Online-Class 08-02-2021

Probability, Statistics and Reliability (MAT3003)

SLOT: B21 + B22 + B23

MODULE - 1

Topic: Conditional Probability & Independent Events

Brief Contents

- Review of Previous Class
- Conditional Probability
- Independent Events
- Multiplication Rule
- Related Application Questions
- Practice Questions

Review of Previous Class

- De Morgan's Laws

$$E^C \cap F^C = (E \cup F)^C$$
$$E^C \cup F^C = (E \cap F)^C$$

This leads to two more properties:

$$P(E^C \cap F^C) = P((E \cup F)^C) = 1 - P(E \cup F)$$
$$P(E^C \cup F^C) = P((E \cap F)^C) = 1 - P(E \cap F)$$

This often is used in conjunction with the rule that the sum of an event and its complement is 1:

$$P(E \cap F) = 1 - P((E \cap F)^C) \qquad \text{Since } P(E) + P(E^C) = 1$$

$$= 1 - P(E^C \cup F^C) \qquad \text{By DeMorgan's law}$$

$$P(E \cup F) = 1 - P((E \cup F)^C) \qquad \text{Since } P(E) + P(E^C) = 1$$

$$= 1 - P(E^C \cap B^F) \qquad \text{By DeMorgan's law}$$

Axioms of Probability

- Here are some basic truths about probabilities that we accept as axioms:

Basic Truth 1: $0 \le P(E) \le 1$
Basic Truth 2: $P(S) = 1$
Basic Truth 3: $P(E) = 1 - P(E^C)$

Conditional Probability

- Let's take a **real-life** example.
- **Probability** of selling a TV on a given normal day maybe only 30%. But if we consider that given day is Diwali, then there are much more chances of selling a TV.

- Quite often we are interested in determining whether two events, *A* and *B*, are related in the sense that knowledge about the occurrence of one, say *B*, alters the likelihood of occurrence of the other, *A*. This requires that we find the **conditional probability**,

$$P(B/A)$$

of event *B* given that event *A* has occurred.

Conditional Probability

The probability of an event B occurring when it is known that some event A has occurred is called a **conditional probability** and is denoted by $P(B|A)$. The symbol $P(B|A)$ is usually read "the probability that B occurs given that A occurs" or simply "the probability of B, given A."

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

Suppose, event B is already over,

A is happening now.

The probability of A given that B is already over, is given by:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Independent Events

- By Definition of conditional probability,   ??

Independent Events

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

Independent Events…contd.

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Multiplicative Rule

- By Definition of conditional probability, we have

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\boldsymbol{P(A \cap B) = \mathbf{P(A)\ P(B/A)}} \qquad (1)$$

This is the *Multiplicative Rule for any two events*.

If the events are independent, then

$$P(B/A) = P(B) \qquad (2)$$

From (1) & (2), we have

$$\boldsymbol{P(A \cap B) = \mathbf{P(A)\ P(B)}} \qquad (3)$$

This is the *Multiplicative Rule for two independent events*.

Multiplicative Rule…contd.

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

Thus, the probability that both A and B occur is equal to the probability that A occurs multiplied by the conditional probability that B occurs, given that A occurs. Since the events $A \cap B$ and $B \cap A$ are equivalent, we can also write

$$P(A \cap B) = P(B \cap A) = P(B)P(A|B).$$

In other words, it does not matter which event is referred to as A and which event is referred to as B.

Multiplicative Rule for More than Two-Events

- The multiplicative rule can be extended to more than two-event situations.

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

Reference: Probability and Statistics for Engineers & Scientists by R.E.Walpole, Pearson (2012).

Question 1 (Conditional Probability)

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane

(a) arrives on time, given that it departed on time, and

(b) departed on time, given that it has arrived on time.

Solution

(a) The probability that a plane arrives on time, given that it departed on time, is

$$P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

(b) The probability that a plane departed on time, given that it has arrived on time, is

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95$$

Question 2 (Independent Event)

Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Reference: Probability and Statistics for Engineers & Scientists by R.E.Walpole, Pearson (2012).

Solution

We shall let A be the event that the first fuse is defective and B the event that the second fuse is defective; then we interpret $A \cap B$ as the event that A occurs and then B occurs after A has occurred.

The probability of first removing a defective fuse is $1/4$; then the probability of removing a second defective fuse from the remaining 4 is $4/19$.

Hence,

$$P(A \cap B) = \left(\frac{1}{4}\right)\left(\frac{4}{19}\right) = \frac{1}{19}.$$

Reference: Probability and Statistics for Engineers & Scientists
by R.E.Walpole, Pearson (2012).

Question 3

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Solution

Let A and B represent the respective events that the fire engine and the ambulance are available. Then

$$P(A \cap B) = P(A)P(B)$$

$$= (0.98)(0.92) = 0.9016.$$

Question 4 (For Students)

A manufacturer of a flu vaccine is concerned about the quality of its flu serum. Batches of serum are processed by three different departments having rejection rates of 0.10, 0.08, and 0.12, respectively.

The inspections by the three departments are sequential and independent.

(a) What is the probability that a batch of serum survives the first departmental inspection but is rejected by the second department?

(b) What is the probability that a batch of serum is rejected by the third department?

Reference: Probability and Statistics for Engineers & Scientists by R.E.Walpole, Pearson (2012).

Solution

Let D_1, D_2, D_3 be the events that the serum would be rejected by departments 1, 2, 3 respectively.

Then $P(D_1) = 0.10$, $P(D_2) = 0.08$, $P(D_3) = 0.12$, and all three events are independent.

$$\text{(a) } P(D_1' \cap D_2) = (1 - P(D_1))P(D_2) = 0.90 \cdot 0.08 = 0.072.$$

$$\text{(b) } P(D_1' \cap D_2' \cap D_3) = (1 - P(D_1))(1 - P(D_2))P(D_3) = 0.90 \cdot 0.92 \cdot 0.12 \approx 0.099.$$

Practice Questions

1. A town has two fire engines operating independently. The probability that a specific engine is available when needed is 0.96.

(a) What is the probability that neither is available when needed?

(b) What is the probability that a fire engine is available when needed?

Let A and B represent the availability of each fire engine.

(a) $P(A' \cap B') = P(A')P(B') = (0.04)(0.04) = 0.0016.$

(b) $P(A \cup B) = 1 - P(A' \cap B') = 1 - 0.0016 = 0.9984.$

2. The probability that an automobile being filled with gasoline also needs an oil change is 0.25; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and the filter need changing is 0.14.

(a) If the oil has to be changed, what is the probability that a new oil filter is needed? Ans: 0.56

(b) If a new oil filter is needed, what is the probability that the oil has to be changed? Ans: 0.35

3. The probability that a doctor correctly diagnoses a particular illness is 0.7. Given that the doctor makes an incorrect diagnosis, the probability that the patient files a lawsuit is 0.9. What is the probability that the doctor makes an incorrect diagnosis and the patient sues? Ans: 0.27

4. The probability that a married man watches a certain television show is 0.4, and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Find the probability that

   (a) a married couple watches the show; Ans: 0.35

   (b) a wife watches the show, given that her husband does; Ans: 0.875

   (c) at least one member of a married couple will watch the show. Ans: 0.55

THANK YOU