0.1 Defining e as a binomial

0.1.1 Lemma

$f(n,i) = \frac{n!}{n^i(n-i)!}$

$f(n,i) = \frac{(n-i)!\prod_{j=n-i+1}^{n} j}{n^i(n-i)!}$

$f(n,i) = \frac{\prod_{j=n-i+1}^{n} j}{n^i}$

$f(n,i) = \frac{\prod_{j=1}^{i}(j+n-i)}{n^i}$

$f(n,i) = \prod_{j=1}^{i} \frac{j+n-i}{n}$

$f(n,i) = \prod_{j=1}^{i}(\frac{n}{n} + \frac{j-i}{n})$

$f(n,i) = \prod_{j=1}^{i}(1 + \frac{j-i}{n})$

$\lim_{n\to\infty} f(n,i) = \lim_{n\to\infty} \prod_{j=1}^{i}(1 + \frac{j-i}{n})$

$\lim_{n\to\infty} f(n,i) = \prod_{j=1}^{i} 1$

$\lim_{n\to\infty} f(n,i) = 1$

0.1.2 Defining e

We know that:

$(a+b)^n = \sum_{i=0}^{n} a^i b^{n-i} \frac{n!}{i!(n-i)!}$

Let's set $b = 1$

$(a+1)^n = \sum_{i=0}^{n} a^i \frac{n!}{i!(n-i)!}$

Let's set $a = \frac{1}{n}$

$(1+\frac{1}{n})^n = \sum_{i=0}^{n} \frac{1}{n^i} \frac{n!}{i!(n-i)!}$

$(1+\frac{1}{n})^n = \sum_{i=0}^{n} \frac{1}{i!} \frac{n!}{n^i(n-i)!}$

$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} \frac{n!}{n^i(n-i)!}$

From the lemma above:

$\lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{i=0}^{\infty} \frac{1}{i!}$

$e = \sum_{i=0}^{\infty} \frac{1}{i!}$

0.1.3 Defining e^x

$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$

$e^x = \lim_{n\to\infty}(1+\frac{1}{n})^{nx}$

$e^x = \lim_{n\to\infty}\sum_{i=0}^{nx}\frac{1}{n^i}\frac{(nx)!}{i!(nx-i)!}$

$e^x = \lim_{n\to\infty}\sum_{i=0}^{nx}\frac{x^i}{i!}\frac{(nx)!}{(nx)^i(nx-i)!}$

From the lemma:

$e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!}$