General linear programming problem

The mathematical structure of the general linear programming problem is as follows:

(A): Minimize / Maximize $f(x_1, x_2, \ldots x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

$X = (x_1, x_2, \ldots x_n) \rightarrow$ n dimensional decision variable

$f: R^n \rightarrow R$.

Subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq (=/\geq)\ b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq (=/\geq)\ b_2$$

$$- \quad - \quad -$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq (=/\geq)\ b_m.$$

$x_i \geq 0, \quad i = 1, 2, \ldots n,$

$m, n \in N$ and $m \neq n$ (in general)

$a_{ij}, b_i, c_j \in R.$

$i = 1, 2, \ldots m$

$j = 1, 2, \ldots n$

Description: The objective function is a linear function which has to be minimized or maximized. It is of the type $c_1x_1 + c_2x_2 - - + c_nx_n$. $x_1, x_2, \dots x_n$ are decision variables. This could be subject to the following condition:

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i, \quad i=1,2,\dots m. \quad \text{———} (1)$$

All the decision variables, $x_j \ge 0$, $j=1,2,\dots n$.
So the decision vector $X = (x_1, \dots x_n)$ has to be determined subject to the conditions those are given in inequality (1). Generally $m \neq n$, m is the number of inequalities, n is the number of the variables.

~~The system (A)~~
(A) can be written in more compact way, utilizing summation sign as follows:

$$(z \equiv f(x_1, x_2, \dots x_n))$$

(B) $\text{Min / Max} \quad z = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij}x_j \le (= / \ge)\ b_i, \quad i=1,2,\dots m.$

and $x_j \ge 0, \quad j=1,2,\dots n.$

(B) can be written in terms of matrix notation:

(D) $\text{Min / Max} \quad z = C^T X$

Subject to, $AX \le (= / \ge)\ B.$

$X \ge 0.$

$C = (c_1, c_2, \dots c_n)$, $(c_j)_{n\times 1}$ $\qquad A = (a_{ij})_{m\times n}$

$B = (b_i)_{m\times 1}$

Basic Terminology:

Let us look at the basic definitions which are related to the linear programming problem.

Feasible solution: Any vector x satisfying all constraints is said to be feasible solution.

Feasible region: The set of all feasible solutions is called feasible region.

Optimal solution: Out of all feasible solution there is one which is called as the optimum solution, so the best feasible solution is called the optimal solution. Now by best, we mean either minimum or maximum depending upon the problem in hand.

Optimum value: The value of the objective function at that optimum solution is called the optimum value.

Modelling real life problem through Linear programming problems formulation:

Real life optimization problems can be modelled as either linear programming problem or non linear programming problems. At present we consider the case for linear programming problem only.

Example 1 : Profit maximization problem:

A company wishes to produce a product for which it has three models to choose from. The labour and material data for each model is given. Supply of raw material is 200 Kg and the available manpower is 150 hours. Formulate the model to determine the daily production to maximize profit using the following data.

	Model A	Model B	Model C
Labour (Hrs/unit)	7	3	6
Material (Kg/unit)	4	~~4~~ 4	5
Profit (Rs/unit)	4	2	3

→ $\boxed{\text{Step 1} \rightarrow \text{Identify decision variables}}$

Let x_1 := no of units to be produced of model A.

x_2 := no of units to be produced of model B

x_3 := no of units to be produced of model C.

$\boxed{\text{Step 2} \rightarrow \text{Identify Constraints}}$

$$7x_1 + 3x_2 + 6x_3 \le 150$$

$$4x_1 + 4x_2 + 5x_3 \le 200$$

$x_i \ge 0, \quad i = 1, 2, 3, \quad x_i \in \cancel{Z}\, I$ - the set of integers

$\boxed{\text{Step 3} \rightarrow \text{ Identify Objective function}}$

Maximize profit $Z = 4x_1 + 2x_2 + 3x_3$

Summarizing we have

Max $Z = 4x_1 + 2x_2 + 3x_3$

Subject to $7x_1 + 3x_2 + 6x_3 \le 150$

$4x_1 + 4x_2 + 5x_3 \le 200,$

$x_i \ge 0, \; i = 1, 2, 3$. and x_i's are integers.

Example 2: Work-scheduling Problem

A post office requires different number of full time employees on different days of the week. The daily requirement is given in the table. Union rules state that each full time employee must work for five consecutive days and then receive two days off. Formulate an LPP so that the post office can minimize the number of full time employees who must be hired.

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Days of the week	No of full time employee required.
1 = Monday	17
2 = Tuesday	13
3 = Wednesday	15
4 = Thursday	19
5 = Friday	14
6 = Saturday	16
7 = Sunday	11

→ Step 1: Idenfication of decision variables

Let x_i := no of employees begining work on day i, for $i = 1, 2, \dots 7$.

All $x_i \geq 0$ for $i = 1, 2, 3, \dots 7$

All x_i are integers.

Step 2: Identifying objective function.

~~Minimize~~

Objective function is the sum of all the employees on all seven days. This should be minimized, because the more you employ, the more cost is incurred.

So,

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$.

Step 3: Identifying constraints

For constraints, for each of the seven days of the week, employee will take 2 days off according to union rule.

$$
\left.\begin{array}{l}
x_1 + \qquad\qquad + x_4 + x_5 + x_6 + x_7 \geq 17; \\
x_1 + x_2 + \qquad\qquad + x_5 + x_6 + x_7 \geq 13 \\
x_1 + x_2 + x_3 + x_4 \quad\ \, x_6 + x_7 \geq 15 \\
x_1 + x_2 + x_3 + x_4 + \qquad\quad + x_7 \geq 19 \\
x_1 + x_2 + x_3 + x_4 + x_5 \qquad\qquad \geq 14 \\
\quad x_2 + x_3 + x_4 + x_5 + x_6 \geq 16 \\
\qquad x_3 + x_4 + x_5 + x_6 + x_7 \geq 11.
\end{array}\right\} (A)
$$

Summarising

Minimize

$$z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

Subject to (A)

and $x_i \geq 0$, $i = 1, 2, \dots 7$.

x_i's are integers.

Ex 3 : Industrial problem

A company has 3 operational department weaving, processing and packing with the capacity to produce 3 different types of clothes those are suiting, shirting and wooden yielding with the profit of Rs 2, Rs 4 and Rs 3 per meters respectively. 1m suiting requires 3 mins in weaving 2 mins in processing, and 1 min in packing. Similarly 1m of shirting requires 4 mins in weaving, 1 min in processing and 3 mins in packing, while 1 m of woolen requires 3 mins in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing department resp. Formulate a LPP to find the product to maximize ~~to~~ the profit.

[Hint : Tabulation

	Suiting	Shirting	Woolens	Available time (mins)
Weaving	3	4	3	3600
Processing	2	1	3	2400
Packaging	1	3	3	4800
Profit	2	4	3	

Let x_1 : no of units (meters) of suiting

x_2 : no of units (meters) of shirting

x_3 : no of units (meters) of woolen]