Pumping Lemma for CFL

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⇒ Pumping Lemma (for CFL) is used to prove that a language is NOT Context Free.

If A is a CFL, then, A has a Pumping Length 'p' such that any string 'S', where $|S| \geq p$ may be divided into 5 pieces $S = uvxyz$ such that the following conditions must be true:

1) uv^ixy^iz is in A for every $i \geq 0$
2) $|vy| > 0$
3) $|vxy| \leq p$

⇒ Following are the steps to prove a language is not CFL: [Using Contradiction]

① Assume that A is Context free
② It has to have a pumping length (say p)
③ All string longer than p can be pumped $|S| \geq p$
④ Now find a string 'S' in A such that $|S| \geq p$
⑤ Divide S into uvxyz
⑥ Show that $uv^ixy^iz \notin A$ for some i
⑦ Then Consider the ways that S can be divided into uvxyz
⑧ Show that none of these can satisfy all the 3 pumping conditions at the same time.
⑨ S can not be pumped == CONTRADICTION

Example -1

$\Rightarrow$ Show that $L = \{ a^N b^N c^N \mid N \geq 0 \}$ is not Context Free.

① Assume that L is Context Free

② L must have a pumping length (say P)

③ Now we take a string S such that $S = a^P b^P c^P$

④ We divide S into parts $uvxyz$

Eg. $P = 4$ So, $S = a^4 b^4 c^4$

Case I: v and y each contains only one type of symbol.

$\underbrace{a}_{u}\underbrace{aa}_{v}\underbrace{abbbbc}_{x}\underbrace{c}_{y}\underbrace{cc}_{z}$

$uv^i x y^i z$ [$i=2$]

$uv^2 x y^2 z$

aaaaaabbbbccccc

$a^6 b^4 c^5 \notin L$

Case II: Either v or y has more than one kind of symbols.

$\underbrace{aa}_{u}\underbrace{aabb}_{v}\underbrace{b}_{x}\underbrace{b}_{y}\underbrace{cccc}_{z}$

$uv^i x y^i z$ [$i=2$]

$uv^2 x y^2 z$

aaaabbaabbbbbcccc

$\notin L$

Example - 2

Show that $L = \{ww \mid w \in \{0,1\}^*\}$ is NOT Context Free.

→ Assume that L is context free

→ L must have a pumping length (say p)

→ Now we ~~the~~ take a string S such that $S = 0^p 1^p 0^p 1^p$

→ we divide S into parts $uvxyz$

Eg. $p = 5$ so, $S = 0^5 1^5 0^5 1^5$

Case I: vxy does not straddle a boundary → doesn't cross the boundary.

$$\underbrace{0\,0\,0\,0\,0^{*}}_{u}\,\underbrace{1\,1\,1\,1\,1}_{vxy}{}^{*}\,\underbrace{0\,0\,0\,0\,0^{*}\,1\,1\,1\,1\,1}_{z}$$

$|vxy| \leq p$

$$uv^i x y^i z \Rightarrow uv^2 x y^2 z$$

0 0 0 0 0 1 11 1 11 1 0 0 0 0 0 1 1 1 1 1

$$\underbrace{0^5 1^7}\,\underbrace{0^5 1^5} \notin L$$

Example 2

Case 2a: vxy straddles the first boundary

$$\underbrace{0\,0\,0}_{u}\,\underbrace{0\,0^*\,1\,1\,1}_{vxy}\,\underbrace{1\,1^*\,0\,0\,0\,0\,0^*\,1\,1\,1\,1\,1}_{z}$$

Length-5 $\leq P$

$$uv^2xy^2z$$

0 0 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1

$$0^7\,1^7\,0^5\,1^5 \notin L$$

Case 2b: vxy straddles the third boundary

$$\underbrace{0\,0\,0\,0\,0^*\,1^*\,1\,1\,1\,1^*\,0\,0\,0}_{u}\,\underbrace{0\,0^*\,1\,1\,1}_{vxy}\,\underbrace{1\,1}_{z}$$

$$uv^2xy^2z$$

0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1

$$0^5\,1^5\,0^7\,1^7 \notin L$$

Example 2

Case 3: vxy straddles the midpoint

$$\underbrace{0\,0\,0\,0\,0\,1\,1}_{u}\,\underbrace{1\,1\,1\,0\,0}_{vxy}\,\underbrace{0\,0\,0\,1\,1\,1\,1\,1}_{z}$$

$$uv^2xy^2z$$

$$0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1$$

$$\underbrace{0^5 1^7}\,\underbrace{0^7 1^5} \notin L$$

L is not Context Free.