not the original field equations but rather the modified Einstein field equations, which include the cosmological constant:

$$G^{\alpha}_{\beta} + \Lambda\delta^{\alpha}_{\beta} = 8\pi T^{\alpha}_{\beta}. \tag{11}$$

It is important to note that this represents a *different theory* than the original equations. One should not consider Λ to be a minute addition to the original theory. Rather it represents a distinctly different formulation of the theory of gravity. This is evident from the two massless vacuum solutions, the Minkowski and de Sitter spacetimes, given in equations (1) and (2), for which the large r behaviour of the spacetimes is distinctly different.

3. Cone Points

Now that I have reviewed some of the general concepts of the theory, I introduce the notion of cone points. Following the procedure undertaken by Pelavas et al., [5], I begin by noting that the line element for a 2-dimensional plane can be written in the following form:

$$ds^2 = d\theta^2 + \theta^2 d\phi^2 \tag{12}$$

where $0 \leq \phi \leq 2\pi$. Next I introduce a scaling factor α with a range of $0 \leq \alpha \leq 1$, such that

$$ds^2 = d\theta^2 + \alpha^2\theta^2 d\phi^2. \tag{13}$$

Rescaling the ϕ variable using $\tilde{\phi} = \alpha\phi$, I obtain the metric

$$ds^2 = d\theta^2 + \theta^2 d\tilde{\phi}^2 \tag{14}$$

with the range $0 \leq \tilde{\phi} \leq 2\pi\alpha$. If one chooses to write α in the following form,

$$\alpha = 1 - \frac{\delta}{2\pi} \tag{15}$$

then the range in $\tilde{\phi}$ can be expressed as $0 \leq \tilde{\phi} \leq 2\pi - \delta$. Thus the metric described by equation (14) is identical in form to that in equation (12) except that a portion of the manifold has been removed. Enforcing that the manifolds under consideration be continuous, I have the result that the geometry described by equation (14) is smooth except at a single point, designated the *cone point*. The manifold is differentiable everywhere but at this single point. In effect, equations (13) and (15) describe a cone of deficit angle δ. This is demonstrated in figure 1. It follows that any 2-surface that exhibits this geometry *locally* gives rise to conical singularities. In the next section, I apply this analysis to the Kerr-(Anti)-de Sitter spacetime.

4. Conical Singularities in Kerr-(Anti)-de Sitter

The Kerr-(Anti)-de Sitter metric is given by equation (3). Akcay and Matzner [4] claim that the factor Ξ ensures that the metric contains no conical singularities near $\theta = 0$ or $\theta = \pi$, as mentioned above. In order to analyze this, the natural first step is to re-write the line element such that the metric no longer contains Ξ:

$$\begin{aligned} ds^2 = &-\frac{\rho^2}{\Delta_r}dr^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 - \frac{\Delta_\theta \sin^2(\theta)}{\rho^2}\left[adt - (r^2+a^2)d\phi\right]^2 \\ &+ \frac{\Delta_r}{\rho^2}\left[dt - a\sin^2(\theta)d\phi\right]^2. \end{aligned} \tag{16}$$

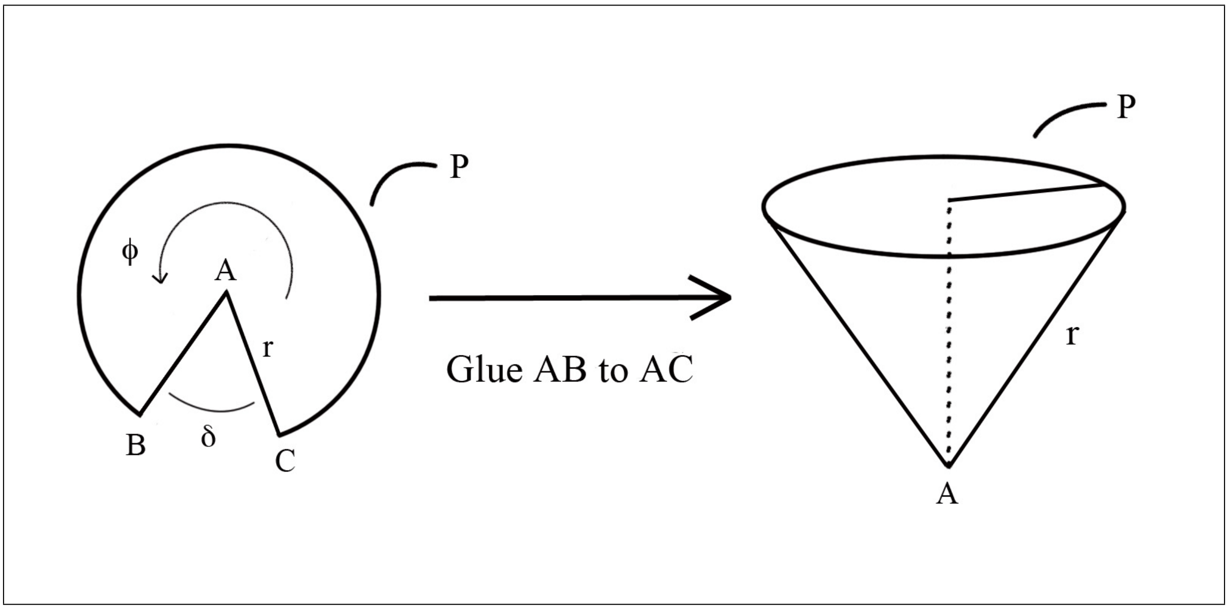

Figure 1: A schematic depicting the formation cone points. The conical structure is apparent if we join the segments AB and AC of the incomplete circle together. The point A becomes the apex of the cone and the arc length of the circle, P, becomes the directrix.

