Homework 7

Ding Zhao
24-677 Special Topics: Linear Control Systems

Due: Oct 22, 2019, 08:30 am. Submit within deadline.

- At the beginning of each question you will find the key words for the knowledge that the exercise will help you to practice.
- Right after each week's recitation, we have a half-hour homework Q&A session. I request you to work on the assignment early and bring your questions to this session to take advantage of this support.
- You can also post your questions on the Piazza. We will try our best to give feedback within 24 hours during the workday and within 48 hours during the weekend. We will keep answering questions until 12 hours before the deadline.
- You need to upload your homework to Gradescope (https://www.gradescope.com/) to be graded. The link is on the panel of CANVAS. If you are not familiar about the tool, post your questions on Piazza or ask during the office hours/homework Q&A sessions. We will use the online submission time as the timestamp.
- We designed an Autograder to provide you instantaneous feedback for most of the questions. Submit **hw7_theory.py** under "Programming Assignment 7" and a your derivations in *.pdf* format to "Homework7". We will manually check all of the answers marked as wrong by Autograder to make sure you get the points you deserve.
- **Note: we do not have programming questions in HW7.**
- **Remember to submit derivations of all questions in your pdf submission**

Exercise 1. *Asymptotic stability and Lyapunov stability.*

For each of the systems given below, determine whether it is Lyapunov stable, whether it is asymptotic stable.

(a)

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

(b)

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

Exercise 2. *Operator Norms*

Find the Norms of the matrices:

1. L_2 norm of : $\begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

2. For A = $\begin{bmatrix} 10 & 2 \\ 0 & -3 \end{bmatrix}$ calculate the Nuclear norm, $||A||_N$ and Frobenius norm, $||A||_F$.

3. L_1, L_2 and L_∞ norm of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Exercise 3. *Operator Norms*

The LASSO cost is given as:

$$\hat{\beta}_{lasso} = \operatorname{argmin}_{\beta \in \mathbb{R}^v} \{K\}$$

where,

$$K = \frac{1}{N}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1$$

Given,

$$X = \begin{bmatrix} -2 & 0 & 5 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}, y = \begin{bmatrix} 3 & 5 \\ 7 & 4 \end{bmatrix}, \lambda = 20 \text{ and } N = 2.$$

Calculate the cost value K. Write calculation steps.

Exercise 4. Given $A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$, find e^{At} for $t = 2$

Exercise 5. *Stability of a CT LTV system.*

Is the homogeneous equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ e^{-3t} & 0 \end{bmatrix} \mathbf{x}$$

for $t_0 \geq 0$, stable i.s.L? Asymptotically stable?
(Autograder will not be used for this question, it shall be graded manually)

Exercise 6.

Prove that,
$$\left\|A^{-1}\right\|_2 = \frac{1}{\sigma_n}.$$

where $\sigma_1 > \sigma_1 > ... > \sigma_n$ are the singular values of A.
(Autograder will not be used for this question, it shall be graded manually.)

Exercise 7. *Stabilizability*

Decompose the state equation

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

to a controllable form. Is the reduced state equation observable, stabilizable, detectable? *(Autograder will not be used for this question, it shall be graded manually.)*

Additional Exercise (Optional)

"William L. Brogan. Modern Control Theory", Example 10.5 (Page 359) and Example 10.6 (page 360).