Artificial Intelligence-C11

Fuzzy Reasoning

Interim Semester 2021-22 BPL

CSE3007-LT-AB306

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Fuzzy Sets: Introduction

- Proposed by Lotfi A. Zadeh
- Generalization of classical set theory
- Uncertainty (Incomplete knowledge, generality, vagueness, ambiguity)
- Effective solving of uncertainty in the problem.
- A classical set is a set with a crisp boundary.
- An element : Belongs to / not belongs to a set (0 or 1)

Eg.: C classical set A of real numbers greater than 6

$$A = \{x \mid x>6\}$$

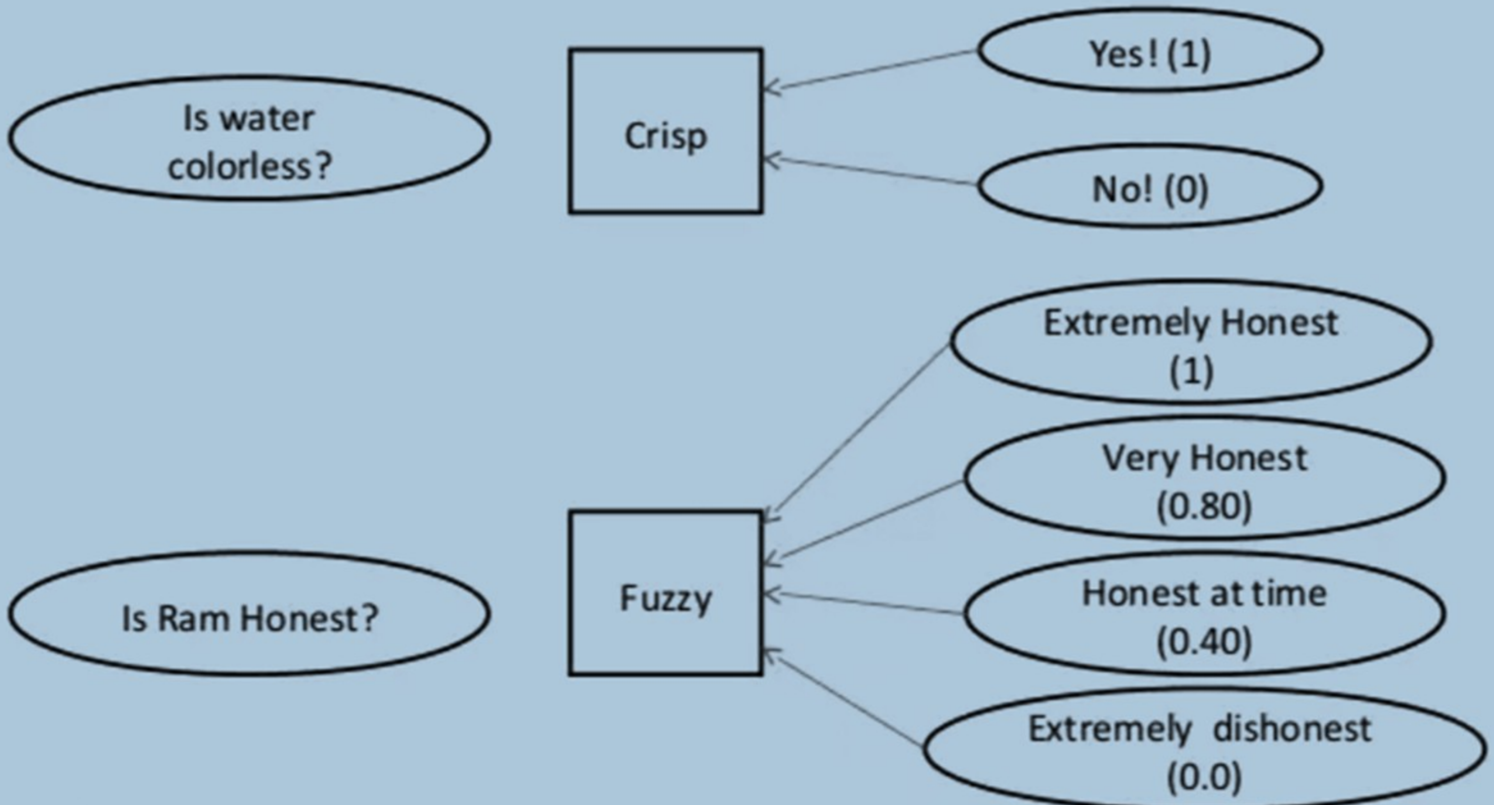
- Fuzzy set many degrees of membership (0 & 1)
- Membership function $\mu_A(x)$ – associated with a fuzzy set A such that the function maps every element of the universe of discourse X to the interval [0,1]

