0.1 Fibonacci sequence

0.1.1 The generating function

Let's use a generating function to create a function for the Fibonacci sequence's cth digit. $F(c) = \sum_{i=c} x^i s_i$

Let's look at it for other starts:

$F(c+k) = \sum_{i=c} x^{i+k} s_{i+k}$

$F(c+k) = \sum_{i=c+k} x^i s_i$

$F(c+1) = \sum_{i=c} x^{i+1} s_{i+1}$

$F(c+2) = \sum_{i=c} x^{i+2} s_{i+2}$

This means

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^i s_i x^2 + \sum_{i=c} x^{i+1} s_{i+1} x$

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} s_i + \sum_{i=c} x^{i+2} s_{i+1}$

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_i + s_{i+1})$

0.1.2 Using the definiton of the Fibonacci sequence

From the definition of the fibonacci sequence, $s_i + s_{i+1} = s_{i+2}$.

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_{i+2})$

$F(c)x^2 + F(c+1)x = F(c+2)$

0.1.3 Reducing the functions

Next, we expand out $F(c+1)$ and $F(c+2)$.

$F(c) - F(c+k) = \sum_{i=c} x^i s_i - \sum_{i=c+k} x^i s_i$

$F(c) - F(c+k) = \sum_{i=c}^{c+k} x^i s_i$

$F(c+k) = F(c) - \sum_{i=c}^{c+k} x^i s_i$

So:

$F(c+1) = F(c) - \sum_{i=c}^{c+1} x^i s_i$

$F(c+1) = F(c) - x^c s_c$

$F(c+2) = F(c) - \sum_{i=c}^{c+2} x^i s_i$

$F(c+2) = F(c) - x^{c+1} s_{c+1} - x^c s_c$

Let's take our previous equation

$F(c)x^2 + F(c+1)x = F(c+2)$

$F(c)x^2 + [F(c) - x^c s_c]x = F(c) - x^{c+1}s_{c+1} - x^c s_c$

$F(c)x^2 + F(c)x - x^{c+1}s_c = F(c) - x^{c+1}s_{c+1} - x^c s_c$

$F(c)[x^2 + x - 1] = x^{c+1}s_c - x^{c+1}s_{c+1} - x^c s_c$

$F(c) = \frac{x^c s_c + x^{c+1}s_{c+1} - x^{c+1}s_c}{1-x-x^2}$

0.1.4 Using the first element in the sequence

For the start of the sequence, $c = 0$, $s_0 = s_1 = 1$.

$F(0) = \frac{x^0 1 + x - x}{1-x-x^2}$

$F(0) = \frac{1}{1-x-x^2}$

Let's factorise this:

$F(0) = \frac{-1}{(x+\frac{1}{2}+\frac{\sqrt{5}}{2})(x+\frac{1}{2}-\frac{\sqrt{5}}{2})}$

We can then use partial fraction decomposition

$\frac{1}{(M+\delta)(M-\delta)} = \frac{1}{2\delta}[\frac{1}{M-\delta} - \frac{1}{M+\delta}]$

To show that

$F(0) = \frac{-1}{\sqrt{5}}[\frac{1}{x+\frac{1}{2}-\frac{\sqrt{5}}{2}} - \frac{1}{x+\frac{1}{2}+\frac{\sqrt{5}}{2}}]$

$F(0) = \frac{-1}{\sqrt{5}}[\frac{\frac{1}{2}+\frac{\sqrt{5}}{2}}{(x+\frac{1}{2}-\frac{\sqrt{5}}{2})(\frac{1}{2}+\frac{\sqrt{5}}{2})} - \frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{(x+\frac{1}{2}+\frac{\sqrt{5}}{2})(\frac{1}{2}-\frac{\sqrt{5}}{2})}]$

$F(0) = \frac{-1}{\sqrt{5}}[\frac{\frac{1}{2}+\frac{\sqrt{5}}{2}}{x(\frac{1}{2}+\frac{\sqrt{5}}{2})-1} - \frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{x(\frac{1}{2}-\frac{\sqrt{5}}{2})-1}]$

$F(0) = \frac{1}{\sqrt{5}}[(\frac{1}{2} + \frac{\sqrt{5}}{2})\frac{1}{1-x(\frac{1}{2}+\frac{\sqrt{5}}{2})} - (\frac{1}{2} - \frac{\sqrt{5}}{2})\frac{1}{1-x(\frac{1}{2}-\frac{\sqrt{5}}{2})}]$

0.1.5 Finishing off

As we know

$\frac{1}{1-x} = \sum_{i=0} x^i$

So

$F(0) = \frac{1}{\sqrt{5}}[(\frac{1}{2} + \frac{\sqrt{5}}{2})\sum_{i=0} x^i(\frac{1}{2} + \frac{\sqrt{5}}{2})^i - (\frac{1}{2} - \frac{\sqrt{5}}{2})\sum_{i=0} x^i(\frac{1}{2} - \frac{\sqrt{5}}{2})^i]$

$F(0) = \frac{1}{\sqrt{5}}[\sum_{i=0} x^i(\frac{1}{2} + \frac{\sqrt{5}}{2})^{i+1} - \sum_{i=0} x^i(\frac{1}{2} - \frac{\sqrt{5}}{2})^{i+1}]$

$F(0) = \frac{1}{\sqrt{5}} \sum_{i=0} x^i [(\frac{1}{2} + \frac{\sqrt{5}}{2})^{i+1} - (\frac{1}{2} - \frac{\sqrt{5}}{2})^{i+1}]$

So the nth number in the sequence (treating $n = 1$ as the first number) is:

$\frac{1}{\sqrt{5}}[(\frac{1}{2} + \frac{\sqrt{5}}{2})^n - (\frac{1}{2} - \frac{\sqrt{5}}{2})^n]$