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0.1 Fibonacci sequence

0.1.1 The generating function

Let's use a generating function to create a function for the Fibonacci sequence's c th digit. $F(c) = \sum_{i=c} x^i s_i$

Let's look at it for other starts:

$$F(c+k) = \sum_{i=c} x^{i+k} s_{i+k}$$

$$F(c+k) = \sum_{i=c+k} x^i s_i$$

$$F(c+1) = \sum_{i=c} x^{i+1} s_{i+1}$$

$$F(c+2) = \sum_{i=c} x^{i+2} s_{i+2}$$

This means

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^i s_i x^2 + \sum_{i=c} x^{i+1} s_{i+1} x$$

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} s_i + \sum_{i=c} x^{i+2} s_{i+1}$$

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_i + s_{i+1})$$

0.1.2 Using the definition of the Fibonacci sequence

From the definition of the fibonacci sequence, $s_i + s_{i+1} = s_{i+2}$.

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_{i+2})$$

$$F(c)x^2 + F(c+1)x = F(c+2)$$

0.1.3 Reducing the functions

Next, we expand out $F(c+1)$ and $F(c+2)$.

$$F(c) - F(c+k) = \sum_{i=c} x^i s_i - \sum_{i=c+k} x^i s_i$$

$$F(c) - F(c+k) = \sum_{i=c}^{c+k} x^i s_i$$

$$F(c+k) = F(c) - \sum_{i=c}^{c+k} x^i s_i$$

So:

$$F(c+1) = F(c) - \sum_{i=c}^{c+1} x^i s_i$$

$$F(c+1) = F(c) - x^c s_c$$

$$F(c+2) = F(c) - \sum_{i=c}^{c+2} x^i s_i$$

$$F(c+2) = F(c) - x^{c+1} s_{c+1} - x^c s_c$$

Let's take our previous equation

$$F(c)x^2 + F(c+1)x = F(c+2)$$

$$F(c)x^2 + [F(c) - x^c s_c]x = F(c) - x^{c+1} s_{c+1} - x^c s_c$$

$$F(c)x^2 + F(c)x - x^{c+1} s_c = F(c) - x^{c+1} s_{c+1} - x^c s_c$$

$$F(c)[x^2 + x - 1] = x^{c+1} s_c - x^{c+1} s_{c+1} - x^c s_c$$

$$F(c) = \frac{x^c s_c + x^{c+1} s_{c+1} - x^{c+1} s_c}{1 - x - x^2}$$

0.1.4 Using the first element in the sequence

For the start of the sequence, $c = 0$, $s_0 = s_1 = 1$.

$$F(0) = \frac{x^0 1 + x - x}{1 - x - x^2}$$

$$F(0) = \frac{1}{1 - x - x^2}$$

Let's factorise this:

$$F(0) = \frac{-1}{(x + \frac{1}{2} + \frac{\sqrt{5}}{2})(x + \frac{1}{2} - \frac{\sqrt{5}}{2})}$$

We can then use partial fraction decomposition

$$\frac{1}{(M+\delta)(M-\delta)} = \frac{1}{2\delta} \left[\frac{1}{M-\delta} - \frac{1}{M+\delta} \right]$$

To show that

$$F(0) = \frac{-1}{\sqrt{5}} \left[\frac{1}{x + \frac{1}{2} - \frac{\sqrt{5}}{2}} - \frac{1}{x + \frac{1}{2} + \frac{\sqrt{5}}{2}} \right]$$

$$F(0) = \frac{-1}{\sqrt{5}} \left[\frac{\frac{1}{2} + \frac{\sqrt{5}}{2}}{(x + \frac{1}{2} - \frac{\sqrt{5}}{2})(\frac{1}{2} + \frac{\sqrt{5}}{2})} - \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{(x + \frac{1}{2} + \frac{\sqrt{5}}{2})(\frac{1}{2} - \frac{\sqrt{5}}{2})} \right]$$

$$F(0) = \frac{-1}{\sqrt{5}} \left[\frac{\frac{1}{2} + \frac{\sqrt{5}}{2}}{x(\frac{1}{2} + \frac{\sqrt{5}}{2}) - 1} - \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{x(\frac{1}{2} - \frac{\sqrt{5}}{2}) - 1} \right]$$

$$F(0) = \frac{1}{\sqrt{5}} \left[\left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right) \frac{1}{1 - x(\frac{1}{2} + \frac{\sqrt{5}}{2})} - \left(\frac{1}{2} - \frac{\sqrt{5}}{2} \right) \frac{1}{1 - x(\frac{1}{2} - \frac{\sqrt{5}}{2})} \right]$$

0.1.5 Finishing off

As we know

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

So

$$F(0) = \frac{1}{\sqrt{5}} \left[\left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right) \sum_{i=0}^{\infty} x^i \left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right)^i - \left(\frac{1}{2} - \frac{\sqrt{5}}{2} \right) \sum_{i=0}^{\infty} x^i \left(\frac{1}{2} - \frac{\sqrt{5}}{2} \right)^i \right]$$

$$F(0) = \frac{1}{\sqrt{5}} \left[\sum_{i=0}^{\infty} x^i \left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{i+1} - \sum_{i=0}^{\infty} x^i \left(\frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{i+1} \right]$$

$$F(0) = \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} x^i \left[\left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{i+1} - \left(\frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{i+1} \right]$$

So the n th number in the sequence (treating $n = 1$ as the first number) is:

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right)^n - \left(\frac{1}{2} - \frac{\sqrt{5}}{2} \right)^n \right]$$