Definition 2.1.1 (Linearly dependent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of V is said to be linear dependent if there exist scalar $a_1, a_2, \ldots, a_n \in F$ not all of them 0 (some of them may be zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$

Definition 2.1.2 (Linearly Independent)

Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of V is said to be linearly independent if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$
$$a_i \in F, 1 \le i \le n \Rightarrow a_i = 0 \text{ for each} 1 \le i \le n$$

An infinite set of vector of V is said to be linearly independent if its every finite subset is linearly independent, otherwise it is linearly dependent.

Example 2.1.3

Find whether the set of vector $v_1 = (1,2,1)$, $v_2 = (3,1,5)$, $v_3 = (3,-4,7)$ is linearly independent or dependent.

Let a_1, a_2, a_3 be three scalars such that

$$\begin{aligned} a_1v_1 + a_2v_2 + a_3v_3 &= 0 \\ \Rightarrow a_1(1,2,1) + a_2(3,1,5) + a_3(3,-4,7) &= 0 \\ (a_1 + 3a_2 + 3a_3, 2a_1 + a_2 - 4a_3, a_1 + 5a_2 + 7a_3) &= 0 \end{aligned}$$

$$\begin{aligned} a_1 + 3a_2 + 3a_3 &= 0 \\ 2a_1 + a_2 - 4a_3 &= 0 \\ a_1 + 5a_2 + 7a_3 &= 0 \end{aligned}$$

The coefficients matrix of these equation is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix}$$

$$= 1(7+20) - 3(14+4) + 3(10-1) = 27 - 54 + 27 = 0$$

and

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$$

$$\therefore \rho(A) = 2$$

i.e., so *the rank of matrix* $A <$ *no. of unknown quantities*.
The system of equations will have $3 - 2 = 1$ non-zero solutions and hence the set of vectors are linearly dependent.

Problem 2.1.4

Show that the set $\{1, x, 1+x+x^2\}$ is linearly independent set of vectors in the vector space of all polynomial over the real number filed.

Let a_1, a_2, a_3 be scalars (real numbers) such that

$$a_1(1) + a_2(x) + a_3(1+x+x^2) = 0$$

We have

$$\begin{aligned}(a_1+a_3) + (a_2+a_3)x + a_3x^2 &= 0\\ a_1+a_3 = 0, a_2+a_3 = 0, a_3 &= 0\\ a_1 = 0, a_2 = 0, a_3 &= 0\end{aligned}$$

Therefore the vectors $1, x, 1+x+x^2$ are linearly independent over the field of real numbers.

Example 2.1.5

Are the vectors $(2, 2, 2, 4), (2, -2, -4, 0), (4, -2, -5, 2), (4, 2, 1, 6)$ linearly independent?

Let a_1, a_2, a_3 and a_4 be four scalars such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

Here

$\alpha_1 = (2, 2, 2, 4), \alpha_2 = (2, -2, -4, 0), \alpha_3 = (4, -2, -5, 2)$ and $\alpha_4 = (4, 2, 1, 6)$

$$\therefore a_1(2, 2, 2, 4) + a_2(2, -2, -4, 0) + a_3(4, -2, -5, 2) + a_4(4, 2, 1, 6) = 0$$
$$(2a_1 + 2a_2 + 4a_3 + 4a_4, 2a_1 - 2a_2 - 2a_3 + 2a_4,$$
$$2a_1 - 4a_2 - 5a_3 + a_4, 4a_1 + 2a_3 + 6a_4) = (0, 0, 0, 0)$$

The coefficient matrix of these equation is

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 2 & -2 & -4 & 0 \\ 4 & -2 & -5 & 2 \\ 4 & 2 & 1 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -3 \\ 0 & -4 & -6 & -2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 3R_2$ and $R_4 \to R_4 - R_2$, we get

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

i.e., so the rank of matrix $A <$ number of unknown quantities.

The system of equations will have $4 - 2 = 2,$ non-zero solutions and hence the set of vectors are linearly dependent. Hence given vectors are not linearly independent.

Example 2.1.6

Show that the vectors (a_1, a_2) and (b_1, b_2) in $V_2(C)$ are L.D. iff $a_1b_2 - a_2b_1 = 0$, where C is the field complex numbers.

