Introduction to Nonlinear Programming

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A general NLP is represented as

$$\begin{gathered} Minimize\ f(X) \\ Subject\ to\ g_i(X) \leq 0, i = 1,2 \cdots m \\ h_j(X) = 0, j = 1,2 \cdots l \\ X^L \leq X \leq X^U \end{gathered}$$

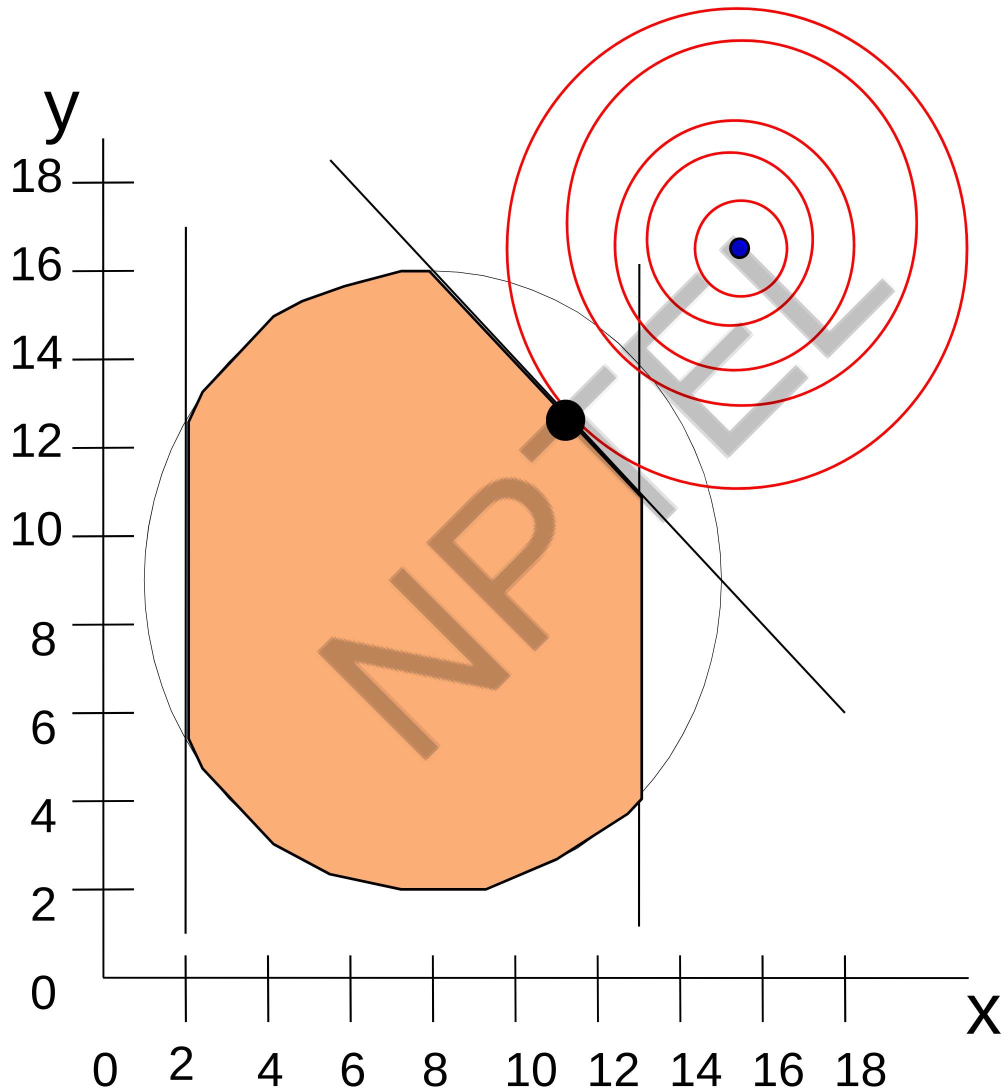
Where, $X = (x_1, x_2 \cdots x_n)^T$ is column vector of *n* real-valued design variables. And X^L and X^U represent explicit lower and upper bounds on the design variables

Example

Let us take a simple example

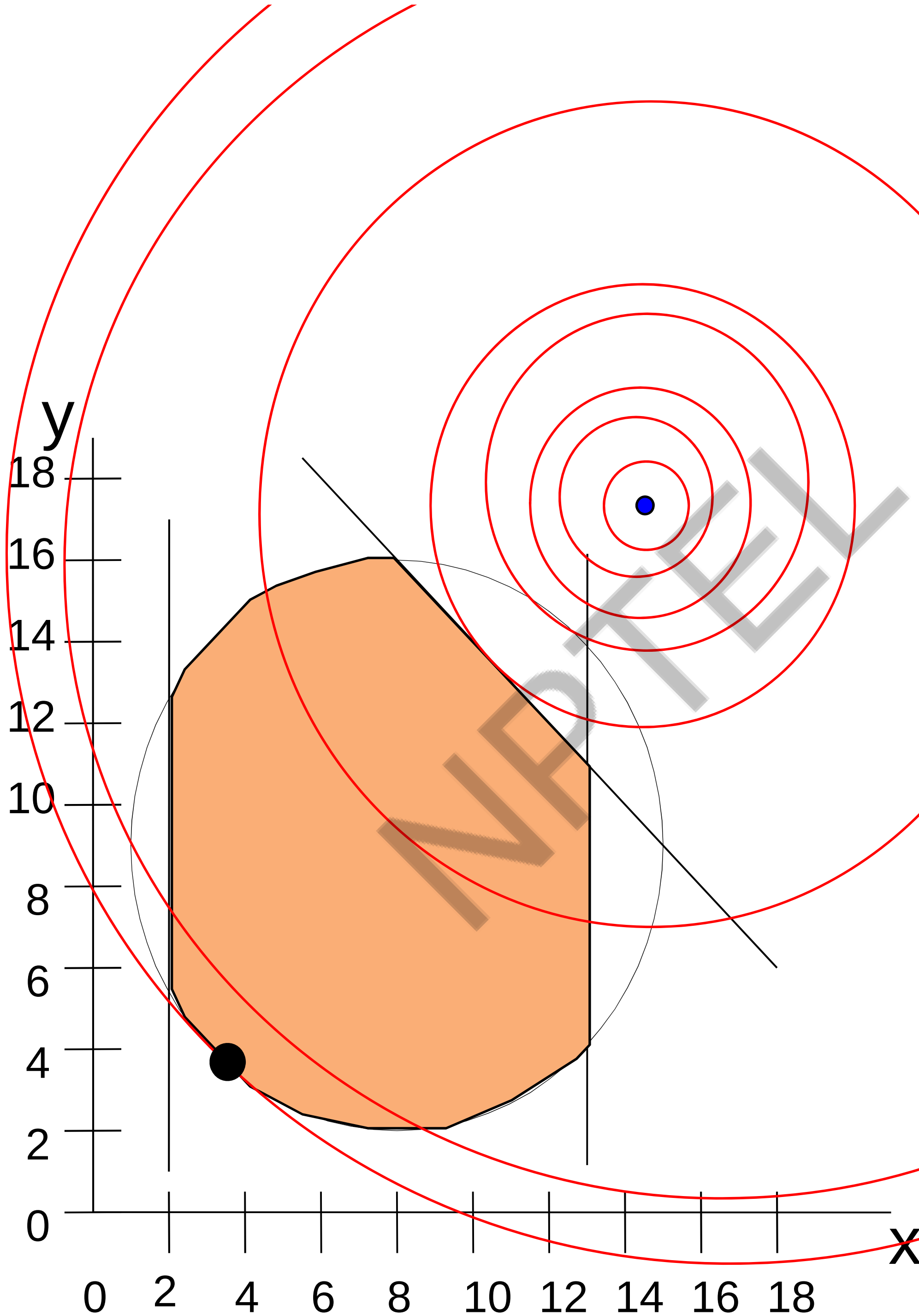
$$\textit{Minimize} \sqrt{(x-14)^2+(y-15)^2}$$
$$\textit{Subject to } (x-8)^2+(y-9)^2 \leq 49$$
$$2 \leq x \leq 13$$
$$x+y \leq 24$$
$$x, y \geq 0.$$

