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24-677 HW3

Q1

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Gram-Schmidt to find orthonormal basis.

let's denote orthonormal basis as u_1, u_2, u_3

$$u_1 = \frac{v_1}{\|v_1\|} \qquad \|v_1\| = \sqrt{1^2+2^2+0^2} = \sqrt{1+4} = \sqrt{5}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$u_2' = v_2 - (u_1 \cdot v_2)\, u_1$$

$$= \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \left(\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} 8 & 1 & 6 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \Big(\underbrace{\frac{8}{\sqrt{5}} + \frac{2}{\sqrt{5}} + 0}_{\frac{10}{\sqrt{5}}} \Big) \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \begin{bmatrix} \overset{2}{\cancel{10}}/\cancel{5} \\ \overset{4}{\cancel{20}}/\cancel{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$$

$$u_2 = \frac{u_2'}{\| u_2' \|} \qquad \| u_2' \| = \sqrt{6^2 + (-3)^2 + 6^2}$$

$$= \sqrt{36 + 9 + 36}$$

$$= \sqrt{81} = 9$$

$$u_2 = \frac{1}{9} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}$$

$$u_3' = v_3 - (u_2 \cdot v_3)u_2 - (u_1 \cdot v_3)u_1$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}$$

$$\underbrace{- \left(\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}}_{0}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left(-\frac{2}{3} \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -4/9 \\ +2/9 \\ +4/9 \end{bmatrix}$$

$$u_3' = \begin{bmatrix} +4/9 \\ -2/9 \\ 5/9 \end{bmatrix}$$

$$u_3 = \frac{u_3'}{\|u_3'\|} \qquad \|u_3'\| = \sqrt{(+4/9)^2 + (-2/9)^2 + (5/9)^2}$$

$$= \sqrt{\frac{16}{81} + \frac{4}{81} + \frac{25}{81}} = \sqrt{\frac{45}{81}}$$

$$\text{vector} = \begin{bmatrix} +4/9 \\ -2/9 \\ 13/9 \end{bmatrix}$$

$$\text{mag} = \boxed{\frac{\sqrt{45}}{9}}$$

$$= \frac{9}{\sqrt{45}} \begin{bmatrix} +4/9 \\ -2/9 \\ 5/9 \end{bmatrix}$$

$$= \begin{bmatrix} +\frac{4}{\sqrt{45}} \\ -\frac{2}{\sqrt{45}} \\ 5/\sqrt{45} \end{bmatrix}$$

Q2

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 5/3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1/3 \end{bmatrix}$$

$$\langle v_1, u \rangle = \left(\begin{bmatrix} 1 & 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = 0$$

let $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$$= u_1 + 0 + \frac{5}{3} u_3 = 0$$

$$0 = u_1 + \frac{5}{3} u_3 \Rightarrow \boxed{u_1 = -\frac{5}{3} u_3}$$

$$\langle v_2, u \rangle = \left(\begin{bmatrix} 0 & 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = 0$$

$$= u_2 - \frac{1}{3} u_3 = 0$$

$$\Rightarrow \boxed{u_2 = \frac{1}{3} u_3}$$

from the equations, we can derive a relationship between the coordinates

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -5/3\, u_3 \\ 1/3\, u_3 \\ u_3 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 1/3 \\ 1 \end{bmatrix}$$

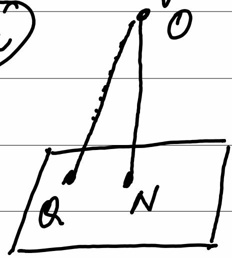
removing fractions

$$\begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$$

Q3 $\quad 4x_1 + 10x_2 - 4x_3 = -8$

origin (0,0,0)

(a)



perpendicular to the plane, vector is:

$\vec{P} = \langle 4, 10, -4 \rangle$

let's assume a point on the plane

$\vec{Q} = \langle -2, 0, 0 \rangle \quad \{x_2 = x_3 = 0$ set in equation$\}$

So for the minimum distance between the plane & the origin is the projection of line OQ onto perpendicular ON.

