

# Contents

<b>1</b>	<b>Elementary algebra</b>	<b>1</b>
1.1	Powers . . . . .	1
1.1.1	Powers . . . . .	1
1.1.2	Binomial expansion . . . . .	3
1.1.3	Difference of two squares . . . . .	3
1.2	Irrational numbers . . . . .	4
1.2.1	Logarithms . . . . .	4
1.2.2	Logarithms for natural numbers . . . . .	5
1.2.3	Logarithms for integers . . . . .	5
1.2.4	Logarithms for rational numbers . . . . .	5
1.3	Equations . . . . .	5
1.3.1	Algebraic equations . . . . .	5
1.4	Single-variable polynomials . . . . .	5
1.4.1	Single-variable polynomials . . . . .	5
1.4.2	Solving quadratic polynomials . . . . .	5
1.4.3	Solving cubic polynomials . . . . .	6
1.5	Multi-variable polynomials . . . . .	6
1.6	Generating functions . . . . .	6
1.6.1	Generating functions . . . . .	6
1.6.2	Fibonacci sequence . . . . .	7
1.7	Other . . . . .	9
1.7.1	Cardinality . . . . .	9

## 1 Elementary algebra

### 1.1 Powers

#### 1.1.1 Powers

##### 1.1.1.1 Exponents and logarithms

Previously we defined addition and multiplication in terms of successive use of the successor function. That is, the definition of addition was:

$$\forall a \in \mathbb{N}(a + 0 = a)$$

$$\forall ab \in \mathbb{N}(a + s(b) = s(a + b))$$

And similarly for multiplication:

$$\forall a \in \mathbb{N}(a.0 = 0)$$

$$\forall ab \in \mathbb{N}(a.s(b) = a.b + a)$$

Additional functions could also be defined, following the same pattern:

$$\forall a \in \mathbb{N}(a \oplus_n 0 = c)$$

$$\forall ab \in \mathbb{N}(a \oplus_n s(b) = (a \oplus_n b) \oplus_{n-1} a)$$

#### 1.1.1.2 Powers

Exponents can also be defined:

#### 1.1.1.3 Axioms

$$\forall a \in \mathbb{N} a^0 = 1$$

$$\forall ab \in \mathbb{N} a^{s(b)} = a^b . a$$

#### 1.1.1.4 Example

So  $2^2$  can be calculated like:

$$2^2 = 2^{s(1)}$$

$$2^{s(1)} = 2.2^1$$

$$2.2^1 = 2.2.2^0$$

$$2.2.2^0 = 2.2.1$$

$$2.2.1 = 4$$

Unlike addition and multiplication, exponentiation is not commutative. That is

$$a^b \neq b^a$$

#### 1.1.1.5 Exponential rules

$$a^b a^c = a^{b+c}$$

$$(a^b)^c = a^{bc}$$

$$(ab)^c = a^c b^c$$

#### 1.1.1.6 Powers of natural numbers

#### 1.1.1.7 Powers of integers

#### 1.1.1.8 Powers of rational numbers

### 1.1.2 Binomial expansion

#### 1.1.2.1 Introduction

How can we expand

$$(a + b)^n, n \in \mathbb{N}$$

We know that:

$$(a + b)^n = (a + b)(a + b)^{n-1}$$

$$(a + b)^n = a(a + b)^{n-1} + b(a + b)^{n-1}$$

Each time this is done, the terms split, and each term is multiplied by either  $a$  or  $b$ . That means at the end there are  $n$  total multiplications.

This can be shown as:

$$(a + b)^n = \sum_{i=1}^n a^i b^{n-i} c_i$$

So we want to identify  $c_i$ .

Each term can be shown as a series of  $n$   $a$ s and  $b$ s. For example:

- $aaba$
- $baaa$

For any of these, there are  $n!$  ways of arranging the sequence, but this includes duplicates. If we were given  $n$  unique terms to multiply there would indeed be  $n!$  different ways this could have arisen, but we can swap  $a$ s and  $b$ s, as they were only generated once. So let's count duplicates.

There are duplicates in the  $a$ s. If there are  $i$   $a$ s, then there are  $i!$  ways of rearranging this. Similarly, if there are  $n - i$   $b$ s, then there are  $(n - i)!$  ways of arranging this.

