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SPACE FILLING CURVES AND THE TOWER OF HANOI

Dane R. CAMP
New Trier High School, USA

ABSTRACT: The author of this article seeks interesting connections between apparently unrelated concepts in order to use them as a motivating tool for students in high school mathematics classes. One example of a rich source of connections is the relationship between the Tower of Hanoi puzzle and various fractal objects. This manuscript demonstrates how two such fractals, both space filling curves, are related to states and solutions of the Tower of Hanoi. First, a brief description of Ian Stewart's graph theoretical depiction of states in the Tower of Hanoi, which leads to Sierpinski's gasket, is reviewed. Then the author demonstrates how a Peano space filling curve can be derived from a Hamiltonian path on Stewart's graph, by going from one initial state to the corresponding terminal state—covering all possible states in between. Finally, the paper closes by describing a mechanism by which a Hilbert space filling curve can be used as an alternative method to derive the same coding scheme for determining a solution as used in Stewart's graph. Though this last object does not contain all possible states, it is a compact geometric object for representing a single optimal solution to the Tower of Hanoi.

Keywords: Tower of Hanoi, Fractal, Space filling, Sierpinski gasket, Peano curve, Hilbert curve.

1. INTRODUCTION

Mathematics is a language of patterns. The more complex and surprising the patterns, the more poetic and lyrical a mathematical discovery seems to be. Fractal geometry is a fascinating dialect of its own, filled with interconnected patterns. The adventure of discovering these connections is akin to the joy of understanding the meaning of a poem or a profound piece of literature for the first time. One surprising vehicle for discovering "fractaleque" patterns is the amazingly innocuous puzzle, invented by the French Mathematician Edouard Lucas in 1883, called the Tower of Hanoi (TOH). Though there are many other interesting fractal patterns embedded in the solution to the Tower of Hanoi [1, 2], here we will explore one special type, namely, space filling curves.

2. THE TOWER OF HANOI (TOH) AND SIERPINSKI'S GASKET

The Tower of Hanoi puzzle consists of 3 pegs and a number of disks of ever increasing sizes. The object is to move all of the disks from one peg to another while obeying two simple rules: 1—only move one disk at a time. 2—never put a larger disk upon a smaller one. Ian Stewart noticed that there is a connection between solutions to the Tower of Hanoi and Sierpinski's gasket [5]. He used graph theory to identify all possible legal moves in the puzzle and, in doing so, any optimal solution.

2.1 TOH for one disk

For example, if the tower consists of only one disk, then there are only three states, depending upon which peg the disks rests. Figure 1 illustrates the states and the corresponding graph.

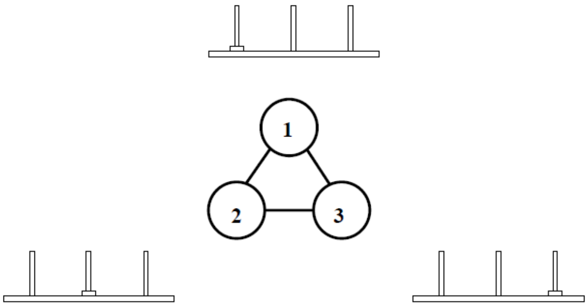

Figure 1: Graph of TOH states for one disk.

2.2 TOH for two disks

If the tower consists of two disks, there are nine possible states. The vertices in the graph are labeled with two digit numbers—the first digit representing the peg upon which the top disk rests and the second digit representing the peg upon which the bottom disk rests. Figure 2 illustrates the states, the corresponding graph, and the appropriate labels.

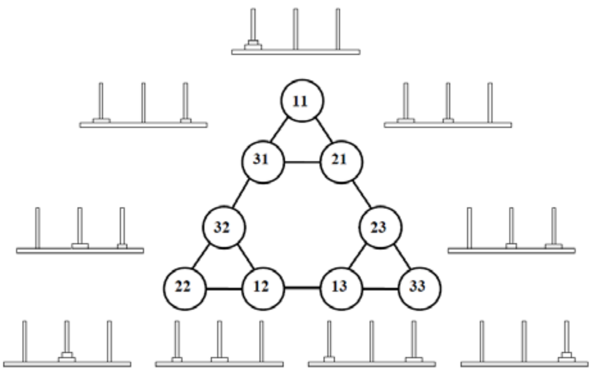

Figure 2: Graph of TOH states for two disks.

