Online-Class 10-02-2021

Probability, Statistics and Reliability (MAT3003)

SLOT: B21 + B22 + B23

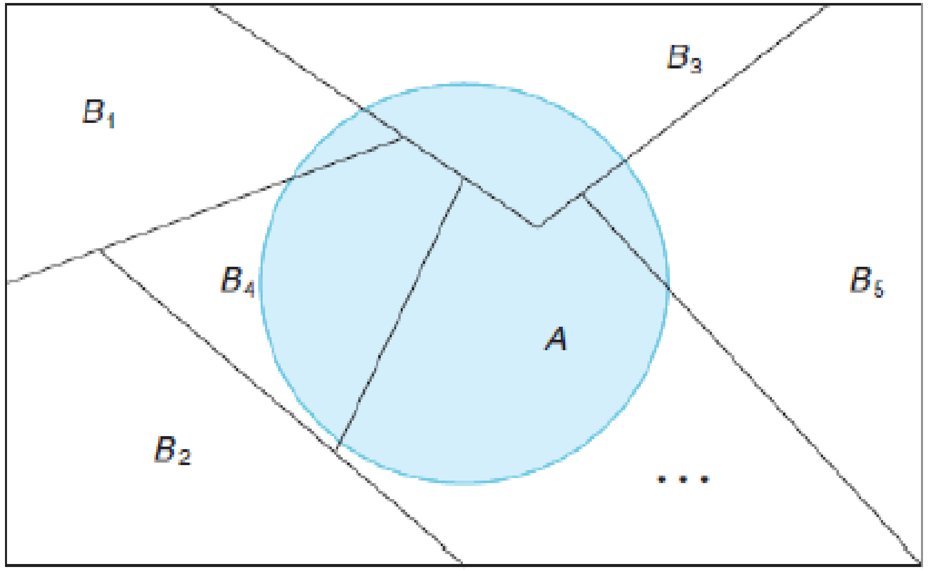
MODULE - 1

Topic: Total Probability Rule, and Bayes’ Theorem

Reference: Probability and Statistics for Engineers & Scientists
by R.E.Walpole, Pearson (2012).

Brief Contents

Total Probability Theorem



If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

Reference: Probability and Statistics for Engineers & Scientists
by R.E.Walpole, Pearson (2012).

Bayes' Theorem (Bayes' Rule or Bayes' Formula)

(Bayes' Rule) If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

Reference: Probability and Statistics for Engineers & Scientists
by R.E.Walpole, Pearson (2012).

Proof:

By the definition of conditional probability,

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)},$$

and then using Total Probability Theorem in the denominator, we have

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)},$$

which completes the proof.

When to Apply Bayes' Theorem?

Apply Bayes' theorem when the following conditions exist.

- The sample space is partitioned into a set of mutually exclusive and exhaustive events $B_1, B_2, \ldots, B_n$.
- Within the sample space, there exists an event A, for which $P(A) \neq 0$.
- The analytical goal is to compute a conditional probability of the form $P(B_r/A)$, for r=1, 2, 3,...
- We know at least one of the two sets of probabilities described below:
  - $P(B_r \cap A)$ for each B_r.
  - $P(B_r)$ and $P(B_r/A)$ for each B_r.

Class Activities on Total Probability and Bayes' Rule

Activity 1:

- Consider B1 and B2 as two mutually exclusive and exhaustive events. Let A be any event in the sample space. Then, write the formula for

  (i) P(A) = ?

  (ii) P(B1/A) = ?

  (iii) P(B2/A) = ?

Activity 2:

- Consider B1 ,B2 and B3 as three mutually exclusive and exhaustive events. Let A be any event in the sample space. Then, write the formula for

  (i) P(A) = ?

  (ii) P(B1/A) = ?

  (iii) P(B2/A) = ?

  (iv) P(B3/A) = ?

Activity 3:

(Law of Total Probability) A population can be divided into two subgroups that occur with probabilities 60% and 40%, respectively. An event *A* occurs 30% of the time in the first subgroup and 50% of the time in the second subgroup. What is the unconditional probability of the event *A,* regardless of which subgroup it comes from? Ans: 0.38

Question 1

In a certain assembly plant, three machines, $B1$, $B2$, and $B3$, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Solution

Consider the following events:

A: the product is defective,

B_1: the product is made by machine B_1,

B_2: the product is made by machine B_2,

B_3: the product is made by machine B_3.

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$
$$= (0.3)(0.02) + (0.45)(0.03) + (0.25)(0.02)$$
$$= 0.006 + 0.0135 + 0.005 = 0.0245.$$

Question 2

Police plan to enforce speed limits by using radar traps at four different locations within the city limits. The radar traps at each of the locations $L1$, $L2$, $L3$, and $L4$ will be operated 40%, 30%, 20%, and 30% of the time.

(a) If a person who is speeding on his way to work has probabilities of 0.2, 0.1, 0.5, and 0.2, respectively, of passing through these locations, what is the probability that the person will receive a speeding ticket? Ans: 0.27

(b) If the person received a speeding ticket on his way to work, what is the probability that he passed through the radar trap located at $L2$? Ans: 0.11

Reference: Probability and Statistics for Engineers & Scientists by R.E.Walpole, Pearson (2012).

Solution

(a) $P = .4(.2) + .3(.1) + .2(.5) + .3(.2) = .08 + .03 + .10 + .06 = .27.$

Question 3 (For Students)

A paint-store chain produces and sells Latex paint and Semi-gloss paint. Based on long-range sales, the probability that a customer will purchase Latex paint is 0.75.

