Formal Proofs

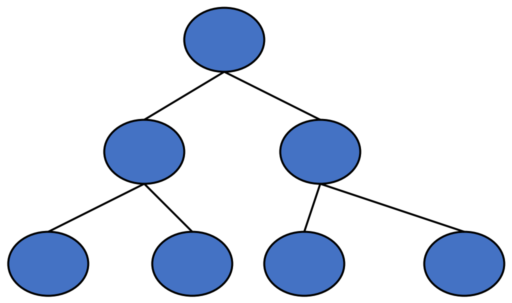
Definitions, Theorems, Lemma and Corollary

- **Definition**: A precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.
- **Theorem**: A mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results.
- **Lemma**: A minor result whose sole purpose is to help in proving a theorem. It is a stepping stone on the path to proving a theorem. It is an intermediate result that we show to prove a larger result.

Definitions, Theorems, Lemma and Corollary

- **Corollary**: A result in which the proof relies heavily on a given theorem. It is a result that follows from an already proven result. ("this is a corollary of Theorem A").

An example:
Theorem: *The height of an n-node binary tree is at least floor(log n)*
Lemma: *Level i of a complete binary tree has 2^i nodes.*
Corollary: *A complete binary tree of height h has 2^{h+1}-1 nodes.*



Proof by contradiction

- We assume that the **theorem is false** and then show that this assumption leads to an **obviously false consequence**, called a contradiction.

Proof by induction

- For each positive integer **n**, let **P(n)** be a mathematical statement that depends on **n**.
- Assume we wish to prove that **P(n)** is true for all positive integers **n**.

STEPS:

Basis: Prove that **P(1)** is true.

Induction step: Prove that for all **n ≥ 1**, the following holds: If **P(n)** is true, then **P(n + 1)** is also true.

- In the induction step, **we choose an arbitrary integer n ≥ 1 and assume that P(n) is true;** this is called the **induction hypothesis**. Then we prove that P(n + 1) is also true.

Proof by induction

- **Theorem**: For all positive integers n, we have

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

- **Proof**:
- We start with the **basis** of the induction. If n = 1, then the left-hand side is equal to 1, and so is the right-hand side. So the theorem is true for n = 1.
- For the **induction** step, let **n ≥ 1** and assume that the theorem is true for **n.**
- We have to prove that the theorem is true for n + 1.

$$1 + 2 + 3 + \ldots + (n+1) = \frac{(n+1)(n+2)}{2}$$

Proof by induction

$$\begin{aligned} 1+2+3+\ldots+(n+1) &= \underbrace{1+2+3+\ldots+n}_{=\frac{n(n+1)}{2}}+(n+1) \\ &= \frac{n(n+1)}{2}+(n+1) \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

- An alternative proof of the theorem: Let S = 1 + 2 + 3 + . . . + n. Then,

$$\begin{array}{ccccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \ldots & + & (n-2) & + & (n-1) & + & n \\ S & = & n & + & (n-1) & + & (n-2) & + & \ldots & + & 3 & + & 2 & + & 1 \\ \hline 2S & = & (n+1) & + & (n+1) & + & (n+1) & + & \ldots & + & (n+1) & + & (n+1) & + & (n+1) \end{array}$$

- Since there are n terms on the right-hand side, we have 2S = n(n+1). This implies that S = n(n + 1)/2.