1 Addition and multiplication

1.1 Addition of natural numbers

1.1.1 Definition

Let's add another function: addition. Defined by:

$\forall a \in \mathbb{N}(a + 0 = a)$

$\forall ab \in \mathbb{N}(a + s(b) = s(a + b))$

That is, adding zero to a number doesn't change it, and $(a+b)+1 = a+(b+1)$.

1.1.2 Example

Let's use this to solve $1 + 2$:

$1 + 2 = 1 + s(1)$

$1 + s(1) = s(1 + 1)$

$s(1 + 1) = s(1 + s(0))$

$s(1 + s(0)) = s(s(1 + 0))$

$s(s(1 + 0)) = s(s(1))$

$s(s(1)) = s(2)$

$s(2) = 3$

$1 + 2 = 3$

All addition can be done iteratively like this.

1.1.3 Commutative property of addition

Addition is commutative:

$x + y = y + x$

1.1.4 Associative property of addition

Addition is associative:

$x + (y + z) = (x + y) + z$

1.2 Multiplication of natural numbers

1.2.1 Definition

Multiplication can be defined by:

$\forall a \in \mathbb{N}(a.0 = 0)$

$\forall ab \in \mathbb{N}(a.s(b) = a.b + a)$

1.2.2 Example

Let's calculate 2.2.

$2.2 = 2.s(1)$

$2.s(1) = 2.1 + 2$

$2.1 + 2 = 2.s(0) + 2$

$2.s(0) + 2 = 2.0 + 2 + 2$

$2.0 + 2 + 2 = 2 + 2$

$2 + 2 = 4$

1.2.3 Commutative property of multiplication

Multiplication is commutative:

$xy = yx$

1.2.4 Associative property of multiplication

Multiplication is associative:

$x(yz) = (xy)z$

1.2.5 Distributive property of multiplication

Multiplication is distributive over addition:

$a(b + c) = ab + ac$

$(a + b)c = ac + bc$

2 Integers

2.1 Subtraction of natural numbers

We have inverse functions for addition. This is subtraction.

For function $\oplus$, its inverse is $\oplus'$, as defined below:

$a \oplus b = c$

$b = c \oplus' a$

$f(a, b) = c \to f^{-1}(c, b) = a$

2.1.1 Subtraction

$a + b = c \to b = c - a$

There is no natural number b that satisfies:

$3 + b = 2$

While addition and multiplication are defined across all natural numbers, subtraction is not.

2.1.2 Properties of subtraction

Subtraction is not commutative:

$x - y \neq y - x$

Subtraction is not associative:

$x - (y - z) \neq (x - y) - z$

2.2 Integers

2.2.1 Defining integers

To extend the number line to negative numbers, we define:

$\forall ab \in \mathbb{N} \exists c(a + c = b)$

For any pair of numbers there exists a terms which can be added to one to get the other.

For $1 + x = 3$ this is another natural number, however for $3 + x = 1$ there is no such number.

Integers are defined as the solutions for any pair of natural numbers.

There are an infinite number of ways to write any integer. -1 can be written as $0 - 1$, $1 - 2$ etc.

The class of these terms form an equivalence class.

2.2.2 Integers as ordered pairs

Integers can be defined as an ordered pair of natural numbers, where the integer is valued at: $a - b$.

For example -1 could be shown as:

$-1 = \{\{0\}, \{0, 1\}\}$

$-1 = \{\{5\}, \{5, 6\}\}$

$(a, b) = a - b$

2.2.3 Converting natural numbers to integers

Natural numbers can be shown as integers by using:

$(n, 0)$

Natural numbers can be converted to integers:

$\{\{a\}, \{a, 0\}\}$

2.2.4 Cardinality of integers

2.3 Ordering of the integers

2.3.1 Ordering integers

Integers are an ordered pair of naturals.

$\{\{x\}, \{x, y\}\}$

For example -4 can be:

$\{\{4\}, \{4, 8\}\}$

$\{\{0\}, \{0, 8\}\}$

We extend the ordering to say:

$\{\{x\}, \{x, y\}\} \leq \{\{s(x)\}, \{s(x), y\}\}$

$\{\{x\}, \{x, s(y)\}\} \leq \{\{x\}, \{x, y\}\}$

So can we define this on an arbitrary pair:

$\{\{a\}, \{a, b\}\} \leq \{\{c\}, \{c, d\}\}$

We know that:

$\{\{a\}, \{a, b\}\} = \{\{s(a)\}, \{s(a), s(b)\}\}$

And either of:

$\{\{a\}, \{a, b\}\} = \{\{0\}, \{0, A\}\}$

$\{\{a\}, \{a, b\}\} = \{\{B\}, \{B, 0\}\}$

$\{\{a\}, \{a, b\}\} = \{\{0\}, \{0, 0\}\}$

As the latter is a case of either of the other 2, we consider only the first 2.