2.3 TOH for three disks

Already it is clear that the graph bears a strange resemblance to Sierpinki's Gasket. For three disks, the corresponding state graph has vertices with three digit labels—the first digit for the smallest disk, the second digit for the medium sized disk, and the third digit for the largest disk as shown in Figure 3.

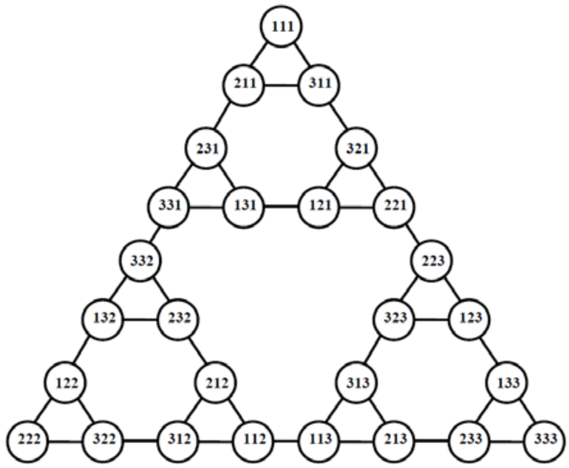

Figure 3: Graph of TOH states for three disks.

2.4 TOH for four disks

You may notice a lot of interesting patterns in how the various labels are related to one another. These patterns are reinforced when an additional disk is added and the graph for four disks is produced in Figure 4. It may seem surprising that a fractal form arises from the solution to the puzzle, however, many other fractal images are connected to the Tower of Hanoi. Here we will investigate two of them, both of which are space filling curves.

3. THE TOWER OF HANOI AND THE PEANO CURVE

Since the connection to the Peano curve [3] requires an even number of disks, we will illustrate the procedure using the graphs for two and four disks respectively. For the case of two disks, first notice that there is a unique Hamiltonian path from any initial state to any corresponding terminal state traversing all possible intermediate states—11 to 33 in the two disk graph—for example. Also observe that the number of edges used in the path correspond to the number of turns in the first iteration of a Peano curve, P_1, as shown in figure 5.

