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**Bachelor Thesis**  
Lexicographic Fréchet Matchings with  
Degenerate Inputs

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Berlin, July 10, 2018

**Abstract**

Ok.

### **Statutory Declaration**

I declare that I have developed and written the enclosed Bachelors Thesis completely by myself, and have not used sources or means without declaration in the text. Any thoughts from others or literal quotations are clearly marked. The Bachelors Thesis was not used in the same or in a similar version to achieve an academic grading or is being published elsewhere.

Anton Begehr  
Berlin, July 10, 2018

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# 1 Introduction

## 1.1 Motivation

!!!TODO!!! compare two paths

## 1.2 Applications

!!!TODO!!! This is a short overview of real world applications of the Fréchet Distance.

### 1.2.1 Molecule Chains (Proteins)

### 1.2.2 Handwriting Recognition

### 1.2.3 Comparing Routes

### 1.2.4 Computer Vision

### 1.2.5 Wifi access point placing

## 1.3 Problem Metaphor (Dog Walking)

The Fréchet distance is often visually explained using the "dog walking"-metaphor, which is described in the following paragraphs.

Imagine two curves:

- Curve  $C_P$  tracing the walking curve of a man
- Curve  $C_Q$  tracing the walking curve of the man's dog

The Fréchet Distance then is the shortest possible length of a leash, the man would need to keep ahold of his dog. Both the man and his dog have to traverse their entire paths and neither are allowed to walk backwards.

Both the man and his dog can independently vary their respective speeds. With the conditions described above, at every point in time, either both the man and his dog are walking forwards along their respective paths or one is idle while the other is walking along his path.

To make calculating the Fréchet distance more feasible by simplification, the arbitrary curves  $C_P$ , on which the man is walking, and  $C_Q$ , on which his dog is walking, are approximated each by a connected chain of line segments, a polygonal chain. The arbitrary curve  $C_P$  is approximated by the polygonal chain  $P$ . The arbitrary curve  $C_Q$  is approximated by the polygonal chain  $Q$ .

## 2 Preliminaries

### 2.1 Polygonal Chains

Given are two polygonal chains  $P$  and  $Q$ . A polygonal chain is defined by a list of points, sorted by the order of the chain, ascending from beginning to end. The first point of the list is the start of the polygonal curve and the last point of the list is the end of the polygonal curve. This order implies the direction of the directional polygonal curve.

$P$  and  $Q$  are polygonal chains consisting of  $n$  and  $m$  points respectively.

$$P_{points} = [P_1, P_2, \dots, P_n]$$

$$Q_{points} = [Q_1, Q_2, \dots, Q_m]$$

Points of  $P$  and  $Q$  are expressed as vectors:

$$P_i = \begin{pmatrix} x_{P_i} \\ y_{P_i} \end{pmatrix}, Q_j = \begin{pmatrix} x_{Q_j} \\ y_{Q_j} \end{pmatrix}$$

For simplification we assume all points  $P_i$  of  $P$  and  $Q_j$  of  $Q$  lie in the  $xy$ -plane, but the results generalize easily to arbitrary dimensions[2].

### 2.2 Parametrisation

We will use an arc-length parametrisation to define the polygonal curves  $P$  and  $Q$  going forward. We are using the arc-length parametrisation, instead of a unit-length parameter, to achieve visualisations with proportional scaled cells in the further course of this examination.

First we calculate the euclidian distances between the points of  $P$  and  $Q$  in the order they appear:

$$d_{P_i} = \|P_{i+1} - P_i\|, 1 \leq i \leq n - 1$$

$$d_{Q_j} = \|Q_{j+1} - Q_j\|, 1 \leq j \leq m - 1$$

We also calculate the distances  $L_{P_i}$  and  $L_{Q_j}$ , that define the distance from beginning of  $P$  and  $Q$  to the point  $P_i$  and  $Q_j$  on the  $P$  and  $Q$  respectively:

$$L_{P_i} = \sum_{k=1}^i d_{P_k}, 1 \leq i \leq n - 1$$

$$L_{Q_j} = \sum_{k=1}^j d_{Q_k}, 1 \leq j \leq m - 1$$

The total lengths of  $P$  and  $Q$  are then defined as  $L_P = L_{P_{n-1}}$  and  $L_Q = L_{Q_{m-1}}$ .

We apply an arc-length parametrisation on  $P$  and  $Q$  using parameters  $x \in [0, L_P]$  and  $y \in [0, L_Q]$  respectively. These parametrisation can be expressed as follows:

$$P(x) = \begin{cases} P_1 + \frac{x}{d_{P1}}(P_2 - P_1) & 0 \leq x \leq L_{P1} \\ P_2 + \frac{x-L_{P1}}{d_{P2}}(P_3 - P_2) & L_{P1} < x \leq L_{P2} \\ \dots & \\ P_{n-2} + \frac{x-L_{P_{n-2}}}{d_{P_{n-1}}}(P_{n-1} - P_{n-2}) & L_{P_{n-2}} < x \leq L_{P_{n-1}} \end{cases}$$

$$Q(y) = \begin{cases} Q_1 + \frac{y}{d_{Q1}}(Q_2 - Q_1) & 0 \leq y \leq L_{Q1} \\ Q_2 + \frac{y-L_{Q1}}{d_{Q2}}(Q_3 - Q_2) & L_{Q1} < y \leq L_{Q2} \\ \dots & \\ Q_{m-2} + \frac{y-L_{Q_{m-2}}}{d_{Q_{m-1}}}(Q_{m-1} - Q_{m-2}) & L_{Q_{m-2}} < y \leq L_{Q_{m-1}} \end{cases}$$

