Example 4.0.18

Let π be the plane in $\mathbb{R}^3$ spanned by vector $x_1 = (1, 2, 2)$ and $x_2 = (-1, 0, 2)$.

1. Using the Gram-Schmidt process find an orthonormal basis for π.
2. Extend it to an orthonormal basis for $\mathbb{R}^3$.

x_1, x_2 is a basis for the plane π. We can extend it to a basis for $\mathbb{R}^3$ by adding one vector from the standard basis. For instance, vectors x_1, x_2, and $x_3 = (0, 0, 1)$ form a basis for R_3 because

$$\begin{vmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0.$$

Using the Gram-Schmidt process, we orthogonalize the basis $x_1 = (1,2,2)$, $x_2 = (-1,0,2)$, $x_3 = (0,0,1)$:

$$\begin{aligned}
v_1 &= x_1 = (1,2,2),\\
v_2 &= x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (-1,0,2) - \frac{(-1.1 + 0.2 + 2.2)}{(1.1 + 2.2 + 2.2)}(1,2,2)\\
&= (-1,0,2) - \frac{3}{9}(1,2,2) = \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right),\\
v_3 &= x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2\\
&= (0,0,1) - \frac{(0.1 + 0.2 + 1.2)}{(1.1 + 2.2 + 2.2)}(1,2,2)\\
&\quad - \frac{(0.\frac{-4}{3} + 0.\frac{-2}{3} + 1.\frac{4}{3})}{(\frac{-4}{3}.\frac{-4}{3} + \frac{-2}{3}.\frac{-2}{3} + \frac{4}{3}.\frac{4}{3})}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)\\
&= (0,0,1) - \frac{2}{9}(1,2,2) - \frac{\frac{4}{3}}{4}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) = \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right).
\end{aligned}$$

Now $v_1 = (1,2,2)$, $v_2 = (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3})$, $v_3 = \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right)$ is an orthogonal basis for R_3 while v_1, v_2 is an orthogonal basis for π. It remains to normalize these vectors.

$$\langle v_1, v_1 \rangle = 9 \Rightarrow \|v_1\| = 3$$
$$\langle v_2, v_2 \rangle = 4 \Rightarrow \|v_2\| = 2$$
$$\langle v_3, v_3 \rangle = \frac{1}{9} \Rightarrow \|v_3\| = \frac{1}{3}$$
$$w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3}(1,2,2),$$
$$w_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}(-2,-1,2),$$
$$w_3 = \frac{v_3}{\|v_3\|} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) = \frac{1}{3}(2,-2,1).$$

w_1, w_2 is an orthonormal basis for π.
w_1, w_2, w_3 is an orthonormal basis for $\mathbb{R}^3$.

Problem 4.0.19

Find the distance from the point $y = (0, 0, 0, 1)$ *to the subspace* $V \subset \mathbb{R}^4$ *spanned by vectors* $x_1 = (1, -1, 1, -1)$, $x_2 = (1, 1, 3, -1)$, *and* $x_3 = (-3, 7, 1, 3)$.

Let us apply the Gram-Schmidt process to vectors x_1, x_2, x_3, y. We should obtain an orthogonal system v_1, v_2, v_3, v_4. The desired distance will be $|v_4|$.
Given

$$x_1 = (1, -1, 1, -1), \qquad x_2 = (1, 1, 3, -1),$$
$$x_3 = (-3, 7, 1, 3), \qquad y = (0, 0, 0, 1).$$

$$
\begin{aligned}
v_1 &= x_1 = (1,-1,1,-1),\\
v_2 &= x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (1,1,3,-1) - \frac{4}{4}(1,-1,1,-1) = (0,2,2,0),\\
v_3 &= x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2\\
&= (-3,7,1,3) - \frac{-12}{4}(1,-1,1,-1) - \frac{16}{8}(0,2,2,0) = (0,0,0,0).
\end{aligned}
$$

The vector x_3 is a linear combination of x_1 and x_2. V is a plane, not a 3-dimensional subspace. We should orthogonalize vectors x_1, x_2, y.

$$\begin{aligned}
\vec{v}_3 &= y - \frac{\langle y, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle y, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \\
&= (0,0,0,1) - \frac{-1}{4}(1,-1,1,-1) - \frac{0}{8}(0,2,2,0) \\
&= \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right). \\
|\vec{v}_3| &= \left|\left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)\right| = \frac{1}{4}|(1,-1,1,3)| \\
&= \frac{\sqrt{1^2 + (-1)^2 + 1^2 + 3^2}}{4} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.
\end{aligned}$$