As a result the number of actual observed instances,  $c_i$ , is:

$$c_i = \frac{n!}{i!(n-i)!}$$

And so:

$$(a + b)^n = \sum_{i=0}^n a^i b^{n-i} \frac{n!}{i!(n-i)!}$$

We can also write this last term as:

$$\binom{n}{i}$$

### 1.1.3 Difference of two squares

#### 1.1.3.1 Differences of two squares

$$(a + b)(a - b) = a^2 - ab + ab - b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

## 1.2 Irrational numbers

### 1.2.1 Logarithms

#### 1.2.1.1 Logarithms

If:

$$c = a^b$$

Then

$$\log_a c = b$$

Product rule:

$$a = c^{\log_c a}$$

$$b = c^{\log_c b}$$

So:

$$ab = c^{\log_c ab}$$

But also:

$$ab = c^{\log_c a} c^{\log_c b}$$

$$ab = c^{\log_c a + \log_c b}$$

So:

$$\log_c a + \log_c b = \log_c ab$$

#### 1.2.1.2 Power rule

$$a = b^{\log_b a}$$

So:

$$a^c = b^{\log_b a^c}$$

And separately:

$$a^c = (b^{\log_b a})^c$$

$$a^c = (b^{c \log_b a})$$

So:

$$c \log_b a = \log_b a^c$$

### **1.2.2 Logarithms for natural numbers**

### **1.2.3 Logarithms for integers**

### **1.2.4 Logarithms for rational numbers**

## **1.3 Equations**

### **1.3.1 Algebraic equations**

#### **1.3.1.1 Introduction**

## **1.4 Single-variable polynomials**

### **1.4.1 Single-variable polynomials**

#### **1.4.1.1 Introduction**

A single-variable polynomial is an equation of the form:

$$\sum_{i=0}^n a_i x^i = 0$$

For example:

- $x = 1$
- $x^2 = 4$
- $x^2 - 3x + 2 = 0$

#### **1.4.1.2 Degrees**

The degree of a polynomial is the highest-order term.

For example  $x^3 + x = 0$  has degree 3.

#### **1.4.1.3 Roots of single-variable polynomials**

A solution to a polynomial is a root.

For example 1 and 2 are roots of  $x^2 - 3x + 2 = 0$

### **1.4.2 Solving quadratic polynomials**

#### **1.4.2.1 Quadratic polynomials**

Quadratic polynomials are of the form  $ax^2 + bx + c = 0$ .

### 1.4.2.2 Solving quadratic polynomials

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### 1.4.2.3 Proof

We can get the two solutions to a quadratic equation from the following manipulation.

$$ax^2 + bx + c = 0$$

$$a[x^2 + \frac{b}{a}x] = -c$$

$$a[(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}] = -c$$

$$a[(x + \frac{b}{2a})^2] = \frac{b^2}{4a} - c$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### 1.4.3 Solving cubic polynomials

#### 1.4.3.1 Cubic polynomials

Cubic polynomials are of the form  $ax^3 + bx^2 + cx + d = 0$ .

#### 1.4.3.2 Solving specific cases

We start by solving when  $b = 0$ , that is:

$$aX^3 + bx + c = 0$$

#### 1.4.3.3 Solving the general case

## 1.5 Multi-variable polynomials

## 1.6 Generating functions

### 1.6.1 Generating functions

#### 1.6.1.1 Definition

A series can be described as:

$$\sum_{i=0}^{\infty} s_i x^i$$

If we know the function equal to this series, we can identify the  $i$ th number.

## 1.6.2 Fibonacci sequence

### 1.6.2.1 The generating function

Let's use a generating function to create a function for the Fibonacci sequence's  $c$ th digit.  $F(c) = \sum_{i=c} x^i s_i$

Let's look at it for other starts:

$$F(c+k) = \sum_{i=c} x^{i+k} s_{i+k}$$

$$F(c+k) = \sum_{i=c+k} x^i s_i$$

$$F(c+1) = \sum_{i=c} x^{i+1} s_{i+1}$$

$$F(c+2) = \sum_{i=c} x^{i+2} s_{i+2}$$

This means

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^i s_i x^2 + \sum_{i=c} x^{i+1} s_{i+1} x$$

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} s_i + \sum_{i=c} x^{i+2} s_{i+1}$$

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_i + s_{i+1})$$

### 1.6.2.2 Using the definition of the Fibonacci sequence

From the definition of the fibonacci sequence,  $s_i + s_{i+1} = s_{i+2}$ .

$$F(c)x^2 + F(c+1)x = \sum_{i=c} x^{i+2} (s_{i+2})$$

$$F(c)x^2 + F(c+1)x = F(c+2)$$

### 1.6.2.3 Reducing the functions

Next, we expand out  $F(c+1)$  and  $F(c+2)$ .