## 2.3 Height Function

To calculate all possible distances between points on  $P$  and  $Q$ , we will define the height function  $\delta$  that maps each point pair  $R$  consisting of one point on  $P$  and one point on  $Q$  to the euclidian distance between the points.

$$R = [0, L_P] \times [0, L_Q]$$

$$(x, y) \in R$$

$$\delta(x, y) := \|P(x) - Q(y)\|$$

## 3 Classical Fréchet Distance

This section introduces the Fréchet distance and gives an overview of the main findings in the 1995 paper “Computing the Fréchet distance between two polygonal curves” by Alt and Godau[1], which later will be essential for computing the lexicographic Fréchet distance.

### 3.1 Traversal of $P$ and $Q$

Let us revisit the "dog walking"-metaphor (Section 1.3) of the man and his dog walking along, or in other words traversing, their paths. The Fréchet distance is a dissimilarity measure that considers the traversal of both polygonal chains  $P$  and  $Q$  with following conditions:

- (A) Both  $P$  and  $Q$  are traversed completely.
- (B) Both  $P$  and  $Q$  are traversed monotonically ascending in the direction of  $P$  and  $Q$  respectively. Meaning a traverser on either  $P$  or  $Q$  can stand idle and go forwards, but may never go backwards.

We traverse both  $P$  and  $Q$  together. This means that at every point in time one of the following cases holds true:

- Both the traverser on  $P$  and the traverser on  $Q$  are moving.
- The traverser on  $P$  is moving and the traverser on  $Q$  is standing idle.
- The traverser on  $Q$  is moving and the traverser on  $P$  is standing idle.
- Neither the traverser on  $P$ , nor the traverser on  $Q$  is moving.

The last case is identical to the end of both traversals. Both traverses have reached the end of their polygonal chains.

We define a traversal of both  $P$  and  $Q$  together as a joint parametrisation  $(\alpha(t), \beta(t))$ . This joint parametrisation represents a continuous curve in the parameter area  $R$ . The curve represented by  $(\alpha(t), \beta(t))$  is a monotonic curve, because of above condition (B) which states that both  $P$  and  $Q$  are traversed only forwards and never backwards.

Revisiting the "dog walking"-metaphor from Section 1.3, the Fréchet distance is the length of the shortest leash, with which it is possible for the man and his dog to traverse their paths completely by only travelling forward. In other words, the Fréchet distance is defined as the largest distance between the traverser on path  $P$  and the traverser on path  $Q$  of the traversal with the smallest maximum distance between the respective traversers.

Using the joint parametrisation  $(\alpha(t), \beta(t))$  as representative for a traversal, next the distance function  $f(t)$  for the joint parametrisation needs to be defined to determine the Fréchet distance. The height function defined in Section 2.3 will be used to define the distance function  $f(t)$  for a joint parametrisation  $(\alpha(t), \beta(t))$ :

$$f(t) = \delta(\alpha(t), \beta(t)) := \|P(\alpha(t)) - Q(\beta(t))\|$$



Now, the Fréchet distance of  $P$  and  $Q$  is the largest value of  $f(t)$  for the joint parameterisations  $(\alpha(t), \beta(t))$  with the smallest maximum of all possible joint parameterisations.

### 3.2 Free-Space Diagram

The free-space diagram  $F_\epsilon$  is the set of all points of  $R$  that lie below the given height  $\epsilon$ :

$$(x, y) \in R$$

$$F_\epsilon = \{(x, y) \mid \|P(x) - Q(y)\| \leq \epsilon\}$$

The fundamental insight of Alt and Godau[1] is that the Fréchet distance of  $P$  and  $Q$  is the smallest  $\epsilon$  for which a monotonic path exists from  $(0, 0)$  to  $(L_P, L_Q)$  in  $F_\epsilon$ . [2]

### 3.3 Decision Problem

The decision problem asks if a height function  $\delta$  can be traversed monotonically while not exceeding the height of a given  $\epsilon$ . In their paper, Alt and Godau[1] define the decision algorithm, which solves the decision problem by using the free-space diagram  $F_\epsilon$ .

In the following example we will see three free-space diagrams  $F_\epsilon$  for the same height function with different  $\epsilon$ .  $\epsilon_F$  denotes the Fréchet distance.

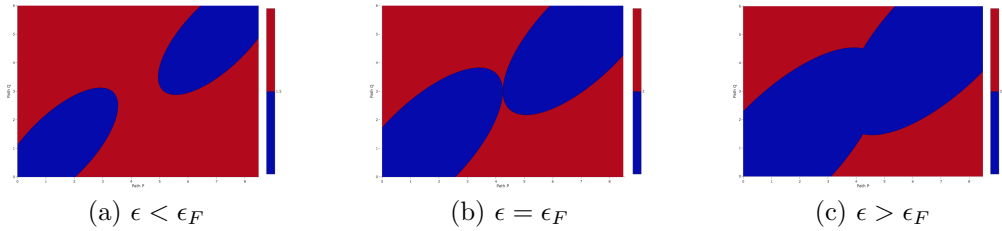


Figure 1: Three free-space diagrams with same height function  $\delta^1$

The following cases are depicted in Figure 1 above:

- (a)  $\epsilon$  is smaller than the Fréchet distance  $\epsilon_F$ . The decision algorithm returns false.

