Simplex Method:

Most real world problems concerns more than two variable. Therefore a linear programming problem of multivariables are too complex for graphical solution. A procedure called the simplex method might be used to find the optimal solution of multivariable linear programming problems.

Simplex Algorithm:

Step 1: Convert the general form of linear programming problem into standard form and also convert the minimization problem into maximization type.

Step 2: Write the initial simplex table and obtain the initial basic feasible solution.

Step 3: Compute $\Delta_j = Z_j - C_j$ for all j.

Step 4: Select the key column with most negative $Z_j - C_j$ value. let $k \in J$, index set for which $Z_j - C_j$ is most negative, x_k is entering variable in basis.

Step 5: Select the key row with minimum non-negative b_i / a_{ij}. If all ratios are negative or infinity, the current solution is unbounded and stop the computation. let $l \in J$ for which b_i/a_{ij} is minimum, x_l is departing basic variable.

Step 6: Identify the key element at the intersection of key row and key column.

(xi) Step 7: Make key element as one and the corresponding other element in that column as zero ~~to~~ and prepare new simplex table with x_k as new basic variable.

Step 8: Compute $z_j - c_j$ for new simplex table. If it is negative for some j, then repeat Step 4 – Step 7. If $z_j - c_j \geq 0$ $\forall j$, then optimal solution is attained.

Example: $\text{Max } z = 60x_1 + 50x_2$

Subject to
$$x_1 + 2x_2 \leq 40$$
$$3x_1 + 2x_2 \leq 60$$
$$x_1, x_2 \geq 0.$$

→ Both constraints are $\leq$ type.
Hence introducing slack variables x_3 and x_4, we reformulate the LPP in the standard form as

$$\text{Max } z = 60x_1 + 50x_2 + 0.x_3 + 0.x_4$$

Subject to
$$x_1 + 2x_2 + x_3 + 0.x_4 = 40$$
$$3x_1 + 2x_2 + 0.x_3 + x_4 = 60$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Here $(c_1, c_2, c_3, c_4) = (60, 50, 0, 0)$

$$a_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad a_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad a_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad a_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We see that the vectors a_3, a_4 form the initial basis and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = B^{-1}$$

$$x_B = B^{-1} b = I \cdot \begin{pmatrix} 40 \\ 60 \end{pmatrix} = \begin{pmatrix} 40 \\ 60 \end{pmatrix}$$

Thus $\begin{pmatrix} x_{B_1} \\ x_{B_2} \end{pmatrix} = \begin{pmatrix} 40 \\ 60 \end{pmatrix}$.

$c_B = (c_{B_1}, c_{B_2}) = (0, 0)$.

Iteration : 1

		c_j		60	50	0	0	Min
C_B	B	x_B	b	a_1	a_2	a_3	a_4	b_i/a_{ij}
0	a_3	x_3	40	1	2	1	0	$40/1 = 40$
0	a_4	x_4	60	$\boxed{3}$	2	0	1	$60/3 = 20 \rightarrow$
		$Z_j - c_j$		-60	-50	0	0	

key element

$Z_j = c_{B_j}^T a_j$

$Z_1 - c_1 = 0 \cdot 1 + 0 \cdot 3 - 60 = -60$

$Z_2 - c_2 = 0 \cdot 2 + 0 \cdot 2 - 50 = -50$

$Z_3 - c_3 = 0 \cdot 1 + 0 \cdot 0 - 0 = 0$

$Z_4 - c_4 = 0 \cdot 0 + 0 \cdot 1 - 0 = 0$

x_1 ~~a_4~~ $\rightarrow$ entering variable

$x_4 \rightarrow$ leaving variable

Iteration : 2

		c_j		60	50	0	0	Min
C_B	B	x_B	b	a_1	a_2	a_3	a_4	b_i/a_{ij}
0	a_3	x_3	20	0	$\boxed{4/3}$	1	$-1/3$	$15 \rightarrow$
60	a_1	x_1	20	1	$2/3$	0	$1/3$	30
		$Z_j - c_j$		0	-10	0	20	

~~$R_1' \rightarrow R_2 - 3R_1$~~

$R_2' \rightarrow R_2\left(\frac{1}{3}\right)$

~~$R_1' \rightarrow R_2' - R_1$~~

$x_2 \rightarrow$ entering variable

$x_3 \rightarrow$ leaving variable

Iteration : 3

			c_j	60	50	0	0	Min b_i/a_{ij}
C_B	B	x_B	b	a_1	a_2	a_3	a_4	
50	a_2	x_2	15	0	1	3/4	−1/4	
60	a_1	x_1	10	1	0	−1/2	1/2	
			$z_j - c_j$	0	0	15/2	35/2	

~~$R_1' \to R_1\left(\frac{3}{4}\right)$~~

~~$R_2' \to R_1' - \frac{3}{2}R_2 - R_1'$~~

~~$R_2' \to R_2 - \frac{2}{3}R_1'$~~

Here $z_j - c_j \geq 0 \;\forall j$

Hence this table gives optimal solution.

