1 Godunov Method

We start with the hyperbolic equations of motion written in vector form,

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F} = \mathbf{S} \tag{1}$$

We integrate this over a cell with volume V, and use the divergence theorem to get the integral equations of motion,

$$\frac{d}{dt}\frac{1}{V}\int dV\,\mathbf{U} + \frac{1}{V}\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \frac{1}{V}\int dV\,\mathbf{S} \tag{2}$$

Define the volume averaged quantities as $\bar{\mathbf{U}} = \frac{1}{V}\int dV\,\mathbf{U}$, so that now,

$$\frac{d}{dt}\bar{\mathbf{U}} + \frac{1}{V}\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \bar{\mathbf{S}} \tag{3}$$

Now integrate in time from $t = 0$ to $t = \Delta t$,

$$\bar{\mathbf{U}}(\Delta t) - \bar{\mathbf{U}} + \frac{1}{V}\int dt\,\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \int dt\,\bar{\mathbf{S}} \tag{4}$$

Up until this point we haven't made any approximations. The trick now is to evaluate the time-averaged boundary fluxes and source terms. We would also like to retain at least second order accuracy in time and space. To do this we'll use a MUSCLE-Hancock scheme with slope limiters and an approximate Riemann solver.

2 MUSCLE-Hancock Scheme

This summary is taken from Toro p.557.

1. Set boundary conditions

2. Set timestep based on CFL condition.

$$\Delta t = C_{cfl}\frac{\Delta x}{S_{\max}} \tag{5}$$

   where $S_{\max}$ is the maximum wave speed. This is typically the faster of advection, sound speeds, viscous speeds, etc.

3. Data reconstruction and boundary extrapolated values. Use the primitive equation,

$$\partial_t \mathbf{W} + \mathbf{A}(\mathbf{W})\partial_x \mathbf{W} = 0 \tag{6}$$

   To evolve the boundary extrapolated values half a timestep,

$$\mathbf{W}_L = \mathbf{W}_i^n + \frac{1}{2}\left[\mathbf{I} - \frac{\Delta t}{\Delta x}\mathbf{A}(\mathbf{W}_i^n)\right]\Delta_i, \tag{7}$$

$$\mathbf{W}_R = \mathbf{W}_{i+1}^n - \frac{1}{2}\left[\mathbf{I} + \frac{\Delta t}{\Delta x}\mathbf{A}(\mathbf{W}_{i+1}^n)\right]\Delta_{i+1}, \tag{8}$$

   where Δ_i are the slopes of the primitive variables to be determined below.

4. Solution of Riemann problem at each interface. The Riemann problem uses $\mathbf{W}^{L,R}$ to determine $\mathbf{W}_{i+1/2,j}(x/t)$ in the x direction. The interface fluxes are then,

$$\mathbf{F}_{i+1/2,j} = \mathbf{F}\left(\mathbf{W}_{i+1/2,j}(0)\right) \qquad \mathbf{G}_{i,j+1/2} = \mathbf{G}\left(\mathbf{W}_{i,j+1/2}(0)\right) \tag{9}$$

   IF your cell is moving with some speed $\mathbf{w} = (w_x, w_y)$ (e.g if you have a Lagrangian mesh) then you would evaluate the fluxes at $x/t = w_x$ and $y/t = w_y$ rather than $x/t = y/t = 0$.

2.1 Slopes and Slope-Limiters

The slopes are,

$$\Delta_i = \frac{1}{2}(1+w)\Delta_{i-1/2} + \frac{1}{2}(1-w)\Delta_{i+1/2} \qquad \Delta_{i+1/2} = \mathbf{U}^n_{i+1} - \mathbf{U}^n_i \tag{10}$$

The simplest limiter to use is the MINBEE/SUBERBEE limiter,

$$\Delta_i = \begin{cases} \max\left[0, \min(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \min(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right], & \Delta_{i+1/2} > 0, \\ \min\left[0, \max(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \max(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right], & \Delta_{i+1/2} < 0 \end{cases} \tag{11}$$

where $\beta = 1, 2$ correspond to the MINBEE and SUPERBEE limiters.

3 HLLC Riemann Solver

The HLLC solver puts the contact wave back into the HLL solver.

1. Get wave speeds S_L, $S_\star$, S_R.
2. Construct $\mathbf{U}^\star_L$ and $\mathbf{U}^\star_R$.
3. Calculate $\mathbf{F}^{hllc}_*$.

