# 1 Partial differentiation

## 1.1 The partial differential operator

### 1.1.1 Differential

When we change the value of an input to a function, we also change the output. We can examine these changes.

Consider the value of a function $f(x)$ at points $x_1$ and $x_2$.

$y_1 = f(x_1)$

$y_2 = f(x_2)$

$y_2 - y_1 = f(x_2) - f(x_1)$

$\frac{y_2-y_1}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Let's define $x_2$ in terms of its distance from x_1:

$x_2 = x_1 + \epsilon$

$\frac{y_2-y_1}{\epsilon} = \frac{f(x_1+\epsilon)-f(x_1)}{\epsilon}$

We define the differential of a function as:

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)-f(x)}{\epsilon}$

If this is defined, then we say the function is differentiable at that point.

### 1.1.2 Differential operator

## 1.2 Differentiating constants, the identity function, and linear functions

### 1.2.1 Constants

$f(x) = c$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)-f(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{c-c}{\epsilon} = 0$

### 1.2.2 $x$

$f(x) = x$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)-f(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{x+\epsilon-x}{\epsilon} = 1$

### 1.2.3 Addition

$f(x) = g(x) + h(g)$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon)+h(x+\epsilon)-g(x)-h(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon)-g(x)}{\epsilon} + \lim_{\epsilon \to 0^+} \frac{h(x+\epsilon)-h(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \frac{\delta g}{\delta x} + \frac{\delta h}{\delta x}$

## 1.3 Partial differentiation is a linear operator

### 1.3.1 Intro

## 1.4 The chain rule, the product rule and the quotient rule

### 1.4.1 Chain rule

$f(x) = f(g(x))$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(g(x+\epsilon))-f(g(x))}{\epsilon}$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon)-g(x)}{g(x+\epsilon)-g(x)} \frac{f(g(x+\epsilon))-f(g(x))}{\epsilon}$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} \frac{g(x+\epsilon)-g(x)}{\epsilon} \frac{f(g(x+\epsilon))-f(g(x))}{g(x+\epsilon)-g(x)}$

$\frac{\delta f}{\delta x} = \lim_{\epsilon \to 0^+} [\frac{g(x+\epsilon)-g(x)}{\epsilon}] \lim_{\epsilon \to 0^+} [\frac{f(g(x+\epsilon))-f(g(x))}{g(x+\epsilon)-g(x)}]$

$\frac{\delta f}{\delta x} = \frac{\delta g}{\delta x} \frac{\delta f}{\delta g}$

### 1.4.2 Product rule

$y = f(x)g(x)$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)g(x+\epsilon)-f(x)g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)g(x+\epsilon)-f(x)g(x+\epsilon)+f(x)g(x+\epsilon)-f(x)g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon)g(x+\epsilon)-f(x)g(x+\epsilon)}{\epsilon} + \lim_{\epsilon \to 0^+} \frac{f(x)g(x+\epsilon)-f(x)g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = \lim_{\epsilon \to 0^+} g(x+\epsilon) \frac{f(x+\epsilon)-f(x)}{\epsilon} + \lim_{\epsilon \to 0^+} f(x) \frac{g(x+\epsilon)-g(x)}{\epsilon}$

$\frac{\delta y}{\delta x} = g(x) \frac{\delta f}{\delta x} + f(x) \frac{\delta g}{\delta x}$

### 1.4.3 Quotient rule

$y = \frac{f(x)}{g(x)}$

$\frac{\delta}{\delta x}y = \frac{\delta}{\delta x}\frac{f(x)}{g(x)}$

$\frac{\delta}{\delta x}y = \frac{\delta}{\delta x}f(x)\frac{1}{g(x)}$

$\frac{\delta}{\delta x}y = \frac{\delta f}{\delta x}\frac{1}{g(x)} - \frac{\delta g}{\delta x}\frac{f(x)}{g(x)^2}$

$\frac{\delta}{\delta x}y = \frac{\frac{\delta f}{\delta x}g(x) - \frac{\delta g}{\delta x}f(x)}{g(x)^2}$

## 1.5 Differentiating natural number power functions

### 1.5.1 Other

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0}\frac{(x+\delta)^n - x^n}{\delta}$

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0}\frac{(\sum_{i=0}^{n} x^i \delta^{n-i}\frac{n!}{i!(n-i)!}) - x^n}{\delta}$