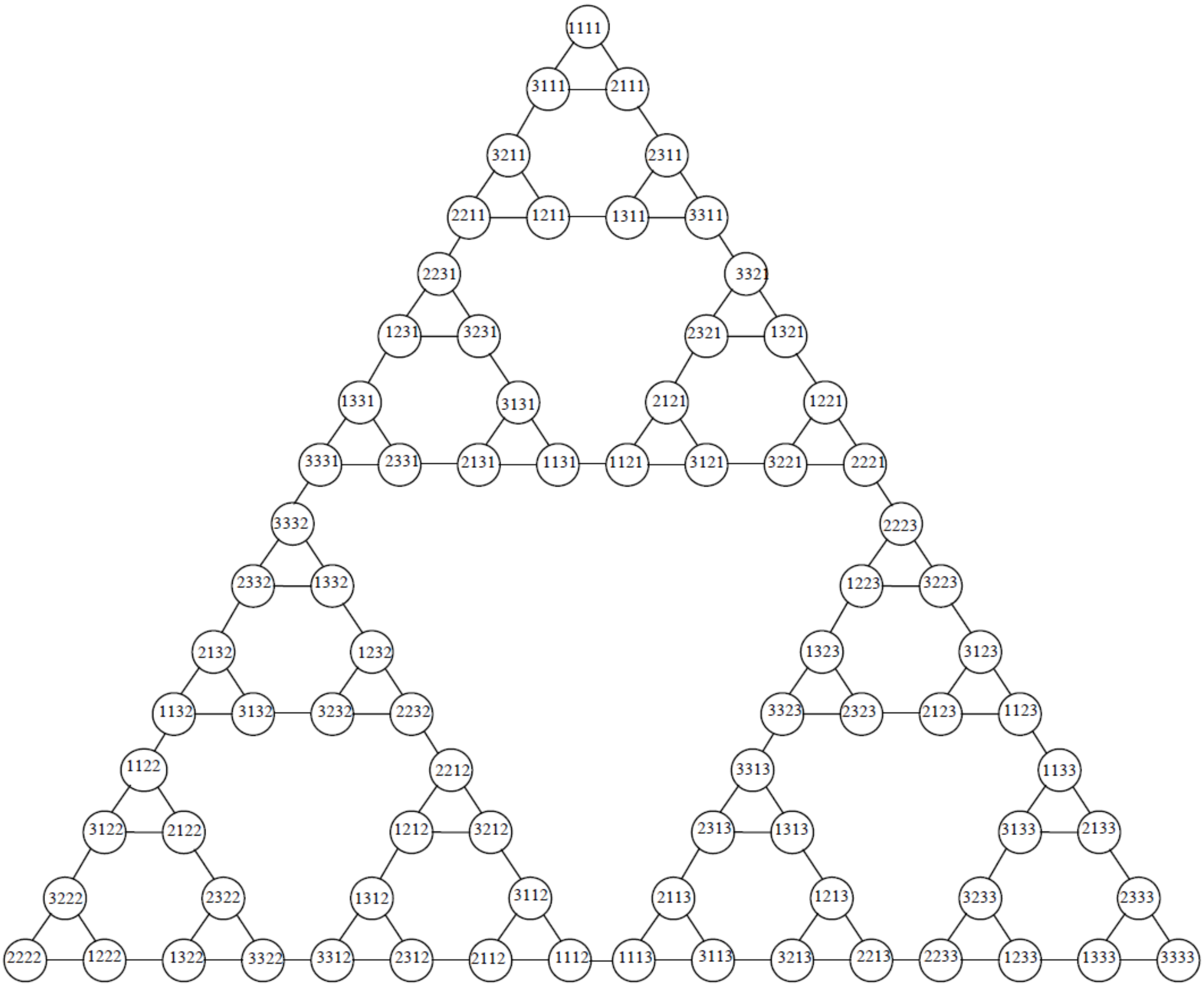

Figure 4: Graph of TOH states for four disks.

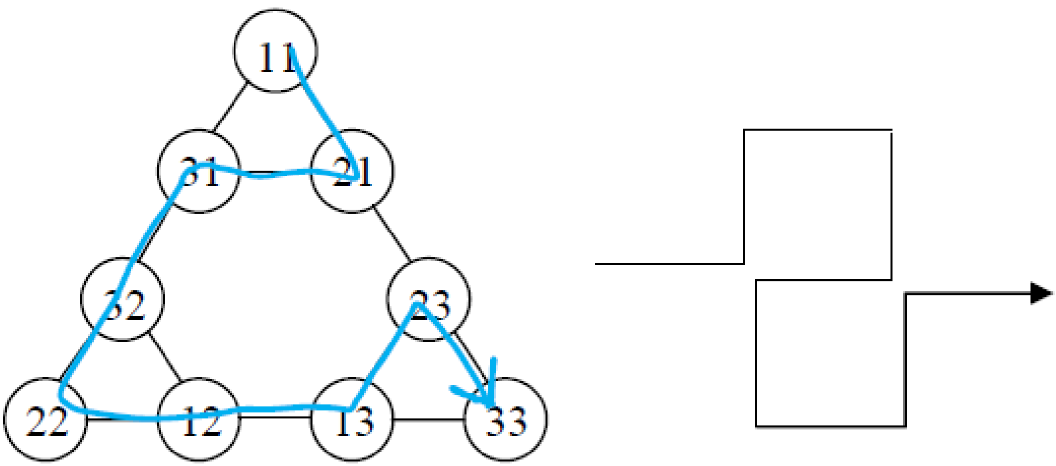

Figure 5: Hamiltonian path for two disks and the first iteration of the Peano curve, P_1.

