TAIL RECURSION TRANSFORMATION

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2016

Associative Binary Operator

some binary operator

old function: $f(\bar{x}) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ c(\bar{x}) * f(\bar{d}(\bar{x}))$ $\quad \bar{x} = (x_1, \ldots, x_n)$ $\quad \bar{d}(\bar{x}) = (d_1(\bar{x}), \ldots, d_n(\bar{x}))$ $\quad n > 0$

$\boxed{\tau_f}$ $\quad \neg a(\bar{x}) \Rightarrow \mu_f(\bar{d}(\bar{x})) \prec_f \mu_f(\bar{x})$

$$f(\bar{x}) = \underbrace{c(\bar{x}) * \left(c(\bar{d}(\bar{x})) * \left(c(\bar{d}^2(\bar{x})) * \ldots * \left(c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x}))\right) \ldots\right)\right)}_{\text{expansion of } f(\bar{x})} \qquad m \geq 0 \qquad \forall j \in \{0, \ldots, m-1\}.\ \neg a(\bar{d}^j(\bar{x})) \qquad a(\bar{d}^m(\bar{x}))$$

condition: $\boxed{\text{ASC}}$ $\quad u * (v * w) = (u * v) * w$ — associativity $\Longrightarrow (\mathcal{U}, *)$ is a semigroup

$$\text{expansion of } f(\bar{x}) = \left(\ldots\left(\left(c(\bar{x}) * c(\bar{d}(\bar{x}))\right) * c(\bar{d}^2(\bar{x}))\right) * \ldots * c(\bar{d}^{m-1}(\bar{x}))\right) * b(\bar{d}^m(\bar{x}))$$

initially, if $\neg a(\bar{x})$, set $r := c(\bar{x})$ and update $\bar{x} := \bar{d}(\bar{x})$

then, while $\neg a(\bar{x})$, update $r := r * c(\bar{x})$ and $\bar{x} := \bar{d}(\bar{x})$

finally, when $a(\bar{x})$, return $r * b(\bar{x})$

calculate $f(\bar{x})$ when $m > 0$

if $a(\bar{x})$ initially, return $b(\bar{x})$

calculate $f(\bar{x})$ when $m = 0$

$\Downarrow$

new function: $f'(\bar{x}, r) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r * b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x}))$ — tail-recursive

$\mu_{f'}(\bar{x}, r) \triangleq \mu_f(\bar{x}) \qquad \prec_{f'} \triangleq \prec_f$

relation between f and f': $\quad f(\bar{x}) = \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), c(\bar{x}))$

$\vdash \boxed{\tau_{f'}}$ $\neg a(\bar{x}) \Rightarrow \mu_{f'}(\bar{d}(\bar{x}), r * c(\bar{x})) \prec_{f'} \mu_{f'}(\bar{x}, r)$ — f' terminates

$$\begin{tikzcd}
\neg a(\bar{x}) \arrow[r, "\tau_f"'] & \mu_f(\bar{d}(\bar{x})) \arrow[r, phantom, "\prec_f"] \arrow[d, equal, "\delta_{\mu_{f'}}"'] & \mu_f(\bar{x}) \arrow[d, equal, "\delta_{\mu_{f'}}"] \\
& \mu_{f'}(\bar{d}(\bar{x}), r * c(\bar{x})) \arrow[r, phantom, "\prec_{f'}"] & \mu_{f'}(\bar{x}, r)
\end{tikzcd}$$

QED

$\vdash \boxed{f'f}$ $f'(\bar{x}, r) = r * f(\bar{x})$ — relation between f' and f (f' in terms of f)

induct f'

base) $a(\bar{x})$

$\xrightarrow{\delta_{f'}} f'(\bar{x}, r) = r * b(\bar{x}) = r * f(\bar{x})$

$\xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x})$

step) $\neg a(x)$

$\xrightarrow{\delta_f} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$

$\xrightarrow{\delta_{f'}} f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \overset{ASC}{=} r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x})$

IH $\rightarrow f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x}))$

QED

$\vdash \boxed{ff'}$ $f(\bar{x}) = \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), c(\bar{x}))$ — relation between f and f' (f in terms of f')

$f(\bar{x}) \overset{\delta_f}{=} \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ c(\bar{x}) * f(\bar{d}(\bar{x})) = \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), c(\bar{x}))$

$f'f \xrightarrow[\bar{x} := \bar{d}(\bar{x})]{r := c(\bar{x})} f'(\bar{d}(\bar{x}), c(\bar{x})) = c(\bar{x}) * f(\bar{d}(\bar{x}))$

QED

wrapper : $\tilde{f}(\bar{x}) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), c(\bar{x})) \implies \vdash \boxed{f\tilde{f}}$ $f(\bar{x}) = \tilde{f}(\bar{x})$

Associative Binary Operator with Left Identity

conditions $\begin{cases} \boxed{\text{ASC}} & u * (v * w) = (u * v) * w \quad \text{— associativity} \\ \boxed{\text{LI}} & b(\bar{x}) * u = u \quad \text{— left identity} \end{cases}$

$$f(\bar{x}) = c(\bar{x}) * \left(\overline{c(\bar{d}(\bar{x})) * \left(\overline{c(\bar{d}^2(\bar{x})) * \ldots * \left(\overline{c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x}))}\right)\ldots}\right)}\right) \quad \text{— expansion of } f\text{, as before}$$

$$\|$$

$$b(\bar{x}) * \left(c(\bar{x}) * \left(\overline{c(\bar{d}(\bar{x})) * \left(\overline{c(\bar{d}^2(\bar{x})) * \ldots * \left(\overline{c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x}))}\right)\ldots}\right)}\right)\right)$$

ASC $\longrightarrow$ $\|$

$$\left(\overline{\ldots\left(\left(\left(\overline{b(\bar{x}) * c(\bar{x})}\right) * c(\bar{d}(\bar{x}))\right) * c(\bar{d}^2(\bar{x}))\right) * \ldots * c(\bar{d}^{m-1}(\bar{x}))}\right) * b(\bar{d}^m(\bar{x}))$$

start with $r := b(\bar{x})$ instead of $r := c(\bar{x})$

one more update $r := r * c(\bar{x})$ than before, no need to update $\bar{x} := \bar{d}(\bar{x})$ initially

no initial split on $a(\bar{x})$, because r starts as $b(\bar{x})$ instead of $c(\bar{x})$

$$\Downarrow$$

new function: $f'(\bar{x}, r) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r * b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x}))$ — as in associative-only case

