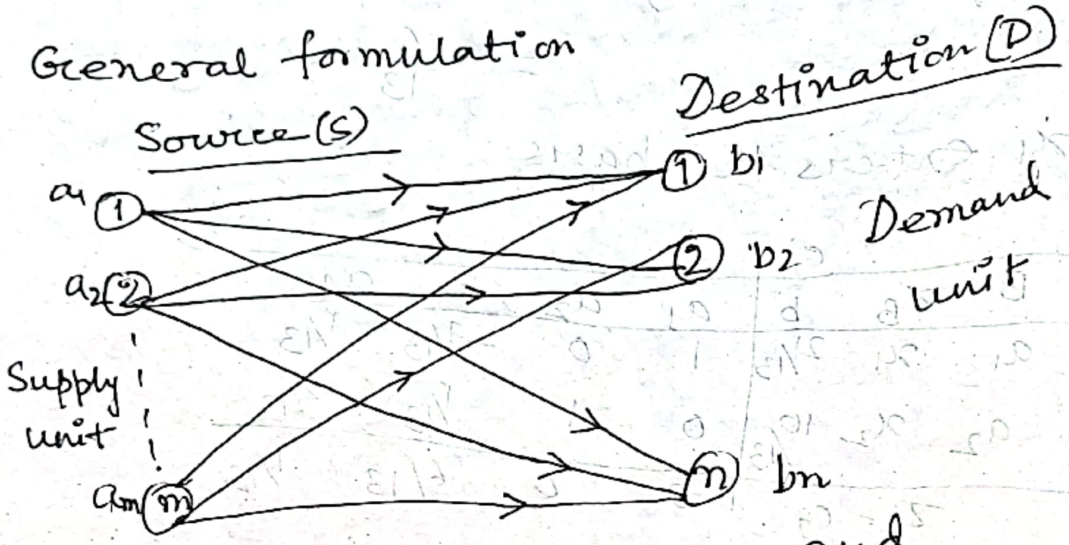
Transportation Problem

Transportation problem is one kind of special case of linear programming problem. Various quantities of single homogeneous comodity are intially stored at various origin. The basic objective of transportation problem is to transport these commodities from various origin to different destination such that total transportation cost is minimum.

General formulation



- Nodes represent source and destination
- Arcs (i, j) represent the amount x_{ij} shipped from ith ~~unit~~ source to j th destination with shipment cost c_{ij}.
- The amount of supply at source i is a_i and amount of demand at destination j is b_j
- Transportation model is based upon the assumption of balancing i.e.

Total demand = Total supply

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$a_i :=$ availability (Row requirement)

$b_j :=$ demand (Column requirement)

Clearly $a_i \geq 0$, $b_j \geq 0$ $\forall\, i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$

For feasibility, we assume

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Let $c_{ij} :=$ cost of shipping one unit from source S_i to destination D_j

$x_{ij} :=$ quantity shipped from source S_i to destination D_j.

In table form, we can write

	D_1	D_2	$\cdots$	D_n	
S_1	c_{11}	c_{12}		c_{1n}	a_1
S_2	c_{21}	c_{22}		c_{2n}	a_2
$\vdots$	$\vdots$				$\vdots$
S_m	c_{m1}	c_{m2}		c_{mn}	a_m
	b_1	b_2	$\cdots$	b_n	

availability

demand

Solution of the above transportation problem takes the form

$$X = \begin{array}{|c|c|c|c|} \hline x_{11} & x_{12} & & x_{1n} \\ \hline x_{21} & x_{22} & & x_{2n} \\ \hline & & & \\ \hline x_{m1} & x_{m2} & & x_{mn} \\ \hline \end{array}$$

$x_{ij} \geq 0 \;\forall\, i, j$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots n$$

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Transportation problem as LPP

Transportation problem is a special case of LPP. Transportation problem can be written as

$$\text{Min } Z = cX$$

$$\text{S.t. } AX = b, \quad X \geq 0.$$

$c = (c_{11}\ c_{12} \ldots c_{ij}, \ldots c_{mn})$, mn component row vector.

$X = (x_{11}, x_{12} \ldots x_{ij}, \ldots x_{mn})'$, mn component column vector.

$b = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots b_n)'$, b is $(m+n)$ component column vector.

$A = (\bar{a}_{11}\ \bar{a}_{12} \ldots \bar{a}_{mn})$ co efficient matrix in which $\bar{a}_{ij}$ is a ~~ele~~column vector associated with x_{ij}.

$$\therefore \text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{S.t. } \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots m.$$

No of constraints $= m+n$
No of variables $= mn$
} for balanced Transportation Problem.