3.1 TOH and P_1

One can't help but think that there must be a way to set up a correspondence between the two objects. It turns out that there is a rather obvious method by considering the parity of the disk and the parity of the move. There are eight moves through the Hamiltonian path and there are two disks, the top—disk 1 and the bottom—disk 2. If the parity of the move and the disk are the same we will record an L, if they are different we will record an R. So, on move 1 disk 1 is moved (same parity L), on move 2 disk 1 is moved again (opposite parity R), on move 3 disk 2 (R), on move 4 disk 1 (R), on move 5 disk 1 (L), on move 6 disk 2 (L), on move 7 disk 1 (L), and on move 8 disk 1 (R). Translating these into Left and Right turns we can see how this sequence connects to the first iteration of the Peano curve [see figure 6].

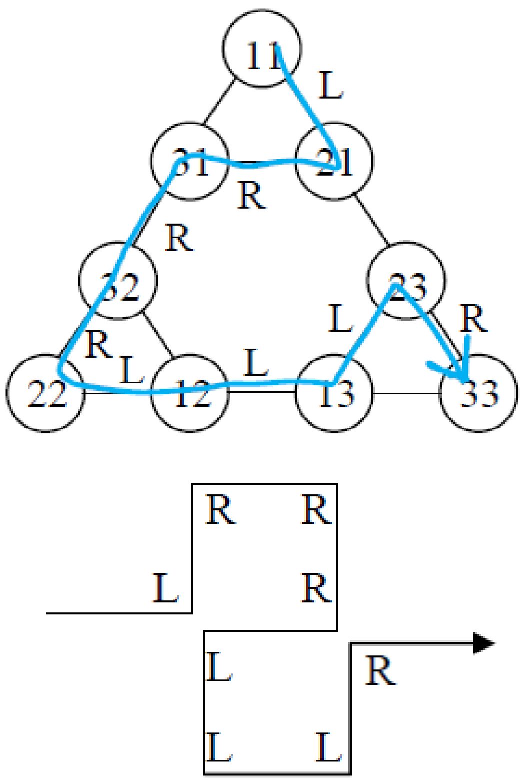

Figure 6: The Hamiltonian path for two disks coded to match P_1.

3.2 TOH and P_2

The Peano curve will have 8*8 + 8 = 72 turns on its next iteration, which will require us to use the Hamiltonian path on the graph for four disks and the same coding scheme as before [see figure 7]. The subsequent iterations of the Peano curve can be derived from the Tower of Hanoi graph with the appropriate number of even disks.

