Contents

1 Trigonometry

1.1 Sine and cosine

1.1.1 Defing sine and cosine using Euler's formula

1.1.1.1 Euler's formula

Previously we showed that:

$e^x = \sum_{i=0}^\infty \frac{x^i}{i!}$

Consider:

$e^{i\theta}$

$e^{i\theta} = \sum_{j=0}^\infty \frac{(i\theta)^j}{j!}$

$e^{i\theta} = [\sum_{j=0}^\infty \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^\infty \frac{(\theta)^{4j+2}}{(4j+2)!}] + i[\sum_{j=0}^\infty \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^\infty \frac{(\theta)^{4j+3}}{(4j+3)!}]$

We then use this to define sin and cos functions.

$\cos(\theta) := \sum_{j=0}^\infty \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^\infty \frac{(\theta)^{4j+2}}{(4j+2)!}$

$\sin(\theta) := \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+3}}{(4j+3)!}$

So:

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

1.1.1.2 Alternative formulae for sine and cosine

We know

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

So

$e^{i\theta} + e^{-i\theta} = cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta)$

$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$

And

$e^{i\theta} - e^{-i\theta} = cos(\theta) + i\sin(\theta) - \cos(\theta) + i\sin(\theta)$

$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

1.1.1.3 Sine and cosine are odd and even functions

Sine is an odd function.

$\sin(-\theta) = -\sin(\theta)$

Cosine is an even function.

$\cos(-\theta) = \cos(\theta)$

1.1.2 De Moive's formula

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Let $\theta = nx$:

$e^{inx} = \cos(nx) + i\sin(nx)$

$(e^{ix})^n = \cos(nx) + i\sin(nx)$

$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$

1.1.3 Expanding sine and cosine

1.1.3.1 Expansion

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

1.1.4 Addition of sine and cosine

1.1.4.1 Adding waves with same frequency

We know that:

$a\sin(bx+c) = a\sin(bx)\cos(c) + a\sin(c)\cos(bx)$

So:

$a\sin(bx+c)+d\sin(bx+e) = a\sin(bx)\cos(c)+a\sin(c)\cos(bx)+d\sin(bx)\cos(e)+d\sin(e)\cos(bx)$

We know that:

$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

So:

$a\sin(bx+c)+d\sin(bx+f) = a\frac{e^{i(bx+c)}-e^{-i(bx+c)}}{2i} + d\frac{e^{i(bx+f)}-e^{-i(bx+f)}}{2i}$

$a\sin(bx+c)+d\sin(bx+f) = \frac{a(e^{i(bx+c)}-e^{-i(bx+c)})+d(e^{i(bx+f)}-e^{-i(bx+f)})}{2i}$

$a\sin(bx+c)+d\sin(bx+f) = \frac{a(e^{ibx}e^{ic}-e^{-ibx}e^{-ic})+d(e^{ibx}e^{if}-e^{-ibx}e^{-if})}{2i}$

$a\sin(bx+c)+d\sin(bx+f) = \frac{(e^{ibx}(ae^{ic}+de^{if})-e^{-ibx}(ae^{-c}+d^{-if})}{2i}$

$a_i\sin(b_ix+c_i)+a_j\sin(b_jx+c_j) = a_i\sin(b_ix+c_i)+a_j\sin(b_ix+c_j)$

$a_i\sin(b_ix+c_i)+a_j\sin(b_jx+c_j) = a_i\sin(b_ix)\cos(c_i)+a_i\sin(c_i)\cos(b_ix)+a_j\sin(b_ix)\cos(c_j)+a_j\sin(c_j)\cos(b_ix)$

1.1.5 Calculus of sine and cosine

1.1.5.1 Unity

Note that with imaginary numbers we can reverse all is. So:

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

$e^{i\theta}e^{-i\theta} = (\cos(\theta)+i\sin(\theta))(\cos(\theta)-i\sin(\theta))$

$e^{i\theta}e^{-i\theta} = \cos(\theta)^2+\sin(\theta)^2$

$e^{i\theta}e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1$

So:

$\cos(\theta)^2 + \sin(\theta)^2 = 1$

Note that if $\cos(\theta)^2 = 0$, then $\sin(\theta)^2 = \pm 1$

That is, if the real part of $e^{i\theta}$ is 0, the imaginary part is ± 1. And visa versa.