<sup>1</sup>View the example here: [https://abegehr.github.io/frechet/?p=\(1\\_2\)\(4\\_5\)\(7\\_2\)&q=\(1\\_3\)\(7\\_3\)](https://abegehr.github.io/frechet/?p=(1_2)(4_5)(7_2)&q=(1_3)(7_3))

- (b)  $\epsilon$  is equal to the Fréchet distance  $\epsilon_F$ . The decision algorithm returns true.
- (c)  $\epsilon$  is larger than the Fréchet distance  $\epsilon_F$ . The decision algorithm returns true.

### 3.4 Classical Critical Events

Alt and Godau also identified, that the critical height  $\epsilon_F$ , where a critical pathway in the free-space diagram closes, occurs in one of three cases. These are the three types of classical critical events:

- (a)  $\epsilon_F$  at  $(0,0) \in F_\epsilon$  or  $(L_P, L_Q) \in F_\epsilon$
- (b) A new passage with height  $\epsilon_F$  opens at border between two cells.
- (c) A new horizontal or vertical passage with height  $\epsilon_F$  opens on borders with cells in between.

In Figure 2 example free-space diagrams for each type of classical critical event are shown.

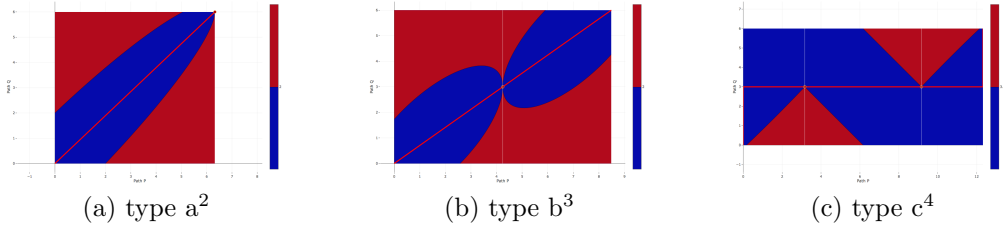


Figure 2: Examples for the three types of classical critical events

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<sup>2</sup>Classical critical event of type a at height  $\epsilon_F$  is at the upper-right corner  $(L_P, L_Q)$ . View the example here: [https://abegehr.github.io/frechet/?p=\(1\\_3\)\(7\\_5\)&q=\(1\\_3\)\(7\\_3\)](https://abegehr.github.io/frechet/?p=(1_3)(7_5)&q=(1_3)(7_3))

<sup>3</sup>Classical critical event of type b at height  $\epsilon_F$  is at vertical border connecting the left to the right cell. View the example here: [https://abegehr.github.io/frechet/?p=\(1\\_2\)\(4\\_5\)\(7\\_2\)&q=\(1\\_3\)\(7\\_3\)](https://abegehr.github.io/frechet/?p=(1_2)(4_5)(7_2)&q=(1_3)(7_3))

<sup>4</sup>Classical critical event of type c at height  $\epsilon_F$  is at vertical border connecting the left to the right cell over the middle cell. View the example here: [https://abegehr.github.io/frechet/?p=\(4\\_3\)\(7\\_4\)\(1\\_4\)\(4\\_3\)&q=\(1\\_3\)\(7\\_3\)](https://abegehr.github.io/frechet/?p=(4_3)(7_4)(1_4)(4_3)&q=(1_3)(7_3))

### 3.5 Computing the Classical Fréchet Distance

We now have the tools to compute the classical Fréchet distance. To compute the classical Fréchet distance, we can apply *Algorithm 2* of Alt and Godau’s paper.[1] The algorithm first determines all classical critical events of type a, b, and c, including their critical  $\epsilon$  and then using a binary search and the decision algorithm, finds the smallest critical  $\epsilon$  for which the height function  $\delta$  is traversable.

## 4 Lexicographic Fréchet Matchings

In his 2014 paper “Lexicographic Fréchet Matchings”, Rote extended upon the findings from Alt and Godau in their paper “Computing the Fréchet distance between two polygonal curves” to establish an algorithm that produces a lexicographic Fréchet matching.

Taking a lexicographic approach roughly means that we want to minimize the time  $T(s)$  during which the height exceeds a threshold  $s$ . [2] We want to spend as little time as possible at the current height and want to descend as quickly as possible.

### 4.1 Steepest Descent

To achieve a lexicographically optimized traversal, Rote makes use of the steepest descent. Therefore he developed a steepest descent algorithm.

Because we chose our input to be two paths consisting of straight line segments, the level set in a cell consists of concentric ellipses. Rote defines two lines for every cell,  $l$  and  $l'$ :

- ( $l$ ) The points where the ellipses have vertical tangents and  $\delta$  has a linear horizontal gradient lie on line  $l$  through the common center.[2]
- ( $l'$ ) The points where the ellipses have horizontal tangents and  $\delta$  has a linear vertical gradient lie on line  $l'$  through the common center.[2]

Depending on whether we are descending from a point on one of the lines or from in between one of the quadrants they span up, the direction of the steepest descent is affected. Consult Rote’s paper “Lexicographic Fréchet Matchings”[2] (Figure 5) for more detail. It is important to note at this point, that the steepest descent for a point is unique.

## 4.2 New Lexicographic Type of Critical Event

Rote also identifies a new lexicographic type of critical event, which does not influence the global Fréchet distance  $\epsilon_F$ , but is essential for achieving a lexicographically correct traversal.

The new lexicographic type of critical event occurs while descending on a steepest descent path. It occurs when we would horizontally or vertically surpass a critical opening of the height at which we currently are.

