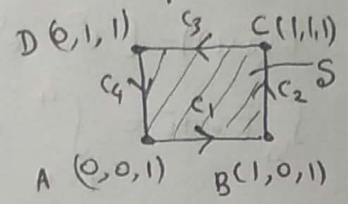
Problems on Stoke's Theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{Curl}\,\bar{F}\cdot\hat{n}\,ds$$

$$= \iint_R \text{Curl}\,\bar{F}(\bar{r}(u,v))\cdot\bar{N}\,du\,dv \quad \text{(in parametric form)}$$

Q(1) Verify Stoke's theorem for
$\bar{F} = [z^2, 5x, 0]$, $S: 0\le x\le 1,\ 0\le y\le 1,\ z=1.$

Sol:- The surface S is the square in $z=1$ plane.



Here $C = C_1\cup C_2\cup C_3\cup C_4$

L.H.S. of Stoke's theorem is $\oint_C \bar{F}\cdot d\bar{r}$.

Since

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{C_1}\bar{F}\cdot d\bar{r} + \int_{C_2}\bar{F}\cdot d\bar{r} + \int_{C_3}\bar{F}\cdot d\bar{r} + \int_{C_4}\bar{F}\cdot d\bar{r} \quad ---- (1)$$

$C_1: \bar{r}(t) = [t, 0, 1],\ 0\le t\le 1$

$\bar{F}(\bar{r}(t)) = [1, 5t, 0]$, $\frac{d\bar{r}}{dt} = [1, 0, 0]$

$$\int_{C_1}\bar{F}\cdot d\bar{r} = \int_{t=0}^{1} \bar{F}(\bar{r}(t))\cdot\frac{d\bar{r}}{dt}\,dt$$

$$= \int_{t=0}^{1} 1\,dt = 1 \quad --- (2)$$

$C_2: \bar{r}(t) = [1, t, 1],\ 0\le t\le 1$

$\bar{F}(\bar{r}(t)) = [1, 5, 0]$, $\frac{d\bar{r}}{dt} = [0, 1, 0]$

$$\int_{C_2}\bar{F}\cdot d\bar{r} = \int_{t=0}^{1} 5\,dt = 5 \quad --- (3)$$

$C_3: \bar{r}(t) = [t, 1, 1]$, 't' varies from '1' to '0'

$\bar{F}(\bar{r}(t)) = [1, 5t, 0]$, $\frac{d\bar{r}}{dt} = [1, 0, 0]$

$$\int_{C_3}\bar{F}\cdot d\bar{r} = \int_{t=1}^{0} 1\,dt = -1 \quad --- (4)$$

$C_4: \bar{r}(t) = [0, t, 1]$, 't' varies from '1' to '0'.

$\bar{F}(\bar{r}(t)) = [1, 0, 0]$, $\frac{d\bar{r}}{dt} = [0, 1, 0]$

$$\int_{C_4}\bar{F}\cdot d\bar{r} = \int_{t=1}^{0} 0\,dt = 0 \quad -(5)$$

Using (2), (3), (4) and (5) in (1), one gets:

$$\oint_C \bar{F}\cdot d\bar{r} = 1+5-1+0 = \underline{\underline{5 = \text{L.H.S.}}}$$

R.H.S. of Stoke's th. is $\iint_R \text{Curl}\,\bar{F}(\bar{r})\cdot\bar{N}\,du\,dv$

Parametric rep. of surface S is given by

$\bar{r}(u,v) = [u, v, 1],\ 0\le u\le 1,\ 0\le v\le 1$

Normal vector $\bar{N} = \bar{r}_u\times\bar{r}_v = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}$

$= [0, 0, 1]$

$$\text{Curl}\,\bar{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z^2 & 5x & 0\end{vmatrix} = [-5, 2z, 5]$$

$\text{Curl}\,\bar{F}(\bar{r}(u,v)) = [-5, 2.1, 5] = [-5, 2, 5]$

Now

$$\iint_R \text{Curl}\,\bar{F}(\bar{r}(u,v))\cdot\bar{N}\,du\,dv$$

$$= \int_{v=0}^{1}\int_{u=0}^{1} 5\,du\,dv = \int_{v=0}^{1} 5[u]_0^1\,dv$$

$$= \int_0^1 5\,dv = \underline{\underline{5 = \text{R.H.S.}}}$$

Hence L.H.S. = R.H.S. <u>Verified.</u>

Point to Remember:

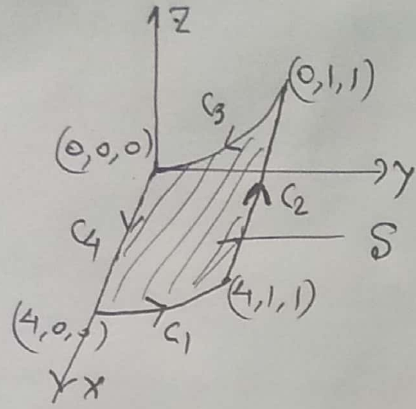
> The equation of a straight line which passes through two given points A and B having position vectors $\bar{a}$ and $\bar{b}$ w.r.t. an origin O is
>
> $\bar{r}(t) = \bar{a} + t(\bar{b}-\bar{a})$

Dr. A. K. Shukla

Q(2) $\bar{F} = [e^{2z}, e^{z}\sin y, e^{z}\cos y]$

$S: z = y^2, \quad 0 \le x \le 4, \quad 0 \le y \le 1$

Sol.



Stoke's theorem

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl} \bar{F}\cdot \hat{n}\, ds$$

$$= \iint_R \operatorname{curl} \bar{F}(\bar{r}(u,v))\cdot \bar{N}\, du\, dv$$

$$\text{L.H.S.} = \oint_C \bar{F}\cdot d\bar{r} = \sum_{i=1}^{4} \int_{C_i} \bar{F}\cdot d\bar{r} \quad \text{--- (1)}$$

$C_1: \; z = y^2, \; \bar{r}(t) = [4, t, t^2]$

$0 \le t \le 1$

$\bar{F}(\bar{r}(t)) = [e^{2t^2}, e^{2t^2}\sin t, e^{t^2}\cos t]$

$\frac{d\bar{r}}{dt} = [0, 1, 2t]$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} \left(e^{2t^2}\sin t + e^{t^2} 2t\cos t\right) dt \quad \text{--- (2)}$$

$C_2: \; \bar{r}(t) = [4,1,1] + t[0-4, 1-1, 1-1]$

$= [4,1,1] + t[-4,0,0]$

$\bar{r}(t) = [4-4t, 1, 1], \quad 0 \le t \le 1$

$\frac{d\bar{r}}{dt} = [-4, 0, 0]$

$\bar{F}(\bar{r}(t)) = [e^{2}, e^{2}\sin 1, e\cos 1]$

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} -4e^2\, dt = -4e^2 \quad \text{--- (3)}$$

$C_3: \; \bar{r}(t) = [0, t, t^2]$, 't' varies from '1' to '0'.

$\frac{d\bar{r}}{dt} = [0, 1, 2t]$

$\bar{F}(\bar{r}(t)) = [e^{2t^2}, e^{2t^2}\sin t, e^{t^2}\cos t]$

$$\int_{C_3} \bar{F}\cdot d\bar{r} = \int_{t=1}^{0} \left(e^{2t^2}\sin t + 2t e^{t^2}\cos t\right) dt \quad \text{--- (4)}$$

$C_4: \; \bar{r}(t) = [t, 0, 0]$, t varies from '0' to '4'.

$\frac{d\bar{r}}{dt} = [1, 0, 0]$

$\bar{F}(\bar{r}(t)) = [e^0, e^0\sin 0, e^0\cos 0] = [1, 0, 1]$

$$\int_{C_4} \bar{F}\cdot d\bar{r} = \int_{t=0}^{4} dt = 4. \quad \text{--- (5)}$$

Now, $\int_C \bar{F}\cdot d\bar{r} = -4e^2 + 4 = \underline{4(1-e^2)}$

Since (2) & (4) cancel each other.

