Ex 1. Minimize $u_1^2 + u_2^2 + u_3^2$

Subject to $u_1 + u_2 + u_3 \geq 100$

$u_1, u_2, u_3 \geq 0$.

→ Decision variables are $u_1, u_2, u_3$

State variables are $x_3 = u_1 + u_2 + u_3 \geq 100$

$x_2 = u_1 + u_2 = x_3 - u_3$

$x_1 = u_1 = x_2 - u_2$

State transformation functions are

$F_3(x_3) = \min_{u_3} (u_3^2 + F_2(x_2))$

$F_2(x_2) = \min_{u_2} (u_2^2 + F_1(x_1))$

$F_1(x_1) = u_1^2 = (x_2 - u_2)^2$

So, $F_2(x_2) = \min_{u_2} (u_2^2 + (x_2 - u_2)^2)$

To minimize, $(u_2^2 + (x_2 - u_2)^2)$,

differentiating it w.r.t $u_2$

i.e. $2u_2 - 2(x_2 - u_2) = 0 \Rightarrow u_2 = x_2/2$

$\therefore F_2(x_2) = \frac{x_2^2}{2}$.

Now $F_3(x_3) = \min_{(u_3)} (u_3^2 + F_2(x_2))$.

$= \min_{(u_3)} \left(u_3^2 + \frac{x_2^2}{2}\right)$.

$= \min_{u_3} (u_3^2 + (x_3 - u_3)^2/2)$

For minimum, $2u_3 - 2(x_3 - u_3)\cdot\frac{1}{2} = 0$

$\Rightarrow u_3 = x_3/3$

$\therefore F_3(x_3) = x_3^2/3, \quad x_3 \geq 100$

Obiviously, $F_3(x_3)$ is the least for $x_3 = 100$

So, minimum of $u_3 = 100/3$

$$u_2 = \frac{1}{2}\left(x_3 - \frac{x_3}{3}\right)$$

$$= \frac{x_3}{3} = \frac{100}{3}$$

$$u_1 = \frac{2 \times 100}{3} - \frac{100}{3} = \frac{100}{3}$$

$\therefore$ $u_1 = u_2 = u_3 = \frac{100}{3}$

$\therefore$ $\text{Min}\ (u_1^2 + u_2^2 + u_3^2) = \frac{100^2}{3^2} + \frac{100^2}{3^2} + \frac{100^2}{3^2}$

$$= \frac{3 \cdot 100^2}{3^2} = \frac{100^2}{3}$$

Prob 1. $\text{Min}\ z = y_1^2 + y_2^2 + y_3^2$

s.t. $y_1 + y_2 + y_3 \geq 15$

$y_1, y_2, y_3 \geq 0$. (Ans: $y_1 = y_2 = y_3 = 5$
$Z_{min} = 75$)

Prob. 2. Determine the value of $u_1, u_2, u_3$ so as to maximize $(u_1 u_2 u_3)$ Subject to $u_1 + u_2 + u_3 = 10$, $u_1, u_2, u_3 \geq 0$.
(Ans: $u_1 = u_2 = u_3 = \frac{10}{3}$
$Z_{max} = \frac{1000}{27}$)

Solution of LPP by Dynamic Programming problem:

The linear programming problem in the general form is

$\text{Max}\ Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

s.t. $a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$

$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$

$\vdots$

$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$

$x_1, x_2, \cdots x_n \geq 0$.

This problem involving $m$ resources and $n$ decision variables can be formulated as as dynamic programming problem as follows.

Each activity $j$ $(j=1,2,\dots n)$ is considered a stage. The problem can be regarded as $n$ stage problem and decision variables are level of activities $x_j$. Since $x_j$ is $x_j\ (\geq 0)$ at stage $j$. Since continuous, each activity has infinite no of alternatives within the feasible space.

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Allocation problems are particular type of LP problems that require allocation of available resources. The constants $b_1, b_2, \dots b_m$ are amounts of available resources.