Figure 3 shows a minimal example of these new lexicographic types of critical events.

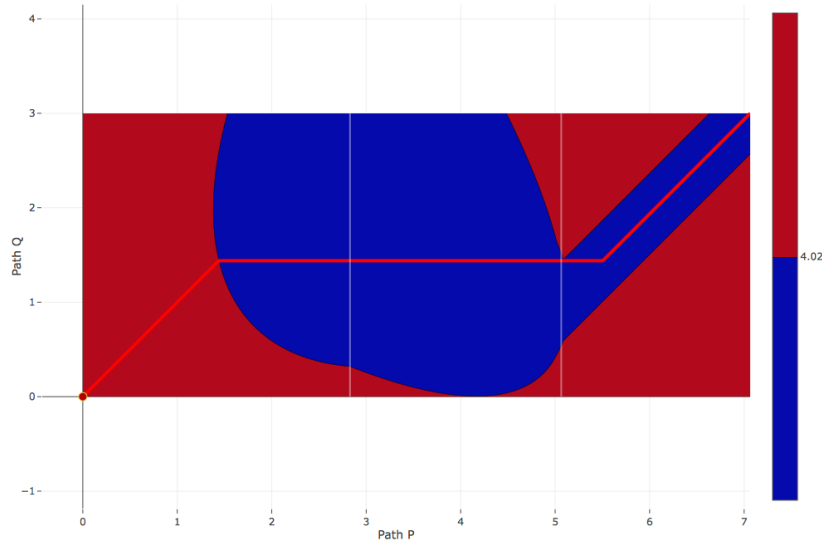


Figure 3: Example of new lexicographic type of critical event<sup>5</sup>

## 4.3 Lexicographic Fréchet Matching Algorithm

We now have the tools needed to understand Rote’s algorithm for lexicographic Fréchet matchings[2] under *general inputs* (not yet regarding degenerate inputs, as described in Section 5). In contrary to the algorithm for computing the classical Fréchet distance, the algorithm for lexicographic Fréchet matching utilizes recursion to allow for a locally correct lexicographic traversal. The algorithm as described here is aligned to the implementation of the online visualization.

<sup>5</sup>New lexicographic critical event of type a below height  $\epsilon_F$  connecting the middle of the left cell to the vertical border between the middle and the right cell. View the example here: [https://abegehr.github.io/frechet/?p=\(3\\_2\)\(5\\_4\)\(3\\_3\)\(5\\_3\)&q=\(2\\_7\)\(5\\_7\)](https://abegehr.github.io/frechet/?p=(3_2)(5_4)(3_3)(5_3)&q=(2_7)(5_7))

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**Algorithm 1** Requirements for Lexicographic Fréchet Matching Algorithm

---

```
1: function DECISIONALGORITHM( $\delta, A, B, \epsilon$ )
2:      $\triangleright$  Implementation of the decision algorithm, that decides if  $\delta$  is
        traversable from  $A$  to  $B$  with  $\epsilon$ .
3:     return true or false
4:
5: function STEEPESTDESCENT( $\delta, A, B, \epsilon_A, \epsilon_B$ )
6:      $\triangleright$  Attempts a steepest descent as far as possible for the higher
        of  $A$  and  $B$ , or both if they reside at equal height. Attempt the steepest
        descent to  $A'$  and  $B'$  until  $A'$  or  $B'$  reach a cell border or a line  $l$  or  $l'$ ,
        they would pass each other vertically or horizontally, or they meet each
        other. Descend to an equal height if possible.
7:     return  $A', B', \epsilon_{A'}, \epsilon_{B'}$ 
8:
9: function REVERSEDESCENT( $\delta, A, B, \epsilon_A, \epsilon_B$ )
10:     $\triangleright$  Reverses steepest descent until a classical critical event of
        type b or c, or a new lexicographic critical event is found. Also returns
        the critical event  $c$ .
11:    return  $A', B', \epsilon_{A'}, \epsilon_{B'}, c$ 
12:
13: function CLASSICALFRÉCHET( $\delta, A, B, \epsilon_A, \epsilon_B$ )
14:     $\triangleright$  Using a
        binary search and the decision algorithm, this function finds the lowest
        traversable classical critical event that is needed to traverse  $\delta$  from  $A$  to
         $B$ . This is the same as the algorithm for the classical Fréchet distance.
15:    return  $\epsilon_F, c$ 
```

