Problem 5.1.1

Find the least square solution of $AX = Y$ *for* $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 & 3+1+1+3 & 5+0+2+3 \\ 3+1+1+3 & 9+1+1+9 & 15+0+2+9 \\ 5+0+2+3 & 15+0+2+9 & 25+0+4+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}$$

$$A^TY = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 3+5+7-3 \\ 9+5+7-9 \\ 15+0+14-9 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 20 \end{bmatrix}$$

$$A^T A X = A^T Y$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 20 \end{bmatrix}$$

$$[A : B] = \left[\begin{array}{ccc|c} 4 & 8 & 10 & 12 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 8 & 10 & 12 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right] \Rightarrow R_1 \rightarrow \frac{R_1}{4} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right] \Rightarrow R_2 \rightarrow R_2 - 8 \times R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 10 & 26 & 38 & 20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 10 & 26 & 38 & 20 \end{array}\right] \Rightarrow R_3 \rightarrow R_3 - 10 \times R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 0 & 6 & 13 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 4 & 6 & -12 \\ 0 & 6 & 13 & -10 \end{array}\right] \Rightarrow R_2 \rightarrow \frac{R_2}{4} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 6 & 13 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 6 & 13 & -10 \end{array}\right] \Rightarrow R_3 - 6 \times R_2 \qquad \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 0 & 4 & 8 \end{array}\right] \Rightarrow R_3 \to \frac{R_3}{4} \qquad \sim \left[\begin{array}{ccc|c} 1 & 2 & 2.5 & 3 \\ 0 & 1 & 1.5 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solving the above triangular system we obtain

$$x_1 = 10 \qquad x_2 = -6 \qquad x_3 = 2$$

Problem 5.1.2

Find the orthogonal projection of $\vec{y}$ *onto span* $\{\vec{u_1}, \vec{u_2}\}$.

$$\vec{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}, \vec{u_1} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

Let $W = Span\{\vec{u_1}, \vec{u_2}\}$.

By the orthogonal decomposition theorem,

$$proj_W \vec{y} = \hat{y} = \left(\frac{\vec{y}.\vec{u_1}}{\vec{u_1}.\vec{u_1}} \right) \vec{u_1} + \left(\frac{\vec{y}.\vec{u_2}}{\vec{u_2}.\vec{u_2}} \right) \vec{u_2}$$

$$\hat{y} = \left(\frac{6(3) + 3(4) + (-2)(0)}{(3)^2 + (4)^2 + (0)^2} \right) \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \left(\frac{6(-4) + 3(3) + (-2)(0)}{(-4)^2 + (3)^2 + (0)^2} \right)$$

$$= \left(\frac{30}{25} \right) \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \left(\frac{15}{25} \right) \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

Problem 5.1.3

Let W be the subspace spanned by $\{\vec{u_1}, \vec{u_2}\}$, and write $\vec{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \vec{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

Since,

$$\vec{u_1}.\vec{u_2} = (-1)(1) + (1)(3) + (1)(-2) = -1 + 3 - 2 = 0.$$

$\{\vec{u_1}, \vec{u_2}\}$ is an orthogonal set.

$$\hat{y} = \left(\frac{\vec{y}.\vec{u_1}}{\vec{u_1}.\vec{u_1}}\right)\vec{u_1} + \left(\frac{\vec{y}.\vec{u_2}}{\vec{u_2}.\vec{u_2}}\right)\vec{u_2}$$

and

$$\vec{z} = \vec{y} - \hat{y}$$

$$\hat{y} = \left(\frac{(-1)(1)+4(1)+(3)(1)}{(1)^2+(1)^2+(1)^2}\right)\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$+\left(\frac{(-1)(-1)+4(3)+(3)(-2)}{(-1)^2+(3)^2+(-2)^2}\right)\begin{bmatrix}-1\\3\\-2\end{bmatrix}$$

$$= 2\begin{bmatrix}1\\1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-1\\3\\-2\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{bmatrix} = \begin{bmatrix}\frac{-5}{2}\\\frac{1}{2}\\2\end{bmatrix}$$

$$\vec{y} = \hat{y} + \vec{z}$$

$$= \begin{bmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{bmatrix} + \begin{bmatrix}\frac{-5}{2}\\\frac{1}{2}\\2\end{bmatrix}$$

Problem 5.1.4

Find the closet point to vector $\vec{y}$ in the subspace W spanned by $\vec{v_1}, \vec{v_2}, \vec{v_3}$. Then find the distance from $\vec{y}$ to W.

$$\vec{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v_1}.\vec{v_2} = (1)(1) + (1)(0) + (0)(1) + (-1)(1) = 0$$
$$\vec{v_1}.\vec{v_3} = (1)(0) + (1)(-1) + (0)(1) + (-1)(-1) = 0$$
$$\vec{v_2}.\vec{v_3} = (1)(0) + (0)(-1) + (1)(1) + (1)(-1) = 0$$

$$
\begin{aligned}
\hat{y} &= \left(\frac{\vec{y}.\vec{v_1}}{\vec{v_1}.\vec{v_1}}\right)\vec{v_1} + \left(\frac{\vec{y}.\vec{v_2}}{\vec{v_2}.\vec{v_2}}\right)\vec{v_2} \\
&= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}
\end{aligned}
$$

Distance from $\vec{y}$ to W is $\|\vec{y} - \hat{y}\|$.

$$
\begin{aligned}
\|\vec{y} - \hat{y}\| &= \sqrt{(3-5)^2 + (4-2)^2 + (5-3)^2 + (6-6)^2} \\
&= \sqrt{12} = 2\sqrt{3}
\end{aligned}
$$