Let $a, b \in C$, then

$$a(a_1, a_2) + b(b_1, b_2) = 0$$
$$i.e., (aa_1 + bb_1, aa_2 + bb_2) = (0, 0)$$

$$\left.\begin{aligned} aa_1 + bb_1 &= 0 \\ aa_2 + bb_2 &= 0 \end{aligned}\right\} \tag{9}$$

The system of equations (9) will possess a non-zero solution iff

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \Rightarrow a_1b_2 - a_2b_1 = 0$$

Thus the given system of vectors is L.D. iff $a_1b_2 - a_2b_1 = 0$.

Problem 2.2.1

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z).$$

Find basis and dimension of it's Range and Null space.

$$\begin{aligned}
N(T) &= \{T(x, y, z) = (0, 0, 0)\} \\
(x + 2y - z, y + z, x + y - 2z) &= (0, 0, 0) \\
x + 2y - z &= 0 \\
y + z &= 0 \\
x + y - 2z &= 0
\end{aligned}$$

$$\begin{aligned}
y &= -z \\
x - 2z - z &= 0 \\
x &= 3z
\end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -z \\ z \end{bmatrix} \Rightarrow z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$N(T) = \{T(x,y,z) = (0,0,0)\} = (3,-1,1)$$

$$R(T) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1, R_3 \Rightarrow R_3 + R_1$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$dim(R(T)) = 2$$

$$Basic = (1, 0, 1)(0, 1, -1)$$

Problem 2.2.2

Let V be vector space 2×2 matrices over R and $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. Let be $T : V \to V$ be linear transform defined by $T(A) = PA$. Find basis and dim of null space of T and Range space of T.

$$N(T) = \{T(A) = 0 : A \in V$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

$$PA = 0$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\begin{bmatrix} a - c & b - d \\ -2a + 2c & -2b + 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} a - c &= 0 \\ -2a + 2c &= 0 \\ a &= c \end{aligned} \qquad\qquad \begin{aligned} b - d &= 0 \\ -2b + 2d &= 0 \\ b &= d \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

To find basis:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(E_1) = PE_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$T(E_2) = PE_2 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$T(E_3) = PE_3 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T(E_4) = PE_4 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$T(E_1) = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}; T(E_2) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}; T(E_3) = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}; T(E_4) = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$R_3 = R_3 + R_1; R_4 = R_4 + R_2$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Range space of T is

$$\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Rank of dimension of range space = 2

Problem 2.2.3

Let w_1 and w_2 be the subspace generated by $(-1,2,1)$, $(2,0,1)$ and $(-8,4,-1)$ in $\mathbb{R}^3(\mathbb{R})$ and w_2 generated by all vectors $(a,0,b)$ $\forall a,b \in \mathbb{R}$. Find basis and dimension of w_1, w_2 and w_1+w_2.

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 1 \\ -8 & 4 & -1 \end{bmatrix}$$

$R_2 = R_2 + 2R_1;\ R_3 = R_3 - 8R_1$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & -12 & -9 \end{bmatrix}$$

$R_3 = R_3 + 3R_2$

$$R(w_1) = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Basis = (-1, 2, 1) \text{ and } (0, 4, 3)$$
$$dim(w_1) = 2$$

$$R(w_2) = (a, 0, b)$$
$$= a\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

$$Basis(w_2) = B_2 = (1, 0, 0), (0, 0, 1)$$
$$dim(w_1) = 2$$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{2}R_2$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 = R_3 + \frac{1}{2}R_4$

$$w_1 + w_2 = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} dim(w_1 + w_2) &= 3 \\ Basis(w_1 + w_2) &= (-1, 2, 1), (0, 4, 3), (0, 0, 1) \end{aligned}$$

$$\begin{aligned} (w_1 \cap w_1) &= dim(w_1) + dim(w_2) - dim(w_1 + w_2) \\ &= 2 + 2 - 3 = 4 - 3 \\ &= 1 \end{aligned}$$

Example 2.2.4

Let M and N be two subspace of R^4

$$M = \{(a,b,c,d) | b+c+d\} \text{ and}$$
$$N = \{(a,b,c,d) | a+b=0, c=2d\}$$

Find basis and dimension of $(i)M$, $(ii)N$ and $(iii)M \cap N$

Problem 2.3.1

Let T be a linear transformation on $V_3(\mathbb{R})$ defined by $T(a, b, c) = (3a, a - b, 2a + b + c)\ \forall (a, b, c) \in V_3(\mathbb{R})$. Is T invertible?. If so, find a rule for T^{-1} as the one which defines T.

