Problem 2.2.1

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z).$$

Find basis and dimension of it's Range and Null space.

$$\begin{aligned} N(T) &= \{T(x, y, z) = (0, 0, 0)\} \\ (x + 2y - z, y + z, x + y - 2z) &= (0, 0, 0) \\ x + 2y - z &= 0 \\ y + z &= 0 \\ x + y - 2z &= 0 \end{aligned}$$

$$\begin{aligned} y &= -z \\ x - 2z - z &= 0 \\ x &= 3z \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -z \\ z \end{bmatrix} \Rightarrow z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$N(T) = \{T(x, y, z) = (0, 0, 0)\} = (3, -1, 1)$$

$$R(T) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1, R_3 \Rightarrow R_3 + R_1$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$dim(R(T)) = 2$$
$$Basic = (1,0,1)(0,1,-1)$$

Problem 2.2.2

Let V be vector space 2×2 matrices over R and $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. Let be $T : V \to V$ be linear transform defined by $T(A) = PA$. Find basis and dim of null space of T and Range space of T.

$$N(T) = \{T(A) = 0 : A \in V$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

$$PA = 0$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\begin{bmatrix} a - c & b - d \\ -2a + 2c & -2b + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} a - c &= 0 \\ -2a + 2c &= 0 \\ a &= c \end{aligned} \qquad\qquad \begin{aligned} b - d &= 0 \\ -2b + 2d &= 0 \\ b &= d \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

To find basis:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(E_1) = PE_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$T(E_2) = PE_2 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$T(E_3) = PE_3 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T(E_4) = PE_4 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$T(E_1) = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}; T(E_2) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}; T(E_3) = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}; T(E_4) = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$R_3 = R_3 + R_1; R_4 = R_4 + R_2$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Range space of T is

$$\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Rank of dimension of range space = 2

Problem 2.2.3

Let w_1 and w_2 be the subspace generated by $(-1, 2, 1)$, $(2, 0, 1)$ and $(-8, 4, -1)$ in $\mathbb{R}^3(\mathbb{R})$ and w_2 generated by all vectors $(a, 0, b)$ $\forall a, b \in \mathbb{R}$. Find basis and dimension of w_1, w_2 and $w_1 + w_2$.

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 1 \\ -8 & 4 & -1 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$; $R_3 = R_3 - 8R_1$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & -12 & -9 \end{bmatrix}$$

$R_3 = R_3 + 3R_2$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Basis = (-1,2,1) \text{ and } (0,4,3)$$
$$dim(w_1) = 2$$

$$R(w_2) = (a,0,b)$$
$$= a\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

$$Basis(w_2) = B_2 = (1,0,0),(0,0,1)$$
$$dim(w_1) = 2$$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{2}R_2$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + \frac{1}{2}R_4$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$dim(w_1 + w_2) = 3$$
$$Basis(w_1 + w_2) = (-1, 2, 1), (0, 4, 3), (0, 0, 1)$$