The three IMPORTANT quantities of Vector Calculus: Gradient, Divergence and Curl

Recall the vector differential operator (del)$\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$.Using this operator, three quantities can be defined.

Quantity Defined	Nature of Input	Nature of Output	Definition
Gradient of f denoted by grad*f* defined by ∇f	Scalar function	Vector Function	$\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$
Divergent of **F** denoted by divF defined by $\nabla.\mathbf{F}$	Vector Function	Scalar function	$\nabla.\mathbf{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\bullet(iF_1 + jF_2 + kF_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
Curl of **F** denoted by Curl **F** defined by $\nabla \text{x}\mathbf{F}$	Vector Function	Vector function	$\nabla \text{x}\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

Physical Meaning of grad *f*, Div F, Curl F:

Gradient: Consider a scalar function, say f(x,y,z), denoting the temperature at various points (x,y,z) of a room. Let f(x,y,z) has continuous first order partial derivatives. Then grad*f* evaluated at a particular point P, has the direction of maximum increase of the function at that point P. If we visualize the surface obtained by setting f(x,y,z) = a constant = say C(of course a level surface of this scalar function), then grad*f* evaluated at a particular point lying on the surface (i.e. at a point satisfying f(x,y,z)=C), gives the normal vector of the surface at that point.

Defn: Given a scalar function f(x,y,z),and a direction say **b,** then the directional derivative of f(x,y,z), in the direction of **b**, denoted by $D_{\mathbf{b}}f$ is defined by $D_{\mathbf{b}}f = \text{grad}f \bullet \frac{b}{|b|}$**.** This gives the rate of change of *f* at P(x,y,z) in the direction of **b.** The rate of change will be maximum if we move along the direction of grad*f* itself.

Divergence: Consider water in a pool. In the pool there are some sources of water (e.g. taps), and some "sinks" of water (e.g. cracks) and some barriers to water movement (e.g. a big stone). The resulting motion of water is a vector field; at each point in the pool, the water has certain velocity vector. The physical significance of the divergence of a vector field is the rate at which "matter" exits or diverges from a given region in space. This change in the density is measured by analyzing the flux across the surface (usually, out flow – inflow). Therefore the divergence of a vector field is a scalar quantity. In the absence of creation or destruction of matter, the density within a region of space can change, only by some external flow into or out of that region. This fundamental property in physics is referred as "principle of continuity" (law of conservation of mass). We see that the divergence to be negative at a sink and positive at a source.

If the fluid is incompressible, then the density ρ is constant and *div* (**V**)=0 and we say that **V** is a divergenceless field.

Curl: The physical significance of the curl of a vector field is the amount of "rotation". It arises in fluid mechanics and elasticity theory. It is also fundamental in the theory of electromagnetism, where it arises in two of the four Maxwell equations. If you throw a leaf into river water, it may travel in a linear manner or sometimes it may exhibit circular motions at some places. If $\nabla \times \mathbf{V} = 0$, then the field is said to be an irrotational field and the field becomes conservative in nature; means in such a field the work done to move a particle from A to B will be independent of path.(i.e.)work done to move a particle from A to B will remain same irrespective of the path chosen.

Some Useful identities:

(1) curl(**u**+**v**) = curl**u** + curl**v** (2) curl(*f***v**) = (grad*f*)x**v** + *f* curl**v** (3) div (**u**x**v**) = **v** • curl**u** – **u** • curl**v** (First order relations)

(4) div(curl**v**) =0 (5) curl(grad*f*) =0 (Second order relations)

Note: For a given vector function **V,** if there exists a scalar function *f* , such that **V** = grad*f* ,then *f* is called the scalar potential of **V.** As curl(grad*f*) is always 0, if a potential exists for a **V**, then curl**V** should be zero. This is the checking condition for the existence of scalar potential for a **V**.