Nonlinear Programming with inequality constraints

Consider the following constrained optimization problem with inequality constraints:

$$\text{Min } f(x)$$

subject to $g_i(x) \leq 0, \quad i = 1, 2, \ldots m$

where $f: R^n \to R$ and $g_i: R^n \to R$ are defined and continuously differentiable f^n.

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Karush-Kuhn Tucker Necessary cond^{n}s:

Let $\bar{x}$ be a local minimum point of the basic constraint problem at which constraint qualification holds. Then there exist multipliers (called KKT multipliers) $\bar{\lambda}_i$, $i = 1, 2, \ldots m$ such that the following cond^{n}s hold.

(i) $\nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0$

(ii) $g_i(\bar{x}) \leq 0, \quad i = 1, 2, \ldots m.$

(iii) $\bar{\lambda}_i g_i(\bar{x}) = 0, \quad i = 1, 2, \ldots m.$

(iv) $\bar{\lambda}_i \geq 0 \ \forall i.$

These conditions are called KKT cond^{n}s.

Sufficient Cond^{n} :

Let $(\bar{x}, \bar{\lambda}_1, \bar{\lambda}_2, \ldots \bar{\lambda}_m)$ satisfy the KKT cond^{n} (1)-(4). Let f and g_i be differentiable convex function. Then x is a global minimizer point of the minimization problem.

Ex. Min $f(x_1, x_2) = 2x_1 + x_2$

S.t. $x_1^2 + x_2^2 \le 4 \Rightarrow g_1(x_1, x_2) = x_1^2 + x_2^2 - 4 \le 0$

$x_1 - x_2 \le 0 \Rightarrow g_2(x_1, x_2) = x_1 - x_2 \le 0$

$\nabla^2 g_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. $\lambda_1 = 2, \lambda_2 = 2 > 0$.

$\therefore g_1$ is strictly convex.

g_2 is also convex.

$\cancel{\lambda_1(\bar{x}_1^2 + \bar{x}_2^2 - 4) + \lambda_2(}$

$\cancel{\nabla f(\bar{x}_1, \bar{x}_2) +} \; \bar{\lambda}_2 \nabla g_2(\bar{x}_1, \bar{x}_2) = 0$

$\nabla f(\bar{x}_1, \bar{x}_2) + \bar{\lambda}_1 \nabla g_1(\bar{x}_1, \bar{x}_2) +$

$\Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \bar{\lambda}_1 \begin{pmatrix} 2\bar{x}_1 \\ 2x_2 \end{pmatrix} + \bar{\lambda}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

$\Rightarrow 2 + \bar{\lambda}_1 \cdot 2\bar{x}_1 + \bar{\lambda}_2 = 0$ —— (1)

$1 + \bar{\lambda}_1 \cdot 2x_2 - \bar{\lambda}_2 = 0$ —— (2)

$\lambda_1 g_1(x) = 0 \Rightarrow \lambda_1(x_1^2 + x_2^2 - 4) = 0$. —— (3)

$\lambda_2 g_2(x) = 0 \Rightarrow \lambda_2(x_1 - x_2) = 0$. —— (4)

$x_1^2 + x_2^2 - 4 \le 0$ —— (5)

$x_1 - x_2 \le 0$ —— (6)

$\lambda_1, \lambda_2 \ge 0$ —— (7).

Case 1: $\lambda_1 = \lambda_2 = 0$. ← not possible

Case 2: $\lambda_2 = 0$, $x_1 = x_2$ — not possible.

Case 3: $x_1^2 + x_2^2 = 4$, $\lambda_2 = 0$

Case 4: $\cancel{\lambda_2 = 0}$ $x_1^2 + x_2^2 = 4$, $x_1 = x_2$

Case 3 $\Rightarrow x_1 \lambda_1 = -1$

$x_2 \lambda_1 = -1/2$

$x_1^2 + x_2^2 = 4 \Rightarrow \left(-\frac{1}{\lambda_1}\right)^2 + \left(-\frac{1}{2\lambda_1}\right)^2 = 4$

$\Rightarrow \frac{4+1}{4\lambda_1^2} = 4$

$\Rightarrow \lambda_1^2 = \frac{16}{5} \Rightarrow \lambda_1 = \frac{\sqrt{5}}{4}$.

$2 + 2x_1 \cdot \frac{\sqrt{5}}{4} = 0$

$\Rightarrow x_1 = -\frac{4}{\sqrt{5}}$

$\therefore x_2 = -\frac{2}{\sqrt{5}}$

which also satisfies (5) - (6) cond^{n}.