Next I expand the quadratic terms to obtain the elements of the metric tensor, $g_{\alpha\beta}$:

$$\begin{aligned} ds^2 = &- \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\Delta_\theta \sin^2(\theta)}{\rho^2} \left[a^2 dt^2 - 2a(r^2 + a^2) dt d\phi + (r^2 + a^2)^2 d\phi^2 \right] \\ &+ \frac{\Delta_r}{\rho^2} \left[dt^2 - 2a \sin^2(\theta) dt d\phi + a^2 \sin^4(\theta) d\phi^2 \right] \end{aligned} \tag{17}$$

$$\begin{aligned} = &- \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 + \left[\frac{\Delta_r a^2 \sin^4(\theta)}{\rho^2} - \frac{\Delta_\theta (r^2 + a^2)^2 \sin^2(\theta)}{\rho^2} \right] d\phi^2 \\ &+ \frac{2a \sin^2(\theta)}{\rho^2} \left[\Delta_\theta (r^2 + a^2) - \Delta_r \right] d\phi dt + \left[\frac{\Delta_r}{\rho^2} - \frac{\Delta_\theta a^2 \sin^2(\theta)}{\rho^2} \right] dt^2. \end{aligned} \tag{18}$$

I now consider 2-surfaces of constant r and t such that $dr = dt = 0$, and so the relevant components of the metric are $g_{\theta\theta}$ and $g_{\phi\phi}$. These can be read from equation (18) as

$$g_{\theta\theta} = -\frac{\rho^2}{\Delta_\theta} \quad , \quad g_{\phi\phi} = \frac{\Delta_r a^2 \sin^4(\theta)}{\rho^2} - \frac{\Delta_\theta (r^2 + a^2)^2 \sin^2(\theta)}{\rho^2}. \tag{19}$$

In this analysis, I consider the region around $\theta = 0$. I use small angle approximations such that $\cos(\theta) \approx 1$ and $\sin(\theta) \approx \theta$. I also note that $\sin^4(\theta) \ll \sin^2(\theta)$ and can be ignored if only terms to second order are considered in the approximation. Further, the terms ρ^2 and Δ_θ take on the following forms:

$$\rho^2 = r^2 + a^2 \quad , \quad \Delta_\theta = 1 + \frac{\Lambda}{3} a^2. \tag{20}$$

The metric of the 2-surface is then given as

$$ds_2^2 = g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 = -\frac{r^2+a^2}{1+\frac{\Lambda}{3}a^2}\left[d\theta^2 + (1+\frac{\Lambda}{3}a^2)^2\theta^2 d\phi^2\right]. \tag{21}$$

It should be immediately apparent to the reader that this result has the same form as the one that I presented in equation (13). Thus, we see that equation (21) appears to suggest that removing Ξ from the metric admits cone points on surfaces of constant r and t in the Kerr-(Anti)-de Sitter geometry. To verify Akcay and Matzner's claim [4], I reintroduce the factor of Ξ and follow the same procedure. This amounts to an additional factor of Ξ^2 in the denominator of the expression for $g_{\phi\phi}$ in equation (19). Writing out the corresponding metric of the 2-surface, I find:

$$ds_2^2 = -\frac{r^2+a^2}{1+\frac{\Lambda}{3}a^2}\left[d\theta^2 + \theta^2 d\phi^2\right]. \tag{22}$$

It does indeed appear to be the case that the factor of Ξ ensures that no cone points occur on surfaces of constant r and t.

At this point, it is important to note that these results are contingent on the range of ϕ. It was implicitly assumed that the range is given as $0 \leq \phi \leq 2\pi$ in both cases, i.e. with Ξ and without Ξ. If this is true, then the metric with Ξ is indeed free of cone points and the metric without Ξ allows for cone points. Thus the claim made by Akcay and Matzner holds. This does mean however that the metric with Ξ cannot be transformed into the metric without Ξ by a transformation of the ϕ coordinate. This is true because a transformation such as rescaling ϕ with Ξ must necessarily also rescale the range in ϕ. Thus, though the factor of Ξ may not appear explicitly in the metric in such a case, it acts implicitly on the geometry, and the spacetime remains free of cone points on surfaces of constant r and t. The range in ϕ could of course be something other than $0 \leq \phi \leq 2\pi$, in which case the situation is different. However there is no reason to expect this to be the case since ϕ represents the standard azimuthal angle and is quoted as having the range that I have used [3][6].

In order to verify this result, I perform a topological analysis of surfaces of constant r and t in Kerr-(Anti)-de Sitter.

5. Topological Analysis of Kerr-(Anti)-de Sitter

The existence of cone points can be verified via the topology of a manifold if one observes the failure of the standard Gauss-Bonnet theorem. The *total curvature* for a 2-surface Σ with metric g_{AB} is given by

$$T_\Sigma = -\frac{1}{2}\iint_\Sigma R\sqrt{g}dx^A dx^B \tag{23}$$

where R is the Ricci scalar and g is the determinant of g_{AB}. Further, the Gauss-Bonnet theorem states that

$$T_\Sigma = 2\pi\chi(\Sigma) \tag{24}$$

where $\chi(\Sigma) = 2 - 2n$ is the Euler characteristic of Σ and n is the number of 'holes' on the manifold. For example, a sphere has $n = 0$, $\chi = 2$, and $T_\Sigma = 4\pi$. The Gauss-Bonnet theorem holds if the manifold is smooth. As mentioned above, if the manifold contains a cone point, it will not be differentiable at that one point. This lack of differentiability results in the failure of the theorem. When evaluating the integral in equation (23), cone points lead to results that cannot be written in the form of equation (24). In particular, T_Σ will not give rise to 2π times an integer.

