COLUMBIA UNIVERSITY

IEOR 4735 : Structured Hybrid Project Report – Fall 2020

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Introduction:

In this project, we will try to price a hybrid derivative contract paying

$$\max\left\{0, \left(\frac{\mathrm{S(T)}}{\mathrm{S(0)}} - \mathrm{k}\right).\left(\left(\frac{\mathrm{L(T-\Delta, T-\Delta, T)}}{\mathrm{L(0, T-\Delta, T)}}\right) - \mathrm{k}'\right)\right\},$$

at expiration T where

- $S(T)$ is the STOXX50E spot price quantoed from EUR into USD
- $L(t, T-\Delta, T)$ is the Δ = 3-month USD LIBOR rate observed at time t between $T-\Delta$ and T
- T is the expiration duration
- k, k' are the given relative strike prices

Assumptions:

In the given payoff above, the stock price S_T is denominated in USD, so we model the stock index price process in terms of USD risk-neutral measure. Note that if the STOXX50E prices were to be quoted in EUR, then price process would simply be a geometric brownian motion in terms of EUR risk neutral measure with the usual drift and volatility rate.

Next, since we use GBM model for STOXX50E prices, the following assumptions hold:

1. Stock prices are log-normally distributed and have constant volatility
2. Stock doesn't pay any dividends
3. No transaction fees or taxes while buying or selling securities
4. Market is arbitrage-free, complete and perfectly liquid
5. The contract can be exercised only on the expiration date

Methodology:

Stochastic equation for modelling a hybrid Equity/FX process:

The first task here is to describe the stock price process as a GBM in terms of the USD risk neutral measure (this needs to be done as it is originally in the EUR currency). This is described the following process:

$$\frac{dS_t}{S_t} = (r_{EUR} - \rho_{sx}\sigma_x\sigma_s)dt + \sigma_s dW^{Q^d}$$

where

r_{EUR} is the EUR interest rate which is assumed to be deterministic,
ρ_{sx} is the correlation of STOXX50E market and USD EUR exchange rate
σ_x is the exchange rate vol of USD EUR exchange rate
σ_s is the vol of STOXX50E price

<u>The EUR/USD exchange rate process is given by below:</u>

$$\frac{dX}{X} = \alpha_x\, dt + \sigma_x\, dW_c$$

where

σ_x is the exchange rate vol of USD EUR exchange rate
α_x is the drift rate of USD EUR exchange rate

