Homework 3
The finite difference method in 2D

Problem 1: Poisson equation

Write a MATLAB code for solving the Poisson equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = S(x,y)$ in 2D over the $[0;1] \times [0;1]$ computational domain for different source terms $S(x,y)$ and compare the numerical solution with the analytical solutions $f_{an}(x,y)$ provided below. The Dirichlet boundary conditions can be derived directly from the analytical solutions.

- Homogeneous source term

$$S(x,y) = A \tag{1}$$

$$f_{an}(x,y) = e^{kx} sin(ky) + \frac{A}{4}\left(x^2 + y^2\right) \tag{2}$$

where A and k are arbitrary numbers.

- Non-homogeneous source term

$$S(x,t) = \left(x^2 + y^2\right) e^{xy} \tag{3}$$

$$f_{an}(x,t) = e^{xy} \tag{4}$$

Problem 2: advection-diffusion equation

Write a MATLAB code for solving the advection-diffusion equation $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = \mathit{\Gamma}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$ over the $[0;1] \times [0;1]$ computational domain, where u, v and $\mathit{\Gamma} > 0$ are arbitrary numbers. The initial condition f_0 is:

$$f(x,y,t=0) = f_0 = C cos\left(B\pi\eta\right) exp\left(\frac{1 - \sqrt{1 + 4A^2\pi^2\Gamma^2}}{2\Gamma}\xi\right) \tag{5}$$

where A and B are arbitrary numbers and $\xi = ux + vy$ and $\eta = vx - uy$. The analytical solution $f_{an}(x,y)$ is given by the following epression:

$$f_{an}(x,y,t) = f_0 e^{-\Gamma\pi^2\left(u^2+v^2\right)\left(B^2-A^2\right)t} \tag{6}$$

The Dirichlet boundary conditions can be derived directly from the analytical solutions. Remember to choose a proper combination of space and time steps, satisfying the stability requirements ($Co < 1$ and $Di < 1/4$). Pay attention to the fact that Dirichlet boundary conditions along the boundaries are time-dependent (see the analytical solution).

An example is given by the following input data:

- Velocity: $u = 1\,m/s$ and $v = 0.5\,m/s$
- Diffusion coefficient: $\Gamma = 0.05\,m^2/s$
- Number of grid points: $n_X = n_Y = 40$
- Time step: $\triangle t = 0.0025\,s$
- Total time of simulation: $\tau = 0.3\,s$
- Arbitrary constants: $A = 1.25$, $B = 0.75$, $C = 1$