0.1 Binomial expansion

0.1.1 Introduction

How can we expand

$(a+b)^n, n \in \mathbb{N}$

We know that:

$(a+b)^n = (a+b)(a+b)^{n-1}$

$(a+b)^n = a(a+b)^{n-1} + b(a+b)^{n-1}$

Each time this is done, the terms split, and each terms is multiplied by either a or b. That means at the end there are n total multiplications.

This can be shown as:

$(a+b)^n = \sum_{i=1}^{n} a^i b^{n-i} c_i$

So we want to identify c_i.

Each term can be shown as a series of n as and bs. For example:

- $aaba$
- $baaa$

For any of these, there are $n!$ ways or arranging the sequence, but this includes duplicates. If we were given n unique terms to multiply there would indeed by $n!$ different ways this could have arisen, but we can swap as and bs, as they were only generated once. So let's count duplicates.

There are duplicates in the as. If there are i as, then there are $i!$ ways of rearranging this. Similarly, if there are $n-i$ bs, then there are $(n-i)!$ ways or arranging this.

As a result the number of actual observed instances, c_i, is:

$c_i = \frac{n!}{i!(n-i)!}$

And so:

$(a+b)^n = \sum_{i=0}^{n} a^i b^{n-i} \frac{n!}{i!(n-i)!}$

We can also write this last term as:

$\binom{n}{i}$