3.1 Wave speeds

Wave speeds are obtained from approximate simple Riemann solvers depending on the left-right states. These solvers are the primitive variable RS (PVRS), the two-rarefaction RS (TRRS), and the two-shock RS (TSRS). If the pressure jump at the interface is less than a user specified ratio (typically, $p_{max}/p_{pmin} < 2$) then the flow is smooth and the PVRS is used to estimate $p_\star$ and $u_\star$. If the pressure jump is larger than this ratio, there is likely either a shock or a rarefaction present. If the interface pressure, $p_\star$, given from the PVRS is less than p_{min}, then the rarefaction solver, TRRS, is used, else the shock solver, TSRS, is used.

The estimates for the three approximate solvers for the pressure and velocity are,

$$p_{pvrs} = \frac{1}{2}(p_L + p_R) - \frac{1}{2}(u_R - u_L)C \tag{12}$$

$$u_{pvrs} = \frac{1}{2}(u_L + u_R) - \frac{1}{2}\frac{p_R - p_L}{C} \tag{13}$$

$$C = \frac{\rho_L + \rho_R}{2}\frac{a_L + a_R}{2} \tag{14}$$

$$p_{trrs} = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/p_L^z + a_r/p_R^z}\right]^z \tag{15}$$

$$u_{trrs} = \frac{P_{LR}u_L/a_L + u_R/a_R + \frac{2(P_{LR}-1)}{\gamma-1}}{P_{LR}/a_L + 1/a_R} \tag{16}$$

$$z = \frac{\gamma - 1}{2\gamma} \qquad P_{LR} = \left(\frac{p_L}{p_R}\right)^z \tag{17}$$

$$p_{tsrs} = \frac{g_L(p_0)p_L + g_R(p_0)p_R - (u_R - u_L)}{g_L(p_0) + g_r(p_0)} \tag{18}$$

$$u_{tsrs} = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\left[(p_{tsrs} - p_R)g_R(p_0) - (p_{tsrs} - p_L)g_L(p_0)\right] \tag{19}$$

$$g_K(p) = \sqrt{\frac{A_K}{p + B_K}} \qquad p_0 = max(0, p_{pvrs}) \tag{20}$$

$$A_K = \frac{2}{\rho_K(\gamma+1)} \qquad B_K = \left(\frac{\gamma-1}{\gamma+1}\right)p_K \tag{21}$$

The estimates for the interface pressure and velocity are then,

$$p_\star, u_\star = \begin{cases} p_{pvrs}, u_{pvrs} & \frac{p_{max}}{p_{min}} < 2 \\ p_{trrs}, u_{trrs} & \frac{p_{max}}{p_{min}} > 2 \,\text{and}\, p_{pvrs} < p_{max} \\ p_{tsrs}, u_{tsrs} & \frac{p_{max}}{p_{min}} > 2 \,\text{and}\, p_{pvrs} > p_{max} \end{cases} \tag{22}$$

Now that we have $p_\star$ and $u_\star$ we can calculate the minimum, maxmimum and intermediate wave speeds as,

$$S_L = u_L - a_L q_L \tag{23}$$

$$S_L = u_R + a_R q_R \tag{24}$$

$$S_\star = \frac{p_R - p_L + \rho_L u_L (S_L - u_L) - \rho_R u_R (S_R - u_R)}{\rho_L(S_L - u_L) - \rho_R(S_R - u_R)} \tag{25}$$

$$q_K = \begin{cases} 1 & p_\star \leq p_K \\ \sqrt{1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_\star}{p_K} - 1\right)} & p_\star > p_K \end{cases} \tag{26}$$

3.2 Star region

Now that we have the wave speeds the conservative left and right states in the starred region are,

$$\mathbf{U}_K^\star = \rho_K \left(\frac{S_K - u_K}{S_K - S_\star}\right) \begin{bmatrix} 1 \\ S_\star \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_\star - u_K)\left[S_\star + \frac{p_K}{\rho_K(S_K - u_K)}\right] \end{bmatrix} \tag{27}$$

Additionally, any passive scalar is advected in the same way as the tangential velocities, i.e

$$(\rho q)_\star^K = \rho_K \left(\frac{S_K - u_K}{S_K - S_\star}\right) q_k \tag{28}$$

3.3 Final flux

Finally, the HLLC flux is,

$$\mathbf{F}_{i+1/2}^{hllc} = \begin{cases} \mathbf{F}_L & 0 \leq S_L \\ \mathbf{F}_L + S_L(\mathbf{U}_\star^L - \mathbf{U}_L) & S_L \leq 0 \leq S_\star \\ \mathbf{F}_R + S_R(\mathbf{U}_\star^R - \mathbf{U}_R) & S_\star \leq 0 \leq S_R \\ \mathbf{F}_R & 0 \geq S_R \end{cases} \tag{29}$$