$\vdash \boxed{f'f}$ $\quad f'(\bar{x}, r) = r * f(\bar{x})$ — as in associative-only case

$\vdash \boxed{ff'}$ $\quad f(\bar{x}) = f'(\bar{x}, b(\bar{x}))$

$$f'f \xrightarrow{r := b(\bar{x})} f'(\bar{x}, b(\bar{x})) = b(\bar{x}) * f(\bar{x}) \overset{\text{LI}}{=} f(\bar{x})$$

QED

wrapper: $\tilde{f}(\bar{x}) \triangleq f'(\bar{x}, b(\bar{x})) \quad \Longrightarrow \quad \vdash \boxed{f\tilde{f}} \quad f(\bar{x}) = \tilde{f}(\bar{x})$

Associative Binary Operator with Right Identity

conditions $\begin{cases} \boxed{ASC} & u * (v * w) = (u * v) * w \quad \text{— associativity} \\ \boxed{RI} & u * b(\bar{x}) = u \quad \text{— right identity} \end{cases}$

$$f(\bar{x}) = c(\bar{x}) * \left(\overline{c(\bar{d}(\bar{x})) * \left(\overline{c(\bar{d}^2(\bar{x})) * \ldots * \left(\overline{c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x}))}\right)\ldots}\right)}\right) \quad \text{— expansion of } f\text{, as before}$$

ASC $\longrightarrow$ $\|$

$$\left(\overline{\ldots\left(\left(\overline{c(\bar{x}) * c(\bar{d}(\bar{x}))}\right) * c(\bar{d}^2(\bar{x}))\right) * \ldots * c(\bar{d}^{m-1}(\bar{x}))}\right) * b(\bar{d}^m(\bar{x}))$$

if $m>0$ $\|$

$$\ldots\left(\left(\overline{c(\bar{x}) * c(\bar{d}(\bar{x}))}\right) * c(\bar{d}^2(\bar{x}))\right) * \ldots * c(\bar{d}^{m-1}(\bar{x}))$$

if $m>0$, finally return r instead of $r * b(\bar{x})$

if $m=0$, initially return $b(\bar{x})$, as in associative-only case

$\Downarrow$

new function: $f'(\bar{x}, r) \triangleq$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $f'(\bar{d}(\bar{x}), r * c(\bar{x}))$ $\qquad \mu_{f'}, \prec_{f'}, \tau_{f'}$ as before

$\vdash \boxed{f'f}$ $\quad f'(\bar{x}, r) = r * f(\bar{x})$

induct f'

base) $a(\bar{x})$: $\delta_{f'}$: $f'(\bar{x}, r) = r \overset{RI}{=} r * b(\bar{x}) = r * f(\bar{x})$; δ_f: $f(\bar{x}) = b(\bar{x})$

step) $\neg a(\bar{x})$: δ_f: $f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$; $\delta_{f'}$: $f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \overset{ASC}{=} r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x})$

IH: $f'(d(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x}))$

QED

$\vdash \boxed{ff'}$ $\quad f(\bar{x}) =$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $f'(\bar{d}(\bar{x}), c(\bar{x}))$ $\qquad$ — as in associative-only case

wrapper: $\tilde{f}(\bar{x}) \triangleq$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $f'(\bar{d}(\bar{x}), c(\bar{x})) \implies \vdash \boxed{f\tilde{f}}$ $\quad f(\bar{x}) = \tilde{f}(\bar{x})$ $\qquad$ — as in associative-only case

Associative Binary Operator with Left and Right Identity

conditions
$$\begin{cases} \boxed{\text{ASC}} & u * (v * w) = (u * v) * w & \text{— associativity} \\ \boxed{\text{LI}} & b(\bar{x}) * u = u & \text{— left identity} \\ \boxed{\text{RI}} & u * b(\bar{x}) = u & \text{— right identity} \end{cases} \implies (\mathcal{U}, *, b_0) \text{ is a monoid}$$

$\vdash \boxed{\text{b.const}}$ $\quad b(\bar{x}) = b(\bar{y})$

$$b(\bar{x}) \underset{RI}{=} b(\bar{x}) * b(\bar{y}) \underset{LI}{=} b(\bar{y}) \implies b_0 \triangleq b(\bar{x}) \quad \text{— constant value of } b$$

QED

$$f(\bar{x}) = c(\bar{x}) * \left(\overline{c(\bar{d}(\bar{x})) * \left(\overline{c(\bar{d}^2(\bar{x})) * \ldots * \left(\overline{c(\bar{d}^{m-1}(\bar{x})) * b(\bar{d}^m(\bar{x}))}\right)\ldots}\right)}\right) \quad \text{— expansion of } f\text{, as before}$$

ASC, LI, RI

$$= \ldots \left(\left(\left(\overline{\overline{\overline{b(\bar{x}) * c(\bar{x})}} * c(\bar{d}(\bar{x}))}\right) * c(\bar{d}^2(\bar{x}))\right) * \ldots * c(\bar{d}^{m-1}(\bar{x}))\right.$$

start with $r := b(\bar{x})$ (LI) — no initial $\bar{x} := \bar{d}(\bar{x})$ (LI) — no final $r * b(\bar{x})$ (RI) — no initial split on $a(\bar{x})$ (LI) — "combine" LI and RI

$\Downarrow$

new function: $f'(\bar{x}, r) \triangleq$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $f'(\bar{d}(\bar{x}), r * c(\bar{x}))$ — as in associative-with-right-identity case

$\vdash \boxed{f'f}$ $\quad f'(\bar{x}, r) = r * f(\bar{x})$ — as in associative-with-right-identity case

$\vdash \boxed{ff'}$ $\quad f(\bar{x}) = f'(\bar{x}, b(\bar{x}))$ — as in associative-with-left-identity case

wrapper: $\tilde{f}(\bar{x}) \triangleq f(\bar{x}, b(\bar{x})) \implies \vdash \boxed{f\tilde{f}}$ $\quad f(\bar{x}) = \tilde{f}(\bar{x})$ — as in associative-with-left-identity case

