Contents

1 Linear forms

1.1 Linear forms

1.1.1 Linear forms

A linear form is a linear map from a vector space to a scalar from the vector space's underlying field.

$\hom(V, F)$

1.1.1.1 Matrix operators

Linear forms can be represented as matrix operators.

$v^T M = f$

Where M has only one column.

1.1.1.2 Stuff

$f(M) = f(v)$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f(\sum_{i=1}^m a_i e_i)$

$f(M) = \sum_{i=1}^m f(a_i e_i)$

$f(M) = \sum_{i=1}^m a_i f(e_i)$

$f(M) = \sum_{i=1}^m a_i f(e_i)$

1.1.1.3 Orthonormal basis

$f(M) = \sum_{i=1}^m a_i$

1.1.2 Dual space

The dual space V^* of vector space V is the set of all linear forms, $\hom(V, F)$.

1.1.2.1 The dual space is itself a vector space

$v \in V$

$f \in F$

$av = f$

$bv = g$

$(a \oplus b)v = f + g$

$(a \oplus b)v = av + bv$

So there is some operation we can do on two members of dual space

Linear in addition. That is, if we have two dual "things", we can define the addition of functions as the operation which results int he outputs being added.

what about linear in scalar? same approach.

Well we define

$c \odot a) = cav$

1.1.2.2 The dual space has the same dimension as the underlying vector space

1.1.3 The dual space forms a vector space

The dual space forms a vector space. We can define addition and scalar multiplication on members of the dual space.

The dimenion of the dual space is the same as the underlying space.

We have defined the dual space. A vector in dual space will have also have components and a basis.

$\mathbf{w} = \sum_i w_i f^j$

So how we describe the components will depend on the choice of basis.

We choose the dual basis, the basis for V^* as:

$\mathbf{e}_i \mathbf{f}^j = \delta_i^j$

If the basis changes, so does the dual basis.

We write the dual basis as e^j

1.2 Bilinear forms

1.2.1 Bilinear forms

A bilinear form takes two vectors and produces a scalar from the underyling field.

This is in contrast to a linear form, which only has one input.

In addition, the function is linear in both arguments.

$\phi(au+x, bv+y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

$\phi(au+x, bv+y) = ab\phi(u, v) + a\phi(u, y) + b\phi(x, v) + \phi(x, y)$

1.2.1.1 Representing bilinear forms

They can be represented as:

$\phi(u, v) = v^T M u$

$f(M) = f([v_1, v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f([\sum_{i=1}^m a_{1i}e_i, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m f([a_{1k}e_k, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m f([a_{1k}e_k, a_{2i}e_i])$

Because this in linear in scalars:

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}f([e_k, e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}e_k^T M e_i$

1.2.1.2 Orthonormal basis and $M = I$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}e_k^T e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}a_{2i}\delta_i^k$

$f(M) = \sum_{i=1}^m a_{1i}a_{2i}$

1.2.2 The dot product

$v^T M u = f$

If the operator is I then we have the dot product.

$v^T u$

1.2.3 Orthogonal vectors

Given a metric M, two vectors v and u are orthogonal if:

$v^T M u = 0$

For example if we have the metric $M = I$, then two vectors are orthogonal if:

$v^T u = 0$

1.2.4 Metric-preserving transformations and isometry groups

If we have a bilinear form we can write the form as:

$u^T M v$

After a transformation P to the vectors it is:

$(Pu)^T M(Pv)$

$u^T P^T M P v$

So the value of the metric will be unaffected if:

$u^T P^T M P v = u^T M v$

$P^T M P = M$

1.2.4.1 Equivalent metrics

Different metrics can produce the same group. For example multiplying the metric by a constant.

$P^T M P = M$

1.2.5 Orthogonal groups $O(n, F)$

1.2.5.1 Recap: Metric-preserving transformations

The bilinear form is:

$u^T M v$

The transformations which preserve this are:

$P^T M P = M$

1.2.5.2 The orthogonal group

If the metic is $M = I$ then the condition is:

$P^T P = I$

$P^T = P^{-1}$

These form the orthogonal group.

We use O instead of P:

$O^T = O^{-1}$

1.2.5.3 Rotations and reflections

The orthogonal group is the rotations and reflections.

