General form of a Linear Programming Problem:

$$\text{Max/Min } z = \sum_{j=1}^{n} c_j x_j$$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots m'$

$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = m'+1, m'+2, \ldots m''$

$\sum_{j=1}^{n} a_{ij} x_j = b_i; \quad i = m''+1, m''+2, \ldots m.$

$x_j \geq 0, \quad j = 1, 2, \ldots n$

Canonical form for maximum type of LPP:

A general linear programming problem of maximization type is said to be in canonical form if

(i) The objective function is of maximization type.

(ii) All the constraints are of less than equal ($\leq$) type.

(iii) All decision variables are non-negative

i.e. $\text{Max } z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots m.$

$x_j \geq 0, \quad j = 1, 2, \ldots n.$

In Matrix form, $\text{Max } z = c^T x$

Subject to $Ax \leq b,$

$x \geq 0$

$A \in R^{m \times n}, \quad c \in R^n, \quad b \in R^m.$

The Canonical form for minimum type LPP :

A general linear programming problem of minimization type is said to be in the ~~canoni~~ canonical form if

(i) The objective function is of minimization type.

(ii) All the constraints are of ($\geq$) greater or equals type.

(iii) All the decision variables are non negative.

i.e. Min $Z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \; i=1,2, \ldots m.$

$x_j \geq 0, \; j=1,2 \ldots n.$

In Matrix form,

Min $z = c^T x$

Subject to $Ax \geq b$

$x \geq 0$

$A \in R^{m \times n}, \; b \in R^m, \; x, c \in R^n.$

Standard form of LPP

A general linear programming problem defined before can be always put into the standard form whose characteristics are the following.

(i) All constraints are equations except for the non negativity of variables which remain inequalities (≥ 0).

(ii) The right hand side element of each constraint equation is non-negative.

(iii) All variables are non-negative.

(iv) The objective function is of maximization or minimization type.

$$\text{Max/Min } z = \sum_{j=1}^{n} c_j x_j .$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \dots m.$$

$$x_j \geq 0, \quad j = 1, 2, \dots n.$$

Matrix form of the above problem

$$\text{(LP)} \quad \left.\begin{array}{ll} \text{Max/Min} & z = c^T x \\ \text{Subject to} & Ax = b, \\ & x \geq 0 \end{array}\right\} \text{(A1)}$$

$$A \in R^{m \times n}, \; b \in R^m, \; c \in R^n .$$

Few terminologies:

(1) Slack variables: A non negative variable which is added in the left hand side of less than or equals to $(\leq)$ type constraint to get an equality type constraint, is called slack variable.

Suppose in general formulation, you have $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, $i = 1, 2, \dots m$

To transform it into the equality form,

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+1} = b_i, \quad i = 1, 2, \dots m, \quad x_{n+1} \geq 0$$

$$\Rightarrow x_{n+1} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad \text{---(1)}$$

$x_{n+1} \geq 0$

This x_{n+1} is called slack variable.

(2) Surplus variables: A non negative variable which is subtracted from the left hand side of greater than or equals to $(\geq)$ type constraints constraint, to get an equality type constraint, is called surplus variable.

Suppose in general formulation,

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = m+1, \dots m'$$

To tranform it into equality,

$$\sum_{j=1}^{n} a_{ij} x_j - x_{n+1} = b_i, \quad x_{n+1} \geq 0,$$

$$\Rightarrow x_{n+1} = \sum_{j=1}^{n} a_{ij} x_j - b_i \quad \text{---(2)}$$

This x_{n+1} is called surplus variable.

Some results on Standardisation Operation

(i) There is a one-one correspondence between the optimal solution of the original linear programming problem and that of the new problem, in the standard form, when slack and surplus variables are introduced if

$$C_{slack} = 0 \text{ and } C_{surplus} = 0.$$

(ii) If S be the set of all feasible solution of $Ax = b$, $x \geq 0$ and if $x' \in S$ minimize the objective function $Z = cx$, then x^* also maximizes the objective function $z' = (-c)x$ over S.

In particular $\text{Min } Z = c^T x = -[\text{Max}(-Z) = -c^T x]$

(iii) In some LPP, there may be possible cases in which the variables are unrestricted in sign i.e. they may be positive, negative or zero. This problem can be recast into standard formulation by replacing each unrestricted variable by two non negative variable.

Thus an unrestricted variable x_j can be written as

$$x_j = x_j' - x_j'', \quad \text{where } x_j', x_j'' \geq 0.$$

The variable x_j is positive, negative or zero according as

$$x_j' > x_j''; \; x_j' < x_j'' \text{ or } x_j' = x_j''. \text{ resp.}$$

Example: Convert the LPP into standard form.

(1) Minimize $z = 3x_1 + x_2 + 2x_3$

Subject to $-2x_1 + 4x_2 \leq 3$

$x_1 + 2x_2 + 3x_3 \geq 5$

$2x_1 + 5x_3 \leq 2$

and $x_1, x_2, x_3 \geq 0$.

$\rightarrow$ This problem can be converted to standard form by transforming inequality constraints into equations by introducing slack and surplus variables. This can be recast into Maximization problem also.

