1 Powers

1.1 Powers

1.1.1 Exponents and logarithms

Previously we defined addition and multiplication in terms of successive use of the sucessor function. That is, the definition of addition was:

$\forall a \in \mathbb{N}(a + 0 = a)$

$\forall ab \in \mathbb{N}(a + s(b) = s(a + b))$

And similarly for multiplication:

$\forall a \in \mathbb{N}(a.0 = 0)$

$\forall ab \in \mathbb{N}(a.s(b) = a.b + a)$

Additional functions could also be defined, following the same pattern:

$\forall a \in \mathbb{N}(a \oplus_n 0 = c)$

$\forall ab \in \mathbb{N}(a \oplus_n s(b) = (a \oplus_n b) \oplus_{n-1} a)$

1.1.2 Powers

Exponents can also be defined:

1.1.3 Axioms

$\forall a \in \mathbb{N} a^0 = 1$

$\forall ab \in \mathbb{N} a^{s(b)} = a^b.a$

1.1.4 Example

So 2^2 can be calculated like:

$2^2 = 2^{s(1)}$

$2^{s(1)} = 2.2^1$

$2.2^1 = 2.2.2^0$

$2.2.2^0 = 2.2.1$

$2.2.1 = 4$

Unlike addition and multiplication, exponention is not commutative. That is

$a^b \neq b^a$

1.1.5 Exponential rules

$a^b a^c = a^{b+c}$

$(a^b)^c = a^{bc}$

$(ab)^c = a^c b^c$

1.1.6 Powers of natural numbers

1.1.7 Powers of integers

1.1.8 Powers of rational numbers

1.2 Binomial expansion

1.2.1 Introduction

How can we expand

$(a+b)^n, n \in \mathbb{N}$

We know that:

$(a+b)^n = (a+b)(a+b)^{n-1}$

$(a+b)^n = a(a+b)^{n-1} + b(a+b)^{n-1}$

Each time this is done, the terms split, and each terms is multiplied by either a or b. That means at the end there are n total multiplications.

This can be shown as:

$(a+b)^n = \sum_{i=1}^n a^i b^{n-i} c_i$

So we want to identify c_i.

Each term can be shown as a series of n as and bs. For example:

- $aaba$
- $baaa$

For any of these, there are $n!$ ways or arranging the sequence, but this includes duplicates. If we were given n unique terms to multiply there would indeed by $n!$ different ways this could have arisen, but we can swap as and bs, as they were only generated once. So let's count duplicates.

There are duplicates in the as. If there are i as, then there are $i!$ ways of rearranging this. Similarly, if there are $n-i$ bs, then there are $(n-i)!$ ways or arranging this.

As a result the number of actual observed instances, c_i, is:

$c_i = \frac{n!}{i!(n-i)!}$

And so:

$(a+b)^n = \sum_{i=0}^{n} a^i b^{n-i} \frac{n!}{i!(n-i)!}$

We can also write this last term as:

$\binom{n}{i}$

1.3 Difference of two squares

1.3.1 Differences of two squares

$(a+b)(a-b) = a^2 - ab + ab - b^2$

$(a+b)(a-b) = a^2 - b^2$

2 Irrational numbers

2.1 Logarithms

2.1.1 Logarithms

If:

$c = a^b$

Then

$log_a c = b$

Product rule:

$a = c^{log_c a}$

$b = c^{log_c b}$

So:

$ab = c^{log_c ab}$

But also:

$ab = c^{log_c a} c^{log_c b}$

$ab = c^{log_c a + log_c b}$

So:

$log_c a + log_c b = log_c ab$

2.1.2 Power rule

$a = b^{log_b a}$

So:

$a^c = b^{log_b a^c}$

And separately:

$a^c = (b^{log_b a})^c$

$a^c = (b^{c log_b a})$

So:

$c log_b a = log_b a^c$

2.2 Logarithms for natural numbers

2.3 Logarithms for integers

2.4 Logarithms for rational numbers

3 Equations

3.1 Algebraic equations

3.1.1 Introduction

4 Single-variable polynomials

4.1 Single-variable polynomials

4.1.1 Introduction

A single-variable polynomial is an equation of the form:

$\sum_{i=0}^{n} a_i x^i = 0$

For example:

- $x = 1$
- $x^2 = 4$
- $x^2 - 3x + 2 = 0$

4.1.2 Degrees

The degree of a polynomial is the highest-order term.

