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1 Set theory

1.1 Introduction to sets

1.1.1 Membership relation

Say we have a preterite $P(x)$ which is true for some values of x. Sets allow us to explore the properties of these values.

We may want to talk about a collection of terms for which $P(x)$ is true, which we call a set.

To do this we need to introduce new axioms, however first we can add (conservative) definitions to help us do this.

We introduce a new relation: membership. If element x is in set s then the following relation is true, otherwise it is false:

$x \in s$

Sets are also terms. In first-order logic they will be included in quantifiers. Indeed, in set theory, we aim to treat everything as sets.

If a term is not a member of another term, we can write this using the non-member relation as follows:

$\forall x \forall s[\neg(x \in s) \leftrightarrow x \notin s]$

1.1.2 Axiom of extensionality

If two sets contain the same elements, they are equal.

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]$

This is an axiom, not a definition, because equality was defined as part of first-order logic.

Note that this is not bidirectional. $x = y$ does not imply that x and y contain the same elements. This is appropraite as $\frac{1}{2} = \frac{2}{4}$ for example, even though they are written differently as sets.

1.1.2.1 Reflexivity of equality

The reflexive property is:

$\forall x(x = x)$

We can replace the instance of y with x:

$\forall x[\forall z(z \in x \leftrightarrow z \in x) \to x = x]$

We can show that the following is true:

$\forall z(z \in x \leftrightarrow z \in x)$

Therefore:

$\forall x[T \rightarrow x = x]$

$x = x$

1.1.2.2 Symmetry of equality

The symmetry property is:

$\forall x \forall y[(x = y) \leftrightarrow (y = x)]$

We know that the following are true:

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow y = x]$

So:

$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow (x = y \land y = x)]$

1.1.2.3 Transitivity of equality

The transitive property is:

$\forall x \forall y \forall z[(x = y \land y = z) \rightarrow x = z]$

1.1.2.4 Substitution for functions

The substitutive property for functions is:

$\forall x \forall y[(x = y) \rightarrow (f(x) = f(y))]$

1.1.2.5 Substitution for formulae

The substitutive property for formulae is:

$\forall x \forall y[((x = y) \land P(x)) \rightarrow P(y)]$

Doesn't this require iterating over predicates? Is this possible in first order logic??

1.1.2.6 Result 1: The empty set is unique

We can now show the empty set is unique.

1.1.2.7 Result 2: Every element of a set exists

If an element did not exist, the set containing it would be equal to a set which did not contain that element.

1.1.2.8 Result 3: Sets are unique

1.1.3 Equivalence classes

We have already ready defined the relationship equality, between terms.

$a = b$.

Sometimes we may wish to talk about a collection of terms which are all equal to each other. This is an equivalence class.

Though we have not yet defined it, integers are example of this. For example -1 can be written as $0 - 1$, $1 - 2$ and so on.

$\forall y forall x x = y \to x \in z$

For all sets, we can call the class of all sets equal to the set an equivalence class.

This does not necessarily exist.

1.2 Defining sets

1.2.1 Axiom schema of specification

1.2.1.1 The axiom schema of unrestricted comprehension

We want to formalise the relationship between the preterite and the set. An obvious way of doing this is to add an axiom for each preterite in our structure that:

$\forall x \exists s[P(x) \leftrightarrow (x \in s)]$

This is known as "unrestricted comprehension" and there are problems with this approach.

Consider a predicate for all terms which are not members of themselves. That is:

$\neg(x \in x)$

This implies the following is true:

$\forall x \exists s[\neg(x \in x) \leftrightarrow (x \in s)]$

As this is true for all x, it is true for $x = s$. So:

$\exists s[\neg(s \in s) \leftrightarrow (s \in s)]$

This statement is false. As we have inferred a false formula, the axiom of unrestricted comprehension does not work. This result is known as Russel's Paradox.

This is an axiom schema rather than an axiom. That is, there is a new axiom for each preterite.

1.2.1.2 Axiom schema of specification

To resolve Russels' paradox, we amend the axiom schema to:

$\forall x \forall a \exists s[(P(x) \land x \in a) \leftrightarrow (x \in s)]$

That is, for every set a, we can define a subset s for each predicate.

This resolves Russel's Paradox. Let's take the same steps on the above formula as in unrestricted comprehension;

$\forall x \forall a \exists s[(\neg(x \in x) \land x \in a) \leftrightarrow (x \in s)]$

$\exists s[(\neg(s \in s) \land s \in s) \leftrightarrow (s \in s)]$

So long as the subsets s are not members of themselves, this holds.