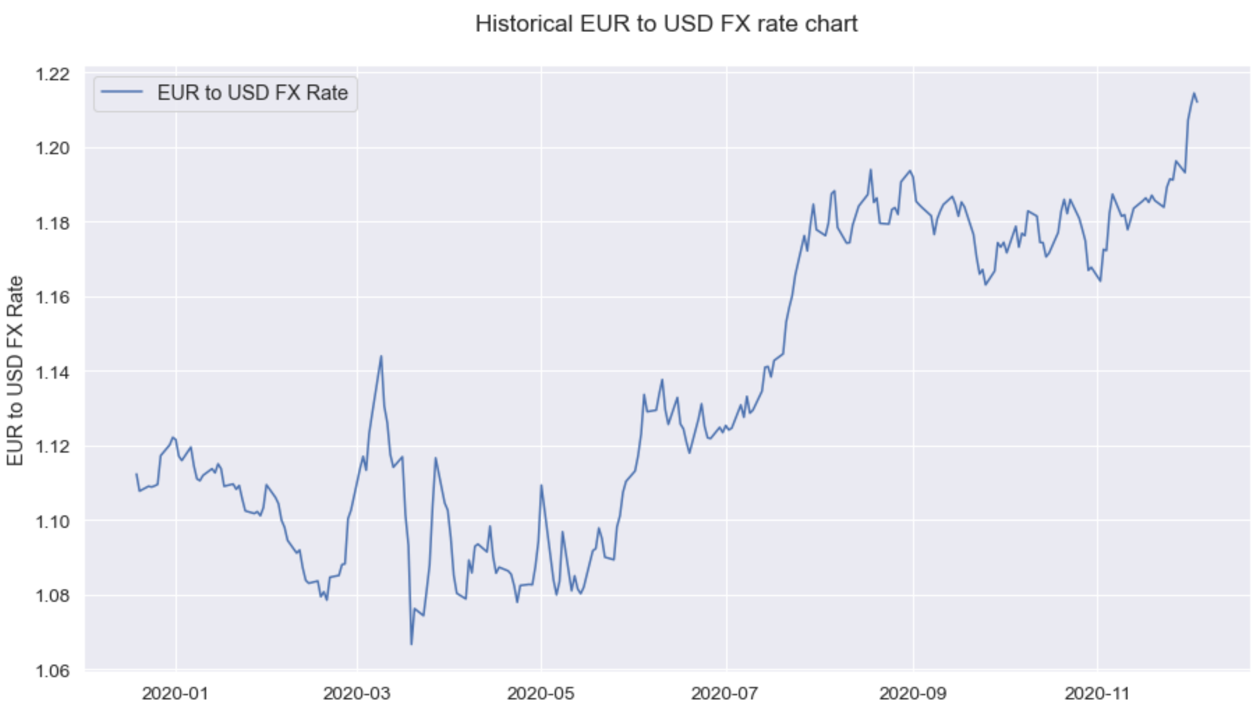
<u>Vasicek short-rate model:</u>

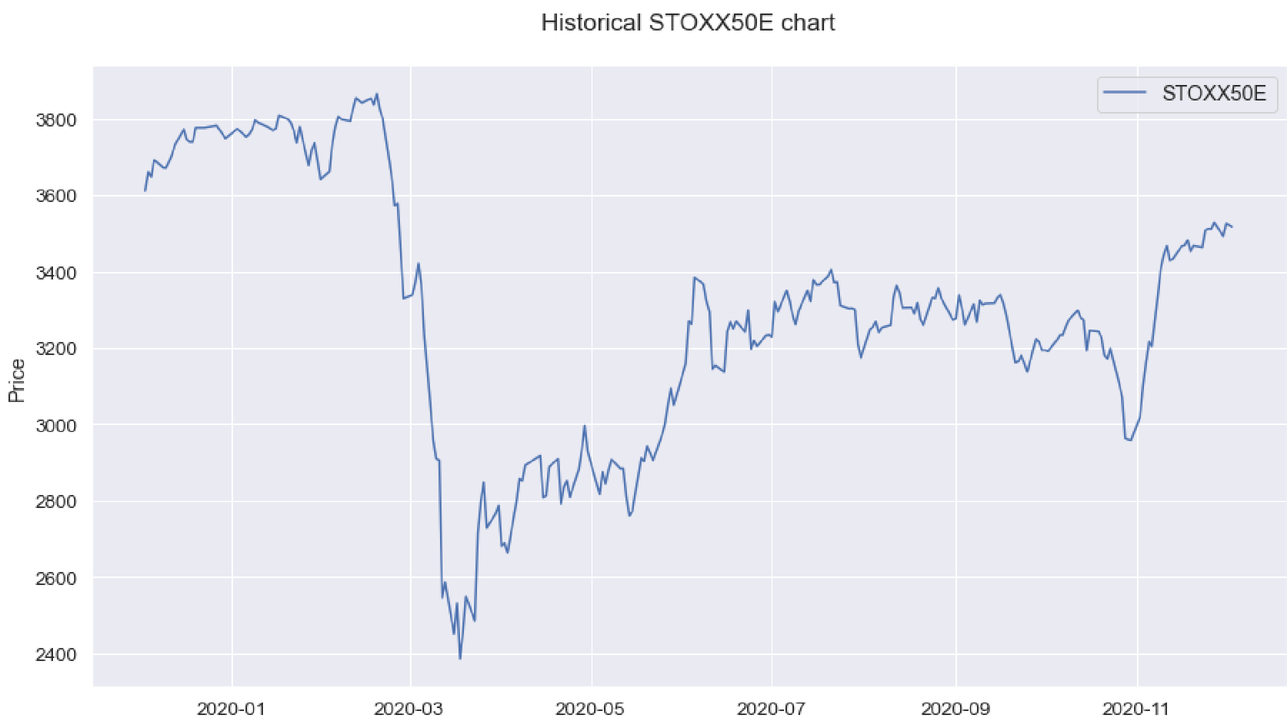
We model USD-3month LIBOR short rate using Vasicek short-rate model:

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dZ_t$$

<u>Model calibration and Simulation:</u>

We use historical data of EUR/USD rates and STOXX50E prices to calibrate the equity/FX model





<u>Equity-FX model calibration:</u>

MLE estimates for GBM model parameters are calibrated as follows:

In the below model,

$$\frac{dS}{S} = (r_f - \rho_{sx}\sigma_x\sigma_s)dt + \sigma_s dW^{Q^d}$$

$$\Rightarrow d\log(S_t) = \nu\, dt + \sigma dW_t$$

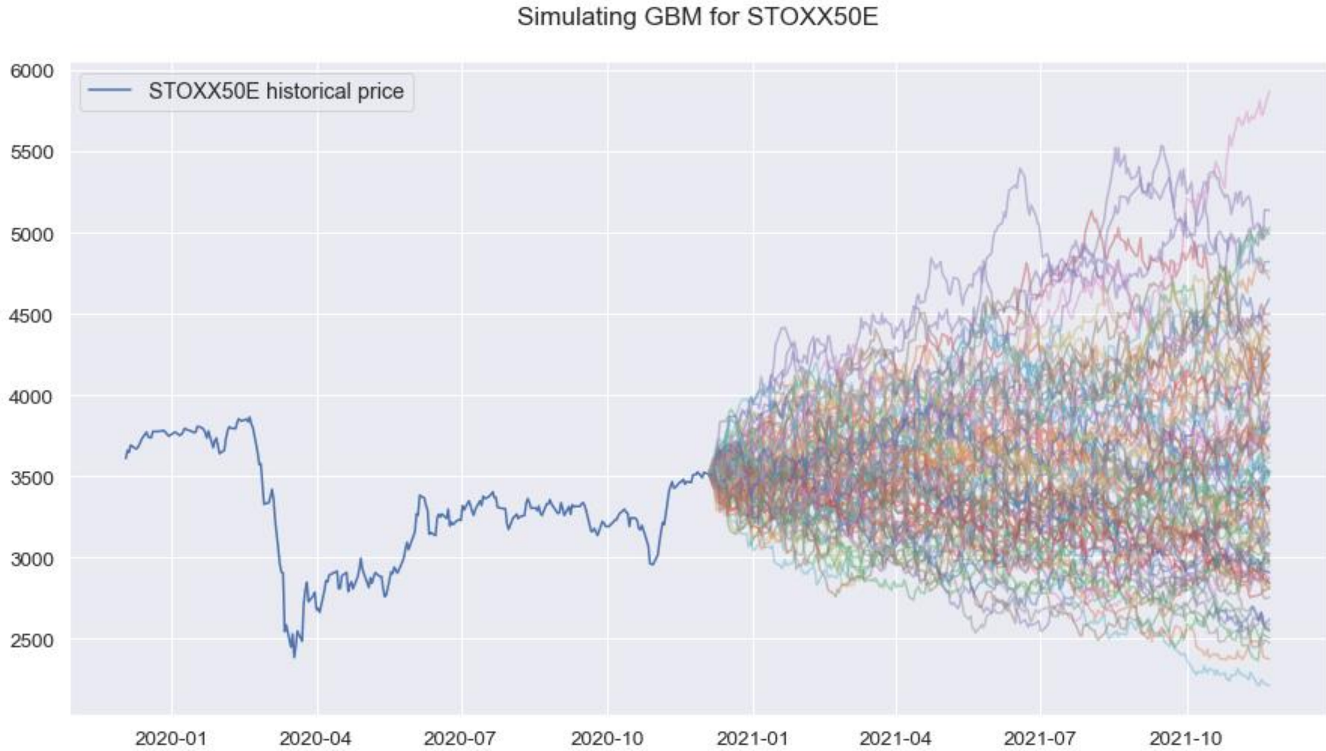
where $\nu = (r_f - \rho_{sx}\sigma_x\sigma_s) - \frac{\sigma_s^2}{2}$ from Ito's calculus. Given we observe the historical stock price in equally spaced intervals, $t_0 < t_1 < t_2 < \cdots \ldots \ldots < t_n,\ where\ \delta = t_i - t_{i-1}$. Let $X_i = \log(S_{t_i}) - \log(S_{t_{i-1}}), i = 1,2,\ldots n$ are independent and identically distributed samples from a Normal distribution with mean $\nu\delta$ and variance $\sigma^2\delta$. The log-likelihood function of a Gaussian process is as follows:

$$\tilde{\nu} = \frac{1}{\delta}\bar{X}\,, \quad \widetilde{\sigma^2} = \frac{1}{n\delta}\,\Sigma_{i=1}^{n}(X_i - X)^2$$

