## Example 4.0.6

Show that the following function defines an inner product on $\mathbb{R}^2$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are define by $\langle u, v\rangle = u_1v_1 + 2u_2v_2$.

(1) To prove Conjugate symmetry:

$$\begin{aligned}\langle u, v\rangle &= u_1v_1 + 2u_2v_2\\ &= v_1u_1 + 2v_2u_2\\ &= \langle v, u\rangle\end{aligned}$$

(2) To prove linearity:
Let $w = (w_1, w_2)$

$$\begin{aligned}\langle u, v+w\rangle &= \langle (u_1, u_2), (v_1 + w_1, v_2 + w_2)\rangle\\ &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2)\\ &= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)\\ &= \langle u, v\rangle + \langle u, w\rangle\end{aligned}$$

(3) Let $c \in \mathbb{R}$, then

$$\begin{aligned} c\langle u, v\rangle &= c(u_1 v_1 + 2u_2 v_2) \\ &= (cu_1)v_1 + 2(cu_2)v_2 \\ &= \langle cu, v\rangle \end{aligned}$$

(4) To prove non-negativity:

$$\langle v, v\rangle = v_1^2 + 2v_2^2 \geq 0$$

Moreover,

$$\begin{aligned} & \langle v, v\rangle = 0 \\ \Leftrightarrow \quad & v_1^2 + 2v_2 = 0 \\ \Leftrightarrow \quad & v_1 = 0,\ v_2 = 0,\ \text{since, } v_1^2 \geq 0, v_2 \geq 0 \\ \Leftrightarrow \quad & v = (0, 0) \end{aligned}$$

## Definition 4.0.7 (Length of a vector)

The length or norm of a vector is the non-negative scalar $\|v\|$ defined by

$$\|v\| = \sqrt{v.v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Suppose

$$\vec{V} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Then,

$$\|v\|^2 = a^2 + b^2$$
$$\|v\| = \sqrt{a^2 + b^2}$$

## Definition 4.0.8 (Distance)

The distance between $\vec{u}$ and $\vec{v}$ can be found by

$$dist(\vec{u}, \vec{v}) = \|u - v\|$$

## Example 4.0.9

Compute $dist(\vec{u},\vec{v})$ for $\begin{bmatrix}3\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\4\end{bmatrix}$.

$$\begin{aligned}
dist(\vec{u},\vec{v}) &= \|\vec{u}-\vec{v}\| \\
u-v &= \begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}-1\\4\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix} \\
\|\vec{u}-\vec{v}\| &= \sqrt{4^2+(-2)^2} \\
&= \sqrt{16+4} \\
&= \sqrt{20}
\end{aligned}$$

## Example 4.0.10

Find the distance between $\vec{u} = \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$.

$$\begin{aligned} dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| &= \sqrt{9^2 + 3^2 + 5^2} \\ &= \sqrt{81 + 9 + 25} \\ &= \sqrt{115} \end{aligned}$$

## Definition 4.0.11

Angle between vectors are defined by

$$\cos \omega = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

## Problem 4.0.12

*Find the angle between* $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

To find angle between the vectors

$$\cos\omega = \frac{x^T y}{\sqrt{x^T x \times y^T y}}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\sqrt{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}}$$

$$\cos\omega = \frac{3}{\sqrt{10}} = 0.9486$$

$$\omega \approx \cos^{-1} 0.9486$$

$$\omega \approx 0.32 \frac{180}{\pi} \approx 18^0$$

## Problem 4.0.13

*Find the angle between* $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To find angle between the vectors

$$\cos\omega = \frac{x^T y}{\sqrt{x^T x \times y^T y}}$$

$$\cos\omega = 0$$

$$\omega \approx \frac{\pi}{2} = 90^0$$