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0}\sum_{i=0}^{n-1} x^i \delta^{n-i-1}\frac{n!}{i!(n-i)!}$

$\frac{\delta}{\delta x}x^n = \lim_{\delta \to 0}x^{n-1}\frac{n!}{(n-1)!(n-n+1)!} + \sum_{i=0}^{n-2} x^i \delta^{n-i-1}\frac{n!}{i!(n-i)!}$

$\frac{\delta}{\delta x}x^n = nx^{n-1}$

## 1.6 L'Hôpital's rule

### 1.6.1 L'Hôpital's rule

If there are two functions which are both tend to 0 at a limit, calculating the limit of their divisor is hard. We can use L'Hopital's rule.

We want to calculate:

$\lim_{x \to c}\frac{f(x)}{g(x)}$

This is:

$\lim_{x \to c}\frac{f(x)}{g(x)} = \lim_{x \to c}\frac{\frac{f(x)-0}{\delta}}{\frac{g(x)-0}{\delta}}$

If:

$\lim_{x \to c}f(x) = \lim_{x \to c}g(x) = 0$

Then

$\lim_{x \to c}\frac{f(x)}{g(x)} = \lim_{x \to c}\frac{\frac{f(x)-f(c)}{\delta}}{\frac{g(x)-f(c)}{\delta}}$

$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$

## 1.7 Rolle's theorem

### 1.7.1 Rolle's theorem

Take a real function $f(x)$ on closed interval $[a, b]$, differentiable on $(a, b, )$, and $f(a) = f(b)$.

Rolle's theorem states that:

$\exists c \in (a, b)(f'(c) = 0)$

Generalised Rolle's theorem states that:

Generalised Rolle's theorem implies Rolle's theorem, so we only need to prove the generalised theorem.

## 1.8 Mean value theorem

### 1.8.1 Mean value theorem

Take a real function $f(x)$ on closed interval $[a, b]$, differentiable on $(a, b, )$.

The mean value theorem states that:

$\exists c \in (a, b)(f'(c) = \frac{f(b)-f(a)}{b-a})$

## 1.9 Elasticity

### 1.9.1 Introduction

We have (x)

$Ef(x) = \frac{x}{f(x)}\frac{\delta f(x)}{\delta x}$

This is the same as:

$Ef(x) = \frac{\delta \ln f(x)}{\delta \ln x}$

## 1.10 Smooth functions

## 1.11 Analytic function

### 1.11.1 Introduction

# 2 Higher-order differentials

## 2.1 Differentiable functions

### 2.1.1 Introduction

A differentiable function is one where the differential is defined at all points on the real line.

All differentiable functions are continuous. Not all continuous functions are differentiable.

### 2.1.2 Differentiability class

We can describe a function with its differentiability class. If a function can be differentiated $n$ times and these differentials are all continous, then the function is class $C^n$.

### 2.1.3 Smooth functions

If a function can be differentiated infinitely many times to produce continous functions, it is $C^\infty$, or smooth.

## 2.2 Critial points

### 2.2.1 Critical points

Where partial derivative are 0.

# 3 Exponentials

## 3.1 Defining $e$ as a binomial

### 3.1.1 Lemma

$f(n,i) = \frac{n!}{n^i(n-i)!}$

$f(n,i) = \frac{(n-i)!\prod_{j=n-i+1}^{n} j}{n^i(n-i)!}$

$f(n,i) = \frac{\prod_{j=n-i+1}^{n} j}{n^i}$

$f(n,i) = \frac{\prod_{j=1}^{i}(j+n-i)}{n^i}$

$f(n,i) = \prod_{j=1}^{i} \frac{j+n-i}{n}$

$f(n,i) = \prod_{j=1}^{i}(\frac{n}{n} + \frac{j-i}{n})$

$f(n,i) = \prod_{j=1}^{i}(1 + \frac{j-i}{n})$

$\lim_{n\to\infty} f(n,i) = \lim_{n\to\infty} \prod_{j=1}^{i}(1 + \frac{j-i}{n})$

$\lim_{n\to\infty} f(n,i) = \prod_{j=1}^{i} 1$

$\lim_{n\to\infty} f(n,i) = 1$

### 3.1.2 Defining $e$

We know that:

$(a+b)^n = \sum_{i=0}^{n} a^i b^{n-i} \frac{n!}{i!(n-i)!}$

Let's set $b = 1$

$(a+1)^n = \sum_{i=0}^{n} a^i \frac{n!}{i!(n-i)!}$

Let's set $a = \frac{1}{n}$

$(1+\frac{1}{n})^n = \sum_{i=0}^{n} \frac{1}{n^i} \frac{n!}{i!(n-i)!}$

$(1+\frac{1}{n})^n = \sum_{i=0}^{n} \frac{1}{i!} \frac{n!}{n^i(n-i)!}$

$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} \frac{n!}{n^i(n-i)!}$

From the lemma above:

$\lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{i=0}^{\infty} \frac{1}{i!}$

$e = \sum_{i=0}^{\infty} \frac{1}{i!}$

### 3.1.3 Defining $e^x$

$e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$

$e^x = \lim_{n\to\infty}(1 + \frac{1}{n})^{nx}$

$e^x = \lim_{n\to\infty} \sum_{i=0}^{nx} \frac{1}{n^i} \frac{(nx)!}{i!(nx-i)!}$

$e^x = \lim_{n\to\infty} \sum_{i=0}^{nx} \frac{x^i}{i!} \frac{(nx)!}{(nx)^i(nx-i)!}$

From the lemma:

$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

## 3.2 Differentiating $e^x$

### 3.2.1 Intro

We have $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \frac{\delta}{\delta x} \sum_{i=0}^{\infty} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=0}^{\infty} \frac{\delta}{\delta x} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=1}^{\infty} \frac{\delta}{\delta x} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=1}^{\infty} \frac{x^{i-1}}{(i-1)!}$

$\frac{\delta}{\delta x} e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

$\frac{\delta}{\delta x} e^x = e^x$

## 3.3 Differentiating exponents, logarithms and power functions

### 3.3.1 Differentiating the natural logarithm

$\frac{\delta}{\delta x} \ln(x) = \lim_{\delta\to 0} \frac{\ln(x+\delta) - ln(x)}{\delta}$

$\frac{\delta}{\delta x} \ln(x) = \lim_{\delta\to 0} \frac{\ln \frac{x+\delta}{x}}{\delta}$

$\frac{\delta}{\delta x} \ln(x) = \lim_{\delta\to 0} \frac{\ln(1+\frac{\delta}{x})}{\delta}$

$\frac{\delta}{\delta x} \ln(x) = \frac{1}{x} \lim_{\delta\to 0} \frac{x}{\delta} \ln(1 + \frac{\delta}{x})$

$\frac{\delta}{\delta x} \ln(x) = \frac{1}{x} \ln(\lim_{\delta\to 0}(1 + \frac{\delta}{x})^{\frac{x}{\delta}})$

$\frac{\delta}{\delta x} \ln(x) = \frac{1}{x} \ln(e)$

$\frac{\delta}{\delta x}\ln(x) = \frac{1}{x}$

### 3.3.2 Differentiating logarithms of other bases

$log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

$log_a(x) = \frac{\ln(x)}{\ln(a)}$

$\frac{\delta}{\delta x}\log_a(x) = \frac{\delta}{\delta x}\frac{\ln(x)}{ln(a)}$

$\frac{\delta}{\delta x}\log_a(x) = \frac{1}{xln(a)}$

### 3.3.3 Exponents

$y = a^x$

$\ln(y) = x\ln(a)$

$\frac{\delta}{\delta x}\ln(y) = \frac{\delta}{\delta x}x\ln(a)$

$\frac{\delta}{\delta x}\ln(y) = \ln(a)$

$\frac{1}{y}\frac{\delta}{\delta x}y = \ln(a)$

$\frac{\delta}{\delta x}a^x = a^x\ln(a)$

### 3.3.4 Power functions

$y = x^n$

$\frac{\delta}{\delta x}y = \frac{\delta}{\delta x}x^n$

$\frac{\delta}{\delta x}y = \frac{\delta}{\delta x}e^{n\ln(x)}$

$\frac{\delta}{\delta x}y = \frac{n}{x}e^{n\ln(x)}$

$\frac{\delta}{\delta x}y = nx^{n-1}$

# 4 Integration

## 4.1 Riemann integral

### 4.1.1 Riemann sums

Given a function $f(x)$ and an interval $[a,b]$, we can divide $[a,b]$ into $n$ sections and calculate:

$\sum_{j=0}^{n(b-a)} f(a + \frac{j}{n})$

This is the Riemann sum.

