1 Godunov Method

We start with the hyperbolic equations of motion written in vector form,

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F} = \mathbf{S} \tag{1}$$

We integrate this over a cell with volume V, and use the divergence theorem to get the integral equations of motion,

$$\frac{d}{dt}\frac{1}{V}\int dV\,\mathbf{U} + \frac{1}{V}\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \frac{1}{V}\int dV\,\mathbf{S} \tag{2}$$

Define the volume averaged quantities as $\bar{\mathbf{U}} = \frac{1}{V}\int dV\,\mathbf{U}$, so that now,

$$\frac{d}{dt}\bar{\mathbf{U}} + \frac{1}{V}\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \bar{\mathbf{S}} \tag{3}$$

Now integrate in time from $t = 0$ to $t = \Delta t$,

$$\bar{\mathbf{U}}(\Delta t) - \bar{\mathbf{U}} + \frac{1}{V}\int dt\,\left(\mathbf{F}^+ - \mathbf{F}^-\right) = \int dt\,\bar{\mathbf{S}} \tag{4}$$

Up until this point we haven't made any approximations. The trick now is to evaluate the time-averaged boundary fluxes and source terms. We would also like to retain at least second order accuracy in time and space. To do this we'll use a MUSCLE-Hancock scheme with slope limiters and an approximate Riemann solver.

2 MUSCLE-Hancock Scheme

This summary is taken from Toro p.557.

1. Set boundary conditions
2. Set timestep based on CFL condition.
$$\Delta t = C_{cfl}\frac{\Delta x}{S_{\max}} \tag{5}$$
where $S_{\max}$ is the maximum wave speed. This is typically the faster of advection, sound speeds, viscous speeds, etc.
3. Data reconstruction and boundary extrapolated values. Use the primitive equation,
$$\partial_t \mathbf{W} + \mathbf{A}(\mathbf{W})\partial_x \mathbf{W} = 0 \tag{6}$$
To evolve the boundary extrapolated values half a timestep,
$$\mathbf{W}_L = \mathbf{W}_i^n + \frac{1}{2}\left[\mathbf{I} - \frac{\Delta t}{\Delta x}\mathbf{A}(\mathbf{W}_i^n)\right]\Delta_i, \tag{7}$$
$$\mathbf{W}_R = \mathbf{W}_{i+1}^n - \frac{1}{2}\left[\mathbf{I} + \frac{\Delta t}{\Delta x}\mathbf{A}(\mathbf{W}_{i+1}^n)\right]\Delta_{i+1}, \tag{8}$$
where Δ_i are the slopes of the primitive variables to be determined below.
4. Solution of Riemann problem at each interface. The Riemann problem uses $\mathbf{W}^{L,R}$ to determine $\mathbf{W}_{i+1/2,j}(x/t)$ in the x direction. The interface fluxes are then,
$$\mathbf{F}_{i+1/2,j} = \mathbf{F}\left(\mathbf{W}_{i+1/2,j}(0)\right) \qquad \mathbf{G}_{i,j+1/2} = \mathbf{G}\left(\mathbf{W}_{i,j+1/2}(0)\right) \tag{9}$$
IF your cell is moving with some speed $\mathbf{w} = (w_x, w_y)$ (e.g if you have a Lagrangian mesh) then you would evaluate the fluxes at $x/t = w_x$ and $y/t = w_y$ rather than $x/t = y/t = 0$.

2.1 Slopes and Slope-Limiters

The slopes are,

$$\Delta_i = \frac{1}{2}(1+w)\Delta_{i-1/2} + \frac{1}{2}(1-w)\Delta_{i+1/2} \qquad \Delta_{i+1/2} = \mathbf{U}_{i+1}^n - \mathbf{U}_i^n \tag{10}$$

The simplest limiter to use is the MINBEE/SUBERBEE limiter,

$$\Delta_i = \begin{cases} \max\left[0, \min(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \min(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right], & \Delta_{i+1/2} > 0, \\ \min\left[0, \max(\beta\Delta_{i-1/2}, \Delta_{i+1/2}), \max(\Delta_{i-1/2}, \beta\Delta_{i+1/2})\right], & \Delta_{i+1/2} < 0 \end{cases} \tag{11}$$

where $\beta = 1, 2$ correspond to the MINBEE and SUPERBEE limiters.

3 HLLC Riemann Solver

The HLLC solver puts the contact wave back into the HLL solver.

1. Get wave speeds S_L, $S_\star$, S_R.
2. Construct $\mathbf{U}_L^\star$ and $\mathbf{U}_R^\star$.
3. Calculate $\mathbf{F}_*^{hllc}$.

3.1 Wave speeds

Wave speeds are obtained from approximate simple Riemann solvers depending on the left-right states. These solvers are the primitive variable RS (PVRS), the two-rarefaction RS (TRRS), and the two-shock RS (TSRS). If the pressure jump at the interface is less than a user specified ratio (typically, $p_{max}/p_{pmin} < 2$) then the flow is smooth and the PVRS is used to estimate $p_\star$ and $u_\star$. If the pressure jump is larger than this ratio, there is likely either a shock or a rarefaction present. If the interface pressure, $p_\star$, given from the PVRS is less than p_{min}, then the rarefaction solver, TRRS, is used, else the shock solver, TSRS, is used.