For proving T is invertible, we need to show only T is one-one and onto.
To prove one-one:
Let

$$\alpha = (a_1, b_1, c_1) \in V_3(\mathbb{R})$$
$$\beta = (a_2, b_2, c_2) \in V_3(\mathbb{R})$$

Then,

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$3a_1 = 3a_2$$
$$a_1 = a_2$$
$$a_1 - b_1 = a_2 - b_2$$
$$a_2 = b_2$$
$$c_1 = c_2$$
$$(a_1, b_1, c_1) = (a_2, b_2, c_2)$$
$$\alpha = \beta$$

$\therefore T$ is one-one.

To prove onto:

T is linear transformation on a finite dimensional vector space $V_3(\mathbb{R})$, where dimension in 3.

$$\Rightarrow \text{Also } T \text{ is one-one}$$
$$\Rightarrow T \text{ must be onto}$$
$$\Rightarrow T \text{ is invertible}$$

$$\text{If } T(a,b,c) = (p,q,r)$$
$$\text{then, } T^{-1}(p,q,r) = (a,b,c)$$
$$T(a,b,c) = (p,q,r)$$
$$(3a, a-b, 2a+b+c) = (p,q,r)$$
$$3a = p$$
$$p = 3a$$
$$a = \frac{p}{3}$$
$$a - b = q$$
$$\frac{p}{3} - b = q$$
$$\frac{p}{3} - q = b$$

$$
\begin{aligned}
2a + b + c &= r \\
2\frac{p}{3} + \left(\frac{p}{3} - q\right) + c &= r \\
c &= r - p + q
\end{aligned}
$$

$$
\begin{aligned}
\because T^{-1}(p, q, r) &= (a, b, c) \\
&= \left(\frac{p}{3}, \frac{p}{3} - q, r - p + q\right)
\end{aligned}
$$

Example 2.3.2

Let T be a linear map on $V_3(R)$ defined by $T(a,b,c) = [3a, a-b, 2a+b+c]$ $\forall a,b,c \in \mathbb{R}$. Is T invertible?. If so find a rule for T^{-1} like one which define T.

For proving T is invertible, we need to show that T is one-one and onto.
To prove one-one:
Let $\alpha = (a_1, b_1, c_1)$, $\beta - (a_2, b_2, c_2)$ be any two elements of $V_3(\mathbb{R})$.

$$T(\alpha) = T(\beta)$$
$$T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$$
$$(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$$
\begin{aligned}
3a_1 &= 3a_2\\
a_1 - b_1 &= a_2 - b_2 + c_2\\
2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2\\
a_1 &= a_2\\
\because a_1 - b_1 &= a_2 - b_2\\
-b_1 &= -b_2\\
b_1 &= b_2\\
\because 2a_1 + b_1 + c_1 &= 2a_2 + b_2 + c_2\\
\because a_1 &= b_1\\
b_1 &= b_2\\
c_1 &= c_2\\
(a_1, b_1, c_1) &= (a_2, b_2, c_2)\\
\alpha &= \beta
\end{aligned}
$$

$$\because T(\alpha) = T(\beta)$$
$$\alpha = \beta$$
$$T := A \to B$$

Hence T is one-one.
To find onto:
Since, T is a linear one-one map on a finite dimensional vector space.
$\Rightarrow T$ is onto.
$\Rightarrow T$ is one-one and onto.
$\Rightarrow T$ is invertible.

