Definition 1.8.1 (Application of Matrices in Cryptography)

In this section you will learn to

1. encode a message using matrix multiplication.
2. decode a coded message using the matrix inverse and matrix multiplication.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Problem 1.8.2

Use matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ *to encode the message:* $ATTACK\ NOW$.

We divide the letters of the message into groups of two.

$$AT \quad TA \quad CK \quad -N \quad OW$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix}; \begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}; \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}; \begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix}; \begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix}; \begin{bmatrix} 20 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \end{bmatrix} \begin{bmatrix} 15 \\ 23 \end{bmatrix} \tag{5}$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A. For example, the product of A with our first matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$$

And the product of A with our second matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$$

Multiplying each matrix in (5) by matrix A, in turn, gives the desired coded message:

$$\begin{bmatrix} 41 \\ 66 \end{bmatrix} \begin{bmatrix} 22 \\ 23 \end{bmatrix} \begin{bmatrix} 25 \\ 36 \end{bmatrix} \begin{bmatrix} 55 \\ 69 \end{bmatrix} \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

Problem 1.8.3

Decode the following message that was encoded using matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix} \begin{bmatrix} 37 \\ 53 \end{bmatrix} \begin{bmatrix} 45 \\ 54 \end{bmatrix} \begin{bmatrix} 74 \\ 101 \end{bmatrix} \begin{bmatrix} 53 \\ 69 \end{bmatrix} \tag{6}$$

We decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example:

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

By multiplying each of the matrices in (6) by the matrix A^{-1}, we get the following.

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix} \begin{bmatrix} 5 \\ 16 \end{bmatrix} \begin{bmatrix} 27 \\ 9 \end{bmatrix} \begin{bmatrix} 20 \\ 27 \end{bmatrix} \begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} K \\ E \end{bmatrix} \begin{bmatrix} E \\ P \end{bmatrix} \begin{bmatrix} - \\ I \end{bmatrix} \begin{bmatrix} T \\ - \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

And the message reads: **KEEP IT UP**.

Problem 1.8.4

Using the matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$*, encode the message:* **ATTACK NOW**.

We divide the letters of the message into groups of three.

$$ATT \quad ACK \quad -NO \quad W - -$$

Note that since the single letter W was left over on the end, we added two spaces to make it into a triplet.

Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix} A \\ T \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} A \\ C \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} - \\ N \\ O \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} W \\ - \\ - \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

So far we have,

$$\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix} \tag{7}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in (7) by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix} \begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix} \begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix} \begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

Problem 1.8.5

Decode the following message

$$\begin{bmatrix}11\\20\\43\end{bmatrix}\begin{bmatrix}25\\10\\41\end{bmatrix}\begin{bmatrix}22\\14\\41\end{bmatrix} \tag{8}$$

that was encoded using matrix

$$B = \begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}.$$

Since this message was encoded by multiplying by the matrix B. We first determine inverse of B.

$$B^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in (8) by the matrix B^{-1} gives the following.

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix} \begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix} \begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain

$$\begin{bmatrix} H \\ O \\ L \end{bmatrix} \begin{bmatrix} D \\ - \\ F \end{bmatrix} \begin{bmatrix} I \\ R \\ E \end{bmatrix}$$

The message reads: **HOLD FIRE**.

$$A = \left[\begin{array}{cc:cc:c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hdashline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc:ccc} 1 & -2 & 0 & 1 & 3 \\ \hdashline 2 & 3 & 5 & 7 & -2 \\ 3 & 1 & 4 & 5 & 9 \\ \hdashline 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc:cc} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hdashline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

Definition 1.3.1 (Square block matrix)

Let M be a block matrix, the M is square block matrix if

1. M is square.
2. The block form a square matrix.
3. The diagonal blocks are also square matrices.

Example 1.3.2

$$A = \left[\begin{array}{cc:cc:c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 1 & 3 \\ \hdashline 2 & 1 & 2 & 2 & 1 \\ \hdashline 3 & 2 & 1 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{array}\right]$$

This is not a square block matrix

$$A = \left[\begin{array}{cc:cc:c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 1 & 3 \\ \hdashline 2 & 1 & 2 & 2 & 1 \\ 3 & 2 & 1 & 4 & 5 \\ \hdashline 5 & 2 & 3 & 1 & 4 \end{array}\right]$$

This is a square block matrix

Definition 1.3.3 (Block diagonal matrix)

Let $M : [A_{ij}]$ be a square block matrix such that non-diagonal blocks are all zero matrices.
i.e. $A_{ij} = 0$ whenever $i \neq j$.
M is called block diagonal matrix, if $M = diag(A_{11}, A_{22}, A_{33}, \ldots, A_{rr})$ or $M = A_{11} \oplus A_{22} \oplus \ldots \oplus A_{rr}$.

$$M = \left[\begin{array}{c:c:c} & 0 & \\ \hdashline 0 & & 0 \\ \hdashline 0 & 0 & \end{array}\right]$$

Remark 1.3.4

Note : $M : diag\left(A_{11}, A_{22}, \ldots, A_{rr}\right)$ *is invertible iff* A_{ii} *is invertible* $\forall i$. *Then* $M^{-1} = diag\left(A_{11}^{-1}, A_{22}^{-1}, \ldots, A_{rr}^{-1}\right)$

Example 1.3.5

$$M = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

Example 1.3.6

Find Block matrix multiplication of

$$A = \begin{bmatrix} 1 & 3 & -2 & 3 & -3 \\ 2 & 4 & -2 & 2 & 1 \\ 0 & -2 & 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

$$A = \left[\begin{array}{ccc:cc} 1 & 3 & -2 & 3 & -3 \\ 2 & 4 & -2 & 2 & 1 \\ \hdashline 0 & -2 & 1 & 1 & 1 \end{array}\right]_{3\times 5}, \quad B = \left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \\ 3 & 3 \\ \hdashline 2 & 2 \\ 1 & 2 \end{array}\right]_{5\times 2}$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}; A = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_1B_1 & A_2B_2 \\ A_3B_1 & A_4B_2 \end{bmatrix}$$

$$A_1B_1 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix}$$

$$A_2B_2 = \begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & 6 \end{bmatrix}$$

$$A_3B_1 = \begin{bmatrix} 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$A_4B_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_1B_1 & A_2B_2 \\ A_3B_1 & A_4B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix} \\ \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ 9 & 4 \\ 2 & 5 \end{bmatrix}$$

Example 1.4.1

Find the elementary matrices of $A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$.

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} = M_1 A$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = M_2M_1A$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} = M_3M_2M_1A$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$U = M_3M_2M_1A$$

Definition 1.6.1 (Cayley Hamilton Theorem)

A matrix satisfies its own characteristic equation. That is, if the characteristic equation of an $n \times n$ matrix A is $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1A + a_0I = 0.$$

Problem 1.6.2

Verify Cayley Hamilton theorem for the following matrix A and hence find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Given

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

The characteristic equation

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$s_1 = 1 - 3 - 2 = -4$$

$$s_2 = \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}$$

$$= (6 - 1)(-2 + 2) + (-3 - 6) \Rightarrow 5 - 9 = -4$$

$$s_3 = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 1(6 - 1) - 2(-6 - 2) - 1(3 + 6) - 1(3 + 6) = 5 + 16 - 9 = 12$$

Then the characteristic equation is,

$$\lambda^3 + 4\lambda^2 - 4\lambda - 12 = 0$$

To verify the Cayley Hamilton theorem in characteristic replace λ by $'A'$, then we have

$$A^3 + 4A^2 - 4A - 12I = 0 \tag{4}$$

$$A.A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-2 & 2-6-1 & -1+2+2 \\ 3-9+2 & 6+9+1 & -3-3-2 \\ 2+3-4 & 4-3-2 & -2+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix}$$

$$A^3 = A^2.A$$

$$= \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix}$$

Let us substitute the value in equation (4)

$$
\begin{aligned}
A^3 + 4A^2 - 4A - 12I
&= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + 4\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + \begin{bmatrix} 20 & -20 & 12 \\ -20 & 64 & -32 \\ 4 & -4 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ 0 & 12 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -4+20-4-12 & 28-20-8-0 & -16+12+4-0 \\ 28-16-12-0 & -64+64+12-12 & 36-32-4-0 \\ 4+4-8-0 & 8-4-4-0 & -8+12+8-12 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Hence, Cayley Hamilton theorem proved.

Consider the characteristic equation

$$A^3 + 4A^2 - 4A - 12I = 0$$

Multiply both side A^{-1}

$$\begin{aligned} A^3A^{-1} + 4A^2A^{-1} - 4AA^{-1} - 12IA^{-1} &= 0 \\ A^2 + 4A - 4I - 12A^{-1} &= 0 \\ A^2 + 4A - 4I &= 12A^{-1} \\ A^{-1} &= \frac{1}{12}\left[A^2 + 4A - 4I\right] \end{aligned}$$

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5+4-4 & -5+8+0 & 3-4-0 \\ -4+12-0 & 16-12+4 & -8+4-0 \\ 1+8-0 & -1+4+0 & 3-8-4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5 & 3 & -1 \\ 8 & 0 & -4 \\ 9 & 3 & -9 \end{bmatrix}$$

Example 4.0.6

Show that the following function defines an inner product on $\mathbb{R}^2$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are define by $\langle u, v\rangle = u_1v_1 + 2u_2v_2$.

(1) To prove Conjugate symmetry:

$$\begin{aligned}\langle u, v\rangle &= u_1v_1 + 2u_2v_2\\ &= v_1u_1 + 2v_2u_2\\ &= \langle v, u\rangle\end{aligned}$$

(2) To prove linearity:
Let $w = (w_1, w_2)$

$$\begin{aligned}\langle u, v+w\rangle &= \langle (u_1, u_2), (v_1 + w_1, v_2 + w_2)\rangle\\ &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2)\\ &= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)\\ &= \langle u, v\rangle + \langle u, w\rangle\end{aligned}$$

(3) Let $c \in \mathbb{R}$, then

$$\begin{aligned} c\langle u, v\rangle &= c(u_1 v_1 + 2u_2 v_2) \\ &= (cu_1)v_1 + 2(cu_2)v_2 \\ &= \langle cu, v\rangle \end{aligned}$$

(4) To prove non-negativity:

$$\langle v, v\rangle = v_1^2 + 2v_2^2 \geq 0$$

Moreover,

$$\begin{aligned} & \langle v, v\rangle = 0 \\ \Leftrightarrow \quad & v_1^2 + 2v_2 = 0 \\ \Leftrightarrow \quad & v_1 = 0,\ v_2 = 0,\ \text{since, } v_1^2 \geq 0, v_2 \geq 0 \\ \Leftrightarrow \quad & v = (0, 0) \end{aligned}$$

Definition 4.0.7 (Length of a vector)

The length or norm of a vector is the non-negative scalar $\|v\|$ defined by

$$\|v\| = \sqrt{v.v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Suppose

$$\vec{V} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Then,

$$\|v\|^2 = a^2 + b^2$$
$$\|v\| = \sqrt{a^2 + b^2}$$

Definition 4.0.8 (Distance)

The distance between $\vec{u}$ and $\vec{v}$ can be found by

$$dist(\vec{u}, \vec{v}) = \|u - v\|$$

Example 4.0.9

Compute $dist(\vec{u}, \vec{v})$ for $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

$$\begin{aligned} dist(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ u - v &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \\ \|\vec{u} - \vec{v}\| &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \end{aligned}$$

Example 4.0.10

Find the distance between $\vec{u} = \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$.

$$
\begin{aligned}
dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| &= \sqrt{9^2 + 3^2 + 5^2} \\
&= \sqrt{81 + 9 + 25} \\
&= \sqrt{115}
\end{aligned}
$$

Definition 4.0.11

Angle between vectors are defined by

$$\cos\omega = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$$

Problem 4.0.12

Find the angle between $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

To find angle between the vectors

$$\cos\omega = \frac{x^T y}{\sqrt{x^T x \times y^T y}}$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\sqrt{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}}$$

$$\cos\omega = \frac{3}{\sqrt{10}} = 0.9486$$

$$\omega \approx \cos^{-1} 0.9486$$

$$\omega \approx 0.32 \frac{180}{\pi} \approx 18^0$$

Problem 4.0.13

Find the angle between $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To find angle between the vectors

$$\cos\omega = \frac{x^T y}{\sqrt{x^T x \times y^T y}}$$

$$\cos\omega = 0$$

$$\omega \approx \frac{\pi}{2} = 90^0$$

Example 4.0.18

Let π be the plane in $\mathbb{R}^3$ spanned by vector $x_1 = (1, 2, 2)$ and $x_2 = (-1, 0, 2)$.

1. Using the Gram-Schmidt process find an orthonormal basis for π.
2. Extend it to an orthonormal basis for $\mathbb{R}^3$.

x_1, x_2 is a basis for the plane π. We can extend it to a basis for $\mathbb{R}^3$ by adding one vector from the standard basis. For instance, vectors x_1, x_2, and $x_3 = (0, 0, 1)$ form a basis for R_3 because

$$\begin{vmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0.$$

Using the Gram-Schmidt process, we orthogonalize the basis $x_1 = (1,2,2)$, $x_2 = (-1,0,2)$, $x_3 = (0,0,1)$:

$$\begin{aligned}
v_1 &= x_1 = (1,2,2),\\
v_2 &= x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (-1,0,2) - \frac{(-1.1+0.2+2.2)}{(1.1+2.2+2.2)}(1,2,2)\\
&= (-1,0,2) - \frac{3}{9}(1,2,2) = \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right),\\
v_3 &= x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2\\
&= (0,0,1) - \frac{(0.1+0.2+1.2)}{(1.1+2.2+2.2)}(1,2,2)\\
&\quad - \frac{(0.\frac{-4}{3}+0.\frac{-2}{3}+1.\frac{4}{3})}{(\frac{-4}{3}.\frac{-4}{3}+\frac{-2}{3}.\frac{-2}{3}+\frac{4}{3}.\frac{4}{3})}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)\\
&= (0,0,1) - \frac{2}{9}(1,2,2) - \frac{\frac{4}{3}}{4}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) = \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right).
\end{aligned}$$

Now $v_1 = (1,2,2)$, $v_2 = (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3})$, $v_3 = \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right)$ is an orthogonal basis for R_3 while v_1, v_2 is an orthogonal basis for π. It remains to normalize these vectors.

$$\langle v_1, v_1\rangle = 9 \Rightarrow \|v_1\| = 3$$
$$\langle v_2, v_2\rangle = 4 \Rightarrow \|v_2\| = 2$$
$$\langle v_3, v_3\rangle = \frac{1}{9} \Rightarrow \|v_3\| = \frac{1}{3}$$
$$w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3}(1,2,2),$$
$$w_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}(-2,-1,2),$$
$$w_3 = \frac{v_3}{\|v_3\|} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) = \frac{1}{3}(2,-2,1).$$

w_1, w_2 is an orthonormal basis for π.
w_1, w_2, w_3 is an orthonormal basis for $\mathbb{R}^3$.

Problem 4.0.19

Find the distance from the point $y = (0,0,0,1)$ *to the subspace* $V \subset \mathbb{R}^4$ *spanned by vectors* $x_1 = (1,-1,1,-1)$, $x_2 = (1,1,3,-1)$, *and* $x_3 = (-3,7,1,3)$.

Let us apply the Gram-Schmidt process to vectors x_1, x_2, x_3, y. We should obtain an orthogonal system v_1, v_2, v_3, v_4. The desired distance will be $|v_4|$.
Given

$$x_1 = (1,-1,1,-1), \qquad x_2 = (1,1,3,-1),$$
$$x_3 = (-3,7,1,3), \qquad y = (0,0,0,1).$$

$$
\begin{aligned}
v_1 &= x_1 = (1,-1,1,-1),\\
v_2 &= x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (1,1,3,-1) - \frac{4}{4}(1,-1,1,-1) = (0,2,2,0),\\
v_3 &= x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2\\
&= (-3,7,1,3) - \frac{-12}{4}(1,-1,1,-1) - \frac{16}{8}(0,2,2,0) = (0,0,0,0).
\end{aligned}
$$

The vector x_3 is a linear combination of x_1 and x_2. V is a plane, not a 3-dimensional subspace. We should orthogonalize vectors x_1, x_2, y.

$$\begin{aligned}
\vec{v}_3 &= y - \frac{\langle y, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle y, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \\
&= (0,0,0,1) - \frac{-1}{4}(1,-1,1,-1) - \frac{0}{8}(0,2,2,0) \\
&= \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right). \\
|\vec{v}_3| &= \left|\left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)\right| = \frac{1}{4}|(1,-1,1,3)| \\
&= \frac{\sqrt{1^2 + (-1)^2 + 1^2 + 3^2}}{4} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.
\end{aligned}$$

Course Objectives:

Linear algebra is one of the most important subjects of pure mathematics and has many applications in electrical, communications and computer science. This course aims at introducing students to the fundamental concepts of linear algebra by starting with linear equations and culminating in abstract vector spaces and linear transformations.