Graph



Note: the optimal solution is not at a corner point. It is where the contour first hits the feasible region

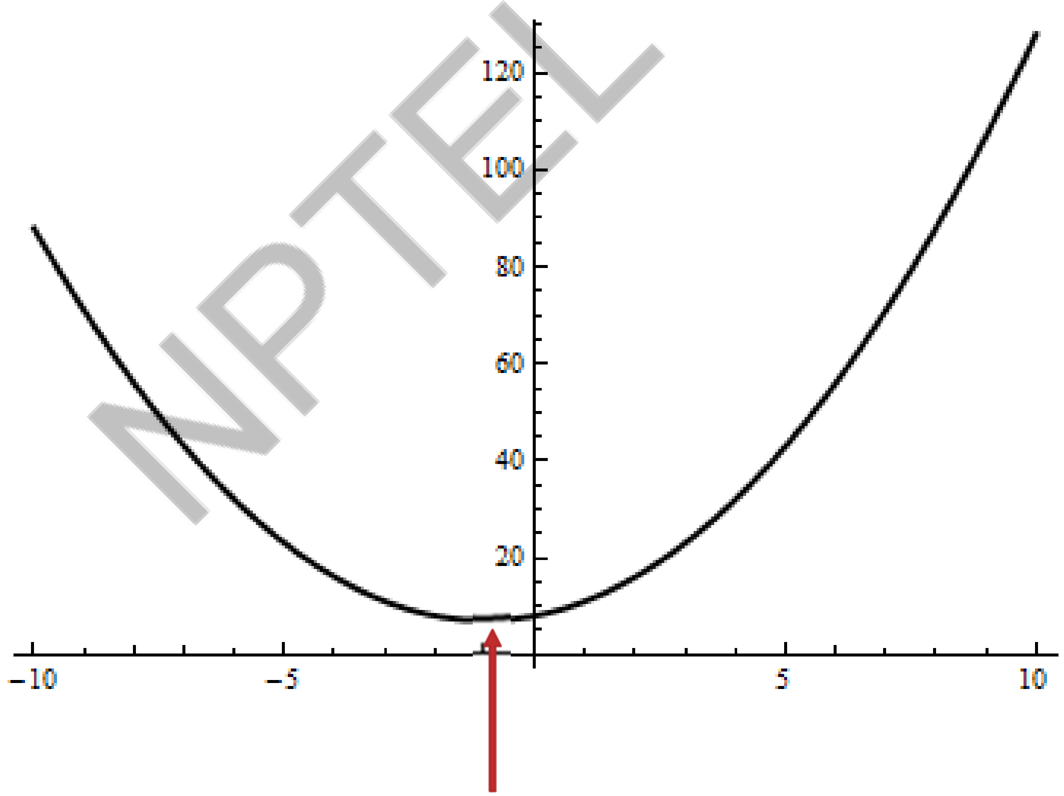
Graph

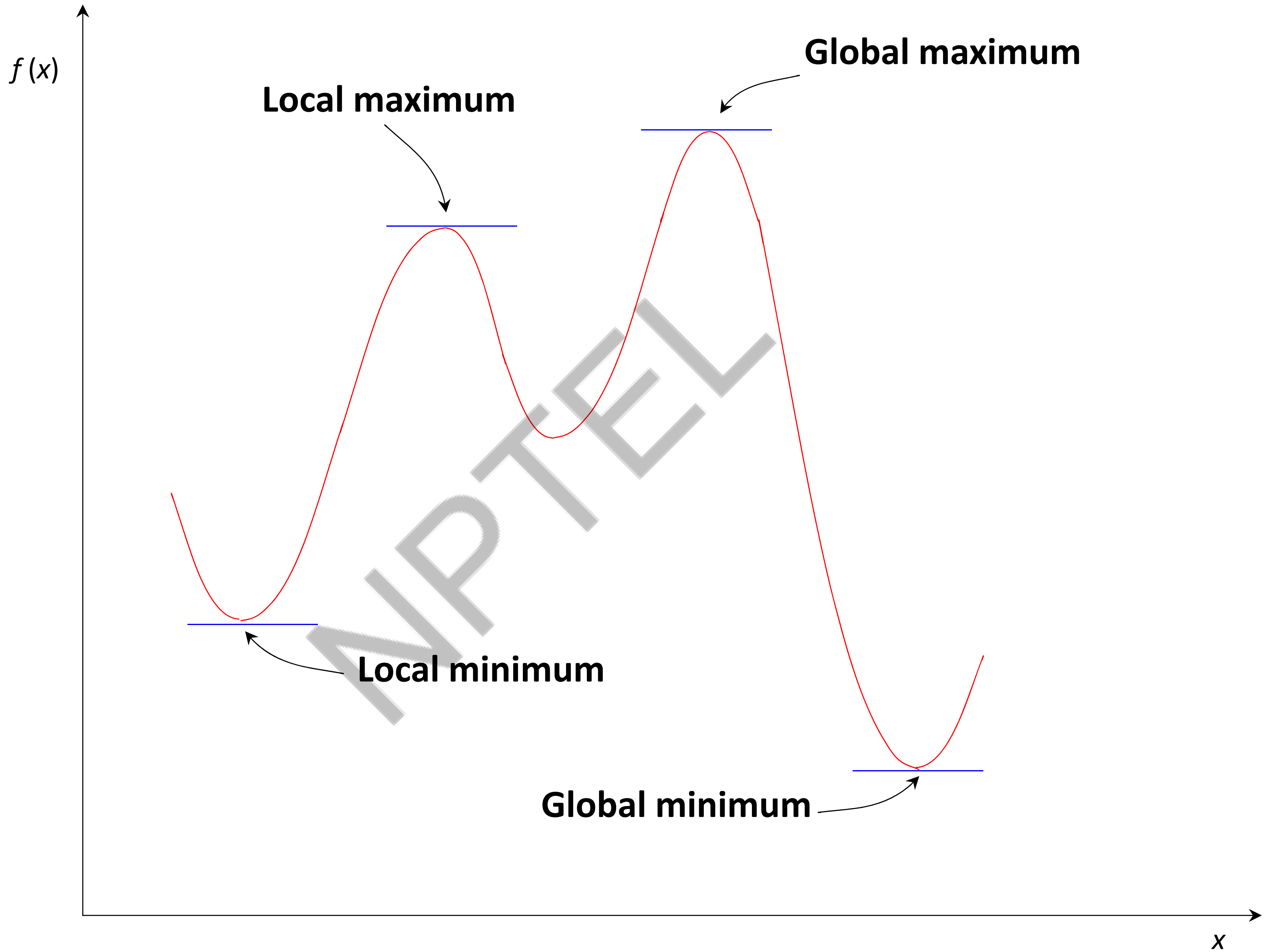


Note: the optimal solution is not at a corner point. Maximum occurs where the contour maximally can reach

