Introduction

Given an undirected connected graph G = (V, E), where G can contain self-loops and multiple edges, we need to find the **number of spanning trees of G**. A spanning tree $H(V_H, E_H)$ is a connected subgraph of graph $G = (V_G, E_G)$ which includes all the vertices of the graph with minimum possible edges i.e $|V_H| = |V_G|$ and $|E_H| = |V_H| - 1$.

Kirchhoff's theorem helps us in finding the number of such spanning trees of a given undirected connected graph efficiently.

Kirchhoff's Theorem

Let G = (V, E) be a connected undirected graph, let us define a matrix A which represents the adjacency matrix representation of the graph G and a degree matrix D.

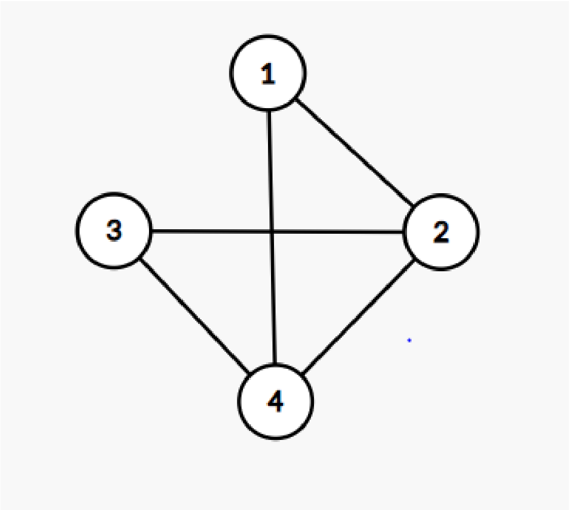
Let |V| = n, the adjacency matrix 'A' of G is a n x n matrix, where A[i][j] equals the number of edges between node 'i' and node 'j'.
The degree matrix 'D' of G is a diagonal n x n matrix, where:
$D_{u,v} = 0$, for u != v
$D_{u,v,} = deg(u)$, for u = v, where deg(u) is the degree of vertex u which equals the number of edges connecting it.

Algorithm:

- Calculate the adjacency matrix A and the degree matrix D.
- Define a new matrix L = D - A, where L is known as the **Laplacian matrix**, where all the cofactors of this matrix are equal. The (i,j) cofactor of the matrix, where i represents the row index and j represents the column index, is defined as the product of $(-1)^{i+j}$ with the determinant of the matrix obtained after removing the i^{th} row and j^{th} column.
- The cofactor of matrix L(any cofactor) represents the number of spanning trees of graph G. The determinant of a matrix can be found using Gaussian Elimination in $O(N^3)$, where N is the number of rows and columns.
  Gaussian Elimination: https://en.wikipedia.org/wiki/Gaussian_elimination

Example:

Let us consider the following graph and find the number of spanning trees for this graph.



The Adjacency matrix(A) for the above graph is:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The Degree matrix(D) for the above graph is:

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Calculating the Laplacian matrix(L):
L = D - A

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Finding the cofactor of matrix corresponding to (1,1) i.e the determinant of the matrix obtained by deleting the first row and first column:
Number of spanning trees = $(-1)^{(1+1)}$.Det(M) where M is:

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Number of spanning trees = 8