Lower and Upper Value of Game –
Minimax principle with pure strategies

Consider the payoff matrix $(a_{ij})_{m\times n}$ of player A. If player A chooses strategy A_i, then he is sure to get $\min_j a_{ij}$, $j = 1, 2, .. n$. Thus he would like to choose that strategy A_i ($i = 1, 2, .. m$) for which $\min_j a_{ij}$ is maximized to get $\max_i(\min_j a_{ij})$. It is denoted by $\underline{a}$ and is called the lower value of the game.

$$\text{Thus } \max_i \left(\min_j a_{ij}\right) = \underline{a} \quad \text{---}(1)$$

On the other hand, if player B chooses strategy B_j, then he/she is sure that the player A will not get more than $\max_i a_{ij}$. Thus B would like to choose the strategy B_j which minimizes the maximum gain to player A, reducing it to $\min_j(\max_i a_{ij})$. It is denoted by $\bar{a}$ and is called the upper value of the game

$$\text{Thus } \min_j \left(\max_i a_{ij}\right) = \bar{a} \quad \text{---}(2)$$

Equations (1) & (2) are called the maximin and minimax criteria, respectively.

If $\underline{a} = \bar{a} = \alpha$ (say), then the game is said to have saddle point and α is the game value. If $\alpha = a_{ij}$, then we conclude that optimal ~~strateg~~ strategy of player A is A_i and that of B is B_j.

Note → A game is said to be fair if both of lower and upper values of the game are equal to zero.

Theorem: If $\underline{a} = \max_i (\min_j a_{ij})$ and $\bar{a} = \min_j (\max_i a_{ij})$ are the lower and upper values of the game resp, then the lower value is always less than or equals to upper value of the game, i.e. $\underline{a} = \max_i (\min_j a_{ij}) \leq \min_j (\max_i a_{ij}) = \bar{a}$

Procedure to determine saddle point:

① Choose the minimum element of each row i (α_i's) of the pay off matrix and write it on the extreme right of that row i.

② Choose the maximum element of each column j (β_j's) of the pay off matrix and write it against the column j.

③ If maximum of α_i's are equal to minimum of β_j's then the common value is the game value and thus we can conclude that the saddle point exists, Otherwise the saddle point does not exist in pure strategies.

(4) When the saddle point does not exist, use mixed strategies to determine the value of the game.

Ex 1. Find the lower and upper values of a game for the following game matrix (pay off matrix for player A). Determine whether the saddle point exists:

player B

$$\text{player A} \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}$$

Sol^n Let us draw the pay off matrix for player A

A \ B	B_1	B_2	α_i
A_1	0	2	0
A_2	-1	4	-1
β_j	0	4	

α_i = minimum value of ith row

β_j = maximum value of jth Column.

$\text{Max}\{\alpha_i\} = 0$ $\min\{\beta_j\} = 0$.

While player A uses maxmin strategy, player B uses minmax Strategy. Both the lower and upper value of game is 0. So the saddle point exist.

Matrix Reduction by Dominance Principle:

Sometimes the size of game's pay off matrix can be reduced by eleminating an inferior course of action among those available, such that the inferior one is never used. Such a course of action is said to be dominated by the other course of action. The other course of action is called the dominating course of action. The principle of reducing of a pay off matrix is called the principle of dominance. This concept is useful for evaluation of two person zero sum games, where a saddle point does not exist.

General rules for dominance

(i) If all the elements of the ith row be less than or equals to the corresponding elements of any other row, say the rth, then rth row dominates ith row, and we discard ith row.

(II) If all the elements of the jth column be greater than or equals to the corresponding elements of any other column, say pth, then the pth column is dominated by jth column and we discard jth column.

In case of row player, the inferior row is discarded while in case of column player, the dominating column is discarded.

(III) If the ith row be dominated by a convex combination of other rows, then the ith row is deleted from the pay off matrix.

(IV) When each entry in the convex linear combination of certain number of pure strategies of player B ~~is less than or equals to~~ (i.e. column in pay off matrix) is less than or equal to corresponding entries of B's jth strategi then jth column is said to be inferior / dominated column and we discard ~~it~~ jth column from pay off matrix.