1.2.5.4 Parameters of the orthogonal group

The orthogonal group depends on the dimension of the vector space, and the underlying field. So we can have:

- $O(n, R)$; and
- $O(n, C)$.

1.2.5.5 We generally refer only to the reals

$O(n)$ means $O(n, R)$.

The generally refer to the reals only.

1.2.6 Indefinite (pseduo) and split orthognal groups $O(n, m, F)$

1.2.6.1 Recap: Metric-preserving transformations

The bilinear form is:

$u^T M v$

The transformations which preserve this are:

$P^T M P = M$

1.2.6.2 The metric

If the metric is:

$M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Then we have the indefinite orthogonal group $O(3, 1)$

1.2.6.3 The split orthogonal group

Where $n = m$ we have the split orthogonal group.

$O(n, n, F)$

1.2.6.4 Signatures

1.2.7 The Lorentz group

The Lorentz group is the $O(1,3)$ group.

1.2.7.1 Symmetries of the Lorentz group

We can do the usual 3 rotations, however there are additional 3 symmetries, making the Lorzentz group 6-dimensional.

These are the Lorentz boosts.

A symmetry has:

$t'^2 - x'2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2$

We consider the case wherewe just boost on x, so $y = y'$ and $z = z'$.

$t'^2 - x'2 = t^2 - x^2$

Or with c:

$c^2 t'^2 - x'2 = t^2 - x^2$

1.3 Sesquilinear forms

1.3.1 Sesquilinear forms

1.3.1.1 Bilinear form recap

A bilinear form takes two vectors and produces a scalar from the underyling field.

The function is linear in addition in both arguments.

$\phi(au + x, bv + y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

The function is also linear in multiplication in both arguments.

$\phi(au + x, bv + y) = ab\phi(u, v) + a\phi(u, y) + b\phi(x, v) + \phi(x, y)$

They can be represented as:

$\phi(u, v) = v^T M u$

1.3.1.2 Sesquilinear forms

Like bilinear forms, sesquilinear are linear in addition:

$\phi(au + x, bv + y) = \phi(au, bv) + \phi(au, y) + \phi(x, bv) + \phi(x, y)$

Sesqulinear forms however are only multiplictively linear in the second argument.

$\phi(au + x, bv + y) = b\phi(au, v) + \phi(au, y) + b\phi(x, v) + \phi(x, y)$

In the first argument they are "twisted"

$\phi(au + x, bv + y) = \bar{a}b\phi(u, v) + \bar{a}\phi(u, y) + b\phi(x, v) + \phi(x, y)$

1.3.1.3 The real field

For the real field, $\bar{b} = b$ and so the sesqulinear form is the same as the bilinear form.

1.3.1.4 Representing sesquilinear forms

We can show the sesquilinear form as v^*Mu

1.3.1.5 Stuff

$f(M) = f([v_1, v_2])$

We introduce e_i, the element vector. This is 0 for all entries except for i where it is 1. Any vector can be shown as a sum of these vectors multiplied by a scalar.

$f(M) = f([\sum_{i=1}^m a_{1i}e_i, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m f([a_{1k}e_k, \sum_{i=1}^m a_{2i}e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m f([a_{1k}e_k, a_{2i}e_i])$

Because this in linear in scalars:

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} f([e_k, e_i])$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* M e_i$

1.3.1.6 Orthonormal basis and $M = I$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* M e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} e_k^* e_i$

$f(M) = \sum_{k=1}^m \sum_{i=1}^m a_{1k}^* a_{2i} \delta_i^k$

$f(M) = \sum_{i=1}^m a_{1i}^* a_{2i}$

1.3.2 Unitary groups $U(n, F)$

1.3.2.1 Metric preserving transformations for sesquilinear forms

For bilinear forms, the transformations which preserved metrics were:

$P^T = P^{-1}$

For sesquilinear they are different:

$u^* M v$

$(Pu)^* M (Pv)$

$u^* P^* M P v$

So we want the matrices where:

$P^* M P = M$

1.3.2.2 The unitary group

The unitary group is where $M = I$

$P^* P = I$

$P^* = P^{-1}$

We refer to these using U instead of P.