In general, it is an arduous task to evaluate the integral in equation (23) since the Ricci scalar is a complicated expression. However, for any 2-surface of revolution, the integral can be evaluated in a simplified manner, as follows. Consider the general metric for such a surface:

$$ds^2 = -A^2(\theta)d\theta^2 - B^2(\theta)d\phi^2. \tag{25}$$

The determinant of the metric and Ricci scalar are given as

$$\sqrt{g} = AB \quad , \quad R = \frac{2}{AB}\frac{d}{d\theta}\left(\frac{\frac{dB}{d\theta}}{A}\right). \tag{26}$$

Then, using $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$, equation (23) becomes

$$T_\Sigma = -\frac{1}{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{2}{AB}\frac{d}{d\theta}\left(\frac{\frac{dB}{d\theta}}{A}\right)ABd\theta d\phi = -2\pi\left[\frac{\frac{dB}{d\theta}}{A}\right]_0^\pi. \tag{27}$$

Now, I apply this method to surfaces of constant t and r on the Kerr-(Anti)-de Sitter spacetime. The corresponding metric is given by $ds_2^2 = g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2$. Comparing to equation (25), it is apparent that $A(\theta) = \sqrt{-g_{\theta\theta}}$ and $B(\theta) = \sqrt{-g_{\phi\phi}}$. The expressions for these elements of the metric tensor are given by

$$g_{\theta\theta} = -\frac{\rho^2}{\Delta_\theta} \quad , \quad g_{\phi\phi} = \frac{\Delta_r a^2\sin^4(\theta)}{\rho^2\Xi^2} - \frac{\Delta_\theta(r^2+a^2)^2\sin^2(\theta)}{\rho^2\Xi^2}. \tag{28}$$

Therefore,

$$A(\theta) = \sqrt{\frac{\rho^2}{\Delta_\theta}} \tag{29}$$

$$B(\theta) = \frac{(r^2+a^2)\sqrt{\Delta_\theta}\sin(\theta)}{\rho\Xi}\sqrt{1-\frac{\Delta_r a^2\sin^2(\theta)}{\Delta_\theta(r^2+a^2)^2}}. \tag{30}$$

The derivative of $B(\theta)$ was obtained using computational methods and evaluated at the appropriate limits. It was found that

$$B'(0) = \frac{\sqrt{\rho^2}}{\sqrt{\Xi}} \tag{31}$$

$$B'(\pi) = -\frac{\sqrt{\rho^2}}{\sqrt{\Xi}} \tag{32}$$

where the primed symbol represents differentiation with respect to θ. Therefore, noting that $A(0) = A(\pi)$,

$$T_\Sigma = -\frac{2\pi}{A(0)}\left[B'(\pi) - B'(0)\right] = -2\pi\sqrt{\frac{\Xi}{\rho^2}}\left(-2\frac{\sqrt{\rho^2}}{\sqrt{\Xi}}\right) = 4\pi. \tag{33}$$

I have obtained the result that, for the Kerr-(Anti)-de Sitter spacetime, surfaces of constant r and t exhibit a spherical topology. Further, the Gauss-Bonnet theorem holds in this analysis, and so there are no conical singularities on the spacetime described by equation (3).

In addition to this, I computed the total curvature for the metric without Ξ, given by equation (18). In effect, this corresponds to removing the factor of Ξ from equation (30). Using the same method as above, the total curvature was found to be

$$T_\Sigma = -\frac{2\pi}{A(0)}\left[B'(\pi) - B'(0)\right] = -2\pi\frac{\Xi}{\sqrt{\rho^2}}\left(-2\sqrt{\rho^2}\sqrt{\Xi}\right) = 4\pi\Xi. \tag{34}$$

This clearly does not satisfy the Gauss-Bonnet theorem, since Ξ is not, in general, an integer. Thus using the metric without Ξ results in the existence of cone points on surfaces of constant r and t. As expected, these results agree with what was demonstrated in Section 4, i.e. that Ξ does in fact prevent the occurrence of cone points on surfaces of constant r and t if the range in ϕ is $0 \leq \phi \leq 2\pi$.

As mentioned in the introduction, the analysis of cone points in the geometry of the Kerr-(Anti)-de Sitter spacetime lead us to consider an alternate feature of the spacetime. By considering the line element for a 2-surface of constant r and t for the metric without Ξ, as given by equation (21), one might naturally consider the question of what happens when $\Xi = 0$. In terms of cone points, this corresponds to a cone of deficit angle 2π, which is problematic since the manifold becomes undefined in this region. In addition to this, as can be seen from equation (3), the Kerr-(Anti)-de Sitter metric with Ξ is poorly behaved when $\Xi = 0$. Thus, regardless of whether the metric contains Ξ or not, something goes wrong when $\Xi = 0$. In trying to analyze and resolve this issue, we came across a claim made by Griffiths and Podolsky that $\Xi = 0$ is associated with a signature change of the metric [3], though they do not go into any more depth than this. Thus, in order to obtain a deeper understanding of the Kerr-(Anti)-de Sitter spacetime, we decided to move away from an analysis of potential cone points in the geometry and instead analyze this claim of signature change in the spacetime. In order to move on to the next section of the analysis, it is necessary to introduce another concept in general relativity: the *signature* of a metric.