Let $f_n(b_1, b_2 \dots b_m)$ be the optimal value of the objective function defined above for stages $x_1, x_2, \dots x_n$ for states $b_1, b_2 \dots b_m$. Using forward computational procedure, the recursive eqn can be written as

$$f_j(b_1, b_2 \dots b_m) = \max_{0 \leq x_j \leq b} \left[ c_j x_j + f_{j-1}(b_1 - a_{1j}x_j,\ b_2 - a_{2j}x_j, \dots b_m - a_{mj}x_j) \right].$$

ie maximum value ~~b~~ of $b$ that $x_j$ can ~~sto~~ assume

$$b = \min\left[\frac{b_1}{a_{1j}}, \frac{b_2}{a_{2j}}, \dots \frac{b_m}{a_{mj}}\right]$$

Ex 1. Use dynamic programming to solve the following LPP.

$$\text{Max } Z = 3x_1 + 5x_2$$

S.t. $x_1 \le 4$

$x_2 \le 6$

$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$.

→ There are two variables, and hence the problem can be treated as a two stage dynamic programming problem. Both $x_1$ and $x_2$ being continuous, represent the infinite no of alternatives within the feasible space. The three constraints can be regarded as three resources, say $b_1, b_2, b_3$ which are to be allocated to $x_1$ and $x_2$ at different stages.

Stage 1.

The optimal value of $f_1(b_1, b_2, b_3)$ at stage 1 is given by

$$f_1(b_1, b_2, b_3) = \max_{0 \le x_1 \le b} [3x_1]$$

where $b_1 = 4$, $b_2 = 6$, $b_3 = 18$.

The feasible value of $x_1$ is non negative and satisfies all the three constraints.

But the maximum value of b that $x_1$ can assume is $= \min\left(\frac{4}{1}, \frac{6}{0}, \frac{18}{3}\right)$

$= 4$

$$\therefore f_1(4, 6, 18) = \max_{0 \le x_1 \le 4} [3x_1]$$

$$= 3 \min\left[4, \frac{18 - 2x_2}{3}\right]$$

where $x_1^* = \min\left[4, \frac{18-2x_2}{3}\right] = 4$

Stage 2

The recursive eqn for optimization of this two stage problem is

$$f_2(4, 6, 18) = \max_{0 \le x_2 \le b} \left[5x_2 + 3x_1\right]$$

$$= \max_{0 \le x_2 \le b} \left[5x_2 + 3\min\left(4, \frac{18-2x_2}{3}\right)\right]$$

But the maximum value of b that $x_2$ can assume is $\min\left(\frac{4}{0}, \frac{6}{1}, \frac{18}{2}\right) = 6$

$$\therefore f_2(4, 6, 18) = \max_{0 \le x_2 \le 6} \left[5x_2 + 3\min\left(4, \frac{18-2x_2}{3}\right)\right]$$

$$\min\left(4, \frac{18-2x_2}{3}\right) = \begin{cases} 4, & \text{if } 0 \le x_2 \le 3 \\ \frac{18-2x_2}{3}, & \text{if } 3 < x_2 \le 6 \end{cases}$$

$$\therefore 5x_2 + 3\min\left(4, \frac{18-2x_2}{3}\right) = \begin{cases} 5x_2 + 12, & \text{if } 0 \le x_2 \le 3 \\ 3x_2 + 18, & \text{if } 3 \le x_2 \le 6 \end{cases}$$

$$f_2(4, 6, 18) = \max \begin{cases} 27, & \text{at } x_2 = 3 \\ 36, & \text{at } x_2 = 6 \end{cases}$$

$$= 36$$

$x_1^* = \min\left[4, \frac{18-2x_2}{3}\right]$

$x_2^* = 6$, $Z_{max} = 36$, $\quad = \min\left\{4, \frac{18-12}{3}\right\} = 2$

Prob 1. Solve the LPP by the method of dynamic programming problem.

$$\text{Max } Z = 2x_1 + 5x_2,$$

$$\text{s.t.} \quad 2x_1 + x_2 \le 430$$

$$2x_2 \le 460$$

$$x_1, x_2 \geq 0$$