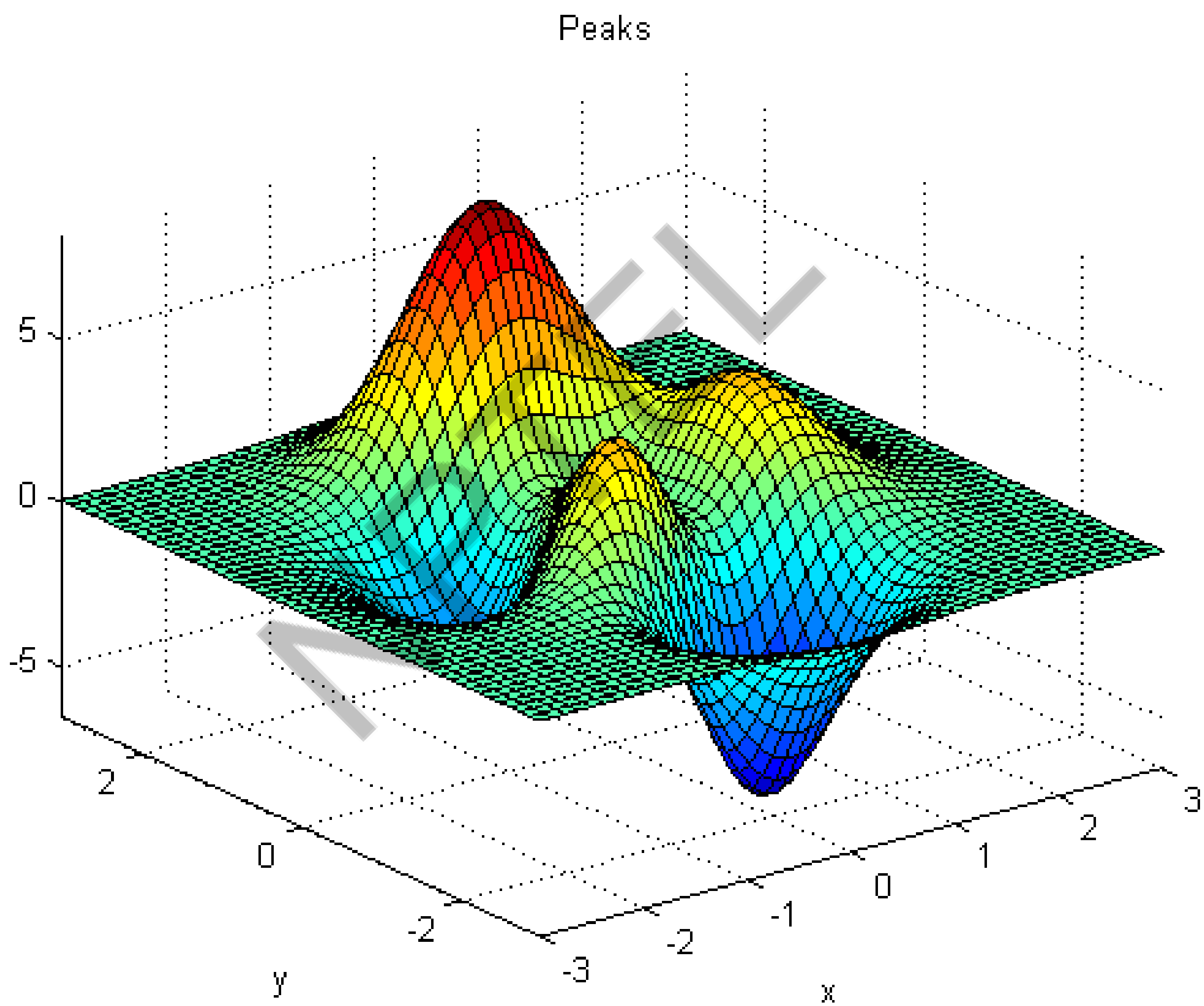
Unconstrained problem

$$Minimize\ x^2 + 2x + 8$$





f (x)
Local maximum
Global maximum
Local minimum
Global minimum
x


Peaks
5
0
-5
2
0
-2
y
-3
-2
-1
0
1
2
3
x

Optimality Criteria

- In finding optimal solution, two questions generally must be addressed:

1. **Static Question.** How can one determine whether a given point x* is the optimal solution?
2. **Dynamic Question.** If x* is not the optimal point, then how does one go about finding a solution that is optimal?

What is a Function?

- Monotonic and unimodal functions

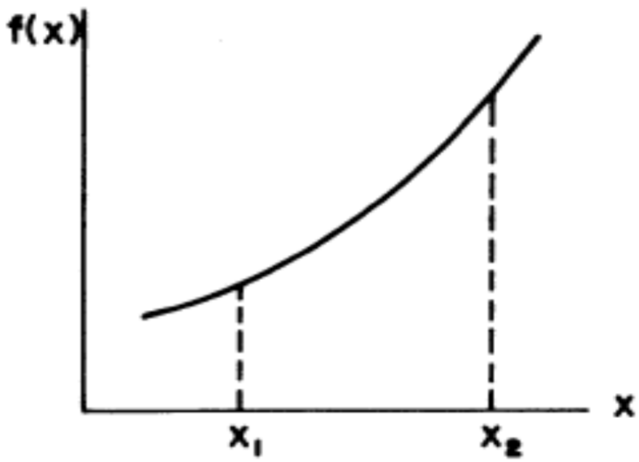
Monotonic:

$for\ any\ two\ points\ x_1 and\ x_2, where\ x_1 \leq\ x_2$
$if\ f(x_1) \leq f(x_2)\ (monotonically\ increasing)$
$if f(x_1) \geq f(x_2)\ (monotonically\ decreasing)$

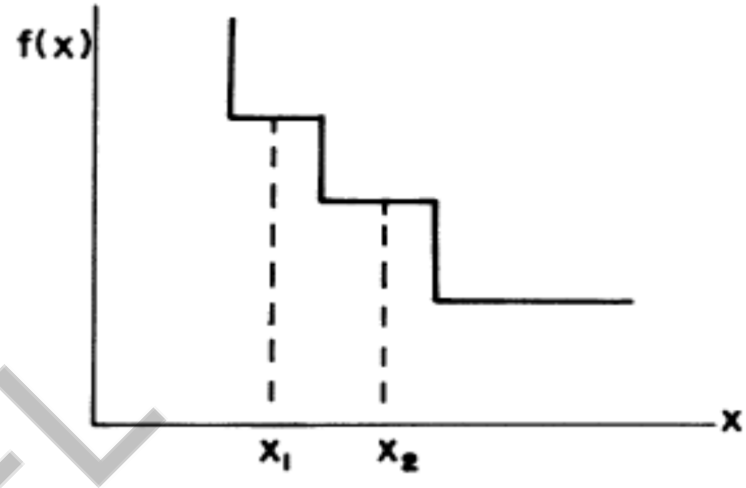
Unimodal:

$f(x) in\ unimodal\ on\ the\ interval\ a \leq x$
$\leq b\ if\ and\ only\ if\ it\ is\ monotic\ on\ either\ of$
$the\ single\ optimal\ point\ x^* in\ the\ interval$

