Recursion and Dynamic Programming

Biostatistics 615/815
Lecture 5

Last Lecture

- Principles for analysis of algorithms
  - Empirical Analysis
  - Theoretical Analysis
- Common relationships between inputs and running time
- Described two simple search algorithms

Recursive refers to …

- A function that is part of its own definition

e.g. $$Factorial(N) = \begin{cases} N \cdot Factorial(N-1) & \text{if N} > 0 \\ 1 & \text{if N} = 0 \end{cases}$$

- A program that calls itself

Key Applications of Recursion

- Dynamic Programming
  - Related to Markov processes in Statistics
- Divide-and-Conquer Algorithms
- Tree Processing

Recursive Function in C

```
int factorial (int N)
  {
  if (N == 0)
     return 1;
  else
     return N * factorial(N - 1);
  }
```

Key Features of Recursions

- Simple solution for a few cases
- Recursive definition for other values
  - Computation of large N depends on smaller N
- Can be naturally expressed in a function that calls itself
  - Loops are sometimes an alternative

A Typical Recursion: Euclid's Algorithm

- Algorithm for finding greatest common divisor of two integers *a* and *b*
  - If *a* divides *b*
    - GCD(a,b) is *a*
  - Otherwise, find the largest integer *t* such that
    - $at + r = b$
    - GCD(a,b) = GCD(r,a)

Euclid's Algorithm in C

```
int gcd (int a, int b)
  {
  if (a == 0)
    return b;

  return gcd(b % a, a);
  }
```

Evaluating GCD(4458, 2099)

```
gcd(2099, 4458)
    gcd(350, 2099)
        gcd(349, 350)
            gcd(1, 349)
                gcd(0, 1)
```

Divide-And-Conquer Algorithms

- Common class of recursive functions
- Common feature
  - Process input
  - Divide input in smaller portions
  - Recursive call(s) process at least one portion
- Recursion may sometimes occur before input is processed

Recursive Binary Search

```
int search(int a[], int value, int start, int stop)
   {
   // Search failed
   if (start > stop)
      return -1;

   // Find midpoint
   int mid = (start + stop) / 2;

   // Compare midpoint to value
   if (value == a[mid]) return mid;

   // Reduce input in half!!!
   if (value < a[mid])
     return search(a, start, mid - 1);
   else
     return search(a, mid + 1, stop);
   }
```

Recursive Maximum

```
int Maximum(int a[], int start, int stop)
   {
   int left, right;

   // Maximum of one element
   if (start == stop)
      return a[start];

   left = Maximum(a, start, (start + stop) / 2);
   right = Maximum(a, (start + stop) / 2 + 1, stop);

   // Reduce input in half!!!
   if (left > right)
      return left;
   else
      return right;
   }
```