The estimates for the three approximate solvers for the pressure and velocity are,

$$p_{pvrs} = \frac{1}{2}(p_L + p_R) - \frac{1}{2}(u_R - u_L)C \tag{12}$$

$$u_{pvrs} = \frac{1}{2}(u_L + u_R) - \frac{1}{2}\frac{p_R - p_L}{C} \tag{13}$$

$$C = \frac{\rho_L + \rho_R}{2}\frac{a_L + a_R}{2} \tag{14}$$

$$p_{trrs} = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/p_L^z + a_r/p_R^z}\right]^z \tag{15}$$

$$u_{trrs} = \frac{P_{LR}u_L/a_L + u_R/a_R + \frac{2(P_{LR}-1)}{\gamma-1}}{P_{LR}/a_L + 1/a_R} \tag{16}$$

$$z = \frac{\gamma-1}{2\gamma} \qquad P_{LR} = \left(\frac{p_L}{p_R}\right)^z \tag{17}$$

$$p_{tsrs} = \frac{g_L(p_0)p_L + g_R(p_0)p_R - (u_R - u_L)}{g_L(p_0) + g_r(p_0)} \tag{18}$$

$$u_{tsrs} = \frac{1}{2}(u_L + u_R) + \frac{1}{2}\left[(p_{tsrs} - p_R)g_R(p_0) - (p_{tsrs} - p_L)g_L(p_0)\right] \tag{19}$$

$$g_K(p) = \sqrt{\frac{A_K}{p + B_K}} \qquad p_0 = max(0, p_{pvrs}) \tag{20}$$

$$A_K = \frac{2}{\rho_K(\gamma+1)} \qquad B_K = \left(\frac{\gamma-1}{\gamma+1}\right) p_K \tag{21}$$

The estimates for the interface pressure and velocity are then,

$$p_\star, u_\star = \begin{cases} p_{pvrs}, u_{pvrs} & \frac{p_{max}}{p_{min}} < 2 \\ p_{trrs}, u_{trrs} & \frac{p_{max}}{p_{min}} > 2 \,\text{and}\, p_{pvrs} < p_{max} \\ p_{tsrs}, u_{tsrs} & \frac{p_{max}}{p_{min}} > 2 \,\text{and}\, p_{pvrs} > p_{max} \end{cases} \tag{22}$$

Now that we have $p_\star$ and $u_\star$ we can calculate the minimum, maxmimum and intermediate wave speeds as,

$$S_L = u_L - a_L q_L \tag{23}$$

$$S_L = u_R + a_R q_R \tag{24}$$

$$S_\star = \frac{p_R - p_L + \rho_L u_L (S_L - u_L) - \rho_R u_R (S_R - u_R)}{\rho_L (S_L - u_L) - \rho_R (S_R - u_R)} \tag{25}$$

$$q_K = \begin{cases} 1 & p_\star \le p_K \\ \sqrt{1 + \frac{\gamma+1}{2\gamma}\left(\frac{p_\star}{p_K} - 1\right)} & p_\star > p_K \end{cases} \tag{26}$$

3.2 Star region

Now that we have the wave speeds the conservative left and right states in the starred region are,

$$\mathbf{U}_K^\star = \rho_K \left(\frac{S_K - u_K}{S_K - S_\star}\right) \begin{bmatrix} 1 \\ S_\star \\ v_K \\ w_K \\ \frac{E_K}{\rho_K} + (S_\star - u_K)\left[S_\star + \frac{p_K}{\rho_K(S_K - u_K)}\right] \end{bmatrix} \tag{27}$$

Additionally, any passive scalar is advected in the same way as the tangential velocities, i.e

$$(\rho q)_\star^K = \rho_K \left(\frac{S_K - u_K}{S_K - S_\star}\right) q_k \tag{28}$$

3.3 Final flux

Finally, the HLLC flux is,

$$\mathbf{F}_{i+1/2}^{hllc} = \begin{cases} \mathbf{F}_L & 0 \le S_L \\ \mathbf{F}_L + S_L(\mathbf{U}_\star^L - \mathbf{U}_L) & S_L \le 0 \le S_\star \\ \mathbf{F}_R + S_R(\mathbf{U}_\star^R - \mathbf{U}_R) & S_\star \le 0 \le S_R \\ \mathbf{F}_R & 0 \ge S_R \end{cases} \tag{29}$$

4 Equations of motion for orthogonal coordinate system

For an orthogonal coordinate system (x_i, x_j, x_k) with diagonal metric $g_{ij} = h_i^2 \delta_{ij}$, scale factors h_i, coordinate vectors $\mathbf{e}_i = h_i \hat{\mathbf{e}}_i$, the volume element is $\Delta V = dv \Delta x_1 \Delta x_2 \Delta x_3$ where $dv \equiv h_1 h_2 h_3$, the surface area elements are, $\Delta S_i = ds_i \Delta x_j \Delta x_k$, and where $ds_i \equiv dv/h_i$, where i, j, k are cyclic indices (so no Einstein summation)

The gradient of a scalar, Φ is,

$$\nabla \Phi = \frac{1}{h_i}\frac{\partial \Phi}{\partial x_i}\hat{\mathbf{x}}_i + \frac{1}{h_j}\frac{\partial \Phi}{\partial x_j}\hat{\mathbf{x}}_j + \frac{1}{h_k}\frac{\partial \Phi}{\partial x_k}\hat{\mathbf{x}}_k \tag{30}$$

The Laplacian is,

$$dv\nabla^2\Phi = \frac{\partial}{\partial x_i}\left(\frac{ds_i}{h_i}\frac{\partial \Phi}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{ds_j}{h_j}\frac{\partial \Phi}{\partial x_j}\right) + \frac{\partial}{\partial x_k}\left(\frac{ds_k}{h_k}\frac{\partial \Phi}{\partial x_k}\right) \tag{31}$$

The divergence of a vector $\mathbf{v}$ is,

$$dv(\nabla\cdot\mathbf{v}) = \frac{\partial}{\partial x_i}(ds_i v_i) + \frac{\partial}{\partial x_j}(ds_j v_j) + \frac{\partial}{\partial x_k}(ds_k v_k) \tag{32}$$