### 4.1.2 Riemann integral

We take the limit of the Riemann sum as $n \to \infty$

$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a + \frac{j}{n})$

### 4.1.3 Linearity

$\int_a^b f(x) + g(x)dx = \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a + \frac{j}{n}) + g(a + \frac{j}{n})$

$\int_a^b f(x) + g(x)dx = \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a + \frac{j}{n}) + \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} g(a + \frac{j}{n})$

$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

### 4.1.4 Continuation

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty} \sum_{j=0}^{n(b-a)} f(a + \frac{j}{n}) + \lim_{n\to\infty} \sum_{j=0}^{n(c-b)} f(b + \frac{j}{n})$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty}[\sum_{j=0}^{n(b-a)} f(a + \frac{j}{n}) + \sum_{j=0}^{n(c-b)} f(b + \frac{j}{n})]$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty}[\sum_{j=0}^{n(b-a)} f(a + \frac{j}{n}) + \sum_{j=n(b-a)}^{n(c-b)+n(b-a)} f(b + \frac{j-n(b-a)}{n})]$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty}[\sum_{j=0}^{n(b-a)} f(a + \frac{j}{n}) + \sum_{j=n(b-a)}^{n(c-a)} f(a + \frac{j}{n})]$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \lim_{n\to\infty}[\sum_{j=0}^{n(c-a)} f(a + \frac{j}{n})$

$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

## 4.2 Definite and indefinite integrals

### 4.2.1 Introduction

## 4.3 Anti-derivative

### 4.3.1 Content

## 4.4 Integration by parts

### 4.4.1 Integration by parts

We have:

$\frac{\delta y}{\delta x} = f(x)g(x)$

We want that in terms of $y$.

We know from the product rule of differentiation:

$y = a(x)b(x)$

Means that:

$\frac{\delta y}{\delta x} = a'(x)b(x) + a(x)b'(x)$

So let's relabel $f(x)$ as $h'(x)$

$\delta$

$\frac{\delta y}{\delta x} = h'(x)g(x)$

$\frac{\delta y}{\delta x} + h(x)g'(x) = h'(x)g(x) + h(x)g'(x)$

$y + \int h(x)g'(x) = \int h'(x)g(x) + h(x)g'(x)$

$y + \int h(x)g'(x) = h(x)g(x)$

$y = h(x)g(x) - \int h(x)g'(x)$

For example:

$\frac{\delta y}{\delta x} = x. \cos(x)$

$f(x) = cos(x)$

$g(x) = x$

$h(x) = sin(x)$

$g'(x) = 1$

So:

$y = x \int cos(x)dx - \int sin(x)dx$

$y = xsin(x) - cos(x) + c$

## 4.5 Integrals

### 4.5.1 Methods of integration

Trigonometric substitution

Inverse function integration function integration

Anti-derivative

Taking the derivative of a function provides another function. The anti-derivative of a function is a function which, when differentiated, provides the original function.

As this function can include any additive constant, there are an infinite number of anti-derivatives for any function.

Integration

Limit of summation

Show same as anti-derivative

Show properties of function same as summation (can take constants out etc)

### 4.5.2 Getting functions from derivatives

$f(c) = f(a) + \int_a^c \frac{\delta}{\delta x} f(x)dx$

Definite integration

Indefinite integration

## 4.6 Fundamental Theorem of Calculus

### 4.6.1 Mean value theorem for integration

Take function $f(x)$. From the extreme value theorem we know that:

$\exists m \in \mathbb{R} \exists M \in \mathbb{R} \forall x \in [a, b](m < f(x) < M)$

### 4.6.2 Fundamental theorem of calculus

From continuation we know that:

$\int_a^{x_1} f(x)dx + \int_{x_1}^{x_1+\delta x} f(x)dx = \int_a^{x_1+\delta x} f(x)dx$

$\int_{x}^{x_1+\delta x} f(x)dx = \int_{a}^{x_1+\delta} f(x)dx - \int_{a}^{x_1} f(x)dx$

Indefinite integrals