Advanced Linear Algebra (MiM 2019, Spring)

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Chapter 1

A Review of Linear Algebra

1.1 Algebraic Fields

Definition 1.1.1. An *algebraic field* or, simply, a *field* is a set $\mathbf{k}$ with fixed elements $0_{\mathbf{k}} = 0$ (*zero*) and $1_{\mathbf{k}} = 1$ (*identity*), where $0 \neq 1$, and two operations: an addition

$$\mathbf{k} \times \mathbf{k} \to \mathbf{k}, \quad (\lambda, \mu) \mapsto \lambda + \mu$$

and a multiplication

$$\mathbf{k} \times \mathbf{k} \to \mathbf{k}, \quad (\lambda, \mu) \mapsto \lambda\mu$$

such that for all $\lambda, \mu, \ldots \in \mathbf{k}$ we have

1. $\lambda + \mu = \mu + \lambda$ (commutativity of the addition);
2. $(\lambda + \mu) + \gamma = \lambda + (\mu + \gamma)$ (associativity of the addition);
3. $\lambda + 0 = \lambda$;
4. for every $\lambda \in \mathbf{k}$ there exists $-\lambda \in \mathbf{k}$ such that $\lambda + (-\lambda) = 0$;
5. $\lambda\mu = \mu\lambda$ (commutativity of the multiplication);
6. $\lambda(\mu\gamma) = (\lambda\mu)\gamma$ (associativity of the multiplication);
7. $1\lambda = \lambda$;
8. for every $\lambda \in \mathbf{k}^* := \mathbf{k} \smallsetminus \{0\}$ there exists $\lambda^{-1} \in \mathbf{k}$ such that $\lambda\lambda^{-1} = 1$;
9. $\lambda(\mu + \gamma) = \lambda\mu + \lambda\gamma$ (distributivity).

In a field $\mathbf{k}$ we introduce the operations: the subtraction

$$\mathbf{k} \times \mathbf{k} \to \mathbf{k}, \quad (\lambda, \mu) \mapsto \lambda - \mu := \lambda + (-\mu)$$

and the division

$$\mathbf{k} \times \mathbf{k}^* \to \mathbf{k}, \quad (\lambda, \mu) \mapsto \lambda\mu^{-1}.$$

Loosely speaking, a field is a set with the zero element 0, the identity element 1, and four arithmetic operation: addition, subdtraction, multiplication, and division with natural properties.

Examples 1.1.2. The basic examples of fields are

- $\mathbb{Q}$ – the field of rational numbers,
- $\mathbb{R}$ – the field of real numbers,
- $\mathbb{C}$ – the field of complex numbers,
- $\mathbb{F}_p = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$ – the field integers modulo p.

Cosider the field $\mathbb{C}$ of complex numbers. The map

$$\bar{\cdot} : \mathbb{C} \to \mathbb{C}, \quad z = a + b\sqrt{-1} \mapsto \overline{z} = a - b\sqrt{-1}$$

is an *automorphism* of $\mathbb{C}$. This means that

- $\overline{0} = 0$, $\overline{1} = 1$;
- $\overline{z+w} = \overline{z} + \overline{w}$;
- $\overline{zw} = \overline{z}\,\overline{w}$.

Complex number $\overline{z}$ is called *conjugate* of z.

A field $\mathbf{k}$ is called *algebraically closed* whenever for every nonconstant polynomial $f(t)$ there is $t_0 \in \mathbf{k}$ such that $f(t_0) = 0$; that is, every nonconstant polynomial over $\mathbf{k}$ has a root in $\mathbf{k}$. If a field $\mathbf{k}$ is algebraically closed, then every nonconstant polynomial can be decomposed into product of polynomials of degree one.

Exercise 1.1.3. Prove that

1. The fields $\mathbb{Q}$ and $\mathbb{R}$ are not algebraically closed.
2. Any finite field is not algebraically closed.

Theorem 1.1.4 (Fundamental Theorem of Algebra)**.** *The field $\mathbb{C}$ of complex numbers is algebraically closed.*

For a proof see complex analysis textbooks.

