Wolfe's modified Simplex

Consider the following QPP:

$$\text{Maximize } z = f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j c_{jk} x_k$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n$$

where $c_{jk} = c_{kj}$ for all j, k and $b_i \geq 0$ for all $i = 1, 2, \ldots m$.

Iterative procedure:

Step 1: First introduce slack variable q_i^2 in the ith constraint $(i = 1, 2, \ldots m)$ and slack variable r_j^2 in the jth non negative constraint $(j = 1, 2, \ldots n)$ to convert these constraints into equality.

Step 2: Construct Lagrangian function

$$L(x, q, r, \lambda, \mu) = f(x) - \sum_{i=1}^{m} \lambda_i \left[\sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2 \right] - \sum_{j=1}^{n} \mu_j \left[-x_j + r_j^2 \right]$$

where $x = (x_1, x_2, \ldots x_n)$, $q = (q_1^2, q_2^2, \ldots q_m^2)$, $r = (r_1^2, r_2^2, \ldots r_n^2)$ and $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_m)$, $\mu = (\mu_1, \mu_2, \ldots \mu_n)$.

Differentiating L partially w.r.t x, q, r, λ, μ and equating to zero, derive Kuhn Tucker necessary conditions.

Step 3: Introduce the non negative artificial variable v_j, $j = 1, 2 \ldots n$ in the Kuhn Tucker conditions

$$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0, \quad j = 1, 2, \ldots n$$

and construct an objective fn

$$Z_v = v_1 + v_2 + \cdots + v_n.$$

Step 4: Obtain an initial BFS to the following LPP: $\text{Min } Z_v = v_1 + v_2 + \cdots + v_n$

subject to $\sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + v_j = -c_j$, $j = 1, 2, \ldots n$.

$$\sum_{j=1}^{n} a_{ij} x_j + q_i^2 = b_i, \quad i = 1, 2, \ldots m$$

$v_j, \lambda_i, \mu_j, x_j \geq 0$, $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$.

We have the complementary slackness condn $\sum_{j=1}^{n} \mu_j x_j + \sum_{i=1}^{m} \lambda_i s_i = 0$, where $s_i = q_i^2$

or $\lambda_i s_i = 0$ and $\mu_j x_j = 0$, $i = 1, 2 \ldots m$
$j = 1, 2, \ldots n$.

Step 5: Apply two phase simplex method in the usual manner to find an optimum soln to the LPP constructed in Step 4. The solution must satisfy the above complementary slackness condn.

Step 6: The optimum soln thus obtained in Step 5 gives the optimum soln of QPP.

Note: ① If the QPP is given in minimization form, then convert it into maximization form with $\leq$ type constraints.

② Modify the simplex algorithm to include the complementary slackness cond^n.

③ The required solution is obtained in Phase 1 of simplex method. As our aim is to obtain a feasible sol^n, we need not to consider Phase II.

④ Phase I ends with the sum of all artificial variables equal to zero, provided that the feasible sol^n of the problem exists.

Ex. Apply Wolfe's method for solving quadratic programming problem.

$$\text{Max } z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

$$\text{s.t.} \quad x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0.$$

Sol^n Step 1: First we write all the constraint inequaties with $\leq$ sign as follows:

$$x_1 + 2x_2 \leq 2, \; -x_1 \leq 0, \; -x_2 \leq 0.$$

Step 2: Now introducing the slack variables q_1^2, r_1^2, r_2^2, our constraints becomes

$$x_1 + 2x_2 + q_1^2 = 2$$

$$-x_1 + r_1^2 = 0$$

$$-x_2 + r_2^2 = 0$$

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Step 3: We construct the Lagrangian function

$L(x_1, x_2, q_1, r_2, \lambda_1, \mu_1, \mu_2) = (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$

$- \lambda_1 (x_1 + 2x_2 + q_1^2 - 2) - \mu_1(-x_1 + r_1^2)$
$- \mu_2(x_2 + r_2^2)$.

