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## 0.1 Calculus of sine and cosine

### 0.1.1 Unity

Note that with imaginary numbers we can reverse all *is*. So:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$e^{-i\theta} = \cos(\theta) - i \sin(\theta)$$

$$e^{i\theta} e^{-i\theta} = (\cos(\theta) + i \sin(\theta))(\cos(\theta) - i \sin(\theta))$$

$$e^{i\theta} e^{-i\theta} = \cos(\theta)^2 + \sin(\theta)^2$$

$$e^{i\theta} e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1$$

So:

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

Note that if  $\cos(\theta)^2 = 0$ , then  $\sin(\theta)^2 = \pm 1$

That is, if the real part of  $e^{i\theta}$  is 0, the imaginary part is  $\pm 1$ . And visa versa.

Similarly if the derivative of the real part of  $e^{i\theta}$  is 0, the imaginary part is  $\pm 1$ . And visa versa.

### 0.1.2 Sine and cosine are linked by their derivatives

Note that these functions are linked in their derivatives.

$$\frac{\delta}{\delta \theta} \cos(\theta) = \sum_{j=0}^{\infty} \frac{(\theta)^{(4j+3)}}{(4j+3)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!}$$

$$\frac{\delta}{\delta \theta} \cos(\theta) = -\sin(\theta)$$

Similarly:

$$\frac{\delta}{\delta \theta} \sin(\theta) = \cos(\theta)$$

### 0.1.3 Both sine and cosine oscillate

$$\frac{\delta^2}{\delta\theta^2} \sin(\theta) = -\sin(\theta)$$

$$\frac{\delta^2}{\delta\theta^2} \cos(\theta) = -\cos(\theta)$$

So for either of:

$$y = \cos(\theta)$$

$$y = \sin(\theta)$$

We know that

$$\frac{\delta^2}{\delta\theta^2} y(\theta) = -y(\theta)$$

Consider  $\theta = 0$ .

$$e^{i \cdot 0} = \cos(0) + i \sin(0)$$

$$1 = \cos(0) + i \sin(0)$$

$$\sin(0) = 0$$

$$\cos(0) = 1$$

Similarly we know that the derivative:

$$\sin'(0) = \cos(0) = 1$$

$$\cos'(0) = -\sin(0) = 0$$

Consider  $\cos(\theta)$ .

As  $\cos(0)$  is static at  $\theta = 0$ , and is positive, it will fall until  $\cos(\theta) = 0$ .

While this is happening,  $\sin(\theta)$  is increasing. As:

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

$\sin(\theta)$  will equal 1 where  $\cos(\theta) = 0$ .

Due to symmetry this will repeat 4 times.

Let's call the length of this period  $\tau$ .

Where  $\theta = \tau * 0$

- $\cos(\theta) = 1$
- $\sin(\theta) = 0$

Where  $\theta = \tau * \frac{1}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = 1$

Where  $\theta = \tau * \frac{2}{4}$

- $\cos(\theta) = -1$
- $\sin(\theta) = 0$

Where  $\theta = \tau * \frac{3}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = -1$

#### 0.1.4 Relationship between $\cos(\theta)$ and $\sin(\theta)$

Note that  $\sin(\theta + \frac{\tau}{4}) = \cos(\theta)$

Note that  $\sin(\theta) = \cos(\theta)$  at

- $\tau * \frac{1}{8}$
- $\tau * \frac{5}{8}$

And that all these answers loop. That is, add any integer multiple of  $\tau$  to  $\theta$  and the results hold.

$$e^{i\theta} = e^{i\theta + n\tau}$$

$$n \in \mathbb{N}$$

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$e^{i\theta} = \cos(\theta + n\tau) + i \sin(\theta + n\tau)$$

$$e^{i\theta} = e^{i(\theta + n\tau)}$$

#### 0.1.5 Calculus of trig

Relationship between cos and sine

$$\sin(x + \frac{\pi}{2}) = \cos(x)$$

$$\cos(x + \frac{\pi}{2}) = -\sin(x)$$

$$\sin(x + \pi) = -\sin(x)$$

$$\cos(x + \pi) = -\cos(x)$$

$$\sin(x + \tau) = \sin(x)$$

$$\cos(x + \tau) = \cos(x)$$