$$F(c) - F(c+k) = \sum_{i=c} x^i s_i - \sum_{i=c+k} x^i s_i$$

$$F(c) - F(c+k) = \sum_{i=c}^{c+k} x^i s_i$$

$$F(c+k) = F(c) - \sum_{i=c}^{c+k} x^i s_i$$

So:

$$F(c+1) = F(c) - \sum_{i=c}^{c+1} x^i s_i$$

$$F(c+1) = F(c) - x^c s_c$$

$$F(c+2) = F(c) - \sum_{i=c}^{c+2} x^i s_i$$

$$F(c+2) = F(c) - x^{c+1}s_{c+1} - x^c s_c$$

Let's take our previous equation

$$F(c)x^2 + F(c+1)x = F(c+2)$$

$$F(c)x^2 + [F(c) - x^c s_c]x = F(c) - x^{c+1}s_{c+1} - x^c s_c$$

$$F(c)x^2 + F(c)x - x^{c+1}s_c = F(c) - x^{c+1}s_{c+1} - x^c s_c$$

$$F(c)[x^2 + x - 1] = x^{c+1}s_c - x^{c+1}s_{c+1} - x^c s_c$$

$$F(c) = \frac{x^c s_c + x^{c+1}s_{c+1} - x^{c+1}s_c}{1 - x - x^2}$$

#### 1.6.2.4 Using the first element in the sequence

For the start of the sequence,  $c = 0$ ,  $s_0 = s_1 = 1$ .

$$F(0) = \frac{x^0 1 + x - x}{1 - x - x^2}$$

$$F(0) = \frac{1}{1 - x - x^2}$$

Let's factorise this:

$$F(0) = \frac{-1}{(x + \frac{1}{2} + \frac{\sqrt{5}}{2})(x + \frac{1}{2} - \frac{\sqrt{5}}{2})}$$

We can then use partial fraction decomposition

$$\frac{1}{(M+\delta)(M-\delta)} = \frac{1}{2\delta} \left[ \frac{1}{M-\delta} - \frac{1}{M+\delta} \right]$$

To show that

$$F(0) = \frac{-1}{\sqrt{5}} \left[ \frac{1}{x + \frac{1}{2} - \frac{\sqrt{5}}{2}} - \frac{1}{x + \frac{1}{2} + \frac{\sqrt{5}}{2}} \right]$$

$$F(0) = \frac{-1}{\sqrt{5}} \left[ \frac{\frac{1}{2} + \frac{\sqrt{5}}{2}}{(x + \frac{1}{2} - \frac{\sqrt{5}}{2})(\frac{1}{2} + \frac{\sqrt{5}}{2})} - \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{(x + \frac{1}{2} + \frac{\sqrt{5}}{2})(\frac{1}{2} - \frac{\sqrt{5}}{2})} \right]$$

$$F(0) = \frac{-1}{\sqrt{5}} \left[ \frac{\frac{1}{2} + \frac{\sqrt{5}}{2}}{x(\frac{1}{2} + \frac{\sqrt{5}}{2}) - 1} - \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{x(\frac{1}{2} - \frac{\sqrt{5}}{2}) - 1} \right]$$

$$F(0) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right) \frac{1}{1 - x(\frac{1}{2} + \frac{\sqrt{5}}{2})} - \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right) \frac{1}{1 - x(\frac{1}{2} - \frac{\sqrt{5}}{2})} \right]$$

#### 1.6.2.5 Finishing off

As we know

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

So

$$F(0) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right) \sum_{i=0}^{\infty} x^i \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^i - \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right) \sum_{i=0}^{\infty} x^i \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^i \right]$$

$$F(0) = \frac{1}{\sqrt{5}} \left[ \sum_{i=0}^{\infty} x^i \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{i+1} - \sum_{i=0}^{\infty} x^i \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{i+1} \right]$$



$$F(0) = \frac{1}{\sqrt{5}} \sum_{i=0} x^i [(\frac{1}{2} + \frac{\sqrt{5}}{2})^{i+1} - (\frac{1}{2} - \frac{\sqrt{5}}{2})^{i+1}]$$

So the  $n$ th number in the sequence (treating  $n = 1$  as the first number) is:

$$\frac{1}{\sqrt{5}} [(\frac{1}{2} + \frac{\sqrt{5}}{2})^n - (\frac{1}{2} - \frac{\sqrt{5}}{2})^n]$$

## 1.7 Other

### 1.7.1 Cardinality

#### 1.7.1.1 Cardinality of cartesian product

What about the cardinality of Cartesian products? So if we have sets:

$$\{1, 2, 3\}$$

$$\{a, b\}$$

We can have the Cartesian product set:

$$\{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$$

We can see that:

$$|A.B| = |A|.|B|$$

#### 1.7.1.2 Cardinality of union and intersection

$$|A \vee B| = |A| + |B| - |A \wedge B|$$

#### 1.7.1.3 Cardinality of powerset

$$|P(s)| = 2^{|s|}$$

#### 1.7.1.4 Cardinality of complement

$$|a \setminus b| = |a| - |a \wedge b|$$

#### 1.7.1.5 Cardinality of even/odd natural numbers

What about the cardinality of even numbers? Well, we can define a bijective function between each:

$$f(n) = 2n$$

Similarly for odd numbers:

$$f(n) = 2n + 1$$

So these both have cardinality  $\aleph_0$ .