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24-677 HW-5

Q1 $x(k+1) = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$ —①

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$ —②

$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $u(k) = 1$

$k = 0, 1, \dots, \infty$

find $y(5)$

$x(1) = Ax(0) + Bu(0)$

$x(2) = Ax(1) + Bu(1)$

$= A(Ax(0) + Bu(0)) + Bu(1)$

$= A^2x(0) + ABu(0) + Bu(1)$

$\vdots$

$x(k) = A^k x(0) + \sum_{m=0}^{k-1} A^{k-m-1} Bu(m)$

$y(k) = CA^k x(0) + \sum_{m=0}^{k-1} CA^{k-m-1} Bu(m)$

To find $y(5)$

$\boxed{K=5}$

$$x(5) = A^5 x(0) + \sum_{m=0}^{4} A^{4-m} B u(m) \quad - (3)$$

$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $A^5 x(0)$ is zero.

so, max A^4 is needed.

$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$ $\qquad A^0 = I$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ 0.5 & 0.5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -0.5 & -1 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ -0.25 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0.5 & 0.5 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} -0.25 & 0 \\ 0 & -0.25 \end{bmatrix}$$

(3) becomes.

$$x(5) = A^5 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.25 & 0 \\ 0 & -0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1) +$$

$$\begin{bmatrix} 0.5 & 0.5 \\ -0.25 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1) + \begin{bmatrix} -0.5 & -1 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1) + \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1)$$

$$+ I \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1)$$

$$x(5) = \begin{bmatrix} -0.25 \\ -0.25 \end{bmatrix} + \begin{bmatrix} 1 \\ -0.25 \end{bmatrix} + \begin{bmatrix} -1.5 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(5) = \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}$$

$$y(5) = CA^5 x(0) + \sum_{m=0}^{4} CA^{4-m} Bu(m)$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= CA^5 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \underbrace{C \sum_{m=0}^{4} A^{4-m} Bu(m)}_{\text{equal to } \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1.25 \\ 0 \end{bmatrix} = \boxed{1.25}$$

$$y(5) = 1.25$$

Q2. $\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 2u(t)$

$T = 1s.$

$A_d = e^{AT}$, $A = M\hat{A}M^{-1}$

eigenvalues of A: $\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$

$\det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$

$(-2-\lambda)(-\lambda) - 1 = 0$

$2\lambda + \lambda^2 - 1 = 0$

$\lambda^2 + 2\lambda - 1 = 0$

$\lambda = -1 + \sqrt{2}, \; -1 - \sqrt{2}$

$0.4142 \qquad -2.4142$

$\hat{A} = \begin{bmatrix} -1-\sqrt{2} & 0 \\ 0 & -1+\sqrt{2} \end{bmatrix}$

$M: \begin{bmatrix} -1-\sqrt{2} & -1+\sqrt{2} \\ 1 & 1 \end{bmatrix}$ $M^{-1} = \begin{bmatrix} \frac{-1}{2\sqrt{2}} & -\frac{1-\sqrt{2}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{-1-\sqrt{2}}{2\sqrt{2}} \end{bmatrix}$

$$A_d = \begin{bmatrix} -1-\sqrt{2} & -1+\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(-1-\sqrt{2})} & 0 \\ 0 & e^{-1+\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{2\sqrt{2}} & -\frac{1-\sqrt{2}}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{-1-\sqrt{2}}{2\sqrt{2}} \end{bmatrix}$$

$$A_d = \begin{bmatrix} 0.2979 & 0.5033 \\ 0.5033 & 1.3046 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B_d = A^{-1}(A_d - I)B$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 0.2979 & 0.5033 \\ 0.5033 & 1.3046 \end{bmatrix} - I \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0.5033 \\ 0.3045 \end{bmatrix}$$

Q3 $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t)$

$x(0) = 0$

u = unit step.

find $y(5)$

$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ eigenvalues of A

$\det[A - \lambda I] = 0$

$\lambda = -1 \pm i$

using cayley-Hamilton :

$e^{\lambda t} = \beta_1 \lambda + \beta_0$

for $\lambda = -1 + i$

$e^{-t+it} = e^{-t}(\cos t + i \sin t)$ — ①

$= \beta_1(-1+i) + \beta_0$

$= -\beta_1 + \beta_1 i + \beta_0$

$= (-\beta_1 + \beta_0) + \beta_1 i$ — ②

compare ① & ②

$\boxed{\beta_1 = e^{-t} \sin t}$

$\beta_0 - \beta_1 = e^{-t} \cos t$

$\beta_0 = e^{-t} \cos t + e^{-t} \sin t$

$\boxed{\beta_0 = e^{-t}(\cos t + \sin t)}$

for $\lambda = -1-i$

$$e^{-t-it} = e^{-t}(\cos t - i\sin t) \quad — ③$$

$$= \beta_1(-1-i) + \beta_0$$

$$= -\beta_1 - \beta_1 i + \beta_0$$

$$= (\beta_0 - \beta_1) - \beta_1 i \quad — ④$$

compare ③ & ④

$$\boxed{\beta_1 = e^{-t}\sin t}$$

$$\beta_0 - \beta_1 = e^{-t}\cos t$$

$$\boxed{\beta_0 = e^{-t}(\cos t + \sin t)}$$

so, $e^{At} = \beta_1 A + \beta_0 = e^{-t}\begin{bmatrix} \sin t + \cos t & \sin t \\ -2\sin t & \cos t - \sin t \end{bmatrix}$

