Introduction to Duality Theory

Every LPP has a corresponding mirror image formulation called the dual. If the original problem has n variables and m constraints, then its dual will have m variables and n constraints.

If a given LPP can be thought of resource allocation model in which the objective is to maximize revenue or profit subject to constraint on the consumption of resources, then its dual corresponds to an LPP which minimizes consumption of resources subject to some profit maximizing constraints.

Some interesting properties of dual LPP are :

(i) Any feasible solution of dual model provides a bound on the objective to the original primal model.

(ii) Optimal solution of dual = Optimal solution of primal.

(iii) Dual of a Dual model is once again the original primal model.

Primal – Dual formulation :

Primal		Dual
Min $c^T x$ Subject to $Ax \geq b$ $x \geq 0$	$\longleftrightarrow$	Max $b^T y$ Subject to $A^T y \leq c^T$ $y \geq 0$
Max $c^T x$ Subject to $Ax \leq b$ $x \geq 0$	$\longleftrightarrow$	Min $b^T y$ Subject to $A^T y \geq c^T$ $y \geq 0$

To be more explicitly, if the primal problem be in the following form :

Maximize $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Subject to
$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$$
$$\cdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$
$$x_1, x_2, \ldots x_n \geq 0.$$

Then its dual will be

Minimize $\quad w = b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$

Subject to

$$a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m \geq c_1$$

$$a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m \geq c_2$$

$$\cdots$$

$$a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m \geq c_n.$$

$$v_1, v_2, \cdots v_m \geq 0.$$

From the above formulations of the primal and dual problems, we observe the following:

(a) Number of variables in the dual is equal to the number of constraints in the primal and vice-versa.

(b) The elements of the requirement vector (not necessarily positive) in one problem are the respective prices in the objective function of the other problem.

(c) The row co-effecients of the primal constraints become the column coeffecient of dual constraints.

(d) One of the problem seeks maximization while other seeks minimization.

(e) If the primal maximization type problem has $\leq$ type constraints, the dual minimization problem has greater equals $\geq$ type constraints.

(f) The variable in both problems are non-negative.

Theorem: If any of the constraints in the primal problem be a perfect equality, then corresponding dual variable is unrestricted.

Theorem: If any variable of the primal problem be unrestricted in sign, then the corresponding constraint of the dual will be an equality.

Example 1: Formulate the dual of the following primal problem.

Maximize $Z = 2x_1 - 6x_2$

Subject to $x_1 - 3x_2 \le 6$

$2x_1 + 4x_2 \ge 8$

$x_1 - 3x_2 \ge -6$

$x_1, x_2 \ge 0$.

Soln → The given problem is in maximization form. We first rewrite this problem into canonical form.

Max $Z = 2x_1 - 6x_2$

S.t. $x_1 - 3x_2 \le 6$

$-2x_1 - 4x_2 \le -8$

$-x_1 + 3x_2 \le 6$, $x_1, x_2 \ge 0$.

Thus the primal is in form

Max $Z = c^T x$

S.t $Ax \le b$, $x \ge 0$

$c = (2 \quad -6)$ $\quad x = [x_1 \quad x_2]$

$$A = \begin{pmatrix} 1 & -3 \\ -2 & -4 \\ -1 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ -8 \\ 6 \end{pmatrix}.$$

There are three constraints and two variables in primal problem.

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Thus the dual problem has 3 variables and 2 constraints. Let v_1, v_2, v_3 be the ~~or~~ variables associated with dual. The corresponding dual formulation will be

$$\text{Minimize} \quad w = b'v = \begin{pmatrix} 6 & -8 & +6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Subject to $A'v \geq c'$, $v \geq 0$

$$A' = \begin{pmatrix} 1 & -2 & -1 \\ -3 & -4 & 3 \end{pmatrix}. \quad c' = \begin{pmatrix} 2 \\ -6 \end{pmatrix}.$$

This can be written as

$$\text{Min } w = 6v_1 - 8v_2 + 6v_3$$

S.t. $v_1 - 2v_2 - v_3 \geq 2$

$-3v_1 - 4v_2 + 3v_3 \geq -6$

$v_1, v_2, v_3 \geq 0$.