Classical set theory	Fuzzy set theory
• Classes of objects with sharp boundaries.	• Classes of objects with un-sharp boundaries.
• A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the location of the set boundaries.	• A fuzzy set is defined by its ambiguous boundaries, i.e., there exists uncertainty about the location of the set boundaries.
• Widely used in digital system design	• Used in fuzzy controllers.

The areas of potential fuzzy implementation are numerous and not just for control:

- Speech recognition
- fault analysis
- decision making
- image analysis
- scheduling

Example



Fuzzy vs crips

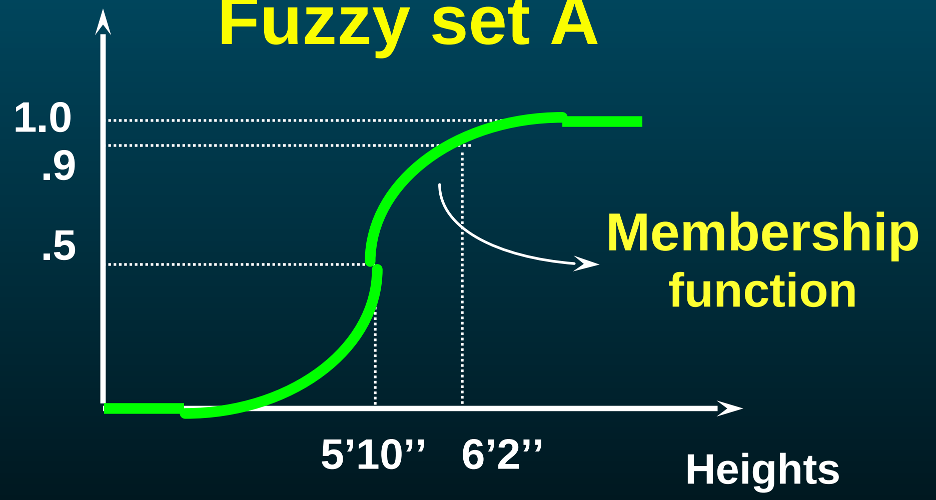
Fuzzy Sets

Sets with fuzzy boundaries

A = Set of tall people

Crisp set A

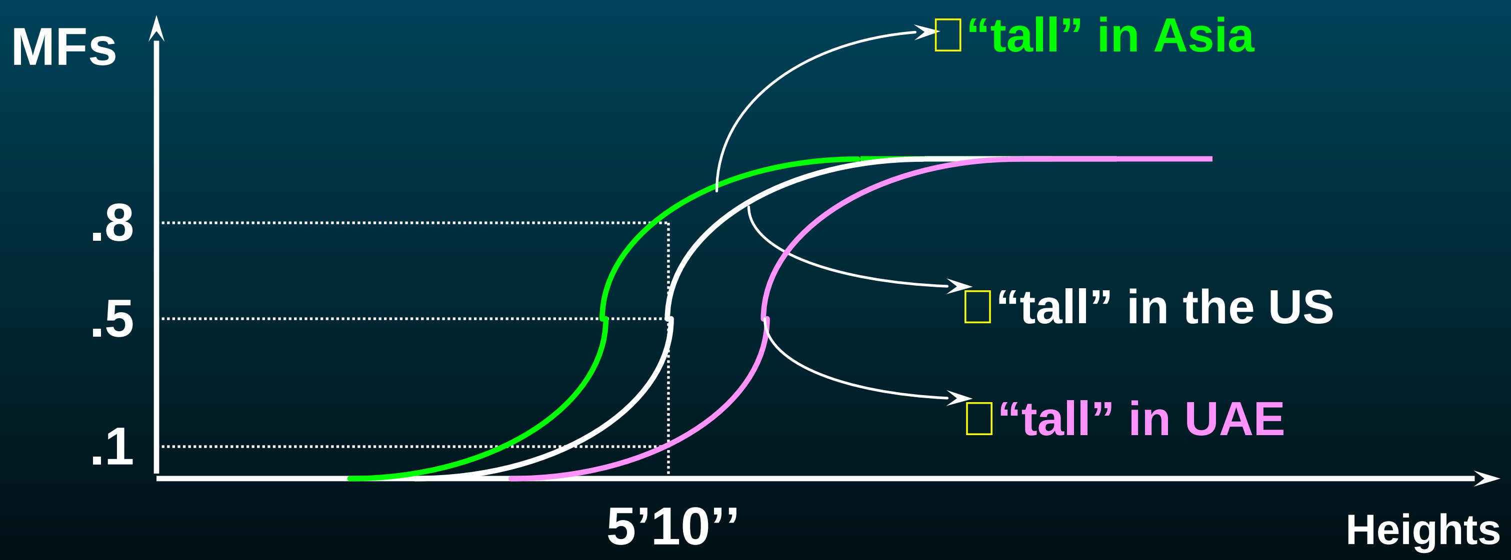
Fuzzy set A



Membership Functions (MFs)

Characteristics of MFs:

- Subjective measures
- Not probability functions



Fuzzy Sets

Formal definition:

