Contents

1 Infinite-dimensional vector spaces

1.1 Real functions as infinite-dimensional vectors

1.1.1 Real functions are vectors

The real function space is a vector space because it is linear in multiplication and addition.

$g(x) = cf(x)$

$h(x) = f(x) + k(x)$

1.2 Endomorphisms of infinite-dimensional vector spaces

1.2.1 Endomorphisms on real functions

We start with our vector $f(x)$.

$h(x) = f(x)g(x)$

The equivalent of the identity matrix is where $g(x) = 1$.

These are similar to endomorphisms where all off diagonal elements are 0.

1.2.1.1 Differentiation

$h(x) = \frac{\delta}{\delta x} f(x)$

1.2.1.2 Integration

$h(x) = \int_{-\infty}^{x} f(z)dz$

1.2.2 Examples of linear operators on real functions

For a function v we can define operators Ov.

Here we consider some examples and their properties.

1.2.2.1 Real multiplication

$Rv = rf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form rI.

1.2.2.2 Multiplication by underlying real number

$Xv = xf(x)$

This operator is hermitian. This is equivalent to a finite operator of the form $M_{ii} = i$ and $M_{ij} = 0$.

1.2.2.3 Differentiation

$Dv = \frac{\delta}{\delta x} f(x)$

While this operator is not hermitian, the following is:

$-iDv = \frac{\delta}{\delta x}[-if(x)]$

1.3 Eigenvalues and eigenvectors of infinite-dimensional vectors

1.3.1 Spectral theorem for infinite-dimensional vector spaces

1.4 Forms on infinite-dimensional vector spaces

1.4.1 Forms on real functions

A form takes two vectors and produces a scalar.

1.4.1.1 Integration as a form

We can use integration to get a bilinear form.

$\int f(x)g(x)dx$

If we instead want a sesquilinear form we can instead use:

$\int f(\bar{x})g(x)dx$

1.4.2 Functionals

Functionals map functions to scalars. They are the 1-forms of infinite-dimensional vector spaces.

If we have a function f, we can write functional $J[f]$.

1.4.2.1 More

We can define neighbourhoods around a function f. For example, taking y to be f with infintesimal changes. to each of the values.

The difference between the functional at both points is

$\delta J = J[y] - J[f]$

1.4.2.2 Extrema

If

$\delta J = J[y] - J[f]$

is the same sign for all y around f, then J has an extremum at f.

1.4.2.3 Functional derivatives

1.4.3 Hilbert space

A complete space with an inner product. That is, a Banach space where the norm is derived from an inner product.

1.5 Calculus of variations

1.5.1 Calculus of variations

Integrate over possible functions?

1.6 Sort

1.6.1 Banach space

A complete normed vector space