Artificial Variable Techniques

In '$\geq$' type inequalities, we have to subtract surplus variable to reformulate the LPP into standard form. In that case, surplus variables donot provide an initial basic feasible solution. So the question may arise how to start the initial table of simplex Method and proceed.

LPP in which constraints may also have $\geq$ or $=$ signs after ensuring that all $b_i \geq 0$ are considered in this section. In such cases, basis matrix cannot be obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called Artificial variable. These variables are fictitions and cannot have any physical meaning. The artificial variable technique is a device to get the starting basic feasible solution, so that simplex procedure may be adopted untill the optional solution is reached. To solve such LPP, there are two methods.

(1) The Big-M method (Charne's method of penalty)

(2) The two phase method

The Big M method:

The following steps are involved in solving a LPP using the Big M method.

Step 1: Express the problem in the standard form.

Step 2: Add non-negative artificial variables to left hand side of the equations corresponding to the constraints of $\geq$ type after surplus variables. However addition of these artificial variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the final solution. This achieved by assigning a very large penalty ($-M$ for maximization and $+M$ for minimization) in the objective function.

Step 3: Solve the modified LPP by Simplex method, untill anyone of the three cases may arise.

(i) If no artificial variable appears in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.

(ii) If atleast one artificial variable presents in the basis at zero level and optimality condition is satisfied, then the current solution is an optimal basic feasible solution (though degenerate solution)

(iii) If atleast one artificial variable appears in the basis at positive level and optimality condition is satisfied, then original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function, since it contains a very large penalty M and is called pseudo optimal solution.

Ex 1. Solve the following LPP using Big M method:

$$\text{Minimize } Z = 2x_1 + x_2$$

Subject to

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0.$$

→ Introducing slack, surplus and artificial variable, the problem becomes

$$\text{Max } Z' = -2x_1 - x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$$

$$3x_1 + x_2 + x_5 = 3$$

$$4x_1 + 3x_2 - x_3 + x_6 = 6$$

$$x_1 + 2x_2 + x_4 = 4$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

x_5, x_6 → artificial variable

x_3 → surplus variable

x_4 → slack variable

x_5, x_6, x_4 forms basis vector.

Table-1

			c_j	-2	-1	0	0	-M	-M	Min ratio
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	
-M	a_5	x_5	3	$\boxed{3}$	1	0	0	1	0	$3/3=1 \rightarrow$
-M	a_6	x_6	6	4	3	-1	0	0	1	$6/4=3/2$
0	a_4	x_4	4	1	2	0	1	0	0	$4/1=4$
		$Z_j - c_j$		$2-7M$	$1-4M$	M	0	0	0	

↑

$z_1 - c_1 = -3M - 4M + 2 = 2 - 7M$

$z_2 - c_2 = -M - 3M + 1 = 1 - 4M$

$z_3 - c_3 = M$ $\qquad x_1 \rightarrow$ entering variable

$z_4 - c_4 = 0$ $\qquad x_5 \rightarrow$ leaving variable

$z_5 - c_5 = 0$

$z_6 - c_6 = 0$

Table-2

			c_j	-2	-1	0	0	-M	-M	Min ratio
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	
-2	a_1	x_1	1	1	1/3	0	0	1/3	0	$1/1/3=3$
-M	a_6	x_6	~~6~~ 2	0	$\boxed{5/3}$	-1	~~-4/3~~ 0	-4/3	1	$2/5/3=6/5 \rightarrow$
0	a_4	x_4	3	0	5/3	0	1	-1/3	0	$3/5/3=9/5$
		$Z_j - c_j$		0	$(1-5M)/3$	M	0	$\frac{7M-2}{3}$	0	

↑

$x_2 \rightarrow$ entering variable

$x_6 \rightarrow$ leaving variable

~~Key row in~~

$R_1' \leftrightarrow R_1 \left(\frac{1}{3}\right)$

for R_2', R_3', new value = old value − $\frac{\text{key corr row value} \times \text{corr key column value}}{\text{Key element}}$

$b_2' = \cancel{6} - \frac{4\times3}{3} = 2$ $\qquad a_{42}' = 0 - \frac{0\times4}{3} = 0$

$a_{12}' = 4 - \frac{3\times4}{3} = 0$ $\qquad a_{52}' = 0 - \frac{1\times4}{3} = -4/3$

$a_{22}' = 3 - \frac{1\times4}{3} = 5/3$ $\qquad a_{62}' = 1 - \frac{0\times4}{3} = 1$