Ex 2. Reduce the following pay off matrix using the principle of dominance.

$$\begin{array}{c|cccc|} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 2 & 3 & 11 & 8 \\ A_2 & 7 & 5 & 2 & 7 \\ A_3 & 6 & 4 & -4 & -9 \\ \hline \end{array}$$

→ Note that each entry in A_2 is greater than corresponding entry in A_3. Using principle of dominance A_3 will be removed to get truncated matrix.

$$\begin{array}{c|cccc|} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 2 & 3 & 11 & 8 \\ A_2 & 7 & 5 & 2 & 7 \\ \hline \end{array}$$

Further convex linear combination of columns of B_1 and B_2 is ~~dom~~ dominated by column B_4. So, B_4 can be removed

$$\begin{array}{c|ccc|} & B_1 & B_2 & B_3 \\ \hline A_1 & 2 & 3 & 11 \\ A_2 & 7 & 5 & 2 \\ \hline \end{array}$$

This is the final truncated matrix.

Two person zero sum game with mixed strategies or linear programming method

Every two person zero sum game can be solved through mixed strategies.

Let us assume that the player A selects the strategy A_i with probability p_i and the player B selects strategy B_j with the probability q_j, $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$. A_i and B_j can be considered events with probabilities p_i and q_j i.e. $P(A_i) = p_i$, $P(B_j) = q_j$

$$
\begin{array}{cc|ccccc|}
 & & q_1 & q_2 \ldots & q_j & \ldots & q_n \\
 & & B_1 & B_2 \ldots & B_j & \ldots & B_n \\
\hline
p_1 & A_1 & a_{11} & a_{12} \ldots & a_{1j} & \ldots & a_{1n} \\
p_2 & A_2 & a_{21} & a_{22} & a_{2j} & \ldots & a_{2n} \\
\vdots & \vdots & \vdots & & & & \\
p_i & A_i & a_{i1} & a_{i2} & a_{ij} & \ldots & a_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
p_m & A_m & a_{m1} & a_{m2} & a_{mj} & & a_{mn} \\
\hline
\end{array}
$$

Payoff matrix for player A in mixed strategies

To obtain the value of the game, we have to determine the value of p_i's and q_j's. We do this by using minimax or maximin criteria.

If B selects pure stragy B_j, then the expected pay off to player A is to find

$$a_{1j} p_1 + a_{2j} p_2 + \cdots + a_{mj} p_j = \sum_{i=1}^{m} a_{ij} p_i$$

Hence player A would like to select p_i's in such a way that it maximizes its smallest pay off. Thus A's problem becomes

$$\underset{p_1, p_2 \cdots p_m}{\text{Max}} \left[\text{Min} \left\{ \sum_{i=1}^{m} a_{i1} p_i , \sum_{i=1}^{m} a_{i2} p_i , \cdots \sum_{i=1}^{m} a_{in} p_i \right\} \right] \quad \text{—— (1)}$$

subject to condition $\sum_{i=1}^{m} p_i = 1$, p_i's ≥ 0 $\forall i$.

Similarly player B would like to select q_j's which minimize the largest expected pay off to A.

$$\text{Thus } \underset{q_1, q_2, \cdots q_n}{\text{Min}} \left[\text{Max} \left\{ \sum_{j=1}^{n} a_{1j} q_j , \sum_{j=1}^{n} a_{2j} q_j \cdots \sum_{j=1}^{n} a_{mj} q_j \right\} \right] \quad \text{—— (2)}$$

Subject to the condⁿ $\sum_{j=1}^{n} q_j = 1$, q_j's ≥ 0 $\forall j$

The value in (1) and (2) are maximin and minimax expected pay off to A. ~~and~~ Let these be denoted by $\underline{a}$ and ~~B resp~~. $\overline{a}$ resp.

$$\underline{a} = \text{Min} \left\{ \sum_{i=1}^{m} a_{i1} p_i , \sum_{i=1}^{m} a_{i2} p_i , \cdots , \sum_{i=1}^{m} a_{in} p_i \right\}$$

Thus $\quad \text{Max } Z_A = \underline{a}$

s.t. $\sum_{i=1}^{m} a_{ij} p_i \geq \underline{a}$, $j = 1, 2 \cdots n$

$\sum p_i = 1$, p_i's ≥ 0.

