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1 Linear endomorphisms

1.1 Endomorphisms of vector spaces

1.1.1 Endomorphisms

An endomorphism maps a vector space onto itself.

$end(V) = \hom(V, V)$

1.1.2 Endomorphisms form a vector space

An endomorphism maps a vector space onto itself.

$end(V) = \hom(V, V)$

Need to show that endomorphism is a vector space

Essentially

$v \in V$

fF

$av = f$

$bv = g$

$(a \oplus b)v = f + g$

$(a \oplus b)v = av + bv$

so there is some operation we can do on two members of endo linear in addition. That is, if we have two dual “things”, we can define the addition of functions as the operation which results int he outputs being added.

what about linear in scalar? same approach.

Well we define

$c \odot a) = cav$

There is a unique endomorphism which results in two other endomorphisms being added together. define this as addition

1.1.3 Dimension of endomorphisms

$\dim(end(V)) = (\dim V)^2$

1.1.4 Basis of endomorphisms

1.1.5 Projections

A projection is a linear map which if applied again returns the original result.

A projection can drop a dimension for example.

1.1.6 Kernels and images

The kernel of a linear operator is the set of vectors such that:

$Mv = 0$

The kernel is also called the nullspace.

This can be shown as $\ker(M)$

The image of a linear operator is the set of vectors w such that:

$Mv = w$.

This can be shown as $\Im(M)$

We also know that:

$span(M) = \ker(M) + \Im(M)$

1.2 Representing endomorphisms with matrices

1.2.1 Matrix representation

1.2.1.1 Representing linear maps as matrices

We previously discsussed morphisms on vector spaces. We can write these as matrices.

Matrices represents transformations of vector spaces

1.2.1.2 Representing vectors as matrices

We can represent vectors as row or column matrices.

$v = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$

$v = \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_m \end{bmatrix}$

1.2.2 Commutation

We define a function, the commuter, between two objects a and b as:

$[a, b] = ab - ba$

For numbers, $ab - ba = 0$, however for matrices this is not generally true.

1.2.3 Commutators and eigenvectors

Consider two matrices which share an eigenvector v.

$Av = \lambda_A v$

$Bv = \lambda_B v$

Now consider:

$ABv = A\lambda_B v$

$ABv = \lambda_A \lambda_B v$

$BAv = \lambda_A \lambda_B v$

If the matrices share all the same eigenvectors, then the matrices commute, and $AB = BA$.

1.2.4 Identity matrix and the Kronecker delta

1.2.5 Matrix additon and multiplication

1.2.5.1 Matrix multiplication

$A = A^{mn}$

$B = B^{no}$

$C = C^{mo} = A.B$

$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$

Matrix multiplication depends on the order. Unlike for real numbers,

$AB \neq BA$

Matrix multiplication is not defined unless the condition above on dimensions is met.

A matrix multiplied by the identity matrix returns the original matrix.

For matrix $M = M^{mn}$

$M = MI^m = I^n M$

1.2.5.2 Matrix addition

2 matricies of the same size, that is with idental dimensions, can be added together.

If we have 2 matrices A^{mn} and B^{mn}

$C = A + B$

$c_{ij} = a_{ij} + b_{ij}$

An empty matrix with 0s of the same size as the other matrix is the identity matrix for addition.

1.2.5.3 Scalar multiplication

A matrix can be multiplied by a scalar. Every element in the matrix is multiplied by this.

$B = cA$

$b_{ij} = ca_{ij}$

The scalar 1 is the identity scalar.

1.2.6 Basis of an endomorphism

1.2.7 Changing the basis

For any two bases, there is a unique linear mapping from of the element vectors to the other.

1.2.8 Transposition and conjugation

1.2.8.1 Transposition

A matrix of dimensions $m * n$ can be transformed into a matrix $n * m$ by transposition.

$B = A^T$

$b_{ij} = aji$

1.2.8.2 Transpose rules

$(M^T)^T = M$

$(AB)^T = B^T A^T$

$(A + B)^T = A^T + B^T$

$(zM)^T = zM^T$

1.2.8.3 Conjugation

With conjugation we take the complex conjugate of each element.

