Problem 1.1.1

Apply Gauss elimination method to solve the equation $x+4y-z=-5$, $x+y-6z=-12$, $3x-y-z=4$.

$$\begin{bmatrix}1 & 4 & -1\\ 1 & 1 & -6\\ 3 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-5\\ -12\\ 4\end{bmatrix}$$

Its augmented matrix is

$$C = \begin{bmatrix}A & : & B\end{bmatrix} = \begin{bmatrix}1 & 4 & -1 & : & -5\\ 1 & 1 & -6 & : & -12\\ 3 & -1 & -1 & : & 4\end{bmatrix}$$

Applying operations:

$$R_2 \Rightarrow R_2 - R_1,$$
$$R_3 \Rightarrow R_3 - 3R_1.$$

$$C \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & -13 & 2 & : & 19 \end{bmatrix}$$

Applying operation $R_3 \Rightarrow 3R_3 - 13R_2$

$$C \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & 0 & 71 & : & 148 \end{bmatrix}$$

which can be written as

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148 \end{bmatrix}$$

$$x + 4y - z = -5 \quad (1)$$
$$-3y - 5z = -7 \quad (2)$$
$$71z = 148 \quad (3)$$

$$z = \frac{148}{71}$$

From equation (2),

$$y = \frac{-1}{3}\left[-7 + 5\left(\frac{148}{71}\right)\right] = \frac{-81}{71}$$

From equation (1),

$$x = \frac{117}{71}$$

$$x = \frac{117}{71}; y = \frac{-81}{71}; z = \frac{148}{71}$$

Example 1.1.2

Using Gauss elimination method find the solutions of $4x - 3y + z = -8$, $-2x + y - 3z = -4$, $x - y + 2z = 3$.

Ans:

$$x = 2$$
$$y = 1$$
$$z = 3$$

Example 1.1.3

Using Gauss elimination method find the solutions of $x + y + z = 3$, $2x + 3y + 7z = 0$, $x + 3y - 2z = 17$.

Ans:

$$x = 1$$
$$y = 4$$
$$z = -2$$

Problem 1.2.1

Apply Gauss Jordan method to solve the equation $x+y+z=9$, $2x-3y+4z=13$, $3x+4y+5z=40$..

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Augmented matrix is given by

$$\begin{aligned} C &= [A : B] \\ &= \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix} \end{aligned}$$

Applying operations $R_2 \Rightarrow R_2 - 2R_1; R_3 \Rightarrow R_3 - 3R_1$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}$$

$R_2 \Leftrightarrow R_3$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & -5 & 2 & : & -5 \end{bmatrix}$$

$R_3 \Rightarrow R_3 + 5R_2$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - R_2$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$R_3 \Rightarrow \frac{1}{12}(R_3)$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_3$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \Rightarrow R_1 + R_3$

$$C \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$x = 1$$

$$y = 3$$

$$z = 5$$

Problem 1.2.2

Apply Gauss-Jordan method to solve the equation

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

$$[A:B] = \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 2 & 10 & 1 & : & 13 \\ 1 & 1 & 5 & : & 7 \end{bmatrix}$$

$R_1 \Leftrightarrow R_2$

$$[A:B] = \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 2 & 10 & 1 & : & 13 \\ 10 & 1 & 1 & : & 12 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1, R_3 \Rightarrow R_3 - 10R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 0 & 8 & -9 & : & -1 \\ 0 & -9 & -49 & : & -58 \end{bmatrix}$$

$R_1 \Rightarrow 8R_1 - R_2, R_3 \Rightarrow 8R_3 + 9R_2$

$$[A:B] = \begin{bmatrix} 8 & 0 & 49 & : & 57 \\ 0 & 8 & -9 & : & -1 \\ 0 & 0 & -473 & : & -473 \end{bmatrix}$$