Course Outcomes:

By the end of the course, the students will be able to

1. solve systems of linear equations
2. understand the concepts of vector spaces and subspaces, basis and dimensions, linear transformations and inner product spaces and their matrix representations
3. use Gram-Schmidt process to obtain orthonormal basis,
4. find the change of basis matrix with respect to two bases of a vector space.

Linear Equations and Matrices Introduction - Gaussian elimination and Gauss Jordan methods - Block matrices - Elementary matrices - permutation matrix - inverse matrices - LDU factorization - Applications to electrical networks and cryptography.

Vector Spaces and Subspaces Vector spaces and subspaces - Linear Independence, Basis and Dimension - Row, Column and Null spaces - Rank and Nullity - Bases for subspaces - Invertibility - Application: Interpolation and Wronskian.

Linear Transformations Definition and Examples - properties - The Range and Kernel - Invertible linear transformations - Isomorphism - Application: Computer graphics - Matrices of linear transformations - Vector space of linear transformations - change of bases - similarity.

Inner Product Spaces Inner products - The lengths and angles of vectors - Matrix representations of inner products - Orthogonal projections - Gram-Schmidt orthogonalization.

Applications of Inner Product Spaces QR factorization - Singular Value Decomposition - Projection - orthogonal projections - relations of fundamental subspaces - Least square solutions - Orthogonal projection matrices.

Text Book(s):

1. Linear Algebra by Jin Ho Kwak and Sungpyo Hong, Second edition, Springer, 2004.
2. Linear Algebra with applications by Steven J. Leon, 8th Edition, Pearson, 2010.

Reference Book(s):

1. Elementary Linear Algebra by Stephen Andrilli and David Hecker, 4th edition, Academic Press, 2010.
2. Introduction to Linear Algebra by Gilbert Strang, 4th edition, Wellesley-Cambridge Press, 2011.
3. Introductory Linear Algebra - An applied first course by Bernard Kolman and David R. Hill, 9th Edition, Pearson education, 2011.
4. Linear Algebra A Modern Introduction by David Poole, 2nd edition, Thomson Learning, 2006

Problem 1.1.1

Apply Gauss elimination method to solve the equation $x + 4y - z = -5$, $x + y - 6z = -12$, $3x - y - z = 4$.

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

Its augmented matrix is

$$C = \begin{bmatrix} A & : & B \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 1 & 1 & -6 & : & -12 \\ 3 & -1 & -1 & : & 4 \end{bmatrix}$$

Applying operations:

$$R_2 \Rightarrow R_2 - R_1,$$
$$R_3 \Rightarrow R_3 - 3R_1.$$

$$C \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & -13 & 2 & : & 19 \end{bmatrix}$$

Applying operation $R_3 \Rightarrow 3R_3 - 13R_2$

$$C \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & 0 & 71 & : & 148 \end{bmatrix}$$

which can be written as

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148 \end{bmatrix}$$

$$
\begin{aligned}
x + 4y - z &= -5 && (1)\\
-3y - 5z &= -7 && (2)\\
71z &= 148 && (3)
\end{aligned}
$$

$$z = \frac{148}{71}$$

From equation (2),

$$y = \frac{-1}{3}\left[-7 + 5\left(\frac{148}{71}\right)\right] = \frac{-81}{71}$$

From equation (1),

$$x = \frac{117}{71}$$

$$x = \frac{117}{71}; y = \frac{-81}{71}; z = \frac{148}{71}$$

Example 1.1.2

Using Gauss elimination method find the solutions of $x + 4y - z = -5$, $x + y - 6z = -12$, $3x - y - z = 4$.

Ans:

$$x = \frac{117}{71}$$
$$y = \frac{-81}{71}$$
$$z = \frac{148}{71}$$

Problem 1.2.1

Apply Gauss Jordan method to solve the equation $x+y+z=9$, $2x-3y+4z=13$, $3x+4y+5z=40$..

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Augmented matrix is given by

$$\begin{aligned} C &= [A : B] \\ &= \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix} \end{aligned}$$

Applying operations $R_2 \Rightarrow R_2 - 2R_1; R_3 \Rightarrow R_3 - 3R_1$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}$$

$R_2 \Leftrightarrow R_3$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & -5 & 2 & : & -5 \end{bmatrix}$$

$R_3 \Rightarrow R_3 + 5R_2$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - R_2$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$R_3 \Rightarrow \frac{1}{12}(R_3)$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_3$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \Rightarrow R_1 + R_3$

$$C \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$x = 1$$

$$y = 3$$

$$z = 5$$

Problem 1.2.2

Apply Gauss-Jordan method to solve the equation

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

$$[A:B] = \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 2 & 10 & 1 & : & 13 \\ 1 & 1 & 5 & : & 7 \end{bmatrix}$$

$R_1 \Leftrightarrow R_2$

$$[A:B] = \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 2 & 10 & 1 & : & 13 \\ 10 & 1 & 1 & : & 12 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1, R_3 \Rightarrow R_3 - 10R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 0 & 8 & -9 & : & -1 \\ 0 & -9 & -49 & : & -58 \end{bmatrix}$$

$R_1 \Rightarrow 8R_1 - R_2, R_3 \Rightarrow 8R_3 + 9R_2$

$$[A:B] = \begin{bmatrix} 8 & 0 & 49 & : & 57 \\ 0 & 8 & -9 & : & -1 \\ 0 & 0 & -473 & : & -473 \end{bmatrix}$$

$R_3 \Rightarrow \frac{R_3}{-473}$

$$[A:B] = \begin{bmatrix} 8 & 0 & 49 & : & 57 \\ 0 & 8 & -9 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - 49R_3, R_2 \Rightarrow R_1 + 9R_3$

$$[A:B] = \begin{bmatrix} 8 & 0 & 0 & : & 8 \\ 0 & 8 & 0 & : & 8 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$$8x = 8$$
$$8y = 8$$
$$1x = 1$$

$$x = 1$$
$$y = 1$$
$$z = 1$$

Definition 4.0.1 (Inner product)

An inner product on V is a map

$$\begin{aligned} \langle .,.\rangle : V \times V &\to \mathbb{F} \\ (u,v) &\to \langle u,v\rangle \end{aligned}$$

with the following four properties.

Linearity in first slot: $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ and $\langle au,v\rangle = a\langle u,v\rangle$ for all $u,v,w \in V$ and $a \in F$;

Positivity: $\langle v,v\rangle \geq 0$ for all $v \in V$;

Positive definiteness: $\langle v,v\rangle = 0$ if and only if $v = 0$;

Conjugate symmetry: $\langle u,v\rangle = \overline{\langle v,u\rangle}$ for all $u,v \in V$.

Definition 4.0.2 (Inner product space)

An inner product space is a vector space over F together with an inner product $\langle .,.\rangle$.

Definition 4.0.3 (Inner product space)

Let $V(F)$ be a vector space where F is either the field of real numbers or the field of complex numbers. An inner product space on V is a function from $V \times V$ into F which assigns to each ordered pair of vectors α, β in V a scalar (α, β) in a such way that:

(1) Conjugate symmetry:

$$(\alpha, \beta) = \overline{(\beta, \alpha)}, \forall \alpha, \beta \in \mathbb{V}.$$

(2) Linearity:

$$[a\alpha + b\beta]x = a\alpha(x) + b\beta(x), \forall \alpha, \beta, x \in \mathbb{V}, a, b \in \mathbb{F}$$

(3) Non-negativity:

$$(\alpha, \alpha) \geq 0$$

and

$$(\alpha, \alpha) = 0 \Rightarrow \alpha = 0, \forall \alpha \in \mathbb{V}.$$

Also the vector space V is then said to be an inner product space with respect to the specified inner product defined on it.

Problem 4.0.4

If $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2) \in v_2\mathbb{R}$, *let us define*

$$(\alpha, \beta) = a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2 \tag{11}$$

we shall show that all the postulates of an inner product hold in (11).

(1) Symmetry [Conjugate symmetry]:

$$\begin{aligned}(\alpha, \beta) &= (\beta, \alpha)\\ (\beta, \alpha) &= b_1a_1 - b_2a_1 - b_1a_2 + 4b_2a_2\\ &= a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2\\ &= (\alpha, \beta)\end{aligned}$$

Hence symmetry is exist.

(2) Linearty:
If $a, b \in \mathbb{R}$

$$\begin{aligned} a\alpha + b\beta &= a(a_1, a_2) + b(b_1, b_2) \\ &= (aa_1, aa_2) + (bb_1, bb_2) \\ &= (aa_1 + bb_1, aa_2 + bb_2) \end{aligned}$$

Let $\gamma = (c_1, c_2) \in V_2\mathbb{R}$, then

$$\begin{aligned} &((a\alpha + b\beta)\gamma) \\ &= [(aa_1 + bb_1, aa_2 + bb_2)(c_1, c_2)] \\ &= (aa_1 + bb_1)c_1 - (aa_2 + bb_2)c_1 - (aa_1 + bb_1)c_2 + 4(aa_2 + bb_2)c_2 \\ &= [aa_1c_1 - aa_2c_1 - aa_1c_2 + 4aa_2c_2] + [bb_1c_1 - bb_2c_1 - bb_1c_2 + 4bb_2c_2] \\ &= a(a_1c_1 - a_2c_1 - a_1c_2 + 4a_2c_2) + b(b_1c_1 - b_2c_1 - b_1c_2 + 4b_2c_2) \\ &= a(\alpha, \gamma) + b(\beta, \gamma) \end{aligned}$$

Hence, linearity satisfied.

(3) Non-negativity:

We have

$$
\begin{aligned}
(\alpha,\alpha) = [(a_1 a_2).(a_1 a_2)] &= a_1a_1 - a_2a_1 - a_1a_2 + 4a_2a_2 \\
&= a_1^2 - 2a_1a_2 + 4a_2^2 \\
&= a_1^2 - 2a_1a_2 + a_1^2 + 3a_2^2 \\
&= (a_1 - a_2)^2 + 3a_2^2 \qquad (12)
\end{aligned}
$$

It is a sum of two non-negative real numbers. Therefore it is ≥ 0.
Thus $(\alpha,\alpha) \geq 0$. Also,

$$
\begin{aligned}
(\alpha,\alpha) &= 0 \\
(a_1 - a_2)^2 + 3a_2^2 &= 0 \\
(a_1 - a_2)^2 &= 0, & 3a_2^2 &= 0 \\
a_1 - a_2 &= 0 & a_2 &= 0 \\
a_1 &= a_2. & a_2 &= 0 \\
a_1 = 0,\ a_2 &= 0 & \alpha &= 0
\end{aligned}
$$

$\therefore$ all the postulates are satisfied. Hence, it is an inner product.

Problem 4.0.5

Show that $V_n(C)$ is an inner product space with inner product define on $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n) \in V_n(C)$ by $(\alpha, \beta) = a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n}$ which is standard inner product on $V_n(F)$

Let $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$ and $\gamma = (c_1 c_2, \cdots, c_n) \in V_n(F)$

(1) Non-negativity:

$$
\begin{aligned}
(\alpha, \alpha) &= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 \geq 0, \text{ Since } |a_1|^2 \geq 0 \\
(\alpha, \alpha) &= 0 \Leftrightarrow |a_1|^2 + ||a_2|^2 + |a_3|^2 + \cdots + |a_n|^2 = 0 \\
&\Rightarrow \text{ each } a_i \qquad = 0 \Rightarrow \alpha = 0
\end{aligned}
$$

(2) Conjucate symmetry:

$$
\begin{aligned}
(\alpha, \beta) &= a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n} \\
\overline{(\beta, \gamma)} &= \overline{b_1\overline{a_1} + b_2\overline{a_2} + \cdots + b_n\overline{a_n}} \\
&= \overline{b_1}\overline{\overline{a_1}} + \overline{b_2}\overline{\overline{a_2}} + \cdots + \overline{b_n}\overline{\overline{a_n}} \\
&= a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n} \\
(\alpha, \beta) &= \overline{(\beta, \gamma)}
\end{aligned}
$$

(3) Linear:

$$\begin{aligned} a\alpha + b\beta &= a(a_1, a_2, \cdots, a_n) + b(b_1, b_2, \cdots, b_n) \\ &= (aa_1 + bb_1, aa_2 + bb_2, \cdots, aa_n + bb_n) \end{aligned}$$

Now,

$$\begin{aligned} (a\alpha + b\beta, \gamma) &= (aa_1 + bb_1)\overline{c_1} + (aa_2 + bb_2)\overline{c_2} + \cdots + (aa_n + bb_n)\overline{c_n} \\ &= a(a_1\overline{c_1} + a_2\overline{c_2} + \cdots + a_n\overline{c_n}) + b(b_1\overline{c_1} + b_2\overline{c_2} + \cdots + b_n\overline{c_n}) \\ &= a(\alpha, \gamma) + b(\beta, \gamma) \end{aligned}$$

Here inner product define by α, β and γ satisfies all three condition. So $V_n(C)$ is inner product space.

Problem 2.3.1

Let T be a linear transformation on $V_3(\mathbb{R})$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)\ \forall (a,b,c) \in V_3(\mathbb{R})$. Is T invertible?. If so, find a rule for T^{-1} as the one which defines T.

For proving T is invertible, we need to show only T is one-one and onto.
To prove one-one:
Let

$$\alpha = (a_1, b_1, c_1) \in V_3(\mathbb{R})$$
$$\beta = (a_2, b_2, c_2) \in V_3(\mathbb{R})$$

Then,

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$
\begin{aligned}
3a_1 &= 3a_2 \\
a_1 &= a_2 \\
a_1 - b_1 &= a_2 - b_2 \\
a_2 &= b_2 \\
c_1 &= c_2 \\
(a_1, b_1, c_1) &= (a_2, b_2, c_2) \\
\alpha &= \beta
\end{aligned}
$$

$\therefore T$ is one-one.

To prove onto:

T is linear transformation on a finite dimensional vector space $V_3(\mathbb{R})$, where dimension in 3.

$$
\begin{aligned}
&\Rightarrow \text{Also } T \text{ is one-one} \\
&\Rightarrow T \text{ must be onto} \\
&\Rightarrow T \text{ is invertible}
\end{aligned}
$$

$$\text{If } T(a,b,c) = (p,q,r)$$
$$\text{then, } T^{-1}(p,q,r) = (a,b,c)$$
$$T(a,b,c) = (p,q,r)$$
$$(3a, a-b, 2a+b+c) = (p,q,r)$$
$$3a = p$$
$$p = 3a$$
$$a = \frac{p}{3}$$
$$a - b = q$$
$$\frac{p}{3} - b = q$$
$$\frac{p}{3} - q = b$$

$$
\begin{aligned}
2a+b+c &= r \\
2\frac{p}{3}+\left(\frac{p}{3}-q\right)+c &= r \\
c &= r-p+q
\end{aligned}
$$

$$
\begin{aligned}
\because T^{-1}(p,q,r) &= (a,b,c) \\
&= \left(\frac{p}{3},\frac{p}{3}-q,r-p+q\right)
\end{aligned}
$$

Example 2.3.2

Let T be a linear map on $V_3(R)$ defined by $T(a,b,c) = [3a, a-b, 2a+b+c]$ $\forall a,b,c \in \mathbb{R}$. Is T invertible?. If so find a rule for T^{-1} like one which define T.

For proving T is invertible, we need to show that T is one-one and onto.
To prove one-one:
Let $\alpha = (a_1, b_1, c_1)$, $\beta - (a_2, b_2, c_2)$ be any two elements of $V_3(\mathbb{R})$.

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$
\begin{aligned}
3a_1 &= 3a_2 \\
a_1 - b_1 &= a_2 - b_2 + c_2 \\
2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2 \\
a_1 &= a_2 \\
\because a_1 - b_1 &= a_2 - b_2 \\
-b_1 &= -b_2 \\
b_1 &= b_2 \\
\because 2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2 \\
\because a_1 &= b_1 \\
b_1 &= b_2 \\
c_1 &= c_2 \\
(a_1, b_1, c_1) &= (a_2, b_2, c_2) \\
\alpha &= \beta
\end{aligned}
$$

$$\because T(\alpha) = T(\beta)$$
$$\alpha = \beta$$
$$T := A \to B$$

Hence T is one-one.
To find onto:
Since, T is a linear one-one map on a finite dimensional vector space.
$\Rightarrow T$ is onto.
$\Rightarrow T$ is one-one and onto.
$\Rightarrow T$ is invertible.

