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Q1 (a) $x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$

$$\Delta(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 \\ -0.5 & 0.5-\lambda \end{bmatrix}$$

$$= (1-\lambda)(0.5-\lambda) = 0$$

$\lambda = 1, 0.5$

all $|\lambda_i| \leq 1$

$\therefore$ stable i.s.L, Not asymptotically stable.

Q1 (b)

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$\Delta(\lambda) = \det \begin{bmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{bmatrix}$$

$$= (-7-\lambda)\left[(-3-\lambda)(1-\lambda) - 4\right] + 2\left[2 - 2\lambda - 4\right] + 6\left[-4 + 2(-3-\lambda)\right]$$

$$= -\lambda^3 - 9\lambda^2 - 23\lambda - 15 = 0$$

$$\lambda = -1, -5, -3$$

all eigenvalues have negative real part

so, Asymptotically stable.

and also stable i.s.L.

Q2(a) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = A$

$$A^TA = \begin{bmatrix} 2 & 1 & -1 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 6 & 2 & 6 \\ 2 & 6 & 2 \\ 6 & 2 & 6 \end{bmatrix}$$

$\det(A^TA - \lambda I) = 0$

$$\det \begin{bmatrix} 6-\lambda & 2 & 6 \\ 2 & 6-\lambda & 2 \\ 6 & 2 & 6-\lambda \end{bmatrix} = 0$$

$\lambda = 13.1231056, \; 4.87689437, \; 1.36175 \times 10^{-15}$

max of λ = $\max\{\lambda\}$

$= 13.1231056$

$\sigma_{max} = 3.622582725$

L_2 norm of a matrix,

$\|A\|_2$ = max singular value.

$\sigma = 3.622582725$

(b) $A = \begin{bmatrix} 10 & 2 \\ 0 & -3 \end{bmatrix}$

Nuclear norm of a matrix is sum of it's singular values.

$$A^T A = \begin{bmatrix} 10 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 10 & 2 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 100 & 20 \\ 20 & 13 \end{bmatrix}$$

$\det(A^TA - \lambda I) = 0$

$$\det\begin{bmatrix} 100-\lambda & 20 \\ 20 & 13-\lambda \end{bmatrix} = 0$$

$(100-\lambda)(13-\lambda) - 20 \times 20 = 0$

$1300 - 100\lambda - 13\lambda + \lambda^2 - 400 = 0$

$\lambda^2 - 113\lambda + 900 = 0$

$\lambda = 104.37744772, \ 8.62255228$

$\sigma = \sqrt{104.37744772}, \ \sqrt{8.62255228}$

$= 10.216528, \ 2.93641827$

Nuclear norm, $\|A\|_N = \sigma_1 + \sigma_2$

$= 13.152946$

Frobenius norm $\|A\|_F =$

$= \sqrt{\sigma_1^2 + \sigma_2^2}$

$= \sqrt{112.999999}$

$= 10.6301458$

Q2 (c)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

L_1 norm of a matrix is equal to the maximum of L1 norm of a columns of the matrix:

$\rightarrow$ $L_1 = \max\{(1+3), (2+4)\}$

$= \max\{4, 6\}$

$L_1 = 6$

$\rightarrow$ $L_2 =$ max singular value.

$$A^TA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

$\det(A^TA - \lambda I) = 0$ $\quad \det\begin{bmatrix} 10-\lambda & 14 \\ 14 & 20-\lambda \end{bmatrix} = 0$

$\lambda = 0.13393125, 29.86606875$

max of λ = 29.86606875

so max singular value = 5.46498570

so L_2 of A = 5.46498570.