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$\underbrace{A}_{\text{Fuzzy set}} = \{(x, \underbrace{\mu_A}_{\substack{\text{Membership}\\\text{function}\\\text{(MF)}}}(x)) \mid x \in \underbrace{X}_{\substack{\text{Universe or}\\\text{universe of discourse}}}\}$$

A fuzzy set is totally characterized by a membership function (MF).

Alternative Notation

A fuzzy set A can be alternatively denoted as follows:

X is discrete → $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

X is continuous → $$A = \int_X \mu_A(x) / x$$

- **Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.**
- **Membership Function (MF) – Maps each element of X to a membership grade (membership value) between 0 and 1**

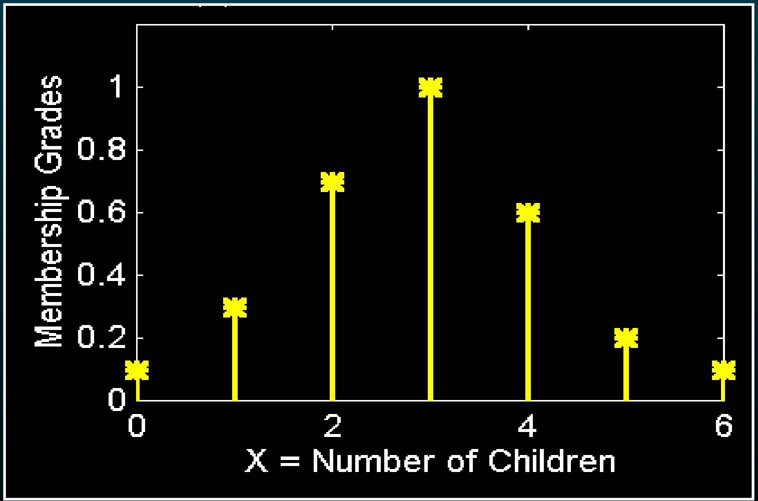
Fuzzy Sets with Discrete Universes

Fuzzy set C = "desirable city to live in"

- X = {SF, Boston, LA} (discrete and nonordered)
- C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

Fuzzy set A = "sensible number of children"

- X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)
- A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}