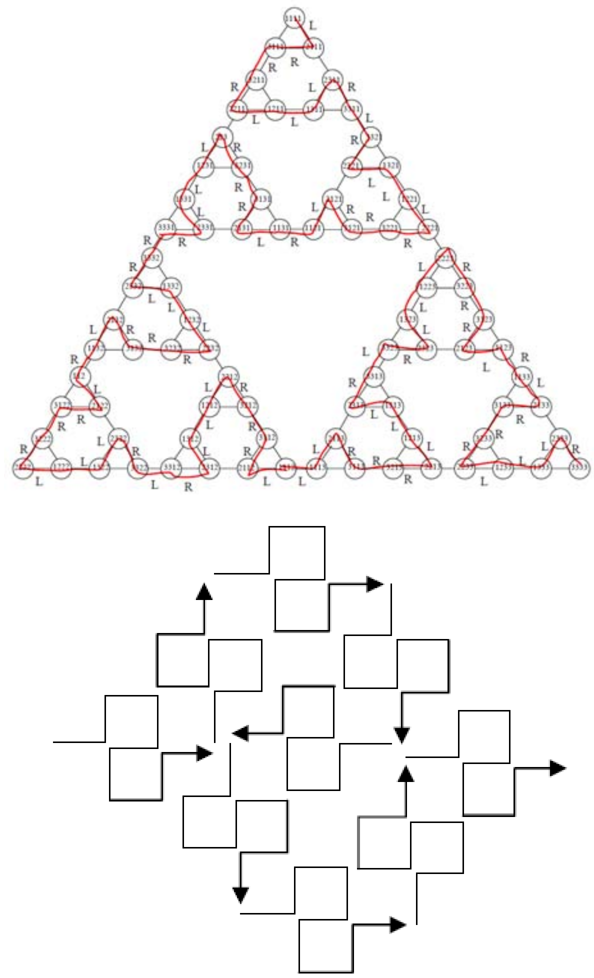

Figure 7: The Hamiltonian path for four disks corresponding to P_2.

4. THE TOWER OF HANOI AND THE HILBERT CURVE

Hilbert's curve is another space filling fractal [4]. There is an interesting connection between Hilbert's curve and solutions to the Tower of Hanoi. This observation also requires an even number of disks. This time, however, we are not interested in examining a path including all possible states in a corresponding graph, but only an *optimal solution* to the puzzle. By this we mean the path along a *side* of Stewart's graph.

4.1 TOH and H_1

For example, with two disks running along the side from 11 to 33 we would have the sequence of vertices: 11, 21, 23, 33, for an optimal solution to the puzzle. We can rearrange the states into a graph corresponding to the first iteration of Hilbert's curve as shown in figure 8.

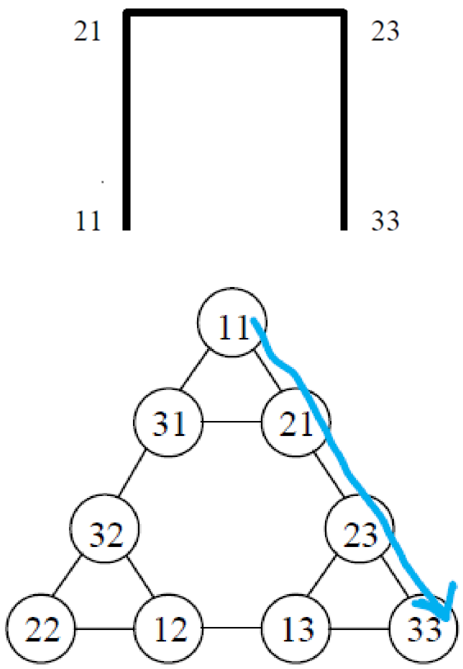

Figure 8: An optimal solution for two disks and the first iteration of Hilbert's curve, H_1.

4.2 TOH and H_2

Next, consider the *top two disks* of the Tower of Hanoi with *four disks* (refer back to figure 4). As the solution runs from 1111 to 3333, this involves the *first two digits* of each label. These can be organized into a graph that corresponds to the second iteration of the Hilbert curve, as shown in figure 9.

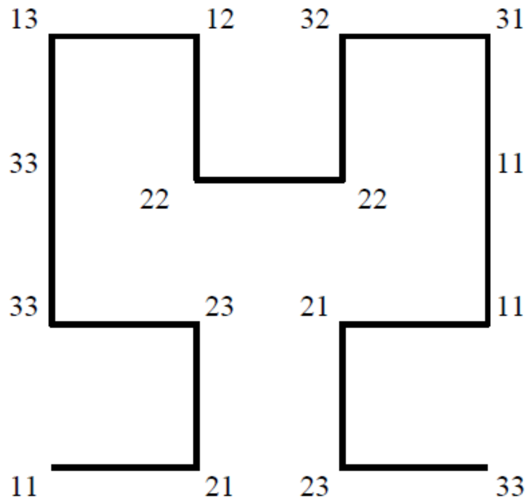

Figure 9: The top two disks for TOH for four disks corresponding to H_2.