6. Signature

The signature of a metric tensor can be obtained by considering a transformation from coordinates x^γ to coordinates $\overline{x}^\gamma$ such that, locally, the metric is diagonalized and the diagonal components are either +1 or -1. The signature is then defined as being the number of diagonal components that are equal to -1 subtracted from the number of diagonal components that are equal to +1. In general relativity, the spacetimes either have a signature of +2 or of -2. In some cases, the signature can be read off from the functional form of the metric tensor or line element. For example, consider the Minkowski spacetime, equation (1):

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2. \tag{35}$$

It can easily be seen that the metric, as written, has a signature of -2. It is actually a matter of convention whether the metric of a spacetime is written down with a signature of -2 or +2. The two situations are entirely equivalent. The Minkowski spacetime could just as easily be written with a signature of +2:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \tag{36}$$

It is also worthy to note that the diagonalization of a metric tensor in general relativity is equivalent to finding a transformation that reduces the spacetime to Minkowski at a particular point. This means that all spacetimes are locally flat.

7. Signature Change in Kerr-(Anti)-de Sitter

Having introduced the concept of signature, I can now examine the signature of the Kerr-(Anti)-de Sitter spacetime. For most of the spacetimes in general relativity, the signature of the metric tensor takes on a single value for all points. This can be verified directly for the Minkowski and de Sitter spacetimes given by equations (1) and (2). Interestingly, as mentioned by Griffiths and Podolsky [3], the signature of Kerr-(Anti)-de Sitter does not take on a single value over the entire spacetime, but in fact changes in certain regions. There is no other known physically interesting spacetime that exhibits this property. In order to examine this, I begin by determining the signature of Kerr-(Anti)-de Sitter. As can be seen from the metric in equation (3), the metric tensor for this geometry is not diagonal due to the quadratic terms that couple the variables ϕ and t. This makes it difficult to obtain the signature of the spacetime, since it cannot be read directly from the metric as in the case of Minkowski or de Sitter. In principle, the signature of Kerr-(Anti)-de Sitter could be obtained by finding a coordinate transformation that locally diagonalizes the metric, as discussed above. However, in general this is practically impossible as the relevant coordinate transformation cannot be obtained by any algorithmic procedure. Instead I aim to give a reasonable argument as to the signature of the spacetime.

Begin by considering the line element for Kerr-(Anti)-de Sitter, given in equation (3), with $\Delta_\theta > 0$ and $\Delta_r > 0$. As mentioned above, the signature is not immediately apparent. In order to obtain an idea of the signature, I set certain parameters in the metric, such as m, a and Λ, to zero.

(i) Setting $a = m = 0$, I obtains the following terms in the metric:

$$\Delta_r = r^2(1 - \frac{\Lambda}{3} r^2) \quad , \quad \Delta_\theta = 1 \quad , \quad \Xi = 1 \quad , \quad \rho^2 = r^2. \tag{37}$$

Taking a look at equation (3), one can see that this trivially reduces to the de Sitter spacetime with a signature of -2, as shown in equation (2). This makes sense since removing a and m is equivalent to removing the black hole from the spacetime. Further setting $\Lambda = 0$ leads to a trivial reduction from de Sitter to Minkowski, again with a signature of -2.

(ii) Setting $a = 0$, I obtain the following terms in the metric:

$$\Delta_r = r^2(1 - \frac{\Lambda}{3}r^2) - 2mr \quad , \quad \Delta_\theta = 1 \quad , \quad \Xi = 1 \quad , \quad \rho^2 = r^2. \tag{38}$$

Performing some simple algebra on equation (3), we find that the metric reduces to the following form:

$$ds^2 = (1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)dt^2 - \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2} - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2. \tag{39}$$

This is the well known Schwarzschild-de Sitter metric for a *non-rotating* black hole of mass m [3]. As before, the signature takes on a value of -2. As with (i), setting $\Lambda = 0$ reduces the metric to the standard Schwarzschild metric, again with signature -2.

One sees that by setting certain relevant parameters to be zero, the Kerr-(Anti)-de Sitter metric reduces to the metrics for well known spacetimes, all with signature -2. Hence it is reasonable to expect that building up the Kerr-(Anti)-de Sitter spacetime from these spacetimes by adding the relevant parameters will not modify the signature. The signature of the metric, as written in equation (3) is thus taken to be -2.

I am now in a position to examine the signature change of Kerr-(Anti)-de Sitter. Recall that when determining the signature to be -2, I had set $\Delta_r > 0$ and $\Delta_\theta > 0$. By examining equation (3), one can see that the terms Δ_r and Δ_θ are the only terms that are not positive definite. This means that, based on their definitions in equation (4), these terms can become negative by varying their respective variables, r and θ. Begin by considering $\Delta_r < 0$ such that $\Delta_r = -|\Delta_r|$. Then equation (3) becomes

$$\begin{aligned} ds^2 = &\frac{\rho^2}{|\Delta_r|} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\Delta_\theta \sin^2(\theta)}{\rho^2} \left[a\frac{dt}{\Xi} - (r^2 + a^2)\frac{d\phi}{\Xi} \right]^2 \\ &- \frac{|\Delta_r|}{\rho^2} \left[\frac{dt}{\Xi} - a \sin^2(\theta) \frac{d\phi}{\Xi} \right]^2 . \end{aligned} \tag{40}$$