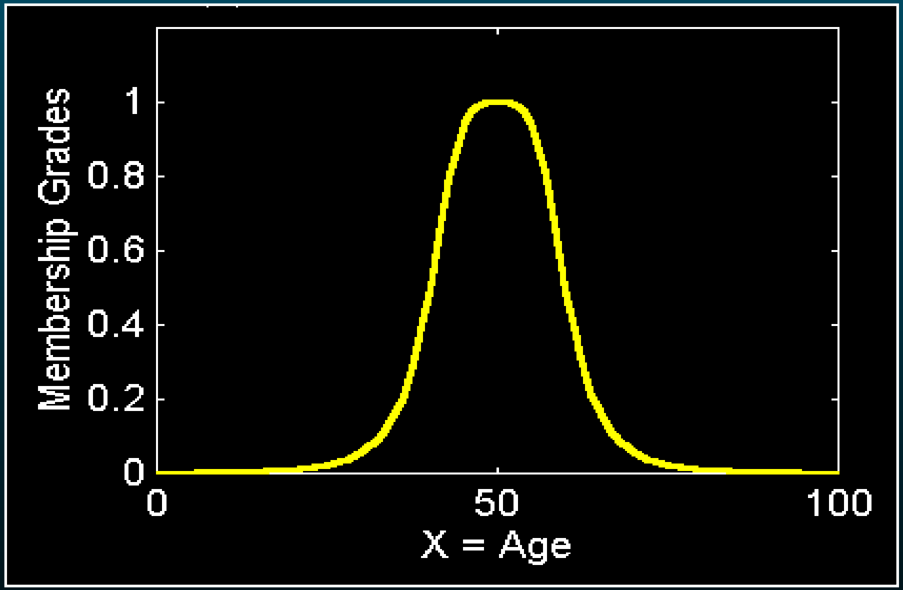


Fuzzy Sets with Cont. Universes

Fuzzy set B = “about 50 years old”

- X = Set of positive real numbers (continuous)
- B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^2}$$

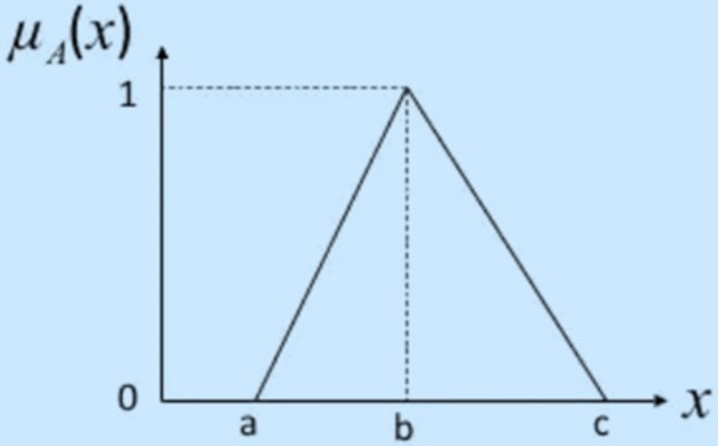


- **Triangular membership function**

  A *triangular* membership function is specified by three parameters {a, b, c}

  a, b and c represent the x coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1)

$$\mu_A(x) = \begin{cases} 0 & if\ x \le a \\ \dfrac{x-a}{b-a} & if\ a \le x \le b \\ \dfrac{c-x}{c-b} & if\ b \le x \le c \\ 0 & if\ x \ge c \end{cases}$$



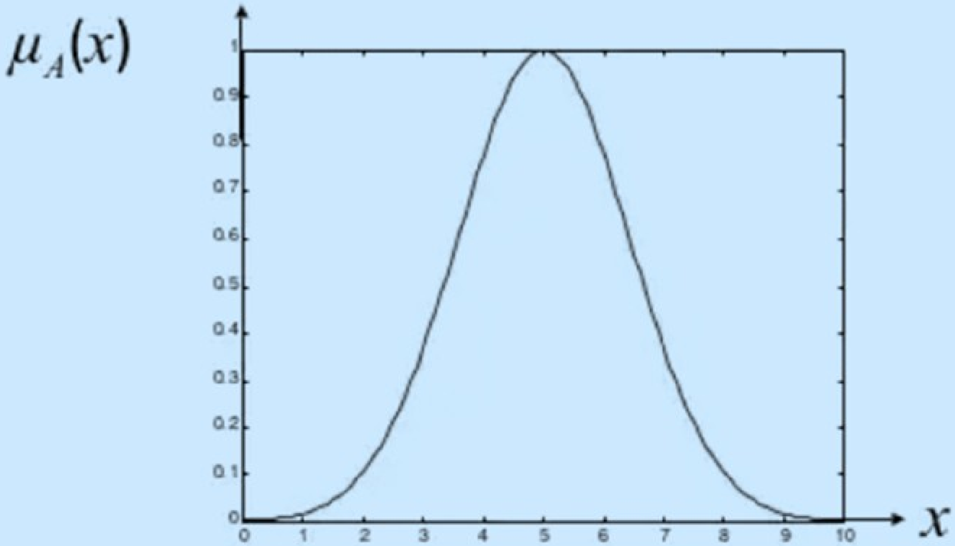
- **Trapezoid membership function**
- A *trapezoidal* membership function is specified by four parameters {a, b, c, d} as follows:

$$\mu_A(x) = \begin{cases} 0 & if\ x \leq a \\ \dfrac{x-a}{b-a} & if\ a \leq x \leq b \\ 1 & if\ b \leq x \leq c \\ \dfrac{d-x}{d-c} & if\ c \leq x \leq d \\ 0 & if\ d \leq x \end{cases}$$

