Problem 2.3.1

Let T be a linear transformation on $V_3(\mathbb{R})$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)\ \forall (a,b,c) \in V_3(\mathbb{R})$. Is T invertible?. If so, find a rule for T^{-1} as the one which defines T.

For proving T is invertible, we need to show only T is one-one and onto.
To prove one-one:
Let

$$\alpha = (a_1, b_1, c_1) \in V_3(\mathbb{R})$$
$$\beta = (a_2, b_2, c_2) \in V_3(\mathbb{R})$$

Then,

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$
\begin{aligned}
3a_1 &= 3a_2 \\
a_1 &= a_2 \\
a_1 - b_1 &= a_2 - b_2 \\
a_2 &= b_2 \\
c_1 &= c_2 \\
(a_1, b_1, c_1) &= (a_2, b_2, c_2) \\
\alpha &= \beta
\end{aligned}
$$

$\therefore T$ is one-one.

To prove onto:

T is linear transformation on a finite dimensional vector space $V_3(\mathbb{R})$, where dimension in 3.

$$
\begin{aligned}
&\Rightarrow \text{Also } T \text{ is one-one} \\
&\Rightarrow T \text{ must be onto} \\
&\Rightarrow T \text{ is invertible}
\end{aligned}
$$

$$
\begin{aligned}
\text{If } T(a,b,c) &= (p,q,r) \\
\text{then, } T^{-1}(p,q,r) &= (a,b,c) \\
T(a,b,c) &= (p,q,r) \\
(3a, a-b, 2a+b+c) &= (p,q,r) \\
3a &= p \\
p &= 3a \\
a &= \frac{p}{3} \\
a-b &= q \\
\frac{p}{3} - b &= q \\
\frac{p}{3} - q &= b
\end{aligned}
$$

$$
\begin{aligned}
2a + b + c &= r \\
2\frac{p}{3} + \left(\frac{p}{3} - q\right) + c &= r \\
c &= r - p + q
\end{aligned}
$$

$$
\begin{aligned}
\because T^{-1}(p,q,r) &= (a,b,c) \\
&= \left(\frac{p}{3}, \frac{p}{3} - q, r - p + q\right)
\end{aligned}
$$

Example 2.3.2

Let T be a linear map on $V_3(R)$ defined by $T(a,b,c) = [3a, a-b, 2a+b+c]$ $\forall a,b,c \in \mathbb{R}$. Is T invertible?. If so find a rule for T^{-1} like one which define T.

For proving T is invertible, we need to show that T is one-one and onto.
To prove one-one:
Let $\alpha = (a_1, b_1, c_1)$, $\beta - (a_2, b_2, c_2)$ be any two elements of $V_3(\mathbb{R})$.

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$
\begin{aligned}
3a_1 &= 3a_2 \\
a_1 - b_1 &= a_2 - b_2 + c_2 \\
2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2 \\
a_1 &= a_2 \\
\because a_1 - b_1 &= a_2 - b_2 \\
-b_1 &= -b_2 \\
b_1 &= b_2 \\
\because 2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2 \\
\because a_1 &= b_1 \\
b_1 &= b_2 \\
c_1 &= c_2 \\
(a_1, b_1, c_1) &= (a_2, b_2, c_2) \\
\alpha &= \beta
\end{aligned}
$$

$$\because T(\alpha) = T(\beta)$$
$$\alpha = \beta$$
$$T := A \to B$$

Hence T is one-one.
To find onto:
Since, T is a linear one-one map on a finite dimensional vector space.
$\Rightarrow T$ is onto.
$\Rightarrow T$ is one-one and onto.
$\Rightarrow T$ is invertible.

Second part:

$$\text{Let } T(a,b,c) = (p,q,r) \tag{10}$$
$$\text{Then } T^{-1}(p,q,r) = (a,b,c)$$

Now

$$\begin{aligned}
T(a,b,c) &= (p,q,r)\\
(3a, a-b, 2a+b+c) &= (p,q,r)\\
3a &= p\\
a &= \frac{p}{3}\\
\because a-b &= q\\
\frac{p}{3}-b &= q\\
\frac{p}{3}-q &= b
\end{aligned}$$

$$\because 2a + b + c = r$$
$$2\frac{p}{3} + \left(\frac{p}{3} - q\right) + c = r$$
$$c = r - p + q$$

Put the value of a, b, c in equation (**??**)

$$T^{-1}(p, , q, r) = \left(\frac{p}{3}, \frac{p}{3} - a, r - p + q\right)$$

or

$$T^{-1}(x, y, z) = \left(\frac{x}{3}, \frac{x}{3} - y, z - x + y\right)$$

which is the rule which defines T^{-1}.

Definition 2.3.3 (Wronskian)

Let f and g be differentiable on $[a, b]$. If Wronskian $W(f, g)(t_0)$ is nonzero for some t_0 in $[a, b]$ then f and g are linearly independent on $[a, b]$. If f and g are linearly dependent then the Wronskian is zero for all t in $[a, b]$.

Problem 2.3.4

Using Wronskian method prove that $\{e^{3x}, e^{5x}\}$ is a linearly independent set on $\mathbb{R}$.

Set $f(x) = e^{3x}$, $g(x) = e^{5x}$. Then,

$$\begin{aligned} W(f(x), g(x)) &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & f''(x) \end{vmatrix} \\ &= \begin{vmatrix} e^{3x} & e^{5x} \\ 3e^{3x} & 5e^{5x} \end{vmatrix} \\ &= 5e^{8x} - 3e^{8x} \\ &= 2e^{8x} \\ &\neq 0 \;\; (\forall x \in \mathbb{R}) \end{aligned}$$

$\because$ The given set $\{e^{3x}, e^{5x}\}$ is linearly independent.

Problem 2.3.5

Using Wronskian method prove that $\{e^{2x}, \cos(x), 2e^{2x}\}$ *is a linearly dependent set on* $\mathbb{R}$.

Set $f(x) = e^{2x}$, $g(x) = \cos x\ h(x) = 2e^{2x}$. Then,

$$W(f(x), g(x), h(x))$$

$$= \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & \cos x & 2e^{2x} \\ 2e^{2x} & -\sin x & 4e^{2x} \\ 4e^{2x} & -\cos x & 8e^{2x} \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} -\sin x & 4e^{e}2x \\ -\cos x & 8e^{2x} \end{vmatrix} - 2e^{2x} \begin{vmatrix} \cos x & 2e^{2x} \\ -\cos x & 8e^{2x} \end{vmatrix} + 4e^{2x} \begin{vmatrix} \cos x & 2e^{2x} \\ -\sin t & 4e^{2x} \end{vmatrix}$$

$$= e^{2x}\left(-8e^{2x}\sin x + 4e^{2x}\cos x\right) - 2e^{2x}\left(8e^{2x}\cos x + 2e^{2x}\cos x\right)$$

$$+ 4e^{2x}\left(4e^{2x}\cos x + 2e^{2x}\sin x\right)$$

$$= e^{2x}\left(-8\sin x + 4\cos x - 20\cos x + 16\cos x + 8\sin x\right)$$
$$= e^{4x}(0)$$
$$= 0 \; (\forall x \in \mathbb{R})$$

Example 2.3.6

Using Wronskian method prove that $\{1, x, x^2\}$ is a linearly dependent set on $\mathbb{R}$.

Ans: $W(f(x), g(x), h(x)) = 2 \neq 0$, So the set is linearly independent.