Contents

1 Univariate calculus

1.1 Partial differentiation

1.1.1 The partial differential operator

1.1.1.1 Differential

When we change the value of an input to a function, we also change the output. We can examine these changes.

Consider the value of a function $f(x)$ at points x_1 and x_2.

$y_1 = f(x_1)$

$y_2 = f(x_2)$

$y_2 - y_1 = f(x_2) - f(x_1)$

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Let's define x_2 in terms of its distance from x_1:

$x_2 = x_1 + \epsilon$

$\frac{y_2 - y_1}{\epsilon} = \frac{f(x_1 + \epsilon) - f(x_1)}{\epsilon}$

We define the differential of a function as:

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) - f(x)}{\epsilon}$

If this is defined, then we say the function is differentiable at that point.

1.1.1.2 Differential operator

1.1.2 Differentiating constants, the identity function, and linear functions

1.1.2.1 Constants

$f(x) = c$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) - f(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{c - c}{\epsilon} = 0$

1.1.2.2 x

$f(x) = x$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) - f(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{x + \epsilon - x}{\epsilon} = 1$

1.1.2.3 Addition

$f(x) = g(x) + h(g)$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon) + h(x+\epsilon) - g(x) - h(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon) - g(x)}{\epsilon} + \lim_{\epsilon \to 0^+} \frac{h(x+\epsilon) - h(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \frac{\delta g}{\delta x} + \frac{\delta h}{\delta x}$

1.1.3 Partial differentiation is a linear operator

1.1.3.1 Intro

1.1.4 The chain rule, the product rule and the quotient rule

1.1.4.1 Chain rule

$f(x) = f(g(x))$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(g(x+\epsilon)) - f(g(x))}{\epsilon}$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon)-g(x)}{g(x+\epsilon)-g(x)} \frac{f(g(x+\epsilon)) - f(g(x))}{\epsilon}$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon)-g(x)}{\epsilon} \frac{f(g(x+\epsilon)) - f(g(x))}{g(x+\epsilon)-g(x)}$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} [\frac{g(x+\epsilon)-g(x)}{\epsilon}] \lim_{\epsilon \to 0^+} [\frac{f(g(x+\epsilon)) - f(g(x))}{g(x+\epsilon)-g(x)}]$

$\frac{\delta f}{\delta x} = \frac{\delta g}{\delta x} \frac{\delta f}{\delta g}$

1.1.4.2 Product rule

$y = f(x)g(x)$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)g(x+\epsilon) - f(x)g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)g(x+\epsilon) - f(x)g(x+\epsilon) + f(x)g(x+\epsilon) - f(x)g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)g(x+\epsilon) - f(x)g(x+\epsilon)}{\epsilon} + \lim_{\epsilon \to 0^+} \frac{f(x)g(x+\epsilon) - f(x)g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} g(x+\epsilon) \frac{f(x+\epsilon) - f(x)}{\epsilon} + \lim_{\epsilon \to 0^+} f(x) \frac{g(x+\epsilon) - g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = g(x) \frac{\delta f}{\delta x} + f(x) \frac{\delta g}{\delta x}$

1.1.4.3 Quotient rule

$y = \frac{f(x)}{g(x)}$

$\frac{\delta}{\delta x} y = \frac{\delta}{\delta x} \frac{f(x)}{g(x)}$

$\frac{\delta}{\delta x} y = \frac{\delta}{\delta x} f(x) \frac{1}{g(x)}$

$\frac{\delta}{\delta x} y = \frac{\delta f}{\delta x} \frac{1}{g(x)} - \frac{\delta g}{\delta x} \frac{f(x)}{g(x)^2}$

$\frac{\delta}{\delta x} y = \frac{\frac{\delta f}{\delta x} g(x) - \frac{\delta g}{\delta x} f(x)}{g(x)^2}$

1.1.5 Differentiating natural number power functions

1.1.5.1 Other

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0} \frac{(x+\delta)^n - x^n}{\delta}$

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0} \frac{(\sum_{i=0}^n x^i \delta^{n-i} \frac{n!}{i!(n-i)!}) - x^n}{\delta}$

