Verification of Stoke's theorem

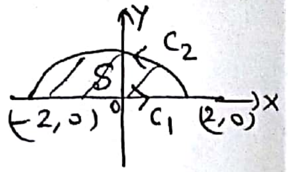
$$\oint_C \bar F\cdot d\bar r = \iint_S (\text{curl}\,\bar F\cdot\hat n)\,ds$$

$$= \iint_R [\text{curl}\,\bar F(\bar r(u,v))\cdot \bar N]\,du\,dv$$

where $\bar N = \bar r_u\times\bar r_v$
$= -(\bar r_v\times \bar r_u)$.

Q① $\bar F=[y^2,-x^2,0]$, $S: x^2+y^2\le 4,\ y\geq 0,\ z=0$.

Sol: R.H.S.



$\bar r(u,v)=[v\cos u,\ v\sin u,\ 0]$
$0\le v\le 2,\ 0\le u\le\pi$

$\bar N = \bar r_u\times\bar r_v = [0,0,-v]$

$\text{curl}\,\bar F = [0,-2x,-2y]$

$\text{curl}\,\bar F(\bar r(u,v)) = [0,-2v\cos u,-2v\sin u]$

$\text{curl}\,\bar F(\bar r(u,v))\cdot\bar N = +2v^2\sin u$

Now,

$$\iint_S (\text{curl}\,\bar F\cdot\hat n)\,ds = \int_{v=0}^{2}\int_{u=0}^{\pi} 2v^2\sin u\,du\,dv$$

$$= \int_{v=0}^{2} 2v^2\,[-\cos u]_{u=0}^{\pi}\,dv$$

$$= 4\left[\frac{v^3}{3}\right]_0^2$$

$$= \frac{32}{3}.$$

The R.H.S. could be $-\frac{32}{3}$ also if one takes $\bar N = \bar r_v\times\bar r_u$.

L.H.S. Let $C = C_1\cup C_2$

$C_1:\ \bar r(t)=[t,0,0],\ -2\le t\le 2$

$\bar F(\bar r(t)) = [0,-t^2,0]$, $\frac{d\bar r}{dt} = [1,0,0]$

$$\int_{C_1}\bar F\cdot d\bar r = \int_{t=-2}^{2} \bar F(\bar r(t))\cdot\frac{d\bar r}{dt}\,dt$$

$$= 0$$

$C_2:\ \bar r(t) = [2\cos t, 2\sin t, 0],\ 0\le t\le\pi$

$\therefore\ \frac{d\bar r}{dt} = [-2\sin t, 2\cos t, 0]$

$\bar F(\bar r(t)) = [4\sin^2 t,\ -4\cos^2 t,\ 0]$

$$\int_{C_2}\bar F\cdot d\bar r = \int_{t=0}^{\pi}\bar F(\bar r(t))\cdot\frac{d\bar r}{dt}\,dt$$

$$= \int_{t=0}^{\pi}(-8\sin^3 t - 8\cos^3 t)\,dt$$

$$= \int_{t=0}^{\pi}\left(\frac{3\sin x-\sin 3x}{4} + \frac{3\cos x+\cos 3x}{4}\right)(-8)\,dt$$

$$= -2\int_{t=0}^{\pi}(3\sin t - \sin 3t)\,dt + 0$$

$$= -2\left[-3\cos t + \frac{\cos 3t}{3}\right]_{t=0}^{\pi}$$

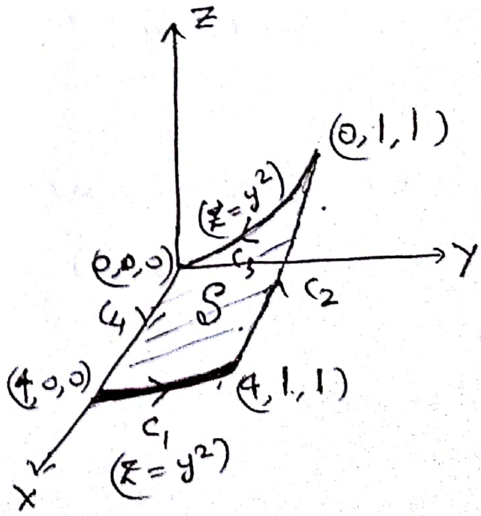
$$= -2\left[6-\frac{2}{3}\right] = -\frac{32}{3}.$$

Now, L.H.S. = R.H.S.