$R_3 \Rightarrow \frac{R_3}{-473}$

$$[A:B] = \begin{bmatrix} 8 & 0 & 49 & : & 57 \\ 0 & 8 & -9 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - 49R_3, R_2 \Rightarrow R_1 + 9R_3$

$$[A:B] = \begin{bmatrix} 8 & 0 & 0 & : & 8 \\ 0 & 8 & 0 & : & 8 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Example 1.2.3

Apply Gauss-Jordan method to solve the equation

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

Solution:

$$x = 1$$
$$y = 4$$
$$z = -2$$

$$8x = 8$$
$$8y = 8$$
$$1x = 1$$

$$x = 1$$
$$y = 1$$
$$z = 1$$

Block matrices

$$A = \left[\begin{array}{cc:cc:c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hdashline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc:ccc} 1 & -2 & 0 & 1 & 3 \\ \hdashline 2 & 3 & 5 & 7 & -2 \\ 3 & 1 & 4 & 5 & 9 \\ \hdashline 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc:cc} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hdashline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

Definition 1.3.1 (Square block matrix)

Let M be a block matrix, the M is square block matrix if

1. M is square.
2. The block form a square matrix.
3. The diagonal blocks are also square matrices.

Example 1.3.2

$$A = \left[\begin{array}{cc:cc:c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 1 & 3 \\ \hdashline 2 & 1 & 2 & 2 & 1 \\ \hdashline 3 & 2 & 1 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{array}\right]$$

This is not a square block matrix

$$A = \left[\begin{array}{cc:cc:c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 1 & 3 \\ \hdashline 2 & 1 & 2 & 2 & 1 \\ 3 & 2 & 1 & 4 & 5 \\ \hdashline 5 & 2 & 3 & 1 & 4 \end{array}\right]$$

This is a square block matrix

Definition 1.3.3 (Block diagonal matrix)

Let $M : [A_{ij}]$ be a square block matrix such that non-diagonal blocks are all zero matrices.
i.e. $A_{ij} = 0$ whenever $i \neq j$.
M is called block diagonal matrix, if $M = diag(A_{11}, A_{22}, A_{33}, \ldots, A_{rr})$ or $M = A_{11} \oplus A_{22} \oplus \ldots \oplus A_{rr}$.

$$M = \left[\begin{array}{c:c:c} & 0 & \\ \hdashline 0 & & 0 \\ \hdashline 0 & 0 & \end{array}\right]$$

Remark 1.3.4

Note : $M : diag\left(A_{11}, A_{22}, \ldots, A_{rr}\right)$ *is invertible iff* A_{ii} *is invertible* $\forall i$. *Then* $M^{-1} = diag\left(A_{11}^{-1}, A_{22}^{-1}, \ldots, A_{rr}^{-1}\right)$

Example 1.3.5

$$M = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

Example 1.3.6

Find Block matrix multiplication of

$$A = \begin{bmatrix} 1 & 3 & -2 & 3 & -3 \\ 2 & 4 & -2 & 2 & 1 \\ 0 & -2 & 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

$$A = \left[\begin{array}{ccc|cc} 1 & 3 & -2 & 3 & -3 \\ 2 & 4 & -2 & 2 & 1 \\ \hline 0 & -2 & 1 & 1 & 1 \end{array}\right]_{3\times 5}, \; B = \left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \\ 3 & 3 \\ \hline 2 & 2 \\ 1 & 2 \end{array}\right]_{5\times 2}$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}; A = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_1B_1 & A_2B_2 \\ A_3B_1 & A_4B_2 \end{bmatrix}$$

$$A_1B_1 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix}$$

$$A_2B_2 = \begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & 6 \end{bmatrix}$$

$$A_3B_1 = \begin{bmatrix} 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$A_4B_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_1B_1 & A_2B_2 \\ A_3B_1 & A_4B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix} \\ \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ 9 & 4 \\ 2 & 5 \end{bmatrix}$$

Example 1.4.1

Find the elementary matrices of $A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$.