4.3 TOH and H_3

Naturally, we can extend the analogy and consider the *top two disks* of the Tower of Hanoi for six disks. The graph is too large to include here, however, we can follow one solution going from 111111 to 333333 by utilizing the first two digits again. The corresponding Hilbert curve is shown in figure 10:

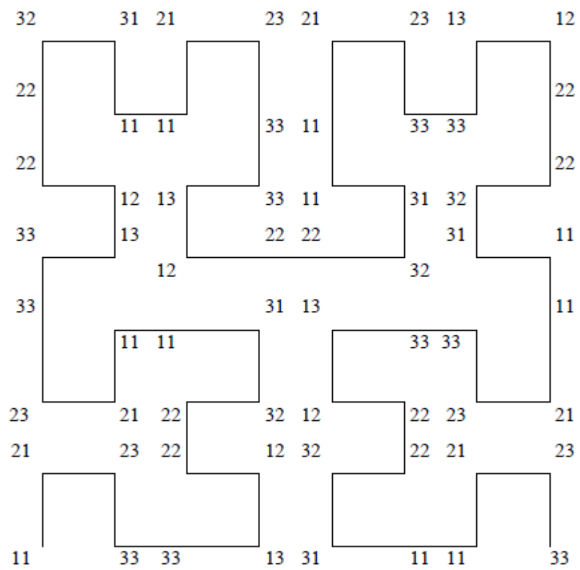

Figure 10: The top two disks of TOH for six disks corresponding to H_3.

4.4 The combined curves

When we put the three graphs together (Figure 11), we have an efficient way of representing an optimal solution to the Tower of Hanoi for 6 disks. The corresponding Sierpinski gasket graph would be very large and include extraneous information, since we are only interested in one optimal solution, not all of the possible states. The optimal solution can be derived as shown in Figure 12 by a rather simple procedure. First, move the top two disks through the first 4 states of H_3—11, 21, 23, 33. Since the bottom four disks remain fixed, this corresponds to states 111111, 211111, 231111, and 331111. Then move the top of the middle two disks from the first to the second states in H_2 while moving to the next (duplicated) state in H_3, so 331111 becomes 332111 (the coupled arrows move together). Next move three states on H_3 (132111, 122111, 222111), one coupled state on H_2 and H_3 (222311), three on H_3 (322311, 312311, 112311), one coupled state on H_2 and H_3 (113311), three on H_3 (213311, 233311, 333311), then change the state on the inner graph, H_1, corresponding to it's top disk while moving on both H_2 and H_3, to 333321 (these are now "triplicated" states coupled together). Continue the process till the optimal solution is complete (133321, 1233321, 223321, 221321, 321321, 311321, 111321, 111221, 211221, 231221, 331221, 332221, 132221, 122221, 222221, 222223, 322223, 312223, 112223, 113223, 213223, 233223, 333223, 333123, 133123 123123, 223123, 221123, 321123, 311123, 111123, 111133, 211133, 231133, 331133, 332133, 132133, 122133, 222133, 222333, 322333, 312333, 112333, 113333, 213333, 233333, and 333333).

4.5 The construction technique

Note that the construction technique for the succeeding iterations of the Hilbert curve correspond nicely to a procedure for determining the move sequence for each pair of disks, as shown in Figure 13. The first and fourth modules are simply rotated versions of the mirror image of the initial figure about a vertical axis (-90º and +90º respectively). The second module has each disk sequence shifting to the left (think of it as adding 2 mod 3) and the third module is a shift to the right (think of it as adding 1 mod 3). The same procedure can be followed for subsequent iterations.

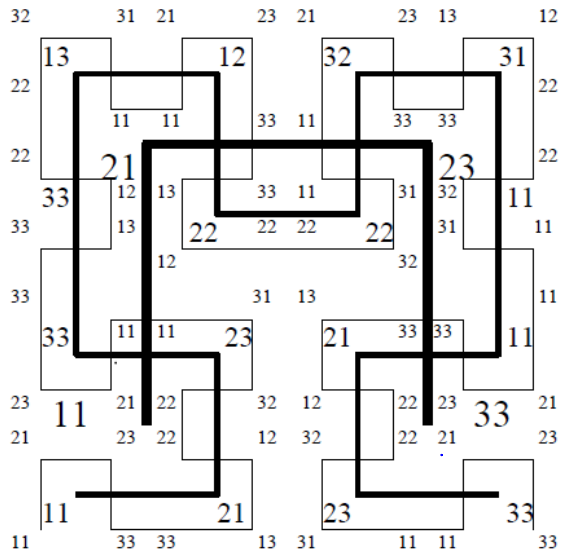

Figure 11: The combined Hilbert curves.