If total supply exceed total demand or vice versa, then

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots m$$

$$\sum_{i=1}^{m} x_{ij} \leq b_j, \quad j = 1, 2, \ldots n.$$

No of basic variables $= (m+n-1)$.

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Initial basic Feasible Solution:

The transportation problem always has a feasible solution. It is always possible to ~~allign~~ assign allocations to different cells to have an initial basic feasible solution which will satisfy all availability and requirements of the problem.

There are some methods to obtain initial basic feasible solution.

(a) North West ~~Corner~~ Corner method.

(b) Matrix Minima method (Least cost method)

(c) Vogel's approximation method (VAM) (Unit penalty method)

(a) North West Corner rule:

Step 1 Allocate maximum allowable amount to the north west corner of the table.

Max allowable amount = Max availablity a_i and requirement b_j

Step 2 Allocation process is continued untill all the available quantities are exhausted.

Example 1: Find the initial BFS of the following Transportation problem

	D_1	D_2	D_3	D_4	a_i
O_1	19	20	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	

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→

[5] 19	[2] 20	50	10	7/2/0
70	[6] 30	[3] 40	60	9/3/0
40	8	[4] 70	[14] 20	18/14/0
5/0	8/6/0	7/4/0	14/0	

$\sum a_i = 7 + 9 + 18 = 34$

$\sum b_j = 5 + 8 + 7 + 14 = 34$

So $\sum a_i = \sum b_j$. T.P is balanced

Total cost $= 5 \times 19 + 2 \times 20 + 6 \times 30 + 3 \times 40 + 4 \times 70 + 20 \times 14$

$= 995$

(B) <u>Matrix Minima (Least cost Entry)</u>

In this method, the cost matrix is inspected ~~thro~~ thoroughly and the cell with minimum cost is chosen for maximum possible allocation.

The row or column whose capacity or requirement is exhausted is deleted from table and process is repeated with shrunken matrix.

If minimum cost cell is not unique then any one of such cell can be selected arbitrarily.

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Ex 2. Find initial basic feasible solⁿ of the following transportation problem

	D_1	D_2	D_3	D_4	a_i
O_1	21	16	25	13	11
O_2	17	18	14	23	13
O_3	32	27	18	41	19
b_j	6	10	12	15	

$\sum_i a_i = \sum_j b_j = 43$

Step 1 →

21	16	25	[11] 13	11/0
17	18	14	23	13
32	27	18	41	19
6	10	12	15/4	

Minimum cost of all cell = 13

Step 2 →

17	18	[12] 14	23	13/1
32	27	18	41	19
6	10	12/0	4	

Step 3 →

[11] 17	18	23	1/0
32	27	41	19
6/5	10	4	

Step 4 →

32	[10] 27	41	19/9
5	10/0	4	

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Step 6 →

[5] 32	41	9/4
5/0	4	

Step 7 →

[4] 41	4/0
4/0	

Solution is

21	16	25	[11] 13	11
[11] 17	18	[12] 14	23	13
[5] 32	[10] 27	18	[4] 41	19
6	10	12	15	

Total cost is

$$11 \times 13 + 1 \times 17 + 12 \times 14 + 5 \times 32 + 10 \times 27 + 4 \times 41$$

$$= 812$$

(c) Vogel Approximation method :

V.A.M is an improved version of the least cost (Matrix Minima) method but it not always produces better Solution.

It not only considers the least cost (c_{ij}) but also the costs which exceeds c_{ij}.

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Step 1: For each row (or column) determine a penalty cost by subtracting smallest unit cost element in the same row (or column)

Step 2: Identify the row or column with maximum penalty. Allocate as much as possible with least unit cost in the selected row or column. If a row or column is satisfied simultaneously only one of the two is crossed out and the remaining row (column) is assigned zero supply (demand)

Step 3: (a) If exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) If one row (column) with positive supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least cost method, stop.

(c) If all the uncrossed out rows and columns have zero basic variables by the least cost method, stop.

(d) Otherwise go to step 1.

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Ex 3. Solve the example 2 using VAM method.

Step 1 :

21	16	25	13	11 (3)
17	18	14	23	13 (3)
32	27	18	41	19 (9)
6 (4)	10 (2)	12 (4)	15 (10)	

Step 2 :

21	16	25	[11] 13	11/0
17	18	14	[4] 23	13/9
32	27	18	41	19
6	10	12	15/4/0	

Step 3

[6] 17	18	14	9/3 (3)
32	27	18	19 (9)
6/0 (15)	10 (9)	12 (4)	

Step 4

18	14	3 (4)
[7] 27	[12] 18	19/7 (9)
10/3 (9)	12/0 (4)	

Step 5

[3] 18	3/0
3/0	

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Solution is

21	16	25	[11] 13	11
[6] 17	[3] 18	14	[4] 23	13
32	[7] 27	[12] 18	41	19
6	10	12	15	

$x_{14} = 11$
$x_{21} = 6$
$x_{22} = 3$
$x_{24} = 4$
$x_{32} = 7$
$x_{33} = 12$

Total cost is
$11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 27 + 12 \times 18$
$= 796$.

