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The Laplace Transform

Integral Transform:- A definite integral such as $\int_a^b K(s,t) f(t)\,dt$ transforms a function f of the variable t into a function F of the variable s. We are particularly interested in an integral transform, where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_0^\infty K(s,t) f(t)\,dt$ is defined as a limit:

$$\int_0^\infty K(s,t) f(t)\,dt = \lim_{b\to\infty} \int_0^b K(s,t) f(t)\,dt \quad (1)$$

If the limit in (1) exists, then we say that the integral exists or is convergent; if the limit does not exist, the integral does not exist and is divergent. The limit in (1) will, in general, exist for only certain values of the variable s.

* The function $K(s,t)$ in (1) is called the kernel of the transform. The choice $K(s,t) = e^{-st}$ as the kernel gives us an especially important integral transform.

Laplace Transform:- Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

is said to be the Laplace transform of f, provided that the integral converges. We denote $L[f(t)]$ as $F(s)$.

Linear property of L:- Let f and g are functions defined for $t \geq 0$. Then

$$L[\alpha f(t) + \beta g(t)] = \int_0^\infty e^{-st}\{\alpha f(t) + \beta g(t)\}\,dt$$

$$= \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt$$

$$= \alpha\, L[f(t)] + \beta\, L[g(t)]$$

$$= \alpha\, F(s) + \beta\, G(s).$$

Transforms of some basic functions:-

(a) $L\{1\} = \int_0^\infty e^{-st}\cdot(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt = \lim_{b\to\infty}\left[\frac{-e^{-st}}{s}\right]_{t=0}^{b} = \frac{1}{s},\ s>0$

(b) $L\{t\} = \int_0^\infty \underset{\text{II}}{e^{-st}}\cdot \underset{\text{I}}{t}\,dt = \left[\frac{-t e^{-st}}{s}\right]_{t=0}^{\infty} + \frac{1}{s}\int_0^\infty e^{-st}\,dt = 0 + \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2},\ s>0$

(c) $L\{e^{at}\} = \int_0^\infty e^{-st}\cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left.\frac{-e^{-(s-a)t}}{(s-a)}\right|_{t=0}^{\infty} = \frac{1}{s-a},\ s>a.$

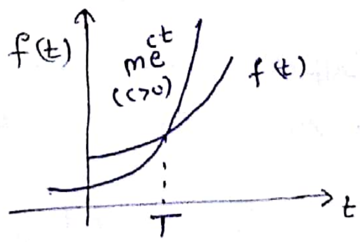
(d) $L\{\sin(at)\} = \frac{a}{s^2+a^2}$, (e) $L\{\cos(at)\} = \frac{s}{s^2+a^2}$

(f) $L\{\sinh at\} = \frac{a}{s^2-a^2}$ (g) $L\{\cosh at\} = \frac{s}{s^2-a^2}$, $s>a$.

Some definitions:-

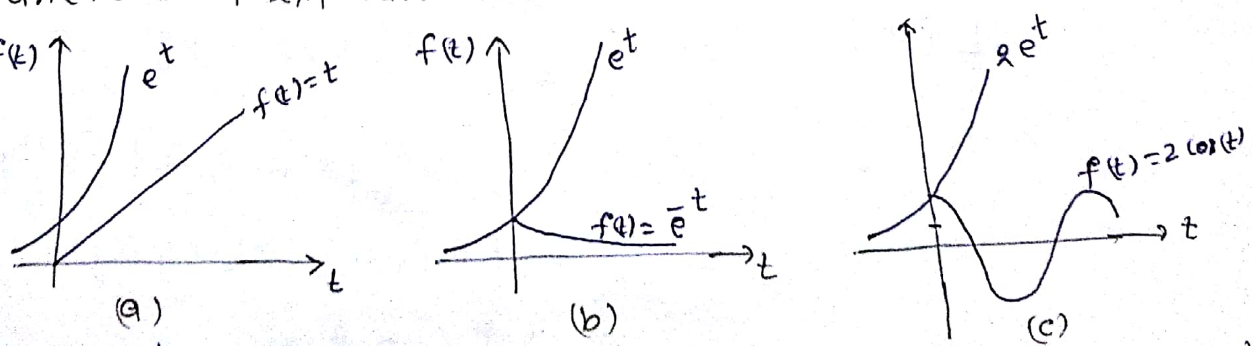
Piecewise Continuous function:- A function f is piecewise continuous on $[0,\infty)$ if, in any interval $0\le a\le t\le b$, there are at most a finite number of points t_k, $k=1,2,\dots,n$ $(t_{k-1}<t_k)$ at which f has finite discontinuities and is continuous on each open interval (t_{k-1},t_k).

