Problem 2.4.1

Transforming a matrix $\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$ *to reduced row echelon form*

$$\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \qquad R_1 \to R_1 \times \frac{1}{5} \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \qquad R_2 \to R_2 + 2R_1 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix} \qquad R_3 \to R_3 + 5R_1 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix} \qquad R_2 \to \frac{5}{21} R_2 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix} \quad R_3 \to \frac{-3}{14} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_2 \to R_2 - \frac{2}{3} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_1 \to R_1 - \frac{8}{5} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_1 \to R_1 - \frac{7}{5} R_2 \quad \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{17}{49} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

Example 2.4.2

Find column space, row space, null space and kernel of

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}.$$

Step (1): Finding $rref(A)$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_1 \to \frac{-1}{3} R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_2 \to R_2 + 2R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix} R_3 \to R_3 + 3R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix} R_2 \to \frac{3}{8} R_2 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & \frac{2}{3} & \frac{7}{3}\\ 0 & 0 & 1 & \frac{5}{4}\\ 0 & 0 & -4 & -5\end{bmatrix} R_3 \to R_3 + 4R_2 \Rightarrow \begin{bmatrix}1 & -3 & \frac{2}{3} & \frac{7}{3}\\ 0 & 0 & 1 & \frac{5}{4}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & \frac{2}{3} & \frac{7}{3}\\ 0 & 0 & 1 & \frac{5}{4}\\ 0 & 0 & 0 & 0\end{bmatrix} R_1 \to R_1 - \frac{-2}{3} R_2 \Rightarrow \begin{bmatrix}1 & -3 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & \frac{5}{4}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}-3 & 9 & -2 & -7\\ 2 & -6 & 4 & 8\\ 3 & -9 & -2 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -3 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & \frac{5}{4}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

To identify row space

$$\begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B_{RS} = \left\{ \begin{pmatrix} 1 \\ -3 \\ 0 \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{5}{4} \end{pmatrix} \right\}$$

To identify column space

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B_{CS} = \left\{ \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} \right\}$$

Check you work

Note: $CS * RS = A$

$$\begin{bmatrix} -3 & -2 \\ 2 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

You can extract the null space quickly by changing the sign of the non-pivot element and adding a pivot where the pivot would line up to an identity matrix but this is how to compute it.

To find Null space and Kernel

The 'Null Space' is the solution to $Ax = 0$.

$$\begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 3x_2 + \frac{3}{2}x_4 &= 0 \\ free : x_2 &= x_2 \\ x_3 + \frac{5}{4}x_4 &= 0 \\ free : x_4 &= x_4 \end{aligned}$$

$$
\begin{aligned}
x_1 &= 3x_2 - \frac{3}{2}x_4 \\
x_2 &= x_2 + 0x_4 \\
x_3 &= 0x_2 - \frac{5}{4}x_4 \\
x_4 &= 0x_2 + x_4
\end{aligned}
$$

$$
x = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} \frac{-3}{2} \\ 0 \\ \frac{-5}{4} \\ 1 \end{pmatrix} x_4,
$$

$$x_2 = 1 \wedge x_4 = 4$$

$$Kernal = B_{NS} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ -5 \\ 4 \end{pmatrix} \right\}$$

Check your work $A * NS = 0;$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 1 & -5 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$Nullspace = \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$