---

---

**Algorithm 2** Lexicographic Fréchet Matching Algorithm

---

```
1: function TRAVERSERECURSIVE( $\delta, A, B, \epsilon_A, \epsilon_B$ )
2:   if  $A = B$  or  $B.x \leq A.x$  or  $B.y \leq A.y$  then
3:      $\triangleright$  Traversal done. Connect A and B.
4:   return CONNECT( $A, B$ )
5:    $\epsilon \leftarrow \max(\epsilon_A, \epsilon_B)$ 
6:   if DECISIONALGORITHM( $\delta, A, B, \epsilon$ ) then
7:      $A', B', \epsilon_{A'}, \epsilon_{B'} \leftarrow$  STEEPESTDESCENT( $\delta, A, B, \epsilon_A, \epsilon_B$ )
8:      $\epsilon' \leftarrow \max(\epsilon_{A'}, \epsilon_{B'})$ 
9:     if DECISIONALGORITHM( $\delta, A', B', \epsilon'$ ) then
10:       $\triangleright$  Traversable after steepest descent.
11:     return TRAVERSERECURSIVE( $\delta, A', B', \epsilon_{A'}, \epsilon_{B'}$ )
12:   else  $\triangleright$  Not traversable. The descent passes a critical event.
13:     while  $A \neq A'$  or  $B \neq B'$  do
14:        $A', B', \epsilon_{A'}, \epsilon_{B'}, c \leftarrow$  REVERSEDESCENT( $\delta, A', B', \epsilon_{A'}, \epsilon_{B'}$ )
15:        $\epsilon' \leftarrow \max(\epsilon_{A'}, \epsilon_{B'})$ 
16:       if DECISIONALGORITHM( $\delta, A', B', \epsilon'$ ) then
17:          $\triangleright$  Traversable with the critical event  $c$ .
18:       return TRAVERSERECURSIVE( $\delta, A', c.B, \epsilon_{A'}, \epsilon_c$ ) +  $c$  +
        TRAVERSERECURSIVE( $\delta, c.A, B', \epsilon_c, \epsilon_{B'}$ )
19:   else  $\triangleright$  A critical event with  $\epsilon_c > \epsilon$  is needed to traverse.
20:      $\epsilon_F, c \leftarrow$  CLASSICALFRÉCHET( $\delta, A, B, \epsilon_A, \epsilon_B$ )
21:   return TRAVERSERECURSIVE( $\delta, A', c.B, \epsilon_{A'}, \epsilon_c$ ) +  $c$  + TRAVERSERECURSIVE( $\delta, c.A, B', \epsilon_c, \epsilon_{B'}$ )
```

---

## 5 Lexicographic Fréchet Distance with Degenerate Inputs

### 5.1 Problem

As Rote describes in Section 7 of his paper “Lexicographic Fréchet Matchings” and as we hinted to in Section 4.3 of this paper, multiple critical events can occur for the same  $\epsilon$ .

Multiple critical events for the same  $\epsilon$  turn out to be problematic, because as can be seen in Algorithm 2, we assume the functions defined in Algorithm 1 return only one critical event, which will be traversed.

This section aims to answer the question of how to handle multiple critical events with the same critical  $\epsilon$  by visualizing the problem and postulating an algorithm that decides which combination of critical events result in a

lexicographic traversal. The basis of this section is Section 7 of Rote’s paper “Lexicographic Fréchet Matchings” [2].

## 5.2 Examples

In this subsection, four examples will be discussed, for which multiple critical events for the same critical  $\epsilon$  require a decision on which of them should be traversed to result in a lexicographic traversal. Instead of providing in-depth detail on solutions, this subsection aims to illustrate the problem through examples.

### 5.2.1 Minimal Example: Traverse Both

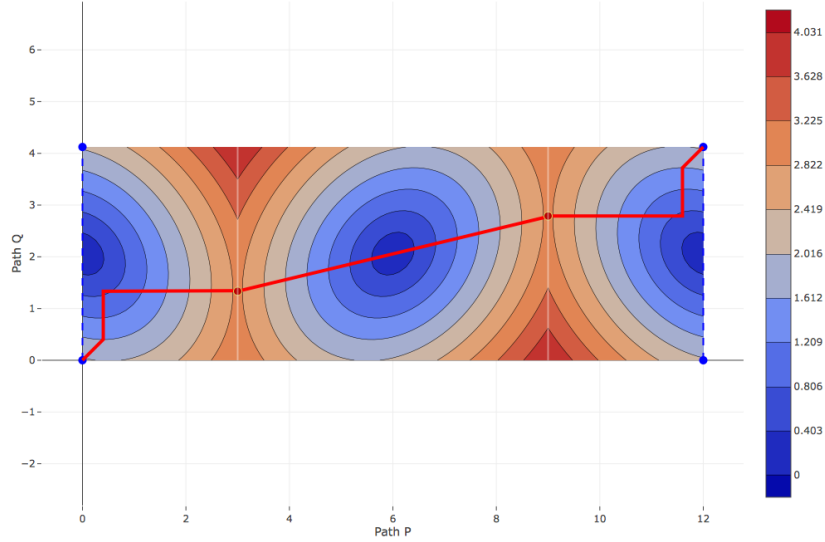


Figure 4: One Path Over Two Critical Events<sup>6</sup>

Figure 4 shows a minimal example with two critical events with the same  $\epsilon$ . There are two classical critical events of type b at  $\epsilon \approx 2.91$  at the two vertical cell-borders. Both critical events need to be traversed.

<sup>6</sup>View the example here: [https://abegehr.github.io/frechet/?p=\(5\\_5\)\(8\\_5\)\(2\\_5\)\(5\\_5\)&q=\(5.5\\_3\)\(4.5\\_7\)](https://abegehr.github.io/frechet/?p=(5_5)(8_5)(2_5)(5_5)&q=(5.5_3)(4.5_7))