1.2.2 Implications of axiom schema of specification

1.2.2.1 All finite subsets exist

Finite subsets. Don't know about infinite subsets

If we can define a subset, by the axiom of specification it exists.

For example if set $\{a, b, c\}$ exists, we can define a preterite to select any subset of this.

For example we can use define a $P(x)$ as $x = a \lor x = b$ to extract the subset $\{a, b\}$.

If a subset is infinitely large,

1.2.2.2 Intersections of finite sets exist

Can prove exists from this axiom

1.2.2.3 If any set exists, the empty set exists

$\forall x \forall a \exists s[(P(x) \land x \in a) \leftrightarrow (x \in s)]$

1.2.3 Set-builder notation

1.2.3.1 Notation

NB. This should be later, the way they are described links to the axiom schema of specification.

We can use short-hand to describe sets.

$\{x \in S : P(x)\}$

This defines a set by a restriction. For example we will later be able to define natural numbers above 5 as:

$\{x \in \mathbb{N} : x > 5\}$

1.2.3.2 Class builder notation

Emuneration can be done through set builder notation too

Can define sets formally! defintion doesn't just affect sets

$\forall x(x \in C \leftrightarrow P(x))$

NB: We're not saying C exists

Can then use examples of equiv class

$\forall x(x \in C \leftrightarrow x = x)$

1.2.4 Empty set

We can use this to define the empty set - the set with no members.

$\varnothing = \{\}$

Using the above definition this is the same as writing:

$\forall x \neg(x \in \varnothing)$

1.2.5 Defining sets by enumeration

We can describe a set by the elements it contains.

$s = \{a, b, c\}$

This is a shorthand way of writing:

$\forall x(x \in s \leftrightarrow (x = a \vee x = b \vee x = c))$

1.2.6 Finite and infinite sets

1.2.6.1 Finite sets

A set is finite is there is a proper subset without a bijection.

Proper subset: $A \subset B$

For example for set $\{a, b, c\}$ There is no subset with a bijection.

1.2.6.2 Infinite sets

For the natural numbers, all natural numbers except 0 is a proper subset, and there is a bijection.

1.2.7 Cardinality

1.2.7.1 Cardinality of finite sets

The cardinality of a set s is shown as $|s|$. It is the number of elements in the set. We define it formally below.

1.2.7.2 Injectives, surjectives and bijectives

Consider 2 sets. If there is an injective from a to b then for every element in a there is a unique element in b.

If this injective exists then we say $|a| \leq |b|$.

Similarly, if there is a surjective, that is for every element in b there is a unique element in a, then $|a| \geq |b|$.

Therefore, if there is a bijection, $|a| = |b|$, and if there is only an injective or a surjective then $|a| < |b|$ or $|a| > |b|$ respectively.

1.2.7.3 Cardinality as a function

Every set has a cardinality. As a result cardinality cannot be a well-defined function, for the same reason there is no set of all sets.

Cardinality functions can be defined on subsets.

1.3 Ordering

1.3.1 Inequalities

1.3.1.1 Less than or equal

Orderings define relations between elements in a set, where one element can preceed the other.

Orderings are antisymmetric. That is, the only case where the relation is satisfied in both directions is if the elements are equal.

$(a \leq b) \wedge (b \leq a) \rightarrow (a = b)$

Orderings are transitive. That is:

$(a \leq b) \wedge (b \leq c) \rightarrow (a \leq c)$

1.3.1.2 Greater than or equal

1.3.1.3 Less than and greater than

The relation $\leq$ is refered to as non-strict.

There is a similar strict relation relation, $<$:

$(a \leq b) \wedge \neg(b \leq a) \rightarrow (a < b)$

1.3.2 Ordered sets

1.3.2.1 Totally ordered sets

A totally ordered set is one where the relation is defined on all pairs:

$\forall a \forall b (a \leq b) \vee (b \leq a)$

Note that totality implies reflexivity.

1.3.2.2 Partially ordered sets (poset)

A partially ordered set, or poset, is one where the relation is defined between each element and itself.

$\forall a (a \leq a)$

That is, every element is related to itself.

These are also called posets.

1.3.2.3 Well-ordering

A well-ordering on a set is a total order on the set where the set contains a minimum number. For example the relation $\leq$ on the natural numbers is a well-ordering because 0 is the minimum.

