Example 4.0.14

Find a vector of unit length which is orthogonal to $(1, 3, 4)$ in $V_3(\mathbb{R})$ with standard inner product.

Let $x = (x_1, x_2, x_3)$ be orthogonal to $y = (1, 3, 4)$ in $V_3(\mathbb{R})$.

$$\begin{aligned} \langle x, y\rangle &= 0 \\ \langle (x_1, x_2, x_3), (1, 3, 4)\rangle &= 0 \\ 1.x_1 + 3.x_2 + 4.x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0 \end{aligned}$$

Put $x_1 = 1$, $x_2 = 1$, we get

$$\begin{aligned} 1 + 3(1) + 4x_3 &= 0 \\ 4x_3 &= -4 \\ x_3 &= -1 \end{aligned}$$

$x = (1, 1, -1)$ is orthogonal to $(1, 3, 4)$.

Orthogonal vector is

$$\left(\frac{x}{\|x\|}\right) = \left(\frac{x_1}{\|x\|}, \frac{x_2}{\|x\|}, \frac{x_3}{\|x\|}\right)$$

$$\frac{x}{\|x\|} = \frac{(1, 1, -1)}{1^2 + 1^2 + (-1)^2} = \frac{1}{3}(1, 1, -1) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$$

Therefore, $\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$ is unit length vector orthogonal to $(1, 3, 4)$.

Problem 4.0.15

Apply gram-schmidt orthogonalization process to construct an orthonormal basis for $V_3(\mathbb{R})$ with the standard inner product for the basis $\{v_1, v_2, v_3\}$ where $v_1 = (1, 0, 1)$, $v_2 = (1, 3, 1)$, $v_3 = (3, 2, 1)$.

Consider a basis $\{v_1, v_2, v_3\}$ of $V_3(\mathbb{R})$, where $v_1 = (1, 0, 1)$, $v_2 = (1, 3, 1)$ and $v_3 = (3, 2, 1)$.
To find w_1,

$$\begin{aligned} w_1 &= v_1 = (1, 0, 1) \\ \|w_1\|^2 &= \langle w_1, w_1 \rangle \\ &= 1.1 + 0.0 + 1.1 \\ &= 1^2 + 0^2 + 1^2 = 2 \\ \|w_1\| &= \sqrt{2} \end{aligned}$$

To find w_2,

$$\begin{aligned}\langle v_2, w_1\rangle &= \langle (1,3,1),(1,0,1)\rangle \\ &= 1.1 + 3.0 + 1.1 \\ &= 2 \\ w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1 \\ &= (1,3,1) - \frac{2}{(\sqrt{2})^2}(1,0,1) \\ &= (1,3,1) - (1,0,1) \\ &= (1-1, 3-0, 1-1) \\ &= (0,3,0)\end{aligned}$$

Therefore,

$$\begin{aligned}\|w_2\|^2 = \langle w_2, w_2\rangle &= \langle (0,3,0),(0,3,0)\rangle \\ &= 0^2 + 3^2 + 0^2 = 9 \\ \|w_2\| &= 3\end{aligned}$$

To find w_3,

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2$$

$$\langle w_3, w_1\rangle = \langle (3,2,1),(1,0,1)\rangle = 3.1 + 2.0 + 1.1 = 3 + 0 + 1 = 4$$

$$\langle w_3, w_2\rangle = \langle (3,2,1),(0,3,0)\rangle = 3.0 + 2.3 + 1.0 = 0 + 6 + 0 = 6$$

$$\begin{aligned} w_3 &= (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9}(0,3,0) \\ &= (3,2,1) - (2,0,1) - (0,2,0) \\ &= (3-2-0, 2-0, 2, 1-1-0) \\ w_3 &= (1,0,0) \end{aligned}$$

Therefore,

$$\begin{aligned} \|w_3\|^2 &= \langle w_3, w_3\rangle \\ &= 1^2 + 0^2 + 0^2 = 1 \\ \|w_3\| &= 1 \end{aligned}$$

The orthogonal basis is

$$\{w_1, w_2, w_3\} = \{(1,0,1), (0,3,0), (1,0,0)\}$$

The orthonormal basis is

$$\begin{aligned}\left\{\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|}\right\} &= \left\{\frac{1}{\sqrt{2}}(1,0,1), \frac{1}{3}(0,3,0), \frac{1}{1}(1,0,0)\right\} \\ &= \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0), (1,0,0)\right\}\end{aligned}$$