

0.1 Raising and lowering indices

We showed that the inner product between two vectors with the same basis can be written as:

$$\langle v, w \rangle = \langle \sum_i e_i v^i, \sum_j f_j w^j \rangle$$

$$\langle v, w \rangle = v^i \overline{w^j} \langle e_i, e_j \rangle$$

Defining the metric as:

$$g_{ij} := \langle e_i, e_j \rangle$$

$$\langle v, w \rangle = v^i \overline{w^j} g_{ij}$$

0.1.1 Metric inverse

We can use this to define the inverse of the metric.

$$g^{ij} := (g_{ij})^{-1}$$

We can use this to raise and lower vectors.

$$v_i := v^j g_{ij}$$

0.1.2 Raising and lowering indices of tensors

If we have tensor:

$$T_{ij}$$

We can define:

$$T_i^k = T_{ij} g^{jk}$$

$$T^{il} = T_{ij} g^{jk} g^{kl}$$

0.1.3 Tensor contraction

If we have:

$$T_{ij} x^j$$

We can contract it to:

$$T_{ij} x^j = v_i$$

Similarly we can have:

$$T^{ij} x_j = v^i$$