Example 3.0.1

If matrix of a linear transform on R^3 relative to basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$. Then find the linear transform matrix T relative to basis $B_1 = \{(0,1,-1),(1,-1,1),(-1,1,0)\}$.

First we find linear transform.
We have

$$[T:B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}^t$$

That is transpose of coefficient matrix. So that

$$T(u_1) = T(1,0,0) = 0(1,0,0) + 1(0,1,0) - 1(0,0,1) = (0,1,-1)$$
$$T(u_2) = T(0,1,0) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1) = (1,0,-1)$$
$$T(u_3) = T(0,0,1) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1) = (1,-1,0)$$

$(x,y,z) \in \mathbb{R}^3$ be any element and B is basis for $\mathbb{R}^3$

$$\therefore (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$
\begin{aligned}
\therefore T(x,y,z) &= xT(1,0,0) + yT(0,1,0) + zT(0,0,1)\\
&= x(0,1,-1) + y(1,0,-1) + z(1,-1,0)\\
T(x,y,z) &= (y+z, x-z, -x-y)
\end{aligned}
$$

which is linear operator T on R^3.
Now we have to find a matrix of T relative basis.

$$B_1 = \{(0,1,-1),(1,-1,1),(-1,1,0)\}$$

Let $(a,b,c) \in \mathbb{R}^3$ be any element
Let

$$
\begin{aligned}
(a,b,c) &= l(0,1,-1) + m(1,-1,1) + n(-1,1,0)\\
(a,b,c) &= (m-n, l-m+n, -l+m)\\
&\Rightarrow a = m-n;\ b = l-m+n;\ c = -l+m
\end{aligned}
$$

Now

$$\begin{array}{c|c|c} l - m + n = b & l - m + n = b & m = a + n \\ l = b + m - n = b + a & n = b - l + m & m = a + b + c \\ l = a + b & n = b + c & \end{array}$$

$$B_1 = \{(0, 1 - 1), (1, -1, 1), (-1, 1, 0)\}$$

$\therefore$ we get

$$(a, b, c) = (a + b)(0, 1, -1) + (a + b + c)(1, -1, 1) + (b + c)(-1, 1, 0)$$

and we have

$$T(x, y, z) = (y + z, x - z, -x - y)$$

Now

$$T(0, 1, -1) = (0, 1, -1) = 1(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0)$$
$$T(1, -1, 1) = (0, 0, 0) = 0(0, 1, -1) + 0(1, -1, 1) + 0(-1, 1, 0)$$
$$T(-1, 1, 0) = (1, -1, 0) = 0(0, 1, -1) + 0(1, -1, 1) + (-1)(-1, 1, 0)$$

$$\therefore [T; B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Example 3.0.2

Let T be linear transform on $\mathbb{R}^2$ and $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ be matrix of T with respect to usual basis of $\mathbb{R}^2$. Then, find that matrix of T with respect to $B_1 = \{(1,2),(5,6)\}$.

Ans:

$$[T : B_1] = \begin{bmatrix} \frac{11}{2} & \frac{41}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Definition 3.1.1 (Isomorphism of a vector space)

Let $U(F)$ and $V(F)$ are two vector spaces then a linear transformation $f : U \to V$ is called Isomorphism, if

1. f is one-one
2. f is onto

Definition 3.1.2 (Isomorphism of a vector space)

$f : U \to V$ is called Isomorphism if

1. f is a linera transform
2. f is one-one
3. f is onto

Problem 3.1.3

Let $f : V_2(\mathbb{R}) \to V_2(\mathbb{R})$ be $f(x, y) = (y, x)$. Prove f is Isomorpism.

To prove one-one:
Let $U, V \in V_2(\mathbb{R})$

$$\begin{aligned} f(u) &= f(v) \\ f(x, y) &= f(p, q) \\ (y, x) &= (q, p) \\ y &= q; x = p \\ (x, y) &= (p, q) \\ u &= v \end{aligned}$$

i.e., f is one-one

To prove onto:
$\forall (x, y) \in V_2(\mathbb{R})$
$\exists (y, x) \in V_2(\mathbb{R})$ such that $f(x, y) = (y, x)$

To prove linear transform:
Let $u, v \in V_2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} f(\alpha u + \beta v) &= f[\alpha(x,y) + \beta(p,q)] \\ &= f[\alpha x + \beta p, \alpha y + \beta q] \\ &= (\alpha y + \beta q, \alpha x + \beta p) \\ &= \alpha(y,x) + \beta(q,p) \\ &= \alpha f(x,y) + \beta(p,q) \\ &= \alpha f(u) + \beta f(v) \end{aligned}$$

So, f is a linear transform, one-one, onto.
i.e., f is an Isomorpism.