Exponential order:- A function f is said to be of exponential order c if there exist constants $c, m>0$ and $T>0$ s.t. $|f(t)|\le m e^{ct}$ $\forall\ t>T$.



f is of exponential order c.

Functions of exponential order 1:-



$|f(t)| = |t| < e^t$ $\qquad |f(t)| = |e^{-t}| < e^t$ $\qquad |f(t)| = |2\cos t| < 2e^t$.

* A function such as $f(t)=e^{t^2}$ is not of exponential order since its graph grows faster than any positive linear power of e for $t>c>0$.

Sufficient conditions for existence of $L[f(t)]$:

Theorem :- If f is piecewise continuous on $[0,\infty)$ and of exponential order c, then $L[f(t)]$ exists for $s>c$.

Proof :- By additive interval property of definite integrals, we can write

$$L[f(t)] = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt = I_1 + I_2 .$$

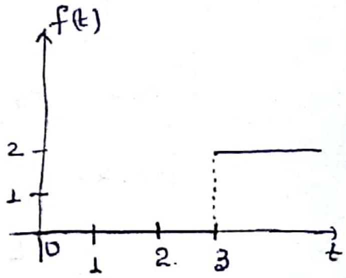
The integral I_1 exists because it can be written as a sum of integrals over intervals on which $e^{-st} f(t)$ is continuous. Now since f is of exponential order, $\exists$ constants, $c, m>0, T>0$ s.t. $|f(t)| \le m e^{ct}$ for $t>T$.
We can then write

$$|I_2| \le \int_T^\infty |e^{-st} f(t)|\,dt \le m\int_T^\infty e^{-st} e^{ct}\,dt = m\,\frac{e^{-(s-c)T}}{(s-c)}, \; s>c$$

Since $\int_T^\infty m\, e^{-(s-c)t}\,dt$ converges, the integral $\int_T^\infty |e^{-st} f(t)|\,dt$ converges by the comparison test for improper integrals. This, implies that I_2 exists for $s>c$. The existence of I_1 and I_2 implies that

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \quad \text{exists for } s>c.$$

Q: Evaluate $L[f(t)]$ where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$



Piecewise continuous function

Sol: The given function is piecewise continuous and of exponential order for $t>0$. Since f is defined in two pieces (or intervals), $L[f(t)]$ is expressed as the sum of two integrals:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 e^{-st}\cdot(0)\,dt + \int_3^\infty e^{-st}\cdot(2)\,dt$$

$$= 0 + \left[\frac{2}{-s} e^{-st}\right]_{t=3}^{\infty} = \frac{2\,e^{-3s}}{s}, \quad s>0.$$

Theorem :- (Behaviour of $F(s) = L[f(t)]$ as $s\to\infty$)
If f is piecewise continuous on $[0,\infty)$ and of exponential order and $F(s) = L[f(t)]$, then $\lim_{s\to\infty} F(s) = 0$.