1.2 Vector Spaces and Subspaces

Definition 1.2.1. Let $\mathbf{k}$ be a field (elements of $\mathbf{k}$ are called *scalars*). A vector space over $\mathbf{k}$ is a set V (elements of V are called *vectors*) with the fixed vector $0 \in V$ (*the zero vector*) and compositions: an addition of vectors

$$V \times V \to V, \quad (v, u) \mapsto v + u \tag{1.2.1}$$

and a multiplication of vectors by scalars

$$\mathbf{k} \times V \to V, \quad (\lambda, v) \mapsto \lambda v$$

such that for all $\lambda, \mu \in \mathbf{k}$, $v, u, w \in V$ we have

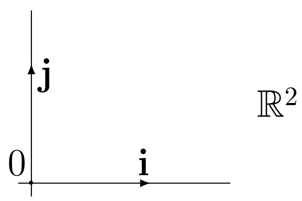
1. $(v + u) + w = v + (u + w)$ (associativity of the addition);
2. $v + u = v + u$ (commutativity of the addition);
3. $v + 0 = v$;
4. for every $v \in V$ there exist $-v \in V$ such that $v + (-v) = 0$;
5. $\lambda(v + u) = \lambda v + \lambda u$ (distributivity);
6. $(\lambda + \mu)v = \lambda v + \mu v$ (distributivity);
7. $(\lambda\mu)v = \lambda(\mu v)$;
8. $1v = v$.

Vector spaces over $\mathbb{R}$ are called *real* and vector spaces over $\mathbb{C}$ are called *complex* vector spaces.

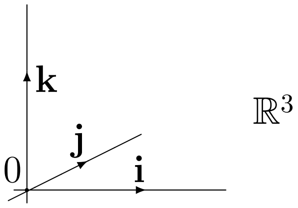
Examples 1.2.2. • The zero space $\{0\}$ (consists of the zero vector only).

- $\mathbb{R}^1$ – the "usual" 1-dimensional line with an origin 0 and a basis $(\mathbf{i})$:

- $\mathbb{R}^2$ – the "usual" 2-dimensional plane with an origin 0 and a basis $(\mathbf{i}, \mathbf{j})$:



- $\mathbb{R}^3$ – the "usual" 3-dimensional space with an origin 0 and a basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$:



- The n-dimensional *coordinate vector space* over a field $\mathbf{k}$,

$$\mathbf{k}^n = \{\begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} \mid a^i \in \mathbf{k}\}.$$

  In particular, the real and the complex n-dimensional coordinate vector spaces.

- The space $\mathbf{k}[t]$ of polynomials in the variable t with coefficients in the field $\mathbf{k}$.

- The space $\mathbf{k}[t]_{\leqslant n}$ of polynomials of degree $\leqslant n$ in the variable t with coefficients in the field $\mathbf{k}$.

- The space $\mathrm{Mat}_{n\times m}(\mathbf{k})$ of $n \times m$ matrices with entries in the field $\mathbf{k}$.

Consider vector $V_1, \ldots, V_n$ spaces over the same field. Then the set-theoretical direct product

$$V_1 \times \ldots \times V_n = \{(v_1, \ldots, v_n) \mid v_i \in V_i\}$$

with the compositions

- $(v_1, \ldots, v_n) + (v_1', \ldots, v_n') := (v_1 + v_1', \ldots, v_n + v_n')$;

- $\lambda(v_1, \ldots, v_n) := (\lambda v_1, \ldots, \lambda v_n)$

is a vector space. It is called the *direct product of* $V_1, \ldots, V_n$ and denoted by $V_1 \times \ldots \times V_n$ (the same as of the set-theoretical direct product). Also it is called the *direct sum* of $V_1, \ldots, V_n$ and denoted by $V_1 \oplus \ldots \oplus V_n$.

Remark 1.2.3. Suppose that J is a set and $\{V_j\}_{j\in J}$ is a set of vector spaces. Then we have the following spaces.

- The *direct product*

$$\prod_{j\in J} V_j := \{\{v_j \in V_j\}_{j\in J}\}$$

  of spaces $\{V_j\}_{j\in J}$. In patricular, if $V_j = V$ for all j, then we have $V^{\times J} := \prod_{j\in J} V$.

- The *direct sum*

$$\bigoplus_{j\in J} V_j := \{\{v_j \in V_j \mid v_j \neq 0 \text{ for finitely many } j\}_{j\in J}\}$$

  of spaces $\{V_j\}_{j\in J}$. In patricular, if $V_j = V$ for all j, then we have $V^{\oplus J} := \oplus_{j\in J} V$.

