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24-677 HW8

Q1

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

$$\dot{x}_1 = a x_1 \quad , \quad \dot{x}_2 = x_1 - x_2$$

Lyapunov's function:

$$V = x_1^2 + x_2^2 > 0 \quad \forall x \neq 0$$

$$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$
$$= 2x_1(ax_1) + 2x_2(x_1 - x_2)$$
$$= 2ax_1^2 + 2x_1x_2 - 2x_2^2$$
$$= 2(ax_1^2 + x_1x_2 - x_2^2)$$

for AS, $\dot{V}(x) < 0$

so, $2(ax_1^2 + x_1x_2 - x_2^2) < 0$

$$(ax_1^2 + x_1x_2 - x_2^2) < 0$$

$$ax_1^2 < x_2^2 - x_1x_2$$

$$\boxed{a < \frac{x_2^2 - x_1x_2}{x_1^2}}$$

$$a < \left(\frac{x_2}{x_1}\right)^2 - \left(\frac{x_2}{x_1}\right)$$

let $\left(\frac{x_2}{x_1}\right) = p$,

$$a < p^2 - p$$

By plotting, we can see that lowest $y = -0.25$ $\{p^2 - p = y\}$

so $a = (-\infty, -0.25)$

Q2 $\dot{x}_1 = x_2 - x_1 x_2^2 = -x_1 x_2^2 + x_2$

$\dot{x}_2 = -x_1^3$

(a)

Using Lyapunov's Indirect (1^{st}) method

$\equiv^m$ point is given by $\dot{x} = 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = 0 \qquad \begin{aligned} x_2 - x_1 x_2^2 &= 0 \quad -(1) \\ -x_1^3 &= 0 \quad -(2) \end{aligned}$$

from (2)

$$\boxed{x_1 = 0}$$

put x_1 in (1)

$$x_2 - 0 = 0$$

$$\boxed{x_2 = 0}$$

linearized system given by:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} -x_2^2 & -2x_1x_2 + 1 \\ -3x_1^2 & 0 \end{bmatrix}_{x_1=0,\, x_2=0}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = 0$$

$$-\lambda(-\lambda) = 0 \qquad \lambda^2 = 0 \qquad , \lambda = 0, 0$$

Not stable

Q 2(b)

$$V(x_1, x_2) = x_1^4 + 2x_2^2 > 0 \quad \forall x \neq 0$$

$$\dot{V} = 4x_1^3 \dot{x}_1 + 4x_2 \dot{x}_2$$

$$= 4x_1^3(x_2 - x_1 x_2^2) + 4x_2(-x_1^3)$$

$$= \cancel{4x_1^3 x_2} - 4x_1^4 x_2^2 - \cancel{4x_1^3 x_2}$$

$$= -4x_1^4 x_2^2 \leq 0$$

as x_1 & x_2 are raised to an even power which will always give no result as 0 or -ve number. So, yes, the system is Lyapunov Direct method <u>stable</u>.

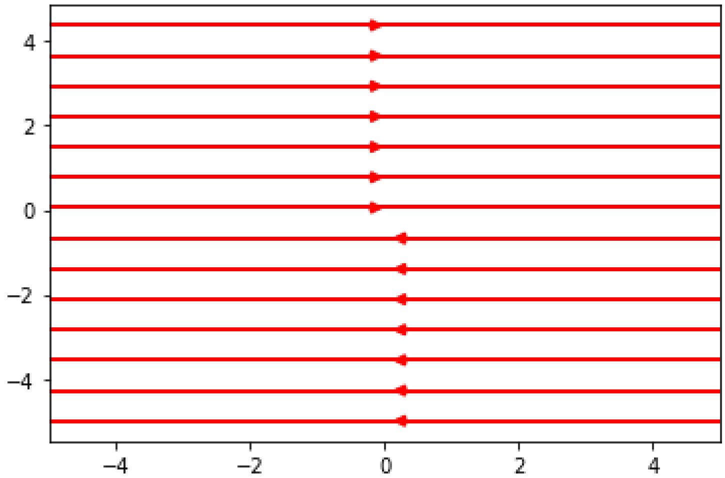
Question 2c

In [15]:

```
from mpl_toolkits import mplot3d
%matplotlib inline
import numpy as np
import matplotlib.pyplot as plt


fig = plt.figure()
x1 = np.linspace(-5,5,100)
x2 = x1

X, Y = np.meshgrid(x1,x2)

U = Y
V = 0*Y

plt.streamplot(X, Y, U, V, density = 0.5, color = 'red', linewidth = 2)
```

Out[15]:

```
<matplotlib.streamplot.StreamplotSet at 0x12b3127f0>
```



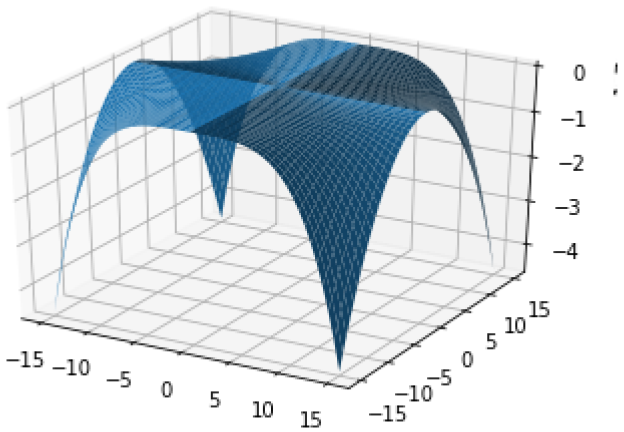
Question 2d

In [5]:

```
fig = plt.figure()

ax = plt.axes(projection='3d')
space = np.linspace(-15, 15, 2000)
x1 = space
x2 = space
X,Y = np.meshgrid(x1,x2)
vdot = -4 * X**4 * Y**2
ax.plot_surface(X, Y, vdot)
```