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0} \sum_{i=0}^{n-1} x^i \delta^{n-i-1} \frac{n!}{i!(n-i)!}$

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0} x^{n-1} \frac{n!}{(n-1)!(n-n+1)!} + \sum_{i=0}^{n-2} x^i \delta^{n-i-1} \frac{n!}{i!(n-i)!}$

$\frac{\delta}{\delta x}x^n = nx^{n-1}$

1.1.6 L'Hôpital's rule

1.1.6.1 L'Hôpital's rule

If there are two functions which are both tend to 0 at a limit, calculating the limit of their divisor is hard. We can use L'Hopital's rule.

We want to calculate:

$\lim_{x\to c} \frac{f(x)}{g(x)}$

This is:

$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{f(x)-0}{\delta}}{\frac{g(x)-0}{\delta}}$

If:

$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

Then

$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{f(x)-f(c)}{\delta}}{\frac{g(x)-f(c)}{\delta}}$

$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$

1.1.7 Rolle's theorem

1.1.7.1 Rolle's theorem

Take a real function $f(x)$ on closed interval $[a, b]$, differentiable on $(a, b,)$, and $f(a) = f(b)$.

Rolle's theorem states that:

$\exists c \in (a, b)(f'(c) = 0)$

Generalised Rolle's theorem states that:

Generalised Rolle's theorem implies Rolle's theorem, so we only need to prove the generalised theorem.

1.1.8 Mean value theorem

1.1.8.1 Mean value theorem

Take a real function $f(x)$ on closed interval $[a, b]$, differentiable on $(a, b,)$.

The mean value theorem states that:

$\exists c \in (a, b)(f'(c) = \frac{f(b)-f(a)}{b-a})$

1.1.9 Elasticity

1.1.9.1 Introduction

We have (x)

$Ef(x) = \frac{x}{f(x)} \frac{\delta f(x)}{\delta x}$

This is the same as:

$Ef(x) = \frac{\delta \ln f(x)}{\delta \ln x}$

1.1.10 Smooth functions

1.1.11 Analytic function

1.1.11.1 Introduction

1.2 Higher-order differentials

1.2.1 Differentiable functions

1.2.1.1 Introduction

A differentiable function is one where the differential is defined at all points on the real line.

All differentiable functions are continuous. Not all continuous functions are differentiable.

1.2.1.2 Differentiability class

We can describe a function with its differentiability class. If a function can be differentiated n times and these differentials are all continous, then the function is class C^n.

1.2.1.3 Smooth functions

If a function can be differentiated infinitely many times to produce continous functions, it is C^∞, or smooth.

1.2.2 Critial points

1.2.2.1 Critical points

Where partial derivative are 0.

1.3 Exponentials

1.3.1 Defining e as a binomial

1.3.1.1 Lemma

$f(n,i) = \frac{n!}{n^i(n-i)!}$

$f(n,i) = \frac{(n-i)!\prod_{j=n-i+1}^{n} j}{n^i(n-i)!}$

$f(n,i) = \frac{\prod_{j=n-i+1}^{n} j}{n^i}$

$f(n,i) = \frac{\prod_{j=1}^{i}(j+n-i)}{n^i}$

$f(n,i) = \prod_{j=1}^{i} \frac{j+n-i}{n}$

$f(n,i) = \prod_{j=1}^{i}(\frac{n}{n} + \frac{j-i}{n})$

$f(n,i) = \prod_{j=1}^{i}(1 + \frac{j-i}{n})$

$\lim_{n\to\infty} f(n,i) = \lim_{n\to\infty} \prod_{j=1}^{i}(1 + \frac{j-i}{n})$

$\lim_{n\to\infty} f(n,i) = \prod_{j=1}^{i} 1$

$\lim_{n\to\infty} f(n,i) = 1$

1.3.1.2 Defining e

We know that:

$(a+b)^n = \sum_{i=0}^{n} a^i b^{n-i} \frac{n!}{i!(n-i)!}$

Let's set $b = 1$

$(a+1)^n = \sum_{i=0}^{n} a^i \frac{n!}{i!(n-i)!}$

Let's set $a = \frac{1}{n}$

$(1+\frac{1}{n})^n = \sum_{i=0}^{n} \frac{1}{n^i} \frac{n!}{i!(n-i)!}$

$(1+\frac{1}{n})^n = \sum_{i=0}^{n} \frac{1}{i!} \frac{n!}{n^i(n-i)!}$

$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} \frac{n!}{n^i(n-i)!}$

From the lemma above:

$\lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{i=0}^{\infty} \frac{1}{i!}$

$e = \sum_{i=0}^{\infty} \frac{1}{i!}$

1.3.1.3 Defining e^x

$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$

$e^x = \lim_{n\to\infty}(1+\frac{1}{n})^{nx}$

$e^x = \lim_{n\to\infty} \sum_{i=0}^{nx} \frac{1}{n^i} \frac{(nx)!}{i!(nx-i)!}$

$e^x = \lim_{n\to\infty} \sum_{i=0}^{nx} \frac{x^i}{i!} \frac{(nx)!}{(nx)^i(nx-i)!}$

From the lemma:

$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

1.3.2 Differentiating e^x

1.3.2.1 Intro

We have $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \frac{\delta}{\delta x} \sum_{i=0}^{\infty} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=0}^{\infty} \frac{\delta}{\delta x} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=1}^{\infty} \frac{\delta}{\delta x} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=1}^{\infty} \frac{x^{i-1}}{(i-1)!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

$\frac{\delta}{\delta x}e^x = e^x$

1.3.3 Differentiating exponents, logarithms and power functions

1.3.3.1 Differentiating the natural logarithm

$\frac{\delta}{\delta x}\ln(x) = \lim_{\delta \to 0} \frac{\ln(x+\delta) - ln(x)}{\delta}$

$\frac{\delta}{\delta x}\ln(x) = \lim_{\delta \to 0} \frac{\ln \frac{x+\delta}{x}}{\delta}$

$\frac{\delta}{\delta x}\ln(x) = \lim_{\delta \to 0} \frac{\ln(1+\frac{\delta}{x})}{\delta}$

$\frac{\delta}{\delta x}\ln(x) = \frac{1}{x}\lim_{\delta \to 0} \frac{x}{\delta}\ln(1+\frac{\delta}{x})$

$\frac{\delta}{\delta x}\ln(x) = \frac{1}{x}\ln(\lim_{\delta \to 0}(1+\frac{\delta}{x})^{\frac{x}{\delta}})$

$\frac{\delta}{\delta x}\ln(x) = \frac{1}{x}\ln(e)$

$\frac{\delta}{\delta x}\ln(x) = \frac{1}{x}$

1.3.3.2 Differentiating logarithms of other bases

$log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

$log_a(x) = \frac{\ln(x)}{\ln(a)}$

$\frac{\delta}{\delta x}\log_a(x) = \frac{\delta}{\delta x}\frac{\ln(x)}{ln(a)}$

$\frac{\delta}{\delta x}\log_a(x) = \frac{1}{xln(a)}$

1.3.3.3 Exponents

$y = a^x$

$\ln(y) = x\ln(a)$

$\frac{\delta}{\delta x}\ln(y) = \frac{\delta}{\delta x}x\ln(a)$

$\frac{\delta}{\delta x}\ln(y) = \ln(a)$

$\frac{1}{y}\frac{\delta}{\delta x}y = \ln(a)$

$\frac{\delta}{\delta x}a^x = a^x\ln(a)$

1.3.3.4 Power functions

$y = x^n$

$\frac{\delta}{\delta x} y = \frac{\delta}{\delta x} x^n$

$\frac{\delta}{\delta x} y = \frac{\delta}{\delta x} e^{n\ln(x)}$

$\frac{\delta}{\delta x} y = \frac{n}{x} e^{n\ln(x)}$

$\frac{\delta}{\delta x} y = nx^{n-1}$

1.4 Integration

1.4.1 Riemann integral

1.4.1.1 Riemann sums

Given a function $f(x)$ and an interval $[a,b]$, we can divide $[a,b]$ into n sections and calculate:

$\sum_{j=0}^{n(b-a)} f(a+\frac{j}{n})$

This is the Riemann sum.

