1 Godunov Method

We start with the hyperbolic equations of motion written in vector form,

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F} = \mathbf{S} \tag{1}$$

We integrate this over a cell with volume V, and use the divergence theorem to get the integral equations of motion,

$$\frac{d}{dt}\frac{1}{V}\int dV\,\mathbf{U} + \frac{1}{V}\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \frac{1}{V}\int dV\,\mathbf{S} \tag{2}$$

Define the volume averaged quantities as $\bar{\mathbf{U}} = \frac{1}{V}\int dV\,\mathbf{U}$, so that now,

$$\frac{d}{dt}\bar{\mathbf{U}} + \frac{1}{V}\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \bar{\mathbf{S}} \tag{3}$$

Now integrate in time from $t = 0$ to $t = \Delta t$,

$$\bar{\mathbf{U}}(\Delta t) - \bar{\mathbf{U}} + \frac{1}{V}\int dt\,\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \int dt\,\bar{\mathbf{S}} \tag{4}$$

Up until this point we haven't made any approximations. The trick now is to evaluate the time-averaged boundary fluxes and source terms. We would also like to retain at least second order accuracy in time and space. To do this we'll use a MUSCLE-Hancock scheme with slope limiters and an approximate Riemann solver.

2 MUSCLE-Hancock Scheme

This summary is taken from Toro p.557.

1. Set boundary conditions
2. Set timestep based on CFL condition.

$$\Delta t = C_{cfl}\frac{\Delta x}{S_{\max}} \tag{5}$$

   where $S_{\max}$ is the maximum wave speed. This is typically the faster of advection, sound speeds, viscous speeds, etc.

3. Data reconstruction and boundary extrapolated values. Use the primitive equation,

$$\partial_t \mathbf{W} + \mathbf{A}(\mathbf{W})\partial_x \mathbf{W} = 0 \tag{6}$$

   To evolve the boundary extrapolated values half a timestep,

$$\mathbf{W}_L = \mathbf{W}_i^n + \frac{1}{2}\left[\mathbf{I} - \frac{\Delta t}{\Delta x}\mathbf{A}(\mathbf{W}_i^n)\right]\Delta_i, \tag{7}$$

$$\mathbf{W}_R = \mathbf{W}_{i+1}^n - \frac{1}{2}\left[\mathbf{I} + \frac{\Delta t}{\Delta x}\mathbf{A}(\mathbf{W}_{i+1}^n)\right]\Delta_{i+1}, \tag{8}$$

   where Δ_i are the slopes of the primitive variables to be determined below.

4. Solution of Riemann problem at each interface. The Riemann problem uses $\mathbf{W}^{L,R}$ to determine $\mathbf{W}_{i+1/2,j}(x/t)$ in the x direction. The interface fluxes are then,

$$\mathbf{F}_{i+1/2,j} = \mathbf{F}\left(\mathbf{W}_{i+1/2,j}(0)\right) \qquad \mathbf{G}_{i,j+1/2} = \mathbf{G}\left(\mathbf{W}_{i,j+1/2}(0)\right) \tag{9}$$

IF your cell is moving with some speed $\mathbf{w} = (w_x, w_y)$ (e.g if you have a Lagrangian mesh) then you would evaluate the fluxes at $x/t = w_x$ and $y/t = w_y$ rather than $x/t = y/t = 0$.

2.1 Slopes and Slope-Limiters

The slopes are,

$$\Delta_i = \frac{1}{2}(1+w)\Delta_{i-1/2} + \frac{1}{2}(1-w)\Delta_{i+1/2} \qquad \Delta_{i+1/2} = \mathbf{U}^n_{i+1} - \mathbf{U}^n_i \tag{10}$$

The simplest limiter to use is the MINBEE/SUBERBEE limiter,

$$\Delta_i = \begin{cases} \max\left[0, \min(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \min(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right], & \Delta_{i+1/2} > 0, \\ \min\left[0, \max(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \max(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right], & \Delta_{i+1/2} < 0 \end{cases} \tag{11}$$

where $\beta = 1, 2$ correspond to the MINBEE and SUPERBEE limiters.

3 HLLC Riemann Solver

The HLLC solver puts the contact wave back into the HLL solver.

1. Get wave speeds S_L, $S_\star$, S_R.
2. Construct $\mathbf{U}^\star_L$ and $\mathbf{U}^\star_R$.
3. Calculate $\mathbf{F}^{hllc}_*$.