Note that the direct product of an infinite number of vector spaces is *not* the direct sum of those vector spaces.

Let V be a vector space over a field $\mathbf{k}$. A subset $W \subset V$ is called a *linear subspace* or, simply, *subspace* of V whenever $W \ni 0$ and

$$w_1, \ldots, w_n \in W,\ \lambda_1, \ldots, \lambda_n \in \mathbf{k} \quad \text{imply} \quad \lambda_1 w_1 + \ldots + \lambda_n w_n \in W.$$

Clearly, a subspace is a vector space over the same field.

Examples 1.2.4. In a vector space V,

1. we always have two subspaces:
   - $\{0\}$ – the *zero subspace*;
   - V itself;
2. vectors $v_1, \ldots, v_n \in V$ span the subspace

$$\operatorname{Span}(v_1, \ldots, v_n) := \{\lambda_1 u_1 + \ldots + \lambda_n u_n \mid \lambda_i \in \mathbf{k}\} \subset V.$$

Example 1.2.5. A system of linear equations

$$\begin{cases} a_1^1\ x_1 + a_1^2\ x_2 + \ldots + a_1^n x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_m^1 x_1 + a_m^2 x_2 + \ldots + a_m^n x_n = 0 \end{cases}, \tag{1.2.2}$$

where $a_i^j \in \mathbf{k}$, defines the subspace

$$\{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \ldots, x_n \text{ satisfy } (1.2.2)\} \subset \mathbf{k}^n.$$

Examples 1.2.6. If $W_1, \ldots, W_n$ are subspaces of a vector space V, then

1. their intersection $W_1 \cap \ldots \cap W_n$ is a subspace of V;
2. their sum

$$W_1 + \ldots + W_n := \{w_1 + \ldots + w_n \mid w_i \in W_i\}$$

   is a subspace of V; the sum $W_1 + \ldots + W_n$ is called *direct* whenever

$$w_1 + \ldots + w_n = 0,\ w_i \in W_i \quad \text{imply} \quad w_i = 0 \text{ for } 1 \leqslant i \leqslant n.$$

In linear algebra, a totally ordered set of vectors is called a *list* of vectors.

Definition 1.2.7. Suppose that (v_i) is a list of vectors of a vector space V. Then

- A finite sum of the form

$$\lambda_{i_1} v_{i_1} + \lambda_{i_2} v_{i_2} + \ldots + \lambda_{i_l} v_{i_l}, \text{ where } \lambda_i \in \mathbf{k}$$

  is called a *linear combination* of the list (v_i); the scalars $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_l}$ are called the *coefficients* of this linear combination.

- The list (v_i) is called *linearly dependent* whenever some nontrivial linear combination of (v_i) is equal to 0.

A list consisting of one vector is linearly dependent whenever this vector is the zero vector. A list of vectors is linearly dependent whenever some vector of the list is a linear combination of the others. In particular, if some vector of the list of vectors is the zero vector, then the list of vectors is linearly dependent.

A list of vectors (e_i) of V is called a *basis* of V whenever every vector $v \in V$ can be uniquely represented as a linear combination of (e_i).

Theorem 1.2.8. *For a vector space (even infinite-dimensional) there exists a basis.*

Remark 1.2.9. A basis of infinite-dimensional space is called a *Hamel's* bases of that space.

Examples 1.2.10. • In $\mathbb{R}^1$ we have the basis $(\mathbf{i})$, in $\mathbb{R}^2$ we have the basis $(\mathbf{i}, \mathbf{j})$, and in $\mathbb{R}^3$ we have the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

- In $\mathbf{k}^n$ we have the *standard basis*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

- In $\mathbf{k}[t]_{\leqslant n}$ we have the basis $(1, t, t^2, \ldots, t^n)$.

- In $\mathbf{k}[t]$ we have the Hamel's basis $(1, t, t^2, \ldots)$.

Lemma 1.2.11. *A list of vectors (e_i) of a vector space V is a basis of V whenever*

1. *every vector $v \in V$ can be represented as a linear combination of (e_i);*

2. *(e_i) is linearly independent.*

Definition 1.2.12. The number of vectors in a basis (e_i) of a vector space V (a natural number or the symbol ∞) is called the *dimension* of V and is denoted by $\dim(V)$.

