What is a Flow network?

A **network** is a directed graph G with vertices V and edges E combined with a function c, which assigns each edge e $\in$ E, a non-negative integer value, the capacity of e. Such a network is called a **flow network**, if we additionally label two vertices, one as the **source** and one as a **sink**.

A **flow** in a flow network is function f, which again assigns each edge e a non-negative integer value, namely the flow.

The function has to fulfill the following conditions:

- The flow of an edge cannot exceed the capacity.

$$f(e) \leq c(e)$$

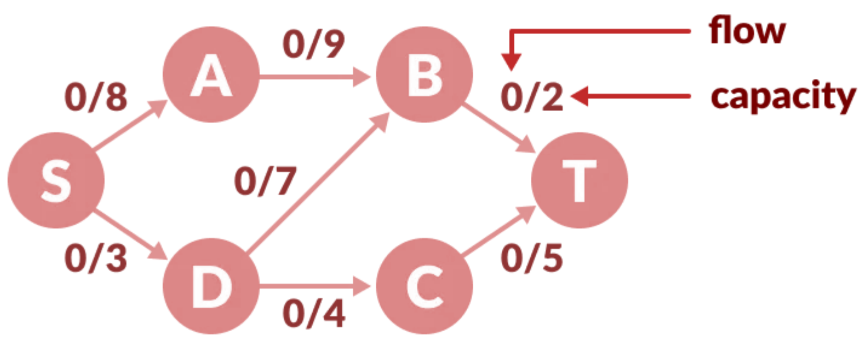
- And the sum of the incoming flow of a vertex u has to be equal to the sum of the outgoing flow of u except in the source and sink vertices.

$$\sum_{(v,u)\in E} f((v,u)) = \sum_{(u,v)\in E} f((u,v))$$

- The source vertex s only has an outgoing flow, and the sink vertex t has only incoming flow.

It is easy to see that the following equation holds.

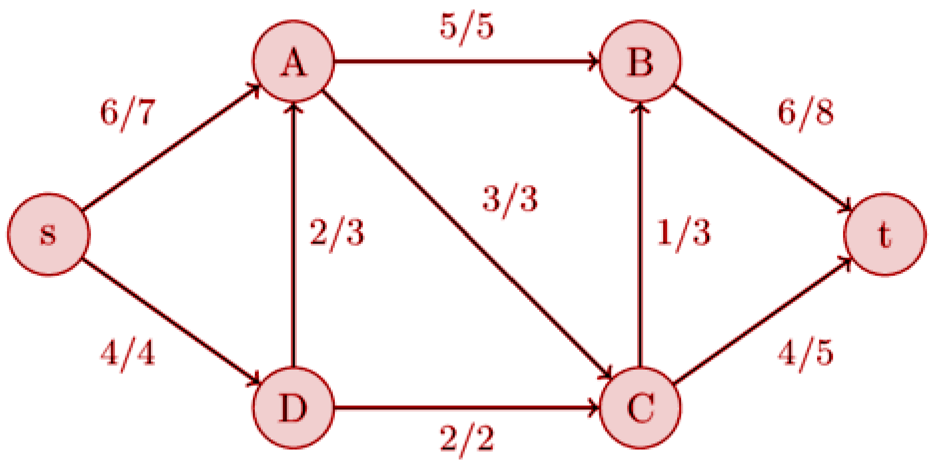
$$\sum_{(s,u)\in E} f((s,u)) = \sum_{(u,t)\in E} f((u,t))$$



Shown here is a flow network, where on each edge a number of the form f/c is written where f and c represent the flow through that edge and the capacity of that edge respectively.

The Max-Flow problem

The value of a flow of a network is the sum of all flows that get produced in source s, or equivalently the sum of the flows that are consumed in the sink t. A **maximal flow** is a flow with the maximal possible value. Finding this maximal flow of a flow network is the problem that we want to solve.



The image above shows the maximal flow in the network which is 10 (The sum of total flows coming out of source vertex).

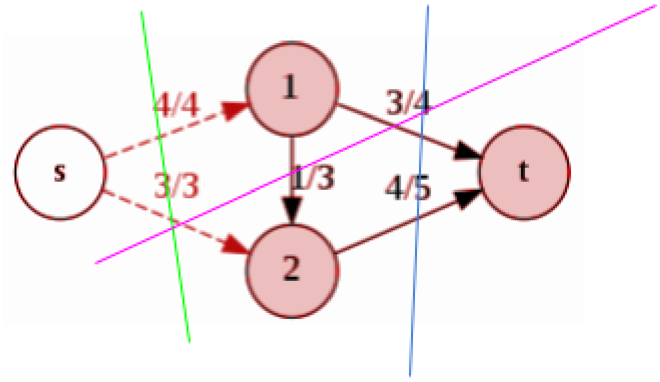
A residual capacity of a directed edge is the capacity minus the flow. It should be noted that if there is a flow along some directed edge (u,v), then the reversed edge has capacity 0 and we can define the flow of it as f((v,u)) = −f((u,v)). This also defines the residual capacity for all reversed edges. From all these edges we can create a residual network, which is just a network with the same vertices and same edges, but we use the residual capacities as capacities.

The Max-Flow Min cut Theorem

An s-t-cut is a partition of the vertices of a flow network into two sets, such that one set includes the source s and the other one includes the sink t. The capacity of an s-t-cut is defined as the sum of capacities of the edges from the source side to the sink side.

Obviously, we cannot send more flow from s to t than the capacity of any s-t-cut. Therefore the maximum flow is bounded by the minimum cut capacity.

The max-flow min-cut theorem says that the capacity of the maximum flow has to be equal to the capacity of the minimum cut.



In the above figure, among all the cuts between s and t, the minimum cut will have a value equal to the maxflow = 7 (green cut).

The other cuts have capacity 10 (pink cut) and 9 (blue cut) respectively.