The divergence of a vector $\mathbf{v}$ is,

$$ds_i\left(\nabla\times\mathbf{v}\right)\cdot\hat{\mathbf{x}}_i = \frac{\partial}{\partial x_j}(h_k v_k) - \frac{\partial}{\partial x_k}(h_j v_j) \tag{33}$$

The divergence of a tensor, $\mathbf{T}$, is,

$$\begin{aligned} dv\left(\nabla\cdot\mathbf{T}\right)\cdot\hat{\mathbf{x}}_i &= \frac{\partial}{\partial x_i}(ds_i T_{ii}) + \frac{\partial}{\partial x_j}(ds_j T_{ij}) + \frac{\partial}{\partial x_k}(ds_k T_{ik}) \\ &\quad + T_{ij}ds_j\frac{1}{h_i}\frac{\partial h_i}{\partial x_j} + T_{ki}ds_k\frac{1}{h_i}\frac{\partial h_i}{\partial x_k} - T_{jj}ds_i\frac{1}{h_j}\frac{\partial h_j}{\partial x_i} - T_{kk}ds_i\frac{1}{h_k}\frac{\partial h_k}{\partial x_i} \end{aligned} \tag{34}$$

We can simplify this further for symmetric tensors, $\mathbf{T}=\mathbf{S}$, and diagonal tensors, $T_{ij} = P\delta_{i,j}$

$$dv\left(\nabla\cdot\mathbf{S}\right)\cdot\hat{\mathbf{x}}_i = \frac{\partial}{\partial x_i}(ds_i S_{ii}) + \frac{1}{h_i}\frac{\partial}{\partial x_j}(h_i ds_j S_{ij}) + \frac{1}{h_i}\frac{\partial}{\partial x_k}(h_i ds_k S_{ik}) - S_{jj}h_k\frac{\partial h_j}{\partial x_i} - S_{kk}h_j\frac{\partial h_k}{\partial x_i} \tag{35}$$

$$dv\left(\nabla\cdot\mathbf{P}\right)\cdot\hat{\mathbf{x}}_i = \frac{\partial}{\partial x_i}(ds_i P) - P\frac{\partial(ds_i)}{\partial x_i} \tag{36}$$

where again the indices ijk are not summed over but instead are cyclic $i\to j\to k$. The point of this form is that if you have a coordinate system where the scale factors only depend on one of the coordinates, then then the non divergence terms for a symmetric tensor will be zero in two of the directions. This is useful for conservation properties. The diagonal tensor non-divergence term evaluates to $-P$.

For the Euler equations we have,

$$\begin{aligned} dv\frac{\partial(\rho v_i)}{\partial t} &+ \frac{\partial}{\partial x_i}\left(ds_i(\rho v_i^2 + P)\right) + \frac{1}{h_i}\frac{\partial}{\partial x_j}(h_i ds_j \rho v_i v_j) + \frac{1}{h_i}\frac{\partial}{\partial x_k}(h_i ds_k \rho v_i v_k) \\ &- \rho v_j^2 h_k\frac{\partial h_j}{\partial x_i} - \rho v_k^2 h_j\frac{\partial h_k}{\partial x_i} - P\frac{\partial(ds_i)}{\partial x_i} \end{aligned} \tag{37}$$

$$dv\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(ds_i\rho v_i) + \frac{\partial}{\partial x_j}(ds_j\rho v_j) + \frac{\partial}{\partial x_k}(ds_k\rho v_k) = 0 \tag{38}$$

$$dv\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}(ds_i(E+P)v_i) + \frac{\partial}{\partial x_j}(ds_j(E+P)v_j) + \frac{\partial}{\partial x_k}(ds_k(E+P)v_k) = 0 \tag{39}$$

$$\tag{40}$$

where $E = P/(\gamma-1) + \rho v^2/2$.

All fluxes are then weighted by the surface area of the cell's face in the update equation,

$$\frac{d}{dt}\frac{1}{V}\int dVQ + \frac{1}{V}\left(S^+F^+ - S^-F^-\right) = \frac{1}{V}\int dVS \tag{41}$$

4.0.1 Cartsian

In cartesian all scale factors are unity, $h=1, ds=1, dv=1$.

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho v_i^2 + P\right) + \frac{\partial}{\partial x_j}(\rho v_i v_j) + \frac{\partial}{\partial x_k}(\rho v_i v_k) = 0 \tag{42}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_j) + \frac{\partial}{\partial x_k}(\rho v_k) = 0 \tag{43}$$

4.0.2 Cylindrical

In cylindrical (r, ϕ, z), the only non-unity scale factors are $h_\phi = ds_r = ds_z = dv = r$

$$r\frac{\partial(\rho v_r)}{\partial t} + \frac{\partial}{\partial r}\left(r\rho v_r^2 + rP\right) + \frac{\partial}{\partial \phi}\left(\rho v_r v_\phi\right) + \frac{\partial}{\partial z}\left(r\rho v_r v_z\right) - \rho v_\phi^2 - P = 0 \tag{44}$$

$$r\frac{\partial(\rho v_\phi)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\rho v_r v_\phi\right) + \frac{\partial}{\partial \phi}\left(\rho v_\phi^2 + P\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(r^2\rho v_\phi v_z\right) = 0 \tag{45}$$

$$r\frac{\partial(\rho v_z)}{\partial t} + \frac{\partial}{\partial r}\left(r\rho v_r v_z\right) + \frac{\partial}{\partial \phi}\left(\rho v_\phi v_z\right) + \frac{\partial}{\partial z}\left(r\rho v_z^2 + rP\right) = 0 \tag{46}$$

$$r\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(r\rho v_r\right) + \frac{\partial}{\partial \phi}\left(\rho v_\phi\right) + \frac{\partial}{\partial z}\left(r\rho v_z\right) \tag{47}$$

