Master's Theorem

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Master's Theorem is used for solving the recurrence relations.
Master's Theorem solves recurrence relations of the form-

$$T(n) = a\,T\left(\frac{n}{b}\right) + \theta\left(n^k \log^p n\right)$$

Master's Theorem

where a >= 1, b > 1, k >= 0 and p is a real number

Solving Recurrence Relations Using Master's Theorem-

In Master's Theorem, we compare 'a' with 'b^k' and then follow the following cases-

Case-01:

If $a > b^k$, then $T(n) = \theta\ (n^{\log_b a})$

Case-02:

If $a = b^k$ and-

- If $p < -1$, then $T(n) = \theta\ (n^{\log_b a})$
- If $p = -1$, then $T(n) = \theta\ (n^{\log_b a}.\log^2 n)$
- If $p > -1$, then $T(n) = \theta\ (n^{\log_b a}.\log^{p+1} n)$

Case-03:

If $a < b^k$ and-

- If $p < 0$, then $T(n) = O(n^k)$
- If $p >= 0$, then $T(n) = \theta(n^k \log^p n)$

Problem-01:

Solve the recurrence relation using Master's Theorem-

$$T(n) = 3T(n/2) + n^2$$

On comparing the given recurrence relation with-

$$T(n) = aT(n/b) + \theta\ (n^k \log^p n)$$

we have-

$$a = 3$$

$$b = 2$$

$$k = 2$$

$$p = 0$$

Now, $a = 3$ and $b^k = 2^2 = 4$.

Clearly, $a < b^k$

So, we follow case-03.

Since $p = 0$, so we have-

$$T(n) = \theta\ (n^k \log^p n)$$

$$T(n) = \theta\ (n^2 \log^0 n)$$

Thus,

$$\boxed{\mathbf{T(n) = \theta\ (n^2)}}$$

Solve the recurrence relation using Master's Theorem-

$$T(n) = 2T(n/2) + n\log n$$

On comparing the given recurrence relation with-

$$T(n) = aT(n/b) + \theta\ (n^k \log^p n)$$

we have-

$$a = 2$$

$$b = 2$$

$$k = 1$$

$$p = 1$$

Now, $a = 2$ and $b^k = 2^1 = 2$.

Clearly, $a = b^k$

So, we follow case-02.

Since $p = 1$, so we have-

$$T(n) = \theta\ (n^{\log_b a} . \log^{p+1} n)$$

$$T(n) = \theta\ (n^{\log_2 2} . \log^{1+1} n)$$

Thus,

$$\boxed{\mathbf{T(n) = \theta\ (n\log^2 n)}}$$

Solve the recurrence relation using Master's Theorem-

$$T(n) = 2T(n/4) + n^{0.51}$$

On comparing the given recurrence relation with-

$$T(n) = aT(n/b) + \theta(n^k \log^p n)$$

we have-

$$a = 2$$

$$b = 4$$

$$k = 0.51$$

$$p = 0$$

Now, $a = 2$ and $b^k = 4^{0.51} = 2.0279$

Clearly, $a < b^k$

So, we follow case-03.

Since $p = 0$, so we have-

$$T(n) = \theta(n^k \log^p n)$$

$$T(n) = \theta(n^{0.51} \log^0 n)$$

Thus,

$$\boxed{\mathbf{T(n) = \theta(n^{0.51})}}$$

Solve the recurrence relation using Master's Theorem-

$$T(n) = \sqrt{2}T(n/2) + \log n$$

On comparing the given recurrence relation with-

$$T(n) = aT(n/b) + \theta\ (n^k \log^p n)$$

we have-

$$a = \sqrt{2}$$

$$b = 2$$

$$k = 0$$

$$p = 1$$

Now, $a = \sqrt{2} = 1.414$ and $b^k = 2^0 = 1$.

Clearly, $a > b^k$

So, we follow case-01.

So we have-

$$T(n) = \theta\ (n^{\log_b a})$$

$$T(n) = \theta\ (n^{\log_2 \sqrt{2}})$$

$$T(n) = \theta\ (n^{1/2})$$

Thus,

$$\boxed{\mathbf{T(n) = \theta\ (\sqrt{n})}}$$

Solve the recurrence relation using Master's Theorem-

$$T(n) = 3T(n/3) + n/2$$

We can write the given recurrence relation as-

$$T(n) = 3T(n/3) + n$$

The reason is because in the general form, we have written θ for function f(n) which hides constants in it.

Now, we can apply Master's Theorem.

On comparing the given recurrence relation with-

$$T(n) = aT(n/b) + \theta\ (n^k \log^p n)$$

we have-

$$a = 3$$

$$b = 3$$

$$k = 1$$

$$p = 0$$

Now, $a = 3$ and $b^k = 3^1 = 3$.

Clearly, $a = b^k$

So, we follow case-02.

Since $p = 0$, we have-

$$T(n) = \theta\ (n^{\log_b a}.\log^{p+1} n)$$

$$T(n) = \theta\ (n^{\log_3 3}.\log^{0+1} n)$$

$$T(n) = \theta\ (n^1.\log^1 n)$$

Thus,

$$\boxed{\mathbf{T(n) = \theta\ (nlogn)}}$$