An inefficient recursion

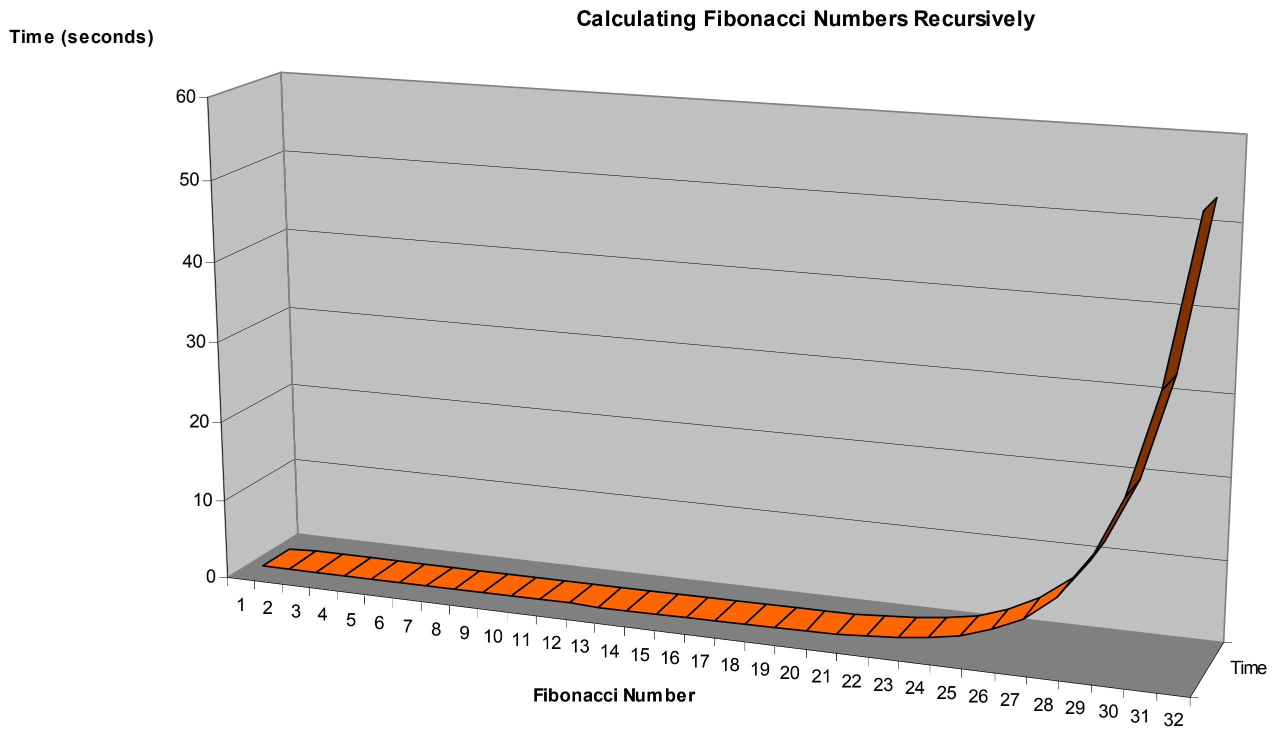
- Consider the Fibonacci numbers

$$Fibonacci(N) = \begin{cases} 0 & \text{if N} = 0 \\ 1 & \text{if N} = 1 \\ Fibonacci(N-1) + Fibonacci(N-2) & \end{cases}$$

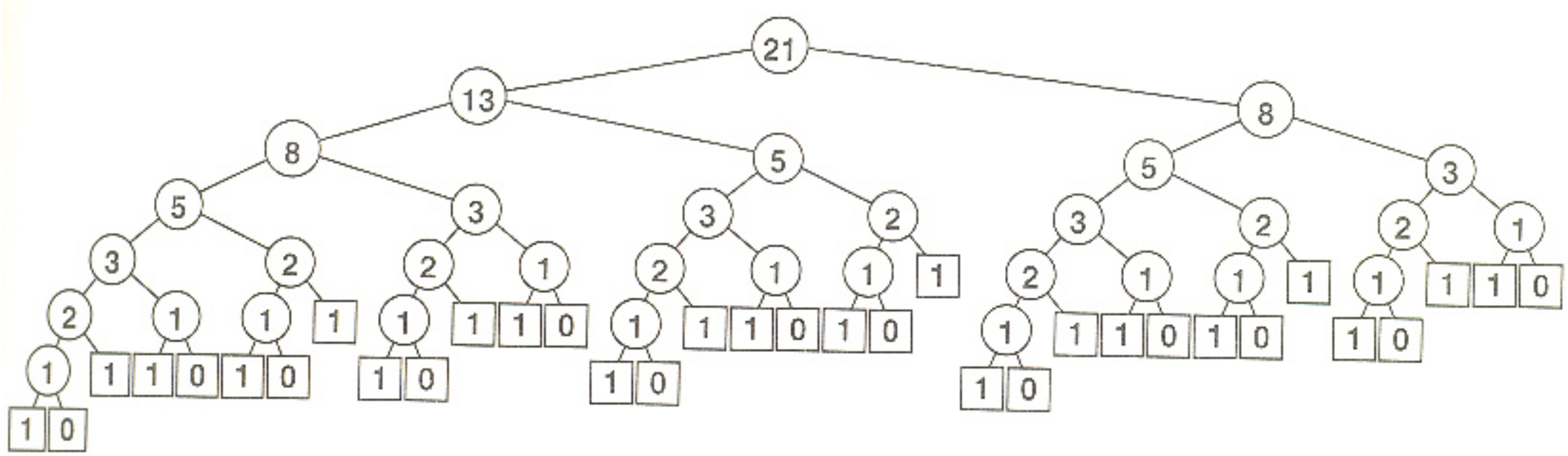
Fibonacci Numbers

```
int Fibonacci(int i)
  {
  // Simple cases first
  if (i == 0)
     return 0;

  if (i == 1)
     return 1;

  return Fibonacci(i - 1) + Fibonacci(i - 2);
  }
```

Terribly Slow!