→ L_∞ of A:

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \quad \text{(max row sum)}$$

$$= \max\{(1+2), (3+4)\}$$

$$= \max\{3, 7\}$$

$$\|A\|_\infty = 7$$

Q3 $\hat{\beta}_{lasso} = \text{argmin}_{\beta \in \mathbb{R}^{v}} \{K\}$

where $K = \frac{1}{N} \|y - X\beta\|_2^2 + \lambda \|\beta\|_1$

Given,

$$X = \begin{bmatrix} -2 & 0 & 5 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 3 & 5 \\ 7 & 4 \end{bmatrix} \quad \lambda = 20 \quad N = 2$$

$$\|\beta\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}| \quad \text{max column sum}$$

$$\|\beta\|_1 = \max\{(1+1+1+2), (2+2+2)\}$$

$$= \max\{5, 6\} = 6$$

$$X\beta = \begin{bmatrix} -2 & 0 & 5 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

$$y - X\beta = \begin{bmatrix} 3 & 5 \\ 7 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

let $Q = y - X\beta = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$$\|y - X\beta\|_2 = \|Q\|_2$$

$$Q^T Q = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$$

$$\det(Q^T Q - \lambda I) = 0$$

$$\det \begin{bmatrix} 20-\lambda & 10 \\ 10 & 5-\lambda \end{bmatrix} = 0$$

$$0 = (20-\lambda)(5-\lambda) - 100$$

$$\lambda = 25, 0$$

singular values $= 5, 0$

$\|Q\|_2$ = max singular value

$= 5$

$$K = \frac{1}{N} \|y - X\beta\|_2^2 + \lambda \|\beta\|_1$$

$N = 2$, $\lambda = 20$, $\|y - X\beta\|_2 = 5$, $\|\beta\|_1 = 6$

$$K = \frac{1}{2}(5)^2 + 20 \times 6$$

$$= \frac{25}{2} + 120 = \frac{25 + 240}{2}$$

$$K = \frac{265}{2} = \boxed{132.5}$$

Q4

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

find e^{At} for $t=2$.

$$e^{2A} = \begin{bmatrix} e^{2\times 2} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2} & 2e^{2} & 2e^{2} & 0 & 0 \\ 0 & 0 & e^{2} & 2e^{2} & 0 & 0 \\ 0 & 0 & 0 & e^{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2} \end{bmatrix}$$

$$e^{2A} = \begin{bmatrix} 54.598 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7.389 & 14.778 & 14.778 & 0 & 0 \\ 0 & 0 & 7.389 & 14.778 & 0 & 0 \\ 0 & 0 & 0 & 7.389 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.135 \end{bmatrix}$$

Q5 $\dot{x} = \begin{bmatrix} -1 & 0 \\ e^{-3t} & 0 \end{bmatrix} x$

for $t_0 \geq 0$

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = e^{-3t} x_1$$

$$\frac{dx_1}{dt} = -x_1 \Rightarrow \frac{dx_1}{x_1} = -dt$$

$$\int_{t_0}^{t} \frac{dx_1}{x_1} = \int_{t_0}^{t} -dt$$

$$\ln x_1 \Big|_{t_0}^{t} = -t \Big|_{t_0}^{t}$$

$$\ln \frac{x_1(t)}{x_1(t_0)} = -t + t_0$$

$$\boxed{x_1(t) = e^{(-t+t_0)} x_1(t_0)}$$

$$\dot{x}_2 = e^{-3t} x_1(t)$$

$$\frac{dx_2}{dt} = e^{-3t} e^{(-t+t_0)} x_1(t_0)$$

$$\int_{t_0}^{t} dx_2 = x_1(t_0)\int_{t_0}^{t} e^{-3t-t+t_0}\,dt$$

$$x_2(t)\Big|_{t_0}^{t} = x_1(t_0)\left(\frac{e^{-4t+t_0}}{-4}\Big|_{t_0}^{t}\right)$$

$$x_2(t) - x_2(t_0) = x_1(t_0)\left(-\frac{e^{-4t+t_0}}{4} + \frac{e^{-3t_0}}{4}\right)$$

$$x_2(t) = x_1(t_0)\left(-\frac{e^{-4t+t_0}}{4} + \frac{e^{-3t_0}}{4}\right) + x_2(t_0)$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} e^{(-t)} \\ \frac{e^0 - e^{-4t}}{4} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ \frac{1 - e^{-4t}}{4} \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} e^{-t} & 0 \\ \frac{1-e^{-4t}}{4} & 1 \end{bmatrix}$$

