1 Introducing complex numbers

1.1 Defining complex numbers

1.1.1 Define as an ordered pair of reals

We have a complete set of real numbers. Do we need any more?

For the real numbers, we showed there were functions on the rational numbers which did not have rational solutions. We can similarly show that there are functions on real numbers which do not have real solutions.

Consider:

$f(x) = \sqrt{x}$

This has no real solution for $x < 0$.

We define:

$i := \sqrt{-1}$

i and $-i$ can be used interchangeably.

$(-i)^2 = (-1)^2 i^2 = i^2 = -1$

Complex numbers can be shown more generally as:

$a + bi$

We define the complex conjugate of

$x = a + bi$

As

$\bar{x} = a - bi$

Note that

$x\bar{x} = (a + bi)(a - bi) = a^2 - b^2$

We can take exponents of imaginary numbers

$c^{i\theta} = a + bi$

We know the opposite is true.

$c^{-i\theta} = a - bi$

So

$c^{i\theta} c^{-i\theta} = (a + bi)(a - bi)$

$1 = a^2 - b^2$

The case where $c = e$ is of particular note. We explore this later.

1.2 Real numbers aren't closed

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2 Operators on complex numbers

2.1 Arithmetic on complex numbers

For each of these we have:

$x = a + bi$

$y = c + di$

Addition is defined as:

$x + y = a + bi + c + di$

$x + y = (a + c) + (b + d)i$

Subtraction is defined as:

$x - y = a + bi - c - di$

$x - y = (a - c) + (b - d)i$

Multiplication is defined as:

$xy = (a + bi)(c + di)$

$xy = ac - bd + adi + bci$

$xy = (ac - bd) + (ad + bc)i$

Division is defined as:

$\frac{x}{y} = \frac{a + bi}{c + di}$

$\frac{x}{y} = \frac{(a + bi)(c - di)}{(c + di)(c - di)}$

$\frac{x}{y} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$

2.2 Complex conjugate

We have $z = a + bi$.

The complex conjugate is:

$\bar{z} = a - bi$

2.3 Absolute value

$|z| = \sqrt{z\bar{z}}$

$|z| = \sqrt{(a+bi)(a-bi)}$

$|z| = \sqrt{a^2 + b^2}$

3 Results

3.1 Roots of unity

3.2 Complex logarithms

3.3 Disks

A disk is the area contained by a circle.

An open disk at (a, b) of radius r is:

$\{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < r^2\}$

For a closed disk it is:

$\{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \leq r^2\}$

3.4 Disks

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3.5 Annulus

An annulus is a disk, which excludes a smaller disk inside the disk

3.6 Punctured disk

If the interior disk is just a point, it is a punctured disk.