Restriction of Operator Properties to a Domain

$D \subseteq \mathcal{U}$ — domain

conditions

$\boxed{Db}$	$a(\bar{x}) \Rightarrow D(b(\bar{x}))$	— domain closure for b
$\boxed{Dc}$	$\neg a(\bar{x}) \Rightarrow D(c(\bar{x}))$	— domain closure for c
$\boxed{D*}$	$D(u) \wedge D(v) \Rightarrow D(u * v)$	— domain closure for $*$
$\boxed{ASC}$	$D(u) \wedge D(v) \wedge D(w) \Rightarrow u * (v * w) = (u * v) * w$	— associativity
$\boxed{LI}$	$a(\bar{x}) \wedge D(u) \Rightarrow b(\bar{x}) * u = u$	— left identity (optional)
$\boxed{RI}$	$a(\bar{x}) \wedge D(u) \Rightarrow u * b(\bar{x}) = u$	— right identity (optional)

$\vdash \boxed{Df}$ $D(f(\bar{x}))$ — domain closure for f

induct on f

base)

$$\begin{tikzcd}
a(\bar{x}) \arrow[r, "\delta_f"] \arrow[dr, bend right, "Db"'] & f(\bar{x}) = b(\bar{x}) \arrow[dr] & \\
& D(b(\bar{x})) \arrow[r] & D(f(\bar{x}))
\end{tikzcd}$$

step)

$$\begin{tikzcd}
\neg a(\bar{x}) \arrow[r, "\delta_f"] \arrow[dr, bend right, "Dc"'] \arrow[dd, bend right, "IH"'] & f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x})) \arrow[drr] & & \\
& D(c(\bar{x})) \arrow[r] & D(c(\bar{x}) * f(\bar{d}(\bar{x}))) \arrow[r] & D(f(\bar{x})) \\
D(f(\bar{d}(\bar{x}))) \arrow[urr, "D*"'] & & &
\end{tikzcd}$$

QED

$Db \wedge Dc \wedge D*$ $\Rightarrow$ all the operands of $*$ in the expansion of $f(\bar{x})$ are in D $\Rightarrow$ ASC and RI apply

but LI applies only if $b(\bar{x})$ is such that $a(\bar{x})$ holds $\Rightarrow$ we cannot use $b(\bar{x})$ for any $\bar{x}$ in general

$\Longrightarrow$

$\Downarrow$ calculate, from any $\bar{x}$, some $b(\bar{x})$ such that $a(\bar{x})$ holds (i.e. go to the "bottom" of the recursion of f):

$\beta(\bar{x}) \triangleq$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $\beta(\bar{d}(\bar{x}))$ $\qquad \mu_\beta(\bar{x}) \triangleq \mu_f(\bar{x}) \qquad \prec_\beta \triangleq \prec_f$

$\vdash \boxed{\tau_\beta}$ $\neg a(\bar{x}) \Rightarrow \mu_\beta(\bar{d}(\bar{x})) \prec_\beta \mu_\beta(\bar{x})$

$$\begin{array}{ccccc} \neg a(\bar{x}) \xrightarrow{\tau_f} & \mu_f(\bar{d}(\bar{x})) & \prec_f & \mu_f(\bar{x}) \\ & \delta_{\mu_\beta} \| & \delta_{\prec_\beta} \| & \| \delta_{\mu_\beta} \\ & \mu_\beta(\bar{d}(\bar{x})) & \prec_\beta & \mu_\beta(\bar{x}) \end{array}$$

QED

$\vdash \boxed{D\beta}$ $D(\beta(\bar{x}))$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b(\bar{x})$, $a(\bar{x}) \xrightarrow{Db} D(b(\bar{x}))$ $\to D(\beta(\bar{x}))$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$, $\neg a(\bar{x}) \xrightarrow{IH} D(\beta(\bar{d}(\bar{x})))$ $\to D(\beta(\bar{x}))$

QED

LI $\Rightarrow$ $\vdash \boxed{LI\beta}$ $D(u) \Rightarrow \beta(\bar{x}) * u = u$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b(\bar{x})$, $D(u) \xrightarrow{LI} b(\bar{x}) * u = u$ $\to \beta(\bar{x}) * u = u$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$, $D(u) \xrightarrow{IH} \beta(\bar{d}(\bar{x})) * u = u$ $\to \beta(\bar{x}) * u = u$

QED

RI $\Rightarrow$ $\vdash \boxed{RI\beta}$ $D(u) \Rightarrow u * \beta(\bar{x}) = u$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b(\bar{x})$, $D(u) \xrightarrow{RI} u * b(\bar{x}) = u$ $\to u * \beta(\bar{x}) = u$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x}))$, $D(u) \xrightarrow{IH} u * \beta(\bar{d}(\bar{x})) = u$ $\to u * \beta(\bar{x}) = u$

QED

new function: $f'(\bar{x}, r) \triangleq \begin{cases} \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r & \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) \quad \Leftarrow RI \\ \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r * b(\bar{x}) & \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) \quad \Leftarrow \neg RI \end{cases}$ — as when $D = \mathcal{U}$

$\vdash \boxed{f'f} \quad D(r) \Rightarrow f'(\bar{x}, r) = r * f(\bar{x})$

induct on f'

- base)
  - RI)
    - $a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x})$
    - $a(\bar{x}) \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = r = r * b(x) = r * f(\bar{x})$
    - $D(r) \xrightarrow[RI]{} r * b(\bar{x}) = r$
  - $\neg RI$)
    - $a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = b(\bar{x})$
    - $a(\bar{x}) \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = r * b(x) = r * f(\bar{x})$
- step)
  - $\neg a(\bar{x}) \xrightarrow{\delta_f} f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$
  - $\neg a(\bar{x}) \xrightarrow{\delta_{f'}} f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) = r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x})$
  - $\neg a(\bar{x}) \xrightarrow{Dc} D(c(\bar{x}))$
  - $D(c(\bar{x})) \to D(r * c(\bar{x}))$
  - $D(r) \xrightarrow{D*} D(r * c(\bar{x}))$
  - $D(r * c(\bar{x})) \Rightarrow f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x}))$
  - $f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x}))$
  - $Df \to D(f(\bar{d}(\bar{x})))$
  - $(r * c(\bar{x})) * f(\bar{d}(\bar{x})) = r * (c(\bar{x}) * f(\bar{d}(\bar{x})))$ (ASC)
  - IH