### 5.2.2 Minimal Example: Traverse Either

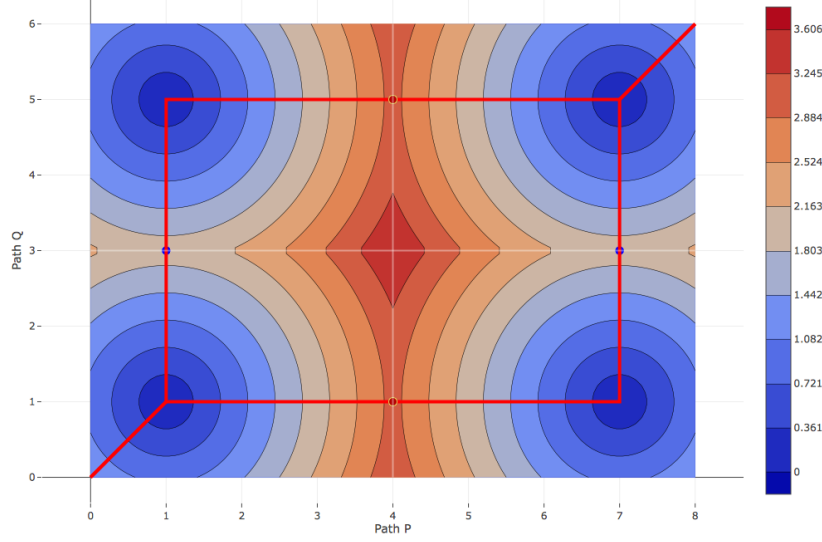


Figure 5: Two Equivalent Paths<sup>7</sup>

Figure 5 shows a minimal example of multiple critical events for the same  $\epsilon$ . There are four critical events to consider. Two classical critical events of type b with equal  $\epsilon = 3$  at the two vertical cell-borders. And two classical critical events of type b with equal  $\epsilon = 2$  at the two horizontal cell-borders. The decision problem shows that  $\epsilon = 3$  is needed to traverse this height function  $\delta$ . Either one of the critical events at  $\epsilon = 3$  need to be traversed.

<sup>7</sup>View the example here: [https://abegehr.github.io/frechet/?p=\(4\\_5\)\(8\\_5\)\(4\\_5\)&q=\(5\\_4\)\(5\\_7\)\(5\\_4\)](https://abegehr.github.io/frechet/?p=(4_5)(8_5)(4_5)&q=(5_4)(5_7)(5_4))



### 5.2.3 Minimal Example: Traverse One

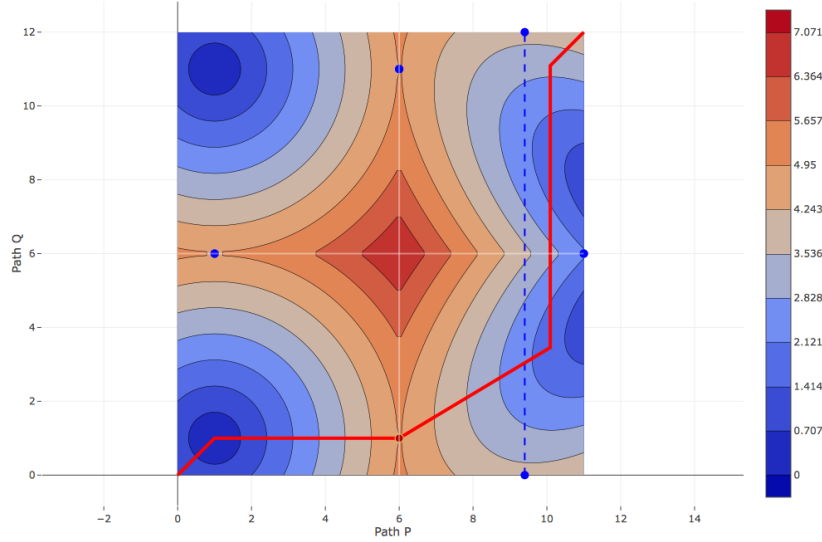


Figure 6: Two Inequivalent Paths<sup>8</sup>

Figure 6 shows three classical critical events of type b at  $\epsilon = 5$  to consider:  $C_1(6|1)$ ,  $C_2(1|6)$ , and  $C_3(6|11)$ . There are two possible paths through these three critical events: traversing just  $C_1$  or traversing  $C_2$  and  $C_3$ . Since the ascends to and descends from  $C_1$ ,  $C_2$ , and  $C_3$  are similar, we choose the path over only  $C_1$ .

### 5.2.4 Complex Example

Figure ?? shows a more complex example of multiple critical events for the same  $\epsilon$ .

## 5.3 Possible Paths

We have seen in the examples, that in some cases we need to traverse a path through multiple of the critical events at equal height  $\epsilon_0$ , while in other cases a path through one critical events is sufficient. To decide which path to traverse, first, we need to find all possible paths through the critical events.

The possible paths through multiple critical events can be represented as a graph. The start node represents the lower left corner  $A$  and the end

<sup>8</sup>View the example here: [https://abegehr.github.io/frechet/?p=\(3\\_2\)\(3\\_8\)\(6\\_4\)&q=\(2\\_3\)\(8\\_3\)\(2\\_3\)](https://abegehr.github.io/frechet/?p=(3_2)(3_8)(6_4)&q=(2_3)(8_3)(2_3))

node represents the upper right corner  $B$  of the current  $\delta$  subsection. Each critical event  $C_k \in C$  of height  $\epsilon_0$  is a node. Edges are directional and denote traversability without another critical event of height  $\epsilon_0$  in between.

To generate this graph of critical traversals, we make use of the free-space diagram and the technique used in the decision algorithm. First we use the free-space diagram represented by  $L_{ij}^F$  and  $B_{ij}^F$  and the decision algorithm (see Alt and Godau [1]) to generate the reachable border bounds  $L_{ij}^R$  and  $B_{ij}^R$  of all cells. Then we use the reachable border bounds and the critical events  $C$  at height  $\epsilon_0$  to generate for each vertical and horizontal cell-border, which subset of critical events  $C$  can reach the border. We call the subset of  $C$ , that reaches the left and bottom border of cell  $(i, j)$ ,  $L_{ij}^C$  and  $B_{ij}^C$  respectively. After computing each  $L_{ij}^C$  and  $B_{ij}^C$ , we then connect critical events in  $L_{ij}^C$  and  $B_{ij}^C$  to reachable critical events starting on the top and right border.