Of those that purchase Latex paint, 60% also purchase Rollers. Only 30% of semi-gloss paint buyers purchase Rollers. A randomly selected buyer purchases a Roller and a can of paint. What is the probability that the paint is Latex?

Solution

L: Latex paint   S: Semiglass paint

R: Roller

$$P(L/R) = \frac{P(R|L)P(L)}{P(R|L)P(L)+P(R|S)P(S)}$$

$$= \frac{(0,60)(0,75)}{(0,60)(0,75)+(0,30)(0,25)}$$

$$= 0,857$$

Question 4

A worker-operated machine produces a defective item with probability 0.01 if the worker follows the machine's operating instructions exactly, and with probability 0.03 if he does not .

- If the worker follows the instructions 90% of time, what proportion of all items produced by the machine will be defective?
- Given that a defective item is produced, what is the conditional probability of the event that the worker exactly follows the machine operating instructions?

Reference: Probability and Statistics for Engineers & Scientists by R.E.Walpole, Pearson (2012).

Solution

D : Machine produces a defective item.
F : Worker follows instructions.
Then, we have following information:

$$\mathbb{P}(D|F) = 0.01 \qquad \mathbb{P}(F) = 0.9$$

$$\mathbb{P}(D|F^c) = 0.03 \qquad \mathbb{P}(F^c) = 0.1$$

According to the law of total probability, we have

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(D|F)\mathbb{P}(F) + \mathbb{P}(D|F^c)\mathbb{P}(F^c) \\ &= 0.01(0.9) + 0.03(0.1) = 0.012.\end{aligned}$$

According to the Bayes' Rule, we have

$$\mathbb{P}(F|D) = \frac{\mathbb{P}(F)\mathbb{P}(D|F)}{\mathbb{P}(D)} = \frac{0.9(0.01)}{0.012} = 0.75$$

Question 5 (For Students)

Whether a grant proposal is funded quite often depends on the reviewers. Suppose a group of research proposals was evaluated by a group of reviewers as to whether the proposals were worthy of funding.

When these same proposals were submitted to a second independent group of reviewers, the decision to fund was reversed in 30% of the cases. If the probability that a proposal is judged worthy of funding by the first group of experts is 0.2, what are the probabilities of these events?

(a) A worthy proposal is approved by both groups.

(b) A worthy proposal is disapproved by both groups.

(c) A worthy proposal is approved by one group.

Reference: Probability and Statistics for Engineers & Scientists by R.E.Walpole, Pearson (2012).

Solution

Define

$$A_1 := \{\text{The worthy proposal is approved by first group}\};$$
$$A_2 := \{\text{The worthy proposal is approved by second group}\};$$
$$D_1 := \{\text{The worthy proposal is disapproved by first group}\};$$
$$D_2 := \{\text{The worthy proposal is disapproved by second group}\}.$$

Then, $\mathbb{P}(A_1) = 0.2$, and $\mathbb{P}(D_1) = 0.8$.

Since the decision to fund was reversed in 30%, we have

$$\mathbb{P}(D_2|A_1) = 0.3 \qquad \mathbb{P}(A_2|D_1) = 0.3$$

According to the complement relationship. we have

$$\mathbb{P}(A_2|A_1) = 0.7 \qquad \mathbb{P}(D_2|D_1) = 0.7$$

Thus,

(a) $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1) = 0.2(0.7) = 0.14$

Practice Questions: Application Oriented Questions

1. (Violent Crime) City crime records show that 20% of all crimes are violent and 80% are nonviolent, involving theft, forgery, and so on. Ninety percent of violent crimes are reported versus 70% of nonviolent crimes.

(a). What is the overall reporting rate for crimes in the city?

(b). If a crime in progress is reported to the police, what is the probability that the crime is violent? What is the probability that it is nonviolent?

(c). Refer to part (b), if a crime in progress is reported to the police, why is it more likely that it is a nonviolent crime? Wouldn't violent crimes be more likely to be reported? Can you explain these results?

2. (Worker Error) A worker-operated machine produces a defective item with probability .01 if the worker follows the machine's operating instructions exactly, and with probability .03 if he does not. If the worker follows the instructions 90% of the time, what proportion of all items produced by the machine will be defective?

3. (Airport Security) Suppose that, in a particular city, airport *A* handles 50% of all airline traffic, and airports *B* and *C* handle 30% and 20%, respectively. The detection rates for weapons at the three airports are 0.9, 0.8, and 0.85, respectively. If a passenger at one of the airports is found to be carrying a weapon through the boarding gate, what is the probability that the passenger is using airport *A*? Airport *C*?

4. (Medical Diagnostics) Medical case histories indicate that different illnesses may produce identical symptoms. Suppose a particular set of symptoms, which we will denote as event *H,* occurs only when any one of three illnesses—*A, B,* or *C*—occurs. (For the sake of simplicity, we will assume that illnesses *A, B,* and *C* are mutually exclusive.) Studies show these probabilities of getting the three illnesses:

$$P(A)=0.01,\ P(B)=0.005,\ \text{and}\ P(C)=0.02.$$

The probabilities of developing the symptoms *H,* given a specific illness, are

$$P(H|A) \quad .90$$

$$P(H|B) \quad .95$$

$$P(H|C) \quad .75$$

Assuming that an ill person shows the symptoms *H,* what is the probability that the person has illness *A*?

THANK YOU