$U^* = U^{-1}$

1.3.2.3 Parameters of the unitary group

The unitary group depends on the dimension of the vector space, and the underlying field. So we can have:

- $U(n, R)$; and
- $U(n, C)$.

1.3.2.4 We generally refer only to the complex

For the $U(n, R)$ we have:

$U^* = U^{-1}$

$U^T = U^{-1}$

This is the condition for the orthogonal group, and so we would instead write $O(n)$.

As a result, $U(n)$ refers to $U(n, C)$.

1.3.2.5 $U(1)$: The circle group

1.4 Inner products

1.4.1 Symmetric matrices

1.4.2 Hermitian (self-adjoint) matrices

A matrix where $M = M^*$

For matrices over the real numbers, these are the same as symmetric matrices.

1.4.2.1 Sesqulinear forms on Hermitian matrices

$\phi(u, v) = u^* M v$

$(u^* M v)^* = v^* M^* u = v^* M u$

$\phi(u, v) = \overline{\phi(v, u)}$

1.4.2.2 The forms on the same vector are always real

$(v^* M v)^* = v^* M^* v = v^* M v$

So we have:

$(v^* M v)^* = v^* M v$

Which is only satisfied for reals.

1.4.2.3 If A and B are Hermitian

If A and B are Hermitian, AB is Hermitian if and only if AB commutes.

$(AB)^* = B^* A^* = BA$

If it commutes then

$(AB)^* = AB$

1.4.2.4 Real eigenvalues

Hermitian matrices have real eigenvalues.

$Hv = \lambda v$

$v^* H v = \lambda v^* v$

$v^* H v = \lambda$

1.4.2.5 Skew-Hermitian matrices

These are also known as anti-Hermitian matrices.

$M^* = -M$

1.4.2.6 If eigenvalues are different, eigenvectors are orthogonal

1.4.3 Pauli matrices

Pauli matrices are 2×2 matrices which are unitary and hermitian.

That is, $P^* = P^{-1}$.

And $P^* = P$.

1.4.3.1 The Pauli matrices

The matrices are:

$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The identity matrix is often considered alongside these as:

$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1.4.3.2 Pauli matrices are their own inverse

$\sigma_i^2 = \sigma_i \sigma_i$

$\sigma_i^2 = \sigma_i \sigma_i^*$

$\sigma_i^2 = \sigma_i \sigma_i^{-1}$

$\sigma_i^2 = I$

1.4.3.3 Determinants and trace of Pauli matrices

$\det \sigma_i = -1$

$Tr(\sigma_i) = 0$

As the sum of eigenvalues is the trace, and the product is the determinant, the eigenvalues are 1 and -1.

1.4.4 Positive-definite matrices

The matrix M is positive definite if for all non-zero vectors the scalar is positive.

$v^T M v$

We know that the outcome is a scalar, so:

$v^T M v = (v^T M v)^T$

$v^T M v = v^T M^T v$

$v^T (M - M^T) v = 0$

1.4.5 Inner products

An inner product is a sesquilinear form with a positive-definite Hermitian matrix.

$\langle u, v \rangle = u^* H v$

If we are using the real field this is the same as:

$\langle u, v \rangle = u^T H v$

Where H is now a symmetric real matrix.

1.4.5.1 Same

$\langle v, v \rangle = v^* H v$

Always positive and real.

1.4.5.2 Properties

$\langle u, v \rangle \langle v, u \rangle = |\langle u, v \rangle|^2$

1.4.6 Cauchy-Schwarz inequality

This states that:

$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \dot{\langle} v, v \rangle$

Consider the vectors u and v. We construct a third vector $u - \lambda v$. We know the length of any vector is non-negative. $0 \leq \langle u - \lambda v, u - \lambda v \rangle$

$0 \leq \langle u, u \rangle + \langle u, -\lambda v \rangle + \langle -\lambda v, u \rangle + \langle -\lambda v, -\lambda v \rangle$

$0 \leq \langle u,u\rangle - \bar{\lambda}\langle u,v\rangle - \lambda\langle v,u\rangle + \lambda\bar{\lambda}\langle v,v\rangle$

We now look for a value of λ to simplify this equation.