$B = \overline{A}$

$b_{ij} = \overline{a}_{ij}$

1.2.8.4 Conjugation rules

$\overline{(\overline{A})} = A$

$\overline{(AB)} = (\overline{A})(\overline{B})$

$\overline{(A + B)} = \overline{A} + \overline{B}$

$\overline{(zM)} = \overline{z}\overline{M}$

1.2.8.5 Conjugate transposition

Like transposition, but with conjucate.

$B = A^*$

$b_{ij} = \bar{a_{ji}}$

Alternatively, and particularly in physics, the following symbol is often used instead.

$(A^*)^T = A^\dagger$

1.2.9 Matrix rank

1.2.9.1 Rank function

The rank of a matrix is the dimension of the span of its component columns.

$rank(M) = span(m_1, m_2, ..., m_n)$

1.2.9.2 Column and row span

The span of the rows is the same as the span of the columns.

1.2.10 Types of matrices

1.2.10.1 Empty matrix

A matrix where every element is 0. There is one for each dimension of matrix.

$A = \begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix}$

1.2.11 Triangular matrix

A matrix where $a_{ij} = 0$ where $i < j$ is upper triangular.

A matrix where $a_{ij} = 0$ where $i > j$ is lower triangular.

A matrix which is either upper or lower triangular is a triangular matrix.

1.2.12 Symmetric matrices

All symmetric matrices are square.

The identity matrix is an example.

A matrix where $a_{ij} = a_{ji}$ is symmetric.

1.2.13 Diagonal matrix

A matrix where $a_ij = 0$ where $i \neq j$ is diagonal.

All diagonal matrices are symmetric.

The identity matrix is an example.

1.3 Automorphisms of vector spaces

1.3.1 Inverse matrices

An invertible matrix implies that if the matrix is multiplied by another matrix, the original matrix can be recovered.

That is, if we have matrix A, there exists matrix A^{-1} such that $AA^{-1} = I$.

Consider a linear map on a vector space.

$Ax = y$

If A is invertible we can have:

$A^{-1}Ax = A^{-1}y$

$x = A^{-1}y$

If we set $y = \mathbf{0}$ then:

$x = \mathbf{0}$

So if there is a non-zero vector x such that:

$Ax = \mathbf{0}$ then A is not invertible.

1.3.2 Left and right inverses

That is, for all matrices A, the left and right inverses of B, B_L^{-1} and B_R^{-1}, are defined such that:

$A(BB_R^{-1}) = A$

$A(B_L^{-1}B) = A$

Left and right inversions are equal

Note that if the left inverse exists then:

$B_L^{-1}B = I$

And if the right inverse exists:

$BB_R^{-1} = I$

Let's take the first:

$B_L^{-1}B = I$

$B_L^{-1}BB_L^{-1} = B_L^{-1}$

$B_L^{-1}BB_L^{-1} - B_L^{-1} = 0$

$B_L^{-1}(BB_L^{-1} - I) = 0$

1.3.3 Inversion of products

$(AB)(AB)^{-1} = I$

$A^{-1}AB(AB)^{-1} = A^{-1}$

$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

1.3.4 Inversion of a diagonal matrix

$DD^{-1} = I$

$D_{ii}D_{ii}^{-1} = 1$

$D_{ii}^{-1} = \frac{1}{D_{ii}}$

1.3.5 Degenerate (singular) matrices

1.3.6 Elementary row operations

Some operations to a matrix can be reversed to arrive at the original matrix. Trivially, multiplying by the identity matrix is reversible.

Similarly, some operations are not reversible. Such as multiplying by the empty matrix.

All matrix operations which can be reversed are combinations of 3 elementary row operations. These are: Swapping rows

$$T_{12} = \begin{bmatrix} 0 & 1 & ... & 0 \\ 1 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}$$

Multiplying rows by a vector

$$D_2(m) = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & m & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}$$

Adding rows to other rows

$$L_{12}(m) = \begin{bmatrix} 1 & 0 & ... & 0 \\ m & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix}$$

1.3.7 Gaussian elimination

1.3.7.1 Simultaneous equations

Matricies can be used to solve simultaneous equations. Condsider the following set of equations.

- $2x + y - z = 8$
- $-3x - y + 2z = -11$
- $-2x + y + 2z = -3$

We can write this in matrix form.

