Contents

1 Algebra over a field, and Lie algebra

1.1 Algebra on a field

1.1.1 Bilinear maps

A bilinear map (or function) is a map from two inputs to an output which preserves addition and scalar multiplication. This is in contrast to a linear map, which only has one input.

In addition, the function is linear in both arguments.

That is if function f is bilinear then:

$X = aM + bN$

$Y = cO + dP$

$f(X, Y) = f(aM + bN, cO + dP)$

$f(X, Y) = f(aM, cO + dP) + f(bN, cO + dP)$

$f(X, Y) = f(aM, cO) + f(aM, dP) + f(bN, cO) + f(bN, dP)$

$f(X, Y) = acf(M, O) + adf(M, P) + bcf(N, O) + bdf(N, P)$

Note that:

$f(X, Y) = f(X + 0, Y)$

$f(X, Y) = f(X, Y) + f(0, Y)$

$(0, Y) = 0$

That is, if any input is 0 in an additative sense, the value of the map must be zero.

1.1.2 Algebra on a field

1.2 Cross products

1.2.1 The cross product

$v \times u$

1.2.1.1 Cross product is a bilinear map

This is a bilinear map from two vectors in $\mathbb{R}^3$ to another vector in the same space.

$V \times V \to V$

1.2.1.2 Calculating the cross product

This is calcualted by:

$u \times v = ||u||||v|| \sin(\theta)n$

The resulting vector is perpendicular to both input vectors.

1.3 Lie groups

1.3.1 Lie groups

1.4 Lie algebra

1.4.1 Lie algebra

Lie groups have symmetries. We can consider only the infintesimal symmetries.

For example the unit circle has many symmetries, but we can consider only those which rotate infintesimally.

1.4.1.1 Example

Take a continous group, such as $U(1)$. Its Lie algebra is all matrices such that their exponential is in the Lie group.

$\mathfrak{u}(1) = \{X \in \mathbb{C}^{1\times 1} | e^{tX} \in U(1) \forall t \in \mathbb{R}\}$

This is satisfied by the matrices where $M = -M^*$. Note that this means the diagonals are all 0.

1.4.1.2 Scale of specific Lie algebra matrices doesn't matter

Because of t.

1.4.1.3 Commutation of Lie group algebra

Consider two members of the Lie algebra: A and B. The commutator is:

A.

The corresponding Lie group member is:

$e^{t(A+B)} = e^{tA}e^{tB}$

While the Lie group multiplication may not commute, the corresponding addition of the Lie algebra does.

1.4.2 The Lie bracket

We can define the Lie bracket from the ring commutator.

We use the Lie bracket, rather than multiplication, as the operator over a field homomorphism.

$[A, B]$

This generates another element in the algebra.

This satisifies:

- Bilinearity: $[xA + yB, C] = x[A, C] + y[B, C]$
- Alternativity: $[A, A] = 0$
- Jacobi identity: $[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$
- Anticommutivity: $[A, B] = -[B, A]$

One option for the Lie bracket is the ring commutor. So that:

$[A, B] = AB - BA$

1.4.3 Commutation of Lie groups

We can measure commutation of Lie groups using:

$ABA^{-1}B^{-1}$

If the group commutes then:

$ABA^{-1}B^{-1} = BA^{-1}B^{-1} = I$

1.4.3.1 Commutation of Lie algebra: COMPLETE THIS

This corresponds to $[A, B] = AB - BA$ in the underlying lie algebra, if we expand.

$A = e^{ta}$

$B = e^{tb}$

$ABA^{-1}B^{-1} = e$

1.5 Lie algebra of specific Lie groups

1.5.1 Lie algebra of $O(n)$

1.5.1.1 $O(n)$ forms a Lie group

1.5.1.2 Lie algebra of $O(n)$

The Lie algebra of (n) is defined as:

$\mathfrak{o}(n) = \{X \in \mathbb{R}^{n\times n} | e^{tX} \in O(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-symmetric matrices where $M = -M^T$. Note that this means the diagonals are all 0.

1.5.2 Lie algebra of $U(n)$

1.5.2.1 $U(n)$ forms a Lie group

1.5.2.2 Lie algebra of $U(n)$

The Lie algebra of (n) is defined as:

$\mathfrak{u}(n) = \{X \in \mathbb{C}^{n\times n} | e^{tX} \in U(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-Hermitian matrices where $M = -M^*$. Note that this means the diagonals are all 0 or pure imaginary.

1.5.3 Lie algebra of $SO(n)$

1.5.3.1 $SO(n)$ forms a Lie group

1.5.3.2 Lie algebra of $SO(n)$

The Lie algebra of (n) is defined as:

$\mathfrak{so}(n) = \{X \in \mathbb{R}^{n\times n} | e^{tX} \in SO(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-symmetric matrices where $M = -M^T$. Note that this means the diagonals are all 0.

1.5.4 Lie algebra of $SU(n)$

1.5.4.1 $SU(n)$ forms a Lie group

1.5.4.2 Lie algebra of $SU(n)$

The Lie algebra of (n) is defined as:

$\mathfrak{su}(n) = \{X \in \mathbb{C}^{n\times n} | e^{tX} \in SU(n) \forall t \in \mathbb{R}\}$

This is satisfied by the skew-Hermitian matrices where $M = -M^*$ and the trace is 0. Note that this means the diagonals are all 0 or pure imaginary.

1.6 Hypercomplex numbers

1.6.1 Hypercomplex numbers

1.6.2 Quaternions

1.6.3 Clifford algebra

1.7 Sort

1.7.1 Projective line in the field