Definition 2.6.1 (Matrices of linear transformations)

We will now take a more algebraic approach to transformations of the plane. As it turns out, matrices are very useful for describing transformations. Whenever we have a 2×2 matrix of real numbers

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we can naturally define a plane transformatiom $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_M(v) = Mv.$$

That is, T_M takes a vector v and multiplies it on the left by the matrix M. If v is the position vector of the point (x, y), then

$$T_M(v) = T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

or equivalently, $T_M(x, y) = (ax + by, cx + dy)$.

Problem 2.6.2

Let

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

1. *Write an expression for* T_M.
2. *Find* $T_M(1,0)$ *and* $T_M(0,1)$.
3. *Find all points* (x,y) *such that* $T_M(x,y) = (1,0)$.

(1) $T_M(x,y) = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+7y \end{bmatrix} = (x+2y, 3x+7y)$.

(2) Using the formula from the previous part,

$$T_M(1,0) = (1,3) \text{ and } T_M(0,1) = (2,7).$$

(3) We have $T_M(x,y) = (x+2y, 3x+7y) = (1,0)$, hence the simultaneous equations

$$x + 2y = 1, 3x + 7y = 0.$$

Solving these equations yields $x = 7, y = -3$; and this is the only solution. So the only point (x,y) such that $T_M(x,y) = (1,0)$ is $(x,y) = (7,-3)$.

Definition 2.6.3 (Linear transformation)

A plane transformation F is linear if either of the following equivalent conditions holds:

1. $F(x,y) = (ax+by, cx+dy)$ for some real a,b,c,d. That is, F arises from a matrix.
2. For any scalar c and vectors v,w, $F(cv) = cF(v)$ and $F(v+w) = F(v)+F(w)$.

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Theorem 2.6.4

For any matrices M and N, $T_M \circ T_N = T_{MN}$.

Problem 2.6.5

Find the matrix for the composition $g \circ f$ of the two linear transformations $f(x,y) = (x+y,y)$ and $g(x,y) = (y,x+y)$.

We have $f = T_M$ and $g = T_N$ where $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. So the matrix of the composition $g \circ f = T_N \circ T_M = T_{NM}$ is the product NM :

$$NM = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Problem 2.6.6

What is the inverse of the transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $F(x, y) = (x + 3y, x + 5y)$*?*

The transformation F is linear and corresponds to the matrix

$$M = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix},$$

which has inverse

$$M^{-1} = \frac{1}{1.5 - 3.1} \begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

The inverse of $F = T_M$ is then $F^{-1} = T_{M^{-1}}$,

$$F^{-1}(x, y) = \left(\frac{5}{2}x - \frac{3}{2}y, \frac{-1}{2}x + \frac{1}{2}y \right).$$

Problem 2.6.7

Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ *that maps* $(1,1)$ *to* $(-1,4)$ *and* $(-1,3)$ *to* $(-7,0)$.

Let M be the matrix of the desired linear transformation. We have

$$M\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix} \text{ and } M\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-7\\0\end{bmatrix}.$$

In fact, we can put these two equations together into a single matrix equation

$$M\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix} = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}$$

which we can then solve for M:

$$M = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4 & -8\\12 & 4\end{bmatrix} = \begin{bmatrix}1 & \\3 & \end{bmatrix}$$

Hence the only such transformation is $T_M(x,y) = (x-2y, 3x+y)$.

Example 2.6.8

Find the linear transformation that sends $(3, 1)$ to $(1, 2)$ and $(-1, 2)$ to $(2, -3)$.