One can see that the only difference between this expression and the original expression is that the first and fourth terms have switched signs. The metric has not undergone an overall sign change, and as such the signature remains -2. For the remainder of the analysis, I consider $\Delta_r > 0$. Physically, this means that I am considering the spacetime outside of the event horizon of the black hole [7]. Next, I consider $\Delta_\theta < 0$ such that $\Delta_\theta = -|\Delta_\theta|$:

$$\begin{aligned} ds^2 = &- \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{|\Delta_\theta|} d\theta^2 + \frac{|\Delta_\theta| \sin^2(\theta)}{\rho^2} \left[a\frac{dt}{\Xi} - (r^2 + a^2)\frac{d\phi}{\Xi} \right]^2 \\ &+ \frac{\Delta_r}{\rho^2} \left[\frac{dt}{\Xi} - a \sin^2(\theta) \frac{d\phi}{\Xi} \right]^2 . \end{aligned} \tag{41}$$

Comparing this expression to equation (3), one sees that the second and third terms have switched sign. However, in contrast to the case of $\Delta_r < 0$ in equation (40), the metric has also undergone an overall sign change in this expression. If the signature in equation (3) is -2 as I have argued that it is, then a reversal of the overall sign of the metric in this fashion means that the signature now takes on the opposite value, i.e. +2. I have found that the signature of the metric changes when Δ_θ changes sign, and so this signature change is only dependent on the variable θ. Recall the expression for Δ_θ from equation (4b):

$$\Delta_\theta = 1 + \frac{\Lambda}{3} a^2 \cos^2(\theta). \tag{42}$$

One should immediately notice from this expression that the condition for signature change cannot be met in universes for which $\Lambda > 0$. It is also apparent that this feature is not present in the Kerr solution to Einstein's original field equations, for which $\Lambda = 0$ and $\Delta_\theta = 1$. Further, the signature change is also dependent on the rotation of the black hole and is consequently not present in the Schwarzschild-de Sitter spacetime, which should be obvious from equation (39). Thus this feature is exclusive to the Kerr-(Anti)-de Sitter spacetime. However, even in Kerr-(Anti)-de Sitter, the signature change is not guaranteed. One requires that

$$\Delta_\theta = 1 + \frac{\Lambda}{3} a^2 \cos^2(\theta) < 0 \tag{43}$$

And so

$$\frac{3}{\Lambda a^2} < -\cos^2(\theta). \tag{44}$$

Due to the form of Δ_θ, for a region of opposite signature to exist, this condition must be met at the extremal values of $\theta = 0$ and $\theta = \pi$, where Δ_θ takes on its lowest values. Therefore,

$$\Lambda a^2 < -3. \tag{45}$$

This gives a relationship between the cosmological constant and the spin of the black hole. This must be satisfied in order to have a region of opposite signature in the Kerr-(Anti)-de Sitter spacetime. The cosmological constant is set by the background cosmology of the universe, and so within a universe with a particular value of Λ, there might exist Kerr-(Anti)-de Sitter black holes with spin a such that condition (45) is met. Further, the 'threshold' condition for which there is no signature change is given by

$$\Lambda a^2 = -3. \tag{46}$$

Coincidentally, this is also the value for which $\Xi = 0$. As mentioned at the end of Section 5, the metric is not well defined for a black hole satisfying this condition. In addition to this, it can be noted that Ξ is related to Δ_θ in the following way:

$$\Xi = \Delta_\theta\big|_{\theta=0,\pi}\,. \tag{47}$$

Therefore condition (45) can equivalently be written as $\Xi < 0$, and so the value of Ξ designates whether or not there exists a region of opposite signature in the spacetime. This is nicely depicted in figure 2. Further, from the figure it is apparent that for any value of Λa^2, there must always exist a region with the original signature, i.e. there always exist values of θ for which $\Delta_\theta > 0$. In particular, this is always true at $\theta = \frac{\pi}{2}$, for which $\Delta_\theta = 1$.