R.H.S.: $\bar{r}(u,v) = [u, v, v^2]$

$0 \le u \le 4, \quad 0 \le v \le 1$

$$\bar{N} = \bar{r}_u \times \bar{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2v \end{vmatrix}$$

$= [0, -2v, 1]$

$\operatorname{curl} \bar{F} = [-2e^{z}\sin y, 2e^{2z}, 0]$

$\operatorname{curl} \bar{F}(\bar{r}(u,v)) = [-2e^{v^2}\sin v, 2e^{2v^2}, 0]$

$$\iint_R \operatorname{curl} \bar{F}(\bar{r}(u,v))\cdot \bar{N}\, du\, dv$$

$$= \int_{v=0}^{1}\int_{u=0}^{4} -4v e^{2v^2}\, du\, dv$$

$$= 4 \cdot \int_{v=0}^{1} -4v e^{2v^2}\, dv$$

$$= 4 \left[-e^{2v^2}\right]_{v=0}^{1}$$

$$= 4(-e^2+1) = \underline{4(1-e^2)}$$

L.H.S. = R.H.S.

Verified.

Dr. A.K. Shukla

Gauss Divergence Theorem

Verify the divergence theorem: $\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_E \text{div}\,\bar{F}\,dv$

(1) $\bar{F} = [e^x, e^y, e^z]$, S is the surface of cube $|x|\le 1, |y|\le 1, |z|\le 1$.

Sol: R.H.S. is done in class and the result is $12(e-e^{-1})$.

We calculate flux of $\bar{F}$ through six surfaces of given cube.

$$\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_{S_1}\bar{F}\cdot\hat{n}\,ds + \iint_{S_2}\bar{F}\cdot\hat{n}\,ds + \iint_{S_3}\bar{F}\cdot\hat{n}\,ds + \iint_{S_4}\bar{F}\cdot\hat{n}\,ds + \iint_{S_5}\bar{F}\cdot\hat{n}\,ds$$
$$+ \iint_{S_6}\bar{F}\cdot\hat{n}\,ds = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \text{ (let)}$$

Let S_1: $x=1$, $|y|\le 1$, $|z|\le 1$, In this case normal is $(\hat{n}) = \hat{i}$

S_2: $x=-1$, $|y|\le 1$, $|z|\le 1$, $\hat{n} = -\hat{i}$

S_3: $y=1$, $|x|\le 1$, $|z|\le 1$, $\hat{n} = \hat{j}$

S_4: $y=-1$, $|x|\le 1$, $|z|\le 1$, $\hat{n} = -\hat{j}$

S_5: $z=1$, $|x|\le 1$, $|y|\le 1$, $\hat{n} = \hat{k}$

S_6: $z=-1$, $|x|\le 1$, $|y|\le 1$, $\hat{n} = -\hat{k}$.

Now,

$$I_1 = \iint_{S_1} \bar{F}\cdot\hat{n}\,ds = \iint_{S_1} [e^x, e^y, e^z]\cdot[1,0,0]\,ds = \int_{z=-1}^{1}\int_{y=-1}^{1} e^x\,dy\,dz$$
$$= \int_{z=-1}^{1}\int_{y=-1}^{1} e^{(1)}\,dy\,dz \qquad [\because x=1 \text{ in } S_1]$$
$$= 4e$$

$$I_2 = \iint_{S_2}\bar{F}\cdot\hat{n}\,ds = \iint_{S_2}[e^x, e^y, e^z]\cdot[-1,0,0]\,ds = \int_{z=-1}^{1}\int_{y=-1}^{1} -e^{-1}\,dy\,dz = -4e^{-1}$$

$$I_3 = \iint_{S_3}\bar{F}\cdot\hat{n}\,ds = \iint_{S_3}[e^x, e^y, e^z]\cdot[0,1,0]\,ds = \int_{z=-1}^{1}\int_{x=-1}^{1} e^{(1)}\,dx\,dz = 4e^{1}.$$

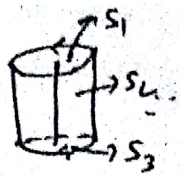
$$I_4 = \iint_{S_4}\bar{F}\cdot\hat{n}\,ds = \int_{z=-1}^{1}\int_{x=-1}^{1} -e^{-1}\,dx\,dz = -4e^{-1}.$$

Similarly, $I_5 = 4e$, $I_6 = -4e^{-1}$.

$$\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = 4e - 4e^{-1} + 4e - 4e^{-1} + 4e - 4e^{-1}$$
$$= 12(e - e^{-1}).$$

Verified.