4.0.3 Spherical

In spherical (r, θ, ϕ), the non-unity scale factors are, $h_\phi = r\sin\theta, h_\theta = r, ds_r = dv = r^2\sin\theta, ds_\phi = r$, and $ds_\theta = r\sin\theta$

$$\begin{aligned} r^2\sin\theta\frac{\partial(\rho v_r)}{\partial t} &+ \frac{\partial}{\partial r}\left(r^2\sin\theta(\rho v_r^2 + P)\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta\rho v_r v_\theta\right) + \frac{\partial}{\partial \phi}\left(r\rho v_r v_\phi\right) \\ &- r\rho v_\theta^2 - r\sin\theta\rho v_\phi^2 - 2Pr\sin\theta = 0 \end{aligned} \tag{48}$$

$$\begin{aligned} r^2\sin\theta\frac{\partial(\rho v_\theta)}{\partial t} &+ \frac{\partial}{\partial r}\left(r^3\sin\theta\rho v_r v_\theta\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta(\rho v_\theta^2 + P)\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(r^2\rho v_\phi v_\theta\right) \\ &- r\cos\theta\rho v_\phi^2 - Pr\cos\theta = 0 \end{aligned} \tag{49}$$

$$r^2\sin\theta\frac{\partial(\rho v_\phi)}{\partial t} + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(r^3\sin^2\theta\rho v_r v_\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(r^2\sin^2\theta\rho v_\theta v_\phi\right) + \frac{\partial}{\partial \phi}\left(r\rho v_\phi^2 + rP\right) = 0 \tag{50}$$

$$r^2\sin\theta\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(r^2\sin\theta\rho v_r\right) + \frac{\partial}{\partial \phi}\left(r\rho v_\phi\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta\rho v_\theta\right) = 0 \tag{51}$$

5 Non-orthogonal

In a non-orthogonal coordinate system with metric g_{ij}, the fluid equations are,

$$\frac{\partial \rho}{\partial t} + \nabla_i\left(\rho V^i\right) = 0 \tag{52}$$

$$\frac{\partial}{\partial t}(\rho V^j) + \nabla_i\left(\rho V^i V^j\right) = -\nabla_i P \tag{53}$$

To mirror what we did for the orthogonal coordinate systems we can write this in a mixed basis,

$$\frac{\partial}{\partial t}(\sqrt{g}\rho) + \frac{\partial}{\partial x^i}\left(\sqrt{g}\rho V^i\right) = 0 \tag{54}$$

$$\frac{\partial}{\partial t}(\sqrt{g}\rho V_j) + \frac{\partial}{\partial x^i}\left(\sqrt{g}T^i_j\right) = -\sqrt{g}\frac{\partial}{\partial x^i}P + \sqrt{g}T^j_k\Gamma^k_{ij} \tag{55}$$

where $\sqrt{g} = \det(g)$, $T^{ij} = \rho V^i V^j$, and

$$T^i_j = g_{jk}T^{ki} = \rho g_{jk}V^k V^i = \rho V_j V^i = \rho V_j g^{ik} V_k \tag{56}$$

Note that $V_i = g_{ij}V^j = v_i/h^i$ where v_i is the physical velocity so that T^i_j is,

$$T^i_j = \frac{\rho}{h^j h_k} v_j g^{ik} v_k \tag{57}$$

$$\frac{\partial}{\partial t}(\sqrt{g}\rho) + \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{h^k}\rho g^{ik} v_k\right) = 0 \tag{58}$$

$$\frac{\partial}{\partial t}(\frac{\sqrt{g}}{h^j}\rho v_j) + \frac{\partial}{\partial x^i}\left(\frac{\sqrt{g}}{h^j h_k}\rho v_j g^{ik} v_k\right) = -\sqrt{g}\frac{\partial}{\partial x^i}P + \frac{\sqrt{g}}{h^k h_j}\rho v_k g^{j\ell} v_\ell \Gamma^k_{ij} \tag{59}$$

6 CTU

1. For each direction in the problem
   (a) Solve for left/right interface states, U_L and U_R for each cell
      i. This entails limiting the left and right slopes and evolving $U_{L,R}$ for $\Delta t/2$ using the fluxes, $F(U_{L,R})$. This can be PCM, PLM, PPM, WENO, etc., for first order, second order, third order, and higher order, respectively.
   (b) Compute the fluxes at the interfaces by solving the Riemann problem $F(U_L, U_R)$.
2. Using the fluxes in each direction, update $U_{L,R}$ in each direction by using the fluxes in the *transverse* directions to evolve $U_{L,R}$ for $\Delta t/2$.
3. Recompute the fluxes in each direction by solving the Riemann problem at the interfaces with the corrected $U_{L,R}$.
4. Update the conserved variables in each cell using the corrected interface fluxes in each direction for a full Δt.