See the algorithm 3 for more detail on how we generate the traversability graph of multiple critical traversals  $C_k$  that are at same height  $\epsilon_0$ .  $p$  and  $q$  is the number of cells between the starting-point  $A$  and the end-point  $b$ .

---

**Algorithm 3** Generate Traversability Graph of multiple critical events at  $\epsilon_0$

---

```

1: function GENERATE TRAVERSAL GRAPH( $A, B, C, \epsilon_0$ )
2:    $\triangleright A$  is start-point of traversal.  $B$  is end-point of traversal.  $C$  holds
   all critical events  $C_k$ .
3:   Add  $A$  and  $B$  as critical events to  $C$ .
4:    $\forall i, j : L_{ij}^C := \{\} \wedge B_{ij}^C := \{\}$ 
5:    $L^R, B^R \leftarrow \text{DECISIONALGORITHM}'(A, B, \epsilon_0)$ 
6:   for  $C_k \in C$  do
7:     Add  $C_k.B$  (end-points) to  $L_{ij}^C$  or  $B_{ij}^C$  where  $C_k.B$  lies.
8:   for  $i := 0$  to  $p$  do determine  $L_{i,0}^C$ 
9:   for  $j := 0$  to  $q$  do determine  $B_{0,j}^C$ 
10:  for  $i := 0$  to  $p$  do
11:    for  $j := 0$  to  $q$  do
12:      for  $C_{k2}$  that starts on right or top border do
13:        for  $C_{k1} \in L_{i,j}^C \cup B_{i,j}^C$  do
14:          if  $C_{k1}$  can reach  $C_{k2}$  then
15:            Add edge  $(C_{k1}, C_{k2})$  to graph.
16:          construct  $L_{i+1,j}^C$  and  $B_{i,j+1}^C$  from  $L_{ij}^C, B_{ij}^C, L_{i+1,j}^R, B_{i,j+1}^R$ 
return graph

```

---

In the above algorithm 3 on line 14, we seek to determine if the critical event  $C_{k1}$  can reach the critical event  $C_{k2}$ . It is known that  $C_{k1}$  can reach

the left or bottom border of the current cell  $(i, j)$ , because  $C_{k1} \in L_{i,j}^C \cup B_{i,j}^C$ . It also is know that  $C_{k2}$  starts at the top or right border of the current cell.

To determine if  $C_{k2}$  is reachable from  $C_{k1}$ , it is important to know if  $C_{k1}$  comes the left or bottom border and if  $C_{k2}$  is on the right or top border, because the origin of  $C_{k1}$  influences the right and top reachability interval. There are four cases to consider:

1.  $C_{k1} \in L_{i,j}^C$  is on left border and  $C_{k2}$  starts on right border.
2.  $C_{k1} \in L_{i,j}^C$  is on left border and  $C_{k2}$  starts on top border.
3.  $C_{k1} \in B_{i,j}^C$  is on bottom border and  $C_{k2}$  starts on right border.
4.  $C_{k1} \in B_{i,j}^C$  is on bottom border and  $C_{k2}$  starts on top border.

The needed reachability intervals of right and top border are as follows:

1. From left to right:  $L_{i,j}^R \cap L_{i+1,j}^R$
2. From left to top: If  $L_{i,j}^R$  is open:  $B_{i,j+1}^R$ . Otherwise closed.
3. From bottom to right: If  $B_{i,j}^R$  is open:  $L_{i+1,j}^R$ . Otherwise closed.
4. From bottom to top:  $B_{i,j}^R \cap B_{i,j+1}^R$

If  $C_{k1}$  does not lie in or on the reachability interval, we discard this potential edge  $(C_{k1}, C_{k2})$ . We also discard the potential edge, if  $C_{k2}$  is not monotonically reachable from  $C_{k1}$ :  $\neg(C_{k1}.B.x \leq C_{k2}.A.x \wedge C_{k1}.B.y \leq C_{k2}.A.y)$ .

This check is not enough as figure **[fig:gengraph'ex]** shows.

## 5.4 Whut

To fulfill our goal of a lexicographic traversal, we want to minimize the time during which the height  $\epsilon$  exceeds a threshold.[2] Concerning our example, we want to minimize the time needed to ascend to and descend from our critical height  $\epsilon = 5$ .

For a critical event  $C_i$ , we will denote the ascent with  $C_i^\uparrow$  and descent with  $C_i^\downarrow$ .

We will now calculate the ascents  $C_1^\uparrow$ ,  $C_2^\uparrow$ , and  $C_3^\uparrow$  as well as the descents  $C_1^\downarrow$ ,  $C_2^\downarrow$ , and  $C_3^\downarrow$ , taking the steepest (monotonic) descent in all six cases.

## 5.5 Postulate Algorithm

Postulate algorithm for choosing a critical event from several with equal epsilon: (visual examples!)

