Introduction

What if I ask you to predict the winner without playing the game? Yes, you can do that by using game theory. Game theory helps you to predict the winner without actually playing the game, but only in **combinatorial games**. Combinatorial games are games that will be played mostly by two players and these games will not have any randomization. For example, games like Chess, Tic Tac Toe, and Game of Nim come under the category of combinatorial games.

Game theory is quite considerable in mathematics.
With some mathematical theorems, we can solve games like 'Game of Nim'. However, it is slightly difficult to solve games like Chess or Tic Tac Toe.

Combinatorial games are of two types:

- **Impartial Games**: The set of moves available from any given position is the same for both players. Ex: Game of Nim.
- **Partizan games**: Each player has a different set of possible moves from a given position. Ex: tic tac toe.

We will have our discussion about game theory with the help of an example problem that is Game of Nim.

Game of Nim

The **'Game of Nim'** is simple. There are 'n' piles of coins, (in between two players) each player can take any positive number of coins, from any of the piles i.e. each player has to take at least 1 coin. The player who makes the last move will win. Both players will play optimally (playing to win).

For example: Consider two players P1 and P2 and initially, there are three piles of coins having 3, 4, and 5 coins respectively.
We assume P1 starts the game and the moves P1 and P2 take are as follows:

- P1 takes 2 coins from the first pile. The coins left in 3 piles are 1,4 and 5.
- P2 takes 3 coins from the third pile. The coins left in 3 piles are 1,4 and 2.
- P1 takes 1 coin from the second pile. The coins left are 1,3 and 2.
- P2 takes 1 coin from the second pile. The coins left are 1,2 and 2.
- P1 removes all the coins from the first pile. The coins left are 0,2 and 2.
- P2 removes 1 coin from the second pile. The coins left are 0,1 and 2.
- P1 removes 1 coin from the third pile. The coins left are 0,1 and 1.
- P2 removes all the coins from the second pile. The coins left are 0,0 and 1.
- Lastly, **P1 removes the last coin from pile 3 and wins the game**.

In this game, P1 won the match but P1 also made the first move. If P2 started the game, then P2 must have won. So the final answer depends on who started the game.

But will it always be like this, that the player who started the game will always win? Consider another scenario where there are again 3 piles but having 1,4 and 5 coins respectively.
Let P1 make the first move. The game goes as follows:

- P1 takes 3 coins from the third pile. The coins left in 3 piles are 1,4 and 2.
- P2 takes 1 coin from the second pile. The coins left in 3 piles are 1,3 and 2.
- P1 takes 1 coin from the second pile. The coins left are 1,2 and 2.
- P2 removes all the coins from the first pile. The coins left are 0,2 and 2.
- P1 removes 1coin from the second pile. The coins left are 0,1 and 2.
- P2 removes 1 coin from the third pile. The coins left are 0,1 and 1.
- P1 removes 1 coin from the second pile. The coins left are 0,0 and 1.
- **P2 removes all the coins from the third pile and wins the game. The coins left are 0,0 and 0**.

So, the second factor on which the winner depends is the initial stage of the piles.
How can we predict the winner before even playing the game?

Nim-Sum: The cumulative XOR value of the number of coins/stones in each pile/heaps at any point of the game is called Nim-Sum at that point.

If both players play optimally i.e., without making any mistakes, then the player who makes the first move is guaranteed to win if the nim-sum of the initial stage of piles is non-zero.

Proof:

- If the XOR sum of 'n' numbers is already zero, then there is no possibility to make the XOR sum zero by a single reduction of a number.

  Example:
  Consider 3 piles having 1, 4, and 5 coins having cumulative XOR 0.

```
0 0 0 1 : 1
0 1 0 0 : 4
0 1 0 1 : 5
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0 0 0 0 : cumulative XOR
```

  Now if you remove any number of coins from a single pile, you will have to toggle a single bit or multiple bits from the binary representation of that pile. It will always result in a non-zero cumulative XOR.

- If the XOR sum of 'n' numbers is non-zero, then there is at least a single approach by which if you reduce a number, the XOR sum is zero.

  Example:
  Consider 3 piles having 3, 4, and 5 coins having non-zero cumulative XOR.

```
0 0 1 1 : 3
0 1 0 0 : 4
0 1 0 1 : 5
-----------
0 0 1 0 : cumulative XOR
```

You can always remove some x number of coins from some pile to make a cumulative XOR of 0.
Like in this example, you can remove 2 coins from the first pile to make the second set bit from right 0. Then there will be an even number of set bits and the cumulative XOR will be 0.

Now, for the Nim Game, two cases can exist:

Case 1: Initially nim sum is zero and P1 makes the first move
Now going by the optimal approach, P1 would have to at least pick a coin and that would make the Nim-Sum non-zero now. Now, in P2's turn, as the nim sum is already non-zero, P2 can always make the nim-sum zero as discussed above. This thing will go on, P1 will make the nim-sum non-zero, and P2 will make nim-sum zero. At last, P2 will remove all the coins from some piles and the binary representation will contain zeros at all positions in all piles and will make P1 lose the game.

Case 2: Initially nim sum is non-zero and P1 makes the first move
Now going by the optimal approach, P1 would make the Nim-Sum zero now (which is possible as the initial Nim sum is non-zero, as mentioned above). Now, in P2's turn, as the nim sum is already zero, whatever number P2 picks, the nim sum would be non-zero and P1 can pick a number to make the nim sum zero again. This will go as long as there are items available in any pile. And P1 will be the one to pick the last item.