Proof: Since f is of exponential order, $\exists$ γ, $m_1>0$ and $T>0$ s.t.

$$|f(t)| < m_1 e^{\gamma t}$$ for $t>T$. Also, since f is piecewise continuous for $0 \le t \le T$, it is necessarily bounded on the interval; i.e.

$$|f(t)| \le m_2 = m_2\, e^{0t}.$$

Let $m = \max\{m_1, m_2\}$ and $c = \max\{0, \gamma\}$, then

$$|F(s)| \le \int_0^\infty e^{-st} |f(t)|\,dt \le m \int_0^\infty e^{-st} e^{ct}\,dt = m \cdot \int_0^\infty e^{-(s-c)t}\,dt = \frac{m}{s-c}, \; s>c.$$

As $s \to \infty$, $|F(s)| \to 0$ and so $F(s) = L[f(t)] \to 0$.

Inverse Transforms :- If $F(s)$ represents the Laplace transform of a function $f(t)$, i.e. $L[f(t)] = F(s)$, we say $f(t)$ is the inverse Laplace transform of $F(s)$ and write $f(t) = L^{-1}\{F(s)\}$.

Exp (i) $L\{1\} = \frac{1}{s}$; $\quad 1 = L^{-1}\{\frac{1}{s}\}$

(ii) $L\{t\} = \frac{1}{s^2}$; $\quad t = L^{-1}\{\frac{1}{s^2}\}$

(iii) $L\{e^{at}\} = \frac{1}{s-a}$; $\quad e^{at} = L^{-1}\{\frac{1}{s-a}\}$, $s>a$

(iv) $L\{t^n\} = \frac{\underline{|n}}{s^{n+1}}$, $n = 1, 2, \dots$; $\quad t^n = L^{-1}\{\frac{\underline{|n}}{s^{n+1}}\}$

(v) $e^{-at} = L^{-1}\{\frac{1}{s+a}\}$, (vi) $\sin at = L^{-1}\{\frac{a}{s^2+a^2}\}$, (vii) $\cos at = L^{-1}\{\frac{s}{s^2+a^2}\}$

(viii) $\sinh at = L^{-1}\{\frac{a}{s^2-a^2}\}$. (ix) $\cosh at = L^{-1}\{\frac{s}{s^2-a^2}\}$

Linear Property of L^{-1} :- The inverse Laplace transform is also a linear transform; i.e. for constants α and β

$$L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha L^{-1}\{F(s)\} + \beta L^{-1}\{G(s)\}.$$

where F and G are the transforms of some functions f and g.

Transforms of Derivatives :- Let $f'(t)$ is continuous for $t \geq 0$, then

$$L[f'(t)] = \int_0^\infty \underset{\mathrm{I}}{e^{-st}} \underset{\mathrm{II}}{f'(t)}\,dt = \left[e^{-st} f(t)\right]_0^\infty + s \int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L[f(t)] \qquad \text{Let } \lim_{t\to\infty} e^{-st} f(t) = 0$$

$$\Rightarrow L[f'(t)] = s\,F(s) - f(0)$$

Similarly, we get

$$L[f''(t)] = \int_0^\infty e^{-st} f''(t)\,dt = \left[e^{-st} f'(t)\right]_0^\infty + s \int_0^\infty e^{-st} f'(t)\,dt$$

$$L[f''(t)] = -f'(0) + s^2 L[f(t)] - s f(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

Theorem: (Transform of a derivative)

If $f, f', --, f^{(n-1)}$ are continuous on $[0,\infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$L[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) -, \dots, - f^{(n-1)}(0).$$

where $F(s) = L[f(t)]$.

Solving Linear ODEs:- Laplace transform ideally suited for solving linear initial-value problem in which the differential equation has constant coefficient. Let the DE is