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} = M_1 A$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = M_2 M_1 A$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} = M_3 M_2 M_1 A$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$U = M_3 M_2 M_1 A$$

The permutation matrices of order two are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and of order three are given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Remark 1.5.1

1. *A permutation matrix is nonsingular, and the determinant is always* ± 1.
2. *A permutation matrix* A *satisfies*

$$AA^T = I,$$

where A^T *is a transpose and* I *is the identity matrix.*

3. I *is a special* P
4. *Every row has one* 1
5. *Every column has* 1

Problem 1.5.2

Find permutation of $A = \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & 7 \end{pmatrix}$$
$$\begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 6 & 2 \end{pmatrix}$$

Remark 1.5.3

- *Multiplication of permutation matrix changes the position of the rows and columns.*

Definition 1.6.1 (Cayley Hamilton Theorem)

A matrix satisfies its own characteristic equation. That is, if the characteristic equation of an $n \times n$ matrix A is $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1A + a_0I = 0.$$

Problem 1.6.2

Verify Cayley Hamilton theorem for the following matrix A and hence find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Given

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

The characteristic equation

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$s_1 = 1 - 3 - 2 = -4$$

$$s_2 = \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}$$

$$= (6 - 1)(-2 + 2) + (-3 - 6) \Rightarrow 5 - 9 = -4$$

$$s_3 = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 1(6 - 1) - 2(-6 - 2) - 1(3 + 6) - 1(3 + 6) = 5 + 16 - 9 = 12$$

Then the characteristic equation is,

$$\lambda^3 + 4\lambda^2 - 4\lambda - 12 = 0$$

To verify the Cayley Hamilton theorem in characteristic replace λ by $'A'$, then we have

$$A^3 + 4A^2 - 4A - 12I = 0 \tag{4}$$

$$A.A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-2 & 2-6-1 & -1+2+2 \\ 3-9+2 & 6+9+1 & -3-3-2 \\ 2+3-4 & 4-3-2 & -2+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix}$$

$$
\begin{aligned}
A^3 &= A^2.A \\
&= \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix}
\end{aligned}
$$

Let us substitute the value in equation (4)

$$
\begin{aligned}
&A^3 + 4A^2 - 4A - 12I \\
&= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + 4\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + \begin{bmatrix} 20 & -20 & 12 \\ -16 & 64 & -32 \\ 4 & -4 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ 0 & 12 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -4+20-4-12 & 28-20-8-0 & -16+12+4-0 \\ 28-16-12-0 & -64+64+12-12 & 36-32-4-0 \\ 4+4-8-0 & 8-4-4-0 & -8+12+8-12 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Hence, Cayley Hamilton theorem proved.

Consider the characteristic equation

$$A^3 + 4A^2 - 4A - 12I = 0$$

Multiply both side A^{-1}

$$\begin{aligned} A^3A^{-1} + 4A^2A^{-1} - 4AA^{-1} - 12IA^{-1} &= 0 \\ A^2 + 4A - 4I - 12A^{-1} &= 0 \\ A^2 + 4A - 4I &= 12A^{-1} \\ A^{-1} &= \frac{1}{12}\left[A^2 + 4A - 4I\right] \end{aligned}$$

$$A^{-1} = \frac{1}{12}\left[\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{12}\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5+4-4 & -5+8+0 & 3-4-0 \\ -4+12-0 & 16-12+4 & -8+4-0 \\ 1+8-0 & -1+4+0 & 3-8-4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5 & 3 & -1 \\ 8 & 0 & -4 \\ 9 & 3 & -9 \end{bmatrix}$$

Definition 1.7.1 (LDU factorization)

$$A = LUD$$

Remark 1.7.2

$$\begin{bmatrix} * & 6th & 5th \\ 1st & * & 4th \\ 2nd & 3rd & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 12th & 11th & 9th \\ 1st & * & 10th & 8th \\ 2nd & 5th & 6th & * \end{bmatrix}$$

