Definition 2.7.1 (Similar matrices)

Let A and B be two square matrices of same order, A is said to be similar to matrix B if there exists a non-singular matrix P, such that

$$B = P^{-1}AP$$

Definition 2.7.2 (Properties of similar matrices)

Similar matrices have same eigen values, eigen vectors, determinant, ranks, nullity, characteristic polynomial and traces.

Definition 2.7.3 (Procedure to find similar matrix)

If A is given

- **Step I** Characteristic polynomial $A - \lambda I$ by using $|A - \lambda I| = 0$.
- **Step II** Find eigen values $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$.
- **Step III** Find eigen vector $v_1, v_2, v_3, \cdots, \lambda_n$ using eigen values.
- **Step IV** Find P by combining all eigen values into one matrix.
- **Step V** Find P^{-1} from P.
- **Step VI** Find $B = P^{-1}AP$

B is called similar matrix.

Problem 2.7.4

Find similar matrix for $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ and λ be eigen value of A then

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \\ |A - \lambda I| &= (2-\lambda)(-2-\lambda) - (-1)(3) \\ &= -4 - 2\lambda + 2\lambda + \lambda^2 + 3 \\ &= \lambda^2 - 1 \end{aligned}$$

To find eigen values:

$$|A - \lambda I| = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda^2 = 1$$
$$\lambda = \pm 1$$

Therefore eigen values are $\lambda_1 = 1$, $\lambda_2 = -1$

To find eigen vectors:

$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

At $\lambda = 1$

$$\begin{bmatrix} 2-1 & 3 \\ -1 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ x_1 &= -3x_2 \\ \text{Let } x_2 &= t \\ \text{then, } x_1 &= -3t \\ v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \end{aligned}$$

At $\lambda = -1$

$$(A - \lambda I)v = 0$$

$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+1 & 3 \\ -1 & -2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 3x_1 + 3x_2 &= 0 \\ x_1 + x_2 &= 0 \\ x_1 &= -x_2 \\ \text{Let } x_2 &= -t \\ x_1 &= -t \\ v_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

To find P matrix,

$$P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$|P| = (-3)(1) - (1)(-1) = -3 + 1 = -2 \neq 0.$$

P is non-singular.

$$P^{-1} = \frac{1}{|A|} Adj(P) = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

$$P^{-1}AP = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -6+3 & -2+3 \\ 3-2 & 1-2 \end{pmatrix}$$

$$= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -3+1 & 1+1 \\ 3-3 & -1+3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To check

$$
\begin{aligned}
|B-\lambda I| &= 0 \\
\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)(-1-\lambda)-0 &= 0 \\
-1-\lambda+\lambda+\lambda^2 &= 0 \\
\lambda^2 &= 1 \\
\lambda &= \pm 1
\end{aligned}
$$

1. Eigen values of B matrix are similar to A matrix. So, $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. $|A| = |B| = -1$
3. $Trace(A) = Trace(B)$ i.e., $2-2=0=1-1$

Example 2.7.5

Find similar matrix for the following matrix

1. $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$