

1 Sine and cosine

1.1 Defining sine and cosine using Euler's formula

1.1.1 Euler's formula

Previously we showed that:

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Consider:

$$e^{i\theta}$$

$$e^{i\theta} = \sum_{j=0}^{\infty} \frac{(i\theta)^j}{j!}$$

$$e^{i\theta} = [\sum_{j=0}^{\infty} \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+2}}{(4j+2)!}] + i[\sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+3}}{(4j+3)!}]$$

We then use this to define sin and cos functions.

$$\cos(\theta) := \sum_{j=0}^{\infty} \frac{(\theta)^{4j}}{(4j)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+2}}{(4j+2)!}$$

$$\sin(\theta) := \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+3}}{(4j+3)!}$$

So:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

1.1.2 Alternative formulae for sine and cosine

We know

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$e^{-i\theta} = \cos(\theta) - i \sin(\theta)$$

So

$$e^{i\theta} + e^{-i\theta} = \cos(\theta) + i \sin(\theta) + \cos(\theta) - i \sin(\theta)$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And

$$e^{i\theta} - e^{-i\theta} = \cos(\theta) + i \sin(\theta) - \cos(\theta) + i \sin(\theta)$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

1.1.3 Sine and cosine are odd and even functions

Sine is an odd function.

$$\sin(-\theta) = -\sin(\theta)$$

Cosine is an even function.

$$\cos(-\theta) = \cos(\theta)$$

1.2 De Moivre's formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Let $\theta = nx$:

$$e^{inx} = \cos(nx) + i \sin(nx)$$

$$(e^{ix})^n = \cos(nx) + i \sin(nx)$$

$$(\cos(x) + i \sin(x))^n = \cos(nx) + i \sin(nx)$$

1.3 Expanding sine and cosine

1.3.1 Expansion

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

1.4 Addition of sine and cosine

1.4.1 Adding waves with same frequency

We know that:

$$a \sin(bx + c) = a \sin(bx) \cos(c) + a \sin(c) \cos(bx)$$

So:

$$a \sin(bx + c) + d \sin(bx + e) = a \sin(bx) \cos(c) + a \sin(c) \cos(bx) + d \sin(bx) \cos(e) + d \sin(e) \cos(bx)$$

We know that:

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

So:

$$a \sin(bx + c) + d \sin(bx + f) = a \frac{e^{i(bx+c)} - e^{-i(bx+c)}}{2i} + d \frac{e^{i(bx+f)} - e^{-i(bx+f)}}{2i}$$

$$\begin{aligned}
a \sin(bx + c) + d \sin(bx + f) &= \frac{a(e^{i(bx+c)} - e^{-i(bx+c)}) + d(e^{i(bx+f)} - e^{-i(bx+f)})}{2i} \\
a \sin(bx + c) + d \sin(bx + f) &= \frac{a(e^{ibx}e^{ic} - e^{-ibx}e^{-ic}) + d(e^{ibx}e^{if} - e^{-ibx}e^{-if})}{2i} \\
a \sin(bx + c) + d \sin(bx + f) &= \frac{(e^{ibx}(ae^{ic} + de^{if}) - e^{-ibx}(ae^{-c} + d^{-if}))}{2i} \\
a_i \sin(b_i x + c_i) + a_j \sin(b_j x + c_j) &= a_i \sin(b_i x + c_i) + a_j \sin(b_i x + c_j) \\
a_i \sin(b_i x + c_i) + a_j \sin(b_j x + c_j) &= a_i \sin(b_i x) \cos(c_i) + a_i \sin(c_i) \cos(b_i x) + \\
a_j \sin(b_i x) \cos(c_j) + a_j \sin(c_j) \cos(b_i x)
\end{aligned}$$

1.5 Calculus of sine and cosine

1.5.1 Unity

Note that with imaginary numbers we can reverse all *is*. So:

$$\begin{aligned}
e^{i\theta} &= \cos(\theta) + i \sin(\theta) \\
e^{-i\theta} &= \cos(\theta) - i \sin(\theta) \\
e^{i\theta} e^{-i\theta} &= (\cos(\theta) + i \sin(\theta))(\cos(\theta) - i \sin(\theta)) \\
e^{i\theta} e^{-i\theta} &= \cos(\theta)^2 + \sin(\theta)^2 \\
e^{i\theta} e^{-i\theta} &= e^{i\theta - i\theta} = e^0 = 1
\end{aligned}$$

So:

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

Note that if $\cos(\theta)^2 = 0$, then $\sin(\theta)^2 = \pm 1$

That is, if the real part of $e^{i\theta}$ is 0, the imaginary part is ± 1 . And visa versa.

