# 1 Real functions as infinite-dimensional vectors

## 1.1 Real functions are vectors

The real function space is a vector space because it is linear in multiplication and addition.

$g(x) = cf(x)$

$h(x) = f(x) + k(x)$

# 2 Endomorphisms of infinite-dimensional vector spaces

## 2.1 Endomorphisms on real functions

We start with our vector $f(x)$.

$h(x) = f(x)g(x)$

The equivalent of the identity matrix is where $g(x) = 1$.

These are similar to endomorphisms where all off diagonal elements are 0.

### 2.1.1 Differentiation

$h(x) = \frac{\delta}{\delta x} f(x)$

### 2.1.2 Integration

$h(x) = \int_{-\infty}^{x} f(z)dz$

## 2.2 Examples of linear operators on real functions

For a function $v$ we can define operators $Ov$.

Here we consider some examples and their properties.

### 2.2.1 Real multiplication

$Rv = rf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form $rI$.

#### 2.2.2 Multiplication by underlying real number

$Xv = xf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form $M_{ii} = i$ and $M_{ij} = 0$.

#### 2.2.3 Differentiation

$Dv = \frac{\delta}{\delta x} f(x)$

While this operator is not hermitian, the following is:

$-iDv = \frac{\delta}{\delta x}[-if(x)]$

# 3 Eigenvalues and eigenvectors of infinite-dimensional vectors

## 3.1 Spectral theorem for infinite-dimensional vector spaces

# 4 Forms on infinite-dimensional vector spaces

## 4.1 Forms on real functions

A form takes two vectors and produces a scalar.

#### 4.1.1 Integration as a form

We can use integration to get a bilinear form.

$\int f(x)g(x)dx$

If we instead want a sesquilinear form we can instead use:

$\int f\bar{(x)}g(x)dx$

## 4.2 Functionals

Functionals map functions to scalars. They are the 1-forms of infinite-dimensional vector spaces.

If we have a function $f$, we can write functional $J[f]$.

#### 4.2.1 More

We can define neighbourhoods around a function $f$. For example, taking $y$ to be $f$ with infintesimal changes. to each of the values.

The difference between the functional at both points is

$\delta J = J[y] - J[f]$

#### 4.2.2 Extrema

If

$\delta J = J[y] - J[f]$

is the same sign for all y around f, then J has an extremum at f.

#### 4.2.3 Functional derivatives

### 4.3 Hilbert space

A complete space with an inner product. That is, a Banach space where the norm is derived from an inner product.

## 5 Calculus of variations

### 5.1 Calculus of variations

Integrate over possible functions?

## 6 Sort

### 6.1 Banach space

A complete normed vector space