3.1 Wave speeds

Wave speeds are obtained from approximate simple Riemann solvers depending on the left-right states. These solvers are the primitive variable RS (PVRS), the two-rarefaction RS (TRRS), and the two-shock RS (TSRS). If the pressure jump at the interface is less than a user specified ratio (typically, $p_{max}/p_{pmin} < 2$) then the flow is smooth and the PVRS is used to estimate $p_\star$ and $u_\star$. If the pressure jump is larger than this ratio, there is likely either a shock or a rarefaction present. If the interface pressure, $p_\star$, given from the PVRS is less than p_{min}, then the rarefaction solver, TRRS, is used, else the shock solver, TSRS, is used.

The estimates for the three approximate solvers for the pressure and velocity are,

$$p_{pvrs} = \frac{1}{2}(p_L + p_R) - \frac{1}{2}(u_R - u_L)C \tag{12}$$

$$u_{pvrs} = \frac{1}{2}(u_L + u_R) - \frac{1}{2}\frac{p_R - p_L}{C} \tag{13}$$

$$C = \frac{\rho_L + \rho_R}{2}\frac{a_L + a_R}{2} \tag{14}$$

$$p_{trrs} = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/p_L^z + a_r/p_R^z}\right]^z \tag{15}$$

$$u_{trrs} = \frac{P_{LR}u_L/a_L + u_R/a_R + \frac{2(P_{LR}-1)}{\gamma-1}}{P_{LR}/a_L + 1/a_R} \tag{16}$$

$$z = \frac{\gamma - 1}{2\gamma} \qquad P_{LR} = \left(\frac{p_L}{p_R}\right)^z \tag{17}$$

$$p_{tsrs} = \frac{g_L(p_0)p_L + g_R(p_0)p_R - (u_R - u_L)}{g_L(p_0) + g_r(p_0)} \tag{18}$$

$$u_{tsrs} = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\left[(p_{tsrs} - p_R)g_R(p_0) - (p_{tsrs} - p_L)g_L(p_0)\right] \tag{19}$$

$$g_K(p) = \sqrt{\frac{A_K}{p + B_K}} \qquad p_0 = max(0, p_{pvrs}) \tag{20}$$

$$A_K = \frac{2}{\rho_K(\gamma + 1)} \qquad B_K = \left(\frac{\gamma - 1}{\gamma + 1}\right)p_K \tag{21}$$

The estimates for the interface pressure and velocity are then,

$$p_\star, u_\star = \begin{cases} p_{pvrs}, u_{pvrs} & \frac{p_{max}}{p_{min}} < 2 \\ p_{trrs}, u_{trrs} & \frac{p_{max}}{p_{min}} > 2 \text{ and } p_{pvrs} < p_{max} \\ p_{tsrs}, u_{tsrs} & \frac{p_{max}}{p_{min}} > 2 \text{ and } p_{pvrs} > p_{max} \end{cases} \tag{22}$$

Now that we have $p_\star$ and $u_\star$ we can calculate the minimum, maxmimum and intermediate wave speeds as,

$$S_L = u_L - a_L q_L \tag{23}$$

$$S_L = u_R + a_R q_R \tag{24}$$

$$S_\star = \frac{p_R - p_L + \rho_L u_L(S_L - u_L) - \rho_R u_R(S_R - u_R)}{\rho_L(S_L - u_L) - \rho_R(S_R - u_R)} \tag{25}$$

$$q_K = \begin{cases} 1 & p_\star \le p_K \\ \sqrt{1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_\star}{p_K} - 1\right)} & p_\star > p_K \end{cases} \tag{26}$$

3.2 Star region

Now that we have the wave speeds the conservative left and right states in the starred region are,

$$\mathbf{U}_K^\star = \rho_K \left(\frac{S_K - u_K}{S_K - S_\star}\right) \begin{bmatrix} 1 \\ S_\star \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_\star - u_K)\left[S_\star + \frac{p_K}{\rho_K(S_K - u_K)}\right] \end{bmatrix} \tag{27}$$

Additionally, any passive scalar is advected in the same way as the tangential velocities, i.e

$$(\rho q)_\star^K = \rho_K \left(\frac{S_K - u_K}{S_K - S_\star}\right) q_k \tag{28}$$

3.3 Final flux

Finally, the HLLC flux is,

$$\mathbf{F}_{i+1/2}^{hllc} = \begin{cases} \mathbf{F}_L & 0 \leq S_L \\ \mathbf{F}_L + S_L(\mathbf{U}_\star^L - \mathbf{U}_L) & S_L \leq 0 \leq S_\star \\ \mathbf{F}_R + S_R(\mathbf{U}_\star^R - \mathbf{U}_R) & S_\star \leq 0 \leq S_R \\ \mathbf{F}_R & 0 \geq S_R \end{cases} \tag{29}$$