- **Gaussian membership function**

$$\mu_A(x,c,s,m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

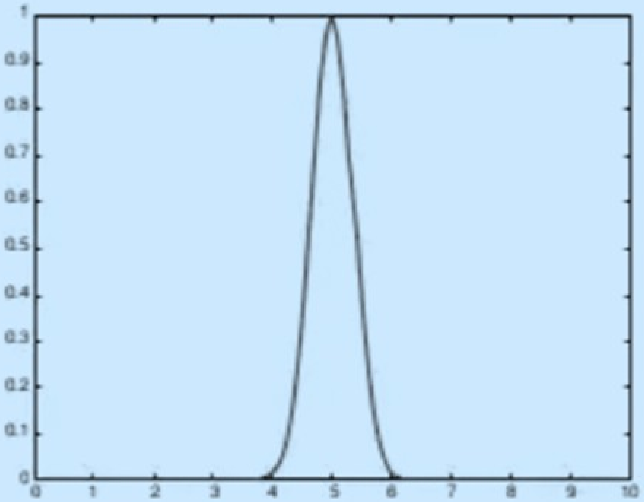
  - *c:* centre
  - *s:* width
  - *m:* fuzzification factor (e.g., *m*=2)



c=5

s=2

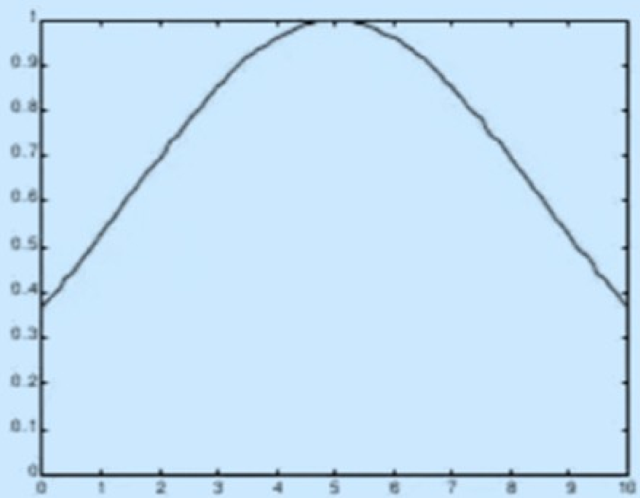
m=2



$c=5$

$s=0.5$

$m=2$



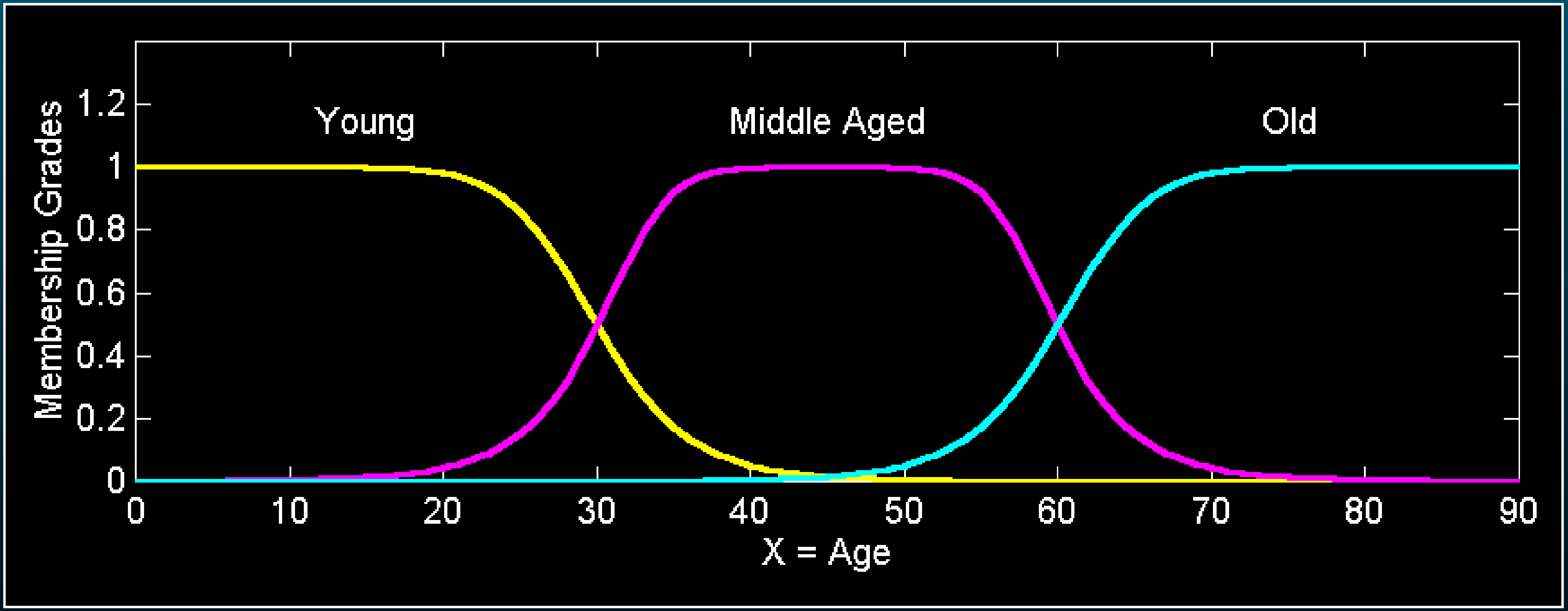
$c=5$

$s=5$

$m=2$

Fuzzy Partition

Fuzzy partitions formed by the linguistic values "young", "middle aged", and "old":



More Definitions

- Fuzzy set is uniquely specified by its MF
- To define membership functions more specifically