Second part:

$$\text{Let } T(a,b,c) = (p,q,r) \tag{10}$$
$$\text{Then } T^{-1}(p,q,r) = (a,b,c)$$

Now

$$\begin{aligned}
T(a,b,c) &= (p,q,r)\\
(3a, a-b, 2a+b+c) &= (p,q,r)\\
3a &= p\\
a &= \frac{p}{3}\\
\because a-b &= q\\
\frac{p}{3}-b &= q\\
\frac{p}{3}-q &= b
\end{aligned}$$

$$\because 2a + b + c = r$$
$$2\frac{p}{3} + \left(\frac{p}{3} - q\right) + c = r$$
$$c = r - p + q$$

Put the value of a, b, c in equation (**??**)

$$T^{-1}(p, , q, r) = \left(\frac{p}{3}, \frac{p}{3} - a, r - p + q\right)$$

or

$$T^{-1}(x, y, z) = \left(\frac{x}{3}, \frac{x}{3} - y, z - x + y\right)$$

which is the rule which defines T^{-1}.

Definition 2.3.3 (Wronskian)

Let f and g be differentiable on $[a, b]$. If Wronskian $W(f, g)(t_0)$ is nonzero for some t_0 in $[a, b]$ then f and g are linearly independent on $[a, b]$. If f and g are linearly dependent then the Wronskian is zero for all t in $[a, b]$.

Problem 2.3.4

Using Wronskian method prove that $\{e^{3x}, e^{5x}\}$ *is a linearly independent set on* $\mathbb{R}$.

Set $f(x) = e^{3x}$, $g(x) = e^{5x}$. Then,

$$\begin{aligned} W(f(x), g(x)) &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & f''(x) \end{vmatrix} \\ &= \begin{vmatrix} e^{3x} & e^{5x} \\ 3e^{3x} & 5e^{5x} \end{vmatrix} \\ &= 5e^{8x} - 3e^{8x} \\ &= 2e^{8x} \\ &\neq 0 \ \ (\forall x \in \mathbb{R}) \end{aligned}$$

$\because$ The given set $\{e^{3x}, e^{5x}\}$ is linearly independent.

Problem 2.3.5

Using Wronskian method prove that $\{e^{2x}, \cos(x), 2e^{2x}\}$ *is a linearly dependent set on* $\mathbb{R}$.

Set $f(x) = e^{2x}$, $g(x) = \cos x\ h(x) = 2e^{2x}$. Then,

$$W(f(x), g(x), h(x))$$

$$= \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & \cos x & 2e^{2x} \\ 2e^{2x} & -\sin x & 4e^{2x} \\ 4e^{2x} & -\cos x & 8e^{2x} \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} -\sin x & 4e^e 2x \\ -\cos x & 8e^{2x} \end{vmatrix} - 2e^{2x} \begin{vmatrix} \cos x & 2e^{2x} \\ -\cos x & 8e^{2x} \end{vmatrix} + 4e^{2x} \begin{vmatrix} \cos x & 2e^{2x} \\ -\sin t & 4e^{2x} \end{vmatrix}$$

$$= e^{2x}\left(-8e^{2x}\sin x + 4e^{2x}\cos x\right) - 2e^{2x}\left(8e^{2x}\cos x + 2e^{2x}\cos x\right)$$

$$+ 4e^{2x}\left(4e^{2x}\cos x + 2e^{2x}\sin x\right)$$

$$
\begin{aligned}
&= e^{2x}\left(-8\sin x + 4\cos x - 20\cos x + 16\cos x + 8\sin x\right)\\
&= e^{4x}(0)\\
&= 0 \ (\forall x \in \mathbb{R})
\end{aligned}
$$

Example 2.3.6

Using Wronskian method prove that $\{1, x, x^2\}$ is a linearly dependent set on $\mathbb{R}$.

Ans: $W(f(x), g(x), h(x)) = 2 \neq 0$, So the set is linearly independent.

Example 2.5.3

If matrix of a linear transform on R^3 relative to basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$. Then find the linear transform matrix T relative to basis $B_1 = \{(0,1,-1),(1,-1,1),(-1,1,0)\}$.

First we find linear transform.
We have

$$[T:B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}^t$$

That is transpose of coefficient matrix. So that

$$T(u_1) = T(1,0,0) = 0(1,0,0) + 1(0,1,0) - 1(0,0,1) = (0,1,-1)$$
$$T(u_2) = T(0,1,0) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1) = (1,0,-1)$$
$$T(u_3) = T(0,0,1) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1) = (1,-1,0)$$

$(x,y,z) \in \mathbb{R}^3$ be any element and B is basis for $\mathbb{R}^3$

$$\therefore (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$\begin{aligned}\therefore T(x,y,z) &= xT(1,0,0)+yT(0,1,0)+zT(0,0,1)\\ &= x(0,1,-1)+y(1,0,-1)+z(1,-1,0)\\ T(x,y,z) &= (y+z, x-z, -x-y)\end{aligned}$$

which is linear operator T on R^3.
Now we have to find a matrix of T relative basis.

$$B_1 = \{(0,1,-1),(1,-1,1),(-1,1,0)\}$$

Let $(a,b,c) \in \mathbb{R}^3$ be any element
Let

$$\begin{aligned}(a,b,c) &= l(0,1,-1)+m(1,-1,1)+n(-1,1,0)\\ (a,b,c) &= (m-n, l-m+n, -l+m)\\ &\Rightarrow a=m-n;\ b=l-m+n;\ c=-l+m\end{aligned}$$

Now

$$
\begin{array}{c|c|c}
l - m + n = b & l - m + n = b & m = a + n \\
l = b + m - n = b + a & n = b - l + m & m = a + b + c \\
l = a + b & n = b + c &
\end{array}
$$

$$B_1 = \{(0, 1 - 1), (1, -1, 1), (-1, 1, 0)\}$$

$\therefore$ we get

$$(a, b, c) = (a + b)(0, 1, -1) + (a + b + c)(1, -1, 1) + (b + c)(-1, 1, 0)$$

and we have

$$T(x, y, z) = (y + z, x - z, -x - y)$$

Now

$$\begin{aligned}
T(0, 1, -1) &= (0, 1, -1) = 1(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0) \\
T(1, -1, 1) &= (0, 0, 0) = 0(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0) \\
T(-1, 1, 0) &= (1, -1, 0) = 0(0, 1, -1) + 0(1, -1, 1) + (-1)(-1, 1, 0)
\end{aligned}$$

$$\therefore [T; B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Example 2.5.4

Let T be linear transform on $\mathbb{R}^2$ and $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ be matrix of T with respect to usual basis of $\mathbb{R}^2$. Then, find that matrix of T with respect to $B_1 = \{(1,2),(5,6)\}$.

Ans:

$$[T : B_1] = \begin{bmatrix} \frac{11}{2} & \frac{41}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Definition 2.5.5 (Isomorphism of a vector space)

Let $U(F)$ and $V(F)$ are two vector spaces then a linear transformation $f : U \to V$ is called Isomorphism, if

1. f is one-one
2. f is onto

Definition 2.5.6 (Isomorphism of a vector space)

$f : U \to V$ is called Isomorphism if

1. f is a linera transform
2. f is one-one
3. f is onto

Problem 2.5.7

Let $f : V_2(\mathbb{R}) \to V_2(\mathbb{R})$ be $f(x,y) = (y,x)$. Prove f is Isomorpism.

To prove one-one:
Let $U, V \in V_2(\mathbb{R})$

$$\begin{aligned} f(u) &= f(v) \\ f(x,y) &= f(p,q) \\ (y,x) &= (q,p) \\ y &= q; x = p \\ (x,y) &= (p,q) \\ u &= v \end{aligned}$$

i.e., f is one-one

To prove onto:

$\forall (x, y) \in V_2(\mathbb{R})$

$\exists (y, x) \in V_2(\mathbb{R})$ such that $f(x, y) = (y, x)$

To prove linear transform:
Let $u, v \in V_2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$$
\begin{aligned}
f(\alpha u + \beta v) &= f[\alpha(x, y) + \beta(p, q)] \\
&= f[\alpha x + \beta p, \alpha y + \beta q] \\
&= (\alpha y + \beta q, \alpha x + \beta p) \\
&= \alpha(y, x) + \beta(q, p) \\
&= \alpha f(x, y) + \beta(p, q) \\
&= \alpha f(u) + \beta f(v)
\end{aligned}
$$

So, f is a linear transform, one-one, onto.
i.e., f is an Isomorpism.

Problem 2.5.8

Let $T : P_2 \to V_3 \to \{(x_1, x_2, x_3)|x_i \in \mathbb{R}\}$ *(*P_2*-set of all polynomials of degree* ≤ 2*)* $\{a_0 + a_1x + a_2x^2|a_0, a_1, a_2 \in \mathbb{R}\}$*. Prove that* T *is Isomorphism.* $T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2)$

To prove T is one-one:

$$T(p_1) = T(p_2)$$
$$T(a_0 + a_1x + a_2x^2) = T(b_0, b_1x + b_2x^2)$$
$$(a_0, a_1, a_2) = (b_0, b_1, b_2)$$
$$a_0 = b_0; a_1 = b_1; a_2 = b_2$$
$$a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$$
$$p_1(x) = p_2(x)$$
$$p_1 = p_2$$

T is one-one.

To prove T is onto:

$T : p_2 \to v_3$. For every $(a_0, a_1, a_2) \in v_3$ we have a polynomial $p = a_0 + a_1x + a_2x^2$ in p_2. Such that

$$T(p) = (a_0, a_1, a_2)$$

T is onto.

T is one-one and onto.

To prove T is linear.

$$\begin{aligned}
&T(\alpha(a_0 + a_1x + a_2x^2) + \beta(b_0 + b_1x + b_2x^2)) \\
&= T((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2) \\
&= (\alpha a_0 + \beta b_0, \alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2) \\
&= (\alpha a_0, \alpha a_1, \alpha a_2) + (\beta b_0, \beta b_1, \beta b_2) \\
&= \alpha(a_0, a_1, a_2) + \beta(b_0, b_1, b_2) \\
T(\alpha p_1 + \beta p_2) &= \alpha T(p_1(x)) + \beta T(p_2(x))
\end{aligned}$$

This proves T is linear.
$\therefore T$ is an isomorphic.
To find its inverse:

$$T^{-1} : v_3 \to p_2$$
$$T^{-1}(a_0, a_1, a_2) = a_0 + a_1x + a_2x^2$$

Example 2.5.9

$T : v_2 \to v_2 \; T(x_1, x_2) = (x_1, -x_2)$

Definition 1.7.1 (LDU factorization)

$$A = LUD$$

Remark 1.7.2

$$\begin{bmatrix} * & 6th & 5th \\ 1st & * & 4th \\ 2nd & 3rd & * \end{bmatrix} and \begin{bmatrix} * & 12th & 11th & 9th \\ 1st & * & 10th & 8th \\ 2nd & 5th & 6th & * \end{bmatrix}$$

Problem 1.7.3

Find LDU of $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$R_2 \Rightarrow -3R_1 + R_2$$
$$R_3 \Rightarrow (-1)R_1 + R_3$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} = E_1 A$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$R_2 \Rightarrow \frac{2}{3}R_3 + R_2$$

$$R_1 \Rightarrow \frac{-1}{3}R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} = E_2E_1A$$

$$E_2 = \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \Rightarrow \frac{1}{2}R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= E_3E_2E_1A = D$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore\ A = LUD = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Definition 2.1.1 (Linearly dependent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, a_n\}$ of vectors of V is said to be linear dependent if there exist scalar $a_1, a_2, \ldots, a_n \in F$ not all of them 0 (some of them may be zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$

Definition 2.1.2 (Linearly Independent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, a_n\}$ of vectors of V is said to be linearly independent if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$
$$a_i \in F, 1 \leq i \leq n \Rightarrow a_i = 0 \text{ for each} 1 \leq i \leq n$$

An infinite set of vector of V is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Example 2.1.3

Find whether the set of vector $v_i = (1,2,1)$, $v_2 = (3,1,5)$, $v_3 = (3,-4,7)$ is linearly independent or dependent.

Let a_1, a_2, a_3 be three scalars such that

$$\begin{aligned} a_1v_1 + a_2v_2 + a_3v_3 &= 0 \\ \Rightarrow a_1(1,2,1) + a_2(3,1,5) + a_3(3,-4,7) &= 0 \\ (a_1 + 3a_2 + 3a_3, 2a_1 + a_2 - 4a_3, a_1 + 5a_2 + 7a_3) &= 0 \end{aligned}$$

$$\begin{aligned} a_1 + 3a_2 + 3a_3 &= 0 \\ 2a_1 + a_2 - 4a_3 &= 0 \\ a_1 + 5a_2 + 7a_3 &= 0 \end{aligned}$$

The coefficients matrix of these equation is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix}$$

$$= 1(7+20) - 3(14+4) + 3(10-1) = 27 - 54 + 27 = 0$$

and

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$$

$$\therefore \rho(A) = 2$$

i.e., so *the rank of matrix* $A <$ *no. of unknown quantities*.
The system of equations will have $3 - 2 = 1$ non-zero solutions and hence the set of vectors are linearly dependent.

Problem 2.1.4

Show that the set $\{1, x, 1+x+x^2\}$ is linearly independent set of vectors in the vector space of all polynomial over the real number filed.

Let a_1, a_2, a_3 be scalars (real numbers) such that

$$a_1(1) + a_2(x) + a_3(1+x+x^2) = 0$$

We have

$$\begin{aligned}(a_1+a_3) + (a_2+a_3)x + a_3x^2 &= 0\\ a_1+a_3 = 0, a_2+a_3 = 0, a_3 &= 0\\ a_1 = 0, a_2 = 0, a_3 &= 0\end{aligned}$$

Therefore the vectors $1, x, 1+x+x^2$ are linearly independent over the field of real numbers.

Example 2.1.5

Are the vectors $(2,2,2,4),(2,-2,-4,0),(4,-2,-5,2),(4,2,1,6)$ linearly independent?

Let a_1, a_2, a_3 and a_4 be four scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

Here

$\alpha_1 = (2,2,2,4), \alpha_2 = (2,-2,-4,0), \alpha_3 = (4,-2,-5,2)$ and $\alpha_4 = (4,2,1,6)$

$$\begin{aligned}
\therefore a_1(2,2,2,4) + a_2(2,-2,-4,0) + a_3(4,-2,-5,2) + a_4(4,2,1,6) = 0\\
(2a_1 + 2a_2 + 4a_3 + 4a_4, 2a_1 - 2a_2 - 2a_3 + 2a_4,\\
2a_1 - 4a_2 - 5a_3 + a_4, 4a_1 + 2a_3 + 6a_4) = (0,0,0,0)
\end{aligned}$$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 2 & -2 & -4 & 0 \\ 4 & -2 & -5 & 2 \\ 4 & 2 & 1 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - R - 1, R_3 \to R_3 - R_1$ and $R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -3 \\ 0 & -4 & -6 & -2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 3R_2$ and $R_4 \to -R_2$, we get

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

i.e., so the rank of matrix $A <$ number of unknown quantities.

The system of equations will have $4-2=2,$ non-zero solutions and hence the set of vectors are linearly dependent. Hence given vectors are not linearly independent.

Example 2.1.6

Show that the vectors (a_1, a_2) and (b_1, b_2) in $V_2(C)$ are L.D. iff $a_1b_2 - a_2b_1 = 0$, where C is the field complex numbers.

Let $a, b \in C$, then

$$a(a_1, a_2) + b(b_1, b_2) = 0$$
$$i.e., (aa_1 + bb_1, aa_2 + bb_2) = (0, 0)$$

$$\left.\begin{aligned} aa_1 + bb_1 &= 0 \\ aa_2 + bb_2 &= 0 \end{aligned}\right\} \tag{9}$$

The system of equations (9) will possess a non-zero solution iff

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \Rightarrow a_1b_2 - a_2b_1 = 0$$

Thus the given system of vectors is L.D. iff $a_1b_2 - a_2b_1 = 0$.

Course Code MAT3002	Applied Linear Algebra	Course Type	LT
		Credits	3

Course Objectives:

- Linear algebra is one of the most important subjects of pure mathematics and has many applications in electrical, communications and computer science. This course aims at introducing students to the fundamental concepts of linear algebra by starting with linear equations and culminating in abstract vector spaces and linear transformations.

Course Outcomes:

By the end of the course, the students will be able to

- solve systems of linear equations
- understand the concepts of vector spaces and subspaces, basis and dimensions, linear transformations and inner product spaces and their matrix representations
- use Gram-Schmidt process to obtain orthonormal basis,
- find the change of basis matrix with respect to two bases of a vector space.

