Contents

1 Topology of finite sets

1.1 Nearness functions

1.1.1 Topologies

1.1.2 Topologies on sets

T is a topology on set X if:

- $X \in T$
- $\varnothing \in T$
- Unions of T are in T
- Intersections of T are in T

1.1.3 Examples of topologies: The trivial topology

The trivial topology contains only the underlying set and the empty set.

1.1.4 Examples of topologies: The discrete toplogy

The discrete toplogy contains all subsets of the underlying set (is this the power set?)

1.2 Neighbourhoods

1.2.1 Neighbourhood topology

We have a set X.

For each element $x \in X$, there is a non-empty set of neighbourhoods $N \in \mathbf{N}(x)$ where $x \in N \subseteq X$ such that:

- If N is a subset of M, M is a neighbourhood.
- The intersection of two neighbourhoods of x is a neighbourhood of x.
- N is a neighbourhood for each point in some $M \subseteq N$

1.2.2 Topological distinguishability

If two points have the same neighbourhoods then they are topologically indistinguishable.

For example in the trivial topology, all points are topologically indistinguishable.

1.2.3 Open sets

U is an open set if it is a neighbourhood for all its points.

1.3 Open and closed sets

1.3.1 Limit points and closure

1.3.1.1 Limit points

A point x in the topological set X is a limit point for $S \subset X$ if every neighbourhood of x contains another point in S.

For example -1 is a limit point for the real numbers where S is $[0, 1]$ (or $(0, 1)$.

1.3.1.2 Closure

The closure of a subset of a topological space is the subset itself along with all limit points.

So the closure of $|x| < 1$ includes -1 and 1.

1.3.2 Boundries and interiors

The boundry of the subset S of a topology is the intersection with the closure of S with the closure of the complement of S.

So the boundry of both $(0, 1)$ and $[0, 1]$ are 0 and 1.

The interior of S is S without the boundry.

So the interior of $(0, 1)$ and $[0, 1]$ are both $(0, 1)$.

1.3.3 Closed sets

The complement of any open set is a closed set.

A set can be open, closed, both or neither.

1.4 Compactness

1.4.1 Covers

A space X is covered by a set of subsets of X, C, if the union of C is X.

1.4.2 Subcover

A subset of C which still covers X is a a subcover.

1.4.3 Open cover

C is an open cover if each member is an open set.

1.4.4 Bases of topologies

Subset B of topology T is a base for T if all elements of T are unions of members of B.

1.4.4.1 Second-countable space

If B is finite then the toplogy is a second-countable space.

1.5 Separation

1.5.1 Connected and separated sets

Two subsets of X in topological space T are separated if each subset is disjoint from the other's closure.

So $[-1, 0)$ and $(0, 1)$ are separated.

$[-1, 0]$ and $(0, 1)$ are not separated.

Sets which are not separated are connected.

1.6 Cartesian products

1.6.1 Box topology

1.6.2 Product topology

1.7 Creating topologies from sets

1.7.1 The trivial topology

A topology which contains just X and $\varnothing$ is the trivial topology.

1.7.2 Discrete topology

1.8 Taxonomy of spaces

1.8.1 Lindelöf space

In a Lindelöf space all open covers have countable subcovers.

This is weaker than compactness, which requires that every open cover has a finite subcover.

1.8.2 Kolmogorov space

In a Kolmogorov (or T_0) space, for every pair of points there is a neighbourhood containing one but not the other.

1.9 Local properties

1.9.1 Local properties

Locally, a topology may have properties which are not present globally.

1.9.2 Locally compact spaces

1.9.3 Locally connected spaces

1.10 TO INF

1.10.1 Hausdorf space

In a Hausdorf (or T_2) space, any two different points have neighbourhoods which are disjoint.

1.10.2 Compact spaces

A space X is compact if each open cover has a finite subcover.

If we can define a cover which does not have a finite subcover, then the space is not compact.

For example an infinite cover could be tend towards $(0, 1)$, eg as $\frac{1}{n}, 1 - \frac{1}{n}$

This covers $(0, 1)$, but there is no finite subcover. As a result $(0, 1)$ is not compact.