The relation $\leq$ on the integers however is not a well-ordering, as there is no minimum number in the set.

1.3.3 Intervals

For a totally ordered set we can define a subset as being all elements with a relationship to a number. For example:

$[a, b] = \{x : a \leq x \wedge x \leq b\}$

This denotes a closed interval. Using the definition above we can also define an open interval:

$(a, b) = \{x : a < x \wedge x < b\}$

1.3.4 Infinitum and supremum

1.3.4.1 Infinitum

Consider a subset S of a partially ordered set T.

The infinitum of S is the greatest element in T that is less than or equal to all elements in S.

For example:

$\inf[0, 1] = 0$

$\inf(0, 1) = 0$

1.3.4.2 Supremum

The supremum is the opposite: the smallest element in T which is greater than or equal to all elements in S.

$\sup[0, 1] = 1$

$\sup(0, 1) = 1$

1.3.4.3 Max and min

If the infinitum of a set S is in S, then the infinimum is the minimum of set S. Otherwise, the minimum is not defined.

$\min[0, 1] = 0$

$\min(0, 1)$ isn't defined.

Similarly:

$\max[0, 1] = 1$

$\max(0,1)$ isn't defined.

1.4 Set algebra

1.4.1 Set union and intersection

We discuss functions. Just because we can write a function of sets which exist, does not mean the results of the functions exist. For that we need axioms discussed later.

1.4.1.1 Union function

We define a function on two sets, $a \vee b$, such that the result contains all elements from either sets.

$\forall a \forall x \forall y[a \in (x \vee y) \leftrightarrow (a \in x \vee a \in y)]$

This is commutative: $a \vee b = b \vee a$

This is associative: $(a \vee b) \vee c = a \vee (b \vee c)$

1.4.1.2 Intersection function

We define a function, $a \wedge b$, on two sets, such that the result contains all elements which are in both.

$\forall a \forall x \forall y[a \in (x \wedge y) \leftrightarrow (a \in x \wedge a \in y)]$

This is commutative: $a \wedge b = b \wedge a$

This is associative: $(a \wedge b) \wedge c = a \vee (b \wedge c)$

1.4.1.3 Distribution of union and intersection

Union is distributive over intersection: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Intersection is distributive over union: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

1.4.2 Complements and disjoint sets

1.4.2.1 Disjoint sets

Sets are disjoint is there is no overlap in their elements. Two sets are s_i and s_j are mutually exclusive if:

$s_i \wedge s_j = \emptyset$

A collection of events s are all mutually exclusive if all pairs are mutually exclusive. That is:

$\forall s_i \in s \forall s_j \in s[s_i \wedge s_j \neq \emptyset \to s_i = s_j]$

1.4.2.2 Complement function

x^C is the completement. It is defined such that:

$\forall x[x \wedge x^C = \varnothing]$

For a set b, the complement with respect to a is all elements in a which are not in b.

$\forall x \in a \forall y \in b[x \in (a \setminus b) \wedge y \in (a \setminus b)]$

$b \wedge (a \setminus b) = \varnothing$

That is, b and $a \setminus b$ are disjoint.

1.4.2.3 Existence of the complement

For two sets a and b we can write $(a \setminus b)$. This is the set of elements of a which are not in b.

Consider the axiom of specification:

$\forall x \forall a \exists s[(P(x) \wedge x \in a) \leftrightarrow (x \in s)]$

We can also write

$\forall x \forall a \forall b \exists s[(x \notin b \wedge x \in a) \leftrightarrow (x \in s)]$

Which provides the complement, s.

1.4.3 Axiom of union

1.4.3.1 Motivation

While we have described various sets, we have not said that they exist. That is, if A and B both exist, then currently we cannot ensure $A \wedge B$ exists, just that it can be described.

The axiom of union enables us to ensure all sets from unions and intersections exist.

1.4.3.2 Axiom of union

$\forall a \exists b \forall c[c \in b \leftrightarrow \exists d(c \in d \wedge d \in a)]$

That is, for every set a, there exists a set b where all the elements in b are the elements of the elements in a.

Here, b is the union of the sets in a.

1.4.4 Boolean algebra

1.4.4.1 Boolean algebra in propositional logic

We previously discussed properties of normal form, and the results from these properties.

If another structure shares these properties then they will also share the results.

1.4.4.2 Sets satisfy the definitions of a boolean algebra

If a mathematical structure has the following properties, it shares the results from normal form, and is a boolean algebra.

