0.1 Calculus of sine and cosine

0.1.1 Unity

Note that with imaginary numbers we can reverse all is. So:

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

$e^{i\theta}e^{-i\theta} = (\cos(\theta) + i\sin(\theta))(\cos(\theta) - i\sin(\theta))$

$e^{i\theta}e^{-i\theta} = \cos(\theta)^2 + \sin(\theta)^2$

$e^{i\theta}e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1$

So:

$\cos(\theta)^2 + \sin(\theta)^2 = 1$

Note that if $\cos(\theta)^2 = 0$, then $\sin(\theta)^2 = \pm 1$

That is, if the real part of $e^{i\theta}$ is 0, the imaginary part is ± 1. And visa versa.

Similarly if the derivative of the real part of $e^{i\theta}$ is 0, the imaginary part is ± 1. And visa versa.

0.1.2 Sine and cosine are linked by their derivatives

Note that these functions are linked in their derivatives.

$\frac{\delta}{\delta\theta}\cos(\theta) = \sum_{j=0}^{\infty} \frac{(\theta)^{(4j+3)}}{(4j+3)!} - \sum_{j=0}^{\infty} \frac{(\theta)^{4j+1}}{(4j+1)!}$

$\frac{\delta}{\delta\theta}\cos(\theta) = -\sin(\theta)$

Similarly:

$\frac{\delta}{\delta\theta}\sin(\theta) = cos(\theta)$

0.1.3 Both sine and cosine oscillate

$\frac{\delta^2}{\delta\theta^2}\sin(\theta) = -\sin(\theta)$

$\frac{\delta^2}{\delta\theta^2}\cos(\theta) = -\cos(\theta)$

So for either of:

$y = \cos(\theta)$

$y = \sin(\theta)$

We know that

$\frac{\delta^2}{\delta\theta^2}y(\theta) = -y(\theta)$

Consider $\theta = 0$.

$e^{i.0} = \cos(0) + i\sin(0)$

$1 = \cos(0) + i\sin(0)$

$\sin(0) = 0$

$\cos(0) = 1$

Similarly we know that the derivative:

$\sin'(0) = \cos(0) = 1$

$\cos'(0) = -\sin(0) = 0$

Consider $\cos(\theta)$.

As $\cos(0)$ is static at $\theta = 0$, and is positive, it will fall until $\cos(\theta) = 0$.

While this is happening, $\sin(\theta)$ is increasing. As:

$\cos(\theta)^2 + \sin(\theta)^2 = 1$

$\sin(\theta)$ will equal 1 where $\cos(\theta) = 0$.

Due to symmetry this will repeat 4 times.

Let's call the length of this period τ.

Where $\theta = \tau * 0$

- $\cos(\theta) = 1$
- $\sin(\theta) = 0$

Where $\theta = \tau * \frac{1}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = 1$

Where $\theta = \tau * \frac{2}{4}$

- $\cos(\theta) = -1$
- $\sin(\theta) = 0$

Where $\theta = \tau * \frac{3}{4}$

- $\cos(\theta) = 0$
- $\sin(\theta) = -1$

0.1.4 Relationship between$\cos(\theta)$ and $\sin(\theta)$

Note that $\sin(\theta + \frac{\tau}{4}) = \cos(\theta)$

Note that $\sin(\theta) = \cos(\theta)$ at

- $\tau * \frac{1}{8}$
- $\tau * \frac{5}{8}$

And that all these answers loop. That is, add any integer multiple of τ to θ and the results hold.

$e^{i\theta} = e^{i\theta + n\tau}$

$n \in \mathbb{N}$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$e^{i\theta} = \cos(\theta + n\tau) + i\sin(\theta + n\tau)$

$e^{i\theta} = e^{i(\theta + n\tau)}$

0.1.5 Calculus of trig

Relationship between cos and sine

$\sin(x + \frac{\pi}{2}) = \cos(x)$

$\cos(x + \frac{\pi}{2}) = -\sin(x)$

$\sin(x + \pi) = -\sin(x)$

$\cos(x + \pi) = -\cos(x)$

$\sin(x + \tau) = \sin(x)$

$\cos(x + \tau) = \cos(x)$