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = g(t), \text{ with } \quad --- (1)$$

$$y(0) = y_0, \quad y'(0) = y_1, \quad \dots, \quad y^{(n-1)}(0) = y_{n-1}.$$

where the a_i, $i = 0, 1, \dots, n$ and $y_0, y_1, \dots, y_{n-1}$ are constants. Taking Laplace transform of eq. (1) we get

$$a_n L[y^n(t)] + a_{n-1} L[y^{(n-1)}(t)] + \dots + a_0 L[y(t)] = L[g(t)]$$

$$\Rightarrow a_n[s^n Y(s) - s^{n-1} y_0 - s^{n-2} y_1 - \dots - y_{n-1}(0)]$$

$$+ a_{n-1}[s^{n-1} Y(s) - s^{n-2} y_0 - , \dots, - y_{n-2}] + \dots + a_0 Y(s) = G(s). \quad — (2)$$

where $L[y(t)] = Y(s)$ and $L[g(t)] = G(s)$. It is clear from eq. (2) that the Laplace transform of a differential equation with constant coefficient becomes an algebraic equation in $Y(s)$. If we solve the general transformed equation (2) for $Y(s)$, we first obtain

$$P(s)\, Y(s) = Q(s) + G(s) \Rightarrow Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)} \quad -- (3)$$

where $P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$, $Q(s)$ is a polynomial in s of degree less than or equal to $(n-1)$ consisting of the various products of the coefficient a_i, $i = 1, \dots n$ and the given initial conditions $y_0, \dots, y_{n-1}$. The solution $y(t)$ of the original initial-value problem is $y(t) = L^{-1}(Y(s))$.

* Equation (2) or (3) is called the subsidiary equation.

Use the Lablace transform to solve the initial-value problems

(i) $\frac{dy}{dt} + 3y = 13 \sin 2t$, $y(0) = 6$ (ii) $y'' - 3y' + 2y = e^{-4t}$, $y(0)=1$, $y'(0)=5$.

Sol. (i) $y' + 3y = 13 \sin 2t$, Taking Laplace transform of both sides, we get

$L[y'] + L[3y] = L[13 \sin 2t]$

$\Rightarrow s Y(s) - y(0) + 3Y(s) = 13 \frac{2}{s^2+4}$ {Subsidiary equation}

$\Rightarrow (s+3) Y(s) = \frac{26}{s^2+4} + 6 \Rightarrow Y(s) = \frac{6}{(s+3)} + \frac{26}{(s^2+4)(s+3)} = \frac{6s^2+50}{(s^2+4)(s+3)}$

Now, we find the partial fractions for $Y(s)$ as follows.

$$\frac{6s^2+50}{(s^2+4)(s+3)} = \frac{A}{(s+3)} + \frac{Bs+C}{(s^2+4)}$$

After solving the above equation, we get $A = 8$, $B = -2$, and $C = 6$.

$\therefore Y(s) = \frac{8}{(s+3)} + \frac{-2s+6}{(s^2+4)}$

$\Rightarrow L[Y(t)] = \frac{8}{s+3} - 2 \frac{s}{s^2+4} + \frac{6}{s^2+4}$

$\Rightarrow y(t) = 8 L^{-1}\left\{\frac{1}{(s+3)}\right\} - 2 L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3 L^{-1}\left\{\frac{2}{s^2+4}\right\}$

$$\boxed{y(t) = 8 e^{-3t} - 2 \cos(2t) + 3 \sin(2t)}$$

Sol. (ii) Using the Laplace transform for given equation, we get

$L[y''(t)] - 3 L[y'(t)] + 2 L[y(t)] = L[e^{-4t}]$

$\Rightarrow s^2 Y(s) - sY(0) - Y'(0) - 3[sY(s) - Y(0)] + 2Y(s) = \frac{1}{s+4}$

$\Rightarrow (s^2 - 3s + 2) Y(s) = s + 2 + \frac{1}{s+4}$ {Subsidiary equation}

$$\Rightarrow Y(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

Resolving $\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$ into partial fractions $\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$

we get $A = -\frac{16}{5}$, $B = \frac{25}{6}$, $C = \frac{1}{30}$.