- Both binary operators are commutitive - $A \wedge B = B \wedge A$ and $A \vee B = B \vee A$
- Both binary operators are associative - $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ and $(A \vee B) \vee C = A \vee (B \vee C)$
- Completements - $A \wedge \neg A = \emptyset$ and $A \vee \neg A = U$
- Absorption - $A \wedge (A \vee B) = A$ and $A \vee (A \wedge B) = A$
- Identity - $A \wedge U = A$ and $A \vee \emptyset = A$
- Distributivity - $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ and $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

These hold for sets, and so boolean algebra holds for sets.

1.4.5 Algebra on a set

1.4.5.1 Standard algebra

An algebra, Σ, on set s is a set of subsets of s such that:

- Closed under intersection: If a and b are in Σ then $a \wedge b$ must also be in Σ
- $\forall ab[(a \in \Sigma \wedge b \in \Sigma) \rightarrow (a \wedge b \in \Sigma)]$
- Closed under union: If a and b are in Σ then $a \vee b$ must also be in Σ.
- $\forall ab[(a \in \Sigma \wedge b \in \Sigma) \rightarrow (a \vee b \in \Sigma)]$

If both of these are true, then the following is also true:

- Closed under complement: If a is in Σ then $s \backslash a$ must also be in Σ

We also require that the null set (and therefore the original set, null's complement) is part of the algebra.

1.5 Natural numbers

1.5.1 Axiom of infinity

The axiom of infinity states that:

$\exists I(\varnothing \in I \wedge \forall x \in I((x \vee \{x\}) \in I))$

There exists a set, called the infinite set. This contains the empty set, and for all elements in I the set also contains the successor to it.

1.5.1.1 Sequential function

Let's define the sequential function:

$s(n) := \{n \vee \{n\}\}$

We can now rewrite the axiom of infinity as:

$\exists \mathbb{N}(\varnothing \in \mathbb{N} \wedge \forall x \in \mathbb{N}(s(x) \in \mathbb{N}))$

1.5.1.2 Zero

This set contains the null set: $\varnothing \in \mathbb{N}$.

Zero is defined as the empty set.

$0 := \{\}$

1.5.1.3 Natural numbers

For all elements in the infinite set, there also exists another element in the infinite set: $\forall x \in \mathbb{N}((x \vee \{x\}) \in \mathbb{N})$.

We then define all sequential numbers as the set of all preceding numbers. So:

$1 := \{0\} = \{\{\}\}$

$2 := \{0, 1\} = \{\{\}, \{\{\}\}\}$

$3 := \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$

1.5.1.4 Existence of natural numbers

Does each natural number exist? We know the infinite set exists, and we also know the axiom schema of specification:

Point is: For each set, all finite subsets exist. PROVE ELSEWHERE

1.5.1.5 From infinite set to natural set

We don't know I is limited to natural numbers. Could contain urelements etc.

1.5.1.6 More

Infinite set axiom written using N. should be I

I could be superset of N, for example set of all natural numbers, and also the set containing the set containing 2.

Can extract N using axiom of specification

We need a way to define the set of natural numbers:

$\forall n(n \in \mathbb{N} \leftrightarrow ([n = \emptyset \vee \exists k(n = k \vee \{k\})]\wedge))$

If we can can define N, we can show it exists from specicication

$\forall x \exists s[P(x) \leftrightarrow (x \in s)]$

$\forall n \exists s[n \in N \leftrightarrow (n \in s)]$

1.5.2 Cardinality of the natural numbers

Consider the infinite set, that is the set of all natural numbers which is defined in ZFC. Clearly there isn't a natural number cardinality of this – we instead write $\aleph_0$.

We call sets with this cardinality, countably infinite.

So:

$|\mathbb{N}| = \aleph_0$

1.5.2.1 Cardinality of natural numbers

We define:

$|\emptyset| = 0$

That, the empty set has a cardinality of 0.

As we define 0 as the empty set, $|0| = 0$.

What is 1? using the definition above we know $|1| > |0|$, so let's say $|1| = 1$, and more generally:

$\forall n \in \mathbb{N}|n| = n$

1.5.3 Ordering

1.5.3.1 Ordering of the natural numbers

For natural numbers we can say that number n preceeds number $s(n)$. That is:

$n \leq s(n)$

Similarly:

$s(n) \leq s(s(n))$

From the transitive property we know that:

$n \leq s(s(n))$

We can continue this to get:

$n \leq s(s(...s(n)..))$

What can we say about an arbitary comparison?