QED

β and LI are not used in f' and $f'f$

LI $\Longrightarrow$ $\vdash \boxed{ff'}$ $f(\bar{x}) = f'(\bar{x}, \beta(\bar{x}))$

$D\beta \longrightarrow D(\beta(\bar{x}))$

$f'f \xrightarrow[r := \beta(\bar{x})]{} f'(\bar{x}, \beta(\bar{x})) = \beta(\bar{x}) * f(\bar{x}) = f(\bar{x})$

$Df \longrightarrow D(f(\bar{x})) \xrightarrow{LI\beta} f(\bar{x})$

QED

wrapper : $\tilde{f}(\bar{x}) \triangleq f'(\bar{x}, \beta(\bar{x})) \Longrightarrow \vdash \boxed{f\tilde{f}}$ $f(\bar{x}) = \tilde{f}(\bar{x})$ — as when $D = \mathcal{U}$

$\neg$LI $\Longrightarrow$ $\vdash \boxed{ff'}$ $f(\bar{x}) =$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $f'(\bar{d}(\bar{x}), c(\bar{x}))$

$f(\bar{x}) \overset{\delta_f}{=}$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $c(\bar{x}) * f(\bar{d}(\bar{x})) =$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $f'(\bar{d}(\bar{x}), c(\bar{x}))$

$f'f \xrightarrow[\bar{x} := \bar{d}(\bar{x})]{r := c(\bar{x})} D(c(\bar{x})) \Longrightarrow f'(\bar{d}(\bar{x}), c(\bar{x})) = c(\bar{x}) * f(\bar{d}(\bar{x}))$

$Dc \longrightarrow \neg a(\bar{x}) \Rightarrow D(c(\bar{x}))$

$\neg a(\bar{x}) \Rightarrow f'(\bar{d}(\bar{x}), c(\bar{x})) = c(\bar{x}) * f(\bar{d}(\bar{x}))$

QED

wrapper : $\tilde{f}(\bar{x}) \triangleq$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $f'(\bar{d}(\bar{x}), c(\bar{x})) \Longrightarrow \vdash \boxed{f\tilde{f}}$ $f(\bar{x}) = \tilde{f}(\bar{x})$ — as when $D = \mathcal{U}$

β is used only if LI holds ; only LIβ is used here, not RIβ

$LI \wedge RI \Rightarrow$

$\vdash \boxed{b.const}\quad a(\bar{x}) \wedge a(\bar{y}) \Rightarrow b(\bar{x}) = b(\bar{y})$

$$\begin{tikzcd}
a(\bar{x}) \arrow[r, "Db"] \arrow[rr, bend left, "RI"] & D(b(\bar{x})) \arrow[r] & b(\bar{x}) * b(\bar{y}) = b(\bar{x}) \arrow[dr] & \\
a(\bar{y}) \arrow[r, "Db"] \arrow[rr, bend right, "LI"'] & D(b(\bar{y})) \arrow[r] & b(\bar{x}) * b(\bar{y}) = b(\bar{y}) \arrow[r] & b(\bar{x}) = b(\bar{y})
\end{tikzcd}$$

QED

$\vdash \boxed{\beta b}\quad a(\bar{y}) \Rightarrow \beta(\bar{x}) = b(\bar{y})$

induct on β

base)
$$\begin{tikzcd}
a(\bar{x}) \arrow[r, "\delta\beta"] \arrow[dr] & \beta(\bar{x}) = b(\bar{x}) \arrow[dr] & \\
a(\bar{y}) \arrow[r, "b.const"'] & b(\bar{x}) = b(\bar{y}) \arrow[r] & \beta(\bar{x}) = b(\bar{y})
\end{tikzcd}$$

step)
$$\begin{tikzcd}
\neg a(\bar{x}) \arrow[r, "\delta\beta"] \arrow[dr] & \beta(\bar{x}) = \beta(\bar{d}(\bar{x})) \arrow[dr] & \\
a(\bar{y}) \arrow[r, "IH"'] & \beta(\bar{d}(\bar{x})) = b(\bar{y}) \arrow[r] & \beta(\bar{x}) = b(\bar{y})
\end{tikzcd}$$

QED

$\vdash \boxed{\beta.const}\quad \beta(\bar{x}) = \beta(\bar{y})$

induct on β

base)
$$\begin{tikzcd}
a(\bar{x}) \arrow[r, "\delta\beta"] \arrow[dr, "\beta b", "\bar{x} := \bar{y}"', "\bar{y} := \bar{x}"] & \beta(\bar{x}) = b(\bar{x}) \arrow[dr] & \\
 & \beta(\bar{y}) = b(\bar{x}) \arrow[r] & \beta(\bar{x}) = \beta(\bar{y})
\end{tikzcd}$$

step)
$$\begin{tikzcd}
\neg a(\bar{x}) \arrow[r, "\delta\beta"] \arrow[dr, "IH"'] & \beta(\bar{x}) = \beta(\bar{d}(\bar{x})) \arrow[dr] & \\
 & \beta(\bar{d}(\bar{x})) = \beta(\bar{y}) \arrow[r] & \beta(\bar{x}) = \beta(\bar{y})
\end{tikzcd}$$

QED

$b_0 \triangleq \beta(\bar{x})$ — constant value of β, i.e. of b under a

$\vdash \boxed{LI_0}\quad D(u) \Rightarrow b_0 * u = u$

$$D(u) \xrightarrow[LI\beta]{} \beta(\bar{x}) * u = u \xrightarrow{\delta b_0} b_0 * u = u$$

QED

$\vdash \boxed{RI_0}\quad D(u) \Rightarrow u * b_0 = u$

$$D(u) \xrightarrow[RI\beta]{} u * \beta(\bar{x}) = u \xrightarrow{\delta b_0} u * b_0 = u$$

