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Q1

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

$\Rightarrow n = 3, \quad P = [B : AB : A^2B]$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$\det(P) = 1(-1) = -1$

So P is full rank i.e rank $= 3$

As rank $= n = 3$

So, the system is controllable.

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

$$CA = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix} = \begin{bmatrix} +1 & 2 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ +1 & 2 & 1 \end{bmatrix}$$

$\det(Q) = 0$

rank $\neq 3 \neq n$

So, the system is not observable.

Q2 $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$

$\Rightarrow n = 3, \qquad P = [B \vdots AB \vdots A^2B]$

$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}$

$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 & -1 & -1 \end{bmatrix}$ $\quad \text{rank}(P) = 3 = n$

So, the system is controllable.

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad C = [1 \ 0 \ 1]$$

$$CA = [1 \ 0 \ 1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} = [1 \ 3 \ -1]$$

$$CA^2 = [1 \ 0 \ 1]\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} = [-1 \ -1 \ 4]$$

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

$$\text{Rank}(Q) = 3 = n$$

So, the system is observable.

Q3 $\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$

$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$

$\lambda = 2$; $B^{(2)} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ rank = 2,

$\lambda = 1$; $B^{(1)} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ rank = 2

so system is controllable

$\lambda = 2$; $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ rank = 2 $\neq$ full rank

$\therefore$ Not observable.

$\lambda = 1$; $\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ rank = 2

Q4

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix} u$$

$$y = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \end{bmatrix} x$$

$\lambda = 1$, $B^{(1)} = \begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}$

for $\lambda = 1$

$$(\lambda I - A) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(\lambda I - A) = 2$$

$$[\lambda I - A, B^{(1)}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & 0 & 0 & b_{21} & b_{22} \\ 0 & 0 & 0 & 1 & 0 & b_{31} & b_{32} \\ 0 & 0 & 0 & 0 & 0 & b_{41} & b_{42} \\ 0 & 0 & 0 & 0 & 0 & b_{51} & b_{52} \end{bmatrix}$$

for this to have full rank.

$$\begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}$$ should have rank $= 3$

But, the matrix can't have rank $= 3$

so, the system is <u>uncontrollable</u>.

$\lambda = 1$

$$C^{(1)} = \begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{21} & c_{23} & c_{25} \\ c_{31} & c_{33} & c_{35} \end{bmatrix}$$ for full rank $= 3$, we can look for some combinations of the elements of $C^{(1)}$.

So, $C^{(1)}$ is observable.

Q5

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$|A - \lambda I| = 0 \qquad n = 4$$

$$\lambda = -0.81,\ 1,\ 0.9053 \pm 1.2837i$$

for $\lambda = -0.81$, $[\lambda I - A : B]$

$$\begin{bmatrix} -0.81 & -1 & 0 & -1 & 0 \\ 1 & -1.81 & -1 & 0 & -1 \\ 0 & 0 & -1.81 & -1 & 0 \\ -2 & 0 & 2 & -0.81 & 2 \end{bmatrix}$$

rank = 4 = n = full rank. ∴ controllable.

for $\lambda = 1$, $[\lambda I - A : B]$

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ -2 & 0 & 2 & 1 & 2 \end{bmatrix}$$

rank = 3 ≠ full rank

∴ uncontrollable.

for $\lambda = 0.9053 + 1.2837i$

$[\lambda I - A \vdots B]$

$$\begin{bmatrix} 0.9053+1.2837i & -1 & 0 & -1 & 0 \\ 1 & -0.0947+1.2837i & -1 & 0 & -1 \\ 0 & 0 & -0.0947+1.2837i & -1 & 0 \\ -2 & 0 & 2 & 0.9053+1.2837i & 2 \end{bmatrix}$$

rank = 4 = full rank = n

$\therefore$ controllable.

for $\lambda = 0.9053 - 1.2837i$ $\quad [\lambda I - A \vdots B]$

$$\begin{bmatrix} 0.9053-1.2837i & -1 & 0 & -1 & 0 \\ 1 & -0.0947-1.2837i & -1 & 0 & -1 \\ 0 & 0 & -0.0947-1.2837i & -1 & 0 \\ -2 & 0 & 2 & 0.9053-1.2837i & 2 \end{bmatrix}$$

rank = 4 = n = full rank

$\therefore$ controllable

Eigenvalue corresponding to the uncontrollable mode are: $\boxed{\lambda = 1}$.

Q6

$$\dot{x} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

$\Rightarrow n = 2$

$$P = [B \vdots AB]$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \quad \text{rank} = 1 < n = 2$$

$$M = \left[\begin{array}{c|c} 1 & 0 \\ 1 & 1 \end{array}\right]$$

$$M^{-1}AM = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} 3 & 4 \\ \hline 0 & -5 \end{array}\right]$$

$$M^{-1}B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix}$$

$$CM = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \left[\begin{array}{c|c} 2 & 1 \end{array}\right]$$

$$\dot{x} = \left[\begin{array}{c|c} 3 & 4 \\ \hline 0 & -5 \end{array}\right] x + \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix} u$$

$$y = \left[\begin{array}{c|c} 2 & 1 \end{array}\right] x$$

using the controllable canonical form, we get the controllable decomposition as:

$$\dot{x} = [3]x + [1]u$$

$$y = [2]x$$

Observable $[2] \rightarrow$ full rank

so, yes the system is observable.

Q7

$$\dot{x} = \overbrace{\begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix}}_{C} x$$

$$AB = A \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{bmatrix}$$

$$A^2B = A[AB] = A \begin{bmatrix} 0 \\ \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 \\ \lambda_1^2 \\ \lambda_2^2 \\ 0 \\ 0 \end{bmatrix}$$

$$A^3B = A[A^2B] = A \begin{bmatrix} 2\lambda_1 \\ \lambda_1^2 \\ \lambda_2^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\lambda_1^2 \\ \lambda_1^3 \\ \lambda_2^3 \\ 0 \\ 0 \end{bmatrix}$$

$$A^4B = A[A^3B] = A\begin{bmatrix} 3\lambda_1^2 \\ \lambda_1^3 \\ \lambda_2^3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4\lambda_1^3 \\ \lambda_1^4 \\ \lambda_2^4 \\ 0 \\ 0 \end{bmatrix}$$

$$P = [B \vdots AB \vdots A^2B \vdots A^3B \vdots A^4B]$$

$$\text{rank}(P) = 3$$

so we know, the first rows & corresponding values are controllable:

$$\dot{\hat{x}}_c = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} B_c$$

$$y_c = [0 \quad 1 \quad 1] C_c$$

$$C_c A_c = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \end{bmatrix}$$

$$C_c A_c^2 = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \lambda_1^2 & \lambda_2^2 \end{bmatrix}$$

Building $Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & \lambda_1^2 & \lambda_2^2 \end{bmatrix}$

rank of $Q = 2$ so, we get controllable and observable system as:

$$\dot{x}_{co} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x_{co} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y_{co} = \begin{bmatrix} 1 & 1 \end{bmatrix} x_{co}$$