What is going on? …



Faster Alternatives

- Certain quantities are recalculated
  - Far too many times!
- Need to avoid recalculation
  - Ideally, calculate each unique quantity once.

Dynamic Programming

- A technique for avoiding recomputation
- Can make exponential running times
- become linear!

Bottom-Up Dynamic Programming

- Evaluate function starting with smallest possible argument value
  - Stepping through possible values, gradually increase argument value
- Store all computed values in an array
- As larger arguments evaluated, precomputed values for smaller arguments can be retrieved

Fibonacci Numbers in C

```
int Fibonacci(int i)
   {
   int fib[LARGE_NUMBER], j;

   fib[0] = 0;
   fib[1] = 1;

   for (j = 2; j <= i; j++)
      fib[j] = fib[j - 1] + fib[j - 2];

   return fib[i];
   }
```

Fibonacci With Dynamic Memory

```
int Fibonacci(int i)
   {
   int * fib, j, result;

   if (i < 2) return i;

   fib = malloc(sizeof(int) * (i + 1));

   fib[0] = 0; fib[1] = 1;
   for (j = 2; j <= i; j++)
      fib[j] = fib[j - 1] + fib[j - 2];

   result = fib[i];
   free(fib);

   return result;
   }
```

Top-Down Dynamic Programming

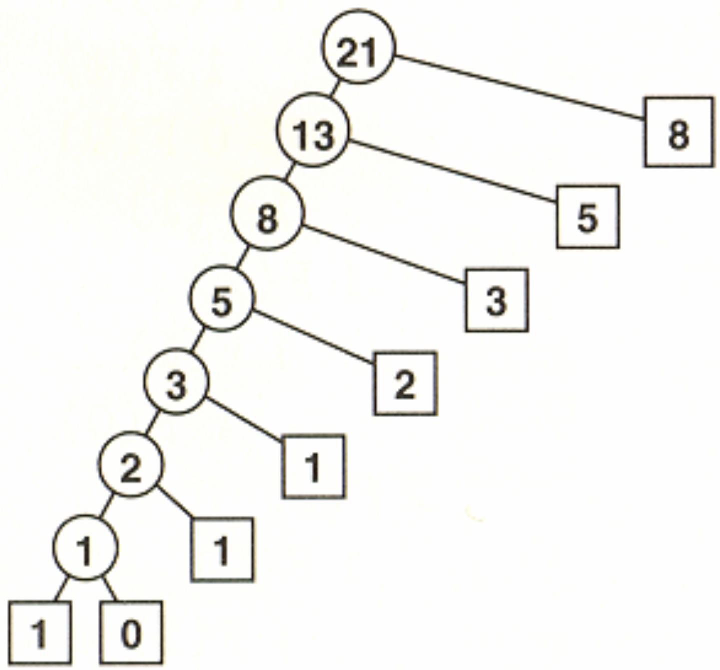
- Save each computed value as final action of recursive function

- Check if pre-computed value exists as the first action

Fibonacci Numbers

```
// Note: saveF should be a global array initialized all zeros

int Fibonacci(int i)
   {
   // Simple cases first
   if (saveF[i] > 0)
      return saveF[i];

   if (i <= 1)
      return i;

   // Recursion
   saveF[i] = Fibonacci(i - 1) + Fibonacci(i - 2);
   return saveF[i];
   }
```

Much less recursion now…



Dynamic Programming Top-down vs. Bottom-up

- In bottom-up programming, programmer has to do the thinking by selecting values to calculate and order of calculation

- In top-down programming, recursive structure of original code is preserved, but unnecessary recalculation is avoided.