$\lambda = \dfrac{\langle u,v\rangle}{\langle v,v\rangle}$

$0 \leq \langle u,u\rangle - \dfrac{\langle v,u\rangle\langle u,v\rangle}{\langle v,v\rangle} - \dfrac{\langle u,v\rangle\langle v,u\rangle}{\langle v,v\rangle} + \dfrac{\langle u,v\rangle}{\langle v,v\rangle}\dfrac{\langle v,u\rangle}{\langle v,v\rangle}\langle v,v\rangle$

$0 \leq \langle u,u\rangle - \dfrac{|\langle u,v\rangle|^2}{\langle v,v\rangle}$

$|\langle u,v\rangle|^2 \geq \langle u,u\rangle\langle v,v\rangle$

1.5 Multilinear forms and determinants

1.5.1 Multilinear forms

1.5.2 Determinants

From invertible matrix section in endo

A matrix can only be inverted if it can be created from a combination of elementary row operations.

How can we identify if a matrix is invertible? We want to create a scalar from the matrix which tells us if this possible. We can this scalar the determinant.

For a matrix A we label the determinant $|A|$, or $\det A$

We propose $|A| = 0$ when the matrix is not invertible.

So how can we identify the function we need to undertake on the matrix?

1.5.2.1 New 1

We know that linear dependence results in determinants of 0.

We can model this as a function on the columns of the matrix.

$\det M = \det([M_1, ..., M_n)$

If there is linear depednence, for example if two columns are the same then:

$\det([M_1, ..., M_i, ..., M_i, ..., M_n]) = 0$

Similarly, if there is a column of 0 then the determinant is 0.

$\det([M_1, ..., 0, ..., M_n]) = 0$

1.5.2.2 New 2

Show linear in addition

How can we identify the determinant of less simple matrices? We can use the multilinear form.

$\sum c_i \mathbf{M}_i = \mathbf{0}$

Where $\mathbf{c} \neq \mathbf{0}$

Or:

$M\mathbf{c} = \mathbf{0}$

1.5.2.3 Rule 1: Columns of matrices can be the input to a multilinear form

A matrix can be shown in terms of its columns. $A = [v_1, ..., v_n]$

$\det A = \det[v_1, ..., v_n]$

$\det A = \sum_{k_1=1}^{m} ... \sum_{k_n=1} \prod_{i=1}^{m} a_{ik_i} \det([e_{k_1}, ..., e_{k_n}])$

1.5.2.4 Multiplying a matrix by a constant multiplies the determinant by the same amount

If a whole row or columns is 0 then:

$\det A = \det[v_1, ..., v_i, ..., v_n]$

$\det A' = \det[v_1, ..., cv_i, ..., v_n]$

$\det A = \det[v_1, ..., v_i, ..., v_n]$

$\det A' = \det[v_1, ..., cv_i, ..., v_n]$

$\det A' = c \det[v_1, ..., v_i, ..., v_n]$

$\det A' = c \det A$

As a result, multiplying a column by 0 makes the determinant 0.

A matrix with a column of 0 therefore has determinant 0

1.5.2.5 Rule 2: A matrix with equal columns has a determinant of 0.

$A = [a_1, ..., a_i, ..., a_i, ..., a_n]$

$D(A) = D([a_1, ..., a_i, ..., a_i, ..., a_n])$

We know from Result 3 that swapping columns reverses the sign. Reversing columns results in the same matrix, so the determinant must be unchanged.

$D(A) = -D(A)$

$D(A) = 0$

1.5.2.6 Linear dependence

If a column is a linear combination of other columns, then the matrix cannot be inverted.

$A = [a_1, ..., \sum_{j \neq i}^n c_j a_j, ..., a_n]$

$\det A = \det([v_1, ..., \sum_{j \neq i}^n c_j v_j, ..., v_n])$

$\det A = \sum_{j \neq i}^n c_j \det([v_1, ..., v_j, ..., v_n])$

$\det A = \sum_{j \neq i}^n c_j \det([v_1, ..., v_j, , ..., v_j, ..., v_n])$

As there is a repeating vector:

$\det A = 0$

1.5.2.7 Swapping columns multiplies the determinant by -1

$A = [v_1, ..., v_i + v_j, ..., v_i + v_j, ..., v_n]$

We know.