For example $x^3 + x = 0$ has degree 3.

4.1.3 Roots of single-variable polynomials

A solution to a polynomial is a root.

For example 1 and 2 are roots of $x^2 - 3x + 2 = 0$

4.2 Solving quadratic polynomials

4.2.1 Quadratic polynomials

Quadratic polynomials are of the form $ax^2 + bx + c = 0$.

4.2.2 Solving quadratic polynomials

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

4.2.3 Proof

We can get the two solutions to a quadratic equation from the following manipulation.

$ax^2 + bx + c = 0$

$a[x^2 + \frac{b}{a}x] = -c$

$a[(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}] = -c$

$a[(x + \frac{b}{2a})^2] = \frac{b^2}{4a} - c$

$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$

$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

4.3 Solving cubic polynomials

4.3.1 Cubic polynomials

Cubic polynomials are of the form $ax^3 + bx^2 + cx + d = 0$.

4.3.2 Solving specific cases

We start by solving when $b = 0$, that is:

$aX^3 + bx + c = 0$

4.3.3 Solving the general case

5 Multi-variable polynomials

6 Generating functions

6.1 Generating functions

6.1.1 Definition

A series can be described as:

$\sum_{i=0}^{\infty} s_i x^i$

If we know the function equal to this series, we can identify the ith number.

6.2 Fibonacci sequence

6.2.1 The generating function

Let's use a generating function to create a function for the Fibonacci sequence's cth digit. $F(c) = \sum_{i=c} x^i s_i$

Let's look at it for other starts:

$F(c+k) = \sum_{i=c} x^{i+k} s_{i+k}$

$F(c+k) = \sum_{i=c+k} x^i s_i$

$F(c+1) = \sum_{i=c} x^{i+1} s_{i+1}$

$F(c+2) = \sum_{i=c} x^{i+2} s_{i+2}$

This means

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^i s_i x^2 + \sum_{i=c} x^{i+1} s_{i+1} x$

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} s_i + \sum_{i=c} x^{i+2} s_{i+1}$

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_i + s_{i+1})$

6.2.2 Using the definiton of the Fibonacci sequence

From the definition of the fibonacci sequence, $s_i + s_{i+1} = s_{i+2}$.

$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_{i+2})$

$F(c)x^2 + F(c+1)x = F(c+2)$

6.2.3 Reducing the functions

Next, we expand out $F(c+1)$ and $F(c+2)$.

$F(c) - F(c+k) = \sum_{i=c} x^i s_i - \sum_{i=c+k} x^i s_i$

$F(c) - F(c+k) = \sum_{i=c}^{c+k} x^i s_i$

$F(c+k) = F(c) - \sum_{i=c}^{c+k} x^i s_i$

So:

$F(c+1) = F(c) - \sum_{i=c}^{c+1} x^i s_i$

$F(c+1) = F(c) - x^c s_c$

$F(c+2) = F(c) - \sum_{i=c}^{c+2} x^i s_i$

$F(c+2) = F(c) - x^{c+1} s_{c+1} - x^c s_c$

Let's take our previous equation

$F(c)x^2 + F(c+1)x = F(c+2)$

$F(c)x^2 + [F(c) - x^c s_c]x = F(c) - x^{c+1} s_{c+1} - x^c s_c$

$F(c)x^2 + F(c)x - x^{c+1} s_c = F(c) - x^{c+1} s_{c+1} - x^c s_c$

$F(c)[x^2 + x - 1] = x^{c+1} s_c - x^{c+1} s_{c+1} - x^c s_c$

$F(c) = \frac{x^c s_c + x^{c+1} s_{c+1} - x^{c+1} s_c}{1 - x - x^2}$

6.2.4 Using the first element in the sequence

For the start of the sequence, $c = 0$, $s_0 = s_1 = 1$.