$a_{32}' = -1 - \frac{0\times4}{3} = -1$

Table-3

C_B	B	x_B	c_j / b	-2 / a_1	-1 / a_2	0 / a_3	0 / a_4	$-M$ / a_5	$-M$ / a_6
-2	a_1	x_1	$3/5$	1	0	$1/5$	0	$3/5$	$-1/5$
-1	a_6	x_2	$6/5$	0	1	$-3/5$	0	$-4/5$	$3/5$
0	a_4	x_4	1	0	0	1	1	1	-1
		$z_j - c_j$		0	0	$1/5$	0	$M-\frac{2}{5}$	$M-\frac{1}{5}$

$R_2' \leftrightarrow R_2\left(\frac{1}{5/3}\right)$

For R_1', R_3', new value $=$ old value $- \dfrac{\text{corr key row value} \times \text{corr key column value}}{\text{key element}}$

$$b_1' = 1 - \frac{\frac{1}{3}\times 2}{5/3} = \frac{3}{5} \qquad a_{41}' = 0 - \frac{0\times 1/3}{5/3} = 0$$

$$a_{11}' = 1 - \frac{0\times 1/3}{5/3} = 1 \qquad a_{51}' = 1/3 - \frac{(-4/3)\times 1/3}{5/3}$$

$$a_{21}' = 1/3 - \frac{1/3 \times 5/3}{5/3} = 0 \qquad = \frac{1}{3} + \frac{4}{15} = \frac{9}{15} = \frac{3}{5}$$

$$a_{31}' = 0 - \frac{(-1)\times 1/3}{5/3} = 1/5 \qquad a_{61}' = 0 - \frac{1\times 1/3}{5/3} = -1/5$$

$$b_3' = 3 - \frac{5/3 \times 2}{5/3} = 1 \qquad a_{43}' = 1 - \frac{0\times 5/3}{5/3} = 1$$

$$a_{13}' = 0 - \frac{0\times 5/3}{5/3} = 0 \qquad a_{53}' = -1/3 - \frac{(-4/3)\times 5/3}{5/3}$$

$$= 1$$

$$a_{23}' = 5/3 - \frac{5/3\times 5/3}{5/3} = 0 \qquad a_{63}' = 0 - \frac{1\times 5/3}{5/3} = -1$$

$$a_{33}' = 0 - \frac{(-1)\times 5/3}{5/3} = 1$$

Since $M > 0$ is very large quantity, so, $z_j - c_j \geq 0 \ \forall j$

mal solution,

$x_1^* = 3/5 \qquad x_2^* = 6/5$

$$z'_{max} = -2\times\frac{3}{5} - \frac{6}{5} = -\frac{12}{5}$$

$$\therefore Z_{min} = -z'_{max} = \frac{12}{5} \text{ Ans.}$$

Example 2. Minimize $Z = 4x_1 + x_2$

Subject to
$$3x_1 + x_2 = 50$$
$$4x_1 + 3x_2 \geq 24$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

$\rightarrow$

Introducing slack variable, surplus variable and artificial variable in LPP and rewriting in maximization form

$$\text{Max } Z' = -4x_1 - x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$$

$$3x_1 + x_2 + x_5 = 50$$
$$4x_1 + 3x_2 - x_3 + x_6 = 24$$
$$x_1 + 2x_2 + x_4 = 3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

$x_3 \rightarrow$ surplus variable $\quad$ $x_5, x_6 \rightarrow$ artificial variable

$x_4 \rightarrow$ slack variable

x_5, x_6, x_4 forms initial basis matrix.

Table - 1

			c_j	-4	-1	0	0	$-M$	$-M$	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min. ratio
$-M$	a_5	x_5	50	3	1	0	0	1	0	$50/3$
$-M$	a_6	x_6	24	4	3	-1	0	0	1	$24/4 = 6$
0	a_4	x_4	3	$\boxed{1}$	2	0	1	0	0	$3/1 = 3 \rightarrow$
			$Z_j - C_j$	$4 - 7M$	$-4M+1$	M	0	0	0	

$\uparrow$

$x_1 \rightarrow$ entering variable

$x_4 \rightarrow$ departing variable

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Table - 2

c_B	B	x_B	c_j / b	-4 / a_1	-1 / a_2	0 / a_3	0 / a_4	-M / a_5	-M / a_6
-M	a_5	x_5	41	0	-5	0	-3	1	0
-M	a_6	x_6	12	0	-5	-1	-4	0	1
-4	a_1	x_1	3	1	2	0	1	0	0
		$z_j - c_j$		0	10M-7	M	7M-4	0	0