Student Outcomes (SO) : a,e,j,k

Module No.	Module Description	Hrs.	SO
1	**Linear Equations and Matrices** Introduction - Gaussian elimination and Gauss Jordan methods – Block matrices - Elementary matrices- permutation matrix - inverse matrices - LDU factorization – Applications to electrical networks and cryptography.	8	a,e,j,k
2	**Vector Spaces and Subspaces** Vector spaces and subspaces – Linear Independence, Basis and Dimension – Row, Column and Null spaces – Rank and Nullity – Bases for subspaces – Invertibility – Application: Interpolation and Wronskian	9	a,e,j,k
3	**Linear Transformations** Definition and Examples – properties - The Range and Kernel – Invertible linear transformations – Isomorphism – Application: Computer graphics - Matrices of linear transformations - Vector space of linear transformations – change of bases – similarity	9	a,e,j,k
4	**Inner Product Spaces**	8	a,e,j,k

	Inner products – The lengths and angles of vectors – Matrix representations of inner products – Orthogonal projections - Gram-Schmidt orthogonalization		
5	**Applications of Inner Product Spaces** QR factorization – Singular Value Decomposition - Projection - orthogonal projections – relations of fundamental subspaces – Least square solutions – Orthogonal projection matrices	**9**	**a,e,j,k**
6	**Guest Lectures by experts on contemporary topics**	**2**	
	Total	**45**	

Mode of Teaching and Learning:

Class room teaching
Use of mathematical softwares (such as MATLAB, MATHEMATICA, SAGE, ETC.) as teaching aid
Minimum of 2 hours lectures by experts on contemporary topics

Mode of Evaluation and assessment: Digital Assignments, Continuous Assessment Tests, Final Assessment Test and unannounced open book examinations, quizzes, student's portfolio generation and assessment, innovative assessment practices

Text Book(s):

1. Linear Algebra by Jin Ho Kwak and Sungpyo Hong, Second edition, Springer, 2004.
2. Linear Algebra with applications by Steven J. Leon, 8th Edition, Pearson, 2010.

Reference Book(s):

1. Elementary Linear Algebra by Stephen Andrilli and David Hecker, 4th edition, Academic Press, 2010.
2. Introduction to Linear Algebra by Gilbert Strang, 4th edition, Wellesley-Cambridge Press, 2011.
3. Introductory Linear Algebra - An applied first course by Bernard Kolman and David R. Hill, 9th Edition, Pearson education, 2011.
4. Linear Algebra A Modern Introduction by David Poole, 2nd edition, Thomson Learning, 2006

Recommendation by the Board of Studies on	22-4-2017
Approval by Academic council on	07-09-2017
Compiled by	Dr.V.Prabhakar & Dr.C.Vijayalakshmi

Problem 1.1.1

Apply Gauss elimination method to solve the equation $x+4y-z=-5$, $x+y-6z=-12$, $3x-y-z=4$.

$$\begin{bmatrix}1 & 4 & -1\\ 1 & 1 & -6\\ 3 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-5\\ -12\\ 4\end{bmatrix}$$

Its augmented matrix is

$$C=\begin{bmatrix}A & : & B\end{bmatrix}=\begin{bmatrix}1 & 4 & -1 & : & -5\\ 1 & 1 & -6 & : & -12\\ 3 & -1 & -1 & : & 4\end{bmatrix}$$

Applying operations:

$$R_2 \Rightarrow R_2 - R_1,$$
$$R_3 \Rightarrow R_3 - 3R_1.$$

$$C \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & -13 & 2 & : & 19 \end{bmatrix}$$

Applying operation $R_3 \Rightarrow 3R_3 - 13R_2$

$$C \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & 0 & 71 & : & 148 \end{bmatrix}$$

which can be written as

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148 \end{bmatrix}$$

$$x + 4y - z = -5 \quad (1)$$
$$-3y - 5z = -7 \quad (2)$$
$$71z = 148 \quad (3)$$

$$z = \frac{148}{71}$$

From equation (2),

$$y = \frac{-1}{3}\left[-7 + 5\left(\frac{148}{71}\right)\right] = \frac{-81}{71}$$

From equation (1),

$$x = \frac{117}{71}$$

$$x = \frac{117}{71}; y = \frac{-81}{71}; z = \frac{148}{71}$$

Example 1.1.2

Using Gauss elimination method find the solutions of $4x - 3y + z = -8$, $-2x + y - 3z = -4$, $x - y + 2z = 3$.

Ans:

$$x = 2$$
$$y = 1$$
$$z = 3$$

Example 1.1.3

Using Gauss elimination method find the solutions of $x+y+z=3$, $2x+3y+7z=0$, $x+3y-2z=17$.

Ans:

$$x = 1$$
$$y = 4$$
$$z = -2$$

Problem 1.2.1

Apply Gauss Jordan method to solve the equation $x+y+z=9$, $2x-3y+4z=13$, $3x+4y+5z=40$..

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Augmented matrix is given by

$$\begin{aligned} C &= [A : B] \\ &= \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & -3 & 4 & : & 13 \\ 3 & 4 & 5 & : & 40 \end{bmatrix} \end{aligned}$$

Applying operations $R_2 \Rightarrow R_2 - 2R_1; R_3 \Rightarrow R_3 - 3R_1$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -5 & 2 & : & -5 \\ 0 & 1 & 2 & : & 13 \end{bmatrix}$$

$R_2 \Leftrightarrow R_3$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & -5 & 2 & : & -5 \end{bmatrix}$$

$R_3 \Rightarrow R_3 + 5R_2$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - R_2$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 12 & : & 60 \end{bmatrix}$$

$R_3 \Rightarrow \frac{1}{12}(R_3)$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 2 & : & 13 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_3$

$$C \sim \begin{bmatrix} 1 & 0 & -1 & : & -4 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \Rightarrow R_1 + R_3$

$$C \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$x = 1$$

$$y = 3$$

$$z = 5$$

Problem 1.2.2

Apply Gauss-Jordan method to solve the equation

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

$$[A : B] = \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 2 & 10 & 1 & : & 13 \\ 1 & 1 & 5 & : & 7 \end{bmatrix}$$

$R_1 \Leftrightarrow R_2$

$$[A : B] = \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 2 & 10 & 1 & : & 13 \\ 10 & 1 & 1 & : & 12 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1, R_3 \Rightarrow R_3 - 10R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 0 & 8 & -9 & : & -1 \\ 0 & -9 & -49 & : & -58 \end{bmatrix}$$

$R_1 \Rightarrow 8R_1 - R_2, R_3 \Rightarrow 8R_3 + 9R_2$

$$[A:B] = \begin{bmatrix} 8 & 0 & 49 & : & 57 \\ 0 & 8 & -9 & : & -1 \\ 0 & 0 & -473 & : & -473 \end{bmatrix}$$

$R_3 \Rightarrow \frac{R_3}{-473}$

$$[A:B] = \begin{bmatrix} 8 & 0 & 49 & : & 57 \\ 0 & 8 & -9 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - 49R_3, R_2 \Rightarrow R_1 + 9R_3$

$$[A:B] = \begin{bmatrix} 8 & 0 & 0 & : & 8 \\ 0 & 8 & 0 & : & 8 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Example 1.2.3

Apply Gauss-Jordan method to solve the equation

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

Solution:

$$x = 1$$
$$y = 4$$
$$z = -2$$

$$8x = 8$$
$$8y = 8$$
$$1x = 1$$

$$x = 1$$
$$y = 1$$
$$z = 1$$

Block matrices

$$A = \left[\begin{array}{cc:cc:c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hdashline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc:ccc} 1 & -2 & 0 & 1 & 3 \\ \hdashline 2 & 3 & 5 & 7 & -2 \\ 3 & 1 & 4 & 5 & 9 \\ \hdashline 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc:cc} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hdashline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

Definition 1.3.1 (Square block matrix)

Let M be a block matrix, the M is square block matrix if

1. M is square.
2. The block form a square matrix.
3. The diagonal blocks are also square matrices.

Example 1.3.2

$$A = \left[\begin{array}{cc:cc:c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 1 & 3 \\ \hdashline 2 & 1 & 2 & 2 & 1 \\ \hdashline 3 & 2 & 1 & 4 & 5 \\ 5 & 2 & 3 & 1 & 4 \end{array}\right]$$

This is not a square block matrix

$$A = \left[\begin{array}{cc:cc:c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 1 & 3 \\ \hdashline 2 & 1 & 2 & 2 & 1 \\ 3 & 2 & 1 & 4 & 5 \\ \hdashline 5 & 2 & 3 & 1 & 4 \end{array}\right]$$

This is a square block matrix

Definition 1.3.3 (Block diagonal matrix)

Let $M : [A_{ij}]$ be a square block matrix such that non-diagonal blocks are all zero matrices.
i.e. $A_{ij} = 0$ whenever $i \neq j$.
M is called block diagonal matrix, if $M = diag(A_{11}, A_{22}, A_{33}, \ldots, A_{rr})$ or $M = A_{11} \oplus A_{22} \oplus \ldots \oplus A_{rr}$.

$$M = \left[\begin{array}{c:c:c} & 0 & \\ \hdashline 0 & & 0 \\ \hdashline 0 & 0 & \end{array}\right]$$

Remark 1.3.4

Note : *$M : diag(A_{11}, A_{22}, \ldots, A_{rr})$ is invertible iff A_{ii} is invertible $\forall i$. Then* $M^{-1} = diag\left(A_{11}^{-1}, A_{22}^{-1}, \ldots, A_{rr}^{-1}\right)$

Example 1.3.5

$$M = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

Example 1.3.6

Find Block matrix multiplication of

$$A = \begin{bmatrix} 1 & 3 & -2 & 3 & -3 \\ 2 & 4 & -2 & 2 & 1 \\ 0 & -2 & 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

$$A = \left[\begin{array}{ccc|cc} 1 & 3 & -2 & 3 & -3 \\ 2 & 4 & -2 & 2 & 1 \\ \hline 0 & -2 & 1 & 1 & 1 \end{array}\right]_{3\times 5}, \quad B = \left[\begin{array}{cc} 1 & 0 \\ 2 & 1 \\ 3 & 3 \\ \hline 2 & 2 \\ 1 & 2 \end{array}\right]_{5\times 2}$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}; A = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_1B_1 & A_2B_2 \\ A_3B_1 & A_4B_2 \end{bmatrix}$$

$$A_1B_1 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix}$$

$$A_2B_2 = \begin{bmatrix} 3 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & 6 \end{bmatrix}$$

$$A_3B_1 = \begin{bmatrix} 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$A_4B_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_1B_1 & A_2B_2 \\ A_3B_1 & A_4B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix} \\ \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ 9 & 4 \\ 2 & 5 \end{bmatrix}$$

Example 1.4.1

Find the elementary matrices of $A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$.

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} = M_1A$$
$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} = M_2 M_1 A$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} = M_3 M_2 M_1 A$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$U = M_3 M_2 M_1 A$$

The permutation matrices of order two are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and of order three are given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Remark 1.5.1

1. *A permutation matrix is nonsingular, and the determinant is always* ± 1.
2. *A permutation matrix* A *satisfies*

$$AA^T = I,$$

   where A^T *is a transpose and* I *is the identity matrix.*
3. I *is a special* P
4. *Every row has one* 1
5. *Every column has* 1

Problem 1.5.2

Find permutation of $A = \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & 7 \end{pmatrix}$$
$$\begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 6 & 2 \end{pmatrix}$$

Remark 1.5.3

- *Multiplication of permutation matrix changes the position of the rows and columns.*

Definition 1.6.1 (Cayley Hamilton Theorem)

A matrix satisfies its own characteristic equation. That is, if the characteristic equation of an $n \times n$ matrix A is $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1A + a_0I = 0.$$

Problem 1.6.2

Verify Cayley Hamilton theorem for the following matrix A and hence find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Given

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

The characteristic equation

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$$s_1 = 1 - 3 - 2 = -4$$

$$s_2 = \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}$$

$$= (6-1)(-2+2) + (-3-6) \Rightarrow 5 - 9 = -4$$

$$s_3 = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 1(6-1) - 2(-6-2) - 1(3+6) - 1(3+6) = 5 + 16 - 9 = 12$$

Then the characteristic equation is,

$$\lambda^3 + 4\lambda^2 - 4\lambda - 12 = 0$$

To verify the Cayley Hamilton theorem in characteristic replace λ by $'A'$, then we have

$$A^3 + 4A^2 - 4A - 12I = 0 \tag{4}$$

$$A.A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-2 & 2-6-1 & -1+2+2 \\ 3-9+2 & 6+9+1 & -3-3-2 \\ 2+3-4 & 4-3-2 & -2+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix}$$

$$
\begin{aligned}
A^3 &= A^2.A \\
&= \begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix}
\end{aligned}
$$

Let us substitute the value in equation (4)

$$A^3 + 4A^2 - 4A - 12I$$

$$= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + 4\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 28 & -16 \\ 28 & -64 & 36 \\ 4 & 8 & -8 \end{bmatrix} + \begin{bmatrix} 20 & -20 & 12 \\ -16 & 64 & -32 \\ 4 & -4 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ 0 & 12 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4+20-4-12 & 28-20-8-0 & -16+12+4-0 \\ 28-16-12-0 & -64+64+12-12 & 36-32-4-0 \\ 4+4-8-0 & 8-4-4-0 & -8+12+8-12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, Cayley Hamilton theorem proved.

Consider the characteristic equation

$$A^3 + 4A^2 - 4A - 12I = 0$$

Multiply both side A^{-1}

$$\begin{aligned}
A^3A^{-1} + 4A^2A^{-1} - 4AA^{-1} - 12IA^{-1} &= 0 \\
A^2 + 4A - 4I - 12A^{-1} &= 0 \\
A^2 + 4A - 4I &= 12A^{-1} \\
A^{-1} &= \frac{1}{12}\left[A^2 + 4A - 4I\right]
\end{aligned}$$

$$A^{-1} = \frac{1}{12}\left[\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{12}\begin{bmatrix} 5 & -5 & 3 \\ -4 & 16 & -8 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 8 & -4 \\ 12 & -12 & 4 \\ 8 & 4 & -8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5+4-4 & -5+8+0 & 3-4-0 \\ -4+12-0 & 16-12+4 & -8+4-0 \\ 1+8-0 & -1+4+0 & 3-8-4 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 5 & 3 & -1 \\ 8 & 0 & -4 \\ 9 & 3 & -9 \end{bmatrix}$$

Definition 1.7.1 (LDU factorization)

$$A = LUD$$

Remark 1.7.2

$$\begin{bmatrix} * & 6th & 5th \\ 1st & * & 4th \\ 2nd & 3rd & * \end{bmatrix} and \begin{bmatrix} * & 12th & 11th & 9th \\ 1st & * & 10th & 8th \\ 2nd & 5th & 6th & * \end{bmatrix}$$

Problem 1.7.3

Find LDU of $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$R_2 \Rightarrow -3R_1 + R_2$$
$$R_3 \Rightarrow (-1)R_1 + R_3$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} = E_1 A$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$R_2 \Rightarrow \frac{2}{3}R_3 + R_2$$

$$R_1 \Rightarrow \frac{-1}{3}R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix} = E_2E_1A$$

$$E_2 = \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \Rightarrow \frac{1}{2}R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{-1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= E_3E_2E_1A = D$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore \; A = LUD = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{3} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Example 1.7.4

Find LDU of $\begin{bmatrix} 4 & -20 & -12 \\ -8 & 45 & 44 \\ 20 & -105 & -79 \end{bmatrix}$

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 1.7.5

Find LDU of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 1.8.1 (Application of Matrices in Cryptography)

In this section you will learn to

1. encode a message using matrix multiplication.
2. decode a coded message using the matrix inverse and matrix multiplication.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Problem 1.8.2

Use matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ *to encode the message:* $ATTACK\ NOW$.

We divide the letters of the message into groups of two.

$$AT \quad TA \quad CK \quad -N \quad OW$$

We assign the numbers to these letters from the above table, and convert each pair of numbers into 2×1 matrices. In the case where a single letter is left over on the end, a space is added to make it into a pair.

$$\begin{bmatrix} A \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix};\begin{bmatrix} T \\ A \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix};\begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix};\begin{bmatrix} - \\ N \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix};\begin{bmatrix} O \\ W \end{bmatrix} = \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

So at this stage, our message expressed as 2×1 matrices is as follows.

