

1 More on sequences

1.1 Limit of a sequence

A sequence converges to a limit if

Can converge to a number ($1/x$)

Can converge to $+/-$ infinity (x)

Otherwise, does not converge ($1, -1, 1, -1, \dots$)

Superior and inferior limits

A bounded increasing sequence converges to least upper bound

1.1.1 Identifying the limit of a sequence

Direct comparison test

Root test

2 Divergent series

2.1 Partial sum

Take a series. We can define the partial sum as:

$$s_k = \sum_{i=1}^k a_i$$

2.2 Cesàro sum

The Cesàro sum is the limit of the average of the first n partial sums.

That is:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n s_k$$

Consider the sequence $\{1, -1, 1, -1, \dots\}$

The partial sum is:

$$s_k = \sum_{i=1}^k a_i$$

$$s_k = k \pmod{2}$$

The Cesàro sum is: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n s_k$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n k \pmod{2}$$

$$\frac{1}{2}$$

2.3 Abel summation

3 Constructing the real numbers

3.1 Cauchy sequences

3.1.1 Cauchy sequence

A cauchy sequence is a sequence such that for an any arbitrarily small number ϵ , there is a point in the sequence where all possible pairs after this are even closer together.

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N} : \forall m, n \in \mathbb{N} > N)(|a_m - a_n| < \epsilon)$$

This last term gives a distance between two entries. In addition to the number line, this could be used on vectors, where distances are defined.

As a example, $\frac{1}{n}$ is a cauchy sequence, $\sum_i \frac{1}{n}$ is not.

3.1.2 Completeness

Cauchy sequences can be defined on some given set. For example given all the numbers between 0 and 1 there are any number of different cauchy sequences converging at some point.

If it is possible to define a cauchy sequence on a set where the limit is not in the set, then the set is incomplete.

For example, the numbers between 0 and 1 but not including 0 and 1 are not complete. It is possible to define sequences which converge to these missing points.

More abstractly, you could have all vectors where $x^2 + y^2 < 1$. This is incomplete (or open) as sequences on these vectors can converge to limits not in the set.

Cauchy sequences are important when considering real numbers. We could define a sequence converging on $\sqrt{2}$, but as this number is not in the set, it is incomplete.

3.2 Incompleteness of the real numbers

3.2.1 The square root of 2 is not a rational number

Let's prove there are numbers which are not rational. Consider $\sqrt{2}$ and let's show that it being rational leads to a contradiction.

$$\sqrt{2} = \frac{x}{y}$$

$$2 = \frac{x^2}{y^2}$$

$$2y^2 = x^2$$

So we know that x^2 is even, and can be shown as $x = 2n$.

$$2y^2 = (2n)^2$$

$$y^2 = 2n^2$$

So y is even. But if both x and y are even, then the fraction was not reduced.

This presents a contraction so the original statement must have been false.

So we know there isn't a rational solution to $\sqrt{2}$.

3.3 Density of rationals and reals

3.3.1 Rationals are dense in reals

3.3.2 Reals are dense in reals

3.3.3 Reals are dense in rationals

3.4 σ -algebra

3.4.1 Review of algebra on a set

An algebra, Σ , on set s is a set of subsets of s such that:

- Closed under intersection: If a and b are in Σ then $a \cap b$ must also be in Σ
- $\forall ab[(a \in \Sigma \wedge b \in \Sigma) \rightarrow (a \cap b \in \Sigma)]$
- Closed under union: If a and b are in Σ then $a \cup b$ must also be in Σ .
- $\forall ab[(a \in \Sigma \wedge b \in \Sigma) \rightarrow (a \cup b \in \Sigma)]$

If both of these are true, then the following is also true:

- Closed under complement: If a is in Σ then $s \setminus a$ must also be in Σ

We also require that the null set (and therefore the original set, null's complement) is part of the algebra.

3.4.2 σ -algebra

A σ -algebra is an algebra with an additional condition:

All countable unions of sets in A are also in A .

This adds a constraint. Consider the real numbers with an algebra of all finite sets.

This contains all finite subsets, and their complements. It does not contain \mathbb{N} .

However a σ -algebra requires all countable unions to be including, and the natural numbers are a countable union.

The power set is a σ -algebra. All other σ -algebras are subsets of the power set.