Similarly if the derivative of the real part of $e^{i\theta}$ is 0, the imaginary part is ± 1 . And visa versa.

1.5.2 Sine and cosine are linked by their derivatives

Note that these functions are linked in their derivatives.

$$\frac{\delta}{\delta\theta} \cos(\theta) = \sum_{j=0}^{\infty} \frac{(\theta)^{(4j+3)}}{(4j+3)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!}$$

$$\frac{\delta}{\delta\theta} \cos(\theta) = -\sin(\theta)$$

Similarly:

$$\frac{\delta}{\delta\theta} \sin(\theta) = \cos(\theta)$$

1.5.3 Both sine and cosine oscillate

$$\frac{\delta^2}{\delta\theta^2} \sin(\theta) = -\sin(\theta)$$

$$\frac{\delta^2}{\delta\theta^2} \cos(\theta) = -\cos(\theta)$$

So for either of:

$$y = \cos(\theta)$$

$$y = \sin(\theta)$$

We know that

$$\frac{\delta^2}{\delta\theta^2} y(\theta) = -y(\theta)$$

Consider $\theta = 0$.

$$e^{i \cdot 0} = \cos(0) + i \sin(0)$$

$$1 = \cos(0) + i \sin(0)$$

$$\sin(0) = 0$$

$$\cos(0) = 1$$

Similarly we know that the derivative:

$$\sin'(0) = \cos(0) = 1$$

$$\cos'(0) = -\sin(0) = 0$$

Consider $\cos(\theta)$.

As $\cos(0)$ is static at $\theta = 0$, and is positive, it will fall until $\cos(\theta) = 0$.

While this is happening, $\sin(\theta)$ is increasing. As:

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

$\sin(\theta)$ will equal 1 where $\cos(\theta) = 0$.

Due to symmetry this will repeat 4 times.

Let's call the length of this period τ .

Where $\theta = \tau * 0$

- $\cos(\theta) = 1$
- $\sin(\theta) = 0$

Where $\theta = \tau * \frac{1}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = 1$

Where $\theta = \tau * \frac{2}{4}$

- $\cos(\theta) = -1$
- $\sin(\theta) = 0$

Where $\theta = \tau * \frac{3}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = -1$

1.5.4 Relationship between $\cos(\theta)$ and $\sin(\theta)$

Note that $\sin(\theta + \frac{\tau}{4}) = \cos(\theta)$

Note that $\sin(\theta) = \cos(\theta)$ at

- $\tau * \frac{1}{8}$
- $\tau * \frac{5}{8}$

And that all these answers loop. That is, add any integer multiple of τ to θ and the results hold.

$$e^{i\theta} = e^{i\theta + n\tau}$$

$$n \in \mathbb{N}$$

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$e^{i\theta} = \cos(\theta + n\tau) + i \sin(\theta + n\tau)$$

$$e^{i\theta} = e^{i(\theta + n\tau)}$$

1.5.5 Calculus of trig

Relationship between cos and sine

$$\sin(x + \frac{\pi}{2}) = \cos(x)$$

$$\cos(x + \frac{\pi}{2}) = -\sin(x)$$

$$\sin(x + \pi) = -\sin(x)$$

$$\cos(x + \pi) = -\cos(x)$$

$$\sin(x + \tau) = \sin(x)$$

$$\cos(x + \tau) = \cos(x)$$

2 Polar coordinates

2.1 Polar co-ordinates

2.1.1 All complex numbers can be shown in polar form

Consider a complex number

$$z = a + bi$$

We can write this as:

$$z = r \cos(\theta) + ir \sin(\theta)$$

2.1.2 Polar forms are not unique

Because the functions loop:

$$ae^{i\theta} = a(\cos(\theta) + i \sin(\theta))$$

$$ae^{i\theta} = a(\cos(\theta + n\tau) + i \sin(\theta + n\tau))$$

$$ae^{i\theta} = ae^{i\theta + n\tau}$$

Additionally:

$$ae^{i\theta} = a(\cos(\theta) + i \sin(\theta))$$

$$ae^{i\theta} = a(\cos(\theta) + i \sin(\theta))$$

$$ae^{i\theta} = -a(\cos(\theta) - i \sin(\theta))$$

$$ae^{i\theta} = -a(\cos(\theta + \frac{\pi}{2}) + i \sin(\theta + \frac{\pi}{2}))$$

2.1.3 Real and imaginary parts of a complex number in polar form

We can extract the real and imaginary parts of this number.

$$Re(z) := r \cos(\theta)$$

$$Im(z) := r \sin(\theta)$$

Alternatively:

$$Re(z) = r \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$Im(z) = r \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

2.2 Moving between polar and cartesian coordinates

All polar numbers can be shown as Cartesian

$$ae^{i\theta} = a(\cos(\theta) + i \sin(\theta))$$

$$ae^{i\theta} = a \cos(\theta) + ia \sin(\theta)$$

$$z = a + bi$$

$$e^{i\theta} =$$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

2.3 Arithmetic of polar coordinates

Addition

$$z_3 = z_1 + z_2$$

$$z_3 = a_1 e^{i\theta_1} + a_2 e^{i\theta_2}$$

$$z_3 = a_1 [\cos(\theta_1) + i \sin(\theta_1)] + a_2 [\cos(\theta_2) + i \sin(\theta_2)]$$

$$z_3 = [a_1 \cos(\theta_1) + a_2 \cos(\theta_2)] + i[a_1 \sin(\theta_1) + a_2 \sin(\theta_2)]$$

Multiplication

$$z_3 = z_1 \cdot z_2$$

$$z_3 = a_1 e^{i\theta_1} a_2 e^{i\theta_2}$$

$$z_3 = a_1 a_2 e^{i(\theta_1 + \theta_2)}$$

$$a_3 = a_1 a_2$$

$$\theta_3 = \theta_1 + \theta_2$$

3 Tangent

3.1 Tan

The $\tan(\theta)$ function is defined as:

$$\tan(\theta) := \frac{\sin(\theta)}{\cos(\theta)}$$

3.1.1 Behaviour around 0

$$\sin(0) = 0$$

$$\cos(0) = 1$$

$$\tan(0) := \frac{\sin(0)}{\cos(0)}$$

$$\tan(0) = \frac{0}{1}$$

$$\tan(0) = 0$$

3.1.2 Behaviour around $\cos(\theta) = 0$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

So $\tan(\theta)$ is undefined where $\cos(\theta) = 0$.

This happens where:

$$\theta = \frac{\tau}{4} + \frac{1}{2}n\tau$$

$$\theta = \frac{1}{4}\tau(1 + 2n)$$

Where $n \in \mathbb{Z}$.

3.1.3 Derivatives

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\frac{\delta}{\delta\theta} \tan(\theta) = \frac{\delta}{\delta\theta} \frac{\sin(\theta)}{\cos(\theta)}$$

$$\frac{\delta}{\delta\theta} \tan(\theta) = \frac{\cos(\theta)}{\cos(\theta)} + \frac{\sin^2(\theta)}{\cos^n(\theta)}$$

$$\frac{\delta}{\delta\theta} \tan(\theta) = 1 + \tan^2(\theta)$$

Note this is always positive. This means:

$$\lim_{\cos(\theta) \rightarrow 0^+} = -\infty$$

$$\lim_{\cos(\theta) \rightarrow 0^-} = \infty$$

3.2 Inverse functions

3.2.1 Inverse trigonometric functions

$$\sin(\arcsin(\theta)) := \theta$$

$$\cos(\arccos(\theta)) := \theta$$

$$\tan(\arctan(\theta)) := \theta$$

3.3 τ

3.3.1 Calculating τ

As we note above, $\sin(\theta) = \cos(\theta)$ at $\theta = \tau * \frac{1}{8}$

This is also where $\tan(\theta) = 1$.