Second part:

$$\text{Let } T(a,b,c) = (p,q,r)$$
$$\text{Then } T^{-1}(p,q,r) = (a,b,c) \tag{10}$$

Now

$$T(a,b,c) = (p,q,r)$$
$$(3a, a-b, 2a+b+c) = (p,q,r)$$
$$3a = p$$
$$a = \frac{p}{3}$$
$$\because a - b = q$$
$$\frac{p}{3} - b = q$$
$$\frac{p}{3} - q = b$$

$$\because 2a + b + c = r$$
$$2\frac{p}{3} + \left(\frac{p}{3} - q\right) + c = r$$
$$c = r - p + q$$

Put the value of a, b, c in equation (10)

$$T^{-1}(p,q,r) = \left(\frac{p}{3}, \frac{p}{3} - a, r - p + q\right)$$

or

$$T^{-1}(x,y,z) = \left(\frac{x}{3}, \frac{x}{3} - y, z - x + y\right)$$

which is the rule which defines T^{-1}.

Definition 2.4.1 (Wronskian)

Let f and g be differentiable on $[a,b]$. If Wronskian $W(f,g)(t_0)$ is nonzero for some t_0 in $[a,b]$ then f and g are linearly independent on $[a,b]$. If f and g are linearly dependent then the Wronskian is zero for all t in $[a,b]$.

Problem 2.4.2

Using Wronskian method prove that $\{e^{3x}, e^{5x}\}$ *is a linearly independent set on* $\mathbb{R}$.

Set $f(x) = e^{3x}$, $g(x) = e^{5x}$. Then,

$$\begin{aligned} W(f(x), g(x)) &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & f''(x) \end{vmatrix} \\ &= \begin{vmatrix} e^{3x} & e^{5x} \\ 3e^{3x} & 5e^{5x} \end{vmatrix} \\ &= 5e^{8x} - 3e^{8x} \\ &= 2e^{8x} \\ &\neq 0 \ \ (\forall x \in \mathbb{R}) \end{aligned}$$

$\because$ The given set $\{e^{3x}, e^{5x}\}$ is linearly independent.

Problem 2.4.3

Using Wronskian method prove that $\{e^{2x}, \cos(x), 2e^{2x}\}$ *is a linearly dependent set on* $\mathbb{R}$.

Set $f(x) = e^{2x}$, $g(x) = \cos x\, h(x) = 2e^{2x}$. Then,

$$W(f(x), g(x), h(x))$$

$$= \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & \cos x & 2e^{2x} \\ 2e^{2x} & -\sin x & 4e^{2x} \\ 4e^{2x} & -\cos x & 8e^{2x} \end{vmatrix}$$

$$= e^{2x} \begin{vmatrix} -\sin x & 4e^e 2x \\ -\cos x & 8e^{2x} \end{vmatrix} - 2e^{2x} \begin{vmatrix} \cos x & 2e^{2x} \\ -\cos x & 8e^{2x} \end{vmatrix} + 4e^{2x} \begin{vmatrix} \cos x & 2e^{2x} \\ -\sin t & 4e^{2x} \end{vmatrix}$$

$$= e^{2x}\left(-8e^{2x}\sin x + 4e^{2x}\cos x\right) - 2e^{2x}\left(8e^{2x}\cos x + 2e^{2x}\cos x\right)$$

$$+ 4e^{2x}\left(4e^{2x}\cos x + 2e^{2x}\sin x\right)$$

$$
\begin{aligned}
&= e^{2x}\left(-8\sin x + 4\cos x - 20\cos x + 16\cos x + 8\sin x\right) \\
&= e^{4x}(0) \\
&= 0 \ (\forall x \in \mathbb{R})
\end{aligned}
$$

Example 2.4.4

Using Wronskian method prove that $\{1, x, x^2\}$ is a linearly dependent set on $\mathbb{R}$.

Ans: $W(f(x), g(x), h(x)) = 2 \neq 0$, So the set is linearly independent.

Problem 2.5.1

Transforming a matrix $\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$ *to reduced row echelon form*