6.1 1D Timestepping algorithm

Let the conserved variables be $U = \{\rho, \rho v_x, \rho v_y, \rho v_z, E\}$, and the flux be $F = \left\{\rho v_x, \rho v_x^2 + P, \rho v_x v_y, \rho v_x v_z, v_x(E + P)\right\}$ so that in 1D,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{60}$$

Let U_i^n represent the volume averaged conserved variables in cell i at time t_n. To go from U_i^n to U_i^{n+1} at time $t_{n+1} = t_n + \Delta t$ we do the following,

1. Reconstruct the values of $U_{i+1/2}$ from the gradient estimates of $U_i - U_{i-1}$ and $U_{i+1} - U_i$. This gives the quantities, U_i^L and U_i^R which are defined at the boundary between cell i and cell $i+1$.
2. Evolve U_i^L and U_i^R for $\Delta t/2$ by the fluxes $F(U_i^L)$ and $F(U_i^R)$,

$$U_i^{L,n+1/2} = U_i^{L,n} - \frac{\Delta t}{2\Delta x}\left(F(U_i^L) - F(U_{i-1}^R)\right) \tag{61}$$

$$U_{i-1}^{R,n+1/2} = U_{i-1}^{R,n} - \frac{\Delta t}{2\Delta x}\left(F(U_i^L) - F(U_{i-1}^R)\right) \tag{62}$$

   The index convention is that $U_i^{L,R}$ lie at the right interface of cell i, so at $i + 1/2$.
3. Solve for the interface flux by solving the 1D Riemann problem $F_i = F(U_i^{L,n+1/2}, U_i^{R,n+1/2})$.
4. Given the left and right fluxes, F_{i-1} and F_i update the volume averaged conserved variables,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_i - F_{i-1}\right) \tag{63}$$

6.2 1D Timestepping algorithm with potential source

With a time independent potential the equations of motion have a source term,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S \tag{64}$$

where $S = \{0, -\rho\partial\Phi/\partial x, -\rho\partial\Phi/\partial y, -\rho\partial\Phi/\partial z, -\rho v \cdot \nabla\Phi\}$ The Reconstruct-Evolve-Solve-Update algorithm given above is essentially the same, the only difference is including the source term in the Evolve and Update steps.

1. Reconstruct the values of $U_{i+1/2}$ from the gradient estimates of $U_i - U_{i-1}$ and $U_{i+1} - U_i$. This gives the quantities, U_i^L and U_i^R which are defined at the boundary between cell i and cell $i+1$.

2. Evolve U_i^L and U_i^R for $\Delta t/2$ by the fluxes $F(U_i^L)$ and $F(U_i^R)$,

$$U_i^{L,n+1/2,*} = U_i^{L,n} - \frac{\Delta t}{2\Delta x}\left(F(U_i^L) - F(U_{i-1}^R\right) \tag{65}$$
$$U_{i-1}^{R,n+1/2,*} = U_{i-1}^{R,n} - \frac{\Delta t}{2\Delta x}\left(F(U_i^L) - F(U_{i-1}^R\right) \tag{66}$$
$$\tag{67}$$

3. Evolve these interface states for $\Delta t/2$ using the source term. For the momenta, this is

$$U_i^{L,n+1/2} = U_i^{L,n+1/2,*} - \frac{\Delta t}{\Delta x}\rho_i^{L,n+1/2,*}\left(\Phi_{i+1/2} - \Phi_i\right) \tag{68}$$
$$U_{i-1}^{R,n+1/2} = U_{i-1}^{R,n+1/2,*} - \frac{\Delta t}{\Delta x}\rho_{i-1}^{R,n+1/2,*}\left(\Phi_i - \Phi_{i-1/2}\right) \tag{69}$$

   where U for the 1D case is ρv_x. Note that this is equivalent to updating the primitive velocity, v_x.

4. Solve for the interface flux by solving the 1D Riemann problem $F_i = F(U_i^{L,n+1/2}, U_i^{R,n+1/2})$.

5. Compute the density at $\Delta t/2$ using the fluxes, F_i.

$$\rho_i^{n+1/2} = \rho_i^n - \frac{\Delta t}{2\Delta x}\left(F_i^\rho - F_{i-1}^\rho\right) \tag{70}$$

   where $F^\rho = \rho v_x$ is the density component of the flux.

6. Given the left and right fluxes, F_{i-1} and F_i, and $\rho_i^{n+1/2}$, update the volume averaged conserved variables. For the momenta,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_i - F_{i-1}\right) - \frac{\Delta t}{\Delta x}S_i \tag{71}$$

   where S_i is zero for the density,

$$\rho_i^{n+1/2}\left(\Phi_{i+1/2} - \Phi_{i-1/2}\right) \tag{72}$$

   for the momenta, and

$$F_i^\rho\left(\Phi_{i+1/2} - \Phi_i\right) + F_{i-1}^\rho\left(\Phi_i - \Phi_{i-1/2}\right) \tag{73}$$

   for the energy.

6.3 2D Timestepping algorithm

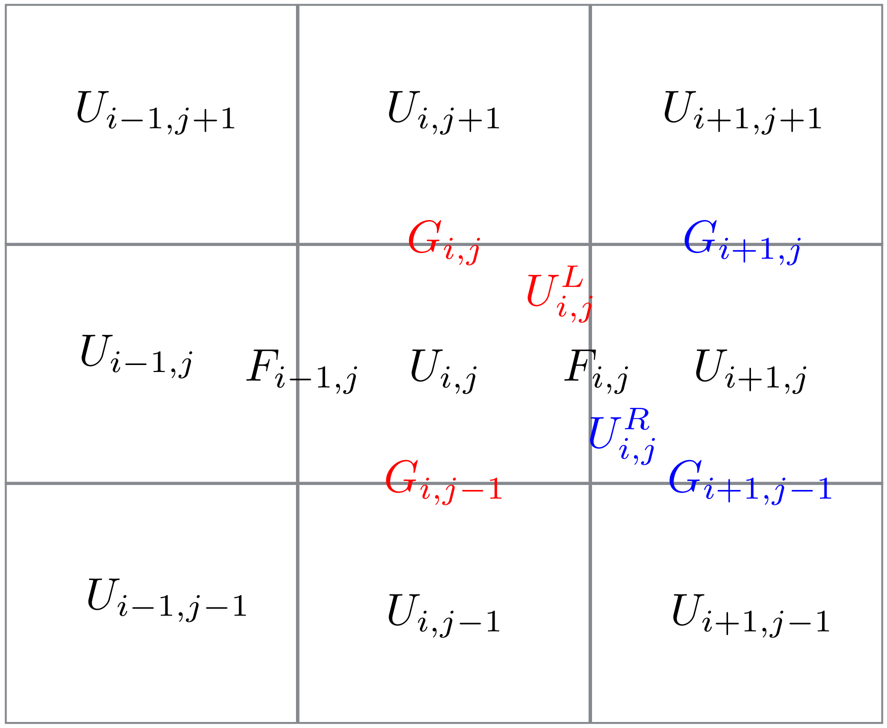
In 2D we have an additional flux in the y direction, $G = \left\{\rho v_y, \rho v_x v_y, \rho v_y^2 + P, \rho v_y v_z, v_y(E+P)\right\}$