$$\arctan(k) = \arctan(a) + \int_a^k \frac{1}{1+y^2} \delta y$$

We start from $a = 0$.

$$\arctan(k) = \arctan(0) + \int_0^k \frac{1}{1+y^2} \delta y$$

We know that one of the results for $\arctan(0)$ is 0.

$$\arctan(k) = \int_0^k \frac{1}{1+y^2} \delta y$$

We want $k = 1$

$$\arctan(1) = \int_0^1 \frac{1}{1+y^2} \delta y$$

$$\frac{\tau}{8} = \int_0^1 \frac{1}{1+y^2} \delta y$$

$$\tau = 8 \int_0^1 \frac{1}{1+y^2} \delta y$$

We know that the $\cos(\theta)$ and $\sin(\theta)$ functions cycle with period τ .

Therefore $\cos(n.\tau) = \cos(0)$

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4 Hyperbolic functions

4.1 Hyperbolic functions

4.1.1 Hyperbolic functions

$$\sinh(\theta) := \sin(i\theta)$$

$$\cosh(\theta) := \cos(i\theta)$$

$$\tanh(\theta) := \tan(i\theta)$$

4.1.2 Inverse trigonometric functions

$$\sinh(\operatorname{arcsinh}(\theta)) := \theta$$

$$\cosh(\operatorname{arccosh}(\theta)) := \theta$$

$$\tanh(\operatorname{arctan}(\theta)) := \theta$$

4.2 Integrals

4.2.1 Cosine and sine

$\arccos(\theta)$, $\arcsin(\theta)$ and difficulty of inverting

In order to determine τ we need inverse functions for $\cos(\theta)$ or $\sin(\theta)$.

These are the $\arccos(\theta)$ and $\arcsin(\theta)$ functions respectively.

However this is not easily calculated. Instead we look for another function.

4.2.2 Calculating $\arctan(\theta)$

So we want a function to inverse this. This is the $\arctan(\theta)$ function.

If $y = \tan(\theta)$, then:

$$\theta = \arctan(y)$$

We know the derivative for $\tan(\theta)$ is:

$$\frac{\delta}{\delta\theta} \tan(\theta) = 1 + \tan^2(\theta)$$

$$\frac{\delta y}{\delta\theta} = 1 + y^2$$

So

$$\frac{\delta\theta}{\delta y} = \frac{1}{1+y^2}$$

$$\frac{\delta}{\delta y} \arctan(y) = \frac{1}{1+y^2}$$

So the value for $\arctan(k)$ is:

$$\arctan(k) = \arctan(a) + \int_a^k \frac{\delta}{\delta y} \arctan(y) \delta y$$

$$\arctan(k) = \arctan(a) + \int_a^k \frac{1}{1+y^2} \delta y$$

What do we know about this function? We know it can map to multiple values of θ because the underlying $\sin(\theta)$ and $\cos(\theta)$ functions also loop.

We know that one of the results for $\arctan(0)$ is 0.

5 Other

5.1 Other functions

5.1.1 Reciprocal trigonometric functions

Standard

$$\csc(\theta) := \frac{1}{\sin(\theta)}$$

$$\sec(\theta) := \frac{1}{\cos(\theta)}$$

$$\cot(\theta) := \frac{1}{\tan(\theta)}$$

Hyperbolic

$$\operatorname{csch}(\theta) := \frac{1}{\sinh(\theta)}$$

$$\operatorname{sech}(\theta) := \frac{1}{\cosh(\theta)}$$

$$\operatorname{coth}(\theta) := \frac{1}{\tanh(\theta)}$$

5.1.2 Inverse trigonometric functions

Reciprocal standard

$$\csc(\operatorname{arccsc}(\theta)) := \theta$$

$$\sec(\operatorname{arcsec}(\theta)) := \theta$$

$$\cot(\operatorname{arccot}(\theta)) := \theta$$

Reciprocal hyperbolic

$$\operatorname{csch}(\operatorname{arccsch}(\theta)) := \theta$$

$$\operatorname{sech}(\operatorname{arcsech}(\theta)) := \theta$$

$$\operatorname{coth}(\operatorname{arccoth}(\theta)) := \theta$$