$\therefore Y(s) = -\frac{16}{5}\left(\frac{1}{s-1}\right) + \frac{25}{6} \cdot \frac{1}{(s-2)} + \frac{1}{30} \cdot \frac{1}{(s+4)}$

$\Rightarrow y(t) = -\frac{16}{5} L^{-1}\left\{\frac{1}{(s-1)}\right\} + \frac{25}{6} L^{-1}\left\{\frac{1}{(s-2)}\right\} + \frac{1}{30} L^{-1}\left\{\frac{1}{s+4}\right\}$

$$\boxed{y(t) = -\frac{16}{5} e^{t} + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}}$$

Operational Properties I :-

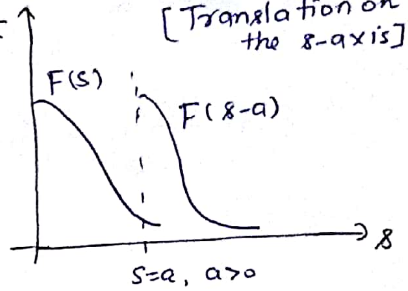
Theorem (First translation theorem)

If $L[f(t)] = F(s)$ and a is any real number then $L[e^{at} f(t)] = F(s-a)$.

Proof: $L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$.

[Translation on the s-axis]

If we consider s a real no., then the graph of $F(s-a)$ is the graph of $F(s)$ shifted on the s-axis by the amount $|a|$. If $a>0$, the graph of $F(s)$ is shifted a units to the right, whereas if $a<0$, the graph is shifted $|a|$ units to the left.



Shifts on s-axis

Inverse form of the above theorem: To compute the inverse of $F(s-a)$ we must recognize $F(s)$, find $f(t)$ by taking the inverse Laplace transform of $F(s)$, and then multiply $f(t)$ by the exponential function e^{at}.

i.e $$L^{-1}[F(s-a)] = e^{at} f(t), \text{ where } f(t) = L^{-1}[F(s)].$$

Problem ① Evaluate (a) $L^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$, (b) $L^{-1}\left\{\frac{s/2 + 5/3}{s^2+4s+6}\right\}$

Problem ② (i) Solve $y'' - 6y' + 9y = t^2 e^{3t}$, $y(0)=2$, $y'(0)=17$

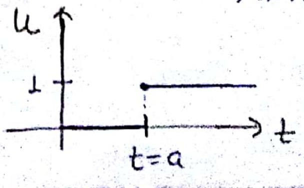
(ii) Solve $y'' + 4y' + 6y = 1 + e^{-t}$, $y(0)=0$, $y'(0)=0$.

Translation on the T-axis :-

Unit step function :- In engineering, one frequently encounters functions that are either "off" or "on". For example, an external force acting on a mechanical system or a voltage impressed on a circuit can be turned off after a period of time. It is convenient, then, to define a special function that is the number 0 (off) up to a certain time $t=a$ and then the number 1 (on) after that time. This function is called the unit ~~st~~ step or the Heaviside function.

The unit step function $u(t-a)$ is

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$



* The unit step function can also be used to write piecewise-defined functions in a compact form. For exp. a general piece-wise-defined function

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$

is same as

$$f(t) = g(t) - g(t)\,u(t-a) + h(t)\,u(t-a)$$

Similarly, a function of the type

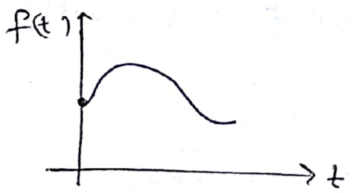
$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$

can be written as

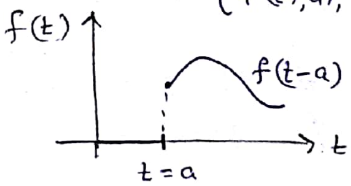
$$f(t) = g(t)\left[u(t-a) - u(t-b)\right].$$

* Consider a general function $y = f(t)$ defined for $t \ge 0$. The piecewise-defined function is

$$f(t-a)\,u(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}$$



$f(t),\ t \ge 0$



$f(t-a)\cdot u(t-a)$

} shifts on t-axis.