Verified.

Q② $\bar F = [e^{2z},\ e^z\sin y,\ e^z\cos y]$

$S:\ z=y^2,\ 0\le x\le 4,\ 0\le y\le 1.$



L.H.S. Let $C = C_1\cup C_2\cup C_3\cup C_4$

$C_1:\ \bar r(t) = [4, t, t^2],\ 0\le t\le 1$

$\frac{d\bar r}{dt} = [0,1,2t]$

$\bar F(\bar r(t)) = [e^{2t^2},\ e^{t^2}\sin t,\ e^{t^2}\cos t]$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} \{e^{t^2}\sin t + 2t\, e^{t^2}\cos t\}\, dt \quad \text{---} \; (1)$$

C_2: $\bar{r}(t) = [t, 1, 1]$, & t from '4' to '0'

$$\bar{F}(\bar{r}(t))\cdot \frac{d\bar{r}}{dt} = [e^2, e\sin 1, e\cos 1]\cdot[1,0,0]$$
$$= e^2$$

$$\therefore \int_{C_2} \bar{F}\cdot d\bar{r} = \int_{t=4}^{0} e^2\, dt = -4e^2 \; \text{---} \; (2)$$

C_3: $\bar{r}(t) = [0, t, t^2]$, t from '1' to '0'

$$\bar{F}(\bar{r}(t)) = [e^{2t^2}, e^{t^2}\sin t, e^{t^2}\cos t]$$

$$\bar{F}(\bar{r}(t))\cdot \frac{d\bar{r}}{dt} = e^{t^2}\sin t + 2t\, e^{t^2}\cos t$$

$$\therefore \int_{C_3} \bar{F}\cdot d\bar{r} = \int_{t=1}^{0} (e^{t^2}\sin t + 2t\, e^{t^2}\cos t)\, dt$$

$$= -\int_{t=0}^{1} (e^{t^2}\sin t + 2t\, e^{t^2}\cos t)\, dt \quad \text{---} \; (3)$$

C_4: $\bar{r}(t) = [t, 0, 0]$, $0 \le t \le 4$,

$$\bar{F}(\bar{r}(t)) = [1, 0, 1], \quad \frac{d\bar{r}}{dt} = [1,0,0]$$

$$\int_{C_4} \bar{F}\cdot d\bar{r} = \int_{t=0}^{4} 1\, dt = 4. \quad \text{---} \; (4)$$

From (1), (2), (3) and (4),

$$\oint_C \bar{F}\cdot d\bar{r} = -4e^2 + 4 = 4(1-e^2).$$

R.H.S.

$\bar{r}(u,v) = [u, v, v^2]$, $0 \le u \le 4$, $0 \le v \le 1$

$$\bar{N} = \bar{r}_u \times \bar{r}_v = [0, -2v, 1]$$

$$\text{curl}\, \bar{F} = [-e^z(\sin y + \cos y), 2e^{2z}, 0]$$

$$\iint_S (\text{curl}\, \bar{F}\cdot \hat{n})\, ds$$

$$= \iint_R [\text{curl}\, \bar{F}(\bar{r}(u,v))\cdot \bar{N}]\, du\, dv$$

$$= \int_{v=0}^{1}\int_{u=0}^{4} [-e^{v^2}(\sin v + \cos v), 2e^{2v^2}, 0] \cdot [0, -2v, 1]\, du\, dv$$

$$= \int_{v=0}^{1}\int_{u=0}^{4} -4v\, e^{2v^2}\, du\, dv$$

$$= -8\int_{v=0}^{1} 2v\, e^{2v^2}\, dv$$

$$= -8 \left[\frac{e^{2v^2}}{2}\right]_{v=0}^{1}$$

$$= -4(e^2 - 1)$$

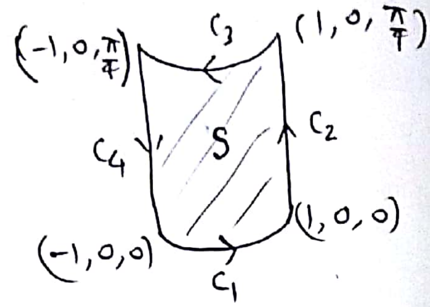
$$= 4(1-e^2).$$

$\therefore$ L.H.S. = R.H.S.