Similarly if the derivative of the real part of $e^{i\theta}$ is 0, the imaginary part is ± 1. And visa versa.

1.1.5.2 Sine and cosine are linked by their derivatives

Note that these functions are linked in their derivatives.

$\frac{\delta}{\delta\theta}\cos(\theta) = \sum_{j=0}^{\infty} \frac{(\theta)^{(4j+3)}}{(4j+3)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!}$

$\frac{\delta}{\delta\theta}\cos(\theta) = -\sin(\theta)$

Similarly:

$\frac{\delta}{\delta\theta}\sin(\theta) = cos(\theta)$

1.1.5.3 Both sine and cosine oscillate

$\frac{\delta^2}{\delta\theta^2}\sin(\theta) = -\sin(\theta)$

$\frac{\delta^2}{\delta\theta^2}\cos(\theta) = -\cos(\theta)$

So for either of:

$y = \cos(\theta)$

$y = \sin(\theta)$

We know that

$\frac{\delta^2}{\delta\theta^2}y(\theta) = -y(\theta)$

Consider $\theta = 0$.

$e^{i.0} = \cos(0) + i\sin(0)$

$1 = \cos(0) + i\sin(0)$

$\sin(0) = 0$

$\cos(0) = 1$

Similarly we know that the derivative:

$\sin'(0) = \cos(0) = 1$

$\cos'(0) = -\sin(0) = 0$

Consider $\cos(\theta)$.

As $\cos(0)$ is static at $\theta = 0$, and is positive, it will fall until $\cos(\theta) = 0$.

While this is happening, $\sin(\theta)$ is increasing. As:

$\cos(\theta)^2 + \sin(\theta)^2 = 1$

$\sin(\theta)$ will equal 1 where $\cos(\theta) = 0$.

Due to symmetry this will repeat 4 times.

Let's call the length of this period τ.

Where $\theta = \tau * 0$

- $\cos(\theta) = 1$
- $\sin(\theta) = 0$

Where $\theta = \tau * \frac{1}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = 1$

Where $\theta = \tau * \frac{2}{4}$

- $\cos(\theta) = -1$
- $\sin(\theta) = 0$

Where $\theta = \tau * \frac{3}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = -1$

1.1.5.4 Relationship between$\cos(\theta)$ and $\sin(\theta)$

Note that $\sin(\theta + \frac{\tau}{4}) = \cos(\theta)$

Note that $\sin(\theta) = \cos(\theta)$ at

- $\tau * \frac{1}{8}$
- $\tau * \frac{5}{8}$

And that all these answers loop. That is, add any integer multiple of τ to θ and the results hold.

$e^{i\theta} = e^{i\theta + n\tau}$

$n \in \mathbb{N}$

$e^{i\theta} = \cos(\theta) + i \sin(\theta)$

$e^{i\theta} = \cos(\theta + n\tau) + i \sin(\theta + n\tau)$

$e^{i\theta} = e^{i(\theta+n\tau)}$

1.1.5.5 Calculus of trig

Relationship between cos and sine

$\sin(x + \frac{\pi}{2}) = \cos(x)$

$\cos(x + \frac{\pi}{2}) = -\sin(x)$

$\sin(x + \pi) = -\sin(x)$

$\cos(x + \pi) = -\cos(x)$

$\sin(x + \tau) = \sin(x)$

$\cos(x + \tau) = \cos(x)$

1.2 Polar coordinates

1.2.1 Polar co-ordinates

1.2.1.1 All complex numbers can be shown in polar form

Consider a complex number

$z = a + bi$

We can write this as:

$z = r\cos(\theta) + ir\sin(\theta)$

1.2.1.2 Polar forms are not unique

Because the functions loop:

$ae^{i\theta} = a(\cos(\theta) + i\sin(\theta))$

$ae^{i\theta} = a(\cos(\theta + n\tau) + i\sin(\theta + n\tau))$

$ae^{i\theta} = ae^{i\theta+n\tau}$

Additionally:

$ae^{i\theta} = a(\cos(\theta) + i\sin(\theta))$

$ae^{i\theta} = a(\cos(\theta) + i\sin(\theta))$

$ae^{i\theta} = -a(\cos(\theta) - i\sin(\theta))$

$ae^{i\theta} = -a(\cos(\theta + \frac{\pi}{2}) + i\sin(\theta + \frac{\pi}{2}))$

1.2.1.3 Real and imaginary parts of a complex number in polar form

We can extract the real and imaginary parts of this number.

$Re(z) := r\cos(\theta)$

$Im(z) := r\sin(\theta)$

Alternatively:

$Re(z) = r\frac{e^{i\theta}+e^{-i\theta}}{2}$

$Im(z) = r\frac{e^{i\theta}-e^{-i\theta}}{2i}$

1.2.2 Moving between polar and cartesian coordinates

All polar numbers can be shown as Cartesian

$ae^{i\theta} = a(\cos(\theta) + i\sin(\theta))$

$ae^{i\theta} = a\cos(\theta) + ia\sin(\theta)$

$z = a + bi$

$e^{i\theta} =$

$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

1.2.3 Arithmetic of polar coordinates

Addition

$z_3 = z_1 + z_2$

$z_3 = a_1 e^{i\theta_1} + a_2 e^{i\theta_2}$

$z_3 = a_1[\cos(\theta_1) + i\sin(\theta_1)] + a_2[\cos(\theta_2) + i\sin(\theta_2)]$

$z_3 = [a_1\cos(\theta_1) + a_2\cos(\theta_2)] + i[a_1\sin(\theta_1) + a_2\sin(\theta_2)]$

Multiplication

$z_3 = z_1.z_2$

$z_3 = a_1 e^{i\theta_1} a_2 e^{i\theta_2}$

$z_3 = a_1 a_2 e^{i(\theta_1+\theta_2)}$

$a_3 = a_1 a_2$

$\theta_3 = \theta_1 + \theta_2$

1.3 Tangent

1.3.1 Tan

The $\tan(\theta)$ function is defined as:

$\tan(\theta) := \frac{\sin(\theta)}{\cos(\theta)}$

1.3.1.1 Behaviour around 0

$\sin(0) = 0$

$\cos(0) = 1$

$\tan(0) := \frac{\sin(0)}{\cos(0)}$

$\tan(0) = \frac{0}{1}$

$\tan(0) = 0$

1.3.1.2 Behaviour around $\cos(\theta) = 0$

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

So $\tan(\theta)$ is undefined where $\cos(\theta) = 0$.

This happens where:

$\theta = \frac{\tau}{4} + \frac{1}{2}n\tau$

$\theta = \frac{1}{4}\tau(1 + 2n)$

Where $n \in \mathbb{Z}$.

1.3.1.3 Derivatives

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

$\frac{\delta}{\delta\theta} \tan(\theta) = \frac{\delta}{\delta\theta} \frac{\sin(\theta)}{\cos(\theta)}$

$\frac{\delta}{\delta\theta} \tan(\theta) = \frac{\cos(\theta)}{\cos(\theta)} + \frac{\sin^2(\theta)}{\cos^n(\theta)}$

$\frac{\delta}{\delta\theta} \tan(\theta) = 1 + \tan^2(\theta)$

Note this is always positive. This means:

$\lim_{\cos(\theta) \to 0^+} = -\infty$

$\lim_{\cos(\theta) \to 0^-} = \infty$

1.3.2 Inverse functions

1.3.2.1 Inverse trigonometric functions

$\sin(\arcsin(\theta)) := \theta$

$\cos(\arccos(\theta)) := \theta$

$\tan(\arctan(\theta)) := \theta$

1.3.3 τ

1.3.3.1 Calculating τ

As we note above, $\sin(\theta) = \cos(\theta)$ at $\theta = \tau * \frac{1}{8}$

This is also where $\tan(\theta) = 1$.