Examples of Useful Settings for Dynamic Programming

- Calculating Binomial Coefficients
- Evaluating Poisson-Binomial Distribution

Binomial Coefficients

- The number of subsets with *k* elements from a set of size *N*

$$\binom{N}{k} = \binom{N-1}{k} + \binom{N-1}{k-1}$$

$$\binom{N}{0} = \binom{N}{N} = 1$$

Bottom-Up Implementation in C

```
int Choose(int N, int k)
   {
   int i, j, M[MAX_N][MAX_N];

   for (i = 1; i <= N; i++)
      {
      M[i][0] = M[i][i] = 1;

      for (j = 1; j < i; j++)
         M[i][j] = M[i - 1][j - 1] + M[i - 1][j];
      }

   return M[N][k];
   }
```

Top-Down Implementation (I)

```
#define MAX_N 30

int choices[MAX_N][MAX_N];

void InitChoose()
    {
    int i, j;

    for (i = 0; i < MAX_N; i++)
        for (j = 0; j < MAX_N; j++)
            choices[i][j] = 0;
    }
```

Top-Down Implementation (II)

```
int Choose(int N, int k)
   {
   // Check if this is an easy case
   if (N == k || k == 0)
      return 1;

   // Or a previously examined case
   if (choices[N][k] > 0)
      return choices[N][k];

   // If neither of the above helps, use recursion
   choices[N][k] = Choose(N – 1, k - 1) + Choose(N – 1, k);

   return choices[N][k];
   }
```

Poisson-Binomial Distribution

- $X_1, X_2, \ldots, X_n$ are Bernoulli random variables
- Probability of success is p_k for X_k
- $\sum_k X_k$ has Poisson-Binomial Distribution

Recursive Formulation

$$P_1(0) = 1 - p_1$$

$$P_1(1) = p_1$$

$$P_j(0) = (1 - p_j)P_{j-1}(0)$$

$$P_j(j) = p_j P_{j-1}(j-1)$$

$$P_j(i) = p_j P_{j-1}(i-1) + (1 - p_j)P_{j-1}(i)$$

Summary

- Recursive functions
  - Arise very often in statistics

- Dynamic programming
  - Bottom-up Dynamic Programming
  - Top-down Dynamic Programming

- Dynamic program is an essential tool for statistical programming

Good Programming Practices

- Today's examples used global variables
  - Variables declared outside a function
  - Accessible throughout the program
- In general, these should be used sparingly
- Two alternatives are:
  - Using **static** variables that are setup on the first call
  - Using C++ to group a function and its data

Function with Built-In Initialization

```
int Choose(int N, int k)
   {
   static int valid = 0, choices[MAX_N][MAX_N], i, j;

   // Check if we need to initialize data
   if (valid == 0)
      {
      for (i = 0; i < MAX_N; i++)
         for (j = 0; j < MAX_N; j++)
             choices[i][j] = 0;
      valid = 1;
      }

   // Check if this is an easy case
   if (N == k || k == 0) return 1;

   // Or a previously examined case
   if (choices[N][k] > 0) return choices[N][k];

   // If neither of the above helps, use recursion
   choices[N][k] = Choose(N - 1, k - 1) + Choose(N - 1, k);

   return choices[N][k];
   }
```

C++ Declarations (for .h file)

```
class Choose
   {
   public:
      // This function is called to initialize data
      // Whenever a variable of type Choose is created
      Choose();

      // This function is called to calculate N_choose_k
      int Evaluate(int N, int k);

   private:
      int choices[MAX_N][MAX_N];
   };
```

C++ Code (for .cpp file)

```
Choose::Choose()
   {
   for (int i = 0; i < MAX_N; i++)
      for (int j = 0; j < MAX_N; j++)
          choices[i][j] = 0;
   }

int Choose::Evaluate()
   {
   // Check if this is an easy case
   if (N == k || k == 0) return 1;

   // Or a previously examined case
   if (choices[N][k] > 0) return choices[N][k];

   // If neither of the above helps, use recursion
   choices[N][k] = Choose(N - 1, k - 1) + Choose(N - 1, k);

   return choices[N][k];
   }
```

Using C++ Code

```
#include "Choose.h"

int main()
    {
    Choose choices;

    // Evaluate 20_choose_10
    choices.Evaluate(20, 10);

    // Evaluate 30_choose_10
    choices.Evaluate(30, 10);

    return 0;
    }
```

Reading

- Sedgewick, Chapters 5.1 – 5.3