Problem 3.1.4

Let $T : P_2 \to V_3 \to \{(x_1, x_2, x_3) | x_i \in \mathbb{R}\}$ *(*P_2*-set of all polynomials of degree* ≤ 2*)* $\{a_0 + a_1x + a_2x^2 | a_0, a_1, a_2 \in \mathbb{R}\}$*. Prove that* T *is Isomorphism.* $T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2)$

To prove T is one-one:

$$\begin{aligned}
T(p_1) &= T(p_2) \\
T(a_0 + a_1x + a_2x^2) &= T(b_0, b_1x + b_2x^2) \\
(a_0, a_1, a_2) &= (b_0, b_1, b_2) \\
a_0 = b_0; a_1 &= b_1; a_2 = b_2 \\
a_0 + a_1x + a_2x^2 &= b_0 + b_1x + b_2x^2 \\
p_1(x) &= p_2(x) \\
p_1 &= p_2
\end{aligned}$$

T is one-one.

To prove T is onto:

$T : p_2 \to v_3$. For every $(a_0, a_1, a_2) \in v_3$ we have a polynomial $p = a_0 + a_1 x + a_2 x^2$ in p_2. Such that

$$T(p) = (a_0, a_1, a_2)$$

T is onto.

T is one-one and onto.

To prove T is linear.

$$\begin{aligned}
&T(\alpha(a_0 + a_1x + a_2x^2) + \beta(b_0 + b_1x + b_2x^2)) \\
&= T((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2) \\
&= (\alpha a_0 + \beta b_0, \alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2) \\
&= (\alpha a_0, \alpha a_1, \alpha a_2) + (\beta b_0, \beta b_1, \beta b_2) \\
&= \alpha(a_0, a_1, a_2) + \beta(b_0, b_1, b_2) \\
T(\alpha p_1 + \beta p_2) &= \alpha T(p_1(x)) + \beta T(p_2(x))
\end{aligned}$$

This proves T is linear.
$\therefore$ T is an isomorphic.
To find its inverse:

$$T^{-1} : v_3 \to p_2$$
$$T^{-1}(a_0, a_1, a_2) = a_0 + a_1x + a_2x^2$$

Example 3.1.5

$T : v_2 \to v_2 \; T(x_1, x_2) = (x_1, -x_2)$

Definition 3.2.1 (Matrices of linear transformations)

We will now take a more algebraic approach to transformations of the plane. As it turns out, matrices are very useful for describing transformations. Whenever we have a 2×2 matrix of real numbers

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we can naturally define a plane transformatiom $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_M(v) = Mv.$$

That is, T_M takes a vector v and multiplies it on the left by the matrix M. If v is the position vector of the point (x, y), then

$$T_M(v) = T_M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

or equivalently, $T_M(x, y) = (ax + by, cx + dy)$.

Problem 3.2.2

Let

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

1. *Write an expression for* T_M.
2. *Find* $T_M(1,0)$ *and* $T_M(0,1)$.
3. *Find all points* (x, y) *such that* $T_M(x, y) = (1, 0)$.

(1) $T_M(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 7y \end{bmatrix} = (x + 2y, 3x + 7y)$.

(2) Using the formula from the previous part,

$$T_M(1,0) = (1,3) \text{ and } T_M(0,1) = (2,7).$$

(3) We have $T_M(x,y) = (x+2y, 3x+7y) = (1,0)$, hence the simultaneous equations

$$x + 2y = 1, 3x + 7y = 0.$$

Solving these equations yields $x = 7, y = -3$; and this is the only solution. So the only point (x,y) such that $T_M(x,y) = (1,0)$ is $(x,y) = (7,-3)$.

Definition 3.2.3 (Linear transformation)

A plane transformation F is linear if either of the following equivalent conditions holds:

1. $F(x,y) = (ax+by, cx+dy)$ for some real a, b, c, d. That is, F arises from a matrix.
2. For any scalar c and vectors v, w, $F(cv) = cF(v)$ and $F(v+w) = F(v) + F(w)$.

Theorem 3.2.4

For any matrices M and N, $T_M \circ T_N = T_{MN}$.

Problem 3.2.5

Find the matrix for the composition $g \circ f$ of the two linear transformations $f(x,y) = (x+y, y)$ and $g(x,y) = (y, x+y)$.

