1 Sequences

1.1 Sequences

1.1.1 Definition

A sequence is an ordered list of terms.

These are commonly maps from natural numbers to real (or complex) numbers.

We can use $a_i = f(i)$ to denote this.

If $f(i)$ is defined on all $i \in \mathbb{N}$ then the sequence is infinite. Otherwise it is finite.

If a sequence is defined on $n \in \mathbb{N}$ and $n \neq 0$ then the sequence must be defined on $n - 1$.

For example $a_0, a_1, a_2, ...$ is a sequence, but $a_1, a_2, ...$ is not.

1.1.2 Monotone sequence

A monotone sequence is one where each element is succeeded ordinally by the next entry.

For example:

$< 1, 2, 3, 6, 7 >$ is monotone

$< 1, 2, 3, 3, 4 >$ is not monotone

An increasing sequence is one where:

$\forall m \in \mathbb{N} \forall n \in \mathbb{N}[m > n \leftrightarrow a_m \geq a_n]$

A strictly increasing sequence is one where:

$\forall m \in \mathbb{N} \forall n \in \mathbb{N}[m > n \leftrightarrow a_m > a_n]$

A decreasing sequence is one where:

$\forall m \in \mathbb{N} \forall n \in \mathbb{N}[m > n \leftrightarrow a_m \leq a_n]$

A strictly decreasing sequence is one where:

$\forall m \in \mathbb{N} \forall n \in \mathbb{N}[m > n \leftrightarrow a_m < a_n]$

All strictly decreasing sequences are decreasing, and all strictly increasing sequences are increasing.

A monotone sequence is one which is either increasing or decreasing.

1.1.3 Subsequences

A subsequence of a sequence is the original sequence with some elements of the original removed, not changing the order.

For example:

$< 1, 3, 5 >$ is a subsequence of $< 2, 1, 3, 4, 7, 5 >$

1.1.4 Bounded sequence

A function $f(x)$ on set X is bounded if:

$\exists M \in \mathbb{R}[\forall x \in X f(x) \leq M]$

A bounded sequence is a special case of a bounded function where:

$X = \mathbb{N}$

That is, a sequence is bounded by M iff:

$\forall n \in \mathbb{R}|f(a_n)| < leM$

2 Series

2.1 Series

2.1.1 Definition

A series is the summation of a sequence. For a series a_n there is a corresponding series:

$s_n = \sum_{i=0}^{n} a_n$

Where:

$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + ... + a_n$

2.1.2 Multiplication of summations

If all members of a sequence are multiplied by a constant, so is each member of the series.

We can take constants out of the series:

$s_n = \sum_{i=0}^{n} a_i$

$s_n = \sum_{i=0}^{n} cb_i$

$s_n = a \sum_{i=0}^{n} b_i$

2.1.3 Summation of constants

If all elements of a sequence are the same, then the series is a multiple of that constant.

$s_n = \sum_{i=0}^{n} a_i$

$s_n = \sum_{i=0}^{n} c$

$s_n = nc$

2.1.4 Addition of summations

Consider a sequence $a_i = b_i + c_i$.

$s_n = \sum_{i=0}^{n} a_i$

$s_n = \sum_{i=0}^{n} (b_i + c_i)$

We can then split this out.

$s_n = \sum_{i=0}^{n} b_i + \sum_{i=0}^{n} c_i$

2.1.5 Summation from a different start point

$\sum_{i=0}^{n} a_i = a_0 + \sum_{i=1}^{n} a_i$

2.1.6 Multiple summations

$\sum_{i=0}^{n} \sum_{j=0}^{m} a_i = n \sum_{j=0}^{m} a_i$

$\sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j = \sum_{i=0}^{n} a_i \sum_{j=0}^{m} b_i$

2.2 Summation of natural numbers

2.2.1 Goal

Let's prove that:

$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

2.2.2 Proof by induction

We use the inference rules Modus Ponens, which says that if X is true, and $X \to Y$ is true, then Y is true.

2.2.3 True for $n = 0$

We know this is true for $n = 0$:

$0 = \frac{0(0+1)}{2}$

$0 = 0$

2.2.4 If it's true for n, it's true for $n + 1$

We can also prove that if it true for n, it is true for $n + 1$.

$\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2}$

$(n + 1) + \sum_{i=0}^{n} i = \frac{n^2+3n+2}{2}$

If it is true for n, then:

$(n + 1) + \frac{n(n+1)}{2} = \frac{n^2+3n+2}{2}$

$\frac{n^2+3n+2}{2} = \frac{n^2+3n+2}{2}$

$1 = 1$

2.2.5 Result

So we know that it is true for $n = 0$, and if it is true for n, then it is true for $n + 1$. As a result it is true for all natural numbers.

3 Products

3.1 Products

A product is a repeated multiplication of a sequence.

$p_n = \prod_{i=0}^{n} s_i$

3.1.1 Multiplication of products

We can take constants out of the product.

$p_n = \prod_{i=0}^{n} ca_i$

$p_n = a^n \sum_{i=0}^{n} a_i$

3.1.2 Products of constants

If $a_i = c$ then the summation is then of the form:

$p_n = \prod_{i=0}^{n} c$

$p_n = c^n \prod_{i=j}^{n} 1$

$p_n = c^n$

3.1.3 Combining products

If a sequence is the product of to other sequences then the product of the sequence is equal to the product of the two individual sequences.

$p_n = \prod_{i=0}^{n} a_i$

$p_n = \prod_{i=0}^{n} b_i c_i$

$p_n = \prod_{i=0}^{n} b_i \prod_{i=0}^{n} c_i$

3.2 Factorials

A factorial is a a product across natural numbers. That is:

$n! := \prod_{i=0}^{n} i$