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{74}$$

Let $U_{i,j}^n$ represent the volume averaged conserved variables in cell (i,j) at time t_n. The first steps are the same as the 1D algorithm,

1. Reconstruct-Evolve-Solve in x direction to obtain $U_{i,j}^{L,R,n}$ and $F_{i,j}^n$.

2. Reconstruct-Evolve-Solve in y direction to obtain $U_{i,j}^{L,R,n}$ and $G_{i,j}^n$.



3. Evolve the left and right states using the *transverse* fluxes,

$$U^{L,n+1/2}_{i+1/2,j} = U^{L,n}_{i+1/2,j} - \frac{\Delta t}{2\Delta y}\left(G^n_{i,j} - G^n_{i,j-1}\right) \tag{75}$$

$$U^{R,n+1/2}_{i+1/2,j} = U^{R,n}_{i+1/2,j} - \frac{\Delta t}{2\Delta y}\left(G^n_{i+1,j} - G^n_{i+1,j-1}\right) \tag{76}$$

$$U^{L,n+1/2}_{i,j+1/2} = U^{L,n}_{i,j+1/2} - \frac{\Delta t}{2\Delta x}\left(F^n_{i,j} - F^n_{i-1,j}\right) \tag{77}$$

$$U^{R,n+1/2}_{i,j+1/2} = U^{R,n}_{i,j+1/2} - \frac{\Delta t}{2\Delta x}\left(F^n_{i,j+1} - F^n_{i-1,j+1}\right) \tag{78}$$

4. Solve the Riemann problem for the corrected interface fluxes, $F^{n+1/2}_{i,j} = F(U^{L,n+1/2}_{i+1/2,j}, U^{R,n+1/2}_{i+1/2,j})$ and $G^{n+1/2}_{i,j} = F(U^{L,n+1/2}_{i,j+1/2}, U^{R,n+1/2}_{i,j+1/2})$.

5. Given the interface fluxes, update the volume averaged conserved variables,

$$U^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x}\left(F^{n+1/2}_{i,j} - F^{n+1/2}_{i-1,j}\right) - \frac{\Delta t}{\Delta y}\left(G^{n+1/2}_{i,j} - G^{n+1/2}_{i,j-1}\right) \tag{79}$$

6.4 2D Timestepping algorithm with Potential

Adding the source term changes the equations of motion to,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S \tag{80}$$

with $S = \{0, -\rho\partial\Phi/\partial x, -\rho\partial\Phi/\partial y, -\rho\partial\Phi/\partial z, -\rho v\cdot\nabla\Phi\}$. The full algorithm is,

1. Reconstruct-Evolve-Solve in x direction to obtain $U^{L,R,n}_{i,j}$ and $F^n_{i,j}$ (see 1D with source above).

2. Reconstruct-Evolve-Solve in y direction to obtain $U_{i,j}^{L,R,n}$ and $G_{i,j}^n$.

3. Evolve the left and right states using the *transverse* fluxes,

$$U_{i,j}^{L,n+1/2,*} = U_{i,j}^{L,n} - \frac{\Delta t}{2\Delta y}\left(G_{i,j}^n - G_{i,j-1}^n\right) \tag{81}$$

$$U_{i-1,j}^{R,n+1/2,*} = U_{i-1,j}^{R,n} - \frac{\Delta t}{2\Delta y}\left(G_{i+1,j}^n - G_{i+1,j-1}^n\right) \tag{82}$$

$$U_{i,j}^{D,n+1/2,*} = U_{i,j}^{D,n} - \frac{\Delta t}{2\Delta x}\left(F_{i,j}^n - F_{i-1,j}^n\right) \tag{83}$$

$$U_{i,j-1}^{U,n+1/2,*} = U_{i,j-1}^{U,n} - \frac{\Delta t}{2\Delta x}\left(F_{i,j+1}^n - F_{i-1,j+1}^n\right) \tag{84}$$

where L, R, U, D correspond to left,right,up,down.

4. Evolve the left and right states using the transverse source terms,

$$U_{i,j}^{L,n+1/2} = U_{i,j}^{L,n+1/2,*} - \frac{\Delta t}{\Delta y}\rho_{i,j}^{L,n+1/2,*}\left(\Phi_{i,j+1/2} - \Phi_{i,j}\right) \tag{85}$$

$$U_{i-1,j}^{R,n+1/2} = U_{i-1,j}^{R,n+1/2,*} - \frac{\Delta t}{\Delta y}\rho_{i-1,j}^{R,n+1/2,*}\left(\Phi_{i,j} - \Phi_{i,j-1/2}\right) \tag{86}$$

$$U_{i,j}^{D,n+1/2} = U_{i,j}^{D,n+1/2,*} - \frac{\Delta t}{\Delta x}\rho_{i,j}^{D,n+1/2,*}\left(\Phi_{i+1/2,j} - \Phi_{i,j}\right) \tag{87}$$

$$U_{i,j-1}^{U,n+1/2} = U_{i,j-1}^{U,n+1/2,*} - \frac{\Delta t}{\Delta x}\rho_{i,j-1}^{U,n+1/2,*}\left(\Phi_{i,j} - \Phi_{i-1/2,j}\right) \tag{88}$$

$$\tag{89}$$

5. Solve the Riemann problem for the corrected interface fluxes, $F_{i,j}^{n+1/2} = F(U_{i,j}^{L,n+1/2}, U_{i,j}^{R,n+1/2})$ and $G_{i,j}^{n+1/2} = F(U_{i,j}^{U,n+1/2}, U_{i,j}^{D,n+1/2})$.