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix};\begin{bmatrix} 20 \\ 1 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix}\begin{bmatrix} 27 \\ 14 \end{bmatrix}\begin{bmatrix} 15 \\ 23 \end{bmatrix} \quad (5)$$

Now to encode, we multiply, on the left, each matrix of our message by the matrix A. For example, the product of A with our first matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \end{bmatrix} = \begin{bmatrix} 41 \\ 61 \end{bmatrix}$$

And the product of A with our second matrix is:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 23 \end{bmatrix}$$

Multiplying each matrix in (5) by matrix A, in turn, gives the desired coded message:

$$\begin{bmatrix} 41 \\ 66 \end{bmatrix} \begin{bmatrix} 22 \\ 23 \end{bmatrix} \begin{bmatrix} 25 \\ 36 \end{bmatrix} \begin{bmatrix} 55 \\ 69 \end{bmatrix} \begin{bmatrix} 61 \\ 84 \end{bmatrix}$$

Problem 1.8.3

Decode the following message that was encoded using matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 21 \\ 26 \end{bmatrix} \begin{bmatrix} 37 \\ 53 \end{bmatrix} \begin{bmatrix} 45 \\ 54 \end{bmatrix} \begin{bmatrix} 74 \\ 101 \end{bmatrix} \begin{bmatrix} 53 \\ 69 \end{bmatrix} \tag{6}$$

We decode this message by first multiplying each matrix, on the left, by the inverse of matrix A given below.

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

For example:

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 21 \\ 26 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

By multiplying each of the matrices in (6) by the matrix A^{-1} , we get the following.

$$\begin{bmatrix} 11 \\ 5 \end{bmatrix} \begin{bmatrix} 5 \\ 16 \end{bmatrix} \begin{bmatrix} 27 \\ 9 \end{bmatrix} \begin{bmatrix} 20 \\ 27 \end{bmatrix} \begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain:

$$\begin{bmatrix} K \\ E \end{bmatrix} \begin{bmatrix} E \\ P \end{bmatrix} \begin{bmatrix} - \\ I \end{bmatrix} \begin{bmatrix} T \\ - \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

And the message reads: **KEEP IT UP**.

Problem 1.8.4

Using the matrix $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, *encode the message:* **ATTACK NOW**.

We divide the letters of the message into groups of three.

$$ATT \quad ACK \quad -NO \quad W--$$

Note that since the single letter W was left over on the end, we added two spaces to make it into a triplet.
Now we assign the numbers their corresponding letters from the table, and convert each triplet of numbers into 3×1 matrices. We get

$$\begin{bmatrix} A \\ T \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} A \\ C \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} - \\ N \\ O \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} W \\ - \\ - \end{bmatrix} = \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix}$$

So far we have,

$$\begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 27 \\ 14 \\ 15 \end{bmatrix} \begin{bmatrix} 23 \\ 27 \\ 27 \end{bmatrix} \tag{7}$$

We multiply, on the left, each matrix of our message by the matrix B. For example,

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix}$$

By multiplying each of the matrices in (7) by the matrix B, we get the desired coded message as follows:

$$\begin{bmatrix} 1 \\ 21 \\ 42 \end{bmatrix} \begin{bmatrix} -7 \\ 12 \\ 16 \end{bmatrix} \begin{bmatrix} 26 \\ 42 \\ 83 \end{bmatrix} \begin{bmatrix} 23 \\ 50 \\ 100 \end{bmatrix}$$

Problem 1.8.5

Decode the following message

$$\begin{bmatrix}11\\20\\43\end{bmatrix}\quad\begin{bmatrix}25\\10\\41\end{bmatrix}\quad\begin{bmatrix}22\\14\\41\end{bmatrix} \tag{8}$$

that was encoded using matrix

$$B = \begin{bmatrix}1 & 1 & -1\\1 & 0 & 1\\2 & 1 & 1\end{bmatrix}.$$

Since this message was encoded by multiplying by the matrix B. We first determine inverse of B.

$$B^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

To decode the message, we multiply each matrix, on the left, by B^{-1}. For example,

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 20 \\ 43 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix}$$

Multiplying each of the matrices in (8) by the matrix B^{-1} gives the following.

$$\begin{bmatrix} 8 \\ 15 \\ 12 \end{bmatrix} \begin{bmatrix} 4 \\ 27 \\ 6 \end{bmatrix} \begin{bmatrix} 9 \\ 18 \\ 5 \end{bmatrix}$$

Finally, by associating the numbers with their corresponding letters, we obtain

$$\begin{bmatrix} H \\ O \\ L \end{bmatrix} \begin{bmatrix} D \\ - \\ F \end{bmatrix} \begin{bmatrix} I \\ R \\ E \end{bmatrix}$$

The message reads: **HOLD FIRE**.

Definition 2.1.1 (Linearly dependent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of V is said to be linear dependent if there exist scalar $a_1, a_2, \ldots, a_n \in F$ not all of them 0 (some of them may be zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$

Definition 2.1.2 (Linearly Independent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of V is said to be linearly independent if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$
$$a_i \in F, 1 \leq i \leq n \Rightarrow a_i = 0 \text{ for each} 1 \leq i \leq n$$

An infinite set of vector of V is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Example 2.1.3

Find whether the set of vector $v_1 = (1,2,1)$, $v_2 = (3,1,5)$, $v_3 = (3,-4,7)$ is linearly independent or dependent.

Let a_1, a_2, a_3 be three scalars such that

$$\begin{aligned} a_1v_1 + a_2v_2 + a_3v_3 &= 0 \\ \Rightarrow a_1(1,2,1) + a_2(3,1,5) + a_3(3,-4,7) &= 0 \\ (a_1 + 3a_2 + 3a_3, 2a_1 + a_2 - 4a_3, a_1 + 5a_2 + 7a_3) &= 0 \end{aligned}$$

$$\begin{aligned} a_1 + 3a_2 + 3a_3 &= 0 \\ 2a_1 + a_2 - 4a_3 &= 0 \\ a_1 + 5a_2 + 7a_3 &= 0 \end{aligned}$$

The coefficients matrix of these equation is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix}$$

$$= 1(7+20) - 3(14+4) + 3(10-1) = 27 - 54 + 27 = 0$$

and

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$$

$$\therefore \rho(A) = 2$$

i.e., so *the rank of matrix* $A <$ *no. of unknown quantities*.
The system of equations will have $3 - 2 = 1$ non-zero solutions and hence the set of vectors are linearly dependent.

Problem 2.1.4

Show that the set $\{1, x, 1+x+x^2\}$ *is linearly independent set of vectors in the vector space of all polynomial over the real number filed.*

Let a_1, a_2, a_3 be scalars (real numbers) such that

$$a_1(1) + a_2(x) + a_3(1+x+x^2) = 0$$

We have

$$\begin{aligned} (a_1+a_3) + (a_2+a_3)x + a_3x^2 &= 0 \\ a_1+a_3 = 0, a_2+a_3 = 0, a_3 &= 0 \\ a_1 = 0, a_2 = 0, a_3 &= 0 \end{aligned}$$

Therefore the vectors $1, x, 1+x+x^2$ are linearly independent over the field of real numbers.

Example 2.1.5

Are the vectors $(2,2,2,4),(2,-2,-4,0),(4,-2,-5,2),(4,2,1,6)$ linearly independent?

Let a_1, a_2, a_3 and a_4 be four scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

Here

$\alpha_1 = (2,2,2,4), \alpha_2 = (2,-2,-4,0), \alpha_3 = (4,-2,-5,2)$ and $\alpha_4 = (4,2,1,6)$

$$\therefore a_1(2,2,2,4) + a_2(2,-2,-4,0) + a_3(4,-2,-5,2) + a_4(4,2,1,6) = 0$$
$$(2a_1 + 2a_2 + 4a_3 + 4a_4, 2a_1 - 2a_2 - 2a_3 + 2a_4,$$
$$2a_1 - 4a_2 - 5a_3 + a_4, 4a_1 + 2a_3 + 6a_4) = (0,0,0,0)$$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 2 & -2 & -4 & 0 \\ 4 & -2 & -5 & 2 \\ 4 & 2 & 1 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -3 \\ 0 & -4 & -6 & -2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 3R_2$ and $R_4 \to R_4 - R_2$, we get

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

i.e., so the rank of matrix $A <$ number of unknown quantities.
The system of equations will have $4 - 2 = 2$, non-zero solutions and hence the set of vectors are linearly dependent. Hence given vectors are not linearly independent.

Example 2.1.6

Show that the vectors (a_1, a_2) and (b_1, b_2) in $V_2(C)$ are L.D. iff $a_1b_2 - a_2b_1 = 0$, where C is the field complex numbers.

Let $a, b \in C$, then

$$a(a_1, a_2) + b(b_1, b_2) = 0$$
$$i.e., (aa_1 + bb_1, aa_2 + bb_2) = (0, 0)$$

$$\left.\begin{aligned} aa_1 + bb_1 &= 0 \\ aa_2 + bb_2 &= 0 \end{aligned}\right\} \tag{9}$$

The system of equations (9) will possess a non-zero solution iff

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \Rightarrow a_1b_2 - a_2b_1 = 0$$

Thus the given system of vectors is L.D. iff $a_1b_2 - a_2b_1 = 0$.

Problem 2.2.1

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z).$$

Find basis and dimension of it's Range and Null space.

$$\begin{aligned} N(T) &= \{T(x, y, z) = (0, 0, 0)\} \\ (x + 2y - z, y + z, x + y - 2z) &= (0, 0, 0) \\ x + 2y - z &= 0 \\ y + z &= 0 \\ x + y - 2z &= 0 \end{aligned}$$

$$\begin{aligned} y &= -z \\ x - 2z - z &= 0 \\ x &= 3z \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -z \\ z \end{bmatrix} \Rightarrow z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$N(T) = \{T(x,y,z) = (0,0,0)\} = (3,-1,1)$$

$$R(T) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1, R_3 \Rightarrow R_3 + R_1$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$dim(R(T)) = 2$$

$$Basic = (1, 0, 1)(0, 1, -1)$$

Problem 2.2.2

Let V be vector space 2×2 matrices over R and $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. Let be $T : V \to V$ be linear transform defined by $T(A) = PA$. Find basis and dim of null space of T and Range space of T.

$$N(T) = \{T(A) = 0 : A \in V$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

$$PA = 0$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} a - c &= 0 \\ -2a + 2c &= 0 \\ a &= c \end{aligned} \qquad \begin{aligned} b - d &= 0 \\ -2b + 2d &= 0 \\ b &= d \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

To find basis:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(E_1) = PE_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$T(E_2) = PE_2 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$T(E_3) = PE_3 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T(E_4) = PE_4 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$T(E_1) = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}; T(E_2) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}; T(E_3) = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}; T(E_4) = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$R_3 = R_3 + R_1; R_4 = R_4 + R_2$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Range space of T is

$$\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Rank of dimension of range space = 2

Problem 2.2.3

Let w_1 and w_2 be the subspace generated by $(-1,2,1)$, $(2,0,1)$ and $(-8,4,-1)$ in $\mathbb{R}^3(\mathbb{R})$ and w_2 generated by all vectors $(a,0,b)$ $\forall a,b \in \mathbb{R}$. Find basis and dimension of w_1, w_2 and w_1+w_2.

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 1 \\ -8 & 4 & -1 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$; $R_3 = R_3 - 8R_1$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & -12 & -9 \end{bmatrix}$$

$R_3 = R_3 + 3R_2$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Basis = (-1, 2, 1) \text{ and } (0, 4, 3)$$
$$dim(w_1) = 2$$

$$R(w_2) = (a, 0, b)$$
$$= a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Basis(w_2) = B_2 = (1, 0, 0), (0, 0, 1)$$
$$dim(w_1) = 2$$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{2}R_2$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + \frac{1}{2}R_4$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$dim(w_1 + w_2) = 3$$
$$Basis(w_1 + w_2) = (-1, 2, 1), (0, 4, 3), (0, 0, 1)$$

$$\begin{aligned}(w_1 \cap w_1) &= dim(w_1) + dim(w_2) - dim(w_1 + w_2) \\ &= 2 + 2 - 3 = 4 - 3 \\ &= 1\end{aligned}$$

Example 2.2.4

Let M and N be two subspace of R^4

$$M = \{(a, b, c, d) | b + c + d\} \text{ and}$$
$$N = \{(a, b, c, d) | a + b = 0, c = 2d\}$$

Find basis and dimension of $(i)M$, $(ii)N$ and $(iii)M \cap N$

Problem 2.3.1

Let T be a linear transformation on $V_3(\mathbb{R})$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)\ \forall (a,b,c) \in V_3(\mathbb{R})$. Is T invertible?. If so, find a rule for T^{-1} as the one which defines T.

For proving T is invertible, we need to show only T is one-one and onto.
To prove one-one:
Let

$$\alpha = (a_1, b_1, c_1) \in V_3(\mathbb{R})$$
$$\beta = (a_2, b_2, c_2) \in V_3(\mathbb{R})$$

Then,

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$3a_1 = 3a_2$$
$$a_1 = a_2$$
$$a_1 - b_1 = a_2 - b_2$$
$$a_2 = b_2$$
$$c_1 = c_2$$
$$(a_1, b_1, c_1) = (a_2, b_2, c_2)$$
$$\alpha = \beta$$

$\therefore T$ is one-one.

To prove onto:

T is linear transformation on a finite dimensional vector space $V_3(\mathbb{R})$, where dimension in 3.

$$\Rightarrow \text{Also } T \text{ is one-one}$$
$$\Rightarrow T \text{ must be onto}$$
$$\Rightarrow T \text{ is invertible}$$

$$
\begin{aligned}
\text{If } T(a,b,c) &= (p,q,r) \\
\text{then, } T^{-1}(p,q,r) &= (a,b,c) \\
T(a,b,c) &= (p,q,r) \\
(3a, a-b, 2a+b+c) &= (p,q,r) \\
3a &= p \\
p &= 3a \\
a &= \frac{p}{3} \\
a - b &= q \\
\frac{p}{3} - b &= q \\
\frac{p}{3} - q &= b
\end{aligned}
$$

$$
\begin{aligned}
2a + b + c &= r \\
2\frac{p}{3} + \left(\frac{p}{3} - q\right) + c &= r \\
c &= r - p + q
\end{aligned}
$$

$$
\begin{aligned}
\because T^{-1}(p, q, r) &= (a, b, c) \\
&= \left(\frac{p}{3}, \frac{p}{3} - q, r - p + q\right)
\end{aligned}
$$

Example 2.3.2

Let T be a linear map on $V_3(R)$ defined by $T(a,b,c) = [3a, a-b, 2a+b+c]$ $\forall a,b,c \in \mathbb{R}$. Is T invertible?. If so find a rule for T^{-1} like one which define T.

For proving T is invertible, we need to show that T is one-one and onto.
To prove one-one:
Let $\alpha = (a_1, b_1, c_1)$, $\beta - (a_2, b_2, c_2)$ be any two elements of $V_3(\mathbb{R})$.

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$
\begin{aligned}
3a_1 &= 3a_2 \\
a_1 - b_1 &= a_2 - b_2 + c_2 \\
2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2 \\
a_1 &= a_2 \\
\because a_1 - b_1 &= a_2 - b_2 \\
-b_1 &= -b_2 \\
b_1 &= b_2 \\
\because 2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2 \\
\because a_1 &= b_1 \\
b_1 &= b_2 \\
c_1 &= c_2 \\
(a_1, b_1, c_1) &= (a_2, b_2, c_2) \\
\alpha &= \beta
\end{aligned}
$$

$$\because T(\alpha) = T(\beta)$$
$$\alpha = \beta$$
$$T := A \to B$$

Hence T is one-one.
To find onto:
Since, T is a linear one-one map on a finite dimensional vector space.
$\Rightarrow T$ is onto.
$\Rightarrow T$ is one-one and onto.
$\Rightarrow T$ is invertible.

Second part:

$$\begin{aligned} \text{Let } T(a,b,c) &= (p,q,r) \\ \text{Then } T^{-1}(p,q,r) &= (a,b,c) \end{aligned} \tag{10}$$

Now

$$\begin{aligned} T(a,b,c) &= (p,q,r) \\ (3a, a-b, 2a+b+c) &= (p,q,r) \\ 3a &= p \\ a &= \frac{p}{3} \\ \because a-b &= q \\ \frac{p}{3} - b &= q \\ \frac{p}{3} - q &= b \end{aligned}$$

$$
\begin{aligned}
\because 2a+b+c &= r \\
2\frac{p}{3}+\left(\frac{p}{3}-q\right)+c &= r \\
c &= r-p+q
\end{aligned}
$$

Put the value of a, b, c in equation (10)

$$T^{-1}(p,q,r)=\left(\frac{p}{3},\frac{p}{3}-a,r-p+q\right)$$

or

$$T^{-1}(x,y,z)=\left(\frac{x}{3},\frac{x}{3}-y,z-x+y\right)$$

which is the rule which defines T^{-1}.