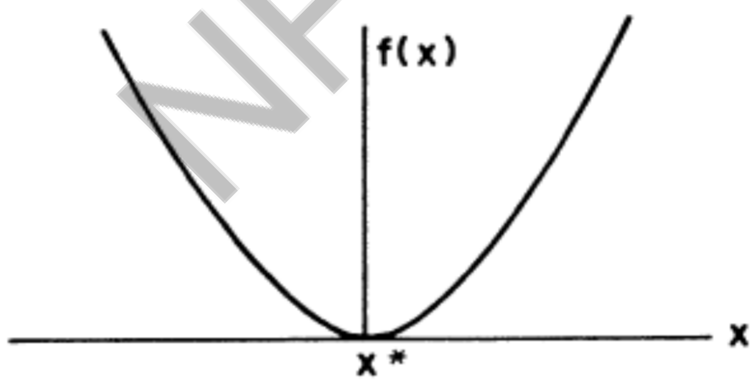
Unimodality **is an extremely important functional property used in optimization.**



A monotonic increasing function



A monotonic decreasing function



An unimodal function

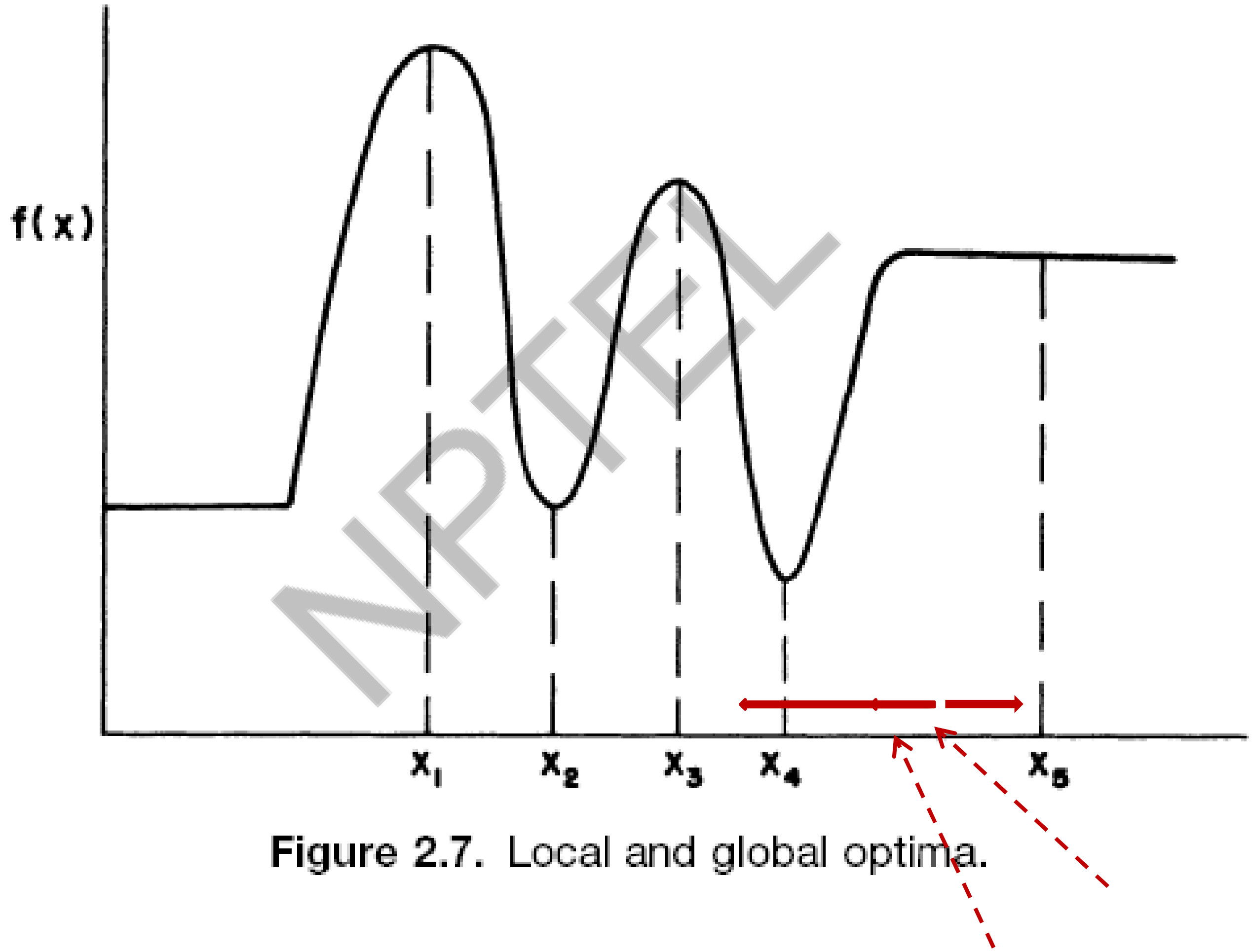


Figure 2.7. Local and global optima.

Basic Philosophy for solving NLP

To produce a sequence of improved approximations to the optimum according to the following scheme

1. Start with an initial trial point X_i
2. Find a suitable direction S_i
3. Find an appropriate step length λ_i
4. Obtain a new approximation

   $$X_{i+1} = X_i + \lambda_i S_i$$

5. Test whether X_{i+1} is optimum

Issues to be addressed

- Nature of the functions : convex/concave
- Modality
- Gradient of functions involved
- Optima are not restricted to extreme points
- Distinguish between local and global optimum
- If the feasible region is disconnected or combination of discrete spaces

Issues to be addressed

- Different starting point leads to different solution
- Difficult to find the feasible starting point
- It is not possible to identify whether the model is infeasible / unbounded
- There are numerous algorithm to solve NLP
- How will you know the function is convex or concave in the region of interest
- You need to know how to use the available different solver

