0.1 Sesquilinear forms

0.1.1 Bilinear form recap

A bilinear form takes two vectors and produces a scalar from the underyling field.

The function is linear in addition in both arguments.

$\phi(au + x, bv + y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

The function is also linear in multiplication in both arguments.

$\phi(au + x, bv + y) = ab\phi(u, v) + a\phi(u, y) + b\phi(x, v) + \phi(x, y)$

They can be represented as:

$\phi(u, v) = v^T Mu$

0.1.2 Sesquilinear forms

Like bilinear forms, sesquilinear are linear in addition:

$\phi(au + x, bv + y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

Sesqulinear forms however are only multiplictively linear in the second argument.

$\phi(au + x, bv + y) = b\phi(au, v) + \phi(au, y) + b\phi(x, v) + \phi(x, y)$

In the first argument they are "twisted"

$\phi(au + x, bv + y) = \bar{a}b\phi(u, v) + \bar{a}\phi(u, y) + b\phi(x, v) + \phi(x, y)$

0.1.3 The real field

For the real field, $\bar{b} = b$ and so the sesqulinear form is the same as the bilinear form.

0.1.4 Representing sesquilinear forms

We can show the sesquilinear form as $v^* Mu$

0.1.5 Stuff

$f(M) = f([v_1, v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f([\sum_{i=1}^{m} a_{1i}e_i, \sum_{i=1}^{m} a_{2i}e_i])$

$f(M) = \sum_{k=1}^{m} f([a_{1k}e_k, \sum_{i=1}^{m} a_{2i}e_i])$

$f(M) = \sum_{k=1}^{m} \sum_{i=1}^{m} f([a_{1k}e_k, a_{2i}e_i])$

Because this in linear in scalars:

$f(M) = \sum_{k=1}^{m} \sum_{i=1}^{m} a_{1k}^{*}a_{2i}f([e_k, e_i])$

$f(M) = \sum_{k=1}^{m} \sum_{i=1}^{m} a_{1k}^{*}a_{2i}e_k^{*}Me_i$

0.1.6 Orthonormal basis and $M = I$

$f(M) = \sum_{k=1}^{m} \sum_{i=1}^{m} a_{1k}^{*}a_{2i}e_k^{*}Me_i$

$f(M) = \sum_{k=1}^{m} \sum_{i=1}^{m} a_{1k}^{*}a_{2i}e_k^{*}e_i$

$f(M) = \sum_{k=1}^{m} \sum_{i=1}^{m} a_{1k}^{*}a_{2i}\delta_i^k$

$f(M) = \sum_{i=1}^{m} a_{1i}^{*}a_{2i}$