$\arctan(k) = \arctan(a) + \int_a^k \frac{1}{1+y^2} \delta y$

We start from $a = 0$.

$\arctan(k) = \arctan(0) + \int_0^k \frac{1}{1+y^2} \delta y$

We know that one of the results for $\arctan(0)$ is 0.

$\arctan(k) = \int_0^k \frac{1}{1+y^2} \delta y$

We want $k = 1$

$\arctan(1) = \int_0^1 \frac{1}{1+y^2} \delta y$

$\frac{\tau}{8} = \int_0^1 \frac{1}{1+y^2} \delta y$

$\tau = 8 \int_0^1 \frac{1}{1+y^2} \delta y$

We know that the $\cos(\theta)$ and $\sin(\theta)$ functions cycle with period τ.

Therefore $cos(n.\tau) = \cos(0)$

1.3.3.2 Calculating τ

As we note above, $\sin(\theta) = \cos(\theta)$ at $\theta = \tau * \frac{1}{8}$

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We know that the $\cos(\theta)$ and $\sin(\theta)$ functions cycle with period τ.

Therefore $cos(n.\tau) = \cos(0)$

1.4 Hyperbolic functions

1.4.1 Hyperbolic functions

1.4.1.1 Hyperbolic functions

$\sinh(\theta) := \sin(i\theta)$

$\cosh(\theta) := \cos(i\theta)$

$\tanh(\theta) := \tan(i\theta)$

1.4.1.2 Inverse trigonometric functions

$\sinh(arcsinh(\theta)) := \theta$

$\cosh(arccosh(\theta)) := \theta$

$\tanh(arctan(\theta)) := \theta$

1.4.2 Integrals

1.4.2.1 Cosine and sine

$\arccos(\theta)$, $\arcsin(\theta)$ and difficulty of inversing

In order to determine τ we need inverse functions for $\cos(\theta)$ or $\sin(\theta)$.

These are the $\arccos(\theta)$ and $\arcsin(\theta)$ functions respectively.

However this is not easily calculated. Instead we look for another function.

1.4.2.2 Calculating $\arctan(\theta)$

So we want a function to inverse this. This is the $\arctan(\theta)$ function.

If $y = \tan(\theta)$, then:

$\theta = \arctan(y)$

We know the derivative for $\tan(\theta)$ is:

$\frac{\delta}{\delta\theta}\tan(\theta) = 1 + \tan^2(\theta)$

$\frac{\delta y}{\delta\theta} = 1 + y^2$

So

$\frac{\delta\theta}{\delta y} = \frac{1}{1+y^2}$

$\frac{\delta}{\delta y}\arctan(y) = \frac{1}{1+y^2}$

So the value for $\arctan(k)$ is:

$\arctan(k) = \arctan(a) + \int_a^k \frac{\delta}{\delta y}\arctan(y)\delta y$

$\arctan(k) = \arctan(a) + \int_a^k \frac{1}{1+y^2}\delta y$

What do we know about this function? We know it can map to multiple values of θ because the underlying $\sin(\theta)$ and $\cos(\theta)$ functions also loop.

We know that one of the results for $\arctan(0)$ is 0.

1.5 Other

1.5.1 Other functions

1.5.1.1 Reciprocal trigonometric functions

Standard

$\csc(\theta) := \frac{1}{\sin(\theta)}$

$\sec(\theta) := \frac{1}{\cos(\theta)}$

$\cot(\theta) := \frac{1}{\tan(\theta)}$

Hyperbolic

$csch(\theta) := \frac{1}{\sinh(\theta)}$

$sech(\theta) := \frac{1}{\cosh(\theta)}$

$coth(\theta) := \frac{1}{\tanh(\theta)}$

1.5.1.2 Inverse trigonometric functions

Reciprocal standard

$\csc(arccsc(\theta)) := \theta$

$\sec(arcsec(\theta)) := \theta$

$\cot(arccot(\theta)) := \theta$

Reciprocal hyperbolic

$csch(arccsch(\theta)) := \theta$

$sech(arcsech(\theta)) := \theta$

$coth(arccoth(\theta)) := \theta$