So this point $\left(-\frac{4}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$ satisfies all KKT cond^{n}s, so it is global minimum.

Note → Without convexity assumptions on f and g, the KKT cond^{n}s are not sufficient for a point to be a local/global minimum.

Ex: Min $-x_1$

s.t $x_1^2 + x_2^2 \le 4$

$-x_1^2 + x_2 \le 0$.

The point (0, 0) satisfies KKT cond^{n}s but it is not a local/global min point.

Quadratic Programming Problems

Quadratic Programming Problems:
A quadratic programming problem (QPP) is the special case of nonlinear optimization problems in which the objective f^n is quadratic and all the constraints are linear.

The general mathematical formulation of a QPP is as follows:

$$(QPP): \quad \text{Min } f(x) = x^T Q x + c^T x.$$
$$\text{s.t. } Ax \leq b$$
$$x \geq 0.$$

where $Q = [q_{ij}]_{n \times n}$ symmetric matrix which is positive semi definite, $c, x \in R^n$, $b \in R^m$ and $A = [a_{ij}]_{m \times n}$

Ex. $\text{Min } f(x) = 3x_1^2 + 4x_2^2 + 2x_1x_2 - 2x_1 - 3x_1$
$$\text{s.t. } 3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0.$$

Equivalently:
$$\text{Min } f(x) = (x_1 \ \ x_2) \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (-2 \ \ -3) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\text{s.t. } \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} +x_1 \\ +x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}$$

Principal minor $D_1 = 3 > 0$
$$D_2 = \begin{vmatrix} 3 & 1 \\ 4 & 4 \end{vmatrix} = 8 > 0$$

Q is positive definite.

KKT conditions for QPP:

The problem QPP can be formulated as.

$$\text{Min } f(x) = x^T Q x + c^T x.$$

$$\text{s.t.} \quad Ax \le b \rightarrow u \in R^m$$

$$\cancel{x \geq 0} \; -x \le 0 \rightarrow v \in R^n$$

KKT Cond^ns are

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0.$$

$$\lambda_i g_i(x) = 0$$

$$g_i(x) \le 0 \quad \forall i$$

$$\lambda_i \geq 0 \quad \forall i$$

In this case of QPP.

$$2x^T Q + c^T + u^T A + v(-I) = 0.$$

$$u^T(Ax - b) - v^T x = 0.$$

$$Ax - b \le 0.$$

$$u, v \geq 0, \; x \geq 0$$

$$\Rightarrow 2Qx + c + A^T u - vI = 0 \quad \left[(2x^T Q)^T = 2Q^T x = 2Qx \right]$$

$$u^T(Ax - b) - v^T x = 0$$

$$\cancel{Ax - b \le 0} \quad Ax - b + s = 0$$

all variables ≥ 0.

$$\Rightarrow \quad 2Qx + c + A^T u - vI = 0$$

$$u^T(-s) - v^T x = 0 \quad [\because Ax - b = -s]$$

$$\Rightarrow u^T s + v^T x = 0.$$

$$\therefore u^T s = 0, \quad v^T x = 0.$$

$$\Rightarrow \left.\begin{array}{l} u_1 s_1 + u_2 s_2 + \ldots + u_m s_m = 0 \\ v_1 x_1 + v_2 x_2 + \ldots + v_n x_n = 0 \end{array}\right\} \quad \begin{array}{l} u_i s_i = 0, \; i = 1, 2, \ldots m \\ v_j x_j = 0, \; j = 1, 2, \ldots n. \end{array}$$

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$\Rightarrow 2Qx + c + A^T u - v I = 0$

$Ax - b + s = 0$

$u_i s_i = 0, \quad i = 1, 2 \ldots m$

$v_j x_j = 0, \quad j = 1, 2 \ldots n.$

or
$$\left(\begin{array}{c} c + 2Qx + A^T u - v = 0 \\ Ax + s = b \\ u_i s_i = 0, \quad i = 1, 2 - m \\ v_j x_j = 0, \quad j = 1, 2 - - n \end{array} \right)$$

The matrix form of KKT cond^ns are

$$\begin{bmatrix} 2Q & A^T & -I_n & 0 \\ A & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \\ v \\ s \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

$u_i s_i = 0, \; v_j x_j = 0 \; \forall \; i, j,$

$u_i, v_j, s_i, x_j \geq 0$

[Note: In case of maximization problem, the function f will be concave and g_i's are convex. Matrix Q will be negative semi definite]