Definition 2.1.1 (Linearly dependent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, a_n\}$ of vectors of V is said to be linear dependent if there exist scalar $a_1, a_2, \ldots, a_n \in F$ not all of them 0 (some of them may be zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$

Definition 2.1.2 (Linearly Independent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, a_n\}$ of vectors of V is said to be linearly independent if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$
$$a_i \in F, 1 \leq i \leq n \Rightarrow a_i = 0 \text{ for each} 1 \leq i \leq n$$

An infinite set of vector of V is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Example 2.1.3

Find whether the set of vector $v_i = (1,2,1)$, $v_2 = (3,1,5)$, $v_3 = (3,-4,7)$ is linearly independent or dependent.

Let a_1, a_2, a_3 be three scalars such that

$$\begin{aligned} a_1v_1 + a_2v_2 + a_3v_3 &= 0 \\ \Rightarrow a_1(1,2,1) + a_2(3,1,5) + a_3(3,-4,7) &= 0 \\ (a_1 + 3a_2 + 3a_3, 2a_1 + a_2 - 4a_3, a_1 + 5a_2 + 7a_3) &= 0 \end{aligned}$$

$$\begin{aligned} a_1 + 3a_2 + 3a_3 &= 0 \\ 2a_1 + a_2 - 4a_3 &= 0 \\ a_1 + 5a_2 + 7a_3 &= 0 \end{aligned}$$

The coefficients matrix of these equation is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix}$$

$$= 1(7+20) - 3(14+4) + 3(10-1) = 27 - 54 + 27 = 0$$

and

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$$

$$\therefore \rho(A) = 2$$

i.e., so *the rank of matrix* $A <$ *no. of unknown quantities*.
The system of equations will have $3 - 2 = 1$ non-zero solutions and hence the set of vectors are linearly dependent.

Problem 2.1.4

Show that the set $\{1, x, 1+x+x^2\}$ *is linearly independent set of vectors in the vector space of all polynomial over the real number filed.*

Let a_1, a_2, a_3 be scalars (real numbers) such that

$$a_1(1) + a_2(x) + a_3(1+x+x^2) = 0$$

We have

$$\begin{aligned} (a_1+a_3) + (a_2+a_3)x + a_3x^2 &= 0 \\ a_1+a_3 = 0, a_2+a_3 = 0, a_3 &= 0 \\ a_1 = 0, a_2 = 0, a_3 &= 0 \end{aligned}$$

Therefore the vectors $1, x, 1+x+x^2$ are linearly independent over the field of real numbers.

Example 2.1.5

Are the vectors $(2, 2, 2, 4), (2, -2, -4, 0), (4, -2, -5, 2), (4, 2, 1, 6)$ linearly independent?

Let a_1, a_2, a_3 and a_4 be four scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

Here

$\alpha_1 = (2, 2, 2, 4), \alpha_2 = (2, -2, -4, 0), \alpha_3 = (4, -2, -5, 2)$ and $\alpha_4 = (4, 2, 1, 6)$

$$\begin{aligned}
\therefore & a_1(2, 2, 2, 4) + a_2(2, -2, -4, 0) + a_3(4, -2, -5, 2) + a_4(4, 2, 1, 6) = 0 \\
& (2a_1 + 2a_2 + 4a_3 + 4a_4, 2a_1 - 2a_2 - 2a_3 + 2a_4, \\
& 2a_1 - 4a_2 - 5a_3 + a_4, 4a_1 + 2a_3 + 6a_4) = (0, 0, 0, 0)
\end{aligned}$$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 2 & -2 & -4 & 0 \\ 4 & -2 & -5 & 2 \\ 4 & 2 & 1 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - R - 1, R_3 \to R_3 - R_1$ and $R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -3 \\ 0 & -4 & -6 & -2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 3R_2$ and $R_4 \to -R_2$, we get

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

i.e., so the rank of matrix $A <$ number of unknown quantities.
The system of equations will have $4 - 2 = 2,$ non-zero solutions and hence the set of vectors are linearly dependent. Hence given vectors are not linearly independent.

Example 2.1.6

Show that the vectors (a_1, a_2) and (b_1, b_2) in $V_2(C)$ are L.D. iff $a_1b_2 - a_2b_1 = 0$, where C is the field complex numbers.

Let $a, b \in C$, then

$$a(a_1, a_2) + b(b_1, b_2) = 0$$
$$i.e., (aa_1 + bb_1, aa_2 + bb_2) = (0, 0)$$

$$\left.\begin{aligned} aa_1 + bb_1 &= 0 \\ aa_2 + bb_2 &= 0 \end{aligned}\right\} \tag{9}$$

The system of equations (9) will possess a non-zero solution iff

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \Rightarrow a_1b_2 - a_2b_1 = 0$$

Thus the given system of vectors is L.D. iff $a_1b_2 - a_2b_1 = 0$.