So, distance $= \left| \dfrac{\vec{OQ} \cdot \vec{P}}{|\vec{P}|} \right|$

$$= \left| \frac{-2 \times 4 + 0 \times 10 + 0 \times -4}{\sqrt{4^2 + 10^2 + (-4)^2}} \right| = \left| \frac{-8}{\sqrt{16+100+16}} \right| = \frac{8}{\sqrt{132}}$$

$$= \underline{\underline{0.6963106238}}$$

Q3 (b). Lets assume a point on the plane:

x_1 = origin x coordinate + $4t$ = $4t$

x_2 = origin y coordinate + $10t$ = $10t$

x_3 = origin z coordinate $-4t$ = $-4t$

So, the point coordinates should now satisfy the equation of the plane.

$$4(4t) + 10(10t) - 4(-4t) = -8$$

$$16t + 100t + 16t = -8$$

$$132t = -8$$

$$t = \frac{-8}{132} = -0.06060606$$

$$x_1 = 4t = 4 \times \frac{-8}{132} = \frac{-32}{132} = -0.242424242$$

$$x_2 = 10t = 10 \times \frac{-8}{132} = \frac{-80}{132} = -0.606060606$$

$$x_3 = -4t = -4 \times \frac{-8}{132} = \frac{+32}{132} = 0.242424242$$

closest point $(-0.242424, -0.606060, 0.242424)$

$$A \quad x = y$$

Q4 (a)

$$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$y \in R(A)$

$\hookrightarrow$ Range space / column space.

$$R(A) = \left[\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \right] \quad y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$2x_1 - x_2 = 1 \quad - (1)$

$-3x_1 + 3x_2 = 0 \Rightarrow -x_1 + x_2 = 0$

$\boxed{x_1 = x_2} \quad - (2)$

$-x_1 + 2x_2 = 1 \quad - (3)$

put (2) in (1)

$2x_1 - x_1 = 1$

$\boxed{x_1 = 1}$

put (2) in (3).

$-x_2 + 2x_2 = 1$

$\boxed{x_2 = 1}$

(1) solution of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2)

$$|A|^2 = |A^T A|$$

$$\begin{bmatrix} 2 & -3 & -1 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 14 & -13 \\ -13 & 14 \end{bmatrix} = 27$$

So the determinent of A will be $\sqrt{|A|^2}$ which will be non-zero.

so, the equations are linearly independent.

So the solution is unique.

3 $y' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix}$

$Ax = y'$

$\begin{bmatrix} 2 & -1 \\ -3 & 3 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$W = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

Row echelon form of A & W is:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\text{Rank}(A) = 2$ $\text{Rank}(W) = 3$

$\text{Rank}(A) \neq \text{Rank}(W)$

Used: emathhelp.net for rowechelon deduction

Q5 1 $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 3$$
$$-x_2 - 2x_3 + 2x_4 = 2$$
$$\boxed{x_4 = 1}$$

$$x_1 + 2x_2 + 3x_3 + 4 = 3$$
$$x_1 + 2x_2 + 3x_3 = 3 - 4 = -1$$

$$-x_2 - 2x_3 + 2 = 2$$
$$-x_2 - 2x_3 = 0$$
$$\boxed{-x_2 = 2x_3}$$

$$x_1 + 2x_2 + 3x_3 = -1$$
$$x_1 - 4x_3 + 3x_3 = -1$$
$$x_1 - x_3 = -1$$
$$\boxed{x_1 = -1 + x_3}$$

$$\overbrace{X = \begin{bmatrix} -1 + x_3 \\ -2x_3 \\ x_3 \\ 1 \end{bmatrix}}^{\text{general solution}}$$

$$\boxed{x_3 = 1} \longrightarrow X = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

(b) Smallest norm (ηp)

$$(\eta p)^2 = (-1+x_3)^2 + (-2x_3)^2 + x_3^2 + 1$$

$$(\eta p)^2 = (1 + x_3^2 - 2x_3) + (4x_3^2) + x_3^2 + 1$$

$$(\eta p)^2 = (4x_3^2 + x_3^2 + x_3^2) - 2x_3 + 2$$

$$(\eta p)^2 = 6x_3^2 - 2x_3 + 2$$

for minimum norm $\dfrac{d(\eta p)^2}{dx_3} = 0$

$$12x_3 - 2 = 0$$

$$12x_3 = 2$$

$$x_3 = \frac{2}{12} = \boxed{\frac{1}{6} = x_3}$$

Q6 (a) $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rank $(A) = 2$

we know the relationship between rank & nullity.

Rank + nullity = no. of columns.

