1 Multivariate space

1.1 Regions

A region is a subset

1.1.1 Type-I regions (y-simple regions)

1.1.2 Type-II regions (x-simple regions)

1.1.3 Elementary regions

An elementary region is a region which is either a type-I region or a type-II region.

1.1.4 Simple regions

A simple region is a region which is both a type-I and a type-II region.

1.2 Curves and closed curves

In a space we can identify a curve between two points. If the input in the real numbers then this curve is unique.

For more general scalar fields this will not be the case. Two points in $\mathbb{R}^2$ could be joined by an infinite number of paths.

A curve can be defined as a function on the real numbers. The curve itself is totally ordered, and homogenous to the real number line.

We can write the curve therefore as:

$r : [a, b] \to C$

Where a and b are the start and end points of the curve, and C is the resulting curve.

1.2.1 Closed curves

If the start and end point of the curve are the same then the curve is closed. We can write this as:

$\oint_C f(r)ds = \int_a^b f(r(t))|r'(t)|dt$

1.3 Surfaces

1.4 Length of a curve

We have a curve from a to b in $\mathbf{R}^n$.

$f : [a, b] \to \mathbf{R}^n$

We divide this into n segments.

The ith cut is at:

$t_i = a + \frac{i}{n}(b - a)$

So the first cut is at:

$t_0 = a$

$t_n = b$

The distance between two sequential cuts is:

$||f(t_i) - f(t_{i-1}||$

The sum of all these differences is:

$L = \sum_{i=1}^n ||f(t_i) - f(t_{i-1})||$

The limit is:

$L = \lim_{n\to\infty} \sum_{i=1}^n ||f(t_i) - f(t_{i-1})||$

1.4.1 Method 1

$L = \lim_{n\to\infty} \sum_{i=1}^n ||f(t_i) - f(t_{i-1})||$

$L = \lim_{n\to\infty} \sum_{i=1}^n ||\frac{f(t_i) - f(t_{i-1})}{\Delta t}||\Delta t$

$L = \lim_{n\to\infty} \sum_{i=1}^n ||f'(t)||\Delta t$

$L = \int_a^b ||f'(t)||dt$

1.4.2 Method 2

$L = \lim_{n\to\infty} \sum_{i=1}^n ||f(t_i) - f(t_{i-1})||$

$L = \lim_{n\to\infty} \sum_{i=1}^n \sqrt{(f(t_i) - f(t_{i-1}))^* M(f(t_i) - f(t_{i-1}))}$

$L = \int_a^b \sqrt{(dt)^T M(dt)}$

2 Scalar fields

2.1 Total differentiation

2.1.1 Scalar fields

A scalar field is a function on an underlying input which produces a real output.

Inputs are not limited to real numbers. In this section we consider functions on vector spaces.

2.1.2 Total differentiation

Consider a multivariate function.

$f(x)$.

We can define:

$\Delta f(x, \Delta x) := f(x + \Delta x) - f(x)$

$\Delta f(x, \Delta x) = \sum_{i=1}^{n} f(x + \Delta x_i + \sum_{j=0}^{i-1} \Delta x_j) - f(x + \sum_{j=0}^{i-1} \Delta x_j)$

$\Delta f(x, \Delta x) = \sum_{i=1}^{n} \Delta x_i \frac{f(x+\Delta x_i + \sum_{j=0}^{i-1} \Delta x_j) - f(x + \sum_{j=0}^{i-1} \Delta x_j)}{\Delta x_i}$

$\frac{\Delta f}{\Delta x_k} = \sum_{i=1}^{n} \frac{\Delta x_i}{\Delta x_k} \frac{f(x+\Delta x_i + \sum_{j=0}^{i-1} \Delta x_j) - f(x + \sum_{j=0}^{i-1} \Delta x_j)}{\Delta x_i}$

$\lim_{\Delta x_k \to 0} \frac{\Delta f}{\Delta x_k} = \sum_{i=1}^{n} \lim_{\Delta x_k \to 0} \frac{\Delta x_i}{\Delta x_k} \frac{f(x+\Delta x_i + \sum_{j=0}^{i-1} \Delta x_j) - f(x + \sum_{j=0}^{i-1} \Delta x_j)}{\Delta x_i}$

$\frac{df}{dx_k} = \sum_{i=1}^{n} \frac{dx_i}{dx_k} \frac{\delta f}{\delta x_i}$

2.1.3 Total differentiation of a univariate function

For a univariate function total differentiation is the same as partial differentiation.

$\frac{df}{dx} = \frac{dx}{dx} \frac{\delta f}{\delta x}$

$\frac{df}{dx} = \frac{\delta f}{\delta x}$

2.2 Del

$\nabla = (\sum_{i=1}^{n} e_i \frac{\delta}{\delta x_i})$

Where e are the basis vectors.

This on its own means nothing. It is similar to the partial differentiation function.

2.3 Gradient

In a scalar field we can calculate the partial derivative at any point with respect to one input.

We may wish to consider these collectively. To do that we use the gradient operator.

We previously introduced the Del operator where:

$\nabla = (\sum_{i=1}^n e_i \frac{\delta}{\delta x_i})$

Where e are the basis vectors.

This on its own means nothing. It is similar to the partial differentiation function.

We now multiply Del by the function. This gives us:

$\nabla f = (\sum_{i=1}^n e_i \frac{\delta f}{\delta x_i})$. This gives us a vector in the underlying vector space.