Using the historical end-of-day prices, we can thus calibrate MLE estimates which are consistent estimators of ν and σ

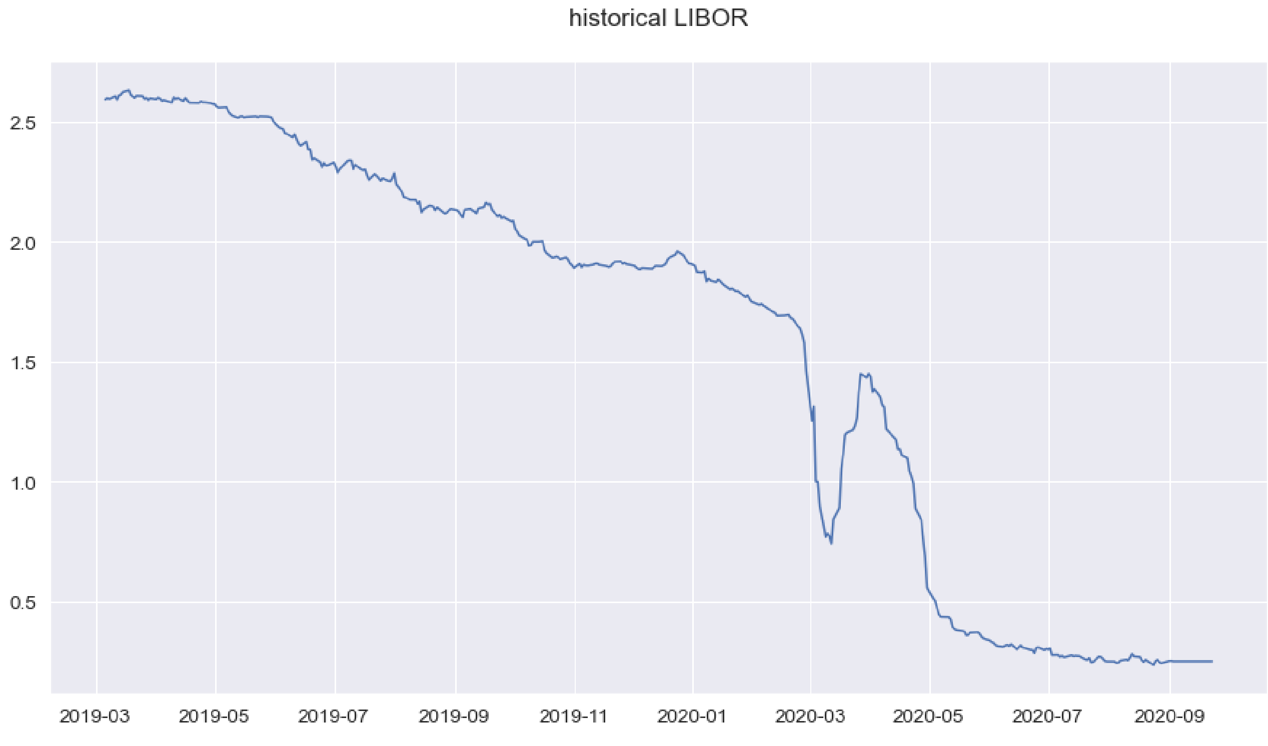
Simulated paths of STOXX50E from the Equity-FX model:

Using the parameters calibrated via maximum likelihood estimation, we simulate 100 paths for forward evolution of STOXX50E prices:



Vasicek short-rate model calibration:

Below is the chart of historical USD 3-month LIBOR rates:



The stochastic differential equation of Vasicek short-rate model follows Ornstein-Uhlenbeck process:

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dZ_t$$

whose closed-form solution is as follows:

$$r_t = r_s \exp\big(-\kappa(t-s)\big) + \ \bar{r}\big(1 - \exp\big(-\kappa(t-s)\big)\big) + \sigma \int_s^t \exp\left(-\kappa(t-u)\right) dZ_u$$

Now, with equidistant time intervals $t_0 < t_1 < t_2 < \cdots \ldots \ldots < t_n,\ where\ \delta t = t_i - t_{i-1}$, MLE estimators of $\bar{r}, \kappa$ and σ are:

$$\bar{r} = (S_1 S_{00} - S_0 S_{01})/(S_0 S_1 - S_0^2 - S_{01} + S_{00})$$

$$\kappa = \frac{\log[\frac{(S_0 - \bar{r})}{(S_1 - \bar{r})}]}{\delta t}$$

$$\sigma^2 = \frac{1}{n\beta(1 - 0.5\kappa\beta)} \Sigma_{i=1}^{n} \left(r_{t_i} - m_{t_{i-1}}(t_i)\right)^2$$