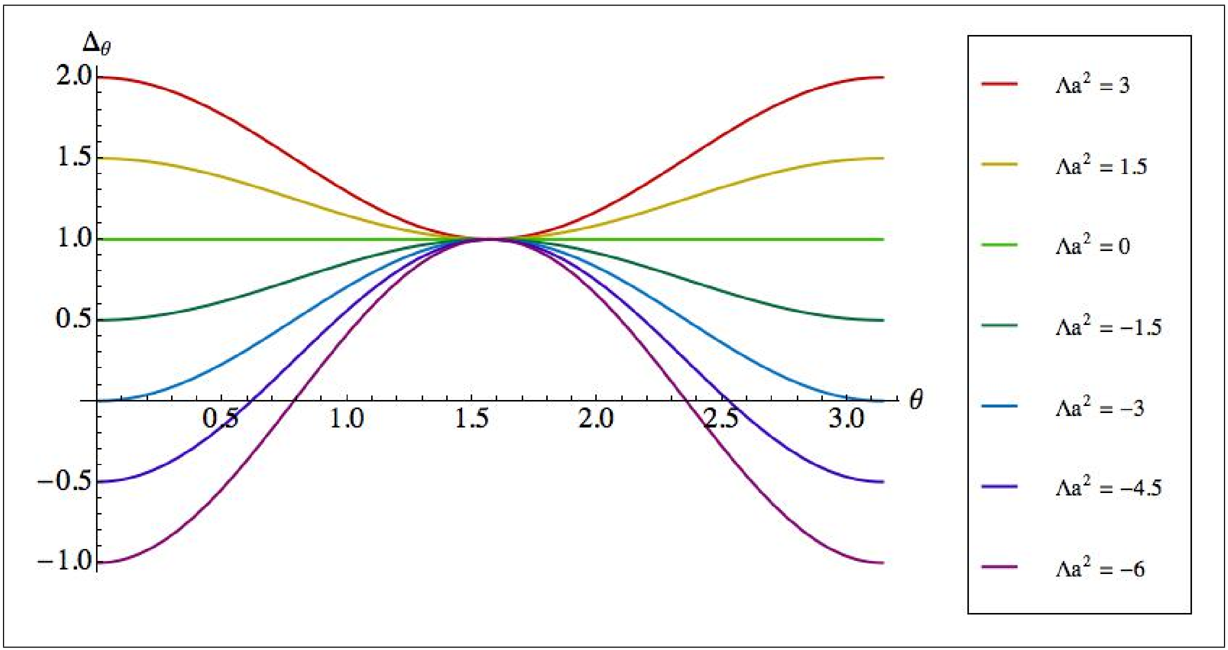

Figure 2: Plots of Δ_θ over the range $0 \leq \theta \leq \pi$ for various couplings between a and Λ.

8. Null θ-Geodesic Analysis

The fact that the Kerr-(Anti)-de Sitter spacetime carries a different signature in different regions is an intriguing and curious property. As mentioned above, no other physically interesting spacetime exhibits this property. Spacetimes in general relativity are usually written down using a single signature convention for the entire manifold. As a result of this, it is unclear what the physical meaning of such a signature change might be. Furthermore, there is little to no research that has been done examining this problem and so one can only speculate at this stage. In order to circumvent the problem, I propose to demonstrate that the regions of opposite signature are entirely disconnected. To do this, one must examine the geodesics of the Kerr-(Anti)-de Sitter spacetime. Recall from Section 2 that particles moving under the influence of a gravitational field travel along geodesics in the corresponding spacetime. The equations that govern this motion are the geodesic equations. For Kerr-(Anti)-de Sitter, these equations are given as [8][9][10]:

$$\rho^4 \dot{r}^2 = \Xi^2 (E\bar{m}^2 \mathcal{P})^2 - \Delta_r (2\mathcal{L} r^2 + K) \tag{48}$$

$$\rho^4 \dot{\theta}^2 = \Delta_\theta (K - 2a^2 \mathcal{L} \cos^2(\theta)) - \frac{\Xi^2 (E\bar{m}\mathcal{T})^2}{\sin^2(\theta)} \tag{49}$$

$$\frac{\rho^2}{\Xi^2}\dot{\phi} = \frac{a}{\Delta_r} E\bar{m}^2 \mathcal{P} - \frac{E\bar{m}}{\Delta_\theta \sin^2\theta}\mathcal{T} \tag{50}$$

$$\frac{\rho^2}{\Xi^2}\dot{t} = \frac{r^2 + a^2}{\Delta_r} E\bar{m}^2 \mathcal{P} - \frac{aE\bar{m}}{\Delta_\theta}\mathcal{T}. \tag{51}$$

Here ρ, Δ_r, Δ_θ, Ξ are as defined previously. $\mathcal{L}$ is the Lagrangian, as defined by equation (8) in Section 2. The term $\bar{m}$ is related to the black hole mass via $\bar{m} = 2m$. The term E is a constant of motion representing the energy per unit mass of the particle. K is Carter's constant, introduced by Carter in 1968 when solving the geodesic equations [6]. The dot $(\dot{\ })$ on the four variables represents differentiation with respect to an affine parameter, τ, which, in the timelike case, can be taken to be the proper time. Finally, the expressions for $\mathcal{P}$ and $\mathcal{T}$ are as follows:

$$\mathcal{P} = \frac{r^2 + a^2}{\bar{m}^2} - \frac{aL_z}{E\bar{m}^2} \tag{52}$$

$$\mathcal{T} = \frac{a\sin^2(\theta)}{\bar{m}} - \frac{L_z}{E\bar{m}} \tag{53}$$

where L_z is the angular momentum per unit mass of the particle in the z direction, i.e. the generalized momentum associated with the variable ϕ. Because the regions of different signature in Kerr-(Anti)-de Sitter are only characterized by their range in θ, they can theoretically only be connected by a particle's motion associated with the θ variable. Thus I only need to consider the geodesic equation for θ. However, one can see from the geodesic equations that equations (48) and (49) are coupled due to ρ^2. Following the route suggested by Hackmann et al. [8], this problem can be resolved by reparameterizing the geodesic affine parameter and writing the equations in terms of a new parameter, λ, such that

$$\frac{d\tau}{d\lambda} = \rho^2. \tag{54}$$

Therefore, one may write

$$\dot{\theta} = \frac{d\theta}{d\tau} = \frac{d\theta}{d\lambda}\frac{d\lambda}{d\tau} = \frac{1}{\rho^2}\frac{d\theta}{d\lambda}. \tag{55}$$

From equation (49),

$$\rho^4\dot{\theta}^2 = \left(\frac{d\theta}{d\lambda}\right)^2 = \Delta_\theta(K - 2a^2\mathcal{L}\cos^2(\theta)) - \frac{\Xi^2(E\bar{m}\mathcal{T})^2}{\sin^2(\theta)}. \tag{56}$$