4 Equations of motion for orthogonal coordinate system

For an orthogonal coordinate system (x_i, x_j, x_k) with diagonal metric $g_{ij} = h_i^2 \delta_{ij}$, scale factors h_i, coordinate vectors $\mathbf{e}_i = h_i \hat{\mathbf{e}}_i$, the volume element is $\Delta V = dv \Delta x_1 \Delta x_2 \Delta x_3$ where $dv \equiv h_1 h_2 h_3$, the surface area elements are, $\Delta S_i = ds_i \Delta x_j \Delta x_k$, and where $ds_i \equiv dv/h_i$, where i, j, k are cyclic indices (so no Einstein summation)

The gradient of a scalar, Φ is,

$$\nabla \Phi = \frac{1}{h_i}\frac{\partial \Phi}{\partial x_i}\hat{\mathbf{x}}_i + \frac{1}{h_j}\frac{\partial \Phi}{\partial x_j}\hat{\mathbf{x}}_j + \frac{1}{h_k}\frac{\partial \Phi}{\partial x_k}\hat{\mathbf{x}}_k \tag{30}$$

The Laplacian is,

$$dv\nabla^2\Phi = \frac{\partial}{\partial x_i}\left(\frac{ds_i}{h_i}\frac{\partial\Phi}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{ds_j}{h_j}\frac{\partial\Phi}{\partial x_j}\right) + \frac{\partial}{\partial x_k}\left(\frac{ds_k}{h_k}\frac{\partial\Phi}{\partial x_k}\right) \tag{31}$$

The divergence of a vector $\mathbf{v}$ is,

$$dv(\nabla\cdot\mathbf{v}) = \frac{\partial}{\partial x_i}(ds_i v_i) + \frac{\partial}{\partial x_j}(ds_j v_j) + \frac{\partial}{\partial x_k}(ds_k v_k) \tag{32}$$

The divergence of a vector $\mathbf{v}$ is,

$$ds_i\left(\nabla\times\mathbf{v}\right)\cdot\hat{\mathbf{x}}_i = \frac{\partial}{\partial x_j}(h_k v_k) - \frac{\partial}{\partial x_k}(h_j v_j) \tag{33}$$

The divergence of a tensor, $\mathbf{T}$, is,

$$\begin{aligned} dv\left(\nabla\cdot\mathbf{T}\right)\cdot\hat{\mathbf{x}}_i = &\frac{\partial}{\partial x_i}(ds_i T_{ii}) + \frac{\partial}{\partial x_j}(ds_j T_{ij}) + \frac{\partial}{\partial x_k}(ds_k T_{ik}) \\ &+ T_{ij}ds_j\frac{1}{h_i}\frac{\partial h_i}{\partial x_j} + T_{ki}ds_k\frac{1}{h_i}\frac{\partial h_i}{\partial x_k} - T_{jj}ds_i\frac{1}{h_j}\frac{\partial h_j}{\partial x_i} - T_{kk}ds_i\frac{1}{h_k}\frac{\partial h_k}{\partial x_i} \end{aligned} \tag{34}$$

We can simplify this further for symmetric tensors, $\mathbf{T}=\mathbf{S}$, and diagonal tensors, $T_{ij} = P\delta_{i,j}$

$$dv\left(\nabla\cdot\mathbf{S}\right)\cdot\hat{\mathbf{x}}_i = \frac{\partial}{\partial x_i}(ds_i S_{ii}) + \frac{1}{h_i}\frac{\partial}{\partial x_j}(h_i ds_j S_{ij}) + \frac{1}{h_i}\frac{\partial}{\partial x_k}(h_i ds_k S_{ik}) - S_{jj}h_k\frac{\partial h_j}{\partial x_i} - S_{kk}h_j\frac{\partial h_k}{\partial x_i} \tag{35}$$

$$dv\left(\nabla\cdot\mathbf{P}\right)\cdot\hat{\mathbf{x}}_i = \frac{\partial}{\partial x_i}(ds_i P) - P\frac{\partial(ds_i)}{\partial x_i} \tag{36}$$

where again the indices ijk are not summed over but instead are cyclic $i\to j\to k$. The point of this form is that if you have a coordinate system where the scale factors only depend on one of the coordinates, then then the non divergence terms for a symmetric tensor will be zero in two of the directions. This is useful for conservation properties. The diagonal tensor non-divergence term evaluates to $-P$.