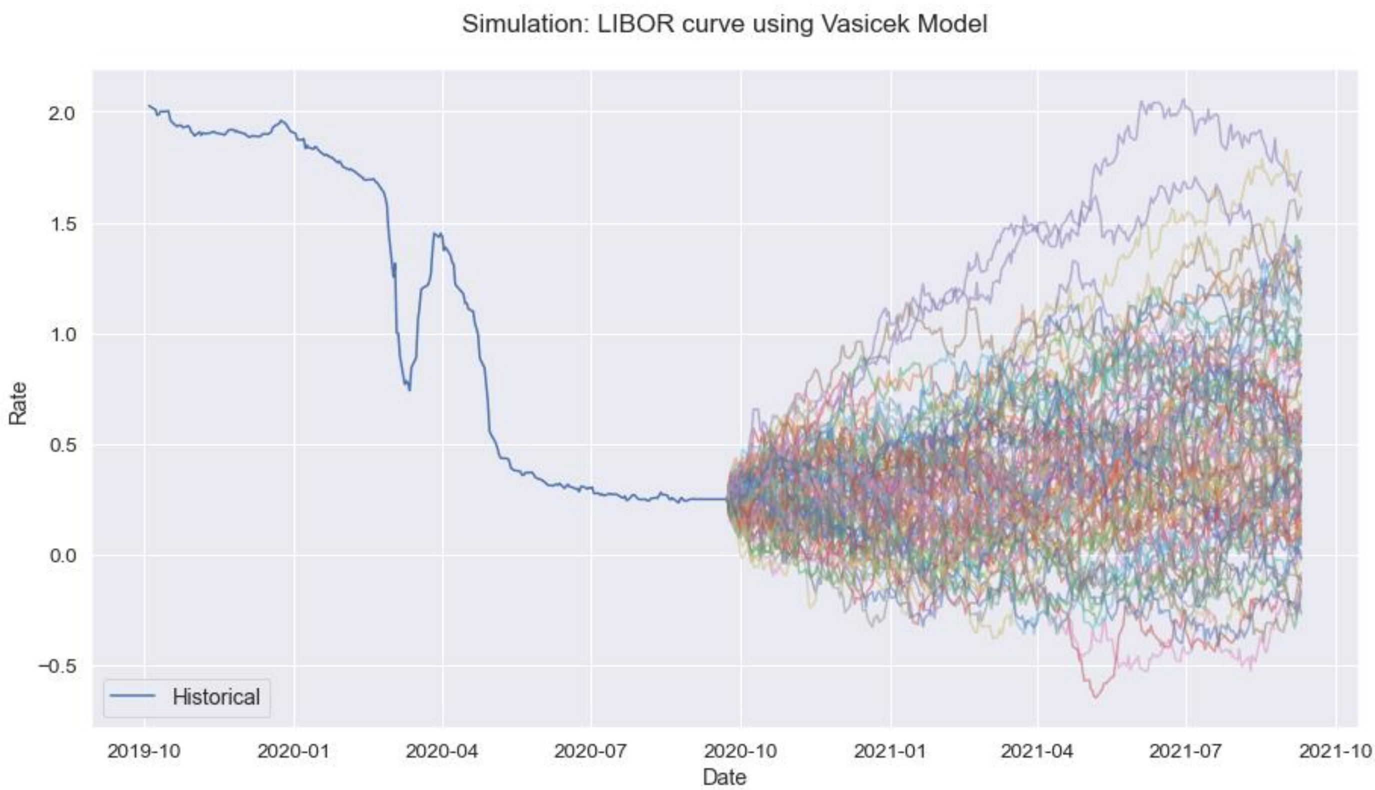
where

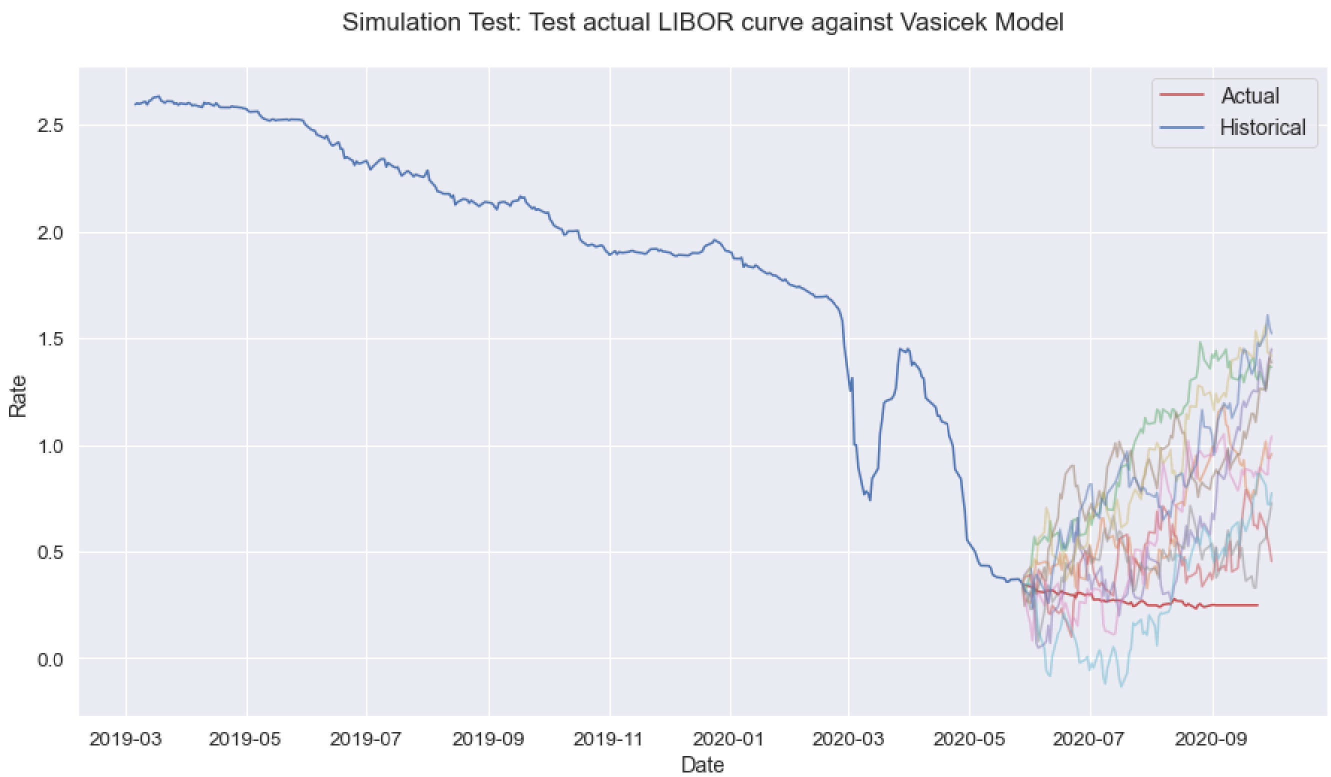
$$S_0 = \frac{1}{n}\ \Sigma_{i=1}^{n} r_{t_{i-1}} \qquad S_1 = \frac{1}{n} \Sigma_{i=1}^{n} r_{t_i} \qquad S_{01} = \frac{1}{n} \Sigma_{i=1}^{n} r_{t_i} r_{t_{i-1}} \quad S_{00} = \frac{1}{n}\ \Sigma_{i=1}^{n} r_{t_{i-1}} r_{t_{i-1}}$$

and

$$\beta = \frac{1}{\kappa}\ (1 - \exp\ (-\kappa\delta t)$$

Simulated 3-month USD LIBOR rates:





Forward Rate model:

$L(0, T-\Delta, T)$ is calculated by bootstrapping procedure which calculates the forward rate under the assumption that the market is arbitrage-free using the following computation:

$$(1+r_1)^{T-\Delta}\big(1+L(0,T-\Delta,T)\big)^{\Delta} = (1+r_2)^{T}$$

which implies

$$L(0,T-\Delta,T) = (DF(0,T-\Delta)/DF(0,T))^{1/\Delta} - 1$$

where

r_1 is the rate for time period $(0, T-\Delta)$

r_2 is the rate for time period $(0, T)$

$DF(0, T-\Delta)$ is the discount factor for time period $(0, T-\Delta)$

$DF(0, T)$ is the discount factor for time period $(0, T)$

Results:

Finally, we run the Monte Carlo simulation of $S(T)$, $L(T-\Delta, T-\Delta, T)$ to obtain price p at time $t = 0$ of the derivative contract which is given by

$$p = E[exp(-rT) * \max\left\{0, \left(\frac{\mathrm{S(T)}}{\mathrm{S(0)}} - \mathrm{k}\right).\left(\left(\frac{\mathrm{L(T-\Delta, T-\Delta, T)}}{\mathrm{L(0, T-\Delta, T)}}\right) - \mathrm{k}'\right)\right\}]$$

Now, the pricing engine takes the following inputs from the user to output p

1. $S(0)$, current STOXX50E price quoted in USD; The day 0 STOXX50E price is taken to be 3571.1 USD in the below examples
2. T, k, k' , time to expiration and relative strike prices respectively; T is fixed at 1 year
3. $\mathrm{L(0, T-\Delta, T)}$, current forward USD-Libor 3m rate between $\mathrm{T}-\Delta$ and T which is obtained from the current 3-month USD Libor curve at time $T-\Delta$

Simulation results and stability of the pricing model for different strikes are as follows:

1.

```
Option Parameters: k= 1.0, k'= 1.0

Option Price (mean):  0.1468
Option Price (standard error):  1.0538
Option Price (95% Confidence Interval): [0.0815, 0.2121]
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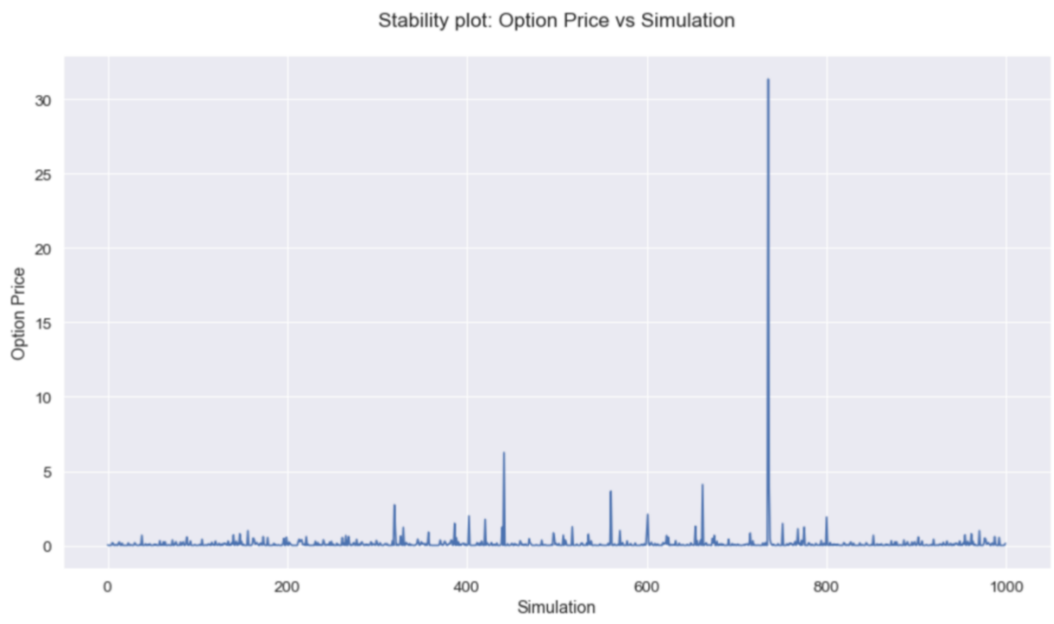

2. **Option Parameters:** k: 0.1, k': 0.1

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Option Price (mean): 0.6595
Option Price (standard error): 2.1749
Option Price (95% Confidence Interval): [0.5247, 0.7943]
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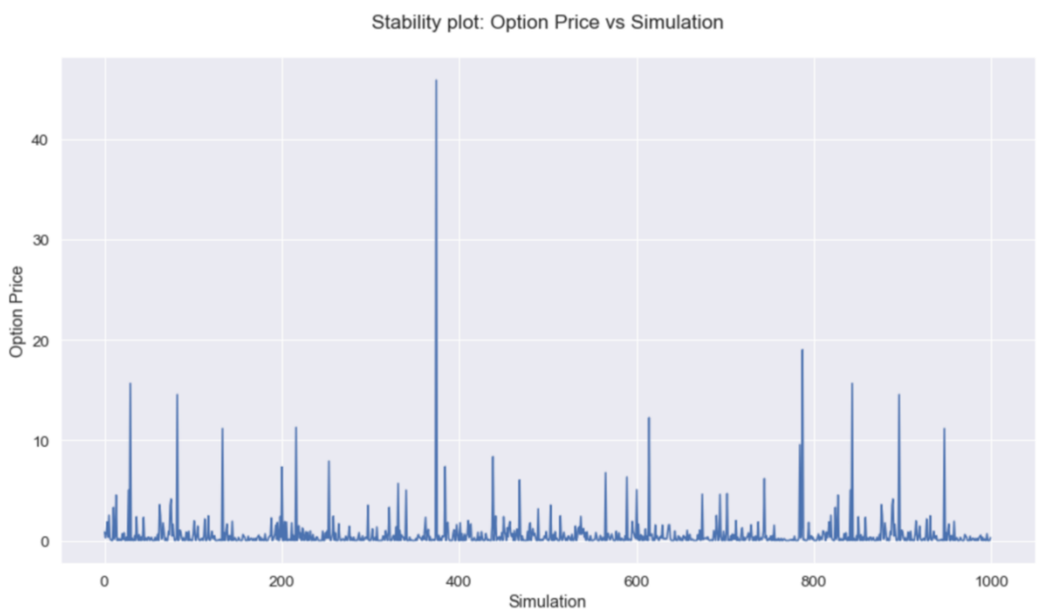

3. **Option Parameters:** k: 0.01, k': 0.01