So,

$$\frac{1}{E^2\bar{m}^2}\left(\frac{d\theta}{d\lambda}\right)^2 = \frac{\Delta_\theta}{E^2\bar{m}^2}(K - 2a^2\mathcal{L}\cos^2(\theta)) - \frac{\Xi^2\mathcal{T}^2}{\sin^2(\theta)}. \tag{57}$$

Next, I rescale a number of parameters in the following way:

$$\bar{a} = \frac{a}{\bar{m}} \quad , \quad \bar{\Lambda} = \Lambda\bar{m}^2 \quad , \quad \bar{L_z} = \frac{L_z}{\bar{m}} \quad , \quad \bar{\mathcal{L}} = \frac{\mathcal{L}}{E^2} \quad , \quad \mathcal{K} = \frac{K}{E^2\bar{m}^2}. \tag{58}$$

I also rescale the parameter λ:

$$\gamma = E\bar{m}\lambda. \tag{59}$$

Using these definitions, the terms Δ_θ, Ξ, and $\mathcal{T}$ are now written as

$$\Delta_\theta = 1 + \frac{\bar{\Lambda}}{3}\bar{a}^2\cos^2(\theta) \tag{60}$$

$$\Xi = 1 + \frac{\bar{\Lambda}}{3}\bar{a}^2 \tag{61}$$

$$\mathcal{T} = \bar{a}\sin^2(\theta) - \frac{\bar{L}_z}{E}. \tag{62}$$

Thus the final decoupled geodesic equation governing the variation in θ is

$$\left(\frac{d\theta}{d\gamma}\right)^2 = \Delta_\theta(\mathcal{K} - 2\bar{a}^2\bar{\mathcal{L}}\cos^2(\theta)) - \frac{\Xi^2\mathcal{T}^2}{\sin^2(\theta)}. \tag{63}$$

Using this equation, one can demonstrate that regions in the spacetime with opposite signature cannot be connected physically. In order to do this, I consider null geodesics. In general relativity, null geodesics are the paths taken by photons in a gravitational field. If null geodesics cannot connect across the regions of different signature, then it is clear that timelike geodesics, which are the paths taken by massive matter, cannot either. Thus it is sufficient to demonstrate that the regions cannot be connected by null geodesics. For null geodesics, the Lagrangian takes on the value of $\bar{\mathcal{L}} = 0$. The θ-motion for null geodesics is then governed by

$$\left(\frac{d\theta}{d\gamma}\right)^2 = \Delta_\theta\mathcal{K} - \frac{\Xi^2\mathcal{T}^2}{\sin^2(\theta)}. \tag{64}$$

Furthermore, due to the positive definite nature of the left-hand side, one requires that

$$\Delta_\theta\mathcal{K} \geq \frac{\Xi^2\mathcal{T}^2}{\sin^2(\theta)} \geq 0. \tag{65}$$

The right-hand inequality holds due to the positive definite character of the middle term. I begin by considering the case that $\Delta_\theta\mathcal{K} = 0$. This is only possible if the following is true:

$$\frac{\Xi^2\mathcal{T}^2}{\sin^2(\theta)} = 0. \tag{66}$$

This condition can be satisfied either if $\Xi = 0$ or if $\mathcal{T} = 0$, so both cases must be examined.

(i) $\underline{\Xi = 0}$**:**

Recalling the expression for Ξ from equation (61), this condition can be written as

$$\bar{\Lambda}\bar{a}^2 = -3. \tag{67}$$

From Section 7, we know that this is the 'threshold' condition for which there is no signature change in the spacetime. This was made evident in figure 2. The important point here is that this condition reflects a very specific black hole in a given universe. For a given Λ, set by the cosmology, this condition is only met for a black hole with the appropriate spin. Furthermore, setting $\Xi = 0$ enforces that Δ_θ becomes

$$\Delta_\theta = 1 + \frac{\bar{\Lambda}}{3}\bar{a}^2\cos^2(\theta) = 1 - \cos^2(\theta) = \sin^2(\theta). \tag{68}$$

Thus, for this case, $\Delta_\theta \mathcal{K} = 0$ is only satisfied when $\theta = 0$ or $\theta = \pi$. Due to the fact that these values are the extremal values of θ and that this is only valid for a very specific black hole, it is not generally true that $\Delta_\theta \mathcal{K} = 0$.

(ii) $\underline{\mathcal{T} = 0}$**:**

Recalling the expression for $\mathcal{T}$, given by equation (62), this condition can be written as

$$\sin^2(\theta) = \frac{\bar{L_z}}{\bar{a}E}. \tag{69}$$

Here $\bar{a}$ is a property of the black hole, but $\bar{L_z}$ and E are properties of the null orbits. This condition actually fixes the allowed values of θ, which I will label θ_r. To be clear, this means that for a given orbit with values $\bar{L_z}$ and E, $\mathcal{T} = 0$ only at the values $\theta = \theta_r$. However, looking at equation (65), one sees that this also fixes the value of Δ_θ, i.e.

$$\Delta_\theta|_{\theta=\theta_r} = 1 + \frac{\bar{\Lambda}}{3}\bar{a}^2\cos^2(\theta_r). \tag{70}$$