<u>Second Translation theorem</u>: If $F(s) = L\{f(t)\}$ and $a > 0$ then

$$L[f(t-a)\,u(t-a)] = e^{-as}F(s).$$

Alternative form:

$$L[g(t)\,u(t-a)] = e^{-as}L[g(t+a)]$$

<u>Proof</u>:- $\int_0^\infty e^{-st}\ \cancel{f(t)a)\,u(t-a)\,dt}$

$$L[f(t-a)\,u(t-a)] = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\,u(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)\,u(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

Let $v = t-a$ $\Rightarrow dt = dv$

$$= \int_0^\infty e^{-s(v+a)} f(v)\,dv = e^{-as}\int_0^\infty e^{-sv} f(v)\,dv$$

$$\boxed{L[f(t-a)\,u(t-a)] = e^{-as}F(s)}$$

$$\Rightarrow \boxed{L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)} \rightarrow \text{Inverse form of above theorem.}$$

Operational Properties II

Derivatives of Transform:- If $F(s) = L\{f(t)\}$ then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s), \quad n = 1, 2, 3, \ldots$$

Transforms of integrals:-

Convolution:- If functions f and g are piecewise continuous on $[0, \infty)$, then a special product, denoted by $f * g$, is defined by the integral

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

and is called the convolution of f and g.

$$\Rightarrow f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = g*f.$$

i.e. Convolution of two functions is commutative.

Convolution Theorem:- If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$L\{f*g\} = L\{f(t)\} \cdot L\{g(t)\} = F(s) \cdot G(s) \quad \text{--- (1)}$$

Inverse Laplace transform of above theorem is

$$L^{-1}\{F(s) \cdot G(s)\} = f*g.$$

Transform of an integral:- When $g(t) = 1$ and $L\{g(t)\} = G(s) = \frac{1}{s}$, the convolution theorem implies that the Laplace transform of the integral of f is

$$L\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}$$

$$\Rightarrow \int_0^t f(\tau)\, d\tau = L^{-1}\left\{\frac{F(s)}{s}\right\}$$

Exp (i) $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s} \cdot F(s)\right\} = \int_0^t \sin(\tau)\, d\tau = 1 - \cos t$

(ii) $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t \int_0^t (\sin(\tau)\, d\tau)\, d\tau = t - \sin(t) = \int_0^t \tau \sin(t-\tau)\, d\tau$ or

(iii) $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t \int_0^t \int_0^t \sin(\tau)\, d\tau\, d\tau\, d\tau = \frac{t^2}{2} - 1 + \cos(t) = \int_0^t \frac{\tau^2}{2} \sin(t-\tau)\, d\tau$ or

Periodic function:- If a function has period T, $T > 0$, then $f(t+T) = f(t)$.

The Laplace transform of a periodic function can be obtained by integration over one period.

Transform of a Periodic function :- If $f(t)$ is piecewise continuous on $[0,\infty)$, of exponential order, and periodic with period T, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Proof :- $\because$ $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt \quad \text{---} (1)$

Let $t = u+T$, then the last integral,

$\because f$ is periodic

$$\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(u+T)} f(u+T)\,du = e^{-sT}\int_0^\infty e^{-su} f(u)\,du$$

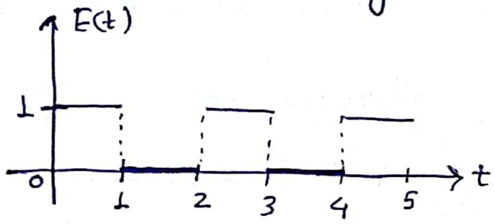
$$= e^{-sT} L\{f(t)\}$$

Using this in (1), we get

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt}$$

Q. Find the Laplace transform of the periodic function shown in figure:

Sol. The function $E(t)$ is called a square wave and has period $T=2$. For one period $E(t)$ can be defined as



$$E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t \le 2 \end{cases}$$

$\because E(t+2) = E(t)$

$$L\{E(t)\} = \frac{1}{1-e^{-s\cdot(2)}} \int_0^2 e^{-st} E(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}\cdot 1\,dt + \int_1^2 e^{-st}\cdot(0)\,dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{1}{1-e^{-2s}} \left\{\frac{1-e^{-s}}{s}\right\} = \frac{1}{s(1+e^{-s})}$$