$Ax = y$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$y = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

1.3.7.2 Augmented matrix

Consider a form for summarising these equations. This is the augmented matrix.

$$(A|y) = \begin{bmatrix} 2 & 1 & -1 & | & -8 \\ -3 & -1 & 2 & | & -11 \\ -2 & 1 & 2 & | & -3 \end{bmatrix}$$

We can take this and recovery our original A and y.

However we can also do things to this augmented matrix which preserve solutions to the set of equations. These are:

Undertaking combinations of these can make it easier to solve the equation. In particular, if we can arrive at the form:

$$(A|y) = \begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

The solutions for x, y, z are a, b, c.

1.3.7.3 Echeleon / triangular form

We first aim for:

$$(A|y) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & a \\ 0 & a_{22} & a_{23} & | & b \\ 0 & 0 & a_{33} & | & c \end{bmatrix}$$

If this cannot be reached there is no single solution. There may be infinite or no solutions.

1.3.7.4 Solving

Once we have the triangular form, we can easily solve.

$$(A|y) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & a \\ 0 & a_{22} & a_{23} & | & b \\ 0 & 0 & a_{33} & | & c \end{bmatrix}$$

$$(A|y) = \begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

This process is back substitution (or forward substitution if the matrix is triangular the other way).

1.3.7.5 Matrix inversion

We can think of the inverse of a matrix as one which which takes a series of reverible operations and does these to a matrix then arriving at the identity matrix.

That is, only the three elementary row operations, and combinations of them, can transform a matrix in a way in which it can be reversed. As such All reversible matricies are combinations of the identity matrix and a series of elementary row operations. The inverse matrix is then those series of row operations, in reverse.

We can find identify an inversion by undertaking gaussian elimination. Each step done on the matrix is done to the identify matrix, reversing the process. The end result is the inverted matrix.

Instead of:

$$(A|y) = \begin{bmatrix} 2 & 1 & -1 & | & -8 \\ -3 & -1 & 2 & | & -11 \\ -2 & 1 & 2 & | & -3 \end{bmatrix}$$

Take:

$$(A|I) = \begin{bmatrix} 2 & 1 & -1 & | & 1 & 0 & 0 \\ -3 & -1 & 2 & | & 0 & 1 & 0 \\ -2 & 1 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

When we solve this we get:

$$(I|A^{-1}) = \begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & | & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & | & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

1.4 Eigenvalues and eigenvectors

1.4.1 Eigenvalues and eigenvectors

Which vectors remain unchanged in direction after a transformation?

That is, for a matrix A, what vectors v are equal to scalar multiplication by λ following the operation of the matrix.

$Av = \lambda v$

1.4.2 Spectrum

The spectrum of a matrix is the set of its eigenvalues.

1.4.3 Eigenvectors as a basis

If eigen vectors space space, we can write

$v = \sum_i \alpha_i |\lambda_i\rangle$

Under what circumstances do they span the entirity?

1.4.4 Calculating eigenvalues and eigenvectors using the characteristic polynomial

The characteristic polynomial of a matrix is a polynomial whose roots are the eigenvalues of the matrix.

We know from the definition of eigenvalues and eigenvectors that:

$Av = \lambda v$

Note that

$Av - \lambda v = 0$

$Av - \lambda Iv = 0$

$(A - \lambda I)v = 0$

Trivially we see that $v = 0$ is a solution.

Otherwise matrix $A - \lambda I$ must be non-invertible. That is:

$Det(A - \lambda I) = 0$

1.4.5 Calculating eigenvalues

For example

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

$Det(A - \lambda I) = (2-\lambda)(2-\lambda) - 1$

When this is 0.

$(2-\lambda)(2-\lambda) - 1 = 0$

$\lambda = 1, 3$

1.4.6 Calculating eigenvectors

You can plug this into the original problem.

For example

$Av = 3v$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

As vectors can be defined at any point on the line, we normalise $x_1 = 1$.

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3x_2 \end{bmatrix}$

Here $x_2 = 1$ and so the eigenvector corresponding to eigenvalue 3 is:

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1.4.7 Traces

The trace of a matrix is the sum of its diagonal components.

$Tr(M) = \sum_i^n m_{ii}$

The trace of a matrix is equal to the sum of its eigenvectors.

1.4.8 Properties of traces

Traces commute

$Tr(AB) = Tr(BA)$

Traces of 1×1 matrices are equal to their component.

