Contents

1 Topological manifolds

1.1 Introduction

1.1.1 Manifolds, charts and atlases

A manifold is a set of points and associated charts.

A chart is a mapping from each point in a subset of the manifold to a point in a vector space.

These charts are invertible. If we are given coordinates, we can identify the point in the manifold it comes from.

1.1.1.1 Example: The sphere

We can map a hemisphere to a subset of R^2. Given a point in R^2 we can identify a specific point on the hemisphere, and given a s specific point on the hemisphere we can identify a point in R^2.

1.1.1.2 Universal charts

If the vector space is flat and non-repeating, then a single chart can be used to map the whole manifold.

1.1.1.3 Atlases

If we have a collection of charts which covers each point needs to be covered at least once, we have an atlas. Each chart needs to be to the same dimensional vector space.

1.1.2 Transition maps

Where two charts overlap we can express the points where the charts overlap as two different coordinates.

We can express the mapping from these coordinates as a function. This is a transition map.

1.1.2.1 Overlapping charts

If two charts cover some of the same points on a manifold then we can define a function for those points where we move from one vector to another.

We can represent moving between charts as:

ab^{-1}

1.1.3 Mapping 2D manifolds to Riemann surfaces

Needs to be orientable and metricisable.

1.2 Dimension theory

1.2.1 Refinement

1.2.2 Ply (order) of a cover

1.2.3 Small inductive dimension

1.2.4 Large inductive dimension

1.2.5 Lebesgue covering dimension

1.3 Paths

1.3.1 Paths and loops

1.3.1.1 Paths

We have the set X. We define a mapping $[0,1] \to X$

If a path exists between any two points, then the space is path-connected.

1.3.1.2 Loops

This is a path which ends on itself.

If $f(0) = f(1)$ then it is a loop.

1.3.2 Holes and genuses

1.3.2.1 Holes

1.3.2.2 Genes

The genus of a topology is the number of holes in the topology.

1.3.3 Path-connect spaces

1.4 Simply-connected 2D manifolds

1.4.1 Elliptic (Riemann sphere)

1.4.2 Parabolic (complex plane)

1.4.3 Hyperbolic (open disk)

1.5 Not simply-connected 2D manifolds

1.5.1 Torus

1.5.2 Hyper-elliptic curves

1.6 Functions between topologies

1.6.1 Functions between topologies

We can define a function from topology to another.

$f(X) = Y$

1.6.1.1 Continuous functions between topologies

If $f(X)$ is continous, then we have a continous function between topologies.

1.6.1.2 Inverse functions between topologies

If $f(X)$ is invertible then there is a inverse mapping.

1.6.2 Homotopy

1.6.3 Homeomorphisms

If there is a mapping which is invertible and continuous, it is a homeomorphism.

1.7 Fibre bundles

1.7.1 Vector bundles

A vector bundle consists of a base manifold (a base space), and a real vector space at each point in the base manifold.

1.7.1.1 Example

For example we can have a base manifold of a circle, and have a 1-dimensional vector space at each point on the circle to create an infinitely extended cylinder.

1.7.2 Bundle projection

This is a projection from any point on any of the fibres, to the underlying base manifold.

1.7.3 Trivial and twisted bundles

1.7.4 Cross-sections and zero-sections of fibre bundles

1.7.5 Trivial bundles and the torus

1.7.5.1 Trivial bundles

1.7.5.2 The torus

$S_1 \times S_1$

1.7.6 Twisted bundles and the Klein bottle

1.7.6.1 Twisted bundles

1.7.6.2 Klein bottles

$S_1 \times S_1$, but twisted

1.7.7 Mobius strips

$S_1 \times$ line segment.

1.8 Other

1.8.1 Submanifolds

Submanifold: subset of manifold which is also manifold

Eg: circle inside a sphere

1.8.2 Boundries and interiors

Around every manifold of dimension n is a boundry of dimension $(n-1)$.

Homeomorphism at boundry: one coordinate always ≥ 0. reduced dimension.

Interior is rest.

1.8.3 Embeddings and immersions

Whitney embedding theroem: all manfiolds can be embedded in R^n space for some n