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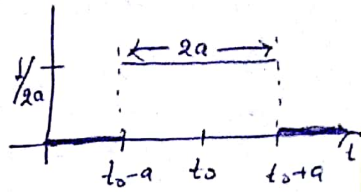
Unit Impulse :- Mechanical systems are often acted on by an external force (or electromotive force in an electrical circuit) of large magnitude that acts only for a very short period of time. The graph of the piecewise defined function

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \geq t_0 + a \end{cases}$$



for $a > 0$, $t > 0$ could serve as a model for such a force. For a small value of a, $\delta_a(t-t_0)$ is essentially a constant function of large magnitude that is "on" for just a very short period of time, around t_0. The function $\delta_a(t-t_0)$ is called a unit impulse, because it possesses the integration property

$$\int_0^\infty \delta_a(t-t_0)\,dt = 1.$$

Dirac Delta function:- In practice it is convenient to work with another type of unit impulse, a "function" that approximates $\delta_a(t-t_0)$ and is defined by the limit

$$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$$

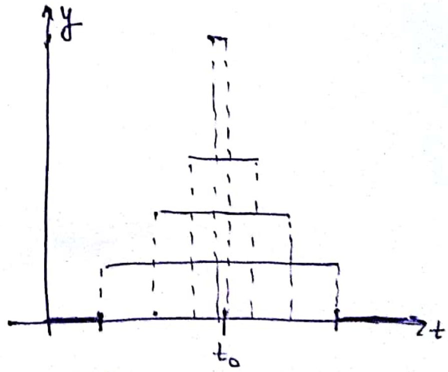
The latter expression, which is not a function at all, can be characterized by the two properties.

(i) $\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$ and (ii) $\int_0^\infty \delta(t-t_0)\,dt = 1.$

The unit impulse $\delta(t-t_0)$ is called the Dirac Delta function.

It is possible to obtain the Laplace transform of the Dirac delta function by the formal assumption that

$$L\{\delta(t-t_0)\} = \lim_{a\to 0} L\{\delta_a(t-t_0)\}.$$



Behaviour of δ_a as $a \to 0$.

Transform of the Dirac Delta function:

For $t_0 > 0$, $\quad L\{\delta(t-t_0)\} = e^{-st_0}$

* Let $t_0 = 0$ then $L\{\delta(t)\} = 1 \Rightarrow L^{-1}[1] = \delta(t)$.

Solve $y'' + y = 4\delta(t-2\pi)$ subject to

(a) $y(0)=1,\ y'(0)=0$ (b) $y(0)=0,\ y'(0)=0$.

The two initial-value problems could serve as models for describing the motion of a mass on a spring moving in a medium in which damping is negligible. At $t=2\pi$ the mass is given a sharp blow. In (a) the mass is released from rest 1 unit below the equilibrium position. ~~Be~~ In (b) the mass is at rest in the equilibrium position.

Sol. (a) $L[y''(t)] + L[y(t)] = 4L\{\delta(t-2\pi)\}$

$$\Rightarrow s^2Y(s) - s + Y(s) = 4e^{-2\pi s} \quad \text{or} \quad Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}$$

$$\therefore y(t) = L^{-1}\left\{\frac{s}{s^2+1}\right\} + 4L^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\}$$

Using IInd translational theorem i.e.
$L[f(t-a)u(t-a)] = e^{-as}F(s),\ a>0$.

$$= \cos(t) + 4\sin\{(t-2\pi)\}\,u(t-2\pi).$$

$$\boxed{y(t) = \cos(t) + 4\sin(t)\,u(t-2\pi)}$$

$$\because u(t-2\pi) = \begin{cases} 0, & 0\le t<2\pi \\ 1, & t\ge 2\pi \end{cases}$$