$2 + \text{nullity} = 3$

$\text{nullity} = 3-2$

$\boxed{\text{nullity} = 1}$

(b) $A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

rank = 3

Rank + nullity = no. of columns

reduced row echelon form,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$3 + \text{nullity} = 3$

$\boxed{\text{nullity} = 0}$

(c) $A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Reduced row echelon form; $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\text{Rank}(A_3) = 3$

$3 + \text{nullity} = \text{number of columns.}$

$3 + \text{nullity} = 4$

$\text{nullity} = 4 - 3$

$\boxed{\text{nullity} = 1}$

used: emathhelp.net to find reduced row echelon form.

Q7. (i) $Ax = x + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$A\alpha x_1 = \alpha x_1 + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{— (1)}$$

$$A\beta x_2 = \beta x_2 + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{— (2)}$$

from (1) & (2)

$$A\alpha x_1 + A\beta x_2 = \alpha x_1 + \beta x_2 + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{— (3)}$$

Now put $(\alpha x_1 + \beta x_2)$ as input

$$A(\alpha x_1 + \beta x_2) = \alpha x_1 + \beta x_2 + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{— (4)}$$

(3) $\neq$ (4)

so non linear operator

(2) $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A\alpha x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $A\beta x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A\alpha x_1 + A\beta x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (1)

Now put $(\alpha x_1 + \beta x_2)$ as input

$A(\alpha x_1 + \beta x_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (2)

(1) = (2)

So linear operator

③ $$Ax = \int_{-\infty}^{\infty} f(\tau)\, x(t-\tau)\, d\tau$$

$$A\alpha x_1 = \int_{-\infty}^{\infty} f(\tau)\, \alpha x_1 (t-\tau)\, d\tau$$

$$A\beta x_2 = \int_{-\infty}^{\infty} f(\tau)\, \beta x_2 (t-\tau)\, d\tau$$

$$A\alpha x_1 + A\beta x_2 = \int_{-\infty}^{\infty} f(\tau)\, \alpha x_1 (t-\tau)\, d\tau + \int_{-\infty}^{\infty} f(\tau)\, \beta x_2 (t-\tau)\, d\tau \quad - ①$$

$$A(\alpha x_1 + \beta x_2) = \int_{-\infty}^{\infty} f(\tau)\, (\alpha x_1 + \beta x_2)(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\, \alpha x_1 (t-\tau)\, d\tau + \int_{-\infty}^{\infty} f(\tau)\, \beta x_2 (t-\tau)\, d\tau \quad - ②$$

① = ②

so linear operator

(4) $$Ax = \begin{bmatrix} x_1 + 2x_3 + 3x_3 \\ x_1 + x_2 + x_3 \\ x_1 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A\alpha y_1 = \begin{bmatrix} \alpha y_{11} + \alpha 2 y_{12} + \alpha 3 y_{13} \\ \alpha y_{11} + \alpha y_{12} + \alpha y_{13} \\ \alpha y_{11} \end{bmatrix}$$

$$A\beta y_2 = \begin{bmatrix} \beta y_{21} + \beta 2 y_{22} + \beta 3 y_{23} \\ \beta y_{21} + \beta y_{22} + \beta y_{23} \\ \beta y_{21} \end{bmatrix}$$

$$A\alpha y_1 + A\beta y_2 = \begin{bmatrix} \alpha y_{11} + \alpha 2 y_{12} + \alpha 3 y_{13} + \beta y_{21} + \beta y_{22}^{2} + \beta 3 y_{23} \\ \alpha y_{11} + \alpha y_{12} + \alpha y_{13} + \beta y_{21} + \beta y_{22} + \beta y_{23} \\ \alpha y_{11} + \beta y_{21} \end{bmatrix}$$

— (1)

$A(\alpha y_1 + \beta y_2) =$

$$\begin{bmatrix} \alpha y_{11} + \alpha 2 y_{12} + \alpha 3 y_{13} + \beta y_{21} + \beta 2 y_{22} + \beta 3 y_{23} \\ \alpha y_{11} + \alpha y_{12} + \alpha y_{13} + \beta y_{21} + \beta y_{22} + \beta y_{23} \\ \alpha y_{11} + \beta y_{22} \end{bmatrix} \quad -(2)$$

$(1) = (2)$

so, linear operator

Discussed with
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