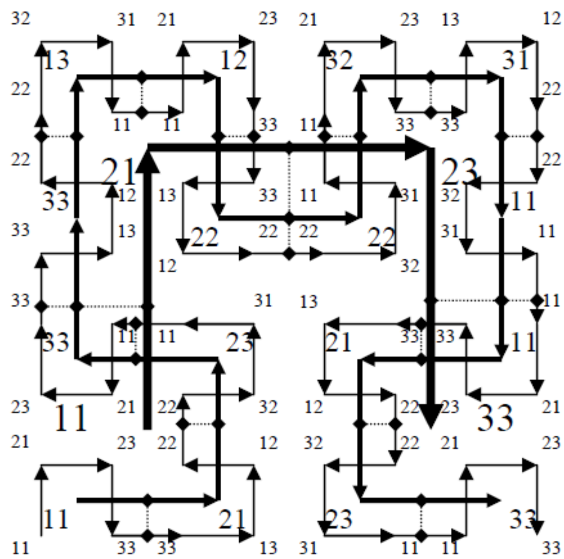

Figure 12: The combined Hilbert curve solution to TOH for six disks.

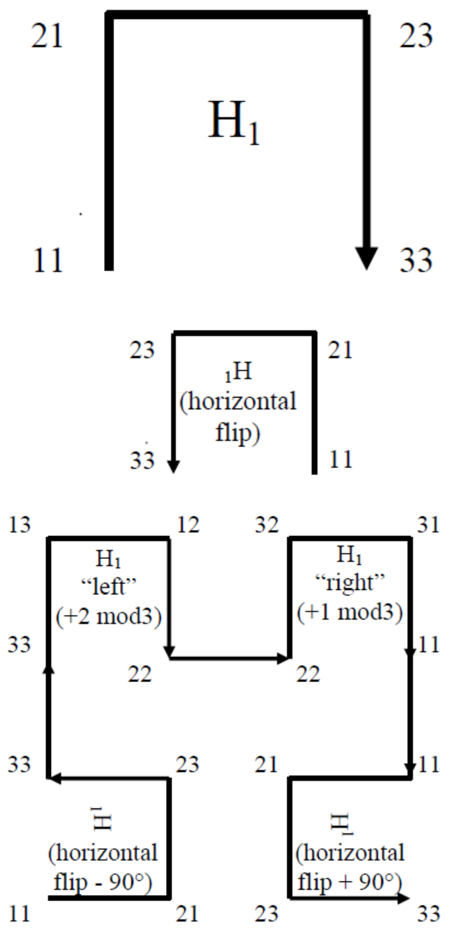

Figure 13: The procedure for generating successive iterations of Hilbert's curve for pairs of disks.

5. CONCLUSIONS

It is fascinating that so many different fractal patterns arise from a rather harmless looking puzzle. This is precisely why the language of mathematics is so powerful—it has the ability to describe patterns that arise from the most unlikely sources and explain how they are interconnected. Perhaps this is why studying mathematics can often be as exciting as reading a mystery novel or spy thriller, one never knows what will happen, even when the plot is familiar and the twists are predictable. The Author of mathematics always seems to throw in a few surprises.

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ABOUT THE AUTHOR

Dane R. CAMP, Dr.(Ed.), is a teacher at the New Trier High School. His e-mail and postal address are as follows: campd@newtrier.k12.il.us Mathematics Department, New Trier High School, 385 Winnetka Avenue, Winnetka, Illinois, 60093, USA.