Online-Class 15-02-2021

Probability, Statistics and Reliability (MAT3003)

SLOT: B21 + B22 + B23

MODULE - 2

Topic: Determination of pdf and cdf for any Random Variable

Reference: Probability and Statistics for Engineers & Scientists
by R.E.Walpole, Pearson (2012).

Probability Density Function pdf $f(x)$ for a CRV

Definition

The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)\, dx$.

Properties of pdf $f(x)$

$P(a \le X \le b)$ or $P(a < X < b)$ is defined as

$$P(a \le X \le b) = \int_a^b f(x)\,\mathrm{d}x.$$

The curve $y = f(x)$ is called *the probability curve* of the RV X.

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Properties of pdf $f(x)$ … $contd.$

When X is a continuous RV

$$P(X = a) = P(a \leq X \leq a) = \int_a^a f(x)\,dx = 0$$

This means that it is almost impossible that a continuous RV assumes a specific value. Hence,
$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b).$

$$P(a < X \leq b) = P(a < X < b) + P(X = b) = P(a < X < b).$$

That is, it does not matter whether we include an endpoint of the interval or not.

This is not true, though, when X is discrete.

Example 1

- Suppose that the error in the reaction temperature, in ◦C, for a controlled laboratory experiment is a continuous random variable *X* having the probability density function:

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \leq 1)$.

Solution

(a) Obviously, $f(x) \geq 0$.

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{2} \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1.$$

(b) $$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3} dx = \left.\frac{x^3}{9}\right|_0^1 = \frac{1}{9}.$$

Cumulative Distribution Function $cdf\ F(x)$

Definition

If X is an RV, discrete or continuous then $P(X \leq x)$ is called the *cumulative distribution function* of X or *distribution function* of X and denoted as $F(x)$.

If X is discrete,

$$F(x) = \sum_{\substack{j \\ x_j \leq x}} P_j$$

If X is continuous,

$$F(x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

Properties of the cdf $F(x)$

1. $F(x)$ is a non-decreasing function of x, i.e., if $x_1 < x_2$, then $F(x_1) \le F(x_2)$.

2. $F(-\infty) = 0$ and $F(\infty) = 1$.

3. If X is a discrete RV taking values $x_1, x_2, \ldots$, where $x_1 < x_2 < x_3 < \ldots < x_{i-1} < x_i < \ldots$, then $P(X = x_i) = F(x_i) - F(x_{i-1})$.

4. If X is a continuous R V, then $\frac{d}{dx} F(x) = f(x)$, at all points where $F(x)$ is differentiable.

5. $P(a < X < b) = F(b) - F(a)$

Reference: Probability and Statistics for Engineers & Scientists
by R.E.Walpole, Pearson (2012).

6. When X is CRV, the following is true:

$$P(a < X \leq b) = P(a < X < b) + P(X = b) = P(a < X < b).$$

That is, it does not matter whether we include an endpoint of the interval or not.

This is not true, though, when X is discrete.

Reference: Probability and Statistics for Engineers & Scientists by R.E.Walpole, Pearson (2012).

Question 1

- The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate b. The DOE has determined that the density function of the winning (low) bid is

$$f(y) = \begin{cases} \frac{5}{8b}, & \frac{2}{5}b \le y \le 2b, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $F(y)$ and use it to determine the probability that the winning bid is less than the DOE's preliminary estimate b.

Solution

For $2b/5 \leq y \leq 2b$,

$$F(y) = \int_{2b/5}^{y} \frac{5}{8b} dy = \frac{5t}{8b}\bigg|_{2b/5}^{y} = \frac{5y}{8b} - \frac{1}{4}.$$

Thus,

$$F(y) = \begin{cases} 0, & y < \frac{2}{5}b, \\ \frac{5y}{8b} - \frac{1}{4}, & \frac{2}{5}b \leq y \leq 2b, \\ 1, & y > 2b. \end{cases}$$

To determine the probability that the winning bid is less than the preliminary bid estimate b, we have

$$P(Y \leq b) = F(b) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}.$$

Question 2

Consider the density function

$$f(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Evaluate k.

(b) Find $F(x)$ and use it to evaluate

$$P(0.3 < X < 0.6).$$

Solution

(a) $1 = k\int_0^1 \sqrt{x}\, dx = \frac{2k}{3}x^{3/2}\Big|_0^1 = \frac{2k}{3}$. Therefore, $k = \frac{3}{2}$.

(b) For $0 \le x < 1$, $F(x) = \frac{3}{2}\int_0^x \sqrt{t}\, dt = t^{3/2}\Big|_0^x = x^{3/2}$.
Hence,

$$F(x) = \begin{cases} 0, & x < 0 \\ x^{3/2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$$P(0.3 < X < 0.6) = F(0.6) - F(0.3) = (0.6)^{3/2} - (0.3)^{3/2} = 0.3004.$$

Practice Questions

1. The proportion of people who respond to a certain mail-order solicitation is a continuous random variable X that has the density function:

$$f(x) = \begin{cases} \frac{2(x+2)}{5}, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that $P(0 < X < 1) = 1$.

(b) Find the probability that more than 1/4 but fewer than 1/2 of the people contacted will respond to this type of solicitation. Ans. 19/80.

2. An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond, is

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \le t < 3, \\ \frac{1}{2}, & 3 \le t < 5, \\ \frac{3}{4}, & 5 \le t < 7, \\ 1, & t \ge 7, \end{cases}$$

find

	Ans.
(a) $P(T = 5)$;	$1/4$.
(b) $P(T > 3)$;	$1/2$.
(c) $P(1.4 < T < 6)$;	$1/2$.
(d) $P(T \le 5 \mid T \ge 2)$.	$\frac{2}{3}$.

THANK YOU