The necessary and sufficient cond^ns for optimum of L are

$\frac{\partial L}{\partial x_1} = 4 - 4x_1 - 2x_2 - \lambda_1 + \mu_1 = 0,$

$\frac{\partial L}{\partial x_2} = 6 - 2x_1 - 4x_2 - 2\lambda_1 + \mu_2 = 0.$

Taking $s_1 = q_1^2$, we have the complementary slackness cond^ns

$\lambda_1 s_1 = 0, \ \mu_1 x_1 = 0, \ \mu_2 x_2 = 0.$

Also $x_1 + 2x_2 + s_1 = 2$ and $x_1, x_2, s_1, \lambda_1, \mu_1, \mu_2 \geq 0$

Step 4: Now, we introduce the artificial variables v_1, v_2 and construct the following LPP:

Max $z^* = -v_1 - v_2$

s.t. $4x_1 + 2x_2 + \lambda_1 - \mu_1 + v_1 = 4$

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + v_2 = 6$

$x_1 + 2x_2 + s_1 = 2$

where all variables are non negative and $\mu_1 x_1 = 0, \ \mu_2 x_2 = 0, \ \lambda_1 s_1 = 0.$

Step 5: The initial simplex table for phase I is shown below.

		c_j		0	0	0	0	0	-1	-1	0
C_B	B	x_B	b	x_1	x_2	λ_1	μ_1	μ_2	v_1	v_2	S_1
-1	v_1	v_1	4	$\boxed{4}$	2	1	-1	0	1	0	0
-1	v_2	v_2	6	2	4	2	0	-1	0	1	0
0	S_1	S_1	2	1	2	0	0	0	0	0	1
		$z_j - c_j$		-6 ↑	-6	-3	1	1	0 ↓	0	0

(Complementary slackness conditions,
$\lambda_1 S_1 = 0$, $\mu_1 x_1 = 0$, $\mu_2 x_2 = 0$).

Since μ_1 is not in the basis, x_1 can be entering vector.

Considering min ratio $= \left\{ \frac{4}{4}, \frac{6}{2}, \frac{2}{1} \right\} = 1 \rightarrow v_1$

Then v_1 will be leaving vector.

Step 6:

		c_j		0	0	0	0	0	-1	-1	0
C_B	B	x_B	b	x_1	x_2	λ_1	μ_1	μ_2	v_1	v_2	S_1
0	x_1	x_1	1	1	1/2	1/4	-1/4	0	1/4	0	0
-1	v_2	v_2	4	0	3	3/2	1/2	-1	-1/2	1	0
0	S_1	S_1	1	0	$\boxed{3/2}$	-1/4	1/4	0	-1/4	0	1
		$z_j - c_j$		0	-3 ↑	-3/2	-1/2	1	3/2	0	0 ↓

Since μ_2 is not in the basis, x_2 will be entering vector.

Considering min ratio $= \left\{ \frac{1}{1/2}, \frac{4}{3}, \frac{1}{3/2} \right\} = \frac{2}{3}$

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Therefore S_1 will be leaving vector

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Step 7 :

C_B	B	x_B	c_j / b	0 / x_1	0 / x_2	0 / λ_1	0 / μ_1	0 / μ_2	-1 / v_1	-1 / v_2	0 / S_1
0	x_1	x_1	2/3	1	0	1/3	$-1/3$	0	1/3	0	$-1/3$
-1	v_2	v_2	2	0	0	$\boxed{2}$	0	-1	0	1	-2
0	x_2	x_2	2/3	0	1	$-1/6$	1/6	0	$-1/6$	0	2/3
		$z_j - c_j$		0	0	-2	0	1	1	0	2

Since S_1 is not in the basis, λ_1 can enter the basis with v_2 as departing vector.

Step 8 :

C_B	B	x_B	b	x_1	x_2	λ_1	μ_1	μ_2	v_1	v_2	S_1
0	x_1	x_1	1/3	1	0	0	$-1/3$	1/6	1/3	$-1/6$	0
0	λ_1	λ_1	1	0	0	1	0	$-1/2$	0	1/2	-1
0	x_2	x_2	5/6	0	1	0	1/6	$-1/12$	$-1/6$	1/12	1/2
		$z_j - c_j$		0	0	0	0	0	1	1	0

Since $z_j - c_j \geq 0 \ \forall j$, this iteration gives optimal sol^n for phase 1. Since $z^* = 0$, the given sol^n is feasible. Thus the required optimal sol^n is $x_1^* = 1/3$, $x_2^* = 5/6$.

$$Z_{max} = 25/6.$$