Definition 2.4.1 (Wronskian)

Let f and g be differentiable on $[a, b]$. If Wronskian $W(f, g)(t_0)$ is nonzero for some t_0 in $[a, b]$ then f and g are linearly independent on $[a, b]$. If f and g are linearly dependent then the Wronskian is zero for all t in $[a, b]$.

Problem 2.4.2

Using Wronskian method prove that $\{e^{3x}, e^{5x}\}$ is a linearly independent set on $\mathbb{R}$.

Set $f(x) = e^{3x}$, $g(x) = e^{5x}$. Then,

$$\begin{aligned} W(f(x), g(x)) &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & f''(x) \end{vmatrix} \\ &= \begin{vmatrix} e^{3x} & e^{5x} \\ 3e^{3x} & 5e^{5x} \end{vmatrix} \\ &= 5e^{8x} - 3e^{8x} \\ &= 2e^{8x} \\ &\neq 0 \;\; (\forall x \in \mathbb{R}) \end{aligned}$$

$\because$ The given set $\{e^{3x}, e^{5x}\}$ is linearly independent.

Problem 2.4.3

Using Wronskian method prove that $\{e^{2x}, \cos(x), 2e^{2x}\}$ *is a linearly dependent set on* $\mathbb{R}$.

Set $f(x) = e^{2x}$, $g(x) = \cos x$ $h(x) = 2e^{2x}$. Then,

$$W(f(x), g(x), h(x))$$
$$= \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$
$$= \begin{vmatrix} e^{2x} & \cos x & 2e^{2x} \\ 2e^{2x} & -\sin x & 4e^{2x} \\ 4e^{2x} & -\cos x & 8e^{2x} \end{vmatrix}$$
$$= e^{2x}\begin{vmatrix} -\sin x & 4e^e 2x \\ -\cos x & 8e^{2x} \end{vmatrix} - 2e^{2x}\begin{vmatrix} \cos x & 2e^{2x} \\ -\cos x & 8e^{2x} \end{vmatrix} + 4e^{2x}\begin{vmatrix} \cos x & 2e^{2x} \\ -\sin t & 4e^{2x} \end{vmatrix}$$
$$= e^{2x}\left(-8e^{2x}\sin x + 4e^{2x}\cos x\right) - 2e^{2x}\left(8e^{2x}\cos x + 2e^{2x}\cos x\right)$$
$$+ 4e^{2x}\left(4e^{2x}\cos x + 2e^{2x}\sin x\right)$$

$$= e^{2x}\left(-8\sin x + 4\cos x - 20\cos x + 16\cos x + 8\sin x\right)$$
$$= e^{4x}(0)$$
$$= 0 \ (\forall x \in \mathbb{R})$$

Example 2.4.4

Using Wronskian method prove that $\{1, x, x^2\}$ is a linearly dependent set on $\mathbb{R}$.

Ans: $W(f(x), g(x), h(x)) = 2 \neq 0$, So the set is linearly independent.

Problem 2.5.1

Transforming a matrix $\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$ *to reduced row echelon form*

$$\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \quad R_1 \to R_1 \times \frac{1}{5} \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \quad R_2 \to R_2 - 2R_1 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix} \quad R_3 \to R_3 - 5R_1 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix} \quad R_2 \to \frac{5}{21}R_2 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix} \quad R_3 \to \frac{-3}{14} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_2 \to R_2 - \frac{2}{3} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_1 \to R_1 - \frac{8}{5} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_1 \to R_1 - \frac{7}{5} R_2 \quad \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{17}{49} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

Example 2.5.2

Find column space, row space, null space and kernel of

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}.$$

Step (1): Finding $rref(A)$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_1 \to \frac{-1}{3} R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_2 \to R_2 - 2R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix} R_3 \to R_3 - 3R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix} R_2 \to \frac{3}{8} R_2 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & -4 & -5 \end{bmatrix} R_3 \to R_3 + 4R_2 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - \frac{-2}{3} R_2 \Rightarrow \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To identify row space

$$\begin{bmatrix}1 & -3 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & \frac{5}{4}\\ 0 & 0 & 0 & 0\end{bmatrix} \Rightarrow B_{RS} = \left\{ \begin{pmatrix}1\\ -3\\ 0\\ \frac{3}{2}\end{pmatrix}, \begin{pmatrix}0\\ 0\\ 1\\ \frac{5}{4}\end{pmatrix} \right\}$$

To identify column space

$$\begin{bmatrix}-3 & 9 & -2 & -7\\ 2 & -6 & 4 & 8\\ 3 & -9 & -2 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -3 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & \frac{5}{4}\\ 0 & 0 & 0 & 0\end{bmatrix} \Rightarrow B_{CS} = \left\{ \begin{pmatrix}-3\\ 2\\ 3\end{pmatrix}, \begin{pmatrix}-2\\ 4\\ -2\end{pmatrix} \right\}$$

Check you work

Note: $CS * RS = A$

$$\begin{bmatrix}-3 & -2\\ 2 & 4\\ 3 & -2\end{bmatrix} \begin{bmatrix}1 & -3 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & \frac{5}{4}\end{bmatrix} = \begin{bmatrix}-3 & 9 & -2 & -7\\ 2 & -6 & 4 & 8\\ 3 & -9 & -2 & 2\end{bmatrix}$$

You can extract the null space quickly by changing the sign of the non-pivot element and adding a pivot where the pivot would line up to an identity matrix but this is how to compute it.

To find Null space and Kernel

The 'Null Space' is the solution to $Ax = 0$.

$$\begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here x_1 and x_3 are pivot variables. So, x_2 and x_4 are free variables.

$$x_1 - 3x_2 + \frac{3}{2}x_4 = 0$$
$$free : x_2 = x_2$$
$$x_3 + \frac{5}{4}x_4 = 0$$
$$free : x_4 = x_4$$

$$x_1 = 3x_2 - \frac{3}{2}x_4$$
$$x_2 = x_2 + 0x_4$$
$$x_3 = 0x_2 - \frac{5}{4}x_4$$
$$x_4 = 0x_2 + x_4$$

$$x = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} \frac{-3}{2} \\ 0 \\ \frac{-5}{4} \\ 1 \end{pmatrix} x_4,$$

$$x_2 = 1 \wedge x_4 = 4$$

$$Kernal = B_{NS} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ -5 \\ 4 \end{pmatrix} \right\}$$

Check your work $A * NS = 0;$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 1 & -5 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$Nullspace = \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2.6.1

Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ *and* $C = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ *be bases for* $\mathbb{R}^2$. *If* $[X]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, *find* $[X]_C$.

$$
\begin{aligned}
[X]_B &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
\Rightarrow X &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 3 \end{bmatrix}
\end{aligned}
$$

$$[X]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

To check

$$-7 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Problem 2.6.2

Let $B = \{u_1, u_2\}$, $B' = \{u_1', u_2'\}$ *for* $\mathbb{R}^2$ *and* $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u_1' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, u_2' = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. *Find the transition matrix from* B *and* B'.

$$\left[\begin{array}{cc|cc} u_1' & u_2' & u_1 & u_2 \end{array}\right] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & -3 & 1 & 0 \end{array}\right] \qquad R_1 \Leftrightarrow R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & -11 & 1 & -2 \end{array}\right] \qquad R_2 \to R_2 - 2R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & \frac{-1}{11} & \frac{2}{11} \end{array}\right] \qquad R_2 \to \frac{-1}{11} R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & \frac{-1}{11} & \frac{2}{11} \end{array}\right] \qquad R_1 \to R_1 - 4R_2$$

Transition matrix P

$$P = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{-1}{11} & \frac{2}{11} \end{bmatrix}$$

Example 3.0.1

If matrix of a linear transform on R^3 relative to basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$. Then find the linear transform matrix T relative to basis $B_1 = \{(0,1,-1),(1,-1,1),(-1,1,0)\}$.

First we find linear transform.
We have

$$[T:B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}^t$$

That is transpose of coefficient matrix. So that

$$T(u_1) = T(1,0,0) = 0(1,0,0) + 1(0,1,0) - 1(0,0,1) = (0,1,-1)$$
$$T(u_2) = T(0,1,0) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1) = (1,0,-1)$$
$$T(u_3) = T(0,0,1) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1) = (1,-1,0)$$

$(x,y,z) \in \mathbb{R}^3$ be any element and B is basis for $\mathbb{R}^3$

$$\therefore (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$\therefore T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$
$$= x(0,1,-1) + y(1,0,-1) + z(1,-1,0)$$
$$T(x,y,z) = (y+z, x-z, -x-y)$$

which is linear operator T on R^3.
Now we have to find a matrix of T relative basis.

$$B_1 = \{(0,1,-1), (1,-1,1), (-1,1,0)\}$$

Let $(a,b,c) \in \mathbb{R}^3$ be any element
Let

$$(a,b,c) = l(0,1,-1) + m(1,-1,1) + n(-1,1,0)$$
$$(a,b,c) = (m-n, l-m+n, -l+m)$$
$$\Rightarrow a = m-n;\ b = l-m+n;\ c = -l+m$$

Now

$$\begin{array}{c|c|c} l-m+n=b & l-m+n=b & m=a+n \\ l=b+m-n=b+a & n=b-l+m & m=a+b+c \\ l=a+b & n=b+c & \end{array}$$

$$B_1 = \{(0, 1 - 1), (1, -1, 1), (-1, 1, 0)\}$$

$\therefore$ we get

$$(a, b, c) = (a + b)(0, 1, -1) + (a + b + c)(1, -1, 1) + (b + c)(-1, 1, 0)$$

and we have

$$T(x, y, z) = (y + z, x - z, -x - y)$$

Now

$$\begin{aligned}
T(0, 1, -1) &= (0, 1, -1) = 1(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0)\\
T(1, -1, 1) &= (0, 0, 0) = 0(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0)\\
T(-1, 1, 0) &= (1, -1, 0) = 0(0, 1, -1) + 0(1, -1, 1) + (-1)(-1, 1, 0)
\end{aligned}$$

$$\therefore [T; B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Example 3.0.2

Let T be linear transform on $\mathbb{R}^2$ and $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ be matrix of T with respect to usual basis of $\mathbb{R}^2$. Then, find that matrix of T with respect to $B_1 = \{(1,2),(5,6)\}$.

Ans:

$$[T : B_1] = \begin{bmatrix} \frac{11}{2} & \frac{41}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Definition 3.1.1 (Isomorphism of a vector space)

Let $U(F)$ and $V(F)$ are two vector spaces then a linear transformation $f : U \to V$ is called Isomorphism, if

1. f is one-one
2. f is onto

Definition 3.1.2 (Isomorphism of a vector space)

$f : U \to V$ is called Isomorphism if

1. f is a linera transform
2. f is one-one
3. f is onto

Problem 3.1.3

Let $f : V_2(\mathbb{R}) \to V_2(\mathbb{R})$ *be* $f(x,y) = (y,x)$. *Prove* f *is Isomorpism.*

To prove one-one:
Let $U, V \in V_2(\mathbb{R})$

$$
\begin{aligned}
f(u) &= f(v) \\
f(x,y) &= f(p,q) \\
(y,x) &= (q,p) \\
y &= q; x = p \\
(x,y) &= (p,q) \\
u &= v
\end{aligned}
$$

i.e., f is one-one

To prove onto:

$\forall (x, y) \in V_2(\mathbb{R})$

$\exists (y, x) \in V_2(\mathbb{R})$ such that $f(x, y) = (y, x)$

To prove linear transform:
Let $u, v \in V_2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} f(\alpha u + \beta v) &= f[\alpha(x,y) + \beta(p,q)] \\ &= f[\alpha x + \beta p, \alpha y + \beta q] \\ &= (\alpha y + \beta q, \alpha x + \beta p) \\ &= \alpha(y,x) + \beta(q,p) \\ &= \alpha f(x,y) + \beta(p,q) \\ &= \alpha f(u) + \beta f(v) \end{aligned}$$

So, f is a linear transform, one-one, onto.
i.e., f is an Isomorpism.

Problem 3.1.4

Let $T : P_2 \to V_3 \to \{(x_1, x_2, x_3) | x_i \in \mathbb{R}\}$ (P_2-set of all polynomials of degree ≤ 2) $\{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{R}\}$. Prove that T is Isomorphism. $T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2)$

To prove T is one-one:

$$T(p_1) = T(p_2)$$
$$T(a_0 + a_1x + a_2x^2) = T(b_0, b_1x + b_2x^2)$$
$$(a_0, a_1, a_2) = (b_0, b_1, b_2)$$
$$a_0 = b_0; a_1 = b_1; a_2 = b_2$$
$$a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$$
$$p_1(x) = p_2(x)$$
$$p_1 = p_2$$

T is one-one.

To prove T is onto:

$T : p_2 \to v_3$. For every $(a_0, a_1, a_2) \in v_3$ we have a polynomial $p = a_0 + a_1 x + a_2 x^2$ in p_2. Such that

$$T(p) = (a_0, a_1, a_2)$$

T is onto.

T is one-one and onto.

To prove T is linear.

$$
\begin{aligned}
& T(\alpha(a_0 + a_1x + a_2x^2) + \beta(b_0 + b_1x + b_2x^2)) \\
&= T((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2) \\
&= (\alpha a_0 + \beta b_0, \alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2) \\
&= (\alpha a_0, \alpha a_1, \alpha a_2) + (\beta b_0, \beta b_1, \beta b_2) \\
&= \alpha(a_0, a_1, a_2) + \beta(b_0, b_1, b_2) \\
T(\alpha p_1 + \beta p_2) &= \alpha T(p_1(x)) + \beta T(p_2(x))
\end{aligned}
$$

This proves T is linear.
$\therefore$ T is an isomorphic.
To find its inverse:

$$
\begin{aligned}
& T^{-1} : v_3 \to p_2 \\
& T^{-1}(a_0, a_1, a_2) = a_0 + a_1x + a_2x^2
\end{aligned}
$$

Example 3.1.5

$T : v_2 \to v_2 \; T(x_1, x_2) = (x_1, -x_2)$

Definition 3.2.1 (Matrices of linear transformations)

We will now take a more algebraic approach to transformations of the plane. As it turns out, matrices are very useful for describing transformations. Whenever we have a 2×2 matrix of real numbers

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we can naturally define a plane transformatiom $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_M(v) = Mv.$$

That is, T_M takes a vector v and multiplies it on the left by the matrix M. If v is the position vector of the point (x, y), then

$$T_M(v) = T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

or equivalently, $T_M(x, y) = (ax + by, cx + dy)$.

Problem 3.2.2

Let

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

1. *Write an expression for* T_M.
2. *Find* $T_M(1,0)$ *and* $T_M(0,1)$.
3. *Find all points* (x,y) *such that* $T_M(x,y) = (1,0)$.

(1) $T_M(x,y) = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+7y \end{bmatrix} = (x+2y, 3x+7y)$.

(2) Using the formula from the previous part,

$$T_M(1,0) = (1,3) \text{ and } T_M(0,1) = (2,7).$$

(3) We have $T_M(x,y) = (x+2y, 3x+7y) = (1,0)$, hence the simultaneous equations

$$x + 2y = 1, 3x + 7y = 0.$$

Solving these equations yields $x = 7, y = -3$; and this is the only solution. So the only point (x,y) such that $T_M(x,y) = (1,0)$ is $(x,y) = (7,-3)$.

Definition 3.2.3 (Linear transformation)

A plane transformation F is linear if either of the following equivalent conditions holds:

1. $F(x,y) = (ax+by, cx+dy)$ for some real a,b,c,d. That is, F arises from a matrix.
2. For any scalar c and vectors v,w, $F(cv) = cF(v)$ and $F(v+w) = F(v) + F(w)$.

Theorem 3.2.4

For any matrices M and N, $T_M \circ T_N = T_{MN}$.

Problem 3.2.5

Find the matrix for the composition $g \circ f$ of the two linear transformations $f(x, y) = (x + y, y)$ and $g(x, y) = (y, x + y)$.

We have $f = T_M$ and $g = T_N$ where $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. So the matrix of the composition $g \circ f = T_N \circ T_M = T_{NM}$ is the product NM :

$$NM = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Problem 3.2.6

What is the inverse of the transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $F(x,y) = (x+3y, x+5y)$*?*

The transformation F is linear and corresponds to the matrix

$$M = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix},$$

which has inverse

$$M^{-1} = \frac{1}{1.5 - 3.1} \begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

The inverse of $F = T_M$ is then $F^{-1} = T_{M^{-1}}$,

$$F^{-1}(x,y) = \left(\frac{5}{2}x - \frac{3}{2}y, \frac{-1}{2}x + \frac{1}{2}y \right).$$

Problem 3.2.7

Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ *that maps* $(1,1)$ *to* $(-1,4)$ *and* $(-1,3)$ *to* $(-7,0)$.