Optimal Solution in Transportation Problem

A Basic feasible solution of an $(m \times n)$ order T.P is nondegenerate if

(i) initial BFS has exactly $m+n-1$ allocations

(ii) Such allocation does not form a closed loop.

Loop in a Transportation problem

An ordered set of four or more cell forms a Loop if

(i) any two adjacent cells in the ordered set lie in the same row or same column.

(ii) any three or more adjacent cells in the ordered set do not lie in the same row or same column

Theorem: A feasible solution to a transport problem is basic if the corresponding cells in TP donot form a loop.

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Modi test for optimality (U-V method)

Determine x_{ij}

$$\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{S.t } a_i - \sum_{j=1}^{n} x_{ij} = 0, \quad i = 1, 2, \ldots m$$

$$b_j - \sum_{i=1}^{m} x_{ij} = 0, \quad j = 1, 2, \ldots n.$$

$$x_{ij} \geq 0 \quad \forall\, i, j$$

It has mn variables and $m+n$ constraint.

Let $u_1, u_2 \ldots u_m$ and $v_1, v_2, \ldots v_n$ be dual variable of the problem and the dual has mn constraints.

$$u_i + v_j \leq c_{ij}, \quad i = 1, \ldots m$$
$$j = 1, 2 \ldots n$$

u_i, v_j are unrestricted sign

$$z_j - c_j = c_B B^{-1} a_j - c_j \quad \forall\, j$$
$$= u_i + v_j - c_{ij}$$

$$\Rightarrow z_{ij} - c_{ij} = u_i + v_j - c_{ij}$$

If u_r, v_s are basic variable then $u_r + v_s = c_{rs}$

If there is no degeneracy, there will be $m+n-1$ dual equations in $m+n$ unknowns for $m+n-1$ basic cells.

We have to assign arbitrary value to one of these variable u_r or v_r.

Then solve the new system of equations of $m+n-1$ equations in $m+n-1$ unknowns.

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Once u_i, v_j are evaluated, Calculate $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$ for net evaluation.

Check if $\Delta_{ij} = -(u_i + v_j) + c_{ij} \geq 0$ or not if yes, the solution is optimal. If no, select the cell with smallest cost. Draw closed loop which (I) begin and ends with starting cell. (II) May pass both of occupied and unoccupied cells. (III) has alternating sign in the route starting with (+) sign in the chosen cell (IV) change direction only at the occupied cell.

A revised ~~amount~~ solution is obtained by considering cells with (−) mark and determine least amount to the cells bearing (+) sign and subtract from cell with negative sign

For the new solution, again evaluate u_i, v_j and Δ_{ij}. If $\Delta_{ij} \geq 0$ stop, otherwise repeat the steps.

Ex 4. Find the optimal solution of the transportation problem

	D_1	D_2	D_3	D_4	
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
	20	40	30	10	

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→ Step 1: Find initial BFS using NWC | Least cost | VAM.

[20] 1	2	[10] 1	4	30
3	[20] 3	[20] 2	[10] 1	50
4	[20] 2	5	9	20
20	40	30	10	

(solved using least cost method)

No of allocated cells = 6 = m+n−1.

So non degenerate.

→ Step 2: Test for optimality.

$C_{ij} = u_i + v_j$ for occupied cells.

$u_1 + v_1 = c_{11} = 1$

$u_1 + v_3 = c_{13} = 1$

$u_2 + v_2 = c_{22} = 3$

$u_2 + v_3 = c_{23} = 2$

$u_2 + v_4 = c_{24} = 1$

$u_3 + v_2 = c_{32} = 2$

We will choose the row with highest no. of allocations.

Choose $u_2 = 0$.

$v_2 = 3$, $v_3 = 2$, $v_4 = 1$, $u_1 = -1$, $u_3 = -1$,

$v_1 = 2$

				u_i
[20] 1	(0) 2	[10] 1	(4) 4	−1
(1) 3	[20] 3	[20] 2	[10] 1	0
(3) 4	[20] 2	(4) 5	(9) 9	−1
v_j 2	3	2	1	

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Compute $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for unoccupied cells

$\Delta_{12} = 2 - (-1 + 3) = 0.$

$\Delta_{14} = 4 - (-1 + 1) = 4$

$\Delta_{21} = 3 - (0 + 2) = 1$

$\Delta_{31} = 4 - (2 - 1) = 3$

$\Delta_{33} = 5 - (2 - 1) = 4$

$\Delta_{34} = 9 - (1 - 1) = 9$

All $\Delta_{ij} \geq 0 \ \forall\, i, j$.