So we can define:

$\{\{a\}, \{a, b\}\} \leq \{\{c\}, \{c, d\}\}$

As any of:

$1 : \{\{0\}, \{0, A\}\} \leq \{\{0\}, \{0, C\}\}$

$2 : \{\{0\}, \{0, A\}\} \leq \{\{D\}, \{D, 0\}\}$

$3 : \{\{B\}, \{B, 0\}\} \leq \{\{0\}, \{0, C\}\}$

$4 : \{\{B\}, \{B, 0\}\} \leq \{\{D\}, \{D, 0\}\}$

Case 1:

$\{\{0\}, \{0, A\}\} \leq \{\{0\}, \{0, C\}\}$

Trivial, depends on relative size of A and C.

Case 2:

$\{\{0\}, \{0, A\}\} \leq \{\{D\}, \{D, 0\}\}$

We can see that:

$\{\{D\}, \{D, A\}\} \leq \{\{D\}, \{D, 0\}\}$

And therefore this holds.

Case 3:

$\{\{B\}, \{B, 0\}\} \leq \{\{0\}, \{0, C\}\}$

We can see that:

$\{\{B\}, \{B, 0\}\} \leq \{\{B\}, \{B, C\}\}$

And therefore this does not hold.

Case 4:

$\{\{B\}, \{B, 0\}\} \leq \{\{D\}, \{D, 0\}\}$

Trivial, like case 1.

2.4 Functions of integers

2.4.1 Addition

Then we can define addition as:

$(a, b) + (c, d) = (a + c, b + d)$

Integer addition can then be defined:

$a + b = \{\{a_1\}, \{a_1, a_2\}\} + \{\{b_1\}, \{b_1, b_2\}\}$

$a + b = \{\{a_1 + b_1\}, \{a_1 + b_1, a_2 + b_2\}\}$

Or:

$a + b = c$

$c_1 = a_1 + b_1$

$c_2 = a_2 + b_2$

2.4.2 Multiplication

Similarly, multiplication can be defined as:

$(a, b).(c, d) = (ac + bd, ad + bc)$

$ab = c$

$c_1 = a_1b_1 + a_2b_2$

$c_2 = a_2b_1 + a_1b_2$

2.4.3 Subtraction

$a - b = c$

$c_1 = a_1 + b_2$

$c_2 = a_2 + b_1$

2.5 Cardinality of the integers

2.5.1 Cardinality of integers

3 Rational numbers

3.1 Division

3.1.1 Introduction

We have inverse functions for multiplication. This is division.

These will not necessarily have solutions for natural numbers or integers.

3.1.2 Division of natural numbers

$a.b = c \rightarrow b = \frac{c}{a}$

3.1.3 Division is not commutative

Division is not commutative:

$\frac{x}{y} \neq \frac{y}{x}$

3.1.4 Division is not associative

$\frac{\frac{x}{y}}{z} \neq \frac{\frac{x}{y}}{z}$

3.1.5 Division is not left distributive

Division is not left distributive over subtraction:

$\frac{a}{b-c} \neq \frac{a}{b} - \frac{a}{c}$

3.1.6 Division is right distributive

Division is right distributive over subtraction:

$\frac{a-b}{c} = \frac{a}{b} - \frac{b}{c}$

3.1.7 Division of integers

3.2 Rational numbers

3.2.1 Defining rational numbers

We previously defined integers in terms of natural numbers. Similarly we can define rational numbers in terms of integers.

$\forall ab \in \mathbb{I}(\check{n}(b=0) \rightarrow \exists c(b.c=a))$

A rational is an ordered pair of integers.

$\{\{a\},\{a,b\}\}$

So that:

$\{\{a\},\{a,b\}\} = \frac{a}{b}$

3.2.2 Converting integers to rational numbers

Integers can be shown as rational numbers using:

$(i,1)$

Integers can then be turned into rational numbers:

$\mathbb{Q} = \frac{a}{1}$

$a = \frac{a_1}{a_2}$

$b = \frac{b_1}{b_2}$

$c = \frac{c_1}{c_2}$

3.2.3 Equivalence classes of rationals

There are an infinite number of ways to write any rational number, as with integers. $\frac{1}{2}$ can be written as $\frac{1}{2}$, $\frac{-2}{-4}$ etc.

