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27-677 HW4

Q1 (a) $A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$(2-\lambda)((2-\lambda)(1-\lambda) - 0) - 0 + 0 = 0$

$(2-\lambda)(2 - 2\lambda - \lambda + \lambda^2) = 0$

$(2-\lambda)(2 - 3\lambda + \lambda^2) = 0$

so,

$2 - \lambda = 0$

$\boxed{\lambda = 2}$

$2 - 3\lambda + \lambda^2 = 0$

$2 - 2\lambda - \lambda + \lambda^2 = 0$

$2(1-\lambda) - \lambda(1-\lambda) = 0$

$(1-\lambda)(2-\lambda) = 0$

$(1-\lambda) = 0$

$\boxed{\lambda = 1}$

$(2-\lambda) = 0$

$\boxed{\lambda = 2}$

$\lambda = (1, 2, 2)$

(b) $A_2 = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(2-\lambda)\Big((1-\lambda)(-1-\lambda) - 3\Big) + 2\Big((-1-\lambda) - 1\Big) + 3\Big(3 - 1 + \lambda\Big) = 0$$

$$(2-\lambda)\left(-1 \cancel{-\lambda} \cancel{+\lambda} + \lambda^2 - 3\right) + 2(-\lambda - 2) + 3(2 + \lambda) = 0$$

$$(2-\lambda)\left(\lambda^2 - 4\right) - 2(\lambda + 2) + 3(2 + \lambda) = 0$$

$$2\lambda^2 - 8 - \lambda^3 + 4\lambda - 2\lambda - 4$$
$$+ 6 + 3\lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$\lambda = -2, 1, 3$$

Ans

① $A_1 = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$$A_1^T A_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2 + 4\times4 & 2\times1 + 4\times2 \\ 1\times2 + 2\times4 & 1\times1 + 2\times2 \end{bmatrix}$$

$$A_1^T A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$$

eigenvalues of $A_1^T A_1$,

$$\det(A_1^T A_1 - \lambda I) = 0$$

$$\det \begin{bmatrix} 20-\lambda & 10 \\ 10 & 5-\lambda \end{bmatrix} = 0$$

$$20-\lambda\,(5-\lambda) - 10(10) = 0$$

$$\cancel{100} - 20\lambda - 5\lambda + \lambda^2 - \cancel{100} = 0$$

$$\lambda^2 - 25\lambda = 0$$

$$\lambda(\lambda - 25) = 0$$

$$\boxed{\lambda = 0, 25}$$

(2) $A_1 A_1^T$

$$= \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 2 + 1\times 1 & 2\times 4 + 1\times 2 \\ 4\times 2 + 2\times 1 & 4\times 4 + 2\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

$$\det(A_1 A_1^T - \lambda I) = 0$$

$$\det \begin{bmatrix} 5-\lambda & 10 \\ 10 & 20-\lambda \end{bmatrix} = 0$$

$$(5-\lambda)(20-\lambda) - 10(10) = 0$$

$$\cancel{5\times 20} - 5\lambda - 20\lambda + \lambda^2 - \cancel{100} = 0$$

$$\lambda^2 - 25\lambda = 0$$

$$\lambda(\lambda - 25) = 0$$

$$\boxed{\lambda = 0, 25}$$

③ Singular values of A_1 are equal to square root of eigen values of $A_1^T A_1$.

So, from Q2 part ①, we know the eigen values of $A_1^T A_1$ are

0 & 25.

To find singular values, we will take square root of the eigenvalues

$$\sqrt{25} = \pm 5$$

singular values $= 0, 5$

④ $A_2 = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

eigenvalues of $A_2^T A_2$

$$= \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1\times -1 + 2\times 2 & -1\times 0 + 2\times -1 & -1\times 1 + 2\times 0 \\ 0\times -1 + -1\times 2 & 0\times 0 + -1\times -1 & 0\times 1 + -1\times 0 \\ -1\times 1 + 0\times 2 & 1\times 0 + 0\times -1 & 1\times 1 + 0\times 0 \end{bmatrix}$$

$$A_2^T A_2 = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\det(A_2^T A_2 - \lambda I) = 0$$

$$\begin{bmatrix} 5-\lambda & -2 & -1 \\ -2 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix}$$

$$(5-\lambda)\big((1-\lambda)(1-\lambda)+0\big)+2\big(-2(1-\lambda)\big)$$
$$-1(1-\lambda) = 0$$

$$(5-\lambda)(1-\lambda-\lambda+\lambda^2) - 4 + 4\lambda - 1 + \lambda = 0$$

$$5 - 10\lambda + 5\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 4 + 4\lambda - 1 + \lambda = 0$$

$$-\lambda^3 + 7\lambda^2 + 5\lambda - 11\lambda = 0$$
$$-\lambda^3 + 7\lambda^2 - 6\lambda = 0$$
$$-\lambda(\lambda^2 - 7\lambda + 6) = 0$$