Let M be the matrix of the desired linear transformation. We have

$$M\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix} \text{ and } M\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-7\\0\end{bmatrix}.$$

In fact, we can put these two equations together into a single matrix equation

$$M\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix} = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}$$

which we can then solve for M:

$$M = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4 & -8\\12 & 4\end{bmatrix} = \begin{bmatrix}1 & \\3 & \end{bmatrix}$$

Hence the only such transformation is $T_M(x,y) = (x-2y, 3x+y)$.

Example 3.2.8

Find the linear transformation that sends $(3, 1)$ to $(1, 2)$ and $(-1, 2)$ to $(2, -3)$.

Definition 3.3.1 (Similar matrices)

Let A and B be two square matrices of same order, A is said to be similar to matrix B if there exists a non-singular matrix P, such that

$$B = P^{-1}AP$$

Definition 3.3.2 (Properties of similar matrices)

Similar matrices have same eigen values, eigen vectors, determinant, ranks, nullity, characteristic polynomial and traces.

Definition 3.3.3 (Procedure to find similar matrix)

If A is given

- Step I Characteristic polynomial $A - \lambda I$ by using $|A - \lambda I| = 0$.
- Step II Find eigen values $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$.
- Step III Find eigen vector $v_1, v_2, v_3, \cdots, \lambda_n$ using eigen values.
- Step IV Find P by combining all eigen values into one matrix.
- Step V Find P^{-1} from P.
- Step VI Find $B = P^{-1}AP$

B is called similar matrix.

Problem 3.3.4

Find similar matrix for $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ and λ be eigen value of A then

$$
\begin{aligned}
A - \lambda I &= \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \\
|A - \lambda I| &= (2-\lambda)(-2-\lambda) - (-1)(3) \\
&= -4 - 2\lambda + 2\lambda + \lambda^2 + 3 \\
&= \lambda^2 - 1
\end{aligned}
$$

To find eigen values:

$$|A - \lambda I| = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda^2 = 1$$
$$\lambda = \pm 1$$

Therefore eigen values are $\lambda_1 = 1$, $\lambda_2 = -1$

To find eigen vectors:

$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

At $\lambda = 1$

$$\begin{bmatrix} 2-1 & 3 \\ -1 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ x_1 &= -3x_2 \\ \text{Let } x_2 &= t \\ \text{then, } x_1 &= -3t \\ v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \end{aligned}$$

At $\lambda = -1$

$$(A - \lambda I)v = 0$$
$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+1 & 3 \\ -1 & -2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 = 0$$
$$x_1 + x_2 = 0$$
$$x_1 = -x_2$$
$$\text{Let } x_2 = -t$$
$$x_1 = -t$$
$$v_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To find P matrix,

$$P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$|P| = (-3)(1) - (1)(-1) = -3 + 1 = -2 \neq 0.$$

P is non-singular.

$$P^{-1} = \frac{1}{|A|} Adj(P) = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -6+3 & -2+3 \\ 3-2 & 1-2 \end{pmatrix} \\ &= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -3+1 & 1+1 \\ 3-3 & -1+3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -2 \\ 0 \end{pmatrix} \end{aligned}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To check

$$
\begin{aligned}
|B-\lambda I| &= 0 \\
\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)(-1-\lambda) - 0 &= 0 \\
-1-\lambda+\lambda+\lambda^2 &= 0 \\
\lambda^2 &= 1 \\
\lambda &= \pm 1
\end{aligned}
$$

1. Eigen values of B matrix are similar to A matrix. So, $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. $|A| = |B| = -1$
3. $Trace(A) = Trace(B)$ i.e., $2-2=0=1-1$

Example 3.3.5

Find similar matrix for the following matrix

1. $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Problem 3.3.6

Find similar matrix for $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Find Matrix Eigenvalues ...

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} (1-\lambda) & -2 & 0 \\ 0 & (2-\lambda) & 0 \\ 0 & 0 & (-2-\lambda) \end{bmatrix} = 0$$

$$(1-\lambda)((2-\lambda)\times(-2-\lambda) - 0\times 0) - (-2)(0\times(-2-\lambda) - 0\times 0) + 0(0\times 0 - (2-\lambda)\times 0) = 0$$
$$(1-\lambda)((-4+\lambda 2) - 0) + 2(0-0) + 0(0-0) = 0$$
$$(1-\lambda)(-4+\lambda 2) + 2(0) + 0(0) = 0$$
$$(-4+4\lambda+\lambda 2-\lambda 3) + 0 + 0 = 0$$
$$(-\lambda 3+\lambda 2+4\lambda-4) = 0$$
$$-(\lambda-1)(\lambda-2)(\lambda+2) = 0$$
$$(\lambda-1) = 0 or (\lambda-2) = 0 or (\lambda+2) = 0$$

$\therefore$ The eigenvalues of the matrix A are given by $\lambda = -2, 1, 2,$

Eigen vector for $\lambda = -2$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector for $\lambda = 1$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector for $\lambda = 2$

$$v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} adj(P)$$

To find $|P|$:

$$|A| = \begin{vmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 0 \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2 \times \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= 0 \times (0 \times 0 - 1 \times 0) - 1 \times (0 \times 0 - 1 \times 1) - 2 \times (0 \times 0 - 0 \times 1)$$

$$= 0 \times (0 + 0) - 1 \times (0 - 1) - 2 \times (0 + 0)$$

$$= 0 \times (0) - 1 \times (-1) - 2 \times (0)$$

$$= 0 + 1 + 0$$

$$= 1$$

To find adjoint of P

$$adj(P) = adj\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$adj(P) = \begin{bmatrix} +(0\times 0 - 1\times 0) & -(0\times 0 - 1\times 1) & +(0\times 0 - 0\times 1) \\ -(1\times 0 - (-2)\times 0) & +(0\times 0 - (-2)\times 1) & -(0\times 0 - 1\times 1) \\ +(1\times 1 - (-2)\times 0) & -(0\times 1 - (-2)\times 0) & +(0\times 0 - 1\times 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0+0) & -(0-1) & +(0+0) \\ -(0+0) & +(0+2) & -(0-1) \\ +(1+0) & -(0+0) & +(0+0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} adj(P) = \frac{1}{1} \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since,

$$B = P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To verify the solution:

$$trace(A) = trace \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = 1 + 2 + (-2) = 1$$

$$trace(B) = trace \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (-2) + 1 + 2 = 1$$

Eigen values of A=$-2, 1, 2$
Eigen values of $B = -2, 1, 2$

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 1 \times \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} - (-2) \times \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} + 0 \times \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$$

$$= 1 \times (2 \times (-2) - 0 \times 0) + 2 \times (0 \times (-2) - 0 \times 0) + 0 \times (0 \times 0 - 2 \times 0$$

$$= 1 \times (-4 + 0) + 2 \times (0 + 0) + 0 \times (0 + 0)$$

$$= 1 \times (-4) + 2 \times (0) + 0 \times (0)$$

$$= -4 + 0 + 0 = -4$$

$$|B| = \begin{vmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -2 \times \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= -2 \times (1 \times 2 - 0 \times 0) + 0 \times (0 \times 2 - 0 \times 0) + 0 \times (0 \times 0 - 1 \times 0)$$

$$= -2 \times (2 + 0) + 0 \times (0 + 0) + 0 \times (0 + 0)$$

$$= -2 \times (2) + 0 \times (0) + 0 \times (0)$$

$$= -4 + 0 + 0 = -4$$

Definition 2.6.1 (Matrices of linear transformations)

We will now take a more algebraic approach to transformations of the plane. As it turns out, matrices are very useful for describing transformations. Whenever we have a 2×2 matrix of real numbers

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we can naturally define a plane transformatiom $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_M(v) = Mv.$$

That is, T_M takes a vector v and multiplies it on the left by the matrix M. If v is the position vector of the point (x, y), then

$$T_M(v) = T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

or equivalently, $T_M(x, y) = (ax + by, cx + dy)$.

Problem 2.6.2

Let

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

1. *Write an expression for* T_M.
2. *Find* $T_M(1,0)$ *and* $T_M(0,1)$.
3. *Find all points* (x,y) *such that* $T_M(x,y) = (1,0)$.

(1) $T_M(x,y) = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+7y \end{bmatrix} = (x+2y, 3x+7y)$.

(2) Using the formula from the previous part,

$$T_M(1,0) = (1,3) \text{ and } T_M(0,1) = (2,7).$$

(3) We have $T_M(x,y) = (x+2y, 3x+7y) = (1,0)$, hence the simultaneous equations

$$x + 2y = 1, 3x + 7y = 0.$$

Solving these equations yields $x = 7, y = -3$; and this is the only solution. So the only point (x,y) such that $T_M(x,y) = (1,0)$ is $(x,y) = (7,-3)$.

Definition 2.6.3 (Linear transformation)

A plane transformation F is linear if either of the following equivalent conditions holds:

1. $F(x,y) = (ax+by, cx+dy)$ for some real a,b,c,d. That is, F arises from a matrix.
2. For any scalar c and vectors v,w, $F(cv) = cF(v)$ and $F(v+w) = F(v) + F(w)$.

Theorem 2.6.4

For any matrices M and N, $T_M \circ T_N = T_{MN}$.

Problem 2.6.5

Find the matrix for the composition $g \circ f$ of the two linear transformations $f(x,y) = (x+y,y)$ and $g(x,y) = (y,x+y)$.

We have $f = T_M$ and $g = T_N$ where $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. So the matrix of the composition $g \circ f = T_N \circ T_M = T_{NM}$ is the product NM :

$$NM = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Problem 2.6.6

What is the inverse of the transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $F(x,y) = (x+3y, x+5y)$*?*

The transformation F is linear and corresponds to the matrix

$$M = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix},$$

which has inverse

$$M^{-1} = \frac{1}{1.5 - 3.1} \begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

The inverse of $F = T_M$ is then $F^{-1} = T_{M^{-1}}$,

$$F^{-1}(x,y) = \left(\frac{5}{2}x - \frac{3}{2}y, \frac{-1}{2}x + \frac{1}{2}y \right).$$

Problem 2.6.7

Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ *that maps* $(1,1)$ *to* $(-1,4)$ *and* $(-1,3)$ *to* $(-7,0)$.

Let M be the matrix of the desired linear transformation. We have

$$M\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix} \text{ and } M\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-7\\0\end{bmatrix}.$$

In fact, we can put these two equations together into a single matrix equation

$$M\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix} = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}$$

which we can then solve for M:

$$M = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4 & -8\\12 & 4\end{bmatrix} = \begin{bmatrix}1 & \\3 & \end{bmatrix}$$

Hence the only such transformation is $T_M(x,y) = (x-2y, 3x+y)$.

Example 2.6.8

Find the linear transformation that sends $(3, 1)$ to $(1, 2)$ and $(-1, 2)$ to $(2, -3)$.

Problem 5.1.1

Find the least square solution of $AX = Y$ *for* $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 & 3+1+1+3 & 5+0+2+3 \\ 3+1+1+3 & 9+1+1+9 & 15+0+2+9 \\ 5+0+2+3 & 15+0+2+9 & 25+0+4+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}$$

$$A^TY = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 3+5+7-3 \\ 9+5+7-9 \\ 15+0+14-9 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 20 \end{bmatrix}$$

$$A^TAX = A^TY$$

$$\begin{bmatrix}1 & 1 & 1 & 1\\3 & 1 & 1 & 3\\5 & 0 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 3 & 5\\1 & 1 & 0\\1 & 1 & 2\\1 & 3 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\3 & 1 & 1 & 3\\5 & 0 & 2 & 3\end{bmatrix}\begin{bmatrix}3\\5\\7\\-3\end{bmatrix}$$

$$\begin{bmatrix}4 & 8 & 10\\8 & 20 & 26\\10 & 26 & 38\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}12\\12\\20\end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c}4 & 8 & 10 & 12\\8 & 20 & 26 & 12\\10 & 26 & 38 & 20\end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 8 & 10 & 12 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right] \Rightarrow R_1 \to \frac{R_1}{4} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right] \Rightarrow R_2 \to R_2 - 8 \times R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 10 & 26 & 38 & 20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 10 & 26 & 38 & 20 \end{array}\right] \Rightarrow R_3 \to R_3 - 10 \times R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 0 & 6 & 13 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 0 & 6 & 13 & -10 \end{array}\right] \Rightarrow R_2 \to \frac{R_2}{4} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 6 & 13 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 6 & 13 & -10 \end{array}\right] \Rightarrow R_3 - 6 \times R_2 \qquad \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right] \Rightarrow R_3 \to \frac{R_3}{4} \qquad \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solving the above triangular system we obtain

$$x_1 = 10 \qquad x_2 = -6 \qquad x_3 = 2$$

Problem 5.1.2

Find the orthogonal projection of $\vec{y}$ *onto span* $\{\vec{u_1}, \vec{u_2}\}$.

$$\vec{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}, \vec{u_1} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

Let $W = Span\{\vec{u_1}, \vec{u_2}\}$.

By the orthogonal decomposition theorem,

$$proj_W \vec{y} = \hat{y} = \left(\frac{\vec{y}.\vec{u_1}}{\vec{u_1}.\vec{u_1}}\right)\vec{u_1} + \left(\frac{\vec{y}.\vec{u_2}}{\vec{u_2}.\vec{u_2}}\right)\vec{u_2}$$

$$\hat{y} = \left(\frac{6(3)+3(4)+(-2)(0)}{(3)^2+(4)^2+(0)^2}\right)\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \left(\frac{6(-4)+3(3)+(-2)(0)}{(-4)^2+(3)^2+(0)^2}\right)$$

$$= \left(\frac{30}{25}\right)\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \left(\frac{15}{25}\right)\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

Problem 5.1.3

Let W be the subspace spanned by $\{\vec{u_1}, \vec{u_2}\}$, and write $\vec{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \vec{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

Since,

$$\vec{u_1}.\vec{u_2} = (-1)(1) + (1)(3) + (1)(-2) = -1 + 3 - 2 = 0.$$

$\{\vec{u_1}, \vec{u_2}\}$ is an orthogonal set.

$$\hat{y} = \left(\frac{\vec{y}.\vec{u_1}}{\vec{u_1}.\vec{u_1}}\right)\vec{u_1} + \left(\frac{\vec{y}.\vec{u_2}}{\vec{u_2}.\vec{u_2}}\right)\vec{u_2}$$

and

$$\vec{z} = \vec{y} - \hat{y}$$

$$\hat{y} = \left(\frac{(-1)(1)+4(1)+(3)(1)}{(1)^2+(1)^2+(1)^2}\right)\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$+\left(\frac{(-1)(-1)+4(3)+(3)(-2)}{(-1)^2+(3)^2+(-2)^2}\right)\begin{bmatrix}-1\\3\\-2\end{bmatrix}$$

$$= 2\begin{bmatrix}1\\1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-1\\3\\-2\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{bmatrix}$$

$$\vec{z} = \vec{y}-\hat{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix}-\begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{bmatrix}=\begin{bmatrix}\frac{-5}{2}\\\frac{1}{2}\\2\end{bmatrix}$$

$$\vec{y} = \hat{y}+\vec{z}$$

$$= \begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{bmatrix}+\begin{bmatrix}\frac{-5}{2}\\\frac{1}{2}\\2\end{bmatrix}$$

Problem 5.1.4

Find the closet point to vector $\vec{y}$ in the subspace W spanned by $\vec{v_1}, \vec{v_2}, \vec{v_3}$. Then find the distance from $\vec{y}$ to W.

$$\vec{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v_1}.\vec{v_2} = (1)(1) + (1)(0) + (0)(1) + (-1)(1) = 0$$
$$\vec{v_1}.\vec{v_3} = (1)(0) + (1)(-1) + (0)(1) + (-1)(-1) = 0$$
$$\vec{v_2}.\vec{v_3} = (1)(0) + (0)(-1) + (1)(1) + (1)(-1) = 0$$

$$\hat{y} = \left(\frac{\vec{y}.\vec{v_1}}{\vec{v_1}.\vec{v_1}}\right)\vec{v_1} + \left(\frac{\vec{y}.\vec{v_2}}{\vec{v_2}.\vec{v_2}}\right)\vec{v_2}$$
$$= \frac{1}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{14}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} - \frac{5}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix}$$

Distance from $\vec{y}$ to W is $\|\vec{y} - \hat{y}\|$.

$$\|\vec{y} - \hat{y}\| = \sqrt{(3-5)^2 + (4-2)^2 + (5-3)^2 + (6-6)^2}$$
$$= \sqrt{12} = 2\sqrt{3}$$

Example 4.0.14

Find a vector of unit length which is orthogonal to $(1,3,4)$ in $V_3(\mathbb{R})$ with standard inner product.