QED

$\vdash \boxed{Db_0}\quad D(b_0)$

$$D\beta \longrightarrow D(\beta(\bar{x})) \xrightarrow{\delta b_0} D(b_0)$$

QED

$D* \wedge Db_0 \wedge ASC \wedge LI_0 \wedge RI_0$

$\Downarrow$

$(D, *, b_0)$ is a monoid

Guards

old function: $f(\bar{x}) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ c(\bar{x}) * f(\bar{d}(\bar{x}))$

$\gamma_{\bar{d}}(\bar{x}) = \gamma_{d_1}(\bar{x}) \wedge \cdots \wedge \gamma_{d_n}(\bar{x})$

$\boxed{\surd f}$ $\gamma_{\gamma_f}(\bar{x}) \wedge \left[\gamma_f(\bar{x}) \Rightarrow \gamma_a(\bar{x}) \wedge \left[a(\bar{x}) \Rightarrow \gamma_b(\bar{x})\right] \wedge \left[\neg a(\bar{x}) \Rightarrow \gamma_c(\bar{x}) \wedge \gamma_{\bar{d}}(\bar{x}) \wedge \gamma_f(\bar{d}(\bar{x})) \wedge \gamma_*(c(\bar{x}), f(\bar{d}(\bar{x})))\right]\right]$

$D \subseteq \mathcal{U}$ — domain

conditions
- $\boxed{GD}$ $\gamma_D = \mathcal{U}$ — D always well-defined
- $\boxed{G*}$ $\gamma_* \supseteq D \times D$ — $*$ well-defined at least over D

new function: $f'(\bar{x}, r) \triangleq \begin{cases} \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r & \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) \quad \Leftarrow RI \\ \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r * b(\bar{x}) & \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) \quad \Leftarrow \neg RI \end{cases}$ — as before

$\gamma_{f'}(\bar{x}, r) \triangleq \left[\gamma_f(\bar{x}) \wedge D(r)\right]$

$\vdash \boxed{\surd f'}$ $\omega_{f'}(\bar{x}, r)$

RI)

$$\omega_{f'}(\bar{x}, r) = \overset{\surd f}{\cancel{\gamma_{\gamma_f}(\bar{x})}} \wedge \overset{GD}{\cancel{\gamma_D(r)}} \wedge \left[\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \overset{\surd f}{\cancel{\gamma_a(\bar{x})}} \wedge \left[\neg a(\bar{x}) \Rightarrow \overset{\surd f}{\cancel{\gamma_{\bar{d}}(\bar{x})}} \wedge \overset{\surd f}{\cancel{\gamma_c(\bar{x})}} \wedge \overset{G*}{\cancel{\gamma_*(r, c(\bar{x}))}} \wedge \overset{\surd f}{\cancel{\gamma_f(\bar{d}(\bar{x}))}} \wedge \overset{D*}{\cancel{D(r * c(\bar{x}))}}\right]\right]$$

$\xrightarrow{Dc} D(c(\bar{x}))$

¬RI)

$$\omega_{f'}(\bar{x}, r) = \overset{\surd f}{\cancel{\gamma_{\gamma_f}(\bar{x})}} \wedge \overset{GD}{\cancel{\gamma_D(r)}} \wedge \left[\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \overset{\surd f}{\cancel{\gamma_a(\bar{x})}} \wedge \left[a(\bar{x}) \Rightarrow \overset{\surd f}{\cancel{\gamma_b(\bar{x})}} \wedge \overset{G*}{\cancel{\gamma_*(r, b(\bar{x}))}}\right] \wedge \left[\neg a(\bar{x}) \Rightarrow \overset{\surd f\ Dc\ G*\ D*}{\cancel{\ldots}}\right]\right]$$

$\xrightarrow{Db} D(b(\bar{x}))$

as with RI

QED

$\beta(\bar{x}) \triangleq \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ \beta(\bar{d}(\bar{x}))$ — as before

$\gamma_\beta(\bar{x}) \triangleq \gamma_f(\bar{x})$

$\vdash \boxed{\surd\beta}\quad \omega_\beta(\bar{x})$

$$\omega_\beta(\bar{x}) = \cancel{\gamma_{\gamma_f}(\bar{x})}^{\surd f} \wedge \left[\gamma_f(\bar{x}) \Rightarrow \cancel{\gamma_a(\bar{x})}^{\surd f} \wedge \left[a(\bar{x}) \Rightarrow \cancel{\gamma_b(\bar{x})}^{\surd f}\right] \wedge \left[\neg a(\bar{x}) \Rightarrow \cancel{\gamma_{\bar{d}}(\bar{x})}^{\surd f} \wedge \cancel{\gamma_f(\bar{d}(\bar{x}))}^{\surd f}\right]\right]$$

QED

wrapper : $\tilde{f}(\bar{x}) \triangleq \begin{cases} f'(\bar{x}, \beta(\bar{x})) & \Leftarrow \text{LI} \\ \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b(\bar{x})\ \underline{\text{else}}\ f'(\bar{d}(\bar{x}), c(\bar{x})) & \Leftarrow \neg\text{LI} \end{cases}$ — as before

$\gamma_{\tilde{f}}(\bar{x}) \triangleq \gamma_f(\bar{x})$

$\vdash \boxed{\surd\tilde{f}}\quad \omega_{\tilde{f}}(\bar{x})$

LI)

$$\omega_{\tilde{f}}(\bar{x}) = \cancel{\gamma_{\gamma_f}(\bar{x})}^{\surd f} \wedge \left[\gamma_f(\bar{x}) \Rightarrow \cancel{\gamma_\beta(\bar{x})}^{\delta\gamma_\beta} \wedge \cancel{\gamma_{f'}(\bar{x}, \beta(\bar{x}))}^{D\beta,\ \delta\gamma_{f'}}\right]$$

