0.1 Bilinear forms

A bilinear form takes two vectors and produces a scalar from the underyling field.

This is in contrast to a linear form, which only has one input.

In addition, the function is linear in both arguments.

$\phi(au + x, bv + y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

$\phi(au + x, bv + y) = ab\phi(u, v) + a\phi(u, y) + b\phi(x, v) + \phi(x, y)$

0.1.1 Representing bilinear forms

They can be represented as:

$\phi(u, v) = v^T M u$

$f(M) = f([v_1, v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f([\sum_{i=1}^m a_{1i} e_i, \sum_{i=1}^m a_{2i} e_i])$

$f(M) = \sum_{k=1}^m f([a_{1k} e_k, \sum_{i=1}^m a_{2i} e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m f([a_{1k} e_k, a_{2i} e_i])$

Because this in linear in scalars:

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k} a_{2i} f([e_k, e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k} a_{2i} e_k^T M e_i$

0.1.2 Orthonormal basis and $M = I$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k} a_{2i} e_k^T e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k} a_{2i} \delta_i^k$

$f(M) = \sum_{i=1}^m a_{1i} a_{2i}$