Algorithm:

1. For every critical event compute reciprocal of derivatives descents and store as critical event's steepness.
2. From set of critical events, generate all possible sequences (ascending monotone).
3. For all sequences sum steepness of their critical events and store as sequence rank.
4. Starting with the smallest-rank sequence, decide if the sequence of critical events is traversable without passing critical events excluded in the sequence.
5. If decision is negative, repeat 4.
6. If decision is positive, repeat 4 for all sequences of equal rank.
7. If multiple sequences of equal rank are found to be valid, traverse all, and decide by comparing.
8. The steepness of the steepest decent.
9. The steepness of critical event that is reached.
10. And choose the path with the steepest summed recents of all paths.
11. If multiple paths with same hight profile are found, return all. Otherwise return the lexicographic optimum.
- 12.

## 5.6 Traversing critical events

Assume we are traversing a heat map generated by two curves. We arrive at a point where we have to decide which critical events need to be traversed for a lexicographic solution. The decision algorithm[1] can be applied for the  $\epsilon$  of each possible critical event. We pick the smallest  $\epsilon$  for which the decision algorithm returns a positive decision. We call it  $\epsilon_0$ . Now two options are feasible:

(A) There is one critical event at height  $\epsilon_0$ .  $|C_{\epsilon_0}| = 1$

(B) There are multiple critical events at height  $\epsilon_0$ .  $|C_{\epsilon_0}| > 1$

In case option A holds true, this critical event will need to be traversed. The traversal-problem is divided into two sub problems, which are both examined recursively and independently.

In case option B holds true, we need to decide which of the critical events  $C_{\epsilon_0}$  need to be traversed and which are not needed for an optimal solution. We will consider all subsets  $C'_{\epsilon_0}$  of  $C_{\epsilon_0}$  with size  $n > 0$ .

### 5.6.1 Traversing multiple critical events with same $\epsilon$

For each subset of critical events at  $\epsilon_0$ ,  $C'_{\epsilon_0} \subseteq C_{\epsilon_0}$  we compute if the critical events can be traversed monotonic. This can be done easily by applying the following procedure. For each critical event  $C'_{\epsilon_0}(i)$ , test if the start- and end-points of all other critical events in  $C'_{\epsilon_0}$  are either lower-left of the starting point of  $C'_{\epsilon_0}(i)$  or upper-right of the ending point of  $C'_{\epsilon_0}(i)$ .

See the a pseudocode for the function here. The function will return *True* if the critical events  $C'_{\epsilon_0}$  are traversable monotonically and *False* if they are not.

---

**Algorithm 4** Critical Events  $C'_{\epsilon_0}$  Monotonic Traversable
 

---

```

1: function  $PLeq(p, q)$ 
2:            $\triangleright$  determines if two points are monotonically traversable
3:   return  $(p_x \leq q_x \text{ and } p_y \leq q_y)$ 
4:
5: function  $MONOTONICTRAVERSABLE(C'_{\epsilon_0})$ 
6:            $\triangleright$  determines if a set of critical events  $C'_{\epsilon_0}$  are monotonically
           traversable
7:   for  $i$  in  $\text{len}(C'_{\epsilon_0})$  do
8:      $c_i \leftarrow C'_{\epsilon_0}(i)$ 
9:      $c_{i,s} \leftarrow c_i.start$ 
10:     $c_{i,e} \leftarrow c_i.end$ 
11:
12:    for  $j$  in  $\text{len}(C'_{\epsilon_0})$  do
13:      if  $j == i$  then continue  $\triangleright c_j$  is skipped
14:
15:       $c_j \leftarrow C'_{\epsilon_0}(j)$ 
16:       $c_{j,s} \leftarrow c_j.start$ 
17:       $c_{j,e} \leftarrow c_j.end$ 
18:
19:      if  $PLeq(c_{j,s}, c_{i,s})$  and  $PLeq(c_{j,e}, c_{i,s})$  then
20:        continue  $\triangleright c_j$  is traversable prior to  $c_i$ 
21:      if  $PLeq(c_{i,e}, c_{j,s})$  and  $PLeq(c_{i,e}, c_{j,e})$  then
22:        continue  $\triangleright c_j$  is traversable after  $c_i$ 
23:
24:      return False  $\triangleright c_i$  and  $c_j$  are not traversable monotonically
25:   return True  $\triangleright$  the critical events  $C'_{\epsilon_0}$  are monotonically traversable

```

---

*MonotonicTraversable* is applied to all subsets of critical events at  $\epsilon_0$ ,  $C'_{\epsilon_0} \subseteq C_{\epsilon_0}$ . We will only consider the critical events  $C'_{\epsilon_0}$ , for which  $C'_{\epsilon_0}$  is monotonically traversable, *i.e.*  $MonotonicTraversable(C'_{\epsilon_0})$  holds True, because other combinations of critical events are not monotonically traversable. We call this monotonically traversable subset  $C''_{\epsilon_0} \subseteq C_{\epsilon_0}$ .

$$C''_{\epsilon_0} := \{C'_{\epsilon_0} \mid MonotonicTraversable(C'_{\epsilon_0})\}$$

## 5.7 Example of Algorithm visualised (visual example!)

## 5.8 Analysing postulated algorithm for handling degenerate inputs

### 5.8.1 Runtime analysis

## 5.9 Real world application

Can you think of one?

# 6 Concluding Perspective

Concluding, all is great! :)

## References

- [1] Helmut Alt and Michael Godau. “Computing the Fréchet distance between two polygonal curves”. In: *International Journal of Computational Geometry & Applications* 5.01n02 (1995), pp. 75–91.
- [2] Günter Rote. “Lexicographic Fréchet Matchings”. In: *30th European Workshop on Computational Geometry*. EuroCG. 2014.