Lemma 1.2.13. *The dimension of a vector space is well-defined.*

We have

$$\dim(\mathbf{k}^n) = n, \quad \dim(\mathbf{k}[t]_{\leqslant n}) = n + 1, \quad \dim(\mathbf{k}[t]) = \infty.$$

Lemma 1.2.14. *Suppose that V is an n-dimensional vector space and a list of vectors (e_i) of V consists of n vectors. Then (e_i) is a basis of V whenever the list (e_i) is linearly independent.*

1.3 Dual Spaces and Linear Maps

Let V be a vector space over a field $\mathbf{k}$. A function $s : V \to \mathbf{k}$ is called *linear* whenever

$$s(\lambda_1 v_1 + \ldots + \lambda_k v_k) = \lambda_1 s(v_1) + \ldots + \lambda_k s(v_k) \text{ for all } \lambda_i \in \mathbf{k}; v_i \in V.$$

Example 1.3.1. An element $x \in \mathbf{k}$ defines the linear function

$$s_x : \mathbf{k}[t]_{\leqslant n} \to \mathbf{k}, \quad s(f) = f(x).$$

The set of all linear functions on V is denoted by V^*; it is a vector space under the following compositions:

1. the sum of $s_1, s_2 \in V^*$ is

$$s_1 + s_2 : V \to \mathbf{k}, \quad (s_1 + s_2)(v) = s_1(v) + s_2(v);$$

2. the product of $\lambda \in \mathbf{k}$ and $s \in V^*$ is

$$\lambda s : V \to \mathbf{k}, \quad (\lambda s)(v) = \lambda s(v).$$

The space V^* is called the *dual* to V vector space.

Remark 1.3.2. For vector spaces with structures there are definitions of their duals and that duals are vector spaces with the same structures. For example, duals of normed, topological, Banach, ... vector spaces are normed, topological, Banach, ... vector spaces respectively.

Suppose that V is a finite-dimensional vector space and $(e_i) = (e_1, \ldots, e_n)$ is a basis of V. A basis $(x^i) = (x^1, \ldots, x^n)$ of V^* is called *dual* to (e_i) whenever

$$x^i(e_j) \quad = \quad \delta^i_j := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Lemma 1.3.3. *There is a unique dual to (e_i) basis (x^i) of V^*. Namely,*

$$x^i(v) = \textit{the } i\textit{th coordinate of } v \textit{ in the basis } (e_i).$$

Corollary 1.3.4. $\dim(V^*) = \dim(V)$.

Example 1.3.5. Consider the space $\mathbf{k}^n$. A row $s = (s_1, \ldots, s_n)$, where $s_i \in \mathbf{k}$, defines the linear function (we denote it by the same symbol)

$$s : \mathbf{k}^n \to \mathbf{k}, \quad \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} \mapsto s_1 a^1 + \ldots + s_n a^n.$$

Conversely, a linear function on $\mathbf{k}^n$ is defined by a uniquely determined row of length n with entries in $\mathbf{k}$. Due to this we identify the dual vector space $(\mathbf{k}^n)^*$ and the space of rows of length n with entries in $\mathbf{k}$. Clearly, the basis

$$\begin{aligned} x^1 &= (1, 0, 0, \dots, 0), \\ x^2 &= (0, 1, 0, \dots, 0), \\ &\dots\dots\dots\dots\dots\dots\dots , \\ x^n &= (0, 0, 0, \dots, 1). \end{aligned}$$

of $\mathbf{k}^{n*}$ is dual to the standard basis of $\mathbf{k}^n$.

Remark 1.3.6. In analysis, when we consider an open subset $U \subset \mathbb{R}^n$ with coordinate functions $(x^1, \dots, x^n)$, the standard notation of the standard basis of the tangent space $T_{x_0}(U)$ of U at $x_0 \in U$ is $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ and the standard notation of the dual basis of $T^*_{x_0}(U)$ is $(dx^1, \dots, dx^n)$.

