Example 2.5.3

If matrix of a linear transform on R^3 relative to basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$. Then find the linear transform matrix T relative to basis $B_1 = \{(0,1,-1),(1,-1,1),(-1,1,0)\}$.

First we find linear transform.
We have

$$[T:B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}^t$$

That is transpose of coefficient matrix. So that

$$T(u_1) = T(1,0,0) = 0(1,0,0) + 1(0,1,0) - 1(0,0,1) = (0,1,-1)$$
$$T(u_2) = T(0,1,0) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1) = (1,0,-1)$$
$$T(u_3) = T(0,0,1) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1) = (1,-1,0)$$

$(x,y,z) \in \mathbb{R}^3$ be any element and B is basis for $\mathbb{R}^3$

$$\therefore (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$
\begin{aligned}
\therefore T(x,y,z) &= xT(1,0,0) + yT(0,1,0) + zT(0,0,1) \\
&= x(0,1,-1) + y(1,0,-1) + z(1,-1,0) \\
T(x,y,z) &= (y+z, x-z, -x-y)
\end{aligned}
$$

which is linear operator T on R^3.
Now we have to find a matrix of T relative basis.

$$B_1 = \{(0,1,-1), (1,-1,1), (-1,1,0)\}$$

Let $(a,b,c) \in \mathbb{R}^3$ be any element
Let

$$
\begin{aligned}
(a,b,c) &= l(0,1,-1) + m(1,-1,1) + n(-1,1,0) \\
(a,b,c) &= (m-n, l-m+n, -l+m) \\
&\Rightarrow a = m-n; \; b = l-m+n; \; c = -l+m
\end{aligned}
$$

Now

$$\begin{array}{c|c|c} l-m+n=b & l-m+n=b & m=a+n \\ l=b+m-n=b+a & n=b-l+m & m=a+b+c \\ l=a+b & n=b+c & \end{array}$$

$$B_1 = \{(0, 1 - 1), (1, -1, 1), (-1, 1, 0)\}$$

$\therefore$ we get

$$(a, b, c) = (a + b)(0, 1, -1) + (a + b + c)(1, -1, 1) + (b + c)(-1, 1, 0)$$

and we have

$$T(x, y, z) = (y + z, x - z, -x - y)$$

Now

$$T(0, 1, -1) = (0, 1, -1) = 1(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0)$$
$$T(1, -1, 1) = (0, 0, 0) = 0(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0)$$
$$T(-1, 1, 0) = (1, -1, 0) = 0(0, 1, -1) + 0(1, -1, 1) + (-1)(-1, 1, 0)$$

$$\therefore [T; B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Example 2.5.4

Let T be linear transform on $\mathbb{R}^2$ and $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ be matrix of T with respect to usual basis of $\mathbb{R}^2$. Then, find that matrix of T with respect to $B_1 = \{(1,2),(5,6)\}$.

Ans:

$$[T : B_1] = \begin{bmatrix} \frac{11}{2} & \frac{41}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Definition 2.5.5 (Isomorphism of a vector space)

Let $U(F)$ and $V(F)$ are two vector spaces then a linear transformation $f : U \to V$ is called Isomorphism, if

1. f is one-one
2. f is onto

Definition 2.5.6 (Isomorphism of a vector space)

$f : U \to V$ is called Isomorphism if

1. f is a linera transform
2. f is one-one
3. f is onto

Problem 2.5.7

Let $f : V_2(\mathbb{R}) \to V_2(\mathbb{R})$ *be* $f(x,y) = (y,x)$. *Prove* f *is Isomorpism.*

To prove one-one:

Let $U, V \in V_2(\mathbb{R})$

$$\begin{aligned} f(u) &= f(v) \\ f(x,y) &= f(p,q) \\ (y,x) &= (q,p) \\ y &= q; x = p \\ (x,y) &= (p,q) \\ u &= v \end{aligned}$$

i.e., f is one-one

To prove onto:
$\forall (x, y) \in V_2(\mathbb{R})$
$\exists (y, x) \in V_2(\mathbb{R})$ such that $f(x, y) = (y, x)$

To prove linear transform:
Let $u, v \in V_2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} f(\alpha u + \beta v) &= f[\alpha(x,y) + \beta(p,q)] \\ &= f[\alpha x + \beta p, \alpha y + \beta q] \\ &= (\alpha y + \beta q, \alpha x + \beta p) \\ &= \alpha(y,x) + \beta(q,p) \\ &= \alpha f(x,y) + \beta(p,q) \\ &= \alpha f(u) + \beta f(v) \end{aligned}$$

So, f is a linear transform, one-one, onto.
i.e., f is an Isomorpism.

Problem 2.5.8

Let $T : P_2 \to V_3 \to \{(x_1, x_2, x_3) | x_i \in \mathbb{R}\}$ *(*P_2*-set of all polynomials of degree* ≤ 2*)* $\{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{R}\}$*. Prove that* T *is Isomorphism.* $T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2)$

To prove T is one-one:

$$
\begin{aligned}
T(p_1) &= T(p_2) \\
T(a_0 + a_1x + a_2x^2) &= T(b_0, b_1x + b_2x^2) \\
(a_0, a_1, a_2) &= (b_0, b_1, b_2) \\
a_0 = b_0; a_1 &= b_1; a_2 = b_2 \\
a_0 + a_1x + a_2x^2 &= b_0 + b_1x + b_2x^2 \\
p_1(x) &= p_2(x) \\
p_1 &= p_2
\end{aligned}
$$

T is one-one.

To prove T is onto:

$T : p_2 \rightarrow v_3$. For every $(a_0, a_1, a_2) \in v_3$ we have a polynomial $p = a_0 + a_1 x + a_2 x^2$ in p_2. Such that

$$T(p) = (a_0, a_1, a_2)$$

T is onto.

T is one-one and onto.

To prove T is linear.

$$
\begin{aligned}
& T(\alpha(a_0 + a_1x + a_2x^2) + \beta(b_0 + b_1x + b_2x^2)) \\
& = T((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2) \\
& = (\alpha a_0 + \beta b_0, \alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2) \\
& = (\alpha a_0, \alpha a_1, \alpha a_2) + (\beta b_0, \beta b_1, \beta b_2) \\
& = \alpha(a_0, a_1, a_2) + \beta(b_0, b_1, b_2) \\
T(\alpha p_1 + \beta p_2) & = \alpha T(p_1(x)) + \beta T(p_2(x))
\end{aligned}
$$

This proves T is linear.
$\therefore$ T is an isomorphic.
To find its inverse:

$$
\begin{aligned}
T^{-1} &: v_3 \to p_2 \\
T^{-1}(a_0, a_1, a_2) &= a_0 + a_1x + a_2x^2
\end{aligned}
$$

Example 2.5.9

$T : v_2 \to v_2 \; T(x_1, x_2) = (x_1, -x_2)$