0.1 Defing sine and cosine using Euler's formula

0.1.1 Euler's formula

Previously we showed that:

$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

Consider:

$e^{i\theta}$

$e^{i\theta} = \sum_{j=0}^{\infty} \frac{(i\theta)^j}{j!}$

$e^{i\theta} = [\sum_{j=0}^{\infty} \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+2}}{(4j+2)!}] + i[\sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+3}}{(4j+3)!}]$

We then use this to define sin and cos functions.

$\cos(\theta) := \sum_{j=0}^{\infty} \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+2}}{(4j+2)!}$

$\sin(\theta) := \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+3}}{(4j+3)!}$

So:

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

0.1.2 Alternative formulae for sine and cosine

We know

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

So

$e^{i\theta} + e^{-i\theta} = cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta)$

$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$

And

$e^{i\theta} - e^{-i\theta} = cos(\theta) + i\sin(\theta) - \cos(\theta) + i\sin(\theta)$

$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

0.1.3 Sine and cosine are odd and even functions

Sine is an odd function.

$\sin(-\theta) = -\sin(\theta)$

Cosine is an even function.

$\cos(-\theta) = \cos(\theta)$