0.1 Axiom of extensionality

If two sets contain the same elements, they are equal.

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]$

This is an axiom, not a definition, because equality was defined as part of first-order logic.

Note that this is not bidirectional. $x = y$ does not imply that x and y contain the same elements. This is appropraite as $\frac{1}{2} = \frac{2}{4}$ for example, even though they are written differently as sets.

0.1.1 Reflexivity of equality

The reflexive property is:

$\forall x(x = x)$

We can replace the intance of y with x:

$\forall x[\forall z(z \in x \leftrightarrow z \in x) \to x = x]$

We can show that the following is true:

$\forall z(z \in x \leftrightarrow z \in x)$

Therefore:

$\forall x[T \to x = x]$

$x = x$

0.1.2 Symmetry of equality

The symmetry property is:

$\forall x \forall y[(x = y) \leftrightarrow (y = x)]$

We know that the following are true:

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]$

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to y = x]$

So:

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to (x = y \wedge y = x)]$

0.1.3 Transitivity of equality

The transitive property is:

$\forall x \forall y \forall z[(x = y \land y = z) \to x = z]$

0.1.4 Substitution for functions

The substitutive property for functions is:

$\forall x \forall y[(x = y) \to (f(x) = f(y))]$

0.1.5 Substitution for formulae

The substitutive property for formulae is:

$\forall x \forall y[((x = y) \land P(x)) \to P(y)]$

Doesn't this require iterating over predicates? Is this possible in first order logic??

0.1.6 Result 1: The empty set is unique

We can now show the empty set is unique.

0.1.7 Result 2: Every element of a set exists

If an element did not exist, the set containing it would be equal to a set which did not contain that element.

0.1.8 Result 3: Sets are unique