We have $f = T_M$ and $g = T_N$ where $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. So the matrix of the composition $g \circ f = T_N \circ T_M = T_{NM}$ is the product NM :

$$NM = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Problem 3.2.6

What is the inverse of the transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x + 3y, x + 5y)$?

The transformation F is linear and corresponds to the matrix

$$M = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix},$$

which has inverse

$$M^{-1} = \frac{1}{1.5 - 3.1} \begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{-3}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

The inverse of $F = T_M$ is then $F^{-1} = T_{M^{-1}}$,

$$F^{-1}(x, y) = \left(\frac{5}{2}x - \frac{3}{2}y, \frac{-1}{2}x + \frac{1}{2}y \right).$$

Problem 3.2.7

Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ *that maps* $(1,1)$ *to* $(-1,4)$ *and* $(-1,3)$ *to* $(-7,0)$.

Let M be the matrix of the desired linear transformation. We have

$$M\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix} \text{ and } M\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-7\\0\end{bmatrix}.$$

In fact, we can put these two equations together into a single matrix equation

$$M\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix} = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}$$

which we can then solve for M:

$$M = \begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix}-1 & -7\\4 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4 & -8\\12 & 4\end{bmatrix} = \begin{bmatrix}1 & \\3 & \end{bmatrix}$$

Hence the only such transformation is $T_M(x,y) = (x-2y, 3x+y)$.

Example 3.2.8

Find the linear transformation that sends $(3, 1)$ to $(1, 2)$ and $(-1, 2)$ to $(2, -3)$.

Definition 3.3.1 (Similar matrices)

Let A and B be two square matrices of same order, A is said to be similar to matrix B if there exists a non-singular matrix P, such that

$$B = P^{-1}AP$$

Definition 3.3.2 (Properties of similar matrices)

Similar matrices have same eigen values, eigen vectors, determinant, ranks, nullity, characteristic polynomial and traces.

Definition 3.3.3 (Procedure to find similar matrix)

If A is given

- Step I Characteristic polynomial $A - \lambda I$ by using $|A - \lambda I| = 0$.
- Step II Find eigen values $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$.
- Step III Find eigen vector $v_1, v_2, v_3, \cdots, \lambda_n$ using eigen values.
- Step IV Find P by combining all eigen values into one matrix.
- Step V Find P^{-1} from P.
- Step VI Find $B = P^{-1}AP$

B is called similar matrix.

Problem 3.3.4

Find similar matrix for $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$.

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ and λ be eigen value of A then

$$\begin{aligned}
A - \lambda I &= \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \\
|A - \lambda I| &= (2-\lambda)(-2-\lambda) - (-1)(3) \\
&= -4 - 2\lambda + 2\lambda + \lambda^2 + 3 \\
&= \lambda^2 - 1
\end{aligned}$$

To find eigen values:

$$
\begin{aligned}
|A - \lambda I| &= 0 \\
\lambda^2 - 1 &= 0 \\
\lambda^2 &= 1 \\
\lambda &= \pm 1
\end{aligned}
$$

Therefore eigen values are $\lambda_1 = 1$, $\lambda_2 = -1$

To find eigen vectors:

$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

At $\lambda = 1$

$$\begin{bmatrix} 2-1 & 3 \\ -1 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ x_1 &= -3x_2 \\ \text{Let } x_2 &= t \\ \text{then, } x_1 &= -3t \\ v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix} \end{aligned}$$

At $\lambda = -1$

$$(A - \lambda I)v = 0$$
$$\begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+1 & 3 \\ -1 & -2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 3x_2 = 0$$
$$x_1 + x_2 = 0$$
$$x_1 = -x_2$$
$$\text{Let } x_2 = -t$$
$$x_1 = -t$$
$$v_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

To find P matrix,

$$P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$|P| = (-3)(1) - (1)(-1) = -3 + 1 = -2 \neq 0.$$

P is non-singular.

$$P^{-1} = \frac{1}{|A|} Adj(P) = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

$$P^{-1}AP = \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} -6+3 & -2+3 \\ 3-2 & 1-2 \end{pmatrix}$$

$$= \frac{-1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -3+1 & 1+1 \\ 3-3 & -1+3 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To check

$$
\begin{aligned}
|B-\lambda I| &= 0 \\
\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} &= 0 \\
(1-\lambda)(-1-\lambda)-0 &= 0 \\
-1-\lambda+\lambda+\lambda^2 &= 0 \\
\lambda^2 &= 1 \\
\lambda &= \pm 1
\end{aligned}
$$

1. Eigen values of B matrix are similar to A matrix. So, $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
2. $|A| = |B| = -1$
3. $Trace(A) = Trace(B)$ i.e., $2-2=0=1-1$

Example 3.3.5

Find similar matrix for the following matrix

1. $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Problem 3.3.6

Find similar matrix for $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Find Matrix Eigenvalues ...

