Grundy Numbers

MEX(Maximum Excludant):
'Mex' of a set of numbers is the smallest non-negative number not present in the set.

Example:

- MEX(ϕ) = 0 .
- MEX({1}) = 0, the smallest non-negative number in the set is 0.
- MEX({0,1,3}) = 2
- MEX({0,1,2,3,4}) = 5

Grundy Number:
The Grundy Number is equal to 0 for a game that is lost immediately by the first player and is equal to **Mex** of the grundy numbers of all possible next positions for any other game.

For example: Consider two players and n stones in a pile. A player can take any number of stones from the pile. The player who is not able to make a move loses the game. Guess the winner.

Now consider n=0, since the first player will not be able to make a move so he loses. So grundy(0) = 0 .
Now consider n=1, player 1 can remove 1 stone and the rest of the game depends on the remaining stones i.e. 0.
So grundy(1) = MEX({grundy(0)}) = MEX(0) = 1.
Now consider n=2, player 1 can remove 1 stone or 2 stones. So the rest of the game depends on the remaining stones that are either n=1 or n=0.
So grundy(2) = MEX({grundy(1), grundy(0)}) = MEX({1, 0}) = 2.
Similarly
grundy(n) = MEX({grundy(n-1), grundy(n-2) … grundy(0)}) = n.

Pseudocode for this example:

```
// create a complex class as follows
function calculateGrundy(n){

        if n == 0
                return 0

        //   create a set grundy to store grundy numbers for value 0 to n-1
        int i = 0
        while (i <= n-1)
                grundy.add(calculateGrundy(i))

        return mex(grundy)
}
```

```
function mex(grundy){
        i = 0

        //   run a loop till you find i, if i is not found return i.
        while(set.contains(grundy))
                i++;
        return i
}
```

Sprague-Grundy Theorem

Let's change the classical nim problem a bit. In a move, P1 or P2 can pick only 1, 2, or 3 stones from the pile. Can we now predict the winner? Yes, we can predict the winner using the Sprague-grundy theorem.

Theorem:
Sprague Grundy theorem states that in a composite game with two players P1 and P2, if P1 and P2 play optimally, then the player starting first is guaranteed to win if the XOR of the grundy numbers of position in each sub-games at the beginning of the game is non-zero.
Example: Consider 3 piles with coins a, b, and c respectively. In a move, the player can only pick 1, 2, or 3 coins from a pile.

We have

$$X = \text{grundy}(a) \wedge \text{grundy}(b) \wedge \text{grundy}(c)$$

The Grundy theorem says that if X is not equal to 0, then the player who made the first move wins, and if X=0, the other player wins.