$\neg$LI)

$$\omega_{\tilde{f}}(\bar{x}) = \cancel{\gamma_{\gamma_f}(\bar{x})}^{\surd f} \wedge \left[\gamma_f(\bar{x}) \Rightarrow \cancel{\gamma_a(\bar{x})}^{\surd f} \wedge \left[a(\bar{x}) \Rightarrow \cancel{\gamma_b(\bar{x})}^{\surd f}\right] \wedge \left[\neg a(\bar{x}) \Rightarrow \cancel{\gamma_{\bar{d}}(\bar{x})}^{\surd f} \wedge \cancel{\gamma_c(\bar{x})}^{\surd f} \wedge \cancel{\gamma_{f'}(\bar{d}(\bar{x}), c(\bar{x}))}^{\surd f,\ \delta\gamma_{f'}}\right]\right]$$

Dc $\quad D(c(\bar{x}))$

QED

if $\tilde{f}$ is not generated, a proof like this establishes that the body of $\tilde{f}$ is guard-verified under $\gamma_f(\bar{x})$:
this theorem is useful to guard-verify terms where a call to f is replaced with the (instantiated) body of $\tilde{f}$

Decomposition of the Old Function

$f(\bar{x}) \triangleq$ <u>if</u> t_1 <u>then</u> t_2 <u>else</u> t_3 $\qquad$ t_1, t_2, t_3 terms without <u>let</u> $\qquad$ f occurs in t_3, not in t_1 or t_2

$a \triangleq \lambda \bar{x}.\, t_1$

$b \triangleq \lambda \bar{x}.\, t_2$

all calls to f in t_3 must be identical : $f(s_1, \ldots, s_n)$ $\qquad$ $s_1, \ldots, s_n$ terms not containing f $\qquad$ $n > 0$

$d_i \triangleq \lambda \bar{x}.\, s_i$

$\tilde{t}_3 \triangleq t_3[f(s_1, \ldots, s_n)/r]$, r fresh variable $\begin{cases} r \in FV(\tilde{t}_3) \text{ , otherwise } f \text{ would not be recursive} \\ \tilde{t}_3 \not\equiv r \text{ , otherwise } f \text{ would be already tail-recursive} \end{cases}$

$C \triangleq \{(c, *) \in (\mathcal{U}^n \to \mathcal{U}) \times (\mathcal{U} \times \mathcal{U} \to \mathcal{U}) \mid \tilde{t}_3 = c(\bar{x}) * r\}$ — candidates for c and $*$

- $C = \emptyset$, e.g. $\tilde{t}_3 \equiv g(x_1, x_2, r)$
- $|C| = 1$, e.g. $\tilde{t}_3 \equiv g(x_1, r)$, $C = \{(id, g)\}$
- $|C| > 1$, e.g. $\tilde{t}_3 \equiv g(h(x_1), r)$, $C = \{(h, g), (id, \lambda(q, r).\, g(h(q), r))\}$

exactly one $(c, *) \in C$ for each term s such that $r \notin FV(s)$ and $FV(\tilde{t}_3[s/q]) \subseteq \{q, r\}$, q fresh variable

s includes all occurrences of $\bar{x}$ in $\tilde{t}_3$

special case : $FV(\tilde{t}_3) = \{r\}$, i.e. $\bar{x}$ do not occur in $\tilde{t}_3$

$\Rightarrow$ any s would do, but then $*$ may ignore its first argument, making ASC, LI, RI less likely

when two such terms s and s' are one a subterm of the other, one is not always better than the other, e.g. :

$$\tilde{t}_3 \equiv -x_1 + r \begin{cases} s \equiv x_1 \Rightarrow * = \lambda(q,r).\, -q + r \text{ is not associative} \\ s' \equiv -x_1 \Rightarrow * = \lambda(q,r).\, q + r \text{ is associative} \end{cases} \Rightarrow \text{larger term is better}$$

$$\tilde{t}_3 \equiv -(-x_1) + r \begin{cases} s \equiv x_1 \Rightarrow * = \lambda(q,r).\, -(-q) + r \text{ is associative} \\ s' \equiv -x_1 \Rightarrow * = \lambda(q,r).\, -q + r \text{ is not associative} \end{cases} \Rightarrow \text{smaller term is better}$$

Special Case : Ground Base Value

old function : $f(\bar{x}) \triangleq \underline{\text{if}}\ t_1\ \underline{\text{then}}\ t_2\ \underline{\text{else}}\ t_3$ — as in discussion about decomposition

$FV(t_2) = \emptyset$ — ground base value $\Rightarrow$ $b \triangleq \lambda \bar{x}.\, b_0$, $b_0 \in \mathcal{U}$ $\Rightarrow$ $f(\bar{x}) = \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ b_0\ \underline{\text{else}}\ c(\bar{x}) * f(\bar{d}(\bar{x}))$

$\vdash \boxed{\beta_0}$ $\beta(\bar{x}) = b_0$

induct on β

base) $a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = b_0$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\beta} \beta(\bar{x}) = \beta(\bar{d}(\bar{x})) = b_0$

IH $\beta(\bar{d}(\bar{x})) = b_0$

QED

$\vdash \boxed{Db_0}$ $D(b_0)$

$D\beta \rightarrow D(\beta(\bar{x})) \xrightarrow{\beta_0} D(b_0)$

QED

LI $\Rightarrow$ $\vdash \boxed{LI_0}$ $D(u) \Rightarrow b_0 * u = u$

$D(u) \xrightarrow{LI\beta} \beta(\bar{x}) * u = u \xrightarrow{\beta_0} b_0 * u = u$

QED

RI $\Rightarrow$ $\vdash \boxed{RI_0}$ $D(u) \Rightarrow u * b_0 = u$

$D(u) \xrightarrow{RI\beta} u * \beta(\bar{x}) = u \xrightarrow{\beta_0} u * b_0 = u$

QED

$D* \wedge Db_0 \wedge ASC \wedge LI_0 \wedge RI_0 \Longrightarrow (D, *, b_0)$ is a monoid

---

f' and $f'f$ are the same as before (they do not use β and LI)

LI $\Rightarrow$ $\vdash \boxed{ff'_0}$ $f(\bar{x}) = f'(\bar{x}, b_0)$

$Db_0 \rightarrow D(b_0)$

$f'f$ $\xrightarrow{r:=b_0}$ $f'(\bar{x}, b_0) = b_0 * f(\bar{x}) = f(\bar{x})$

$Df \rightarrow D(f(\bar{x}))$ $\xrightarrow{LI_0}$

QED

wrapper : $\tilde{f}(\bar{x}) \triangleq f'(\bar{x}, b_0) \Longrightarrow \vdash \boxed{f\tilde{f}}$ $f(\bar{x}) = \tilde{f}(\bar{x})$ — as before