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W.l.o.g let $a_A > 0$. Dividing the constraint by a ~~A~~ and let $x_i = p_i/a$, then

$$\text{Max } Z_A = a$$

$$\text{s.t. } \sum_{i=1}^{m} a_{ij} x_i \geq 1, \quad j = 1, 2, \ldots n$$

$$\text{Further } \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \frac{p_i}{a} = \frac{1}{a_A} \sum_{i=1}^{m} p_i = \frac{1}{a}$$

Ex

$$\text{Thus } \sum_{i=1}^{m} x_i = x_1 + x_2 + \cdots + x_m = \frac{1}{a}.$$

Max $Z_A = a$ is equivalent to $\min Z_A' = \frac{1}{a}$

Thus A's problem can be written

$$\left.\begin{aligned} &\text{Min } Z_A' = x_1 + x_2 + \cdots + x_m \\ &\text{s.t. } \sum_{i=1}^{m} a_{ij} x_i \geq 1. \\ &\qquad x_i \geq 0, \quad i = 1, 2, \ldots m. \end{aligned}\right\} (3)$$

Similarly for B's problem, let $\bar{a} > 0$.

$$\left.\begin{aligned} &\text{Max } Z_B = y_1 + y_2 + \cdots + y_n \\ &\text{s.t. } \sum_{j=1}^{n} a_{ij} y_j \leq 1, \quad i = 1, 2, \ldots m \\ &\qquad y_j \geq 0, \quad j = 1, 2, \ldots n \end{aligned}\right\} (4)$$

Ex. Solve the following game using method of LPP

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \\ & \quad y_1 \quad\ \ y_2 \end{array}$$

→

		A \ B	B_1 (q_1, y_1)	B_2 (q_2, y_2)	α
x_1	p_1	A_1	4	-1	-1
x_2	p_2	A_2	3	5	3
		β	4	5	

Lower value of game $\underline{\alpha} = \max(-1, 3) = 3$

Upper value of game $\bar{\alpha} = \min(4, 5) = 4$

Therefore value of game lies between 3 and 4.

Let us assume that player A uses strategies A_i's with probabilities p_i, $\sum p_i = 1$, $i = 1, 2$

Player B uses strategies B_j, with probability q_j, $\sum q_j = 1$.

Player B's problem is given as

$$\text{Max } Z_B = y_1 + y_2 = \frac{1}{V}$$

s.t.

$$4y_1 - y_2 \le 1$$

$$3y_1 + 5y_2 \le 1,$$

$$y_j \ge 0, \quad y_j = q_j / V.$$

Introducing slack variables y_3, y_4,

$$\text{Max } z_B = y_1 + y_2 + 0 \cdot y_3 + 0 \cdot y_4 = \frac{1}{V}$$

$$\text{s.t.} \quad 4y_1 - y_2 + y_3 = 1$$

$$3y_1 + 5y_2 + y_4 = 1.$$

$$y_j \geq 0.$$

	c_j		1	1	0	0	
C_B	x_B	b	y_1	y_2	y_3	y_4	Min ratio
0	y_3	1	4	-1	1	0	--
0	y_4	1	3	$\boxed{5}$	0	1	$1/5 \rightarrow$
	$z_j - c_j$		-1	$-1 \uparrow$	0	0	
0	y_3	$6/5$	$3/5$	0	1	$1/5$	$6/3 = 2$
1	y_2	$1/5$	$\boxed{3/5}$	1	0	$1/5$	$1/3 \rightarrow$
	$z_j - c_j$		$-2/5 \uparrow$	0	0	$1/5$	
0	y_3	1	0	-1	1	0	
1	y_1	$1/3$	1	$5/3$	0	$1/3$	
	$z_j - c_j$		0	$2/3$	0	$1/3$	

$$z_j - c_j \geq 0.$$

. Optimal value $= y_1 + y_2 = \frac{1}{3} = \frac{1}{V}$

$$V = 3.$$

$$y_j = q_j / V \Rightarrow q_1 = 3 \times \frac{1}{3} = 1$$

$$q_2 = 0.$$

Ai's best strategies appear in $Z_j - C_j$ column under slack variables y_3 and y_4.

Thus $x_1 = 0$, $x_2 = \frac{1}{3}$.

$x_i = p_i / V$, $p_1 = 0$, $p_2 = 1$

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