Verified.

Q③ $\bar{F} = [0, 0, x\cos 2z]$.

$S: x^2+y^2=1,\ y \geq 0,\ 0 \leq z \leq \frac{\pi}{4}$.



L.H.S. Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$

$C_1: \bar{r}(t) = [\cos t, \sin t, 0],\quad 0 \leq t \leq \pi$

$$\frac{d\bar{r}}{dt} = [-\sin t, \cos t, 0]$$

$$\bar{F}(\bar{r}(t)) = [0, 0, \cos t \cos(0)] = [0, 0, \cos t].$$

$$\therefore \int_{C_1} \bar{F}\cdot d\bar{r} = \int_{t=0}^{\pi} (0)\, dt = 0.$$

$C_2: \bar{r}(t) = [1, 0, t],\quad 0 \leq t \leq \frac{\pi}{4}$

$$\frac{d\bar{r}}{dt} = [0, 0, 1]$$

$$\bar{F}(\bar{r}(t)) = [0, 0, \cos(2t)]$$

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{t=0}^{\pi/4} \cos 2t\, dt = \frac{1}{2}.$$

$C_3: \bar{r}(t) = [\cos t, -\sin t, \frac{\pi}{4}],\quad 0 \leq t \leq \pi$

[Pl. Note: Sense of rotation is clockwise that is why rep. is different from C_1.]

$$\frac{d\bar{r}}{dt} = [-\sin t, -\cos t, 0],\quad 0 \leq t \leq \pi$$

$$\bar{F}(\bar{r}(t)) = [0, 0, \cos t \cos(\tfrac{\pi}{2})] = [0, 0, 0]$$

$$\int_{C_3} \bar{F}\cdot d\bar{r} = \int_{t=0}^{\pi} (0)\, dt = 0.$$

$C_4: \bar{r}(t) = [-1, 0, t]$, t from '$\frac{\pi}{4}$' to '0'

$$\frac{d\bar{r}}{dt} = [0, 0, 1]$$

$$\bar{F}(\bar{r}(t)) = [0, 0, -\cos 2t]$$

$$\int_{C_4} \bar{F}\cdot d\bar{r} = \int_{t=\pi/4}^{0} -\cos 2t\, dt = \int_0^{\pi/4} \cos 2t\, dt = \frac{1}{2}.$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \frac{1}{2} + 0 + 0 + \frac{1}{2} = 1.$$

R.H.S.

$\bar{r}(u,v) = [\cos u, \sin u, v]\quad 0 \leq u \leq \pi,\ 0 \leq v \leq \frac{\pi}{4}$

$$\bar{N} = \bar{r}_u \times \bar{r}_v = -(\bar{r}_v \times \bar{r}_u) = [\cos u, \sin u, 0]$$

$$\text{curl } \bar{F} = [0, -\cos 2z, 0]$$

$$\text{Curl } \bar{F}(\bar{r}(u,v)) = [0, -\cos 2v, 0]$$

$$\iint_S (\text{curl } \bar{F}\cdot \hat{n})\, ds = \iint_R [\text{curl } \bar{F}(\bar{r}(u,v))\cdot \bar{N}]\, du\, dv$$

$$= \int_{u=0}^{\pi}\int_{v=0}^{\pi/4} -\sin u \cos 2v\, dv\, du$$

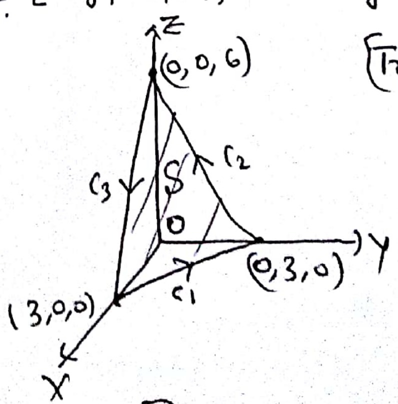
$$= -1.$$

R.H.S. could also be 1 if we take $\bar{N} = \bar{r}_v \times \bar{r}_u$.

$\therefore$ L.H.S. = R.H.S.

Verified.

Q④ $\bar{F} = [-y^2, z, x]$, $S: 2x+2y+z=6$



[Try yourself]

Ans: -9