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## 0.1 Defining sine and cosine using Euler's formula

### 0.1.1 Euler's formula

Previously we showed that:

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Consider:

$$e^{i\theta}$$

$$e^{i\theta} = \sum_{j=0}^{\infty} \frac{(i\theta)^j}{j!}$$

$$e^{i\theta} = [\sum_{j=0}^{\infty} \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+2}}{(4j+2)!}] + i[\sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+3}}{(4j+3)!}]$$

We then use this to define sin and cos functions.

$$\cos(\theta) := \sum_{j=0}^{\infty} \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+2}}{(4j+2)!}$$

$$\sin(\theta) := \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+3}}{(4j+3)!}$$

So:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

### 0.1.2 Alternative formulae for sine and cosine

We know

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$e^{-i\theta} = \cos(\theta) - i \sin(\theta)$$

So

$$e^{i\theta} + e^{-i\theta} = \cos(\theta) + i \sin(\theta) + \cos(\theta) - i \sin(\theta)$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And

$$e^{i\theta} - e^{-i\theta} = \cos(\theta) + i \sin(\theta) - \cos(\theta) + i \sin(\theta)$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

### **0.1.3 Sine and cosine are odd and even functions**

Sine is an odd function.

$$\sin(-\theta) = -\sin(\theta)$$

Cosine is an even function.

$$\cos(-\theta) = \cos(\theta)$$