The class of these terms form an equivalence class.

We can show these are equal:

$\frac{a}{b} = \{\{a\},\{a,b\}\}$

$\frac{ca}{cb} = \{\{a\},\{a,b\}\}$

$\frac{ca}{cb} = \{\{ca\},\{ca,cb\}\}$

$\{\{a\},\{a,b\}\} = \{\{ca\},\{ca,cb\}\}$

3.3 Ordering of rationals

3.4 Functions of rational numbers

3.4.1 Rational addition

Then we can define addition as:

$(a,b)+(c,d)=(a.d+b.c,b.d)$

$a+b=c$

$c_1=a_1b_2+a_2b_1$

$c_1=a_2b_2$

3.4.2 Rational subtraction

$a-b=c$

$c_1=a_1b_2-a_2b_1$

$c_1=a_2b_2$

3.4.3 Rational multiplication

Similarly, multiplication can be defined as:

$(a,b).(c,d)=(a.c,b.d)$

$ab=c$

$c_1=a_1b_1$

$c_1=a_2b_2$

3.4.4 Rational division

$\frac{a}{b}=c$

$c_1=a_1b_2$

$c_1=a_2b_1$

3.5 Cardinality of the rationals

3.5.1 Cardinality of rational numbers

We can see rational numbers as cartesian products of integers. That is:

$\mathbb{Q} = Z.Z$

We can order the rational numbers like so:

$\{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2} \frac{3}{1} \dots\}$

These can be mapped from natural numbers, so there is a bijunctive function.

So:

$|\mathbb{Q}| = |\mathbb{Z}.\mathbb{Z}| = |\mathbb{N}| = \aleph_0$

As: $|\mathbb{Z}.\mathbb{Z}| = |\mathbb{Z}|^2$

$|\mathbb{N}|^n = \mathbb{N}$

3.6 Fraction rules

3.6.1 Addition

$\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$

3.6.2 Multiplication

$\frac{A}{B}\frac{C}{D} = \frac{AC}{BD}$

3.6.3 b Scaler addition

$C + \frac{A}{B} = \frac{BC+A}{B}$

3.6.4 Scaler multiplication

$C\frac{A}{B} = \frac{AC}{B}$

3.6.5 Other

$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$

$\frac{A}{B} = \frac{AC}{BC}$

3.7 Partial fraction decomposition

We have: $\frac{1}{A.B}$

We want this in the form of:

$\frac{a}{A} + \frac{b}{B}$

First, lets define M as the mean of these two numbers, and define $\delta = M - B$. Then:

$\frac{1}{AB} = \frac{1}{(M+\delta)(M-\delta)} = \frac{a}{M+\delta} + \frac{b}{M-\delta}$

We can rearrange the latter two to find:

$1 = a(M - \delta) + b(M + \delta)$

Now we need to find values of a and b to choose.

Let's examine a.

$a = \frac{1-b(M+\delta)}{M-\delta}$

$a = -\frac{bM+b\delta-1}{M-\delta}$

$a = -\frac{bM+b\delta-1}{M-\delta}$

For this to divide neatly we need both the numerator to be a constant multiplier of the denominator. This means the ratio the multiplier for the left hand side of the denominator is equal to the right:

$\frac{bM}{M} = \frac{b\delta-1}{-\delta}$

$b = \frac{b\delta-1}{-\delta}$

$b = \frac{1}{2\delta}$

We can do the same for a.

$a = -\frac{1}{2\delta}$

We can plug these back into our original formula:

$\frac{1}{(M+\delta)(M-\delta)} = \frac{-\frac{1}{2\delta}}{M+\delta} + \frac{\frac{1}{2\delta}}{M-\delta}$

$\frac{1}{(M+\delta)(M-\delta)} = \frac{1}{2\delta}[\frac{1}{M-\delta} - \frac{1}{M+\delta}]$

3.8 Density of the rationals

3.8.1 Rationals are dense in rationals

For any pair of rationals, there is another rational between them:

$a = \frac{p}{q}$

$b = \frac{m}{n}$

Where $b > a$.

We define a new rational:

$c = \frac{a+b}{2}$

$c = \frac{pn+qm}{2qn}$

This is a rational number.

We can write:

$a = \frac{2pn}{2qn}$

$b = \frac{2qm}{2qn}$

As $b > a$ we know $2qm > 2pn$

So: $a < c < b$