Problem Formulations & Graphical Solution

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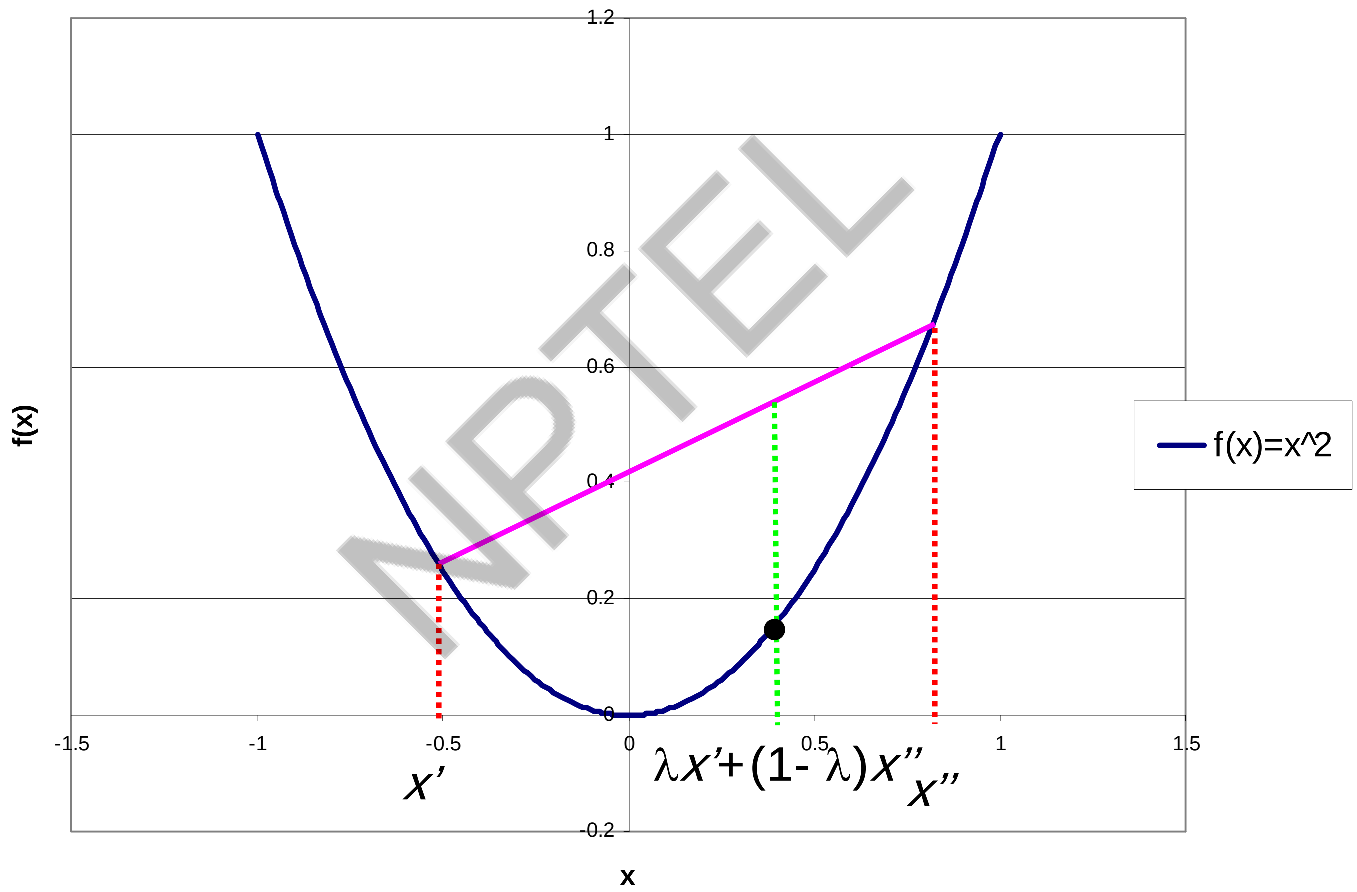
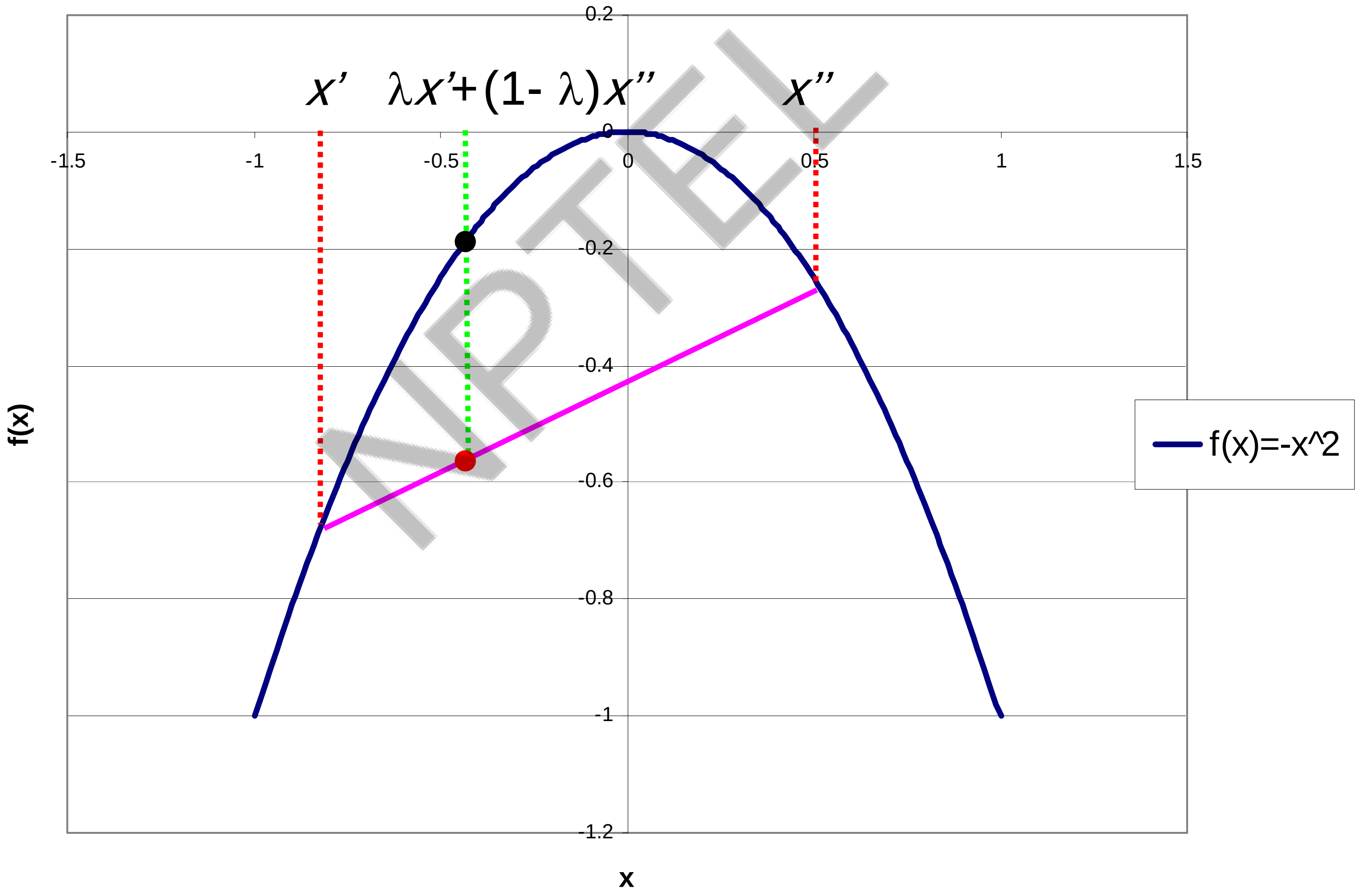
Concave Upward or Convex downward or Convex