$\det A = 0$

$\det A = \det([a_1, ..., a_i, ..., a_i, ..., a_n]) + \det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) + \det([a_1, ..., a_j, ..., a_j, ..., a_n])$

So:

$\det([a_1, ..., a_i, ..., a_i, ..., a_n]) + \det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) + \det([a_1, ..., a_j, ..., a_j, ..., a_n]) = 0$

As 2 of these have equal columns these are equal to 0.

$\det([a_1, ..., a_i, ..., a_j, ..., a_n]) + \det([a_1, ..., a_j, ..., a_i, ..., a_n]) = 0$

$\det([a_1, ..., a_i, ..., a_j, ..., a_n]) = -\det([a_1, ..., a_j, ..., a_i, ..., a_n])$

1.5.2.8 Calculating the determinant

We have

$\det A = \sum_{k_1=1}^m ... \sum_{k_n=1} \prod_{i=1}^m a_{ik_i} \det([e_{k_1}, ..., e_{k_n}])$

So what is the value of the determinant here?

We know that the determinant of the identity matrix is 1.

We know that the determinant of a matrix with identical columns is 0.

We know that swapping columns multiplies the determinant by -1.

Therefore the determinants where the values of k are not all unique are 0.

The determinants of the others are either -1 or 1 depending on how many swaps are required to restore to the identity matrix.

This is also shown as the Leibni formula.

$\det A = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$

1.5.3 Properties of determinants

1.5.3.1 Identity

$\det I = 1$

1.5.3.2 Multiplication

$\det(AB) = \det A \det B$

1.5.3.3 Inverse

$\det(M^{-1}) = \dfrac{1}{\det M}$

We know this because:

$\det(MM^{-1}) = \det I = 1$

$\det M \det M^{-1} = 1$

$\det(M^{-1}) = \dfrac{1}{\det M}$

1.5.3.4 Complex cojugate

$\det(M^*) = \overline{\det M}$

1.5.3.5 Transpose

$\det(M^T) = \det M$

1.5.3.6 Addition

$\det(A + B) = \det A + \det B$

1.5.3.7 Scalar multiplication

$\det cM = c^n \det M$

1.5.3.8 Determinants and eigenvalues

The determinant is equal to the product of the eigenvalues.

1.5.4 Determinants of 2x2 and 3x3 matrices

1.5.4.1 The determinant of a 2x2 matrix

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|M| = ad - bc$

1.5.4.2 The determinant of a 3x3 matrix

$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$|M| = aei + bfg + cdh - ceg - dbi - afh$

1.6 Special groups

1.6.1 Special orthogonal groups $SO(n, F)$

The special orthogonal group, $SO(n, F)$, is the subgroup of the orthogonal group where $|M| = 1$.

As a result it includes only the rotation operators, not the flip operators.

$SO(3)$ is rotations in 3d space.

$SO(2)$ is rotations in 2d space.

1.6.1.1 Determinant of the orthogonal group

The orthogonal group has determinants of -1 or 1.

$O^T = O^{-1}$

$\det(O^T) = \det(O^{-1})$

$\det O = \frac{1}{\det O}$

$\det O = \pm 1$

1.6.2 Special unitary groups $SU(n, F)$

The special unitary group, $SU(n, F)$, is the subgroup of $U(n, F)$ where the determinants are 1.

That is, $|M| = 1$

1.6.2.1 The determinant of unitary matrices

The determinant of the unitary matrices is:

$\det U^* = \det U^{-1}$

$(\det U)^* = \dfrac{1}{\det U}$

$(\det U)^* \det U = 1$

$||\det U|| = 1$

1.6.3 Special lnear groups $SL(n, F)$

The special linear group, $SL(n, F)$, is the subgroup of $GL(n, F)$ where the determinants are 1.

That is, $|M| = 1$

These are endomorphisms, not forms.

1.7 Sort

1.7.1 Normal matrices

$M^*M = MM^*$

All symmetrix matrices are normal

All hermetitian matrices (inc subset symmetric) are normal

Normal matrix never defective