$\alpha(\bar{x}) \triangleq$ <u>if</u> $a(\bar{x})$ <u>then</u> $\bar{x}$ <u>else</u> $\alpha(\bar{d}(\bar{x}))$ $\qquad \alpha : \mathcal{U}^n \to \mathcal{U}^n$ — calculate $\bar{x}_0$ such that $a(\bar{x}_0)$ holds, from any $\bar{x}$

$\mu_\alpha(\bar{x}) \triangleq \mu_f(\bar{x}) \qquad <_\alpha \triangleq <_f$

$\vdash \boxed{\tau_\alpha}$ $\neg a(\bar{x}) \Rightarrow \mu_\alpha(\bar{d}(\bar{x})) <_\alpha \mu_\alpha(\bar{x})$

$$\begin{array}{ccccc} \neg a(\bar{x}) \xrightarrow{\tau_f} & \mu_f(\bar{d}(\bar{x})) & <_f & \mu_f(\bar{x}) \\ & \delta_{\mu_\alpha} \| & \delta_{<_\alpha} \| & \| \delta_{\mu_\alpha} \\ & \mu_\alpha(\bar{d}(\bar{x})) & <_\alpha & \mu_\alpha(\bar{x}) \end{array}$$

QED

$\vdash \boxed{a\alpha}$ $a(\alpha(\bar{x}))$

induct on α

base) $a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \bar{x} \to a(\alpha(\bar{x}))$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \alpha(\bar{d}(\bar{x})) \to a(\alpha(\bar{x}))$

$\neg a(\bar{x}) \xrightarrow{IH} a(\alpha(\bar{d}(\bar{x}))) \to a(\alpha(\bar{x}))$

QED

$\surd f \Rightarrow \vdash \boxed{\gamma_f \alpha}$ $\gamma_f(\bar{x}) \Rightarrow \gamma_f(\alpha(\bar{x}))$

induct on α

base) $a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \bar{x} \to \gamma_f(\alpha(\bar{x}))$

$\gamma_f(\bar{x}) \to \gamma_f(\alpha(\bar{x}))$

step) $\neg a(\bar{x}) \xrightarrow{\delta_\alpha} \alpha(\bar{x}) = \alpha(\bar{d}(\bar{x})) \to \gamma_f(\alpha(\bar{x}))$

$\gamma_f(\bar{x}) \xrightarrow{\surd f} \gamma_f(\bar{d}(\bar{x})) \xrightarrow{IH} \gamma_f(\alpha(\bar{d}(\bar{x}))) \to \gamma_f(\alpha(\bar{x}))$

QED

$b = \lambda \bar{x}.\, b_0 \implies \gamma_b \triangleq \lambda \bar{x}.\, \gamma_{b_0} \quad , \quad \gamma_{b_0} \in \mathcal{U}$

$\vdash \boxed{\gamma_b.\text{const}} \quad \gamma_b(\bar{x}) = \gamma_b(\bar{y})$

$\gamma_b(\bar{x}) \underset{\delta_{\gamma_b}}{=} \gamma_{b_0} \underset{\delta_{\gamma_b}}{=} \gamma_b(\bar{y})$

QED

$\surd f \implies \vdash \boxed{Gb} \quad \gamma_f(\bar{x}) \Rightarrow \gamma_b(\bar{x})$

$$\begin{tikzcd}
\gamma_f(\bar{x}) \arrow[r, "\gamma_f\alpha"'] & \gamma_f(\alpha(\bar{x})) \arrow[dr] & & \\
 & & \gamma_b(\alpha(\bar{x})) \arrow[r, equal, "\gamma_b.\text{const}"] & \gamma_b(\bar{x}) \\
\alpha\alpha \arrow[r] & \alpha(\alpha(\bar{x})) \arrow[urr, "\surd f"'] & &
\end{tikzcd}$$

QED

$\surd f \wedge LI \implies \vdash \boxed{\widetilde{\surd f}} \quad \omega_{\tilde{f}}(\bar{x})$

$\omega_{\tilde{f}}(\bar{x}) = \cancel{\gamma_{\gamma_f}(\bar{x})}^{\surd f} \wedge [\gamma_f(\bar{x}) \overset{Gb}{\Rightarrow} \cancel{\gamma_b(\bar{x})} \wedge \cancel{\gamma_{f'}(\bar{x}, b_0)}^{Db_0\ \delta_{\gamma_{f'}}}]$ — $b_0 = b(\bar{x})$

QED

summary of this special case (ground base value): b_0 can be used instead of β (when LI holds)

Extension of Operator Associativity and Closure outside the Domain

conditions
- $\boxed{Db}$ $a(\bar{x}) \Rightarrow D(b(\bar{x}))$ — as before
- $\boxed{D*'}$ $D(u * v)$ — unconditional version of $D*$
- $\boxed{ASC'}$ $u * (v * w) = (u * v) * w$ — unconditional version of ASC
- $\boxed{LI}$ $a(\bar{x}) \wedge D(u) \Rightarrow b(\bar{x}) * u = u$ — as before, but required (not optional)
- $\boxed{RI}$ $a(\bar{x}) \wedge D(u) \Rightarrow u * b(\bar{x}) = u$ — as before, and still optional

$(Db \wedge D*' \wedge ASC')$ is neither weaker nor stronger than $(Db \wedge Dc \wedge D* \wedge ASC)$

$D*'$ and ASC' may be satisfied when $*$ fixes its arguments to be in D

$\vdash \boxed{Df}$ $D(f(\bar{x}))$ — as before, but slightly different proof

$$
\begin{tikzcd}
a(\bar{x}) \arrow[r, "\delta_f"] \arrow[dr, bend right, "Db"'] & f(\bar{x}) = b(\bar{x}) \arrow[dr] & \\
 & D(b(\bar{x})) \arrow[r] & D(f(\bar{x})) \\
\neg a(\bar{x}) \arrow[r, "\delta_f"] & f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x})) \arrow[dr] & \\
D*' \arrow[r, "u := c(\bar{x})", "v := f(\bar{d}(\bar{x}))"'] & D(c(\bar{x}) * f(\bar{d}(\bar{x}))) \arrow[r] & D(f(\bar{x}))
\end{tikzcd}
$$