2. Max $z = 2x_1 + 3x_2 + x_3$

Subject to
$4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 5x_3 = 4$

$x_1; x_2, x_3 \geq 0$.

Soln

→ First we convert the problem into canonical form.

Max $Z = 2x_1 + 3x_2 + x_3$

S.t. $4x_1 + 3x_2 + x_3 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -6$

$x_1 + 2x_2 + 5x_3 \leq 4$

$-x_1 - 2x_2 - 5x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$.

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Written in matrix form

$$\text{Max } z = c^T x = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

s.t. $Ax \leq b$, $x \geq 0$

where $A = \begin{pmatrix} 4 & 3 & 1 \\ -4 & -3 & -1 \\ 1 & 2 & 5 \\ -1 & -2 & -5 \end{pmatrix}$ $\quad b = \begin{pmatrix} 6 \\ -6 \\ 4 \\ -4 \end{pmatrix}$

The primal has four constraints and three variables. So the dual has three constraints and four variables.

The corresponding dual will be

$$\text{Min } w = b'v = \begin{pmatrix} 6 & -6 & 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

s.t. $A'v \geq c'$, $v \geq 0$

$A' = \begin{pmatrix} 4 & -4 & 1 & -1 \\ 3 & -3 & 2 & -2 \\ 1 & -1 & 5 & -5 \end{pmatrix}$ $\quad c' = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$

In explicitly

$$\text{Min } w = 6v_1 - 6v_2 + 4v_3 - 4v_4$$

s.t. $4v_1 - 4v_2 + v_3 - v_4 \geq 2$

$3v_1 - 3v_2 + 2v_3 - 2v_4 \geq 3$

$v_1 - v_2 + 5v_3 - 5v_4 \geq 1$

~~The new vari~~ $\quad v_1, v_2, v_3, v_4 \geq 0$.

Let $v_1 - v_2 = y$, $\quad v_3 - v_4 = u$.

$$\text{Min } w = 6y + 4u$$

s.t. $4y + u \geq 2$

$3y + 2u \geq 3$

$y + 5u \geq 1$,

y, u are unrestricted in sign since it is the difference between non-negative variable

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3. Max $z = 2x_1 + 3x_2 + 4x_3$

Subject to $x_1 - 5x_2 + 3x_3 = 7$

$2x_1 - 5x_2 \le 3$

$3x_2 - x_3 \ge 5$

$x_1, x_2 \ge 0$, x_3 is unrestricted in sign.

Sol^n → Since x_3 is unrestricted in sign, $x_3 = x_3' - x_3''$

In ~~standard~~ canonical form,

Max $z = 2x_1 + 3x_2 + 4x_3' - 4x_3''$

s.t. $x_1 - 5x_2 + 3(x_3' - x_3'') \le 7$

$-x_1 + 5x_2 - 3(x_3' - x_3'') \le -7$

$2x_1 - 5x_2 \le 3$

$-3x_2 + (x_3' - x_3'') \le -5$

$x_1, x_2, x_3', x_3'' \ge 0$.

The dual problem will be

Min $w = 7v_1 - 7v_2 + 3v_3 - 5v_4$

S.t. $v_1 - v_2 + 2v_3 \ge 2$

$-5v_1 + 5v_2 - 5v_3 - 3v_4 \ge 3$

$3v_1 - 3v_2 + v_4 \ge 4$

$-3v_1 + 3v_2 - v_4 \ge -4$,

$v_1, v_2, v_3, v_4 \ge 0$

Let $v_1 - v_2 = u$

Min $w = 7u + 3v_3 - 5v_4$

$u + 2v_3 \ge 2$

$-5u - 5v_3 - 3v_4 \ge 3$

$\left.\begin{array}{l} 3u + v_4 \ge 4 \\ -3u - v_4 \ge -4 \end{array}\right\} \Rightarrow 3u + v_4 = 4$

$v_3, v_4 \ge 0$

u is unrestricted in sign

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Important result for solving primal-Dual problem:

Primal problem	Dual Problem	Conclusion
(i) Feasible solution	Feasible solution	Finite optimal for both exists
(ii) No feasible solution	Feasible sol^{n}	Dual objective f^{n} unbounded
(iii) Feasible sol^{n}	No feasible sol^{n}	Primal objective f^{n} unbounded
(iv) No feasible sol^{n}	No feasible sol^{n}	No solution exists

Duality and Simplex Method:

Suppose that an optimal solution to the dual problem has been obtained by the application of simplex method.