For the Euler equations we have,

$$\begin{aligned} dv\frac{\partial(\rho v_i)}{\partial t} &+ \frac{\partial}{\partial x_i}\left(ds_i(\rho v_i^2 + P)\right) + \frac{1}{h_i}\frac{\partial}{\partial x_j}(h_i ds_j \rho v_i v_j) + \frac{1}{h_i}\frac{\partial}{\partial x_k}(h_i ds_k \rho v_i v_k) \\ &- \rho v_j^2 h_k\frac{\partial h_j}{\partial x_i} - \rho v_k^2 h_j\frac{\partial h_k}{\partial x_i} - P\frac{\partial(ds_i)}{\partial x_i} \end{aligned} \tag{37}$$

$$dv\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(ds_i\rho v_i) + \frac{\partial}{\partial x_j}(ds_j\rho v_j) + \frac{\partial}{\partial x_k}(ds_k\rho v_k) = 0 \tag{38}$$

$$dv\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}(ds_i(E+P)v_i) + \frac{\partial}{\partial x_j}(ds_j(E+P)v_j) + \frac{\partial}{\partial x_k}(ds_k(E+P)v_k) = 0 \tag{39}$$

$$\tag{40}$$

where $E = P/(\gamma-1) + \rho v^2/2$.

All fluxes are then weighted by the surface area of the cell's face in the update equation,

$$\frac{d}{dt}\frac{1}{V}\int dVQ + \frac{1}{V}\left(S^+F^+ - S^-F^-\right) = \frac{1}{V}\int dVS \tag{41}$$

4.0.1 Cartsian

In cartesian all scale factors are unity, $h=1, ds=1, dv=1$.

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho v_i^2 + P\right) + \frac{\partial}{\partial x_j}(\rho v_i v_j) + \frac{\partial}{\partial x_k}(\rho v_i v_k) = 0 \tag{42}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_j) + \frac{\partial}{\partial x_k}(\rho v_k) = 0 \tag{43}$$

4.0.2 Cylindrical

In cylindrical (r, ϕ, z), the only non-unity scale factors are $h_\phi = ds_r = ds_z = dv = r$

$$r\frac{\partial(\rho v_r)}{\partial t} + \frac{\partial}{\partial r}\left(r\rho v_r^2 + rP\right) + \frac{\partial}{\partial \phi}\left(\rho v_r v_\phi\right) + \frac{\partial}{\partial z}\left(r\rho v_r v_z\right) - \rho v_\phi^2 - P = 0 \tag{44}$$

$$r\frac{\partial(\rho v_\phi)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\rho v_r v_\phi\right) + \frac{\partial}{\partial \phi}\left(\rho v_\phi^2 + P\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(r^2\rho v_\phi v_z\right) = 0 \tag{45}$$

$$r\frac{\partial(\rho v_z)}{\partial t} + \frac{\partial}{\partial r}\left(r\rho v_r v_z\right) + \frac{\partial}{\partial \phi}\left(\rho v_\phi v_z\right) + \frac{\partial}{\partial z}\left(r\rho v_z^2 + rP\right) = 0 \tag{46}$$

$$r\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(r\rho v_r\right) + \frac{\partial}{\partial \phi}\left(\rho v_\phi\right) + \frac{\partial}{\partial z}\left(r\rho v_z\right) \tag{47}$$

4.0.3 Spherical

In spherical (r, θ, ϕ), the non-unity scale factors are, $h_\phi = r\sin\theta, h_\theta = r, ds_r = dv = r^2\sin\theta, ds_\phi = r$, and $ds_\theta = r\sin\theta$

$$\begin{aligned} r^2\sin\theta\frac{\partial(\rho v_r)}{\partial t} &+ \frac{\partial}{\partial r}\left(r^2\sin\theta(\rho v_r^2 + P)\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta\rho v_r v_\theta\right) + \frac{\partial}{\partial \phi}\left(r\rho v_r v_\phi\right) \\ &- r\rho v_\theta^2 - r\sin\theta\rho v_\phi^2 - 2Pr\sin\theta = 0 \end{aligned} \tag{48}$$

$$\begin{aligned} r^2\sin\theta\frac{\partial(\rho v_\theta)}{\partial t} &+ \frac{\partial}{\partial r}\left(r^3\sin\theta\rho v_r v_\theta\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta(\rho v_\theta^2 + P)\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(r^2\rho v_\phi v_\theta\right) \\ &- r\cos\theta\rho v_\phi^2 - Pr\cos\theta = 0 \end{aligned} \tag{49}$$

$$r^2\sin\theta\frac{\partial(\rho v_\phi)}{\partial t} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r^3\sin^2\theta\rho v_r v_\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(r^2\sin^2\theta\rho v_\theta v_\phi\right) + \frac{\partial}{\partial \phi}\left(r\rho v_\phi^2 + rP\right) = 0 \tag{50}$$

$$r^2\sin\theta\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(r^2\sin\theta\rho v_r\right) + \frac{\partial}{\partial \phi}\left(r\rho v_\phi\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta\rho v_\theta\right) = 0 \tag{51}$$

5 CTU