Problem 1.7.3

Find LDU of $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$R_2 \Rightarrow -3R_1 + R_2$$
$$R_3 \Rightarrow (-1)R_1 + R_3$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} = E_1 A$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$R_2 \Rightarrow \frac{2}{3}R_3 + R_2$$
$$R_1 \Rightarrow \frac{-1}{3}R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} = E_2E_1A$$

$$E_2 = \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \Rightarrow \frac{1}{2}R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= E_3 E_2 E_1 A = D$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore \; A = LUD = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Example 1.7.4

Find LDU of $\begin{bmatrix} 4 & -20 & -12 \\ -8 & 45 & 44 \\ 20 & -105 & -79 \end{bmatrix}$

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 1.7.5

Find LDU of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 1.8.1 (Application of Matrices in Cryptography)

In this section you will learn to

1. encode a message using matrix multiplication.
2. decode a coded message using the matrix inverse and matrix multiplication.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Problem 1.8.2

Use matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *to encode the message:* $ATTACK\ NOW$.

We divide the letters of the message into groups of two.

$$AT \quad TA \quad CK \quad -N \quad OW$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}; \begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}; \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}; \begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix}; \begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix}; \begin{bmatrix} 20 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \end{bmatrix} \begin{bmatrix} 15 \\ 23 \end{bmatrix} \quad (5)$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A. For example, the product of A with our first matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$$

And the product of A with our second matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$$

Multiplying each matrix in (5) by matrix A, in turn, gives the desired coded message:

$$\begin{bmatrix} 41 \\ 66 \end{bmatrix} \begin{bmatrix} 22 \\ 23 \end{bmatrix} \begin{bmatrix} 25 \\ 36 \end{bmatrix} \begin{bmatrix} 55 \\ 69 \end{bmatrix} \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

Problem 1.8.3

Decode the following message that was encoded using matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix} \begin{bmatrix} 37 \\ 53 \end{bmatrix} \begin{bmatrix} 45 \\ 54 \end{bmatrix} \begin{bmatrix} 74 \\ 101 \end{bmatrix} \begin{bmatrix} 53 \\ 69 \end{bmatrix} \tag{6}$$

We decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example:

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

By multiplying each of the matrices in (6) by the matrix A^{-1} , we get the following.

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix} \begin{bmatrix} 5 \\ 16 \end{bmatrix} \begin{bmatrix} 27 \\ 9 \end{bmatrix} \begin{bmatrix} 20 \\ 27 \end{bmatrix} \begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} K \\ E \end{bmatrix} \begin{bmatrix} E \\ P \end{bmatrix} \begin{bmatrix} - \\ I \end{bmatrix} \begin{bmatrix} T \\ - \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

And the message reads: **KEEP IT UP**.

Problem 1.8.4

Using the matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$*, encode the message:* **ATTACK NOW**.

We divide the letters of the message into groups of three.

$$ATT \quad ACK \quad -NO \quad W--$$

Note that since the single letter W was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix} A \\ T \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} A \\ C \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} - \\ N \\ O \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} W \\ - \\ - \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

So far we have,

$$\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix} \tag{7}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in (7) by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix} \begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix} \begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix} \begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

Problem 1.8.5

Decode the following message

$$\begin{bmatrix}11\\20\\43\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix} \tag{8}$$

that was encoded using matrix

$$B = \begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}.$$

Since this message was encoded by multiplying by the matrix B. We first determine inverse of B.

$$B^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in (8) by the matrix B^{-1} gives the following.

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix} \begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix} \begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain

$$\begin{bmatrix} H \\ O \\ L \end{bmatrix} \quad \begin{bmatrix} D \\ - \\ F \end{bmatrix} \quad \begin{bmatrix} I \\ R \\ E \end{bmatrix}$$

The message reads: **HOLD FIRE**.