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### 0.1 Axiom of extensionality

If two sets contain the same elements, they are equal.

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

This is an axiom, not a definition, because equality was defined as part of first-order logic.

Note that this is not bidirectional.  $x = y$  does not imply that  $x$  and  $y$  contain the same elements. This is appropriate as  $\frac{1}{2} = \frac{2}{4}$  for example, even though they are written differently as sets.

#### 0.1.1 Reflexivity of equality

The reflexive property is:

$$\forall x (x = x)$$

We can replace the instance of  $y$  with  $x$ :

$$\forall x [\forall z (z \in x \leftrightarrow z \in x) \rightarrow x = x]$$

We can show that the following is true:

$$\forall z (z \in x \leftrightarrow z \in x)$$

Therefore:

$$\forall x [T \rightarrow x = x]$$

$$x = x$$

#### 0.1.2 Symmetry of equality

The symmetry property is:

$$\forall x \forall y [(x = y) \leftrightarrow (y = x)]$$

We know that the following are true:

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow y = x]$$

So:

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow (x = y \wedge y = x)]$$

### 0.1.3 Transitivity of equality

The transitive property is:

$$\forall x \forall y \forall z [(x = y \wedge y = z) \rightarrow x = z]$$

### 0.1.4 Substitution for functions

The substitutive property for functions is:

$$\forall x \forall y [(x = y) \rightarrow (f(x) = f(y))]$$

### 0.1.5 Substitution for formulae

The substitutive property for formulae is:

$$\forall x \forall y [(x = y) \wedge P(x) \rightarrow P(y)]$$

Doesn't this require iterating over predicates? Is this possible in first order logic??

### 0.1.6 Result 1: The empty set is unique

We can now show the empty set is unique.

### 0.1.7 Result 2: Every element of a set exists

If an element did not exist, the set containing it would be equal to a set which did not contain that element.

### 0.1.8 Result 3: Sets are unique