Therefore, in order to have $\Delta_\theta \mathcal{K} = 0$, one requires that $\Delta_\theta|_{\theta=\theta_r} = 0$. This requires that

$$\cos^2(\theta_r) = -\frac{3}{\bar{\Lambda}\bar{a}^2}. \tag{71}$$

This means that θ_r must simultaneously solve equations (69) and (71). Notice that equation (71) can only be satisfied if $\Lambda < 0$. Using the fact that $\cos^2(\theta)$ and $\sin^2(\theta)$ sum to unity, one can write

$$\frac{\bar{L_z}}{\bar{a}E} - \frac{3}{\bar{\Lambda}\bar{a}^2} = 1, \tag{72}$$

which can be rearranged to obtain

$$E = \frac{\bar{\Lambda}\bar{a}}{\bar{\Lambda}\bar{a}^2 + 3}\bar{L_z}. \tag{73}$$

For a given black hole with spin a and mass m in a given universe with cosmological constant Λ, this provides a relationship between the energy and angular momentum of a null orbit, such that θ_r simultaneously solves equations (69) and (71). Thus, for $\mathcal{T} = 0$, $\Delta_\theta \mathcal{K} = 0$ only for the specific values of $\theta = \theta_r$ on specific orbits satisfying equation (73). Needless to say, $\Delta_\theta \mathcal{K} = 0$ is thus not generally true.

Following the examination of these two cases, one sees that in general $\Delta_\theta \mathcal{K} > 0$. Now, if $\mathcal{K} > 0$, one has $\Delta_\theta > 0$, which corresponds to a region of -2 signature in the convention I used. Otherwise $\mathcal{K} < 0$ and there is a region of signature +2 with $\Delta_\theta < 0$. This means that for a null geodesic to connect regions of opposite signature, Carter's constant must change its sign. However, Carter's constant is *fixed* along a geodesic and so cannot undergo a sign change [6]. It follows that the regions of opposite signature in Kerr-(Anti)-de Sitter cannot be connected by a null geodesic. As mentioned above, this implies that they can neither be connected by timelike geodesics. It is possible that the regions may be connected by spacelike paths, but this is unclear. Regardless of this, spacelike geodesics are physically irrelevant due to the fact that spacelike particles, which move faster than the speed of light, have not been observed in nature.

9. Conclusion

In the above analysis, I began by examining the claim suggested by Akcay and Matzner [4] that the factor of Ξ in the Kerr-(Anti)-de Sitter metric prevents the occurrence of conical singularities at the poles on surfaces of constant r and t. I performed this analysis using two different methods: one involving an examination of the metric around the poles and the second involving a calculation of the total curvature. These methods were performed first using the Kerr-(Anti)-de Sitter metric with Ξ, given by equation (3), and subsequently using the metric without Ξ. Indeed, using both methods, I found that there are no cone points on surfaces of constant r and t in the Kerr-(Anti)-de Sitter spacetime when Ξ is included, provided that the range in the variable ϕ is given by $0 \leq \phi \leq 2\pi$. When the factor of Ξ is removed, cone points *do* occur. However these two cases constitute *distinct* spacetimes. One cannot be transformed into the other by means of a simple coordinate transformation, such as a rescaling of ϕ. It should be noted that this result does not suggest that there are no conical singularities in the Kerr-(Anti)-de Sitter spacetime as a whole. For instance, it can be shown that there exist cone points on the ergosphere (the surface below which frame dragging must occur) in the regular ($\Lambda = 0$) Kerr spacetime [5]. Thus it is likely that cone points exist on the ergosphere in Kerr-(Anti)-de Sitter as well. An examination of this topic would require a much more detailed analysis of the spacetime.

The analysis of conical singularities on the spacetime lead to a claim, made by Griffiths and Podolsky [3], that suggested that the condition $\Xi = 0$ is associated with a change in the signature of the metric. It was found that this is in fact true, and that $\Xi = 0$ separates spacetimes that exhibit signature change from those that don't. In particular, for spacetimes such that $\Xi < 0$, there exist regions in the spacetimes where the signature is inverted. The regions occur when $\Delta_\theta < 0$. However, though these regions of opposite signature exist, their physical meaning is unclear. In particular, there is no existing research about what might happen if a particle were to move from one region to the other. In order to circumvent this problem, I demonstrated that the two regions cannot be connected physically. This was accomplished by determining that a null geodesic originating in one region cannot move into the region of opposite signature. To do so would violate the geodesic equations. Furthermore, if null geodesics cannot connect the two regions, one can reasonably say that timelike paths cannot either. A proof of complete disconnection between the two regions would require that spacelike paths can neither connect the regions. However this is a more involved procedure and it is unclear whether it can be accomplished. An examination of the spacelike case would certainly have to be included in future work. As a final remark, it is interesting to consider the potential implications of these regions of distinct signature. Recall that, as mentioned in Section 6, there is no preference in general relativity as to whether a spacetime is written down with a signature of +2 or -2, i.e. either convention is equally valid. This still applies in Kerr-(Anti)-de Sitter, and so both the region with signature +2 and that with signature -2 are physical. The fact that these regions are entirely disconnected and cannot be bridged by any particle paths means that an observer in one of the regions would obtain no information from an observer in the other region. In effect, it is as though two distinct spacetimes are contained within the Kerr-(Anti)-de Sitter metric. There is no other physical spacetime that exhibits such an interesting property. As mentioned above, the Kerr-(Anti)-de Sitter spacetime for a rotating black hole is a spacetime of fundamental importance to astrophysics. And yet the work done here demonstrates that there is still much more to be discovered about this geometry. This is a testament to the complexity and intricacy of the general theory of relativity and it demonstrates that it is a subject well worth exploring.

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