1 Real functions

1.1 Real functions

Consider a function

$y = f(x)$

$f(x)$ is a real function if:

$\forall x \in \mathbb{R} f(x) \in \mathbb{R}$

1.2 Support

$fX \to R$

Support of f is $x \in X$ where $f(x) \neq 0$

1.3 Monotonic functions

Calculus stationary points finding and monotonic functions

1.4 Even and odd functions

1.4.1 Defining odd and even functions

An even function is one where:

$f(x) = f(-x)$

An odd function is one where:

$f(x) = -f(-x)$

1.4.2 Functions which are even and odd

If a function is even and odd:

$f(x) = f(-x) = -f(-x)$

$f(x) = -f(x)$

Then $f(x) = 0$.

1.4.3 Scaling odd and even functions

Scaling an even function provides an even function.

$h(x) = c.f(x)$

$h(-x) = c.f(-x)$

$h(-x) = c.f(x)$

$h(-x) = h(x)$

Scaling an odd function provides an odd function.

$h(x) = c.f(x)$

$-h(-x) = -c.f(-x)$

$-h(-x) = c.f(x)$

$-h(-x) = h(x)$

1.4.4 Adding odd and even functions

Note than 2 even functions added together makes an even function.

$h(x) = f(x) + g(x)$

$h(x) = f(-x) + g(-x)$

$h(-x) = f(x) + g(x)$

$h(x) = h(-x)$

And adding 2 odd functions together makes an odd function.

$h(x) = f(x) + g(x)$

$h(x) = -f(-x) - g(-x)$

$-h(-x) = f(x) + g(x)$

$-h(-x) = h(x)$

1.4.5 Multiplying odd and even functions

Multiplying 2 even functions together makes an even function.

$h(x) = f(x)g(x)$

$h(-x) = f(-x)g(-x)$

$h(-x) = f(x)g(x)$

$h(-x) = h(x)$

Multiplying 2 odd functions together makes an even function.

$h(x) = f(x)g(x)$

$h(-x) = f(-x)g(-x)$

$h(-x) = (-1).(-1.)f(x)g(x)$

$h(-x) = h(x)$

1.5 Concave and convex functions

1.5.1 Convex functions

A convex function is one where:

$\forall x_1, x_2 \in \mathbb{R} \forall t \in [0,1][f(tx_1 + (1-t)x_2 \leq tf(x_1) + (1-t)f(x_2)]$

That is, for any two points of a function, a line between the two points is above the curve.

A function is strictly convex if the line between two points is strictly above the curve:

$\forall x_1, x_2 \in \mathbb{R} \forall t \in (0,1)[f(tx_1 + (1-t)x_2 < tf(x_1) + (1-t)f(x_2)]$

An example is $y = x^2$.

1.5.2 Concave functions

A concave function is an upside down convex function. The line between two points is below the curve.

$\forall x_1, x_2 \in \mathbb{R} \forall t \in [0,1][f(tx_1 + (1-t)x_2 \geq tf(x_1) + (1-t)f(x_2)]$

A function is strictly concave if the line between two points is strictly below the curve:

$\forall x_1, x_2 \in \mathbb{R} \forall t \in (0,1)[f(tx_1 + (1-t)x_2 > tf(x_1) + (1-t)f(x_2)]$

An example is $y = -x^2$.

1.5.3 Affine functions

If a function is both concave and convex, then the line between two points must be the function itself. This means the function is an affine function.

$y = cx$

2 Limits

2.1 Limits of real functions

2.1.1 Limit operator

For a function $f(x)$,

$\lim_{x\to a} f(x) = L$

We can say that L is the limit if:

$\forall \epsilon > 0 \exists \delta > 0 \forall x [0 < |x - p| < \delta \to |f(x) - L| < \epsilon]$

2.2 Limit superior and limit inferior

If a sequence does not converge, but stays between two points, then lim sup is upper bound, lim inf is lower bound.

2.3 Big O and little-o notation

2.3.1 Big O notation

In big O notation we are interested in t he size of a function as it getes larger. We ignore constant multiples.

$cx \in O(x)$

And addition of constants.

$cx + b \in O(x)$

If there are two terms and one is larger, we keep the largest.

$x + x^2 \in O(x^2)$

More generally we write:

$f(x) \in O(g(x))$

2.3.2 Little-o notation

3 Continuous functions

3.1 Continous functions

A function is continuous if:

$\lim_{x\to c} f(x) = f(c)$

For example a function $\frac{1}{x}$ is not continuous as the limit towards 0 is negative infinity. A function like $y = x$ is continous.

More strictly, for any $\epsilon > 0$ there exists

$\delta > 0$

$c - \delta < x < c + \delta$

Such that

$f(c) - \epsilon < f(x) < f(c) + \epsilon$

This means that our function is continuous at our limit c, if for any tiny range around $f(c)$, that is $f(c) - \epsilon$ and $f(c) + \epsilon$, there is a range around c, that is $c - \delta$ and $c + \delta$ such that all the value of $f(x)$ at all of these points is within the other range.

3.1.1 Limits

Why can't we use rationals for analysis?

If discontinous at not rational number, it can still be continous for all rationals.

Eg f(x)=-1 unless xˆ2>2, where f(x)=1

Continous for all rationals, because rationals dense in reals.

But can't be differentiated.

3.2 Reals or rationals for analysis

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3.3 Boundedness theorem

If $f(x)$ is closed and continuous in $[a, b]$ then $f(x)$ is bounded by m and M. That is:

$\exists m \in \mathbb{R} \exists M \in \mathbb{R} \forall x \in [a, b](m < f(x) < M$

3.4 Intermediate value theorem

Take a real function $f(x)$ on closed interval $[a, b]$, continuous on $[a, b,]$.

IVT says that for all numbers u between $f(a)$ and $f(b)$, there is a corresponding value c in $[a, b]$ such that $f(c) = u$.

That is:

$\forall u \in [min(f(a), f(b)), max(f(a), f(b))]\exists c \in [a, b](f(c) = u)$

3.5 Extreme value theorem

We can expand the boundedness theorem such that m and M are functions of $f(x)$ in the bound $[a, b]$. That is:

$\exists m \in \mathbb{R}\exists M \in \mathbb{R}\forall x \in [a, b](m < f(x) < M)$