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} (1-\lambda) & -2 & 0 \\ 0 & (2-\lambda) & 0 \\ 0 & 0 & (-2-\lambda) \end{bmatrix} = 0$$

$$\begin{aligned}
(1-\lambda)((2-\lambda)\times(-2-\lambda) - 0\times 0) - (-2)(0\times(-2-\lambda) - 0\times 0) + 0(0\times 0 - (2-\lambda)\times 0) &= 0 \\
(1-\lambda)((-4+\lambda 2) - 0) + 2(0-0) + 0(0-0) &= 0 \\
(1-\lambda)(-4+\lambda 2) + 2(0) + 0(0) &= 0 \\
(-4+4\lambda+\lambda 2-\lambda 3) + 0 + 0 &= 0 \\
(-\lambda 3+\lambda 2+4\lambda-4) &= 0 \\
-(\lambda-1)(\lambda-2)(\lambda+2) &= 0 \\
(\lambda-1) = 0 or (\lambda-2) = 0 or (\lambda+2) &= 0
\end{aligned}$$

$\therefore$ The eigenvalues of the matrix A are given by $\lambda = -2, 1, 2,$

Eigen vector for $\lambda = -2$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector for $\lambda = 1$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector for $\lambda = 2$

$$v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} adj(P)$$

To find $|P|$:

$$\begin{aligned}
|A| &= \begin{vmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&= 0 \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2 \times \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\
&= 0 \times (0 \times 0 - 1 \times 0) - 1 \times (0 \times 0 - 1 \times 1) - 2 \times (0 \times 0 - 0 \times 1) \\
&= 0 \times (0 + 0) - 1 \times (0 - 1) - 2 \times (0 + 0) \\
&= 0 \times (0) - 1 \times (-1) - 2 \times (0) \\
&= 0 + 1 + 0 \\
&= 1
\end{aligned}$$

To find adjoint of P

$$adj(P) = adj \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$adj(P) = \begin{bmatrix} +(0\times 0 - 1\times 0) & -(0\times 0 - 1\times 1) & +(0\times 0 - 0\times 1) \\ -(1\times 0 - (-2)\times 0) & +(0\times 0 - (-2)\times 1) & -(0\times 0 - 1\times 1) \\ +(1\times 1 - (-2)\times 0) & -(0\times 1 - (-2)\times 0) & +(0\times 0 - 1\times 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} +(0+0) & -(0-1) & +(0+0) \\ -(0+0) & +(0+2) & -(0-1) \\ +(1+0) & -(0+0) & +(0+0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} adj(P) = \frac{1}{1} \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since,

$$B = P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To verify the solution:

$$trace(A) = trace\begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = 1 + 2 + (-2) = 1$$

$$trace(B) = trace\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (-2) + 1 + 2 = 1$$

Eigen values of A=$-2, 1, 2$
Eigen values of $B = -2, 1, 2$

$$
|A| = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 1 \times \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} - (-2) \times \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} + 0 \times \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}
$$

$$
\begin{aligned}
&= 1 \times (2 \times (-2) - 0 \times 0) + 2 \times (0 \times (-2) - 0 \times 0) + 0 \times (0 \times 0 - 2 \times 0 \\
&= 1 \times (-4 + 0) + 2 \times (0 + 0) + 0 \times (0 + 0) \\
&= 1 \times (-4) + 2 \times (0) + 0 \times (0) \\
&= -4 + 0 + 0 = -4
\end{aligned}
$$

$$
|B| = \begin{vmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -2 \times \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} + 0 \times \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}
$$

$$
\begin{aligned}
&= -2 \times (1 \times 2 - 0 \times 0) + 0 \times (0 \times 2 - 0 \times 0) + 0 \times (0 \times 0 - 1 \times 0) \\
&= -2 \times (2 + 0) + 0 \times (0 + 0) + 0 \times (0 + 0) \\
&= -2 \times (2) + 0 \times (0) + 0 \times (0) \\
&= -4 + 0 + 0 = -4
\end{aligned}
$$