Definition 4.0.1 (Inner product)

An inner product on V is a map

$$\langle ., . \rangle : V \times V \to \mathbb{F}$$
$$(u, v) \to \langle u, v \rangle$$

with the following four properties.

Linearity in first slot: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and $\langle au, v \rangle = a \langle u, v \rangle$ for all $u, v, w \in V$ and $a \in F$;

Positivity: $\langle v, v \rangle \geq 0$ for all $v \in V$;

Positive definiteness: $\langle v, v \rangle = 0$ if and only if $v = 0$;

Conjugate symmetry: $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$.

Definition 4.0.2 (Inner product space)

An inner product space is a vector space over F together with an inner product $\langle ., . \rangle$.

Definition 4.0.3 (Inner product space)

Let $V(F)$ be a vector space where F is either the field of real numbers or the field of complex numbers. An inner product space on V is a function from $V \times V$ into F which assigns to each ordered pair of vectors α, β in V a scalar (α, β) in a such way that:

(1) Conjugate symmetry:

$$(\alpha, \beta) = \overline{(\beta, \alpha)}, \forall \alpha, \beta \in \mathbb{V}.$$

(2) Linearity:

$$[a\alpha + b\beta]x = a\alpha(x) + b\beta(x), \forall \alpha, \beta, x \in \mathbb{V}, a, b \in \mathbb{F}$$

(3) Non-negativity:

$$(\alpha, \alpha) \geq 0$$

and

$$(\alpha, \alpha) = 0 \Rightarrow \alpha = 0, \forall \alpha \in \mathbb{V}.$$

Also the vector space V is then said to be an inner product space with respect to the specified inner product defined on it.

Problem 4.0.4

If $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2) \in v_2\mathbb{R}$, *let us define*

$$(\alpha, \beta) = a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2 \tag{11}$$

we shall show that all the postulates of an inner product hold in (11).

(1) Symmetry [Conjugate symmetry]:

$$\begin{aligned}
(\alpha, \beta) &= (\beta, \alpha) \\
(\beta, \alpha) &= b_1a_1 - b_2a_1 - b_1a_2 + 4b_2a_2 \\
&= a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2 \\
&= (\alpha, \beta)
\end{aligned}$$

Hence symmetry is exist.

(2) Linearty:

If $a, b \in \mathbb{R}$

$$\begin{aligned} a\alpha + b\beta &= a(a_1, a_2) + b(b_1, b_2) \\ &= (aa_1, aa_2) + (bb_1, bb_2) \\ &= (aa_1 + bb_1, aa_2 + bb_2) \end{aligned}$$

Let $\gamma = (c_1, c_2) \in V_2\mathbb{R}$, then

$$\begin{aligned} &((a\alpha + b\beta)\gamma) \\ &= [(aa_1 + bb_1, aa_2 + bb_2)(c_1, c_2)] \\ &= (aa_1 + bb_1)c_1 - (aa_2 + bb_2)c_1 - (aa_1 + bb_1)c_2 + 4(aa_2 + bb_2)c_2 \\ &= [aa_1c_1 - aa_2c_1 - aa_1c_2 + 4aa_2c_2] + [bb_1c_1 - bb_2c_1 - bb_1c_2 + 4bb_2c_2] \\ &= a(a_1c_1 - a_2c_1 - a_1c_2 + 4a_2c_2) + b(b_1c_1 - b_2c_1 - b_1c_2 + 4b_2c_2) \\ &= a(\alpha, \gamma) + b(\beta, \gamma) \end{aligned}$$

Hence, linearity satisfied.

(3) Non-negativity:
We have

$$\begin{aligned}(\alpha,\alpha) &= [(a_1a_2).(a_1a_2)] = a_1a_1 - a_2a_1 - a_1a_2 + 4a_2a_2\\ &= a_1^2 - 2a_1a_2 + 4a_2^2\\ &= a_1^2 - 2a_1a_2 + a_1^2 + 3a_2^2\\ &= (a_1-a_2)^2 + 3a_2^2 \end{aligned} \tag{12}$$

It is a sum of two non-negative real numbers. Therefore it is ≥ 0.
Thus $(\alpha,\alpha) \geq 0$. Also,

$$\begin{aligned}(\alpha,\alpha) &= 0\\ (a_1-a_2)^2 + 3a_2^2 &= 0\\ (a_1-a_2)^2 &= 0, & 3a_2^2 &= 0\\ a_1 - a_2 &= 0 & a_2 &= 0\\ a_1 &= a_2. & a_2 &= 0\\ a_1 = 0,\ a_2 &= 0 & \alpha &= 0\end{aligned}$$

$\therefore$ all the postulates are satisfied. Hence, it is an inner product.

Problem 4.0.5

Show that $V_n(C)$ is an inner product space with inner product define on $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n) \in V_n(C)$ by $(\alpha, \beta) = a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n}$ which is standard inner product on $V_n(F)$

Let $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$ and $\gamma = (c_1 c_2, \cdots, c_n) \in V_n(F)$

(1) Non-negativity:

$$(\alpha, \alpha) = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 \geq 0, \text{ Since } |a_1|^2 \geq 0$$
$$(\alpha, \alpha) = 0 \Leftrightarrow |a_1|^2 + ||a_2|^2 + |a_3|^2 + \cdots + |a_n|^2 = 0$$
$$\Rightarrow \text{ each } a_i \quad = 0 \Rightarrow \alpha = 0$$

(2) Conjucate symmetry:

$$
\begin{aligned}
(\alpha, \beta) &= a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n} \\
\overline{(\beta, \gamma)} &= \overline{b_1\overline{a_1} + b_2\overline{a_2} + \cdots + b_n\overline{a_n}} \\
&= \overline{b_1\overline{a_1}} + \overline{b_2\overline{a_2}} + \cdots + \overline{b_n\overline{a_n}} \\
&= a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n} \\
(\alpha, \beta) &= \overline{(\beta, \gamma)}
\end{aligned}
$$

(3) Linear:

$$
\begin{aligned}
a\alpha + b\beta &= a(a_1, a_2, \cdots, a_n) + b(b_1, b_2, \cdots, b_n) \\
&= (aa_1 + bb_1, aa_2 + bb_2, \cdots, aa_n + bb_n)
\end{aligned}
$$

Now,

$$
\begin{aligned}
(a\alpha + b\beta, \gamma) &= (aa_1 + bb_1)\overline{c_1} + (aa_2 + bb_2)\overline{c_2} + \cdots + (aa_n + bb_n)\overline{c_n} \\
&= a(a_1\overline{c_1} + a_2\overline{c_2} + \cdots + a_n\overline{c_n}) + b(b_1\overline{c_1} + b_2\overline{c_2} + \cdots + b_n\overline{c_n}) \\
&= a(\alpha, \gamma) + b(\beta, \gamma)
\end{aligned}
$$

Here inner product define by α, β and γ satisfies all three condition. So $V_n(C)$ is inner product space.