Out[5]:

```
<mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x1299bb780>
```



Q3(a)

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{B} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 5 & 5 \end{bmatrix}}_{C} x(k) \qquad D = 0$$

for BIBO stability,

$$G_D(z) = C(zI - A)^{-1}B + D$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} Z-1 & 0 \\ 0.5 & Z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} \frac{1}{(Z-1)(Z-0.5)} \begin{bmatrix} Z-0.5 & 0 \\ -0.5 & Z-1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} 1/(Z-1) & 0 \\ \frac{-0.5}{(Z-1)(Z-0.5)} & \frac{1}{(Z-0.5)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{(Z-1)} \\ \frac{-0.5}{(Z-1)(Z-0.5)} - \frac{1}{(Z-0.5)} \end{bmatrix}$$

$$= \frac{5}{(z-1)} - \frac{2.5}{(z-1)(z-0.5)} - \frac{5}{(z-0.5)}$$

$$= \frac{5(z-0.5)}{(z-1)(z-0.5)} - \frac{2.5}{(z-1)(z-0.5)} - \frac{5(z-1)}{(z-1)(z-0.5)}$$

$$= \frac{5(z-0.5) - 2.5 - 5(z-1)}{(z-1)(z-0.5)}$$

$$= \frac{5(\cancel{z}-0.5-\cancel{z}+1) - 2.5}{(z-1)(z-0.5)}$$

$$= \frac{2.5 - 2.5}{(z-1)(z-0.5)} = 0$$

Stable as output doesn't depend on the input.

Q3 (b)

$$\dot{x} = \overbrace{\begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}}^{B} u$$

$$\dot{y} = \overbrace{\begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}}^{C} x \qquad D = 0$$

for BIBO stable:

$$G_c(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{s^2+2s-7}{\nabla_2} & \nabla_1 & \frac{2(3s+11)}{\nabla_2} \\ \frac{2}{s^2+8s+15} & \frac{1}{s+3} & -\frac{2}{s^2+8s+15} \\ \nabla_1 & \nabla_1 & \frac{s+5}{s^2+4s+3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

where $\nabla_1 = -\dfrac{2}{s^2+4s+3}$

$\nabla_2 = s^3 + 9s^2 + 23s + 15$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+3} & \frac{1}{s+5} \\ \frac{1}{s+3} & -\frac{1}{s+5} \\ \frac{1}{s+3} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ \frac{1}{s+3} & 0 \end{bmatrix}$$

Pole $= -3$

So we have a negative real pole

$\therefore$ BIBO stable.

Q4 $\frac{U(s)}{Y(s)} = \frac{s+3}{s^2+3s+2} = G(s)$

$$G(s) = \frac{0s^2+s+3}{s^2+3s+2}$$

compare with:

$$G(s) = \frac{b_n s^n + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} x + [0] u$$

Q5

$$G_1(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix}$$

$$D = G_1(\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

we know

$$G = G_{sp} + D$$

$$\Rightarrow G_{sp} = G - D$$

$$= \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$G_{sp} = \begin{bmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{1}{s+3} & \frac{-1}{s+1} \end{bmatrix}$$

$$d(s) = s(s+1)(s+3) = (s^2+s)(s+3)$$

$$= s^3 + 3s^2 + s^2 + 3s = s^3 + 4s^2 + 3s + 0$$

$$G_{sp} = \frac{1}{(s^3+4s^2+3s)} \begin{bmatrix} (s+1)(s+3) & 2s(s+3) \\ s(s+1) & -s(s+3) \end{bmatrix}$$

$$= \frac{1}{(s^3+4s^2+3s+0)} \begin{bmatrix} s^2+4s+3 & 2s^2+6s \\ s^2+s & -s^2-3s \end{bmatrix}$$

$\boxed{\rho = 2}$

$$N_1 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \qquad N_2 = \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix} \qquad N_3 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Q6 System 1:

$$\dot{x} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 2 & 2 \end{bmatrix}}_{C} x$$

$D = 0$

system 2:

$$\dot{x} = \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{B} u \qquad y = \underbrace{\begin{bmatrix} 2 & 0 \end{bmatrix}}_{C} x$$

$D = 0$

for system 1: $G(s) = C(sI - A)^{-1}B + D$

$$(sI - A)^{-1} = \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}^{-1} = \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-1 & 1 \\ 0 & s-2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(s-2)} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{(s-1)} \end{bmatrix}$$

$$C(sI - A)^{-1}B = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{(s-1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} \\ 0 \end{bmatrix} = \boxed{\frac{2}{s-2}}$$

for system2: $G(s) = C(sI-A)^{-1}B + D$

$$(sI-A)^{-1} = \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}^{-1} = \frac{1}{(s-2)(s+1)} \begin{bmatrix} s+1 & 0 \\ -1 & s-2 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s-2} & 0 \\ \frac{-1}{(s-2)(s+1)} & \frac{1}{s+1} \end{bmatrix}$$

$$C(sI-A)^{-1}B = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} & 0 \\ \frac{-1}{(s-2)(s+1)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{s-2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{\frac{2}{s-2}}$$

Yes, the two systems are equivalent.

for system 1: $n = 2$

$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$P = [B \vdots AB]$

$AB = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ rank of P is not full rank. so not controllable

so, system 1 is not minimal realization.

for system 2: $n = 2$

$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P = [B \vdots AB]$

$AB = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$ rank $= 2$ = full rank.

so system 2 is controllable.

$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 \end{bmatrix}$

$$CA = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$ rank = 1 ≠ not full rank

So, system 2 is not observable

∴ system 2 is also not minimal realization.