$$f(x) = x^2$$



Concave Downward or Convex Upward or Concave

$$f(x) = -x^2$$



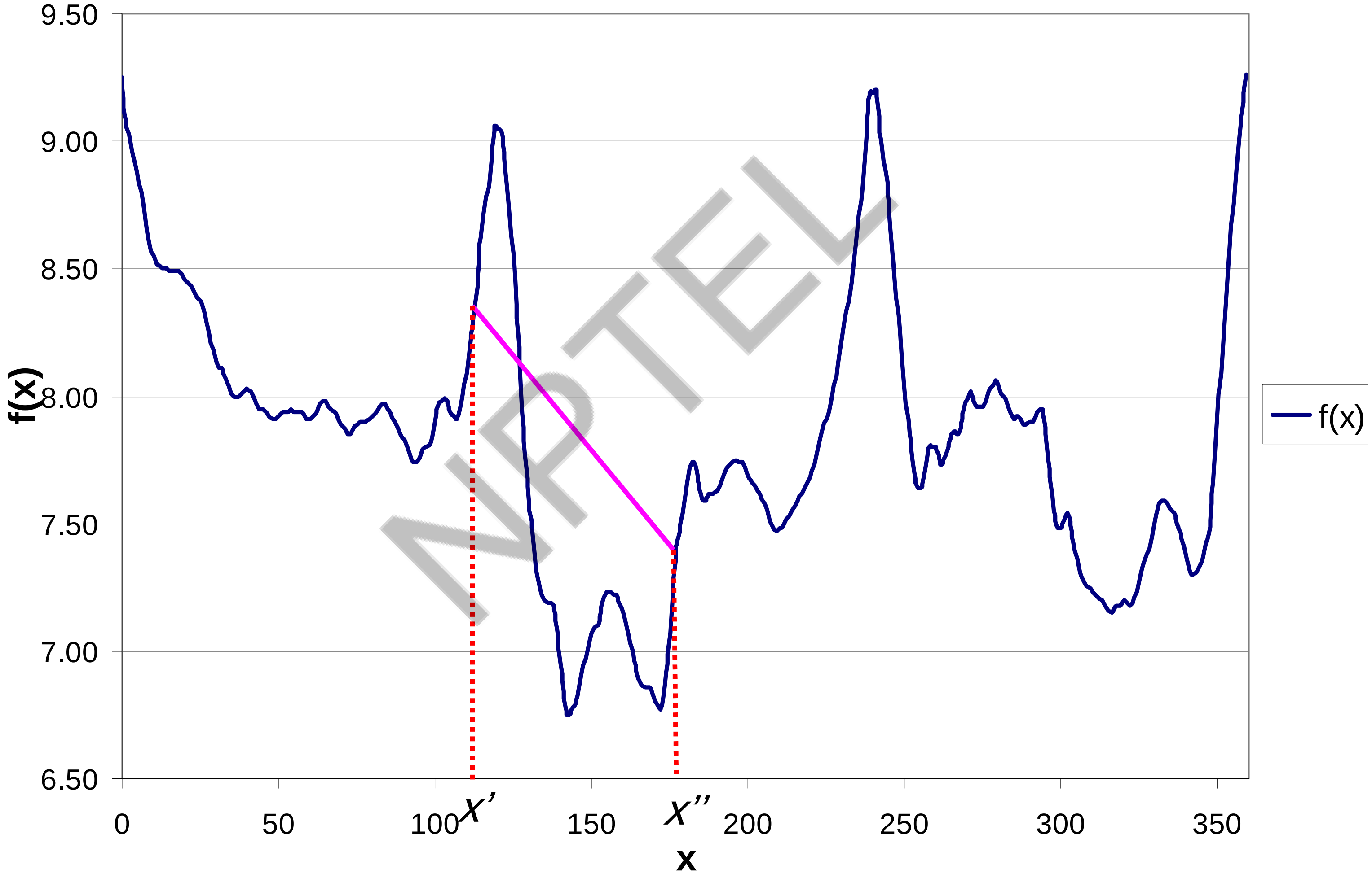
Convex and Concave Function

- The function f is a *convex* function if

$$f(\lambda x' + (1-\lambda)x'') \le \lambda f(x') + (1-\lambda) f(x'')$$

- The function f is a *concave* function if

$$f(\lambda x' + (1-\lambda)x'') \ge \lambda f(x') + (1-\lambda) f(x'')$$



9.50
9.00
8.50
8.00
7.50
7.00
6.50
f(x)
0
50
100
150
200
250
300
350
x
X'
X''
f(x)
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Important Fact

- Minimization of a Convex Function over Convex Sets any local minimum is a global minimum.

- Maximization of a Concave Function over Convex Sets any local maximum is a global maximum.

Local Optimality

A function of one variable is said to have a *relative or local minimum* at $x = x^*$ if $f(x^*) \leq f(x^* + h)$ for all sufficiently small positive and negative value of h.

A function of one variable is said to have a *relative or local maximum* at $\mathrm{x} = x^*$ if $f(x^*) \geq f(x^* + h)$ for all sufficiently small positive and negative value of h.

Global Optimality

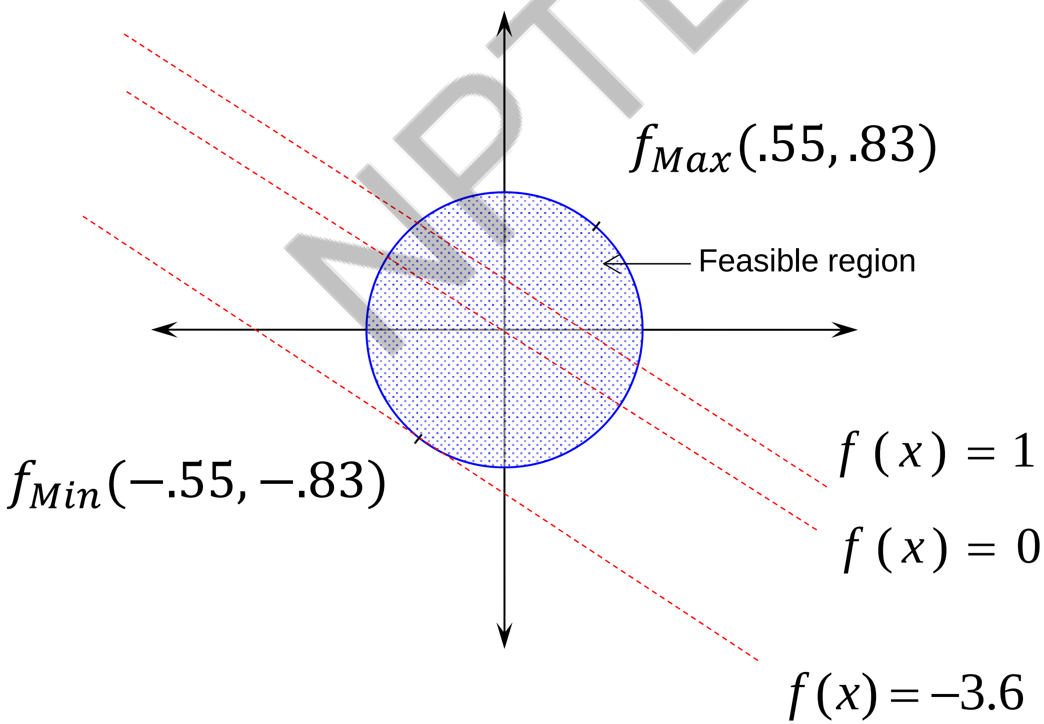
A function of one variable is said to have a *global minimum* at $x = x^*$ if $f(x^*) \leq f(x)$ for all x in the domain of the function.

A function of one variable is said to have a *global maximum* at $x = x^*$ if $f(x^*) \geq f(x)$ for all x in the domain of the function.

If the function increasing (or decreasing) on either side of the point x^*, then x^* is the *point of inflection*.

