Contents

1 Tensors

1.1 Element-wise notation

1.1.1 Einstein summation convention

A vector can be written as a sum of its components.

$v=\sum_i e_i v^i$

The Einstein summation convention is to remove the $\sum_i$ symbols where they are implicit.

For example we would instead write the vector as:

$v=e_i v^i$

1.1.1.1 Adding vectors

$v+w=(\sum_i e_i v^i)+(\sum_i f_i w^i)$

$v + w = \sum_i (e_i v^i + f_i w^i)$

$v + w = e_i v^i + f_i w^i$

If the bases are the same then:

$v + w = e_i (v^i + w^i)$

1.1.1.2 Scalar multiplication

$cv = c \sum_i e_i v^i$

$cv = \sum_i c e_i v^i$

$cv = c e_i v^i$

1.1.1.3 Matrix multiplication

$AB_{ik} = \sum_j A_{ij} B_{jk}$

$AB_{ik} = A_{ij} B_{jk}$

1.1.1.4 Inner products

$\langle v, w \rangle = \langle \sum_i e_i v^i, \sum_j f_j w^j \rangle$

$\langle v, w \rangle = \sum_i v^i \langle e_i, \sum_j f_i w^j \rangle$

$\langle v, w \rangle = \sum_i \sum_j v^i \overline{w^j} \langle e_i, f_j \rangle$

If the two bases are the same then:

$\langle v, w \rangle = \sum_i \sum_j v^i \overline{w^j} \langle e_i, e_j \rangle$

We can define the metric as:

$g_{ij} := \langle e_i, e_j \rangle$

$\langle v, w \rangle = v^i \overline{w^j} g_{ij}$

1.1.2 Covariant and contravariant bases

In element form we write a vector as:

$v = e_i v^i$

The indices are raised and lowered to reflect whether the value is covariant or contravariant.

v^i is contravariant. If the basis moves one way, it moves the other.

e_i is covariant. If the basis moves, it moves with it.

1.2 Tensor product

1.2.1 Tensor product

We have spaces V and W over field F. If we have a linear operation which takes a vector from each space and returns a scalar from the underlying field, it is an element of the tensor product of the two spaces.

For example if we have two vectors:

$v = e_i v^i$

$w = e_j w^j$

A tensor product would take these and return a scalar.

There are three types of tensor products:

- Both are from the vector space
- $T_{ij} v^i w^j$
- $T_{ij} \in V \otimes W$
- Both are from the dual space
- $T^{ij} v_i w_j$
- $T_{ij} \in V^* \otimes W^*$
- One is from each space
- $T_i^j v^i w_j$
- $T_{ij} \in V \otimes W^*$

As a vector space, we can add together tensor products, and do scalar multiplication.

1.2.1.1 Homomorphisms

We can define homomorphisms in terms of tensor products.

$Hom(V) = V \otimes V^*$

T_j^i

We use the dual space for the second argument. This is because it ensures that changes to the bases do not affect the maps.

$w^j = T_i^j v^i$

1.2.2 Raising and lowering indices

We showed that the inner product between two vectors with the same basis can be written as:

$\langle v, w \rangle = \langle \sum_i e_i v^i, \sum_j f_j w^j \rangle$

$\langle v, w \rangle = v^i \overline{w^j} \langle e_i, e_j \rangle$

Defining the metric as:

$g_{ij} := \langle e_i, e_j \rangle$

$\langle v, w \rangle = v^i \overline{w^j} g_{ij}$

1.2.2.1 Metric inverse

We can use this to define the inverse of the metric.

$g^{ij} := (g_{ij})^{-1}$

We can use this to raise and lower vectors.

$v_i := v^j g_{ij}$

1.2.2.2 Raising and lowering indices of tensors

If we have tensor:

T_{ij}

We can define:

$T_i^k = T_{ij} g^{jk}$

$T^{il} = T_{ij} g^{jk} g^{kl}$

1.2.2.3 Tensor contraction

If we have:

$T_{ij} x^j$

We can contract it to:

$T_{ij} x^j = v_i$

Similarly we can have:

$T^{ij} x_j = v^i$

1.2.3 Kronecker delta

Consider matrix multiplication AI.

We have:

$AI_{ik} = A_{ij}I_{jk}$

We write this instead as:

$AI_{ik} = A_{ij}\delta_{jk}$

Where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$.

1.2.4 Tensors form a vector space

1.2.4.1 Recap

1.2.4.2 Tensors form a vector space

1.2.4.3 Dimension of a tensor

1.2.4.4 Basis of a tensor

1.3 Tensors

1.3.1 Tensor valence

1.3.2 Tensor inverses

For second order tensors we have:

- T^i_j
- T_{ij}
- T^{ij}

For each of these we can define an inverse:

- $T^j_i U^k_j = \delta^k_i$
- $T_{ij}U^{jk} = \delta^k_i$
- $T^{ij}U_{jk} = \delta^k_i$

1.3.2.1 Notation for inverses

If we have $T_{ij}U^{jk} = \delta_i^k$, we can instead write:

$T_{ij}T^{jk} = \delta_i^k$

1.3.3 Tensor contraction

We have a vector $v \in V$ and $w \in V^*$.