(2) $\bar{F} = [\cos y,\ \sin x,\ \cos z]$, $\quad S: \ x^2+y^2 \le 4, \quad |z| \le 2.$



Sol: R.H.S of divergence theorem is evaluated in class.

L.H.S: $\because \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_{S_1} \bar{F}\cdot\hat{n}\,ds + \iint_{S_2} \bar{F}\cdot\hat{n}\,ds + \iint_{S_3} \bar{F}\cdot\hat{n}\,ds$
$= I_1 + I_2 + I_3.$

$S_1: \ x^2+y^2 \le 4, \quad z = 2, \quad \hat{n} = \hat{k}$

$S_2: \ x^2+y^2 = 4, \quad |z| \le 2$, [Write the parametric form and then do]

$S_3: \ x^2+y^2 \le 4, \quad z = -2, \quad \hat{n} = -\hat{k}$

$$I_1 = \iint_{S_1} \bar{F}\cdot\hat{k}\,ds = \iint_D \cos(2)\,ds = \cos 2 \cdot \text{area of circle} = 4\pi(\cos 2).$$

$$I_3 = \iint_{S_3} \bar{F}\cdot(-\hat{k})\,ds = \iint_D -\cos(2)\,ds = -4\pi\cos(2).$$

$I_2 = \iint_{S_2} \bar{F}\cdot\hat{n}\,ds$; Here it is not trivial to get the normal!

$\bar{r}(u,v) = [2\cos u,\ 2\sin u,\ v], \quad 0 \le u \le 2\pi, \quad -2 \le v \le 2.$

$\bar{r}_u \times \bar{r}_v = \bar{N} = [2\cos u,\ 2\sin u,\ 0]$

$\bar{F}\cdot\bar{N} = [\cos(2\sin u),\ \sin(2\cos u),\ \cos v]\cdot[2\cos u,\ 2\sin u,\ 0]$

$\bar{F}\cdot\bar{N} = 2\cos(2\sin u)\cdot\cos u + 2\sin(2\cos u)\sin u + 0$

$$I_2 = \int_{v=-2}^{2}\int_{u=0}^{2\pi}\{2\cos(2\sin u)\cos u + 2\sin(2\cos u)\sin u\}\,du\,dv$$

$= 0.$

$$\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = 4\pi\cos(2) + (-4\pi)\cos(2) + 0$$

$= 0.$

Verified.

(3) $\bar{F} = [x^3, y^3, z^3]$. S is sphere $x^2+y^2+z^2=9$;

Sol: R.H.S. we evaluated in the class. [Ans: 972π]

L.H.S: The parametric form of S is

$\bar{r}(u,v) = [3\sin v\cos u,\ 3\sin v\sin u,\ 3\cos v]$, $0\le u\le 2\pi$, $0\le v\le\pi$.

$\bar{r}_u = [-3\sin v\sin u,\ 3\sin v\cos u,\ 0]$, $\bar{r}_v = [3\cos v\cos u,\ 3\cos v\sin u,\ -3\sin v]$

$\vec{N} = \bar{r}_u \times \bar{r}_v = [-9\sin^2 v\cos u,\ -9\sin^2 v\sin u,\ -9\sin v\cos u]$.

$\bar{F}(\bar{r}(u,v)) = [27\sin^3 v\cos^3 u,\ 3\times 9\sin^3 v\sin^3 u,\ 3\times 9\cos^3 v]$

$\therefore\ \bar{F}\cdot\vec{N} = -3^5(\sin^5 v\cos^4 u + \sin^4 v\sin^4 u + \sin v\cos u\cos^3 v)$

Now the surface integral,

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iint_D \bar{F}(\bar{r}(u,v))\cdot\vec{N}\,du\,dv$$

$$= -3^5\int_{v=0}^{\pi}\int_{u=0}^{2\pi}(\sin^5 v\cos^4 u + \sin^4 v\sin^4 u + \sin v\cos^3 v\cos u)\,du\,dv$$

[The job is tedious here but divergence theorem makes it easy]

Please integrate and check whether you are getting 972π. [If it is -972π then we take $\vec{N} = \bar{r}_v\times\bar{r}_u$].

