Homework 1
The finite difference method in 1D

Introduction

The following exercises consist in implementing a proper MATLAB code for solving the 1D advection-diffusion equation with different initial and boundary conditions. The computational domain has always length $L = 1\,m$, starting from initial coordinate $x_0 = 0$. The discretization of advective and diffusion terms has to be carried out using a centered, 2nd order scheme, while time integration via the forward (explicit) Euler method.
We suggest to start with the following data, which satisfy the stability requirements ($Co < 1$ and $Di < 1/2$):

- Velocity: $u = 0.1\,m/s$
- Diffusion coefficient: $\Gamma = 0.01\,m^2/s$
- Number of grid points: $n_P = 21$
- Time step: $\triangle t = 0.1\,s$
- Total time of simulation: $\tau = 20\,s$

Analytical solutions are provided in order to verify the numerical implementation.

Problem 1

Write a MATLAB code for solving the diffusion equation $\frac{\partial f}{\partial t} = \mathit{\Gamma} \frac{\partial^2 f}{\partial x^2}$, with the given initial and boundary conditions:

$$\begin{cases} f(x,0) = sin(n\pi x) \\ f(x=0,t) = 0 \\ f(x=1,t) = 0 \end{cases} \tag{1}$$

where n is an arbitrary integer number (suggested $n = 2$). Compare your numerical solution with the analytical solution:

$$f(x,t) = e^{-\Gamma n^2 \pi^2 t} sin(n\pi x) \tag{2}$$

Problem 2

Write a MATLAB code for solving the diffusion equation $\frac{\partial f}{\partial t} = \mathit{\Gamma} \frac{\partial^2 f}{\partial x^2}$, with the given initial and boundary conditions:

$$\begin{cases} f(x,0) = \sqrt{\frac{A}{B}} e^{-\frac{(x-1)^2}{4\Gamma B}} \\ f(x=0,t) = \sqrt{\frac{A}{t+B}} e^{-\frac{1}{4\Gamma(t+B)}} \\ f(x=1,t) = \sqrt{\frac{A}{t+B}} \end{cases} \tag{3}$$

where A and B are arbitrary real numbers (suggested $A = 2$ and $B = 3$). Compare your numerical solution with the analytical solution:

$$f(x,t) = \sqrt{\frac{A}{t+B}} e^{-\frac{(x-1)^2}{4\Gamma(t+B)}} \tag{4}$$

Note that the boundary condition changes in time at both ends. This type of solution is used for accuracy verification of high-order schemes.

Problem 3

Write a MATLAB code for solving the advection-diffusion equation $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = \mathit{\Gamma}\frac{\partial^2 f}{\partial x^2}$, with the given initial and boundary conditions:

$$\begin{cases} f(x,0) = x \\ f(x=0,t) = 0 \\ f(x=1,t) = 1 \end{cases} \tag{5}$$

The steady-state solution (i.e. for a sufficiently long time) is:

$$f(x) = \frac{1 - e^{xRe}}{1 - e^{Re}} \tag{6}$$

where $Re = u/\mathit{\Gamma}$. Compare your numerical solution at steady-state with the analytical solution above.

Problem 4

Write a MATLAB code for solving the following advection-diffusion equation including a source term:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = \mathit{\Gamma}\frac{\partial^2 f}{\partial x^2} + C\pi\left[u\, cos(\pi x) + \pi\mathit{\Gamma}\, sin(\pi x)\right] \tag{7}$$

with the given initial and boundary conditions:

$$\begin{cases} f(x,0) = x \\ f(x=0,t) = 0 \\ f(x=1,t) = 1 \end{cases} \tag{8}$$

where C is an arbitrary constant (suggested $C = 1.5$). The steady-state solution (i.e. for a sufficiently long time) is:

$$f(x) = \frac{1 - e^{xRe}}{1 - e^{Re}} + C\, sin(\pi x) \tag{9}$$

where $Re = u/\mathit{\Gamma}$. Compare your numerical solution at steady-state with the analytical solution above.