The new recast problem is

$$\text{Max } z' = (-z) = -3x_1 - x_2 - 2x_3 \quad +0.x_4+0.x_5+0.x_6$$

Subject to

$$-2x_1 + 4x_2 + 0.x_3 + x_4 = 3$$

$$x_1 + 2x_2 + 3x_3 + 0.x_4 - x_5 = 5$$

$$2x_1 + 0.x_2 + 5x_3 + 0.x_4 + 0.x_5 + x_6 = 2$$

$x_j \geq 0, \quad j = 1, 2, 3, 4, 5, 6.$

x_4, x_6 are slack variables and x_5 is surplus variable.

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(2) Maximize $Z = 3x_1 + 4x_2 - 4x_3 + 2x_4 + 9x_5$

Subject to

$$3x_1 - 7x_2 - 9x_3 + x_4 - 2x_5 \le 6$$
$$2x_1 + 5x_2 + 4x_3 - 3x_4 + x_5 \le 8$$
$$x_1 + 2x_2 - 5x_3 - 2x_4 + 11x_5 \le 10$$

$x_1, x_2, x_4 \ge 0$ and x_3, x_5 are unrestricted in sign.

→ We write here the unrestricted variable as

$$x_3 = x_3' - x_3'' \text{ and } x_5 = x_5' - x_5''$$
$$x_3', x_3'', x_5', x_5'' \ge 0.$$

Thus the given problem reduces to

$$\text{Max } Z = 3x_1 + 4x_3 - 4x_3' + 4x_3'' + 2x_4 + 9x_5' - 9x_5''$$

+0.x_6+0.x_7+0.x_8

S.t.

$$3x_1 - 7x_2 - 9x_3' + 9x_3'' + x_4 - 2x_5' + 2x_5'' + x_6 = 6$$

$$2x_1 + 5x_2 + 4x_3' - 4x_3'' - 3x_4 + x_5' - x_5'' + 0.x_6 + x_7 = 8$$

$$x_1 + 2x_2 - 5x_3' + 5x_3'' - 2x_4 + 11x_5' - 11x_5'' + 0.x_6 + 0.x_7 + x_8 = 10$$

$$x_1, x_2, x_3', x_3'', x_4, x_5', x_5'', x_6, x_7, x_8 \ge 0.$$

where x_6, x_7, x_8 are slack variables.

Exercise

(1) Max $Z = 2x_1 + 3x_2 + x_3$

S.t. $x_1 + x_2 - 2x_3 \ge -5$

$-6x_1 + 5x_2 - 3x_3 = 12$

$12x_1 - 5x_2 + 5x_3 \le 13$

$x_1, x_2, x_3 \ge 0$

(2) Max $Z = 3x_1 + 2x_2 + 7x_3$

S.t. $x_1 + x_2 + 2x_3 \le 40$

$2x_1 + x_2 - 3x_3 \ge 50$

$|5x_2 + 8x_3| \le 60$

$x_1, x_2, x_3 \ge 0.$

-60<=5x_2+8x_3<=60

--5x_2-8x_3<=60, 5x_2+8x_3<=60

Some important results on LPP :

Let us consider the standard linear programming problem (A1). Next we consider the following results which will hold for (A1).

Theorem 1: The set of all feasible solutions of a linear programming problem is a convex set.

Corollary 1: If an LPP has two feasible solution, then it has an infinite number of feasible solution as any convex combination of two feasible solution is a feasible solution.

Theorem 2: The objective function of a linear programming problem assumes its optimal value at an extreme point of the convex set of all feasible solution.

Theorem 3: A basic feasible solution to a linear programming problem corresponds to an extreme point of the convex set of all feasible solution.

Theorem 4: Every extreme point of the convex set of all feasible solution of the system (A1): $Ax = b$, $x \geq 0$ corresponds to a basic feasible solution.

Fundamental theorem ~~on~~ of LPP :
consider the linear programming problem in its standard form (A1) :

$$\text{Max } z = cx$$
$$\text{Subject to } Ax = b, \; x \geq 0 .$$

A is an $m \times n$ matrix and given by
$$A = [a_1 \; a_2 \; \cdots \; a_n]$$
where a_j is an m component column vector given by
$$a_j = [a_{1j}, a_{2j}, \cdots a_{mj}], \; j = 1, 2, \cdots n.$$

The fundamental theorem states that
If the linear programming problem admits of an optimal solution, then the optimal solution will coincide with atleast one basic feasible solution of the system.

Theorem: If there be a ~~basic~~ feasible solution to a set of m simultaneous equations $Ax = b$, $x \geq 0$ in n unknowns ($n > m$) and if $r(A) = m$, then there is a basic feasible solution to the set.

Optimality condition: (i) If for a basic feasible solution x_B of a linear programming problem
$$\text{Max } z = cx$$
$$\text{S.t } Ax = b, \; x \geq 0,$$
we have $z_j - c_j \geq 0$ for every column a_j of A, then x_B is an optimal solution.

(ii) If for a basic feasible sol. x_B of
$$\text{Min } z = cx$$
$$\text{S.t } Ax = b, \; x \geq 0$$
we have $z_j - c_j \leq 0$ for every column a_j of A, then x_B is optimal sol.