Contents

1 Real numbers

1.1 More on sequences

1.1.1 Limit of a sequence

A sequence converges to a limit if

Can converge to a number (1/x)

Can converge to +/- infinity (x)

Otherwise, does not converge (1,-1,1,-1...)

Superior and inferior limits

A bounded increasing sequence converges to least upper bound

1.1.1.1 Identifying the limit of a sequence

Direct comparison test

Root test

1.2 Divergent series

1.2.1 Partial sum

Take a series. We can define the partial sum as:

$s_k = \sum_{i=1}^{k} a_i$

1.2.2 Cesàro sum

The Cesàro sum is the limit of the average of the first n partial sums.

That is:

$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k$

Consider the sequence $\{1, -1, 1, -1, ...\}$

The partial sum is:

$s_k = \sum_{i=1}^{k} a_i$

$s_k = k \mod (2)$

The Cesàro sum is: $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k$

$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k \mod (2)$

$\frac{1}{2}$

1.2.3 Abel summation

1.3 Constructing the real numbers

1.3.1 Cauchy sequences

1.3.1.1 Cauchy sequence

A cauchy sequence is a sequence such that for an any arbitrarily small number ϵ, there is a point in the sequence where all possible pairs after this are even closer together.

$(\forall \epsilon > 0)(\exists N \in \mathbb{N} : \forall m, n \in \mathbb{N} > N)(|a_m - a_n| < \epsilon)$

This last term gives a distance between two entries. In addition to the number line, this could be used on vectors, where distances are defined.

As a example, $\frac{1}{n}$ is a cauchy sequence, $\sum_i \frac{1}{n}$ is not.

1.3.1.2 Completeness

Cauchy sequences can be defined on some given set. For example given all the numbers between 0 and 1 there are any number of different cauchy sequences converging at some point.

If it is possible to define a cauchy sequence on a set where the limit is not in the set, then the set is incomplete.

For example, the numbers between 0 and 1 but not including 0 and 1 are not complete. It is possible to define sequences which converge to these missing points.

More abstractly, you could have all vectors where $x^2 + y^2 < 1$. This is incomplete (or open) as sequences on these vectors can converge to limits not in the set.

Cauchy sequences are important when considering real numbers. We could define a sequence converging on $\sqrt{2}$, but as this number is not in the set, it is incomplete.

1.3.2 Incompleteness of the real numbers

1.3.2.1 The square root of 2 is not a rational number

Let's prove there are numbers which are not rational. Consider $\sqrt{2}$ and let's show that it being rational leads to a contradiction.

$\sqrt{2} = \frac{x}{y}$

$2 = \frac{x^2}{y^2}$

$2y^2 = x^2$

So we know that x^2 is even, and can be shown as $x = 2n$.

$2y^2 = (2n)^2$

$y^2 = 2n^2$

So y is even. But if both x and y are even, then the fraction was not reduced.

This presents a contraction so the original statement must have been false.

So we know there isn't a rational solution to $\sqrt{2}$.

1.3.3 Density of rationals and reals

1.3.3.1 Rationals are dense in reals

1.3.3.2 Reals are dense in reals

1.3.3.3 Reals are dense in rationals

1.3.4 σ-algebra

1.3.4.1 Review of algebra on a set

An algebra, Σ, on set s is a set of subsets of s such that:

- Closed under intersection: If a and b are in Σ then $a \wedge b$ must also be in Σ
- $\forall ab[(a \in \Sigma \wedge b \in \Sigma) \rightarrow (a \wedge b \in \Sigma)]$
- Closed under union: If a and b are in Σ then $a \vee b$ must also be in Σ.
- $\forall ab[(a \in \Sigma \wedge b \in \Sigma) \rightarrow (a \vee b \in \Sigma)]$

If both of these are true, then the following is also true:

- Closed under complement: If a is in Σ then $s \backslash a$ must also be in Σ

We also require that the null set (and therefore the original set, null's complement) is part of the algebra.

1.3.4.2 σ-algebra

A σ-algebra is an algebra with an additional condition:

All countable unions of sets in A are also in A.

This adds a constraint. Consider the real numbers with an algebra of all finite sets.

This contains all finite subsets, and their complements. It does not contain $\mathbb{N}$.

However a σ-algebra requires all countable unions to be including, and the natural numbers are a countable union.

The power set is a σ-algebra. All other σ-algebras are subsets of the power set.