$$\boxed{\lambda = 0}$$

$$\lambda^2 - 7\lambda + 6 = 0$$
$$\lambda^2 - 6\lambda - \lambda + 6 = 0$$
$$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$$
$$(\lambda - 6)(\lambda - 1) = 0$$

$$\boxed{\lambda = 0, 1, 6}$$

$$\boxed{\lambda = 6, 1}$$

singular value: $\sqrt{0, 1, 6}$

$\sqrt{1} = 1$ $\sqrt{6} = \pm 2.449489$

$0, 1, 2.4494$

$\therefore 1, 2.45$

Q3 Characteristics polynomial

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} -\alpha_1 - \lambda & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$(-\alpha_1 - \lambda)\left(-\lambda(\lambda^2)\right) + \alpha_2(\lambda^2) - \alpha_3(-\lambda)$$

$$+ \alpha_4(1) = 0$$

$$(\alpha_1 + \lambda)(\lambda^3) + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0$$

$$\alpha_1\lambda^3 + \lambda^4 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0$$

$$\lambda^4 + \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 = 0$$

Q3(b) $\lambda^4 = -\alpha_1\lambda^3 - \alpha_2\lambda^2 - \alpha_3\lambda - \alpha_4$

So, we write the equation in matrix form.

$$\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$$

from the above, we can compare & see the matrix equation is in the form. $Av = \lambda v$

A is given in the question

from above we can generalised eigen vector is

$$\begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$$

let's normalize: $\dfrac{1}{\sqrt{1^2+\lambda^2+\lambda^4+\lambda^6}} \begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$

:

Q4

$$A_1 = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\det(A_1 - \lambda I) = 0$ to find eigenvalues of A_1

$$\det \begin{bmatrix} 1-\lambda & 4 & 8 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\left((2-\lambda)(3-\lambda)\right) = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\boxed{\lambda = 1, 2, 3}$$

for $\lambda = 1$

$$0 = \begin{bmatrix} 1-\lambda & 4 & 8 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} v_1 = \begin{bmatrix} 0 & 4 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} v_1$$

$$\begin{bmatrix} 0 & 4 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4 v_{12} + 8 v_{13} = 0$$

$$\boxed{v_{12} = 0}$$

$$\boxed{2 v_{13} = 0}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 2$

$$\begin{bmatrix} 1-\lambda & 4 & 8 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} v_2 = \begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{23} = 0$$

$$-v_{21} + 4 v_{22} + \cancel{8 v_{23}}^{\,0} = 0$$

$$v_{21} = 4 v_{22}$$

$$v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 3$,

$$\begin{bmatrix} 1-\lambda & 4 & 8 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} v_3 = \begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-v_{32} = 0$$

$$-2v_{31} + \cancel{4v_{32}}^{\,0} + 8v_{33} = 0$$

$$\cancel{2}v_{31} = \overset{4}{\cancel{8}}v_{33} \qquad v_3 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$v_{31} = 4v_{33}$$

$$M = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det(M) = 1 \neq 0$$

For Jordan form, we know,

$$A = M \Lambda M^{-1} \text{ so, } \quad \Lambda = M^{-1} A M$$

$$\begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 & 12 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$

$$\det(A_2 - \lambda I) = 0$$

$$\det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix} = 0$$

$$-\lambda\left((-\lambda)(-3-\lambda) + 4\right) - 1(2) = 0$$

$$-\lambda(3\lambda + \lambda^2 + 4) - 2 = 0$$

$$-3\lambda^2 - \lambda^3 - 4\lambda - 2 = 0$$

$$-\lambda^3 - 3\lambda^2 - 4\lambda - 2 = 0$$

$$\lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$$

$$\lambda = -1, -1-i, -1+i$$

for $\lambda = -1$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix} v_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row reduced echelon form,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

for $\lambda = -1-i$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix} v_2 = \begin{bmatrix} 1+i & 1 & 0 \\ 0 & 1+i & 1 \\ -2 & -4 & -2+i \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} -2 & -4 & -2+i \\ 0 & 1+i & 1 \\ 1+i & 1 & 0 \end{bmatrix} \xrightarrow[R_3 - (1+i)R_1]{R_1 \to \frac{R_1}{2}} \begin{bmatrix} 1 & 2 & -1-\frac{i}{2} \\ 0 & 1+i & 1 \\ 0 & -1-2i & \frac{-3-i}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1-\frac{i}{2} \\ 0 & 1 & \frac{1-i}{2} \\ 0 & 1+i & 1 \end{bmatrix}$$

echelon form.

