Assignment Problems

In practical field, we sometimes face with a type of problem which consists in assigning men to offices, jobs to machines, classes in a school to rooms or problems to different research teams etc in which the assignees ~~progress~~ posses varying degree of efficiency, called cost of effectiveness. The basic assumption of this problem is that one person can perform one job at a time. An assignment plan is optimal if it minimizes the total cost of effectiveness or maximizes the profit of performing all the jobs.

		Facilities					
		1	2	3	- - -	m	a_i
Jobs	1	c_{11}	c_{12}	c_{13}	- - -	c_{1m}	1
	2	c_{21}	c_{22}	c_{23}	- - -	c_{2m}	1
	3						
	⋮						⋮
	m	c_{m1}	c_{m2}	c_{m3}	- - -	c_{mm}	1
	b_j	1	1	1	1 - - -	1	m

If c_{ij} be the cost of assigning ith job to the jth facility, then we can represent the cost of effectiveness matrix in the above tableau form.

The tableau represents that only one unit is available for one facility. The job assignment is to be made in such a way that each job can be associated with one and only facility.

Mathematical formulation:

Determine $x_{ij} \geq 0$, $i, j = 1, 2 \cdots m$ which optimizes the total cost

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}$$

Subject to $\sum_{j=1}^{m} x_{ij} = 1, \quad i = 1, 2, \cdots m. \quad \text{--- (1)}$

$\sum_{i=1}^{m} x_{ij} = 1, \quad j = 1, 2, \cdots m \quad \text{--- (2)}$

The requirement $x_{ij} \geq 0$ in the assignment has the form

$$x_{ij} \begin{cases} 1, & \text{if ith job be assigned to jth facility} \\ 0, & \text{otherwise.} \end{cases}$$

Constraints (1) assures that only one job is assigned to a person and the constraints (2) ensure that only person should be assigned with one job.

Solution of assignment problem:

Theorem: If a constant be added to any row and/or any column of the cost matrix of an assignment problem, then the resulting assignment problem has the same optimal solution as the original problem.

Theorem: If all $c_{ij} \geq 0$ and we can find a set $x_{ij} = x^*_{ij}$ such that $\sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x^*_{ij} = 0$ (minimization)

then this solution is optimal.

Theorem: If K be the maximum no. of zeros which can be assigned, then there exists a set of K lines which will cover all the zeros.

Compututional procedure of Hungarian Method : reductions.

Phase 1 : Row and column reductions

Step 1 → Subtract minimum value of each element from the ~~ent~~ each entries of that row.

Step 2 → Subtract the minimum value of each column from the entries of that column.

Phase 2 : Optimization of the problem:

Step 1 → Draw a minimum no of lines to cover all the zeros of the matrix.

Procedure

(a) Row scanning : (i) Starting from the first row, ask the following question. Is there exactly one zero in that row? If yes, mark a square around that zero entry and draw a vertical line passing through the zero otherwise skip that row.

(ii) After scanning the last row, check whether all the zeros are covered with lines. If yes, go to step 2, otherwise do column scanning.

(b) Column scanning: (i) Start from the first column. ask the following question, Is there exactly one zero in that column, If yes, mark a square around the zero entry and draw a horizontal line passing through that zero otherwise skip that column.

(ii) After scanning the last column, check whether all the zeros are covered with lines.

Step 2 → Check whether the no. of square marked is equal to the no. of rows of the matrix. If yes, go to step 5, otherwise go to step 3.

Step 3 → Identify the minimum value of the undeleted cell values.

a) add the minimum undeleted cell value at the intersection points of the present matrix.

b) Subtract the minimum undeleted cell values from the all undeleted cell values.

c) all the other entries remain same

Step 4 → Go to Step 1.

Step 5 → Treat the solution as marked by the squares of the original matrix.

Ex. Solve the following assignment problem using Hungarian method, the matrix entry represents the processing time in hours.

Operator

		1	2	3	4	5
Jobs	1	9	11	14	11	7
	2	6	15	13	13	10
	3	12	13	6	8	8
	4	11	9	10	12	9
	5	7	12	14	10	14

Phase 1

Step 1 →

	1	2	3	4	5	Row minimum
→ 1	9	11	14	11	7	7
2	6	15	13	13	10	6
3	12	13	6	8	8	6
4	11	9	10	12	9	9
5	7	12	14	10	14	7

Step 2 →

	1	2	3	4	5
1	2	4	7	4	0
2	0	9	7	7	4
3	6	7	0	2	2
4	2	0	1	3	0
5	0	5	7	3	7
Column min →	0	0	0	2	0

(Subtracting row minimum from each row)

Step3 →

	1	2	3	4	5
1	2	4	7	2	0
2	0	9	7	5	4
3	6	7	0	0	2
4	2	0	1	1	0
5	0	5	7	1	7

(Subtracting column min from each column)

Scanned with CamScanner

Phase 2

Step 1 →

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 2 & 4 & 7 & 2 & \boxed{0} \\
2 & \boxed{0} & 9 & 7 & 5 & 4 \\
3 & 6 & 7 & \boxed{0} & 0 & 2 \\
4 & 2 & \boxed{0} & 0 & 1 & 0 \\
5 & 0 & 5 & 7 & 1 & 7
\end{array}
$$

(Row Scanning & Column Scanning)

4 squares ≠ 5 rows / columns

optimality has no reached.

Step 2 →

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 2 & 4 & 6 & 1 & \boxed{0} \\
2 & \boxed{0} & 9 & 6 & 4 & 4 \\
3 & 7 & 8 & \boxed{0} & 0 & 3 \\
4 & 2 & \boxed{0} & 0 & 0 & 0 \\
5 & 0 & 5 & 6 & \boxed{0} & 7
\end{array}
$$

~~Step 3~~ No of Squares = 5 = no of rows.

Solution is optimal

Job	Operator	Time
1	→ 5	7
2	→ 1	6
3	→ 3	6
4	→ 2	9
5	→ 4	10

Total processing hours = 7+6+6+9+10

= 38 units