1 Power series

1.1 Power series

of the form:

$\sum_{n=0} a_n(x-c)^n$

1.1.1 Smoothness of power series

Power series are all smooth. That is, they are infinitely differentiable.

2 Taylor series

2.1 Taylor series

$f(x)$ can be estimated at point c by identifying its repeated differentials at point c.

The coefficients of an infinate number of polynomials at point c allow this.

$f(x) = \sum_{i=0}^{\infty} a_i(x-c)^i$

$f'(x) = \sum_{i=1}^{\infty} a_i(x-c)^{i-1}i$

$f''(x) = \sum_{i=2}^{\infty} a_i(x-c)^{i-2}i(i-1)$

$f^j(x) = \sum_{i=j}^{\infty} a_i(x-c)^{i-j}\frac{i!}{(i-j)!}$

For $x = c$ only the first term in the series is non-zero.

$f^j(c) = \sum_{i=j}^{\infty} a_i(c-c)^{i-j}\frac{i!}{(i-j)!}$

$f^j(c) = a_i j!$

So:

$a_j = \frac{f^j(c)}{j!}$

So:

$f(x) = \sum_{i=0}^{\infty}(x-c)^i\frac{f^i(c)}{i!}$

2.2 Convergence

If $x = c$ then the power series will be equal to a_0.

For other values the power series may not converge.

2.2.1 Cauchy-Hadamard theorem

Radius of convergence:

$\frac{1}{R} = \lim\sup_{n\to\infty}(|a_n|^{\frac{1}{n}})$

2.3 Maclaurin series

A Taylor series around $c = 0$.

$f(x) = \sum_{i=0}^{\infty}(x-c)^i \frac{f^i(c)}{i!}$

$f(x) = \sum_{i=0}^{\infty}(x)^i \frac{f^i(0)}{i!}$

For example, for:

$f(x) = (1-x)^{-1}$

$f^i(0) = i!$

So, around $x = 0$:

$f(x) = \sum_{i=0}^{\infty}(x)^i$

2.4 Fourier transforms

2.4.1 Taylor series of matrices

We can also use Taylor series to evaluate functions of matrices.

Consider e^M

We can evaluate this as:

$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$

2.5 Analytic functions

(root test, direct comparison test, rate of convergence, radius of convergence)

3 Fourier analysis

3.1 Representing wave functions

Wave function are of the form:

$\cos(ax + b)$

$\sin(ax + b)$

We can use the following identities:

- $\cos(x) = \sin(x + \frac{\tau}{8})$
- $\sin(-x) = -\sin(x)$
- $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

So we can write any function as:

3.1.1 Using e

3.2 Harmonics

3.3 Fourier series

3.3.1 Fourier series

Motivation: we have a function we want to display as another sort of function.

More specifically, a function can be shown as a combination of sinusoidal waves.

To frame this let's imagine a sound wave, with values $f(t)$ for all time values t. We can imagine this as a summation of sinusoidal functions. That is:

$f(t) = \sum_{n=0}^{\inf} a_n cos(nw_0 t)$

We want to get another function $F(\xi)$ for all frequencies ξ.

3.3.2 Combinations of wave functions

We can add sinusoidal waves to get new waves.

For example

$s_N(x) = 2\sin(x + 3) + \sin(-4x) + \frac{1}{2}\cos(x)$

3.3.3 As a summation of series

We can simplify arbitrary series using the following identities:

$\cos(x) = \sin(x + \frac{\tau}{8})$

$\sin(-x) = -\sin(x)$

So we have:

$s(x) = 2\sin(x+3) - \sin(4x) + \frac{1}{2}\sin(x + \frac{\tau}{8})$

We can put this into the following format:

$s(x) = \sum_{i=1}^{m} a_i \sin(b_i x + c_i)$

Where:

$a = [2, -1, \frac{1}{2}]$

$b = [1, 4, 1]$

$c = [3, 0, \frac{\tau}{8}]$

3.3.4 Ordering by b

We can move terms around to get:

$s(x) = \sum_{i=1}^{m} a_i \sin(b_i x + c_i)$

Where:

$a = [2, \frac{1}{2}, -1]$

$b = [1, 1, 4]$

$c = [3, \frac{\tau}{8}, 0]$

3.3.5 Adding waves with same frequency

We know that:

$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

So:

$\sin(b_i x + c_i) = \sin(b_i x)\cos(c_i) + \sin(c_i)\cos(b_i x)$

If 2 terms have the same value for b_i, then:

$a_i \sin(b_i x + c_i) + a_j \sin(b_j x + c_j) = a_i \sin(b_i x + c_i) + a_j \sin(b_i x + c_j)$

$a_i \sin(b_i x + c_i) + a_j \sin(b_j x + c_j) = a_i \sin(b_i x)\cos(c_i) + a_i \sin(c_i)\cos(b_i x) + a_j \sin(b_i x)\cos(c_j) + a_j \sin(c_j)\cos(b_i x)$

So we now get for:

$s(x) = \sum_{i=1}^{m} a_i \sin(b_i x + c_i)$

$a = [, -1]$

$b = [, 4]$

$c = [, 0]$

3.4 Fourier transforms

3.4.1 Fourier transform

$\hat{f}(\Xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \Xi} dx$

3.4.2 Inverse Fourier transform

$f(x) = \int_{-\infty}^{\infty} \hat{f}(\Xi) e^{2\pi i x \Xi} d\Xi$

3.4.3 Fourier inversion theorem