Graphical Solution

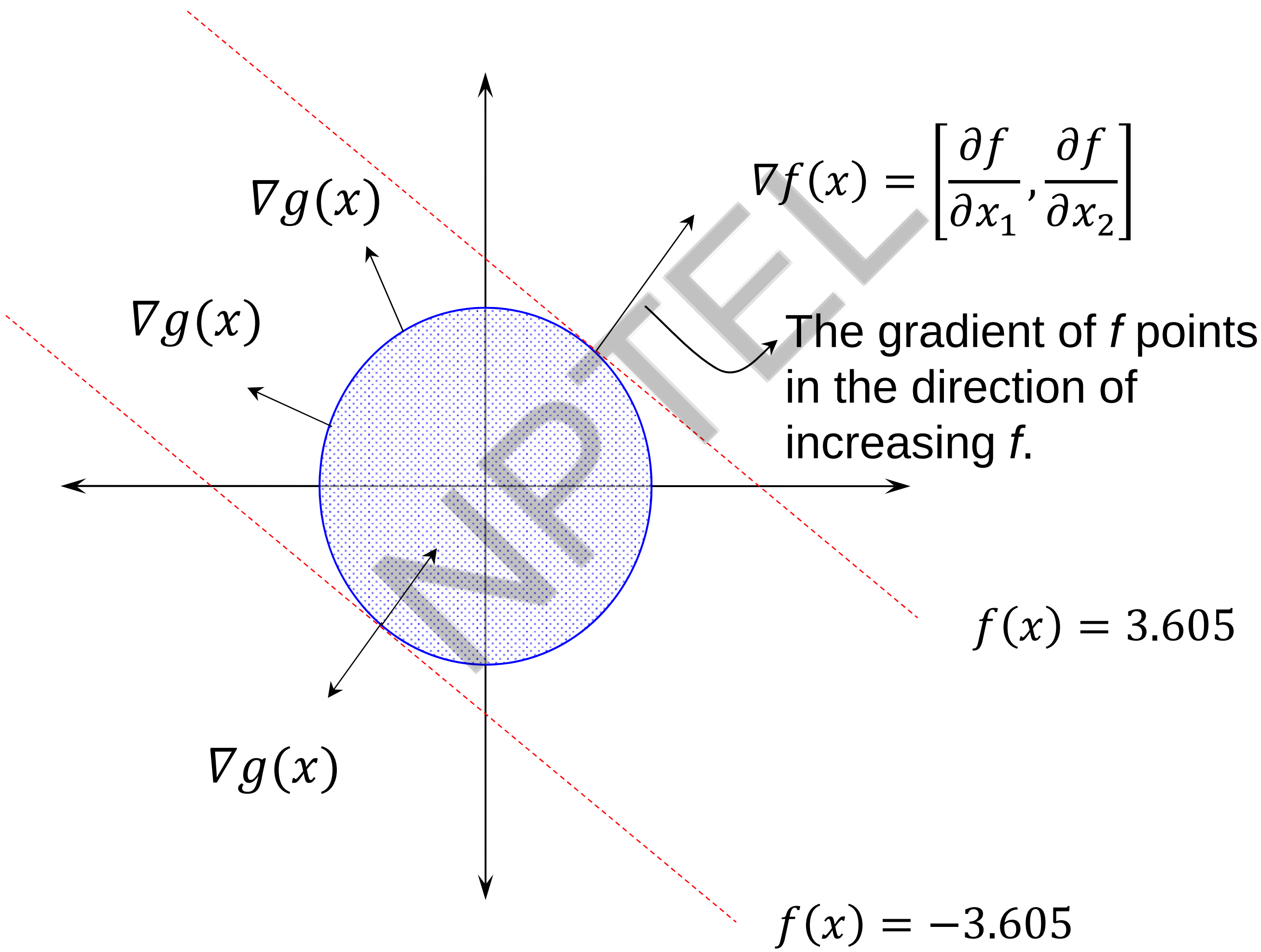
$$Minimize\ 2x + 3y$$
$$Subject\ to\ x^2 + y^2 = 1, x, y \geq 0$$



$f(x) = 3.605$

$f(x) = -3.605$

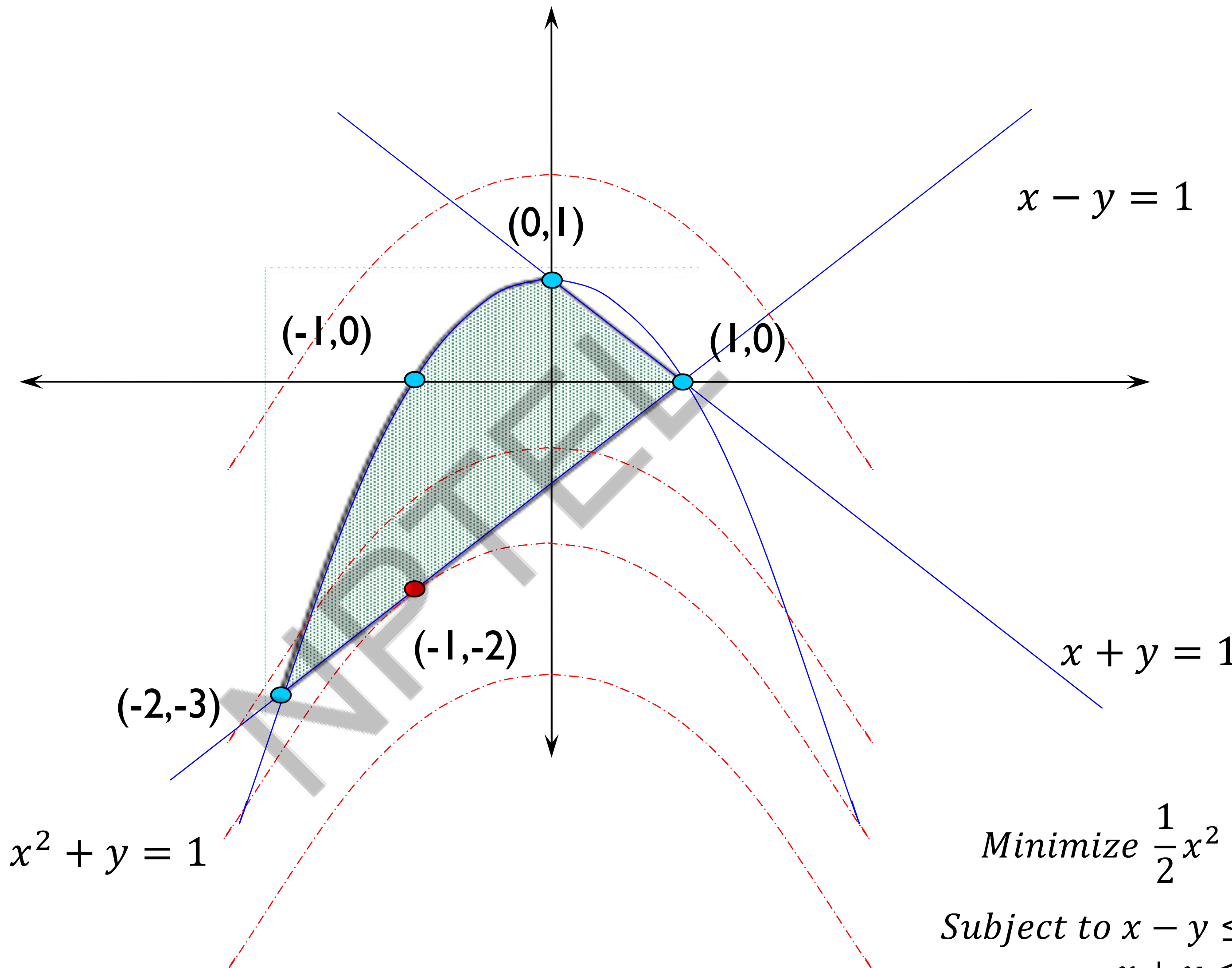
Graphical Solution



Problem 2

$$\textit{Minimize } \frac{1}{2}x^2 + y$$

$$\begin{aligned} \textit{Subject to } x - y &\le 1 \\ x + y &\le 1 \\ x^2 + y &\le 1 \\ x, y &\ge 0 \end{aligned}$$



$$\begin{aligned} \text{Minimize } & \frac{1}{2}x^2 + y \\ \text{Subject to } & x - y \le 1 \\ & x + y \le 1 \\ & x^2 + y \le 1 \\ & x, y \ge 0 \end{aligned}$$

Classification based on :