QED

$\beta(\bar{x}) \triangleq$ <u>if</u> $a(\bar{x})$ <u>then</u> $b(\bar{x})$ <u>else</u> $\beta(\bar{d}(\bar{x}))$ — as before

$\vdash \boxed{D\beta}$ $D(\beta(\bar{x}))$ — as before, same proof

$\vdash \boxed{LI\beta}$ $D(u) \Rightarrow \beta(\bar{x}) * u = u$ — as before, same proof

$RI \Rightarrow \vdash \boxed{RI\beta}$ $D(u) \Rightarrow u * \beta(\bar{x}) = u$ — as before, same proof

$RI \Rightarrow (D, *, b_0)$ is a monoid — as before, same lemmas and proofs

new function: $f'(\bar{x}, r) \triangleq \begin{cases} \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r & \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) \quad \Leftarrow RI \\ \underline{\text{if}}\ a(\bar{x})\ \underline{\text{then}}\ r * b(\bar{x}) & \underline{\text{else}}\ f'(\bar{d}(\bar{x}), r * c(\bar{x})) \quad \Leftarrow \neg RI \end{cases}$ — as before

$\vdash \boxed{f'f}$ $D(r) \Rightarrow f'(\bar{x}, r) = r * f(\bar{x})$ — as before, but slightly different proof

induct on f'

- base)
  - RI)
    - $a(\bar{x})$ $\xrightarrow{\delta_f}$ $f(\bar{x}) = b(\bar{x})$
    - $a(\bar{x})$ $\xrightarrow{\delta_{f'}}$ $f'(\bar{x}, r) = r = r * b(x) = r * f(\bar{x})$
    - $D(r) \xrightarrow[RI]{} r * b(\bar{x}) = r$
  - $\neg$RI)
    - $a(\bar{x})$ $\xrightarrow{\delta_f}$ $f(\bar{x}) = b(\bar{x})$
    - $a(\bar{x})$ $\xrightarrow{\delta_{f'}}$ $f'(\bar{x}, r) = r * b(x) = r * f(\bar{x})$
- step)
  - $\neg a(\bar{x})$
    - $\xrightarrow{\delta_f}$ $f(\bar{x}) = c(\bar{x}) * f(\bar{d}(\bar{x}))$
    - $\xrightarrow{\delta_{f'}}$ $f'(\bar{x}, r) = f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x})) \overset{ASC'}{=} r * (c(\bar{x}) * f(\bar{d}(\bar{x}))) = r * f(\bar{x})$
    - $D*' \longrightarrow D(r * c(\bar{x}))$
    - IH: $D(r * c(\bar{x})) \Rightarrow f'(\bar{d}(\bar{x}), r * c(\bar{x})) = (r * c(\bar{x})) * f(\bar{d}(\bar{x}))$

QED

$\vdash \boxed{ff'}$ $f(\bar{x}) = f'(\bar{x}, \beta(\bar{x}))$ — as before, same proof

the proof of ff' when LI does not hold needs Dc to use an instance of $f'f$

$\Rightarrow$ LI is required here; otherwise $(Db \wedge Dc \wedge D*' \wedge ASC')$ is stronger (i.e, less satisfiable) than $(Db \wedge Dc \wedge D* \wedge ASC)$

conditions
$$\begin{cases} \boxed{GD} & \gamma_D = \mathcal{U} \\ \boxed{G*} & \gamma_* \supseteq D \times D \\ \boxed{GDc} & \gamma_f(\bar{x}) \wedge \neg a(\bar{x}) \Rightarrow D(c(\bar{x})) \end{cases}$$

— as before

— as before

— weaker version of Dc, conditioned by the guard of f

$\gamma_{f'}(\bar{x}, r) \triangleq [\gamma_f(\bar{x}) \wedge D(r)]$ — as before

$\vdash \boxed{\surd f'}$ $\omega_{f'}(\bar{x}, r)$ — as before, but slightly different proof

RI)

$$\omega_{f'}(\bar{x}, r) = \cancel{\gamma_{\gamma_f}(\bar{x})}^{\surd f} \wedge \cancel{\gamma_D(r)}^{GD} \wedge [\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \cancel{\gamma_a(\bar{x})}^{\surd f} \wedge [\neg a(\bar{x}) \Rightarrow \cancel{\gamma_{\bar{a}}(\bar{x})}^{\surd f} \wedge \cancel{\gamma_c(\bar{x})}^{\surd f} \wedge \cancel{\gamma_*(r, c(\bar{x}))}^{G*} \wedge \cancel{\gamma_f(\bar{d}(\bar{x}))}^{\surd f} \wedge \cancel{D(r * c(\bar{x}))}^{D*'}]]$$

$\xrightarrow{GDc} D(c(\bar{x}))$

¬RI)

$$\omega_{f'}(\bar{x}, r) = \cancel{\gamma_{\gamma_f}(\bar{x})}^{\surd f} \wedge \cancel{\gamma_D(r)}^{GD} \wedge [\gamma_f(\bar{x}) \wedge D(r) \Rightarrow \cancel{\gamma_a(\bar{x})}^{\surd f} \wedge [a(\bar{x}) \Rightarrow \cancel{\gamma_b(\bar{x})}^{\surd f} \wedge \cancel{\gamma_*(r, b(\bar{x}))}^{G*}] \wedge [\neg a(\bar{x}) \Rightarrow \cancel{\ldots}^{\surd f\ GDc\ G*\ D*'}]]$$

$\xrightarrow{Db} D(b(\bar{x}))$

as with RI

QED

$\gamma_\beta(\bar{x}) \triangleq \gamma_f(\bar{x})$ — as before

$\vdash \boxed{\surd\beta}$ — as before, same proof

$\gamma_{\tilde{f}}(\bar{x}) \triangleq \gamma_f(\bar{x})$ — as before

$\vdash \boxed{\surd\tilde{f}}$ — as before, same proof (only LI case)

special case of a ground base value: everything is the same as before