0.1 Cosets and normal subgroups

A coset is defined between a group and a subgroup of the group.

For a group G, and its subgroup H:

- The left coset is $\{gH\}$
- The right coset is $\{Hg\}$

For $\forall g \in G$.

For abellian groups, the left and right cosets are the same.

The left and right cosets can also be the same, even if the group G is not abelian.

0.1.1 Normal subgroups

If the left and right cosets are the same then H is a normal subgroup.

0.1.2 Cosets divide a group.

Consider two left cosets, aH and bH, with a common element.

This means that $ah_i = bh_j$.

We can use this to get:

$a = bh_j h_i^{-1}$

$b = ah_i h_j^{-1}$

We know that:

$ah \in aH$

$bh \in bH$

So:

$bh_j h_i^{-1} h \in aH$

$ah_i h_j^{-1} h \in bH$

And so:

$bH \subset aH$

$aH \subset bH$

Therefore:

$aH = bH$

0.1.3 Example 1

Consider the group $\{-1, 1\}, \times$

For the subgroup $\{1\}, \times$, the left coset is $\{gH\} = \{1, -1\}$.

The right coset is the same.

0.1.4 Example 2

Consider the group of integers and addition: $(Z, +)$

For subgroup $(mZ, +)$, the left and right cosets are the same because the group is abelian.

The coset of the subgroup is the subgroup multiplied by each element in G.

This is mZ, $mZ + 1$, $mZ + 2$ and so on.

Once we reach $mZ + m$ this has looped, and is already a coset, so we only need the sets upto $mZ + m - 1$.