$$\phi(t,t_0) = X(t)\,X^{-1}(t_0)$$

$$= \begin{bmatrix} e^{-t} & 0 \\ \frac{1-e^{-4t}}{4} & 1 \end{bmatrix} \begin{bmatrix} e^{-t_0} & 0 \\ \frac{1-e^{-4t_0}}{4} & 1 \end{bmatrix}^{-1}$$

$$X^{-1}(t_0) = \frac{1}{e^{-t_0}} \begin{bmatrix} 1 & 0 \\ \frac{e^{-4t_0}-1}{4} & e^{-t_0} \end{bmatrix}$$

$$X^{-1}(t_0) = \begin{bmatrix} 1/e^{-t_0} & 0 \\ \frac{e^{-4t_0}-1}{4e^{-t_0}} & 1 \end{bmatrix}$$

$$\phi(t,t_0) = X(t)\;X^{-1}(t_0)$$

$$= \begin{bmatrix} \frac{e^{-t}}{e^{-t_0}} & 0 \\ \frac{e^{-4t_0}-e^{-4t}}{4e^{-t_0}} & 1 \end{bmatrix}$$

$\|\phi(t,t_0)\|_\infty$ = max of row sum.

$$= \frac{e^{-4t_0}-e^{-4t}}{4e^{-t_0}} + 1$$

$t \to \infty, \quad t_0 \geq 0, \quad t \geq t_0$

$$\to \frac{e^{-3t_0}}{4} + 1$$

so finite

$\therefore$ stable i.s.L

and not asymptotically stable.

Q6 $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$

where $\sigma_1 > \sigma_2 > \ldots > \sigma_n$ are the singular values of A.

we know SVD of $A = U \Sigma V^T$

Euclidean norm of a matrix is:

$\|A\|_2 = \max\{\sigma\}$

σ is singular values of A

If we take inverse of A and take SVD:

$A = U \Sigma V^T$

$A^{-1} = (U \Sigma V^T)^{-1}$

$= V \Sigma^{-1} U^T$

$\|A^{-1}\|_2 = \max\{\sigma^{-1}\}$

as singular values of A^{-1} are inverse of singular values of A

σ^{-1} is max when σ is min.

So, we can prove that

$$||A^{-1}||_2 = \frac{1}{\sigma_n}$$

where σ_n is the minimum singular value of A.

Q7

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{C} x$$

$$P = [B \vdots AB \vdots A^2B]$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \qquad \text{rank}(P) = 2$$

Eigenvalue of A:

$\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} -\lambda & -1 & 1 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} = 0$$

$\lambda = -2, -1, 0$

for $\lambda = -2$

$$(A - \lambda I) = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

for $\lambda = -1$

$$(A - \lambda I) = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

for $\lambda = 0$

$$(A - \lambda I) = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$M = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ $\det(M) \neq 0$

so $\hat{A} = M^{-1} A M$

$$M^{-1} = \begin{bmatrix} 0.5 & -1 & 0.5 \\ -1 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\hat{C} = CM = \begin{bmatrix} -1 & 0 & 3 \end{bmatrix}$$

$\hat{B}^{-2} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ $\neq$ full rank
$\Rightarrow$ uncontrollable

$\hat{B}^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ = full rank
$\Rightarrow$ controllable

$\hat{B}^{0} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ = full rank
$\Rightarrow$ controllable

Also, Rank of P was 2. so -2 is uncontrollable.

Rearranging :

$$\dot{x} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -2 \end{array}\right] x + \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|c} 3 & 0 & -1 \end{array}\right] x$$

$$\dot{x}_c = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x_c + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u_c$$

$$y_c = \begin{bmatrix} 3 & 0 \end{bmatrix} x_c$$

for observability $Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

rank = 1

not full rank.

So, <u>not observable</u>.

The system is <u>stabilizable</u> as every mode is controllable in the controllable form of the system expressed above.

Yes, it is <u>detectable</u> as $\lambda = -1$ is i.s.L.