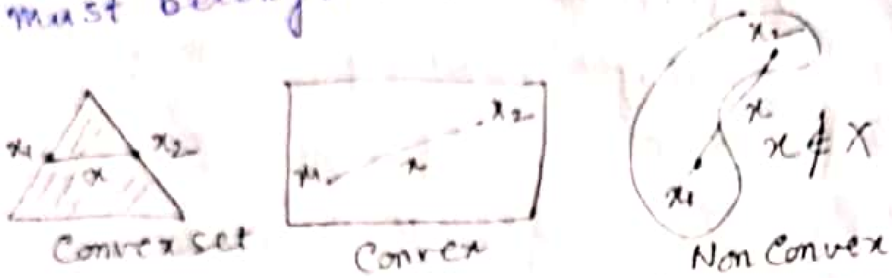
Geometry of LPP

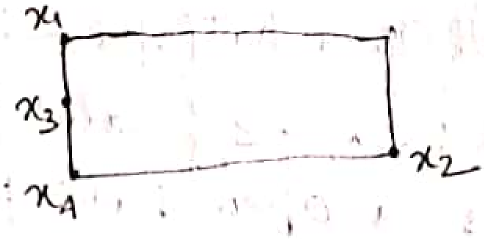
Some terminologies:

→ Convex Set: A set $X \subseteq R^n$ is said to be convex if for any two points $x_1, x_2 \in X$, the line segment joining x_1, x_2 will also lie in the set. Mathematically if $x_1, x_2 \in X$, then there exists $\lambda_1 \in [0,1]$ such that $x = \lambda_1 x_1 + (1-\lambda_1)x_2$, $0 \le \lambda_1 \le 1$ must belongs to X.



→ Line Segment: Let $X_1 = (x_1^1, x_2^1, \ldots x_n^1)$ and $X_2 = (x_1^2, x_2^2, \ldots, x_n^2) \in R^n$.
The line segment joining this X_1 and X_2 is a collection of points of the form X where $X = (x_1, x_2, \ldots x_n)$ which will satisfy the equation $X = \lambda X_1 + (1-\lambda)X_2$, where $0 \le \lambda \le 1$

→ Extreme point: A point x is said to be an extreme point of a convex set if and only if there does not exist any two points x_1, x_2 where $x_1 \ne x_2$ which satisfy $x = \lambda_1 x_1 + (1-\lambda_1)x_2$ where $0 < \lambda_1 < 1$. i.e. x is not a convex combination of x_1 and x_2.



x_1 is an extreme point of this rectangle, similarly for the same other point x_2. If we consider the line segment joining some other two points x_1, x_2, x_4 will not also lie there. x_3 is not an extreme point. If we consider another point x_4, and join x_1 and x_4. Then x_3 will always lie on that line.

→ Hyperplane: Suppose $x = (x_1, x_2, \ldots x_n) \in R^n$ represents a point. If this point satisfies $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$, then this equation represents one hyperplane for a given values of c_i's and z. Therefore for some values of $c_1, c_2, \ldots c_n$ and some values of z, the objective function will represent one hyperplane.

You may have parallel hyperplane also. i.e. $c_1 x = z_1$, $c_2 x = z_2$. These two hyper planes will be parallel if $c_1 = \lambda c_2$, $\lambda \neq 0$.

In other sense, two hyperplane will be parallel if they have same unit normal.

$\therefore \frac{c_1}{c_2} = \text{constant}$.

→ Closed and Open halfspace:

$H_+^o = \{x : cx > z\}$ and $\{x : cx < z\}$

We call it as a open halfspace.

$H_+^c = \{x : cx \geq z\}$ and $\{x : cx \leq z\}$

We call it as a closed halfspace.

→ Convex Polyhedron: If we consider finite number of linearly independent vectors, then convex combination of all of these linearly independent vectors is known as a convex polyhedral. So the convex combination of finite number of linearly independent vectors form a convex polyhedron. If $x_1, x_2, \ldots x_n$ are linearly independent vectors, then we could find a set

$$X = \{x : x = \sum_{i=1}^{n} \lambda_i x_i \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}$$

is known as one convex polyhedron.

Geometrical interpretation:

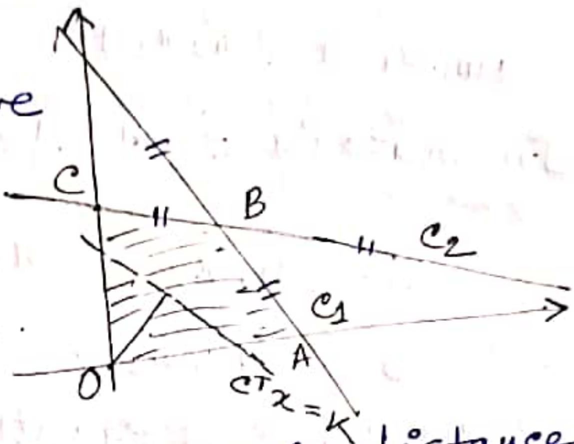
(i) Constraints: Each constraint defines one half spaces. Half spaces means it may be open half space or it may be closed half space.