6. Calculate $\rho_i^{n+1/2}$ with the fluxes,

$$\rho_i^{n+1/2} = \rho_i^n - \frac{\Delta t}{2\Delta x}\left(F_{i,j}^{\rho,n+1/2} - F_{i-1,j}^{\rho,n+1/2}\right) - \frac{\Delta t}{2\Delta y}\left(G_{i,j}^{\rho,n+1/2} - G_{i,j-1}^{\rho,n+1/2}\right) \tag{90}$$

7. Given the interface fluxes and $\rho_i^{n+1/2}$ update the volume averaged conserved variables,

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i,j}^{n+1/2} - F_{i-1,j}^{n+1/2}\right) - \frac{\Delta t}{\Delta y}\left(G_{i,j}^{n+1/2} - G_{i,j-1}^{n+1/2}\right) - \frac{\Delta t}{\Delta x}S_{i,x} - \frac{\Delta t}{\Delta y}S_{i,j,y} \tag{91}$$

where S_i is zero for the density,

$$S_{i,j,x} = \rho_i^{n+1/2}\left(\Phi_{i+1/2,j} - \Phi_{i-1/2,j}\right) \qquad S_{i,j,y} = \rho_i^{n+1/2}\left(\Phi_{i,j+1/2} - \Phi_{i,j-1/2}\right) \tag{92}$$

for the momenta, and

$$S_{i,j,x} = F_{i,j}^{\rho}\left(\Phi_{i+1/2,j} - \Phi_{i,j}\right) + F_{i-1,j}^{\rho}\left(\Phi_{i,j} - \Phi_{i-1/2,j}\right) \tag{93}$$

$$S_{i,j,y} = G_{i,j}^{\rho}\left(\Phi_{i,j+1/2} - \Phi_{i,j}\right) + G_{i,j-1}^{\rho}\left(\Phi_{i,j} - \Phi_{i,j-1/2}\right) \tag{94}$$

for the energy.

7 Reconstructions

7.1 PCM

7.2 PLM

7.2.1 Without characteristics

1. Compute differences in primitive variables,

$$\Delta q_L = q_i - q_{i-1} \qquad \Delta q_R = q_{i+1} - q_i \tag{95}$$

   where w_i is a primitive variable at cell i. The centered difference is the average of the left and right differences, $\Delta q_C = (\Delta q_L + \Delta q_R)/2$.

2. Limit the slopes such that the face values are,

$$q_{L,R} = q_i \mp \frac{1}{2}\phi(r_i)\Delta q_R \tag{96}$$

   where $r = \Delta q_L/\Delta q_R$ and $\phi(r)$ is the generalized minmod limiter,

$$\phi(r) = \max\left(0, \min\left(\theta r, \frac{1}{2}(r+1), \theta\right)\right) \tag{97}$$

   where $1 \le \theta \le 2$ is a parameter which is more dissipative for higher values. In the limit of $\Delta_R = 0$, $r \to \infty$ and $\phi \to \theta$.

3. The face values, $q_{L,R}$, are then evolved for half of a timestep,

$$q_{L,R} = q_{L,R} + \frac{\Delta t}{2\Delta x}\left(F(q_L) - F(q_R)\right) \tag{98}$$

7.2.2 With characteristics

The eigenvalues are $\lambda = (u - c_s, u, u, u, u + c_s)$ and the left and right eigenmatrices are,

$$\mathbf{L} = \begin{pmatrix} 0 & -\rho/(2c_s) & 0 & 0 & 1/(2c_s^2) \\ 1 & 0 & 0 & 0 & -1/c_s^2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \rho/(2c_s) & 0 & 0 & 1/(2c_s^2) \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -c_s/\rho & 0 & 0 & 0 & c_s/\rho \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ c_s^2 & 0 & 0 & 0 & c_s^2 \end{pmatrix} \tag{99}$$

1. Compute differences in primitive variables,

$$\Delta q_L = q_i - q_{i-1} \qquad \Delta q_R = q_{i+1} - q_i \tag{100}$$

   where w_i is a primitive variable at cell i. The centered difference is the average of the left and right differences, $\Delta q_C = (\Delta q_L + \Delta q_R)/2$.

2. Project the differences into differences of the characteristics, ξ.

$$\delta\xi_{L,R} = \begin{pmatrix} -0.5\left(\rho_i \Delta u_{L,R}/c_s - \Delta p_{L,R}/c_s^2\right) \\ \Delta\rho_{L,R} - \Delta p_{L,R}/c_s^2 \\ \Delta v_{L,R} \\ \Delta w_{L,R} \\ 0.5\left(\rho_i \Delta u_{L,R}/c_s + \Delta p_{L,R}/c_s^2\right) \end{pmatrix} \tag{101}$$

   Note that the $\delta\xi_C = (\delta\xi_L + \delta\xi_R)/2$.

