Definition 1.6.1 (Cayley Hamilton Theorem)

A matrix satisfies its own characteristic equation. That is, if the characteristic equation of an $n \times n$ matrix A is $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1A + a_0I = 0.$$

Problem 1.6.2

Verify Cayley Hamilton theorem for the following matrix A and hence find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Given

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

The characteristic equation

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$s_1 = 1 - 3 - 2 = -4$$

$$s_2 = \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}$$

$$= (6-1)(-2+2) + (-3-6) \Rightarrow 5 - 9 = -4$$

$$s_3 = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 1(6-1) - 2(-6-2) - 1(3+6) - 1(3+6) = 5 + 16 - 9 = 12$$

Then the characteristic equation is,

$$\lambda^3 + 4\lambda^2 - 4\lambda - 12 = 0$$

To verify the Cayley Hamilton theorem in characteristic replace λ by $'A'$, then we have

$$A^3 + 4A^2 - 4A - 12I = 0 \tag{4}$$

$$A.A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-2 & 2-6-1 & -1+2+2 \\ 3-9+2 & 6+9+1 & -3-3-2 \\ 2+3-4 & 4-3-2 & -2+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix}$$

$$A^3 = A^2.A$$

$$= \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix}$$

Let us substitute the value in equation (4)

$$A^3 + 4A^2 - 4A - 12I$$

$$= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + 4 \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 12 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + \begin{bmatrix} 20 & -20 & 12 \\ -20 & 64 & -32 \\ 4 & -4 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ 0 & 12 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4+20-4-12 & 28-20-8-0 & -16+12+4-0 \\ 28-16-12-0 & -64+64+12-12 & 36-32-4-0 \\ 4+4-8-0 & 8-4-4-0 & -8+12+8-12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, Cayley Hamilton theorem proved.

Consider the characteristic equation

$$A^3 + 4A^2 - 4A - 12I = 0$$

Multiply both side A^{-1}

$$\begin{aligned} A^3A^{-1} + 4A^2A^{-1} - 4AA^{-1} - 12IA^{-1} &= 0 \\ A^2 + 4A - 4I - 12A^{-1} &= 0 \\ A^2 + 4A - 4I &= 12A^{-1} \\ A^{-1} &= \frac{1}{12}\left[A^2 + 4A - 4I\right] \end{aligned}$$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5+4-4 & -5+8+0 & 3-4-0 \\ -4+12-0 & 16-12+4 & -8+4-0 \\ 1+8-0 & -1+4+0 & 3-8-4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5 & 3 & -1 \\ 8 & 0 & -4 \\ 9 & 3 & -9 \end{bmatrix}$$