Contents

0.1 Determinants

From invertible matrix section in endo

A matrix can only be inverted if it can be created from a combination of elementary row operations.

How can we identify if a matrix is invertible? We want to create a scalar from the matrix which tells us if this possible. We can this scalar the determinant.

For a matrix A we label the determinant $|A|$, or $\det A$

We propose $|A| = 0$ when the matrix is not invertible.

So how can we identify the function we need to undertake on the matrix?

0.1.1 New 1

We know that linear dependence results in determinants of 0.

We can model this as a function on the columns of the matrix.

$\det M = \det([M_1, ..., M_n)$

If there is linear depednence, for example if two columns are the same then:

$\det([M_1, ..., M_i, ..., M_i, ..., M_n]) = 0$

Similarly, if there is a column of 0 then the determinant is 0.

$\det([M_1, ..., 0, ..., M_n]) = 0$

0.1.2 New 2

Show linear in addition

How can we identify the determinant of less simple matrices? We can use the multilinear form.

$\sum c_i \mathbf{M}_i = \mathbf{0}$

Where $\mathbf{c} \neq \mathbf{0}$

Or:

$M\mathbf{c} = \mathbf{0}$

0.1.3 Rule 1: Columns of matrices can be the input to a multilinear form

A matrix can be shown in terms of its columns. $A = [v_1, ..., v_n]$

$\det A = \det[v_1, ..., v_n]$

$\det A = \sum_{k_1=1}^{m} ... \sum_{k_n=1} \prod_{i=1}^{m} a_{ik_i} \det([e_{k_1}, ..., e_{k_n}])$

0.1.4 Multiplying a matrix by a constant multiplies the determinant by the same amount

If a whole row or columns is 0 then:

$\det A = \det[v_1, ..., v_i, ..., v_n]$

$\det A' = \det[v_1, ..., cv_i, ..., v_n]$

$\det A = \det[v_1, ..., v_i, ..., v_n]$

$\det A' = \det[v_1, ..., cv_i, ..., v_n]$

$\det A' = c \det[v_1, ..., v_i, ..., v_n]$

$\det A' = c \det A$

As a result, multiplying a column by 0 makes the determinant 0.

A matrix with a column of 0 therefore has determinant 0

0.1.5 Rule 2: A matrix with equal columns has a determinant of 0.

$A = [a_1, ..., a_i, ..., a_i, ..., a_n]$

$D(A) = D([a_1, ..., a_i, ..., a_i, ..., a_n])$

We know from Result 3 that swapping columns reverses the sign. Reversing columns results in the same matrix, so the determinant must be unchanged.

$D(A) = -D(A)$

$D(A) = 0$

0.1.6 Linear dependence

If a column is a linear combination of other columns, then the matrix cannot be inverted.

$A = [a_1, ..., \sum_{j \neq i}^{n} c_j a_j, ..., a_n]$

$\det A = \det([v_1, ..., \sum_{j \neq i}^{n} c_j v_j, ..., v_n])$

$\det A = \sum_{j \neq i}^{n} c_j \det([v_1, ..., v_j, ..., v_n])$

$\det A = \sum_{j \neq i}^{n} c_j \det([v_1, ..., v_j, , ..., v_j, ..., v_n])$

As there is a repeating vector:

$\det A = 0$

0.1.7 Swapping columns multiplies the determinant by -1

$A = [v_1, ..., v_i + v_j, ..., v_i + v_j, ..., v_n]$

We know.

$\det A = 0$

$\det A = \det([a_1, ..., a_i, ..., a_i, ..., a_n]) + \det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) + \det([a_1, ..., a_j, ..., a_j, ..., a_n])$

So:

$\det([a_1, ..., a_i, ..., a_i, ..., a_n]) + \det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) + \det([a_1, ..., a_j, ..., a_j, ..., a_n]) = 0$

As 2 of these have equal columns these are equal to 0.

$\det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) = 0$

$\det([a_1, ..., a_i, ..., a_j, ..., a_n]) = -\det([a_1, ..., a_j, ..., a_i, ..., a_n])$

0.1.8 Calculating the determinant

We have

$\det A = \sum_{k_1=1}^{m} \cdots \sum_{k_n=1} \prod_{i=1}^{m} a_{ik_i} \det([e_{k_1}, ..., e_{k_n}])$

So what is the value of the determinant here?

We know that the determinant of the identity matrix is 1.

We know that the determinant of a matrix with identical columns is 0.

We know that swapping columns multiplies the determinant by -1.

Therefore the determinants where the values of k are not all unique are 0.

The determinants of the others are either -1 or 1 depending on how many swaps are required to restore to the identity matrix.

This is also shown as the Leibni formula.

$\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$