$a \leq b$

We know that either:

- $a = b$
- $b = s(s(...s(a)...))$
- $a = s(s(...s(b)...))$

In the first case the relation holds.

In the second case the relation holds.

In the third case the relation does not hold, but antisymmetry holds.

As this is is then defined on any pair, the order on natural numbers is total.

As there is a minimum, 0, the relation is also well-ordered.

However if this does not hold then the following instead holds:

1.6 Subsets and powersets

1.6.1 Subset relation

1.6.1.1 Subset

If all terms which are members of term x are also members of term y, then x is a subset of y.

$\forall x \forall y[(\forall z(z \in x \rightarrow z \in y)) \leftrightarrow (x \subseteq y)]$

1.6.1.2 Proper subset

If two sets are equal, then each is a subset of the other. A proper subset is one which is a subset, and not equal to the other set.

$\forall x \forall y[((\forall z(z \in x \to z \in y))) \wedge (x \neq y) \leftrightarrow (x \subset y)]$

1.6.2 Powerset function

The power set of s, $P(s)$, contains all subsets of s.

$\forall x x \subseteq s \leftrightarrow x \in P(x)$

Do all subsets exist?? show elsewhere.

1.6.3 Cantors theorem

The cardinality of the powerset is strictly greater than the cardinality of the underlying set.

That is, $|P(s)| < |s|$.

This applies to finite sets and infinite sets. In particular, this means that the powerset of the natural numbers is bigger than the natural numbers.

1.6.3.1 Proof

If one set is at least as big as another, then then is a surjection from that set to the other.

That is, if we can prove that there is no surjection from a set to its powerset, then we have proved the theorem.

We consider $f(s)$. If there is a surjection, then for every subset of s there should be a mapping from s to that subset.

We take set s and have the powerset of this, $P(s)$.

Consider the set:

$A = \{x \in s | x \notin f(x)\}$

That is, the set of all elements of s which do not map to the surjection.

1.7 Tuples

1.7.1 Tuples

We can get a list of sets in an order. A 2-tuple is an ordered pair:

(a, b)

We can write an ordered pair of a and b as:

$\{\{a\}, \{a, b\}\}$

Ordred pair definition, and tuple

$(a, b) = (c, d) \leftrightarrow (a = c \wedge b = d)$

This is the characteristic property.

1.7.2 Axiom of pairing

For any pair of sets, x and y there is another set z which containing only x and y.

$\forall x \forall y \exists z \forall a [a \in z \leftrightarrow a = x \vee a = y]$

1.7.2.1 For each set, there exists a set containing only that set

Take the axiom, but replace all instance of y with x.

$\forall x \exists z \forall a [a \in z \leftrightarrow a = x \vee a = x]$

$\forall x \exists z \forall a [a \in z \leftrightarrow a = x]$

1.7.2.2 For any finite number of sets, there is a set containing only those sets

1.7.2.3 For any finite number of sets, there is a set containing the intersection of those sets

1.7.3 Cartesian product

The cartesian product takes two sets, and creates a set containing all ordered pairs of a and b.

$a \times b$

1.7.4 Direct sums

1.8 Functions

1.8.1 Constructing functions

1.8.1.1 Use of ordered pairs

We can define this as a set of ordered pairs.

$\{\{a\},\{a,b\}\}$

1.8.2 Domains and ranges

1.8.2.1 Domain

All values on which the function can be called

$\forall x(f(x)=y) \to P(y))$

1.8.2.2 Image

$\forall x((\exists y f(x)=y) \to P(y))$

Outputs of a function.

AKA: Range

The image of x is $f(x)$.

1.8.2.3 Preimage

The preimage of y is all x where $f(x)=y$.

1.8.2.4 Codomain

Sometimes the image is a subset of another set. For example a function may map onto natural numbers above 0. Natural numbers above 0 would be the image, and the natural numbers would be the codomain.

1.8.2.5 Example

$f(n)=s(n)$

Domain is: $\mathbb{N}$

Codomain is also: $\mathbb{N}$

Image is $\mathbb{N} \wedge n \neq 0$

1.8.2.6 Describing functions

If function f maps from set X to set Y we can write this as:

$f : X \to Y$

1.8.3 Axiom of regularity

The axiom of regularity states that:

$\forall x[x \neq \varnothing \rightarrow \exists y \in x(y \wedge x) = \varnothing]$

That is, for all non-empty sets, there is an element of the set which is disjoint from the set itself.

This means that no set can be a member of itself.