1 Nearness functions

1.1 Topologies

1.2 Topologies on sets

T is a topology on set X if:

- $X \in T$
- $\varnothing \in T$
- Unions of T are in T
- Intersections of T are in T

1.3 Examples of topologies: The trivial topology

The trivial topology contains only the underlying set and the empty set.

1.4 Examples of topologies: The discrete toplogy

The discrete toplogy contains all subsets of the underlying set (is this the power set?)

2 Neighbourhoods

2.1 Neighbourhood topology

We have a set X.

For each element $x \in X$, there is a non-empty set of neighbourhoods $N \in \mathbf{N}(x)$ where $x \in N \subseteq X$ such that:

- If N is a subset of M, M is a neighbourhood.
- The intersection of two neighbourhoods of x is a neighbourhood of x.
- N is a neighbourhood for each point in some $M \subseteq N$

2.2 Topological distinguishability

If two points have the same neighbourhoods then they are topologically indistinguishable.

For example in the trivial topology, all points are topologically indistinguishable.

2.3 Open sets

U is an open set if it is a neighbourhood for all its points.

3 Open and closed sets

3.1 Limit points and closure

3.1.1 Limit points

A point x in the topological set X is a limit point for $S \subset X$ if every neighbourhood of x contains another point in S.

For example -1 is a limit point for the real numbers where S is $[0, 1]$ (or $(0, 1)$.

3.1.2 Closure

The closure of a subset of a topological space is the subset itself along with all limit points.

So the closure of $|x| < 1$ includes -1 and 1.

3.2 Boundries and interiors

The boundry of the subset S of a topology is the intersection with the closure of S with the closure of the complement of S.

So the boundry of both $(0, 1)$ and $[0, 1]$ are 0 and 1.

The interior of S is S without the boundry.

So the interior of $(0, 1)$ and $[0, 1]$ are both $(0, 1)$.

3.3 Closed sets

The complement of any open set is a closed set.

A set can be open, closed, both or neither.

4 Compactness

4.1 Covers

A space X is covered by a set of subsets of X, C, if the union of C is X.

4.2 Subcover

A subset of C which still covers X is a a subcover.

4.3 Open cover

C is an open cover if each member is an open set.

4.4 Bases of topologies

Subset B of topology T is a base for T if all elements of T are unions of members of B.

4.4.1 Second-countable space

If B is finite then the toplogy is a second-countable space.

5 Separation

5.1 Connected and separated sets

Two subsets of X in topological space T are separated if each subset is disjoint from the other's closure.

So $[-1, 0)$ and $(0, 1)$ are separated.

$[-1, 0]$ and $(0, 1)$ are not separated.

Sets which are not separated are connected.

6 Cartesian products

6.1 Box topology

6.2 Product topology

7 Creating topologies from sets

7.1 The trivial topology

A topology which contains just X and $\varnothing$ is the trivial topology.

7.2 Discrete topology

8 Taxonomy of spaces

8.1 Lindelöf space

In a Lindelöf space all open covers have countable subcovers.

This is weaker than compactness, which requires that every open cover has a finite subcover.

8.2 Kolmogorov space

In a Kolmogorov (or T_0) space, for every pair of points there is a neighbourhood containing one but not the other.

9 Local properties

9.1 Local properties

Locally, a topology may have properties which are not present globally.

9.2 Locally compact spaces

9.3 Locally connected spaces

10 TO INF

10.1 Hausdorf space

In a Hausdorf (or T$_2$) space, any two different points have neighbourhoods which are disjoint.

10.2 Compact spaces

A space X is compact if each open cover has a finite subcover.

If we can define a cover which does not have a finite subcover, then the space is not compact.

For example an infinite cover could be tend towards $(0, 1)$, eg as $\dfrac{1}{n}, 1 - \dfrac{1}{n}$

This covers $(0, 1)$, but there is no finite subcover. As a result $(0, 1)$ is not compact.