$Tr(M) = m_{11}$

1.4.9 Trace trick

If we want to manipulate the scalar:

$v^T M v$

We can use properties of the trace.

$v^T M v = Tr(v^T M v)$

$v^T M v = Tr([v^T][Mv])$

$v^T M v = Tr([Mv][v^T])$

$v^T M v = Tr(M v v^T)$

1.5 Matrix operations

1.5.1 Matrix powers

For a square matrix M we can calculate $MMMM...$, or M^n where $n \in \mathbb{N}$.

1.5.2 Powers of diagonal matrices

Generally, calculating a matrix to an integer power can be complicated. For diagonal matrices it is trivial.

For a diagonal matrix $M = D^n$, $m_{ij} = d_{ij}^n$.

1.5.3 Matrix exponentials

The exponential of a complex number is defined as:

$e^x = \sum \frac{1}{j!} x^j$

We can extend this definition to matrices.

$e^X := \sum \frac{1}{j!} X^j$

The dimension of a matrix and its exponential are the same.

1.5.4 Matrix logarithms

If we have $e^A = B$ where A and B are matrices then we can say that A is matrix logarithm of B.

That is:

$\log B = A$

The dimensions of a matrix and its logarithm are the same.

1.5.5 Matrix square roots

For a matrix M, the square root $M^{\frac{1}{2}}$ is A where $AA = M$.

This does not necessarily exist.

Square roots may not be unique.

Real matrices may have no real square root.

1.6 Matrix decomposition

1.6.1 Similar matrices

In hermitian, show all symmtric matrices are hermitian

For a diagonal matrix, eigenvalues are the diagonal entries?

Similar matrix:

$M = P^{-1}AP$

M and A have the same eigenvalues. If A diagonal, then entries are eigenvalues.

1.6.2 Defective and diagonalisable matrices

1.6.3 Diagonalisable matrices and eigendecomposition

If matrix M is diagonalisable if there exists matrix P and diagonal matrix A such that:

$M = P^{-1}AP$

1.6.3.1 Diagonalisiable matrices and powers

If these exist then we can more easily work out matrix powers.

$M^n = (P^{-1}AP)^n = P^{-1}A^nP$

A^n is easy to calculate, as each entry in the diagonal taken to the power of n.

1.6.3.2 Defective matrices

Defective matrices are those which cannot be diagonalised.

Non-singular matries can be defective or not defective, for example the identiy matrix.

Singular matrices can also be defective or not defective, for example the empty matrix.

1.6.3.3 Eigen-decomposition

Consider an eigenvector v and eigenvalue λ of matrix M.

We known that $Mv = \lambda v$.

If M is full rank then we can generalise for all eigenvectors and eigenvalues:

$MQ = Q\Lambda$

Where Q is the eigenvectors as columns, and Λ is a diagonal matrix with the corresponding eigenvalues. We can then show that:

$M = Q\Lambda Q^{-1}$

This is only possible to calculate if the matrix of eigenvectors is non-singular. Otherwise the matrix is defective.

If there are linearly dependent eigenvectors then we cannot use eigen-decomposition.

1.6.4 Using the eigen-decomposition to invert a matrix

This can be used to invert M.

We know that:

$M^{-1} = (Q\Lambda Q^{-1})^{-1}$

$M^{-1} = Q^{-1}\Lambda^{-1}Q$

We know Λ can be easily inverted by taking the reciprocal of each diagonal element. We already know both Q and its inverse from the decomposition.

If any eigenvalues are 0 then Λ cannot be inverted. These are singular matrices.

1.6.5 Spectral theorem for finite-dimensional vector spaces

1.7 The linear groups

1.7.1 General linear groups $GL(n, F)$

The general linear group, $GL(n, F)$, contains all $n \odot n$ invertible matrices M over field F.

The binary operation is multiplication.

1.7.2 Endomorphisms as group actions

We can view each member of the group g as a homomorphim on s.

Where s is a vector space V, the representation on each group member is an invertible square matrix.

If the set we use is the vector space V, then we can represent each group element with a square matrix acting on V.

Faithful means ab holds for repesentation too.

Representation theory. groups defined by ab=c. if we can match each eleemnt to amatrix where this holds we have represented the matrix.

1.7.3 Representing finite groups

Finite groups can all be represented with square matrices.

1.7.4 Representing compact groups