Support

Core

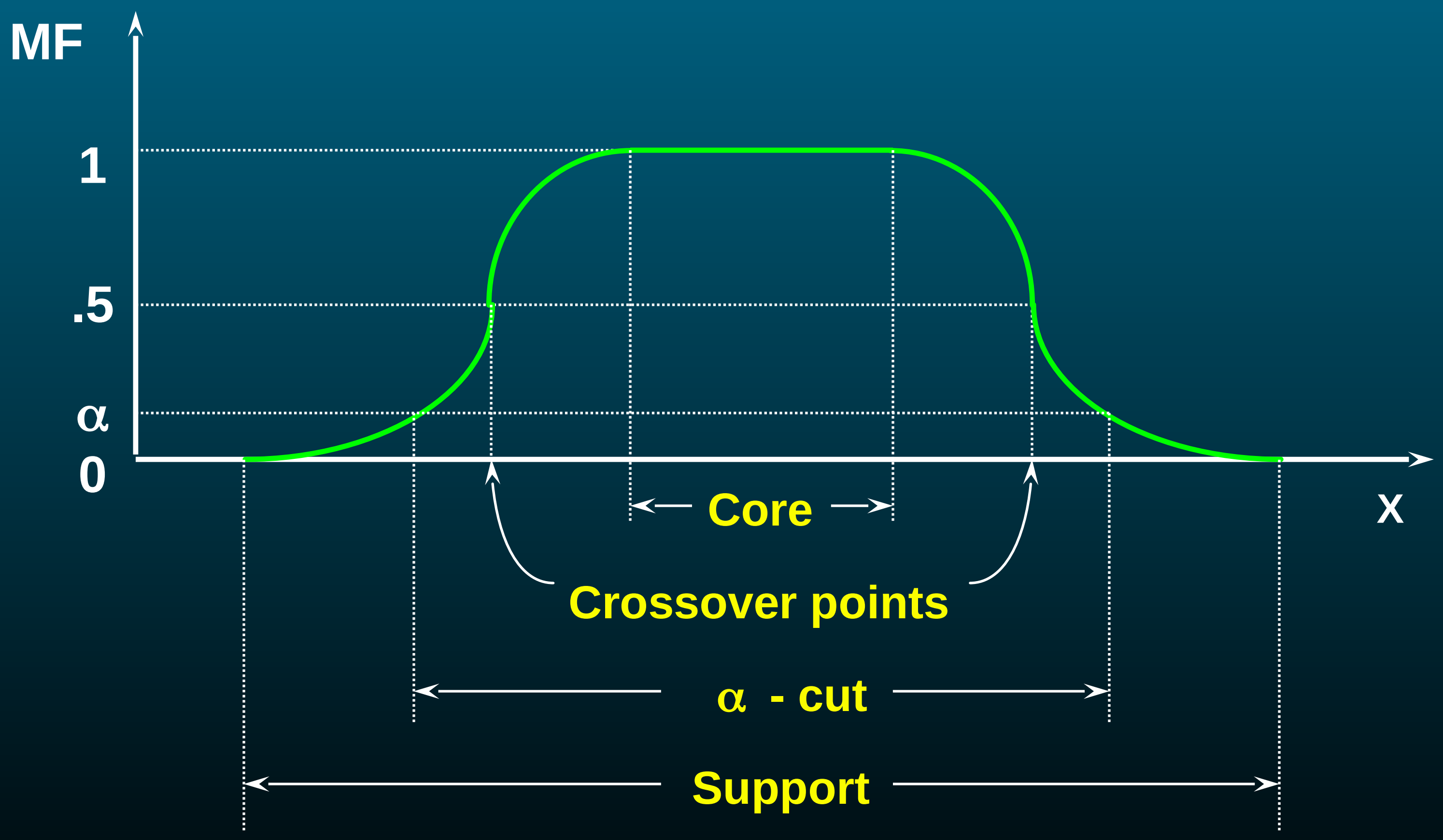
Normality

Crossover points

Fuzzy singleton

α-cut, strong α-cut

MF Terminology



MF Terminology (Cont'd)

The **support** of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$:

$$\text{support}(A) = \{x | \mu_A(x) > 0\}.$$

The **core** of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$:

$$\text{core}(A) = \{x | \mu_A(x) = 1\}.$$

A fuzzy set A is **normal** if its core is nonempty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.

A **crossover point** of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$:

$$\text{crossover}(A) = \{x | \mu_A(x) = 0.5\}.$$

A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a **fuzzy singleton**.

MF Terminology (Cont'd)

The **α-cut** or **α-level set** of a fuzzy set A is a crisp set defined by

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}.$$

Strong α-cut or **strong α-level set** are defined similarly:

$$A'_\alpha = \{x \mid \mu_A(x) > \alpha\}.$$

Let A be a fuzzy set.