$F(0) = \frac{x^0 1 + x - x}{1 - x - x^2}$

$F(0) = \frac{1}{1 - x - x^2}$

Let's factorise this:

$F(0) = \frac{-1}{(x + \frac{1}{2} + \frac{\sqrt{5}}{2})(x + \frac{1}{2} - \frac{\sqrt{5}}{2})}$

We can then use partial fraction decomposition

$\frac{1}{(M+\delta)(M-\delta)} = \frac{1}{2\delta}[\frac{1}{M-\delta} - \frac{1}{M+\delta}]$

To show that

$F(0) = \frac{-1}{\sqrt{5}}[\frac{1}{x + \frac{1}{2} - \frac{\sqrt{5}}{2}} - \frac{1}{x + \frac{1}{2} + \frac{\sqrt{5}}{2}}]$

$F(0) = \frac{-1}{\sqrt{5}}[\frac{\frac{1}{2} + \frac{\sqrt{5}}{2}}{(x + \frac{1}{2} - \frac{\sqrt{5}}{2})(\frac{1}{2} + \frac{\sqrt{5}}{2})} - \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{(x + \frac{1}{2} + \frac{\sqrt{5}}{2})(\frac{1}{2} - \frac{\sqrt{5}}{2})}]$

$F(0) = \frac{-1}{\sqrt{5}}[\frac{\frac{1}{2} + \frac{\sqrt{5}}{2}}{x(\frac{1}{2} + \frac{\sqrt{5}}{2}) - 1} - \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{x(\frac{1}{2} - \frac{\sqrt{5}}{2}) - 1}]$

$F(0) = \frac{1}{\sqrt{5}}[(\frac{1}{2} + \frac{\sqrt{5}}{2})\frac{1}{1 - x(\frac{1}{2} + \frac{\sqrt{5}}{2})} - (\frac{1}{2} - \frac{\sqrt{5}}{2})\frac{1}{1 - x(\frac{1}{2} - \frac{\sqrt{5}}{2})}]$

6.2.5 Finishing off

As we know

$\frac{1}{1-x} = \sum_{i=0} x^i$

So

$F(0) = \frac{1}{\sqrt{5}}[(\frac{1}{2} + \frac{\sqrt{5}}{2})\sum_{i=0} x^i(\frac{1}{2} + \frac{\sqrt{5}}{2})^i - (\frac{1}{2} - \frac{\sqrt{5}}{2})\sum_{i=0} x^i(\frac{1}{2} - \frac{\sqrt{5}}{2})^i]$

$F(0) = \frac{1}{\sqrt{5}}[\sum_{i=0} x^i(\frac{1}{2} + \frac{\sqrt{5}}{2})^{i+1} - \sum_{i=0} x^i(\frac{1}{2} - \frac{\sqrt{5}}{2})^{i+1}]$

$F(0) = \frac{1}{\sqrt{5}}\sum_{i=0} x^i[(\frac{1}{2} + \frac{\sqrt{5}}{2})^{i+1} - (\frac{1}{2} - \frac{\sqrt{5}}{2})^{i+1}]$

So the nth number in the sequence (treating $n = 1$ as the first number) is:

$\frac{1}{\sqrt{5}}[(\frac{1}{2} + \frac{\sqrt{5}}{2})^n - (\frac{1}{2} - \frac{\sqrt{5}}{2})^n]$

7 Other

7.1 Cardinality

7.1.1 Cardinality of cartesian product

What about the cardinality of Cartesian products? So if we have sets:

$\{1, 2, 3\}$

$\{a, b\}$

We can have the Cartesian product set:

$\{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$

We can see that:

$|A.B| = |A|.|B|$

7.1.2 Cardinality of union and intersection

$|A \vee B| = |A| + |B| - |A \wedge B|$

7.1.3 Cardinality of powerset

$|P(s)| = 2^{|s|}$

7.1.4 Cardinality of complement

$|a \setminus b| = |a| - |a \wedge b|$

7.1.5 Cardinality of even/odd natural numbers

What about the cardinality of even numbers? Well, we can define a bijective function between each:

$f(n) = 2n$

Similarly for odd numbers:

$f(n) = 2n + 1$

So these both have cardinality $\aleph_0$.