3. Limit the differenes to obtain $\delta\xi$. Use the same general minmod limiter with $\theta = 2$.

4. Given the limited characteristic differences project back to the primitive differences,

$$\Delta q = \begin{pmatrix} \delta\xi_0 + \delta\xi_1 + \delta\xi_4 \\ (c_s/\rho)(\delta\xi_4 - \delta\xi_0) \\ \delta\xi_2 \\ \delta\xi_3 \\ c_s^2(\delta\xi_0 + \delta\xi_4) \end{pmatrix} \tag{102}$$

   Note that the transverse velocities and passive scalars are limited in the same way as PLM without characteristics, so in practice the characteristic differences for these quantities do not need to be stored in memory.

5. Given the limited primitive slopes, use linear interpolation to calculate the face values,

$$q_{L,R} = q_i \mp \frac{1}{2}\Delta q \tag{103}$$

   where the minus sign is for the left face.

6. To ensure the face values lie between q_{i-1}, q_i, and q_{i+1}, they are limited again,

$$q_L = \min(\max(q_i, q_{i-1}), \max(\min(q_i, q_{i-1}), q_L)) \tag{104}$$
$$q_R = \min(\max(q_i, q_{i+1}), \max(\min(q_i, q_{i+1}), q_R)) \tag{105}$$
$$\tag{106}$$

7. Given the limited face values, the final primitive difference is,

$$\Delta q = q_R - q_L \tag{107}$$

8. Given Δq, the primitives are evolved for half of a timestep in a predictor-corrector fashion. For the preditor step, we evolve with the minimum and maximum wave speeds,

$$q_L^* = q_L + \frac{\Delta t}{2\Delta x}\min(\lambda_{\min}, 0)\Delta q \tag{108}$$
$$q_R^* = q_R - \frac{\Delta t}{2\Delta x}\max(\lambda_{\max}, 0)\Delta q \tag{109}$$

   where $\lambda_{\min,\max}$ are the minimum and maximum wave speeds. The corrector step removes contributions from waves which do not reach the faces in half of a timestep,

$$w_R = w_R^* + \alpha_- \left(-\frac{\rho\Delta u}{2c_s} + \frac{\Delta p}{2c_s^2}\right)\begin{pmatrix} 1 \\ -c_s/\rho \\ 0 \\ 0 \\ c_s^2 \end{pmatrix} + \alpha_0 \begin{pmatrix} \Delta\rho - \Delta p/c_s^2 \\ 0 \\ \Delta v \\ \Delta w \\ 0 \end{pmatrix} + \alpha_+ \left(\frac{\rho\Delta u}{2c_s} + \frac{\Delta p}{2c_s^2}\right)\begin{pmatrix} 1 \\ c_s/\rho \\ 0 \\ 0 \\ c_s^2 \end{pmatrix} \tag{110}$$

   where $\alpha_{0,\pm} = \Delta t/(2\Delta x)(\lambda_{\max} - \lambda_{0,\pm})$ and $w_R = (\rho_R, u_R, v_R, w_R, p_R)$. Terms are only included if their corresponding $\lambda > 0$. The left face is the same, but with $\lambda_{\max} \to \lambda_{\min}$ and only terms with $\lambda < 0$ included. Note that $\lambda_\pm$ will always be opposite signs (unless $u = c_s$ in which case $\lambda_- = 0$ and $\lambda_+ > 0$), so only one needs to be included. If the HLL/C Riemann solver is used, then *all* waves with non-zero wavespeeds are added.

9. The final $w_{L,R}$ values are converted to their conservative values.

7.3 PPM

7.3.1 With characteristics

1. The first steps are the same as steps 1.-4. from PLM. These steps are done for both cell i, cell $i-1$, and cell $i+1$, so that at the end of the steps we have three primitive differences, $\Delta q_i, \Delta q_{i-1}, \Delta q_{i+1}$. This necessarily requires information from cells $i-2$ and $i+2$.

2. Instead of linear interpolation we use parabolic interpolation to estimate the left and right face values of cell i,

$$q_L = \frac{q_i + q_{i-1}}{2} - \frac{\Delta q_i + \Delta q_{i-1}}{6} \tag{111}$$

$$q_R = \frac{q_i + q_{i+1}}{2} - \frac{\Delta q_i + \Delta q_{i+1}}{6} \tag{112}$$

3. Next, limit these face values so that they lie between the neighboring values.

4. These are limited with three conditions,

$$\begin{cases} q_L = q_R = q_i & (q_R - q_i)(q_i - q_L) \leq 0 \\ q_L = 3q_i - 2q_R & (q_R - q_L)(q_L - (3q_i - 2q_R)) < 0 \\ q_R = 3q_i - 2q_L & (q_R - q_L)(3q_i - 2q_L - q_R) < 0 \end{cases} \tag{113}$$

and then the results are limited again with the minmod limiter from PLM,

$$q_L = \min(\max(q_i, q_{i-1}), \max(\min(q_i, q_{i-1}), q_L)) \tag{114}$$

$$q_R = \min(\max(q_i, q_{i+1}), \max(\min(q_i, q_{i+1}), q_R)) \tag{115}$$

$$\tag{116}$$

These interpolated face values then define the new slope, $\Delta q = q_R - q_L$.

5. The face values are evolved in time with a predictor-corrector scheme as in PLM. The prediction step is,

$$q_L^* = q_L - \frac{1}{2}\alpha_{\min}\left[\Delta q + q_6\left(1 + \frac{2}{3}\alpha_{\min}\right)\right] \tag{117}$$

$$q_R^* = q_R - \frac{1}{2}\beta_{\max}\left[\Delta q - q_6\left(1 - \frac{2}{3}\beta_{\max}\right)\right] \tag{118}$$

$$\tag{119}$$

where $q_6 = 6q_i + 3\Delta q$.