$$\begin{bmatrix} 1 & 2 & -1-\frac{i}{2} \\ 0 & 1 & \frac{1-i}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1/2 \\ \frac{i-1}{2} \\ 1 \end{bmatrix}$$

for $\lambda = -1+i$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix} v_3 = \begin{bmatrix} 1-i & 1 & 0 \\ 0 & 1-i & 1 \\ -2 & -4 & -3+1-i \end{bmatrix}$$

row reduced echelon form:

$$\begin{bmatrix} 1 & 0 & -i/2 \\ 0 & 1 & \frac{1+i}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} i/2 \\ \frac{-1-i}{2} \\ 1 \end{bmatrix}$$

As number of eigenvectors (null space of eigenvalues) is equal to number of eigenvalues

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1+i \end{bmatrix}$$

(c) $A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\det(A_3 - \lambda I) = 0$

$\det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$

$(1-\lambda)\left[((1-\lambda)(2-\lambda))\right] = 0$

$(\lambda-1)(\lambda-1)(\lambda-2) = 0$

$\lambda = 1, 1, 2$

for $\lambda = 1$,

$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} v_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$v_1 = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right]$$

for $\lambda = 2$,

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} v_2 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

number of eigenvectors is equal to number of eigenvalues so,

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(d) $A_Y = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$

$$\det(A_Y - \lambda I) = 0$$

$$\det \begin{bmatrix} -\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{bmatrix} = 0$$

$$-\lambda\left((20-\lambda)(-20-\lambda) + 25 \times 16\right) = 0$$

$$-\lambda\left((-\lambda)^2 - (20)^2 + 25 \times 16\right) = 0$$

$$-\lambda\left(\lambda^2 - \cancel{400} + \cancel{400}\right) = 0$$

$$\lambda^3 = 0$$

$$\lambda = 0, 0, 0$$

for $\lambda = 0$,

$$\begin{bmatrix} -\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{bmatrix} v_1 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced row echelon form

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

number of eigenvectors $\neq$ number of eigenvalues.
or multiplicity in this case.

So, # of non-zero = multiplicity − # of eigen vector

$3 - 1 = 2$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Used Rrefcalculator.com for reduced row echelon form.

Q5 $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\left(-\lambda(1-\lambda)\right) = 0$$

$$\lambda = 0, 1, 1$$

Characteristic equation

$$(1-\lambda)(-\lambda + \lambda^2) = 0$$

$$-\lambda + \lambda^2 + \lambda^2 - \lambda^3 = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

Substitute λ with A.

$$A^3 - 2A^2 + A = 0$$

$$A^3 = 2A^2 - A$$

multiply with A^7.

$$A^7 A^3 = 2A^7A^2 - A^7A$$

$$\boxed{A^{10} = 2A^9 - A^8} \quad -① $$

Similarly, $\boxed{A^{103} = 2A^{102} - A^{101}} \quad -②$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

we can calculate A_3 from equation,

$$A^3 = 2A^2 - A$$

$$= \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so we can see the trend that element (1,3) is $(n-1)$ of the A^n matrix so general equation

$$A^n = \begin{bmatrix} 1 & 1 & (n-1) \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{103} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

for e^{At}

$$f(\lambda) = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0 = e^{\lambda t} \quad -①$$

we have eigenvalues as $\boxed{0, 1, 1}$

put $\lambda = 0$ in the eqn. $-①$

$$f(0) = \beta_2 0 + \beta_1 0 + \beta_0 = 1$$

$$\boxed{\beta_0 = 1} \quad -②$$

differentiate wrt λ

$$\frac{d f(\lambda)}{d\lambda} = 2\lambda\beta_2 + \beta_1 = t e^{\lambda t}$$

put $\lambda = 1$

$$2\beta_2 + \beta_1 = t e^{t} \quad -③$$

put $\lambda = 1$, in equation ①

$$\beta_2 + \beta_1 + \beta_0 = e^{t}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ te^t \\ e^t \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ te^t \\ e^t \end{bmatrix}$$

$$\beta_0 = 1$$
$$\beta_1 = -2 - te^t + 2e^t$$
$$\beta_2 = 1 + te^t - e^t$$

$$e^{At} = \beta_0 + \beta_1 A + \beta_2 A^2$$
$$= 1[I] + (-2 - te^t + 2e^t)A + (1 + te^t - e^t)A^2$$

for $t = 1$,

$$\begin{bmatrix} 2.71828183 & 1.71828183 & 1 \\ 0 & 1 & 1.71828183 \\ 0 & 0 & 2.71828183 \end{bmatrix}$$

Q6

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)\Big((2-\lambda)(1-\lambda)\Big) = 0$$

$$\lambda = 1, 2, 2$$

for $\lambda = 1$,

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} v_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{11} = 0$$
$$v_{12} + v_{13} = 0$$
$$v_{12} = -v_{13}$$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

for $\lambda = 2$,

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{21} + v_{23} = 0$$
$$v_{21} = -v_{23}$$

$$v_2 = \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right]$$

we have 3 eigenvectors and 3 eigenvalues
so, the Diagonalized matrix is:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$