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w_i \mathbf{f}^i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w_i \mathbf{f}^i]$

$\mathbf{wv} = \sum_i \sum_j [v^i \mathbf{e}_i][w_j \mathbf{f}^j]$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{f}^j$

We use the dual basis so:

$\mathbf{wv} = \sum_i \sum_j v^i w_j \mathbf{e}_i \mathbf{e}^j$

$\mathbf{wv} = \sum_i \sum_j v^i w_j \delta_i^j$

We can see that this value is unchanged when there is a change in basis.

What if these were both from V?

$\mathbf{v} = \sum_i v^i \mathbf{e}_i$

$\mathbf{w} = \sum_i w^i \mathbf{e}_i$

$\mathbf{wv} = [\sum_i v^i \mathbf{e}_i][\sum_i w^i \mathbf{e}_i]$

$\mathbf{wv} = \sum_i \sum_j v^i w^j \mathbf{e}_i \mathbf{e}_i$

This term is dependent on the basis, and so we do not contract.

So if we have $v_i w^i$, we can contract, because the result (calculated from the components) does not depend on the basis.

But if we have $v_i w_i$, the result (calcualted from the components) will change depending on the choice of basis.

We define a new object

$c = \sum_i w^i v_i$

This new term, c, does not depend on i, and so we have contracted the index.

1.3.4 Symmetric and antisymmetric tensors

Consider a tensor, e g T_{abc}.

In general, this is not symmetric, that is:

$T_{abc} \neq T_{bac}$

1.3.4.1 Symmetric part of a tensor

We can write the symmetric part of this with regard to a and b.

$T_{(ab)c} = \frac{1}{2}(T_{abc} + T_{bac})$

Clearly, $T_{(ab)c} = T_{(ba)c}$

1.3.4.2 Antisymmetric part of a tensor

We can also have an an antisymmetric part with regard to a and b.

$T_{[ab]c} = \frac{1}{2}(T_{abc} - T_{bac})$

Clearly, $T_{[ab]c} = -T_{[ba]c}$

1.3.4.3 Tensors as sums of their symmetric and antisymmetric parts

$T_{(ab)c} + T_{[ab]c} = \frac{1}{2}(T_{abc} + T_{bac}) + \frac{1}{2}(T_{abc} - T_{bac})$

$T_{(ab)c} + T_{[ab]c} = T_{abc}$

1.4 Higher-order tensors

1.4.1 Higher-order tensors

We can create higher order tensors products. For example

$V \otimes V \otimes V \otimes V^* \otimes V^*$

We write elements of these as:

$T^{i_1,\dots,i_p}_{j_1,\dots,j_q}$

We can map from matrix to matrix etc higher dimensional

Matrix has A: a_ij

Tensor can have T: t_ijk for example

0 rank tensor: scalar

1 rank tensor: vector

2 rank tensor: matrix

page on covariance and contravariance and type (p,q)

1.5 Sort

1.5.1 Outer product

1.5.1.1 The outer product is a bilinear map

This is a bilinear map from two vectors from the same vector space to another vector space.

$V \times V \to V$

1.5.1.2 Calculating the outer product

$u \otimes v = w$

$w_{ij} = u_i v_j$

1.5.1.3 The dimensions of the tensor outer product

$\dim(V \otimes W) = \dim V \times \dim W$

1.5.1.4 Outer product on the complex numbers

1.5.1.5 Relation between the dot product and outer product

The dot product in the trace of the outer product.

1.5.2 Kronecker product

The Kronecker product takes the concept of the outer product and applies to to matrices.

We can essentially repace every element in the matrix on the left with the element multiplied by the entire matrix on the right.

Like outer products, Kronecker products are written as:

$u \otimes v = w$

1.5.3 Dot product

1.5.3.1 Dot product is a bilinear form

This is a bilinear form, a mapping from two vectors in the same vector space to the underlying field.

$V \times V \to F$

1.5.3.2 Calculating the dot product

This is calculated by multiplying each matching element, and summing the results.

$u \cdot v = \sum_{i=1}^{n} u_i v_i$

1.5.3.3 Dot product on the complex numbers

Properties don't hold. Can get zero vectors from non-zero inputs. Get complex numbers from dot product on itself.

Inner products better deal with complex number fields. However they are not bilinear maps.

1.5.4 Homomorphism as a tensor product

1.5.5 Tensors

A tensor is an element of a tensor product.