(II) Feasible region: This feasible region is nothing but the convex polyhedron. It is defined as the intersection of the halfspaces. Halfspaces are nothing but the constraints. Feasible region is the region which is bounded by these constraints and it is a convex polyhedron.

(III) Objective function: Objective function represents a hyperplane of the form $z = cx$ for given values of c_i's and z.

One of the objective function can be written as

$z = c^T x = k$, say.



If we draw a line $c^T x = k$, k is distance of line from the origin.

For maximization problem, I can move hyperplane for the objective function away from the origin. For minimization problem, this line will come closer to our origin.

(IV) In figure, we take c_1 and c_2 as constraints with equality signs. Since the variables are non negative, we consider only positive quadrant.

Extreme points will be the bound of intersection of the boundary of these four lines. So, Extreme points are O, A, B, C.

→ Basic Solution:

Consider ~~a~~ the following system of linear simultaneous equation:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

In matrix form $AX = b$, $A = (a_{ij})_{m\times n}$, $m \neq n$.

$$X = (x_1, x_2, \cdots x_n)^T$$

$$b = (b_1, b_2, \cdots, b_m)^T$$

So ~~you~~ we have m equations and n variables

let $m < n$. $rank(A) = m$.

Therefore there will exist one matrix B of order $m\times m$ such that $rank(B) = m$.

According to definition, the matrix $[B]_{m\times m}$ will be non singular.

Since $rank(A) = m$, let $a_1, a_2, \cdots a_m$ are linearly independent columns of A.

So $Ax = Bb$ can be transformed to a system with m equations and m unknowns which is represented by $[B]_{m\times m}$.

So, this is equivalent to a matrix B with m equations and m variables and the remaining $(n-m)$ variables will be zero. This matrix $[B]_{m \times m}$ is called as basis matrix and corresponding m decision variables which are attached to m linearly independent columns, are known as basic variables and the remaining $n-m$ variables are known as non basic variables.

We can say that $x_B = (x_{B_1}, x_{B_2}, \dots x_{B_m})$ are the basic variables of the system obtained from $Ax = b$. The solution, obtained from this system will be $\begin{pmatrix} x_B \\ 0 \end{pmatrix}$. So $\begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is known as the basic solution of the system $Ax = b$.

This obtained basic solution is not unique. This is not the only solution. If we have m equations and n variables, then we can obtain maximum $^{n}C_m = \frac{n!}{m!\,(n-m)!}$ number of basic solution by making all the possible combinations

Example :

Find the basic solution of the following system and hence find the solution.

$$x_1 + x_2 + x_3 = 9$$

$$2x_1 - 4x_2 + 3x_3 = 4.$$

→ If we compare this system with $Ax = b$, then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -4 & 3 \end{pmatrix}. \quad b = \begin{pmatrix} 9 \\ 4 \end{pmatrix}.$$

There are 3 variables, 2 equations. So the number of maximum basic solution will be $^3C_2 = \frac{3!}{2!\,1!}$

$$= 3.$$

$\text{rank}(A) = 2 = \text{rank}(A|b)$.

We construct from $A = (a_1 \ a_2 \ a_3)$,

$$B_1 = (a_1 \ a_2) = \begin{pmatrix} 1 & 1 \\ 2 & -4 \end{pmatrix}$$

$$B_2 = (a_1 \ a_3) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$B_3 = (a_2 \ a_3) = \begin{pmatrix} 1 & 1 \\ -4 & 3 \end{pmatrix}.$$

So the basic solution x_{B_i} will take the form $x_{B_i} = B_i^{-1} b$.

$$x_{B_1} = B_1^{-1} b = \begin{pmatrix} 20/3 \\ 7/3 \end{pmatrix}$$

$$x_{B_2} = B_2^{-1} b = \begin{pmatrix} 23 \\ -14 \end{pmatrix}$$

$x_{B_3} = B_3^{-1} b = \begin{pmatrix} 23/7 \\ 40/7 \end{pmatrix}$.

Since the basic solution x_{B_1} corresponds to (a_1, a_2), So a_3 will be 0.

So, one soln can be $\begin{pmatrix} 20/3 \\ 7/3 \\ 0 \end{pmatrix}$.

For x_{B_2}, it corresponds to $(a_1\ a_3)$,

So a_2 will be 0.

So one soln will be $\begin{pmatrix} 23 \\ 0 \\ -14 \end{pmatrix}$.

For x_{B_3}, it corresponds to $(a_2\ a_3)$,

So a_1 will be ~~t~~ 0.

So one solution will be $\begin{pmatrix} 0 \\ 23/7 \\ -14/7 \end{pmatrix}$.

→ Basic Feasible Solution: A feasible solution to a linear programming problem, which is also basic, is called the basic feasible solution. Basic Feasible Solution are of two types:

(i) Degenerate: A basic feasible solution is called degenerate if value of atleast one basic variable is zero.

(ii) Non degenerate: A basic feasible solution is called non degenerate if all values of m basic are non zero and positive.

Ex: Find all the basic feasible solution of the system and classify.

$$2x_1 + 6x_2 + 3x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2.$$