Note: In case of sphere $x^2+y^2+z^2=a^2$ when $\operatorname{div}\bar{F}$ is not constant then we transform (x,y,z)- co-ordinates to (r,θ,ϕ)- co-ordinates system. It is called spherical polar co-ordinates. Let $\bar{F}(x,y,z)$ and S is sphere given above. Then

$$\iiint_V \operatorname{div}\vec{F}(x,y,z)\,dV = \iiint_V \operatorname{div}\bar{F}(x,y,z)\,dx\,dy\,dz \quad \text{— (1)}$$

Replace

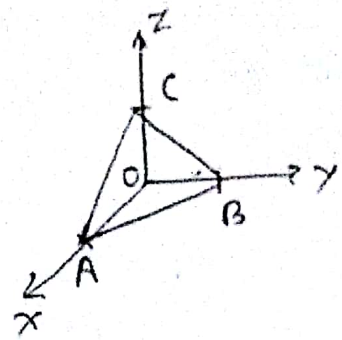
(2) — $\begin{cases} x = r\sin\phi\cos\theta, \\ y = r\sin\phi\sin\theta, \\ z = r\cos\phi \end{cases}$ $0\le r\le a$, $0\le\theta\le 2\pi$, $0\le\phi\le\pi$

$\therefore\ \boxed{dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi}$

Calculate $|J| = r^2\sin\phi$.

In (1) replace x, y, z by (2) and substitute the limits for r, θ and ϕ then integrate.

(4) $\bar{F} = [4x,\ x^2 y,\ -x^2 z]$. S the surface of tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.



Sol: $\because$ $\operatorname{div} \bar{F} = 4 + x^2 - x^2 = 4$.

$$\therefore \iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_E \operatorname{div}\bar{F}\, dV$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (4)\, dz\, dy\, dx$$

$$= \frac{2}{3} \quad [\text{Please integrate and verify.}]$$

It has four surfaces: AOB (S_1), BOC (S_2), AOC (S_3) and ABC (S_4)

S_1: $z=0$, $\hat{n} = \hat{k}$, $\because \bar{F}\cdot\hat{k} = -x^2 z = -x^2(0) = 0$, $\iint_{S_1} \bar{F}\cdot\hat{n}\, ds = 0$

S_2: $x=0$, $\hat{n} = \hat{i}$ $\because \bar{F}\cdot\hat{n} = 4x = 4(0) = 0$, $\therefore \iint_{S_2} \bar{F}\cdot\hat{n}\, ds = 0$

S_3: $y=0$, $\hat{n} = \hat{j}$ $\because \bar{F}\cdot\hat{n} = \bar{F}\cdot\hat{j} = x^2 y = x^2(0) = 0$, $\therefore \iint_{S_3} \bar{F}\cdot\hat{n}\, ds = 0$

S_4: $x+y+z = 1$, $\hat{n} = \hat{i}+\hat{j}+\hat{k}$.

or $\hat{n} = [1,1,1]$

$\boxed{\text{Or you can write the parametric form: } \bar{r}(u,v) = [u,\ v,\ 1-u-v],\ 0 \le u \le 1,\ 0 \le v \le 1-u. \text{ then find } \bar{N} = \bar{r}_u \times \bar{r}_v = [1,1,1]}$

$$\iint_{S_4} \bar{F}\cdot\hat{n}\, ds = \int_{u=0}^{1}\int_{v=0}^{1-u} \{[4u,\ u^2 v,\ -u^2(1-u-v)]\cdot[1,1,1]\}\, dv\, du$$

$$= \int_{u=0}^{1}\int_{v=0}^{1-u} \{4u + u^2 v - u^2(1-u-v)\}\, dv\, du$$

$$= \frac{2}{3} \quad [\text{Verify}]$$