0.1 Tensor contraction

We have a vector $v \in V$ and $w \in V^*$.

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w_i \mathbf{f}^i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w_i \mathbf{f}^i]$

$\mathbf{wv} = \sum_i \sum_j [v^i \mathbf{e}_i][w_j \mathbf{f}^j]$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{f}^j$

We use the dual basis so:

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{e}^j$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \delta_i^j$

We can see that this value is unchanged when there is a change in basis.

What if these were both from V?

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w^i \mathbf{e}_i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w^i \mathbf{e}_i]$

$\mathbf{wv} = \sum_i \sum_j v^i w^j \mathbf{e}_i \mathbf{e}_i$

This term is dependent on the basis, and so we do not contract.

So if we have $v_i w^i$, we can contract, because the result (calculated from the components) does not depend on the basis.

But if we have $v_i w_i$, the result (calcualted from the components) will change depending on the choice of basis.

We define a new object

$c = \sum_i w^i v_i$

This new term, c, does not depend on i, and so we have contracted the index.