$$\begin{bmatrix} 5 & 7 & 8 & 5 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \quad R_1 \to R_1 \times \frac{1}{5} \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 2 & 7 & 6 & 3 \\ 5 & 8 & 4 & 3 \end{bmatrix} \quad R_2 \to R_2 - 2R_1 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 5 & 8 & 4 & 3 \end{bmatrix} \quad R_3 \to R_3 - 5R_1 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & \frac{21}{5} & \frac{14}{5} & 1 \\ 0 & 1 & -4 & -2 \end{bmatrix} \quad R_2 \to \frac{5}{21}R_2 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 1 & -4 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & \frac{-14}{3} & \frac{-47}{21} \end{bmatrix} \quad R_3 \to \frac{-3}{14} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{21} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_2 \to R_2 - \frac{2}{3} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & \frac{8}{5} & 1 \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_1 \to R_1 - \frac{8}{5} R_3 \quad \Rightarrow \begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{5} & 0 & \frac{57}{245} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix} \quad R_1 \to R_1 - \frac{7}{5} R_2 \quad \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{17}{49} \\ 0 & 1 & 0 & \frac{-4}{49} \\ 0 & 0 & 1 & \frac{47}{98} \end{bmatrix}$$

Example 2.5.2

Find column space, row space, null space and kernel of

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}.$$

Step (1): Finding $rref(A)$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_1 \to \frac{-1}{3} R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} R_2 \to R_2 - 2R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 3 & -9 & -2 & 2 \end{bmatrix} R_3 \to R_3 - 3R_1 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix} R_2 \to \frac{3}{8} R_2 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & -4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & -4 & -5 \end{bmatrix} R_3 \to R_3 + 4R_2 \Rightarrow \begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & \frac{2}{3} & \frac{7}{3} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - \frac{-2}{3} R_2 \Rightarrow \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To identify row space

$$\begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B_{RS} = \left\{ \begin{pmatrix} 1 \\ -3 \\ 0 \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{5}{4} \end{pmatrix} \right\}$$

To identify column space

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B_{CS} = \left\{ \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} \right\}$$

Check you work

Note: $CS * RS = A$

$$\begin{bmatrix} -3 & -2 \\ 2 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

You can extract the null space quickly by changing the sign of the non-pivot element and adding a pivot where the pivot would line up to an identity matrix but this is how to compute it.

To find Null space and Kernel

The 'Null Space' is the solution to $Ax = 0$.

$$\begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here x_1 and x_3 are pivot variables. So, x_2 and x_4 are free variables.

$$x_1 - 3x_2 + \frac{3}{2}x_4 = 0$$
$$free : x_2 = x_2$$
$$x_3 + \frac{5}{4}x_4 = 0$$
$$free : x_4 = x_4$$

$$
\begin{aligned}
x_1 &= 3x_2 - \frac{3}{2}x_4 \\
x_2 &= x_2 + 0x_4 \\
x_3 &= 0x_2 - \frac{5}{4}x_4 \\
x_4 &= 0x_2 + x_4
\end{aligned}
$$

$$
x = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} \frac{-3}{2} \\ 0 \\ \frac{-5}{4} \\ 1 \end{pmatrix} x_4,
$$

$$x_2 = 1 \wedge x_4 = 4$$

$$Kernal = B_{NS} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ -5 \\ 4 \end{pmatrix} \right\}$$

Check your work $A * NS = 0;$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 1 & -5 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$Nullspace = \begin{bmatrix} 1 & -3 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2.6.1

Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ *and* $C = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ *be bases for* $\mathbb{R}^2$. *If* $[X]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, *find* $[X]_C$.

$$
\begin{aligned}
[X]_B &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\
\Rightarrow X &= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 3 \end{bmatrix}
\end{aligned}
$$

$$[X]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

To check

$$-7 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Problem 2.6.2

Let $B = \{u_1, u_2\}$, $B' = \{u'_1, u'_2\}$ *for* $\mathbb{R}^2$ *and* $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, u'_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. *Find the transition matrix from* B *and* B'.

$$\left[\begin{array}{cc|cc} u_1' & u_2' & u_1 & u_2 \end{array}\right] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & -3 & 1 & 0 \end{array}\right] \qquad R_1 \Leftrightarrow R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & -11 & 1 & -2 \end{array}\right] \qquad R_2 \to R_2 - 2R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & \frac{-1}{11} & \frac{2}{11} \end{array}\right] \qquad R_2 \to \frac{-1}{11} R_2$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & \frac{-1}{11} & \frac{2}{11} \end{array}\right] \qquad R_1 \to R_1 - 4R_2$$

Transition matrix P

$$P = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ \frac{-1}{11} & \frac{2}{11} \end{bmatrix}$$