$$x(t) = e^{A(t-t_0)}\underbrace{x(t_0)}_{0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

we have $t_0 = 0$ & $x(t_0) = 0$ {initial cond}

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$= \int_0^t e^{-(t-\tau)} \begin{bmatrix} \sin(t-\tau)+\cos(t-\tau) & \sin(t-\tau) \\ -2\sin(t-\tau) & \cos(t-\tau)-\sin(t-\tau) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau$$

$$x(t) = \int_0^t e^{-(t-\tau)} \begin{bmatrix} 2\sin(t-\tau)+\cos(t-\tau) \\ -3\sin(t-\tau)+\cos(t-\tau) \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 1.5 - 0.5e^{-t}(\sin t + 3\cos t) \\ e^{-t}(2\sin t + \cos t) - 1 \end{bmatrix}$$

$$x(5) = \begin{bmatrix} 1.5004 \\ -1.011 \end{bmatrix}$$

$$y(5) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(5)$$

$$= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1.5004 \\ -1.011 \end{bmatrix} \boxed{= -0.0323}$$

Q4 solve for $x(3)$

$$x = [x_1 \quad x_2 \quad x_3]^T$$

$$x_1(k+1) = \frac{1}{2}x_1(k) - \frac{1}{2}x_2(k) + x_3(k)$$

$$x_2(k+1) = \frac{1}{2}x_2(k) + 2x_3(k)$$

$$x_3(k+1) = \frac{1}{2}x_3(k)$$

$$x(0) = [2 \quad 4 \quad 6]^T$$

$$x(k+1) = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$x(1) = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$x(1) = \begin{bmatrix} 5 \\ 14 \\ 3 \end{bmatrix}$$

$$x(2) = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 14 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 13 \\ 1.5 \end{bmatrix}$$

$$x(3) = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1/2 & 2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1.5 \\ 13 \\ 1.5 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -5.75 \\ 9.5 \\ 0.75 \end{bmatrix}}}$$

Q5 $\dot{x} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} x$

$\dot{x}_1 = -\sin t \, x_1$

$\dot{x}_2 = -\cos t \, x_2$

$\frac{dx_1}{dt} = -\sin t \, x_1 \Rightarrow \frac{dx_1}{x_1} = -\sin t \, dt$

$\int_{t_0}^{t} \frac{dx_1}{x_1} = \int_{t_0}^{t} -\sin t \, dt$

$\ln x_1 \Big|_{t_0}^{t} = \cos t \Big|_{t_0}^{t}$

$\frac{x_1(t)}{x_1(t_0)} = e^{(\cos t - \cos t_0)}$

$\boxed{x_1(t) = x_1(t_0) \, e^{(\cos t - \cos t_0)}}$

$\dot{x}_2 = -\cos t \, x_2$

$\frac{dx_2}{dt} = -\cos t \, x_2 \Rightarrow \frac{dx_2}{x_2} = -\cos t \, dt$

$\int_{t_0}^{t} \frac{dx_2}{x_2} = \int_{t_0}^{t} -\cos t \, dt$

$$\ln x_2 \Big|_{t_0}^{t} = -\sin t \Big|_{t_0}^{t}$$

$$\frac{x_2(t)}{x_2(t_0)} = e^{\sin t_0 - \sin t}$$

$$\boxed{x_2(t) = x_2(t_0)\, e^{\sin t_0 - \sin t}}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} e^{\cos t - \cos t_0} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ e^{\sin t_0 - \sin t} \end{pmatrix}$$

$$X(t) = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{\sin t_0 - \sin t} \end{bmatrix}$$

$$\phi(t, t_0) = X(t) X^{-1}(t_0)$$

$$= \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{\sin t_0 - \sin t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{\sin t_0 - \sin t} \end{bmatrix}$$

Q6.
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

$$\dot{x} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

multiply both sides by e^{-At}

$$e^{-At}\dot{x} - e^{-At}Ax = 0$$

from here we can say that
$$\frac{d}{dt}\left(e^{-At}x\right) = 0$$

from this statement, we can say $e^{-At}x$ doesn't change over time. i.e it is constant so,
$$e^{-At}x(t) - e^{-At_0}x(t_0) = 0$$
$$x(t) = \frac{e^{-At_0}x(t_0)}{e^{-At}}$$

If we put $t_0 = 0$
$$x(t) = \frac{e^{-A0}x(0)}{e^{-At}} = e^{At}x(0)$$

so $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

so for e^{At}

eigenvalues of A:

$\det(A - \lambda I) = 0$

$\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$ $\qquad\qquad$ $\lambda = \sqrt{-1} = i$

Cayley - Hamilton:

$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$

$e^{it} = \alpha_0 + \alpha_1 i$

we know $\quad e^{i\theta} = \cos\theta + i\sin\theta$ applying that to the eqx.

$\cos(t) + i\sin(t) = \alpha_0 + \alpha_1 i$

$\Rightarrow \alpha_0 = \cos(t)$

$\alpha_1 = \sin(t)$

$\therefore \quad e^{\lambda t} = \cos t + \lambda \sin t$

from C-H,

$$e^{At} = \cos t + \sin t\,[A]$$

$$e^{At} = \begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix} + \begin{bmatrix} 0 & \sin t \\ -\sin t & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

So, we get:

$$x(t) = e^{At} x(0)$$

$$\Rightarrow \boxed{x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)}$$