- Constraints
- Nature of Design Variable
- Nature of Equations involved
- Permissible Values of Design Variables
- Randomness involved in Design Variables
- Separability of Functions
- Number of Objective Functions

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Types of Optimization Problems

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Classification of Optimization Problems

Classification can be based on:

- **Constraints**
- **Nature of the design variables**
- **Physical structure of the problem**
- **Nature of the equations involved**
- **Permissible values of the design variables**
- **Deterministic nature of the variables**
- **Separability of the functions**

Classification of Optimization Problems

- **Constraints**
  - ✓ Constrained optimization problem
  - ✓ Unconstrained optimization problem

- **Nature of the design variables**
  - ✓ Static optimization problems
  - ✓ Dynamic optimization problems

Classification of Optimization Problems

- **Physical structure of the problem**
  - ✓ Optimal control problems
  - ✓ Non-optimal control problems

- **Nature of the equations involved**
  - ✓ Nonlinear programming problem
  - ✓ Geometric programming problem
  - ✓ Quadratic programming problem
  - ✓ Linear programming problem

Classification of Optimization Problems

- **Permissible values of the design variables**
  - ✓ Integer programming problems
  - ✓ Real valued programming problems

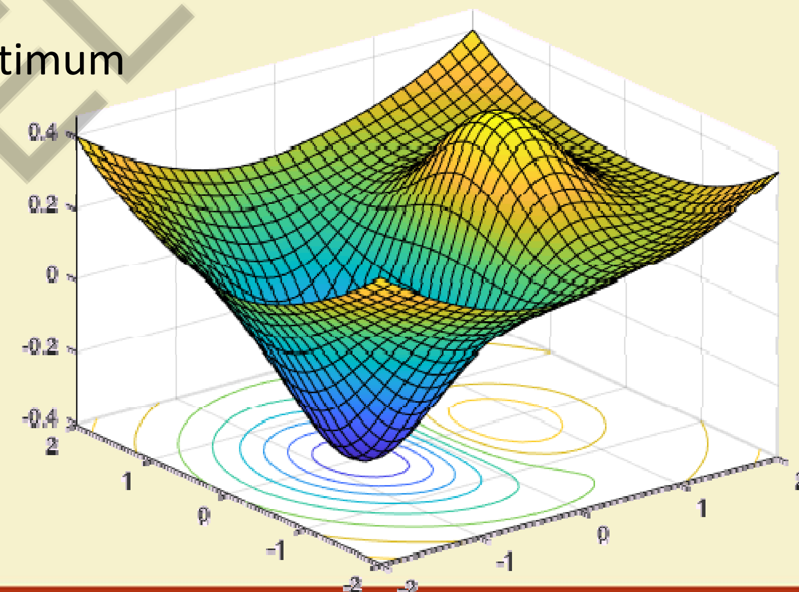
- **Deterministic nature of the variables**
  - ✓ Stochastic programming problem
  - ✓ Deterministic programming problem

Classification of Optimization Problems

- **Separability of the functions**
  - ✓ Separable programming problems
  - ✓ Non-separable programming problems
- **Number of the objective functions**
  - ✓ Single objective programming problem
  - ✓ Multiobjective programming problem

Unconstrained General Optimization Problem

- ✓ **Objective:** Find minimum of $F(\boldsymbol{x})$ where $\boldsymbol{x}$ is a vector of design variables
- ✓ We may know lower and upper bounds for optimum
- ✓ No constraints involved



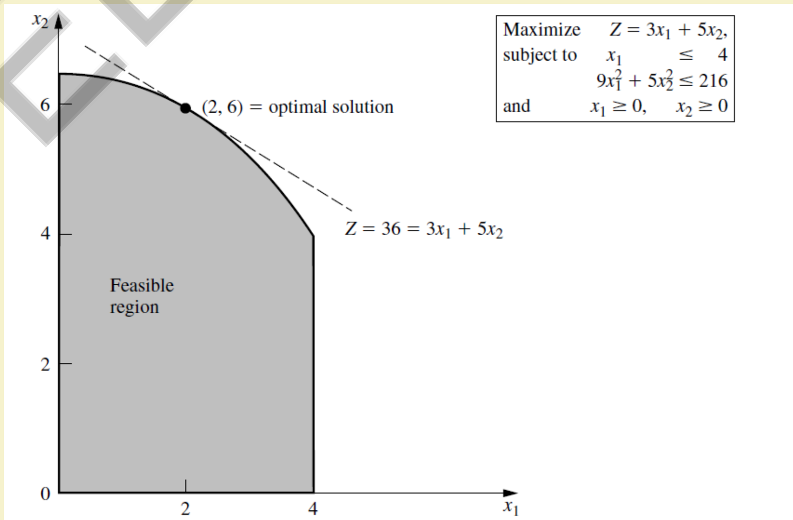
Constrained General Optimization Problem

- ✓ **Objective:** Find minimum of $F(\boldsymbol{x})$ where $\boldsymbol{x}$ is a vector of design variables subject to a set of constraints
- ✓ General format for Minimization problem:

$$Minimize\ F(\boldsymbol{x})$$

$$subject\ to\ G_i(\boldsymbol{x}) \geq 0$$

$$\boldsymbol{x} \geq 0, \quad i = 1, \ldots n$$



Quadratic Programming Problem

- ✓ **A quadratic programming problem** is a nonlinear programming problem with a quadratic objective function and linear constraints. It is usually formulated as follows:

$$F(\mathbf{X}) = c + q^T X + \frac{1}{2} X^T Q X$$

$$= c + \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{ij} x_i x_j$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i = b_j, \quad j = 1, 2, \cdots, m$$

$$x_i \geq 0, \quad i = 1, 2, \cdots, n$$

where c, q_i, Q_{ij}, a_{ij} and b_j are constants.

$$Minimize\ 2{x_1}^2 + 2{x_2}^2 - 2x_1x_2 - 4x_1 - 6x_2 + 8$$

$$Subject\ to\ x_1 + x_2 \le 2$$

$$x_1 + 5x_2 \le 5, -x_1 \le 0, -x_2 \le 0$$

Integer Programming Problem

- ✓ If some or all of the design variables $x_1, x_2, .., xn$ of an optimization problem are restricted to take on only integer (or discrete) values, the problem is called an ***integer programming problem.***

- ✓ ***General form:*** $maximize\ c^T x$

$$subject\ to\ Ax \leq b$$
$$x \geq 0, x \in \mathbb{Z}^n$$

where c, b are vectors and A is a matrix whose all entries are integers.

Separable Programming Problem

- ✓ A function $f(x)$ is said to be ***separable*** if it can be expressed as the sum of n single variable functions, $f_1(x_1)$, $f_2(x_2)$,….,$f_n(x_n)$, that is,

$$f(\mathbf{X}) = \sum_{i=1}^{n} f_i x_i$$

- ✓ A ***separable programming problem*** is one in which the objective function and the constraints are separable.

Find **X** which minimizes $f(\mathbf{X}) = \sum_{i=1}^{n} f_i(x_i)$

subject to

$$g_j(\mathbf{X}) = \sum_{i=1}^{n} g_{ij}(x_i) \le b_j, \qquad j = 1,2,\cdots,m$$

where b_j is constant

Example

Minimize $x_1^2 + x_2^2 + x_3^2$

Subject to $x_1 + x_2 + x_3 \geq 15, \quad x_1, x_2, x_3 \geq 0$

Example

Maximize $x_1 x_2 x_3$

Subject to $x_1 + x_2 + x_3 = 5, \quad x_1, x_2, x_3 \geq 0$