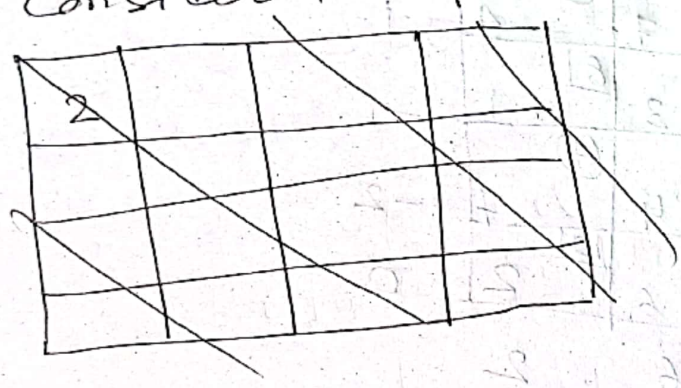
Solution is optimal. Sol^n is

$x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{24} = 10$,

$x_{32} = 20$,

Minimum cost $= 180$.

Ex 5. Consider the problem with initial B.F.S.



[5] 2	7		4	5
3	3	[8]	1	8
5	[7] 4		7	7
[2] 1	[2] 6	[10]	2	14
7	9		18	

Find the optimal Solution

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$c_{ij} = u_i + v_j$ for occupied cells.

$u_1 + v_1 = 2$

$u_2 + v_3 = 1$

$u_3 + v_2 = 7$

$u_4 + v_1 = 1$

$u_4 + v_2 = 6$

$u_4 + v_3 = 2$

Choose $u_4 = 0$, $v_1 = 1$

$v_2 = 6$

$v_3 = 2$

$u_1 = 1$

$u_2 = -1$

$u_3 = -2$

			u_i
$\boxed{5}$ 2	(0) 7	(1) 4	1
(3) 3	(-2) 3	$\boxed{8}$ 1	-1
(6) 5	$\boxed{7}$ 4	(7) 7	-2
$\boxed{2}$ 1	$\boxed{2}$ 6	$\boxed{10}$ 2	0
v_j 1	6	2	

Calculate $\Delta_{ij} = -(u_i + v_j) + c_{ij}$ for unoccupied cell.

$\Delta_{12} = 7 - (6+1) = 0$

$\Delta_{13} = 4 - (1+2) = 1$

$\Delta_{21} = 3 - (1-1) = 3$

$\Delta_{22} = 3 - (6-1) = -2$

$\Delta_{31} = 5 - (-2+1) = 6$

$\Delta_{33} = 7 - (-2+2) = 7$

So, $\Delta_{22} < 0$

The soln is not optimal.

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Loop formation

5		
	(-2) +2	8 -2
	7	
2	2 -2	10 +2

New allocated cells

				u_i
5 / 2	(2) / 7	(1) / 4	5	1
(3) / 3	2 / 3	6 / 1	8	-1
(5) / 5	7 / 4	(3) / 7	7	0
2 / 1	(2) / 6	12 / 2	14	0
v_j 1	4	2		
	9	18		

Compute $u_i + v_j = c_{ij}$ for occupied cells

$u_1 + v_1 = 2$

$u_2 + v_2 = 3$

$u_2 + v_3 = 1$

$u_3 + v_2 = 4$

$u_4 + v_1 = 1$

$u_4 + v_3 = 2$

Choose $u_4 = 0$, $v_1 = 1$, $v_3 = 2$. $u_1 = 1$, $u_2 = -1$, $v_2 = 4$, $u_3 = 0$

(*) → Optimal sol^n is
$x_{11} = 5$ $x_{22} = 2$,
$x_{23} = 6$, $x_{32} = 7$
$x_{41} = 2$, $x_{43} = 12$
Total cost is
$5 \times 2 + 2 \times 3 + 6 \times 1 + 7 \times 4 + 2 \times 1 + 12 \times$
$= 76$.

Compute $\Delta_{ij} = c_{ij} - (u_i + v_j)$

$\Delta_{12} = 7 - (1 + 4) = 2$

$\Delta_{13} = 4 - (1 + 2) = 1$

$\Delta_{21} = 3 - (-1 + 1) = 3$

$\Delta_{31} = 5 - (7 + 7) = 5$

(*) →

$\Delta_{33} = 7 - (0 + 4) = 3$

$\Delta_{42} = 6 - (0 + 4) = 2$

$\Delta_{ij} \geq 0 \ \forall \ i, j$.

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