This is the gradient.

2.4 Directional derivative

We have a function, $f(\mathbf{x})$.

Given a vector v, we can identify by how much this scalar function changes as you move in that direction.

$\nabla_v f(x) := \lim_{\delta \to 0} \dfrac{f(\mathbf{x} + \delta \mathbf{v}) - f(\mathbf{x})}{\delta}$

The directional derivative is the same dimension as underlying field.

2.4.1 Other

Differentiation of scalar field, df, can be defined as a vector field where grad is 0. can differ with orientation, scale

2.5 Line integral of scalar fields

2.6 Double integral of scalar fields

2.7 Surface integral of scalar fields

2.8 Gradient theorem

In a scalar field, the line integral of the gradient field is the difference between the value of the scalar field at the start and end points.

This generalises the fundamental theorem of calclulus.

2.9 Green's theorem

We have a curve C on a plane.

Inside this is region D.

We have two functions: $L(x,y)$ and $M(x,y)$ defined on the region and curve.

$\oint_C (Ldx + Mdy) = \int\int_D (\frac{\delta M}{\delta x} - \frac{\delta L}{\delta y})dxdy$

2.10 Differential forms

2.10.1 Type-I

For type-I, we can integrate over y, then integrate over x.

2.10.2 Type-II

For type-II, we can integrate over x, then integrate over y.

3 Vector fields

3.1 Jacobian matrix

If we have n inputs and m functions such that:

$f_i(\mathbf{x})$

The Jacobian is a matrix where:

$J_{ij} = \frac{\delta f_i}{\delta x_j}$

3.2 Divergence

This takes a vector field and produces a scalar field.

It is the dot product of the vector field with the del operator.

$divF = \nabla.F$

Where $\nabla = (\sum_{i=1}^n e_i \frac{\delta}{\delta x_i})$

$divF = \sum_{i=1}^n e_i \frac{\delta F_i}{\delta x_i}$

3.2.1 Divergence as net flow

Divergence can be thought of as the net flow into a point.

For example, if we have a body of water, and a vector field as the velocity at any given point, then the divergence is 0 at all points.

This is because water is incompressible, and so there can be no net flows.

Areas which flow out are sources, while areas that flow inwards are sinks.

3.2.2 Solenoidal vector fields

If there is no divergence, then the vector field is called solenoidal.

3.3 Scalar potential

Given a vector field $\mathbf{F}$ we may be able to identify a scalar field P such that:

$\mathbf{F} = -\nabla P$

3.4 Non-uniqueness of scalar potentials

Scalar potentials are not unique.

If P is a scalar potential of $\mathbf{F}$, then so is $P + c$, where c is a constant.

3.4.1 Conservative vector fields

Not all vector fields have scalar potentials. Those that do are conservative.

For example if a vector field is the gradient of a scalar height function, then the height is a scalar potential.

If a vector field is the rotation of water, there will not be a scalar potential.

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3.5 Curl

The curl of a vector field is defined as:

$curl\mathbf{F} = \nabla \times \mathbf{F}$

Where: $\nabla = (\sum_{i=1}^{n} e_i \frac{\delta}{\delta x_i})$

And: $\mathbf{x} \times \mathbf{y} = |||\mathbf{x}||||\mathbf{y}|| \sin(\theta)\mathbf{n}$

The curl of a vector field is another vector field.

The curl measures the rotation about a given point. For example if a vector field is the gradient of a height map, the curl is 0 at all points, however for a rotating body of water the curl reflects the rotation at a given point.

3.5.1 Divergence of the curl

If we have a vector field $\mathbf{F}$, the divergence of its curl is 0:

$\nabla.(\nabla \times \mathbf{F}) = 0$

3.6 Vector potential

Given a vector field $\mathbf{F}$ we may be able to identify another vector field A such that:

$\mathbf{F} = \nabla \times \mathbf{A}$

3.6.1 Existence

We know that the divergence of the curl for any vector field is 0, so this applies to A:

$\nabla.(\nabla \times \mathbf{A}) = 0$

Therefore:

$\nabla.\mathbf{F} = 0$

This means that if there is a vector potential of **F**, then **F** has no divergence.

3.6.2 Non-uniqueness of vector potentials

Vector potentials are not unique.

If **A** is a vector potential of **F**, then so is $\mathbf{A} + \nabla c$, where c is a scalar field and ∇c is its gradient.

3.6.3 Conservative vector fields

Not all vector fields have scalar potentials. Those that do are conservative.

For example if a vector field is the gradient of a scalar height function, then the height is a scalar potential.

If a vector field is the rotation of water, there will not be a scalar potential.

3.7 Line integral of vector fields

We may wish to integrate along a curve in a vector field.

We previously showed that we can write a curve as a function on the real line:

$r : [a, b] \to C$

The integral is therefore the sum of the function at all points, with some weighting. We write this:

$\int_C f(r)ds = \lim_{\Delta srightarrow0} \sum_{i=0}^{n} f(r(t_i))\Delta s_i$

In a vector field we use

$\int_C f(r)ds = \int_a^b f(r(t)).r'(t)dt$

3.8 Double integral of vector fields

3.9 Surface integral for vector fields

4 Stoke's theorem

4.1 The divergence theorem

4.2 Stoke's theorem