Rule 1: If the primal (dual) variable be related a slack or surplus variable in the dual problem, then its optimal solution is directly read off from the net evaluation row of the optimal dual simplex table as the net evaluation corresponding to this slack and/or surplus variable.

Rule 2: If the primal variable be related to an artificial variable in the dual problem, then its optimal value is directly read off from the net evaluation row of the optimal dual simplex table as the net evaluation relating to this artificial variable after putting the

the penalty cost M equal to zero.

Rule 3: If either problem (primal or dual) has unbounded solution, then the other will have no feasible sol^n.

Example: Construct the dual of the following LPP and solve both the primal and dual:

$$\text{Maximize } Z = 3x_1 + 4x_2$$

$$\text{Subject to } \quad x_1 + x_2 \le 12$$

$$2x_1 + 3x_2 \le 21$$

$$x_1 \le 8$$

$$x_2 \le 6$$

$$x_1, x_2 \geq 0.$$

→ The dual of the primal is

$$\text{Min } w = 12v_1 + 21v_2 + 8v_3 + 6v_4$$

$$\text{s.t.} \quad v_1 + 2v_2 + v_3 + 0v_4 \geq 3$$

$$v_1 + 3v_2 + 0v_3 + v_4 \geq 4$$

$$v_1, v_2, v_3, v_4 \geq 0.$$

Introducing two surplus variables v_5, v_6 to the constraints,

$$\text{Min } w = 12v_1 + 21v_2 + 8v_3 + 6v_4 + 0v_5 + 0v_6$$

$$\text{s.t.} \quad v_1 + 2v_2 + v_3 + 0v_4 - v_5 = 3$$

$$v_1 + 3v_2 + 0v_3 + v_4 - v_6 = 4$$

Here v_3, v_4 forms initial basis.

C_B	B	v_B	c_j / b	12 / a_1	21 / a_2	8 / a_3	6 / a_4	0 / a_5	0 / a_6	Min Ratio
8	a_3	v_3	3	1	2	1	0	-1	0	$3/2$
6	a_4	v_4	4	1	$\boxed{3}$	0	1	0	-1	$4/3 \rightarrow$
			$z_j - c_j$	2	13 ↑	0	0	-8	-6	

This is minimization problem and hence the vector with maximum most positive $(z_j - c_j)$ will enter the basis. The optimality will reach when $z_j - c_j \leq 0 \; \forall j$.

In the next table, $v_2 \rightarrow$ entering variable

$v_4 \rightarrow$ leaving variable.

C_B	B	v_B	c_j / b	12 / a_1	21 / a_2	8 / a_3	6 / a_4	0 / a_5	0 / a_6
8	a_3	v_3	$1/3$	$1/3$	0	1	$-2/3$	-1	$+2/3$
21	a_2	v_2	$4/3$	$1/3$	1	0	$1/3$	0	$-1/3$
			$z_j - c_j$	$-7/3$	0	0	$-\frac{13}{3}$	-8	$-5/3$

So, $z_j - c_j \leq 0 \; \forall j$. Optimal solution of the dual is $v_1^* = 0$, $v_2^* = 4/3$, $v_3^* = 1/3$, $v_4^* = 0$.

$W_{min} = 12 \times 0 + 21 \times \frac{4}{3} + 8 \times \frac{1}{3} + 6$

$= \frac{92}{3}$.

Optimal soln of the primal will be $z_j - c_j$ of columns of surplus variables v_5, v_6 with changed sign of dual optimal table

$\therefore \; x_1^* = 8, \quad x_2^* = \frac{5}{3}, \quad Z_{max} = 3 \times 8 + 4 \times \frac{5}{3}$

$= \frac{92}{3}$

So $Z_{max} = W_{min}$.