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Option Price (mean): 0.759
Option Price (standard error): 2.3028
Option Price (95% Confidence Interval): [0.6163, 0.9017]
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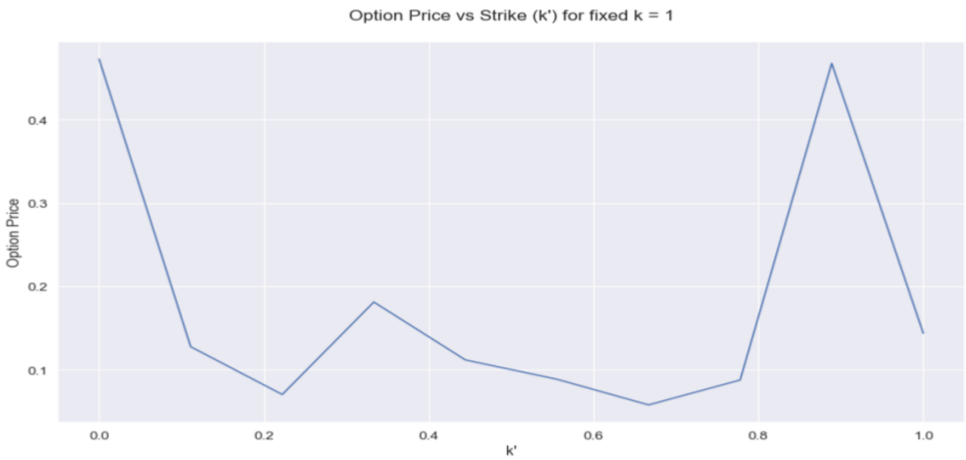
Change of Option Price against different k', for some fixed k.



<u>Conclusion:</u>

We find that the model is not very sensitive to variation in input strikes but for some iterations we observe spikes in the option price which may be due to extreme values for the predicted Stock and/or Interest rates. But on the other hand, we achieved a tight 95% confidence interval for our option price estimate.

We consider extending our project to include variance reduction techniques like antithetic variables, control variate, importance sampling, stratified sampling etc., to reduce the variance of our estimates to gain more confidence in our option price. We would include correlated asset prices, macro trends, industry factors datasets, and various other datasets to improve our prediction accuracy and explore other short rate models like Hull-white, CIR, CIR++, HJM, LFM, and LSM models in our code repository. To know more about our project, please visit here.

References:

[1] Application of Maximum Likelihood Estimation to Stochastic Short Rate Models by K. Ferguson and E. Platen

[2] "Statistical analysis and Time Series" Lecture notes on Maximum Likelihood Estimation by Prof. Agostino Capponi

[3] "Foundations of Financial Engineering" by Prof. Dylan Possamai

Code Repository:

https://github.com/abhishekprog0/StructuredPricingEngine