$b_1' = 50 - \frac{3\times3}{1} = \cancel{44}\,41$

$a_{11}' = 3 - \frac{3\times1}{1} = 0$

$a_{21}' = 1 - \frac{3\times2}{1} = -5$

$a_{31}' = 0 - \frac{0\times3}{1} = 0$

$a_{41}' = 0 - \frac{1\times3}{1} = -3$

$a_{51}' = \cancel{0}\,1 - \frac{0\times3}{1} = 1$

$a_{61}' = 0 - \frac{0\times3}{1} = 0$

$b_2' = 24 - \frac{3\times4}{1} = 12$

$a_{12}' = 4 - \frac{\cancel{3}\,1\times4}{1} = 0$

$a_{22}' = 3 - \frac{4\times2}{1} = -5$

$a_{32}' = \cancel{0}\,-1 - \frac{0\times4}{1} = -1$

$a_{42}' = 0 - \frac{4\times1}{1} = -4$

$a_{52}' = 0 - \frac{0\times4}{1} = 0$

$a_{62}' = 1 - \frac{4\times0}{1} = 1$

Since $M > 0$, a large quantity, so, $z_j - c_j \geq 0 \;\forall j$, so optimality condition is reached. Here the artificial variables x_5, x_6 are present in the basis in positive level (i.e. $x_5 = 41$, $x_6 = 12$). So this problem has no feasible solution.

Problem Set

I ①

$$\text{Max } Z = -3x_1 - 2x_2$$

Subject to

$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \leq 3.$$
$$x_1, x_2 \geq 0.$$

[Ans : $x_1^* = 4$, $x_2^* = 3$, $Z_{max} = -18$]

②

$$\text{Min } Z = 4x_1 + 3x_2$$

Subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

[Ans : $x_1^* = 4$, $x_2^* = 2$, $Z_{min} = 22$]

③

$$\text{Max } Z = 5x_1 + 3x_2$$

Subject to

$$2x_1 + 5x_2 \geq 1500$$
$$3x_1 + x_2 \geq 1200$$
$$\cancel{x_1 + 2x_2 \geq 10}$$
$$x_1, x_2 \geq 0$$

Problem of degeneracy: An LPP is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value. If the minimum ratio of two or more basic variables are co-incidentally same, that both variables want to leave the basis, then we say that the problem has degeneracy. We the problem reduce the degeneracy problem by interchanging the positions of basic variables and compute the minimum of the ratios of basic variables to entering variables

Ex 1. Max $Z = 3x_1 + 9x_2$

subject to $x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$.

Introducing slack variables $x_3 \geq 0, x_4 \geq 0$, the problem becomes:

Max $Z = 3x_1 + 9x_2 + 0x_3 + 0x_4$

subject to $x_1 + 4x_2 + x_3 = 8$

$x_1 + 2x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \geq 0$.

x_3, x_4 forms initial basis.

		c_j		3	9	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	Min ratio
0	a_3	x_3	8	1	4	1	0	$8/4 = 2 \rightarrow$
0	a_4	x_4	4	1	2	0	1	$4/2 = 2 \rightarrow$
		$z_j - c_j$		-3	-9 $\uparrow$	0	0	

Since min ratio is 2 for the key column for both elements, so, both slack variables x_3, x_4 may leave the basis. This is an indication for existence of degeneracy in given LPP. So, we will arrange a_1, a_2, a_3, a_4 in such a way that initial identity (basis) matrix appears first. Thus the initial simplex table becomes

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			C_j	0	0	3	9	Min ratio (a_3/a_2)
C_B	B	x_B	b	a_3	a_4	a_1	a_2	
0	a_3	x_3	8	1	0	1	4	1/4
0	a_4	x_4	4	0	1	1	$\boxed{2}$	0/2 →
			$Z_j - C_j$	0	0	−3	−9 ↑	

2 → key element x_2 → entering variable

x_4 → leaving variable

			C_j	0	0	3	9
C_B	B	x_B	b	a_3	a_4	a_1	a_2
0	a_3	x_3	0	1	−2	−1	0
9	a_2	x_2	2	0	1/2	1/2	1
			$Z_j - C_j$	0	9/2	3/2	0

$Z_j - C_j \geq 0$. Hence optimality reached.

$x_1^* = 0$, $x_2^* = 2$, $Z_{max} = 3 \cdot 0 + 9 \cdot 2$

$= 18$.