Let $x = (x_1, x_2, x_3)$ be orthogonal to $y = (1,3,4)$ in $V_3(\mathbb{R})$.

$$\begin{aligned}
\langle x, y \rangle &= 0 \\
\langle (x_1, x_2, x_3), (1,3,4) \rangle &= 0 \\
1.x_1 + 3.x_2 + 4.x_3 &= 0 \\
x_1 + 3x_2 + 4x_3 &= 0
\end{aligned}$$

Put $x_1 = 1$, $x_2 = 1$, we get

$$\begin{aligned}
1 + 3(1) + 4x_3 &= 0 \\
4x_3 &= -4 \\
x_3 &= -1
\end{aligned}$$

$x = (1,1,-1)$ is orthogonal to $(1,3,4)$.

Orthogonal vector is

$$\left(\frac{x}{\|x\|}\right) = \left(\frac{x_1}{\|x\|}, \frac{x_2}{\|x\|}, \frac{x_3}{\|x\|}\right)$$

$$\frac{x}{\|x\|} = \frac{(1, 1, -1)}{1^2 + 1^2 + (-1)^2} = \frac{1}{3}(1, 1, -1) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$$

Therefore, $\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$ is unit length vector orthogonal to $(1, 3, 4)$.

Problem 4.0.15

Apply gram-schmidt orthogonalization process to construct an orthonormal basis for $V_3(\mathbb{R})$ with the standard inner product for the basis $\{v_1, v_2, v_3\}$ where $v_1 = (1, 0, 1)$, $v_2 = (1, 3, 1)$, $v_3 = (3, 2, 1)$.

Consider a basis $\{v_1, v_2, v_3\}$ of $V_3(\mathbb{R})$, where $v_1 = (1, 0, 1)$, $v_2 = (1, 3, 1)$ and $v_3 = (3, 2, 1)$.
To find w_1,

$$\begin{aligned} w_1 &= v_1 = (1, 0, 1) \\ \|w_1\|^2 &= \langle w_1, w_1 \rangle \\ &= 1.1 + 0.0 + 1.1 \\ &= 1^2 + 0^2 + 1^2 = 2 \\ \|w_1\| &= \sqrt{2} \end{aligned}$$

To find w_2,

$$\begin{aligned}
\langle v_2, w_1 \rangle &= \langle (1,3,1), (1,0,1) \rangle \\
&= 1.1 + 3.0 + 1.1 \\
&= 2 \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 \\
&= (1,3,1) - \frac{2}{(\sqrt{2})^2}(1,0,1) \\
&= (1,3,1) - (1,0,1) \\
&= (1-1, 3-0, 1-1) \\
&= (0,3,0)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|w_2\|^2 &= \langle w_2, w_2 \rangle = \langle (0,3,0), (0,3,0) \rangle \\
&= 0^2 + 3^2 + 0^2 = 9 \\
\|w_2\| &= 3
\end{aligned}$$

To find w_3,

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$$

$$\langle w_3, w_1 \rangle = \langle (3,2,1), (1,0,1) \rangle = 3.1 + 2.0 + 1.1 = 3 + 0 + 1 = 4$$

$$\langle w_3, w_2 \rangle = \langle (3,2,1), (0,3,0) \rangle = 3.0 + 2.3 + 1.0 = 0 + 6 + 0 = 6$$

$$\begin{aligned} w_3 &= (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9}(0,3,0) \\ &= (3,2,1) - (2,0,1) - (0,2,0) \\ &= (3-2-0, 2-0, 2, 1-1-0) \\ w_3 &= (1,0,0) \end{aligned}$$

Therefore,

$$\begin{aligned} \|w_3\|^2 &= \langle w_3, w_3 \rangle \\ &= 1^2 + 0^2 + 0^2 = 1 \\ \|w_3\| &= 1 \end{aligned}$$

The orthogonal basis is

$$\{w_1, w_2, w_3\} = \{(1,0,1), (0,3,0), (1,0,0)\}$$

The orthonormal basis is

$$\begin{aligned}\left\{\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|}\right\} &= \left\{\frac{1}{\sqrt{2}}(1,0,1), \frac{1}{3}(0,3,0), \frac{1}{1}(1,0,0)\right\} \\ &= \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0), (1,0,0)\right\}\end{aligned}$$

Problem 2.2.1

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by*

$$T(x,y,z) = (x+2y-z, y+z, x+y-2z).$$

Find basis and dimension of it's Range and Null space.

$$\begin{aligned} N(T) &= \{T(x,y,z) = (0,0,0)\} \\ (x+2y-z, y+z, x+y-2z) &= (0,0,0) \\ x+2y-z &= 0 \\ y+z &= 0 \\ x+y-2z &= 0 \end{aligned}$$

$$\begin{aligned} y &= -z \\ x-2z-z &= 0 \\ x &= 3z \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -z \\ z \end{bmatrix} \Rightarrow z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$N(T) = \{T(x,y,z) = (0,0,0)\} = (3,-1,1)$$

$$R(T) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1,\ R_3 \Rightarrow R_3 + R_1$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$dim(R(T)) = 2$$

$$Basic = (1, 0, 1)(0, 1, -1)$$

Problem 2.2.2

Let V be vector space 2×2 matrices over R and $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. Let be $T : V \to V$ be linear transform defined by $T(A) = PA$. Find basis and dim of null space of T and Range space of T.

$$N(T) = \{T(A) = 0 : A \in V$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

$$PA = 0$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\begin{bmatrix} a - c & b - d \\ -2a + 2c & -2b + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} a - c &= 0 \\ -2a + 2c &= 0 \\ a &= c \end{aligned} \qquad\qquad \begin{aligned} b - d &= 0 \\ -2b + 2d &= 0 \\ b &= d \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

To find basis:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(E_1) = PE_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$T(E_2) = PE_2 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$T(E_3) = PE_3 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T(E_4) = PE_4 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$T(E_1) = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}; T(E_2) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}; T(E_3) = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}; T(E_4) = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$R_3 = R_3 + R_1; R_4 = R_4 + R_2$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Range space of T is

$$\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Rank of dimension of range space = 2

Problem 2.2.3

Let w_1 and w_2 be the subspace generated by $(-1,2,1)$, $(2,0,1)$ and $(-8,4,-1)$ in $\mathbb{R}^3(\mathbb{R})$ and w_2 generated by all vectors $(a,0,b)$ $\forall a,b \in \mathbb{R}$. Find basis and dimension of w_1, w_2 and w_1+w_2.

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 1 \\ -8 & 4 & -1 \end{bmatrix}$$

$R_2 = R_2 + 2R_1$; $R_3 = R_3 - 8R_1$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & -12 & -9 \end{bmatrix}$$

$R_3 = R_3 + 3R_2$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Basis = (-1,2,1) \text{ and } (0,4,3)$$
$$dim(w_1) = 2$$

$$R(w_2) = (a,0,b)$$
$$= a\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

$$Basis(w_2) = B_2 = (1,0,0),(0,0,1)$$
$$dim(w_1) = 2$$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{2}R_2$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + \frac{1}{2} R_4$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$dim(w_1 + w_2) = 3$$
$$Basis(w_1 + w_2) = (-1, 2, 1), (0, 4, 3), (0, 0, 1)$$

Problem 2.4.1

Transforming a matrix $\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$ *to reduced row echelon form*

$$\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \qquad R_1 \to R_1 \times \frac{1}{5} \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \qquad R_2 \to R_2 + 2R_1 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix} \qquad R_3 \to R_3 + 5R_1 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix} \qquad R_2 \to \frac{5}{21} R_2 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix} \qquad R_3 \to R_3 + R_2 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix} \qquad R_3 \to \frac{-3}{14} R_3 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \qquad R_2 \to R_2 - \frac{2}{3} R_3 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \qquad R_1 \to R_1 - \frac{8}{5} R_3 \qquad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \qquad R_1 \to R_1 - \frac{7}{5} R_2 \qquad \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{17}{49} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

Example 2.4.2

Find column space, row space, null space and kernel of

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}.$$

Step (1): Finding $rref(A)$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_1 \to \frac{-1}{3}R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_2 \to R_2 + 2R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix} R_3 \to R_3 + 3R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix} R_2 \to \frac{3}{8}R_2 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & -4 & -5 \end{bmatrix} R_3 \to R_3 + 4R_2 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - \frac{-2}{3} R_2 \Rightarrow \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To identify row space

$$\begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B_{RS} = \left\{ \begin{pmatrix} 1 \\ -3 \\ 0 \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{5}{4} \end{pmatrix} \right\}$$

To identify column space

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B_{CS} = \left\{ \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} \right\}$$

Check you work

Note: $CS * RS = A$

$$\begin{bmatrix} -3 & -2 \\ 2 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

You can extract the null space quickly by changing the sign of the non-pivot element and adding a pivot where the pivot would line up to an identity matrix but this is how to compute it.

To find Null space and Kernel

The 'Null Space' is the solution to $Ax = 0$.

$$\begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 3x_2 + \frac{3}{2}x_4 &= 0 \\ free : x_2 &= x_2 \\ x_3 + \frac{5}{4}x_4 &= 0 \\ free : x_4 &= x_4 \end{aligned}$$

$$
\begin{aligned}
x_1 &= 3x_2 - \frac{3}{2}x_4 \\
x_2 &= x_2 + 0x_4 \\
x_3 &= 0x_2 - \frac{5}{4}x_4 \\
x_4 &= 0x_2 + x_4
\end{aligned}
$$

$$
x = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} \frac{-3}{2} \\ 0 \\ \frac{-5}{4} \\ 1 \end{pmatrix} x_4,
$$

$$x_2 = 1 \wedge x_4 = 4$$

$$Kernal = B_{NS} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ -5 \\ 4 \end{pmatrix} \right\}$$

Check your work $A * NS = 0;$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 1 & -5 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$Nullspace = \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 2.7.1 (Similar matrices)

Let A and B be two square matrices of same order, A is said to be similar to matrix B if there exists a non-singular matrix P, such that

$$B = P^{-1}AP$$

Definition 2.7.2 (Properties of similar matrices)

Similar matrices have same eigen values, eigen vectors, determinant, ranks, nullity, characteristic polynomial and traces.

Definition 2.7.3 (Procedure to find similar matrix)

If A is given

- Step I Characteristic polynomial $A - \lambda I$ by using $|A - \lambda I| = 0$.
- Step II Find eigen values $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$.
- Step III Find eigen vector $v_1, v_2, v_3, \cdots, \lambda_n$ using eigen values.
- Step IV Find P by combining all eigen values into one matrix.
- Step V Find P^{-1} from P.
- Step VI Find $B = P^{-1}AP$

B is called similar matrix.

Problem 2.7.4

Find similar matrix for $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ and λ be eigen value of A then

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (2-\lambda)(-2-\lambda) - (-1)(3)$$

$$= -4 - 2\lambda + 2\lambda + \lambda^2 + 3$$

$$= \lambda^2 - 1$$

To find eigen values:

$$|A - \lambda I| = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda^2 = 1$$
$$\lambda = \pm 1$$

Therefore eigen values are $\lambda_1 = 1$, $\lambda_2 = -1$

To find eigen vectors:

$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

At $\lambda = 1$

$$\begin{bmatrix} 2-1 & 3 \\ -1 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ x_1 &= -3x_2 \\ \text{Let } x_2 &= t \\ \text{then, } x_1 &= -3t \\ v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \end{aligned}$$

At $\lambda = -1$

$$(A - \lambda I)v = 0$$

$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+1 & 3 \\ -1 & -2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\text{Let } x_2 = -t$$

$$x_1 = -t$$

$$v_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To find P matrix,

$$P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$|P| = (-3)(1) - (1)(-1) = -3 + 1 = -2 \neq 0.$$

P is non-singular.

$$P^{-1} = \frac{1}{|A|} Adj(P) = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

$$P^{-1}AP = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -6+3 & -2+3 \\ 3-2 & 1-2 \end{pmatrix}$$

$$= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -3+1 & 1+1 \\ 3-3 & -1+3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To check

$$
\begin{aligned}
|B - \lambda I| &= 0 \\
\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)(-1-\lambda) - 0 &= 0 \\
-1 - \lambda + \lambda + \lambda^2 &= 0 \\
\lambda^2 &= 1 \\
\lambda &= \pm 1
\end{aligned}
$$

1. Eigen values of B matrix are similar to A matrix. So, $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. $|A| = |B| = -1$
3. $Trace(A) = Trace(B)$ i.e., $2 - 2 = 0 = 1 - 1$

Example 2.7.5

Find similar matrix for the following matrix

1. $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Problem 2.7.6

Find similar matrix for $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Find Matrix Eigenvalues ...

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} (1-\lambda) & -2 & 0 \\ 0 & (2-\lambda) & 0 \\ 0 & 0 & (-2-\lambda) \end{bmatrix} = 0$$

$$\begin{aligned} (1-\lambda)((2-\lambda)\times(-2-\lambda) - 0\times 0) - (-2)(0\times(-2-\lambda) - 0\times 0) + 0(0\times 0 - (2-\lambda)\times 0) &= 0 \\ (1-\lambda)((-4+\lambda 2) - 0) + 2(0-0) + 0(0-0) &= 0 \\ (1-\lambda)(-4+\lambda 2) + 2(0) + 0(0) &= 0 \\ (-4 + 4\lambda + \lambda 2 - \lambda 3) + 0 + 0 &= 0 \\ (-\lambda 3 + \lambda 2 + 4\lambda - 4) &= 0 \\ -(\lambda - 1)(\lambda - 2)(\lambda + 2) &= 0 \\ (\lambda - 1) = 0 or (\lambda - 2) = 0 or (\lambda + 2) &= 0 \end{aligned}$$

$\therefore$ The eigenvalues of the matrix A are given by $\lambda = -2, 1, 2,$

Eigen vector for $\lambda = -2$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector for $\lambda = 1$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector for $\lambda = 2$

$$v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} adj(P)$$

To find $|P|$:

$$\begin{aligned}
|A| &= \begin{vmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&= 0 \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2 \times \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\
&= 0 \times (0 \times 0 - 1 \times 0) - 1 \times (0 \times 0 - 1 \times 1) - 2 \times (0 \times 0 - 0 \times 1) \\
&= 0 \times (0 + 0) - 1 \times (0 - 1) - 2 \times (0 + 0) \\
&= 0 \times (0) - 1 \times (-1) - 2 \times (0) \\
&= 0 + 1 + 0 \\
&= 1
\end{aligned}$$

To find adjoint of P

$$adj(P) = adj \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$adj(P) = \begin{bmatrix} +(0\times 0 - 1\times 0) & -(0\times 0 - 1\times 1) & +(0\times 0 - 0\times 1) \\ -(1\times 0 - (-2)\times 0) & +(0\times 0 - (-2)\times 1) & -(0\times 0 - 1\times 1) \\ +(1\times 1 - (-2)\times 0) & -(0\times 1 - (-2)\times 0) & +(0\times 0 - 1\times 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0+0) & -(0-1) & +(0+0) \\ -(0+0) & +(0+2) & -(0-1) \\ +(1+0) & -(0+0) & +(0+0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} adj(P) = \frac{1}{1} \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since,

$$B = P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To verify the solution:

$$trace(A) = trace \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = 1 + 2 + (-2) = 1$$

$$trace(B) = trace \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (-2) + 1 + 2 = 1$$

Eigen values of A=$-2, 1, 2$
Eigen values of $B = -2, 1, 2$

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 1 \times \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} - (-2) \times \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} + 0 \times \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$$

$$= 1 \times (2 \times (-2) - 0 \times 0) + 2 \times (0 \times (-2) - 0 \times 0) + 0 \times (0 \times 0 - 2 \times 0)$$

$$= 1 \times (-4 + 0) + 2 \times (0 + 0) + 0 \times (0 + 0)$$

$$= 1 \times (-4) + 2 \times (0) + 0 \times (0)$$

$$= -4 + 0 + 0 = -4$$

$$|B| = \begin{vmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -2 \times \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= -2 \times (1 \times 2 - 0 \times 0) + 0 \times (0 \times 2 - 0 \times 0) + 0 \times (0 \times 0 - 1 \times 0)$$

$$= -2 \times (2 + 0) + 0 \times (0 + 0) + 0 \times (0 + 0)$$

$$= -2 \times (2) + 0 \times (0) + 0 \times (0)$$

$$= -4 + 0 + 0 = -4$$