$A = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 1.0), (x_5, 0.7), (x_6, 0.2)\}$

$\text{support}(A) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$\text{core} = \{x_4\}$

The α-cuts of the fuzzy set A are:

$A_{0.1} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$A_{0.2} = \{x_2, x_3, x_4, x_5, x_6\}$

$A_{0.5} = \{x_2, x_3, x_4, x_5\}$

$A_{0.7} = \{x_3, x_4, x_5\}$

$A_{0.8} = \{x_3, x_4\}$

$A_{1.0} = \{x_4\}$

The Strong α-cuts of the fuzzy set A are:

$A_{0.1} = \{ x_2, x_3, x_4, x_5, x_6\}$

$A_{0.2} = \{x_2, x_3, x_4, x_5\}$

$A_{0.5} = \{ x_3, x_4, x_5\}$

$A_{0.7} = \{x_3, x_4\}$

$A_{0.8} = \{x_4\}$

Set-Theoretic Operations

Subset:

$$A \subseteq B \Leftrightarrow \mu_A \leq \mu_B$$

Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

Union:

$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Let A and B be two fuzzy sets.

A = {(1, 0.1), (2, 0.5), (3, 0.8), (4, 1)} and

B = {(1, 1), (2, 0.8), (3, 0.4), (4, 0.1)}

Perform union and intersection operations on those fuzzy sets.

Answer:

A ∪ B = {(1,max(0.1, 1)), (2,max(0.5, 0.8)), (3,max(0.8, 0.4)), (4,max(1, 0.1))}

= {(1, 1), (2, 0.8), (3, 0.8), (4, 1)}

A∩B = {(1,min(0.1, 1)), (2,min(0.5, 0.8)), (3,min(0.8, 0.4)), (4,min(1, 0.1))}

= {(1, 0.1), (2, 0.5), (3, 0.4), (4, 0.1)}

Fuzzy System

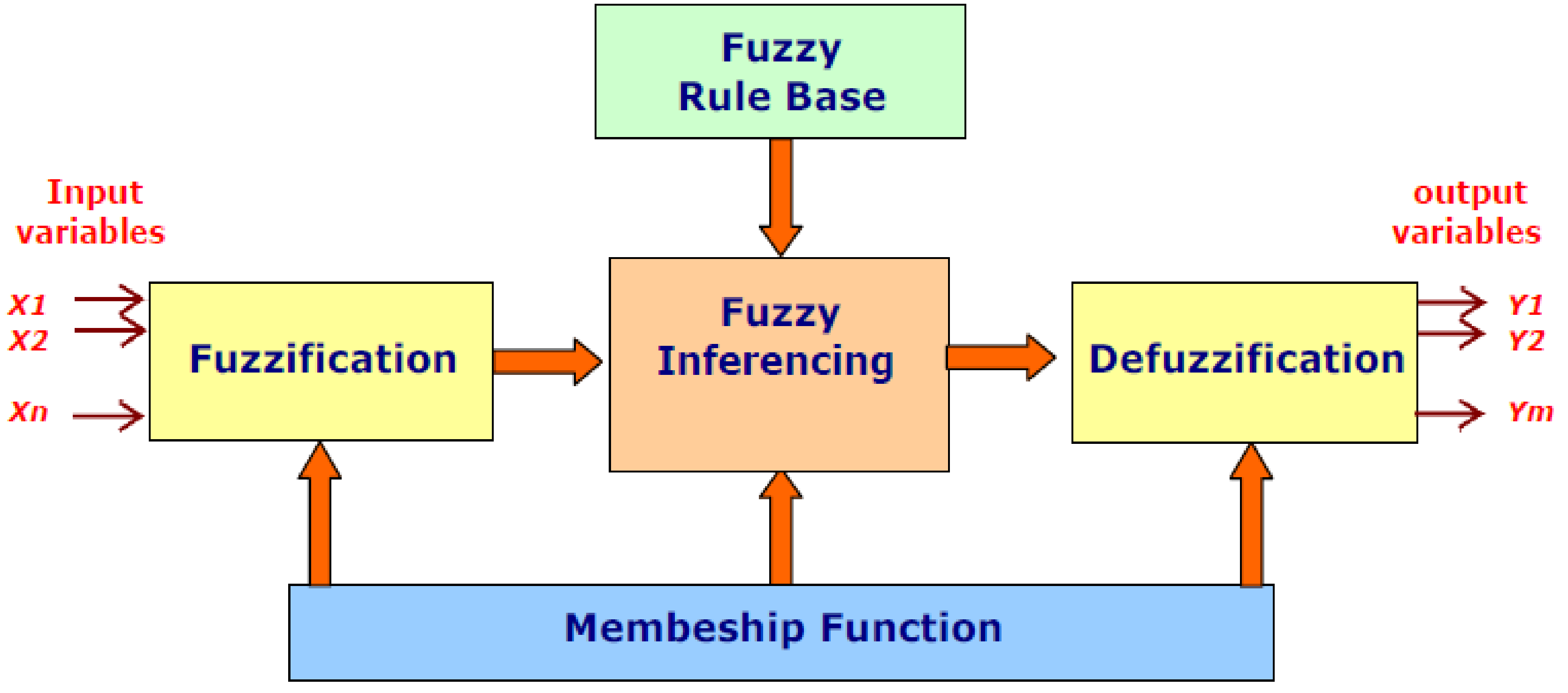


Fig. Elements of Fuzzy System

Fuzzy System elements

- **Input Vector** : $X = [x_1, x_2, \ldots x_n]^T$ are crisp values, which are transformed into fuzzy sets in the fuzzification block.
- **Output Vector** : $Y = [y_1, y_2, \ldots y_m]^T$ comes out from the defuzzification block, which transforms an output fuzzy set back to a crisp value.
- **Fuzzification** : a process of transforming crisp values into grades of membership for linguistic terms, "far", "near", "small" of fuzzy sets.
- **Fuzzy Rule base** : a collection of propositions containing linguistic variables; the rules are expressed in the form:

  If (x is A) AND (y is B) .... THEN (z is C)

  where **x, y** and **z** represent variables (e.g. distance, size) and **A, B** and **Z** are linguistic variables (e.g. \`far', \`near', \`small').
- **Membership function** : provides a measure of the degree of similarity of elements in the universe of discourse **U** to fuzzy set.
- **Fuzzy Inferencing** : combines the facts obtained from the Fuzzification with the rule base and conducts the Fuzzy reasoning process.
- **Defuzzyfication:** Translate results back to the real world values.

References:

J-S R Jang and C-T Sun, Neuro-Fuzzy and Soft Computing, Prentice Hall, 1997

S. Rajasekaran and G.A. Vijayalaksmi Pai ,Neural Network, Fuzzy Logic, and Genetic Algorithms - Synthesis and Applications, (2005), Prentice Hall.