Optimal solution is

$x_1^* = 10, \quad x_2^* = 15 \quad Z_{max} = 60 \times 10 + 50 \times 15$

$= 1350.$

~~$\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{4/2}$~~

~~$20 - \frac{2}{3} \cdot 15$~~

Note →

Row Operations in Simplex ~~Second~~ table

(i) Divide the second row of the 1st table by key element

(ii) New value in 1st row ~~= Old value − $\frac{\text{corr keyrow}}{\text{va}}$~~

$= \text{old value} - \dfrac{\text{corr key row value} \times \text{corr key column value}}{\text{Key element}}$

Second Iteration

$b_1' = 40 - \dfrac{1 \times 60}{3} = 20$

$a_{11}' = 1 - \dfrac{1 \times 3}{3} = 0.$

$a_{21}' = 2 - \dfrac{1 \times 2}{3} = 4/3$

$a_{31}' = 1 - \dfrac{1 \times 0}{3} = 1$

$a_{41}' = 0 - \dfrac{1 \times 1}{3} = -1/3$

Final iteration

$b_2'' = 20 - \dfrac{\frac{2}{3} \times 20}{4/3} = 10$

$a_{12}' = 1 - \dfrac{2/3 \times 0}{4/3} = 1$

$a_{22}' = 2/3 - \dfrac{2/3 \times 4/3}{4/3} = 0$

$a_{32}' = 0 - \dfrac{2/3 \times 1}{4/3} - 1/2$

$a_{42}' = \frac{1}{3} - \dfrac{2/3 \times (-1/3)}{4/3}$

$= \frac{1}{2}$

Example 2: Max $z = x_1 + x_2 + 3x_3$

Subject to $3x_1 + 2x_2 + x_3 \leq 3$

$2x_1 + x_2 + 2x_3 \leq 2$

and $x_1, x_2, x_3 \geq 0$.

→ Introducing slack variables x_4, x_5 and put the problem into standard form, we obtain,

Max $z = x_1 + x_2 + 3x_3 + 0 \cdot x_4 + 0 \cdot x_5$

Subject to $3x_1 + 2x_2 + x_3 + x_4 + 0 \cdot x_5 = 3$

$2x_1 + x_2 + 2x_3 + 0 \cdot x_4 + x_5 = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$.

$a_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $a_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $a_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $a_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$a_5 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $c = (1, 1, 3, 0, 0)$

So, the vectors a_4, a_5 give the initial basis ~~..~~ matrix and x_4, x_5 are corresponding basic variables and initial basic solution is $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 3, x_5 = 2$

Table - 1

		c_j		1	1	3	0	0	Min
c_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Ratio (x_B/a_{i3})
0	a_4	x_4	3	3	2	1	1	0	$3/1 = 3$
0	a_5	x_5	2	2	1	$\boxed{2}$	0	1	$2/2 = 1$ →
		$z_j - c_j$		-1	-1	-3 ↑	0	0	

x_3 ~~a_3~~ → entering variable

x_5 ~~a_5~~ → leaving variable

Table-2

		c_j		1	1	3	0	0
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5
0	a_4	x_4	2	2	3/2	0	1	-1/2
3	a_3	x_3	1	1	1/2	1	0	1/2
	$z_j - c_j$			2	1/2	0	0	3/2

$$b_1' = 3 - \frac{1 \times 2}{2} = 2$$

$$a_{11}' = 3 - \frac{1 \times 2}{2} = 2$$

$$a_{21}' = 2 - \frac{1 \times 1}{2} = 3/2$$

$$a_{31}' = 1 - \frac{1 \times 2}{2} = 0$$

$$a_{41}' = 1 - \frac{1 \times 0}{2} = 1$$

$$a_{51}' = 0 - \frac{1 \times 1}{2} = -1/2$$

All $z_j - c_j \geq 0 \quad \forall j$

Hence the required optimal solution is

$x_1^* = 0, \ x_2^* = 0, \ x_3^* = 1$. $\quad Z_{max} = 1 \times 0 + 1 \times 0 + 3 \times 1 = 3$.

Problem Set

(1) Max $z = 2x_1 + x_2 - 3x_3 + 5x_4$

Subject to

$$x_1 + 2x_2 + 3x_3 + 4x_4 \leq 40$$
$$2x_1 - x_2 + x_3 + 2x_4 \leq 8$$
$$4x_1 - 2x_3 + x_3 - x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

[Ans: $x_1^* = 0, \ x_2^* = 6, \ x_3^* = 0, \ x_4^* = 7, \ Z_{max} = 41$]

(2) Minimize $z = x_1 - 3x_2 + 2x_3$

Subject to
$$3x_1 - x_2 + 3x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0.$$

[Ans: $x_1^* = 4$, $x_2^* = 5$, $x_3^* = 0$, $z_{min} = -11$]

(3) Maximize $z = 3x_1 + 2x_2 + 5x_3$

Subject to
$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420,$$
$$x_1, x_2, x_3 \ge 0.$$

[Ans. $x_1^* = 0$, $x_2^* = 100$, $x_3^* = 230$, $z_{max} = 1350$]

(4) Maximize $Z = 3x_1 + 5x_2 + 4x_3$

Subject to
$$2x_1 + 3x_2 \le 8$$
$$2x_1 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0.$$

[Ans: $x_1^* = 89/41$, $x_2^* = 50/41$, $x_3^* = 62/41$

$z_{max} = 765/41$.]