Contents

1 Differentiable manifolds

1.1 Introduction

1.1.1 Differentiable transition maps

1.1.1.1 Transition map recap

Given two charts with an overlap, we have a transition mapping between the two charts of the overlap, where the mapping corresponds to a position on the manifold.

1.1.1.2 Differentiable transition maps

If this mapping is differentiable, we have a differentiable manifold.

1.1.1.3 Smooth manifolds

If transition maps are smooth (C^∞) then the manifold is smooth.

1.1.2 Differentiable and smooth manifolds

1.2 Tangent space

1.2.1 Tangent space and tangent vectors

Take a topological space: can all subsets in the toplogy be mapped to n dimensional space? if so, manifold

For this we need openness: a graph for example isn't open and so isn't a manifold

We also need the same number of dimensions at each point

Isn't always the case. eg two circles conneceted by a line is not a manifold. it's 2d in circles, 1d on line (and 3d at connections)

We have a homeomorphism from each point in the toplogy to an n dimensional coordinate system

We also have homeomorphisms of transformation maps, between different points on the topology

The vector space from the homeomorphism is tangent to the manifold at that point. the set of all tangents forms a tangent space

Interior: M; boundry M Tangent on a manifold:

The tangent space of manifold M at point p is denoted TM_p.

If we have a normal field

$v = v^i e_i$

Then we can differentiate wrt a direction x.

$$\frac{\delta}{\delta x} v = \frac{\delta}{\delta x} v^i e_i$$

$$\frac{\delta}{\delta x} v = e_i \frac{\delta v^i}{\delta x}$$

Because the basis does not change.

If the basis does change we instead have:

$$\frac{\delta}{\delta x} v = \frac{\delta}{\delta x} v^i e_i$$

$$\frac{\delta}{\delta x} v = e_i \frac{\delta v^i}{\delta x} + v^i \frac{\delta e_i}{\delta x}$$

General point. basis can vary across manifold

After this basis diff

1.2.1.1 Tangent space as vector bundle

1.2.1.2 Christoffel symbols (page)

Christoffel symbols are connections.

1.2.1.3 The torsion tensor (own page)

Torsion tensor is

$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$

If torsion is 0, then the connection is symmetric.

1.2.1.4 Basis of tangent space

We can use as the basis for tangent space:

$\{\frac{(\delta}{\delta x^1})_p, (\frac{\delta}{\delta x^2})_p, ...\}$

This means we can write a tangent vector as:

$u = u^i(\frac{\delta}{\delta x^i})_p$

1.2.1.5 Basis of contant space

We can use as the basis for the contangent space:

$\{dx^1, dx^2, ...\}$

1.2.1.6 Metric on the tangent space (to Riemann)

1.2.1.7 Basis of metric (to Riemann)

The metric depends on the basis too:

$g_{ij}(p) = g((\frac{\delta}{\delta x^i})_p, (\frac{\delta}{\delta x^j})_p)$

The metric on two tangent vectors is defined on the components.

$g = g_{ij}(p)u^i v^j$

1.2.2 Cotangent space and cotangent vectors

1.3 Connections

1.3.1 Transport

1.3.2 Covariant derivative

Essentially as we move across path, we are changing the basis.

We can look at how basis vector change as we translate

We can define as basis as:

$e_i = \frac{\delta x}{\delta x_i}$

How to measure transport

If we take a vector and move it around a curved surface and return it to the same point, it may not face the same way

Eg if you're on equator, move east, north, south to equator, you'll face diffrent direction

This is true on smaller movements of a curved surface

We can use this to measure curvature of a manifold without coordinates

1.3.3 New

covariant derivative. how does change in field compare to parallel transport from curent position?

_v (X)=_t0

We have point p. We can compare how field in tangent space varies in direction of v.

we don't define basis as each point, but rather how basis changes as you move along a curve

1.3.4 Affine connections

If we have a tangent vector at one point of the manifold, we can map it to a tangent vector at a nearby point on the manifold.

We can use chain rule. so we can have coordinate maps where there is no overlap.

1.3.4.1 Smooth connections

1.3.4.2 Affine connection

We have a vector in a tangent space

We have a curve on the manifold from the start point

As we "roll" the tangent, there is a unique vector in each new tangent, determined by transition map

These are affine transformations

Given two points, what path? what transfomration? if curuved then different paths will given different transformation.

1.3.5 Parallel transport

Move and therefore change basis, but components are the same.

1.4

1.4.1 Orientability of surfaces