1.4.1.2 Riemann integral

We take the limit of the Riemann sum as $n \to \infty$

$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a+\frac{j}{n})$

1.4.1.3 Linearity

$\int_a^b f(x) + g(x)dx = \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a+\frac{j}{n}) + g(a+\frac{j}{n})$

$\int_a^b f(x) + g(x)dx = \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a+\frac{j}{n}) + \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} g(a+\frac{j}{n})$

$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

1.4.1.4 Continuation

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a+\frac{j}{n}) + \lim_{n\to\infty} \sum_{j=0}^{n(c-b)} f(b+\frac{j}{n})$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty} [\sum_{j=0}^{n(b-a)} f(a+\frac{j}{n}) + \sum_{j=0}^{n(c-b)} f(b+\frac{j}{n})]$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty} [\sum_{j=0}^{n(b-a)} f(a+\frac{j}{n}) + \sum_{j=n(b-a)}^{n(c-b)+n(b-a)} f(b+\frac{j-n(b-a)}{n})]$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty} [\sum_{j=0}^{n(b-a)} f(a+\frac{j}{n}) + \sum_{j=n(b-a)}^{n(c-a)} f(a+\frac{j}{n})]$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty}[\sum_{j=0}^{n(c-a)} f(a+\frac{j}{n})$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

1.4.2 Definite and indefinite integrals

1.4.2.1 Introduction

1.4.3 Anti-derivative

1.4.3.1 Content

1.4.4 Integration by parts

1.4.4.1 Integration by parts

We have:

$\frac{\delta y}{\delta x} = f(x)g(x)$

We want that in terms of y.

We know from the product rule of differentiation:

$y = a(x)b(x)$

Means that:

$\frac{\delta y}{\delta x} = a'(x)b(x) + a(x)b'(x)$

So let's relabel $f(x)$ as $h'(x)$

δ

$\frac{\delta y}{\delta x} = h'(x)g(x)$

$\frac{\delta y}{\delta x} + h(x)g'(x) = h'(x)g(x) + h(x)g'(x)$

$y + \int h(x)g'(x) = \int h'(x)g(x) + h(x)g'(x)$

$y + \int h(x)g'(x) = h(x)g(x)$

$y = h(x)g(x) - \int h(x)g'(x)$

For example:

$\frac{\delta y}{\delta x} = x.\cos(x)$

$f(x) = cos(x)$

$g(x) = x$

$h(x) = sin(x)$

$g'(x) = 1$

So:

$y = x \int cos(x)dx - \int sin(x)dx$

$y = xsin(x) - cos(x) + c$

1.4.5 Integrals

1.4.5.1 Methods of integration

Trigonometric substitution

Inverse function integration function integration

Anti-derivative

Taking the derivative of a function provides another function. The anti-derivative of a function is a function which, when differentiated, provides the original function.

As this function can include any additive constant, there are an infinite number of anti-derivatives for any function.

Integration

Limit of summation

Show same as anti-derivative

Show properties of function same as summation (can take constants out etc)

1.4.5.2 Getting functions from derivatives

$f(c) = f(a) + \int_a^c \frac{\delta}{\delta x} f(x)dx$

Definite integration

Indefinite integration

1.4.6 Fundamental Theorem of Calculus

1.4.6.1 Mean value theorem for integration

Take function $f(x)$. From the extreme value theorem we know that:

$\exists m \in \mathbb{R} \exists M \in \mathbb{R} \forall x \in [a, b](m < f(x) < M)$

1.4.6.2 Fundamental theorem of calculus

From continuation we know that:

$\int_a^{x_1} f(x)dx + \int_{x_1}^{x_1+\delta x} f(x)dx = \int_a^{x_1+\delta x} f(x)dx$

$\int_x^{x_1+\delta x} f(x)dx = \int_a^{x_1+\delta} f(x)dx - \int_a^{x_1} f(x)dx$

Indefinite integrals