0.1 Axiom schema of specification

0.1.1 The axiom schema of unrestricted comprehension

We want to formalise the relationship between the preterite and the set. An obvious way of doing this is to add an axiom for each preterite in our structure that:

$\forall x \exists s[P(x) \leftrightarrow (x \in s)]$

This is known as “unrestricted comprehension” and there are problems with this approach.

Consider a predicate for all terms which are not members of themselves. That is:

$\text{ň}(x \in x)$

This implies the following is true:

$\forall x \exists s[\text{ň}(x \in x) \leftrightarrow (x \in s)]$

As this is true for all x, it is true for $x = s$. So:

$\exists s[\text{ň}(s \in s) \leftrightarrow (s \in s)]$

This statement is false. As we have inferred a false formula, the axiom of unrestricted comprehension does not work. This result is known as Russel’s Paradox.

This is an axiom schema rather than an axiom. That is, there is a new axiom for each preterite.

0.1.2 Axiom schema of specification

To resolve Russels’ paradox, we amend the axiom schema to:

$\forall x \forall a \exists s[(P(x) \wedge x \in a) \leftrightarrow (x \in s)]$

That is, for every set a, we can define a subset s for each predicate.

This resolves Russel’s Paradox. Let’s take the same steps on the above formula as in unrestricted comprehension;

$\forall x \forall a \exists s[(\text{ň}(x \in x) \wedge x \in a) \leftrightarrow (x \in s)]$

$\exists s[(\text{ň}(s \in s) \wedge s \in s) \leftrightarrow (s \in s)]$

So long as the subsets s are not members of themselves, this holds.