Let V and U be vector spaces over a field $\mathbf{k}$. A map $\varphi : V \to U$ is called *linear* whenever

$$s(\lambda_1 v_1 + \ldots + \lambda_k v_k) = \lambda_1 s(v_1) + \ldots + \lambda_k s(v_k) \text{ for all } \lambda_i \in \mathbf{k}, v_i \in V.$$

Linear maps also called *linear transformations.* The set of linear maps of a vector space V to a vector space U is denoted by $\mathrm{Hom}(V, U)$. If $V = U$, then $\mathrm{Hom}(V, V)$ is also denoted by $\mathrm{End}(V)$; elements of $\mathrm{End}(V)$ are called *operators.* Clearly,

$$\mathrm{Hom}(V, \mathbf{k}^1) \;\simeq\; V^*.$$

Note that $\mathrm{Hom}(V, U)$ is a vector space under the following compositions:

1. the sum $\varphi_1, \varphi_2 \in \mathrm{Hom}(V, U)$ is
$$\varphi_1 + \varphi_2 : V \to U, \quad (\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v);$$
2. the product $\lambda \in \mathbf{k}$ and $\varphi \in \mathrm{Hom}(V, U)$ is
$$\lambda\varphi : V \to U, \quad (\lambda\varphi)(v) = \lambda\varphi(v).$$

Example 1.3.7. A matrix

$$\Phi = \begin{pmatrix} \Phi^1_1 & \Phi^1_2 & \dots & \Phi^1_n \\ \Phi^2_1 & \Phi^2_2 & \dots & \Phi^2_n \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^m_1 & \Phi^m_2 & \dots & \Phi^m_n \end{pmatrix} \in \mathrm{Mat}_{m\times n}(\mathbf{k})$$

defines the linear map (we denote it by the same symbol)

$$\Phi : \mathbf{k}^n \to \mathbf{k}^m, \qquad \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{pmatrix} \mapsto \begin{pmatrix} \Phi^1_1 & \Phi^1_2 & \dots & \Phi^1_n \\ \Phi^2_1 & \Phi^2_2 & \dots & \Phi^2_n \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^m_1 & \Phi^m_2 & \dots & \Phi^m_n \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{pmatrix}.$$

Conversely, a linear map of the form $\mathbf{k}^n \to \mathbf{k}^m$ is defined by a uniquely determined $m \times n$ matrix. So, we identify

$$\mathrm{Hom}(\mathbf{k}^n, \mathbf{k}^m) = \mathrm{Mat}_{m\times n}(\mathbf{k}).$$

Example 1.3.8. Suppose that V, U are vector spaces, $\varphi : V \to U$ is a linear map, and W is a subspace of V. Then we have the linear map

$$\varphi|_W : W \to U, \quad \varphi|_W(w) = \varphi(w);$$

it is called the *restriction of* φ *to* W.

Let V, U be vector spaces and $\varphi : V \to U$ be a linear map. Denote

$$\mathrm{Ker}(\varphi) := \{v \in V \mid \varphi(v) = 0\} \quad \text{(the } \textit{kernel} \text{ of } \varphi\text{)},$$
$$\mathrm{Im}(\varphi) := \{\varphi(v) \mid v \in V\} \quad \text{(the } \textit{image} \text{ of } \varphi\text{)}.$$

Clearly, $\mathrm{Ker}(\varphi)$ is a subspace of V and $\mathrm{Im}(\varphi)$ is a subspace of U. The dimension $\dim(\mathrm{Im}(\varphi))$ is called the *rank* of φ and is denoted by $\mathrm{rk}(\varphi)$. If V is finite-dimensional, then

$$\dim(\mathrm{Im}(\varphi)) + \dim(\mathrm{Ker}(\varphi)) = \dim(V).$$

Exercise 1.3.9. Consider the following vector spaces and linear maps:

$$V_0 \xrightarrow{\varphi_1} V_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_n} V_n$$

Prove that

$$\dim(\mathrm{Ker}(\varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1)) \leqslant \sum_{1\leqslant i\leqslant n} \dim(\mathrm{Ker}(\varphi_i)).$$

A linear map $\varphi : V \to U$ is called an *isomorphism* whenever φ is bijective. In this case the set-theoretical inverse map $\varphi^{-1} : U \to V$ is a linear map.

Lemma 1.3.10. *Suppose that V and U are finite-dimensional vector spaces. Then a linear map $\varphi V \to U$ is an isomorphism whenever* $\dim(V) = \dim(U)$ *and* $\mathrm{Ker}(\varphi) = 0$.

Quotient spaces.

Let V be a vector space and W be a subspace of V. Define the *quotient space* V/W in the following way:

- (as a set) $V/W := \{v + W \mid v \in V\}$, the zero vector is $0 + W = W$;
- (addition of vectors) $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$;
- (multiplication of vectors by scalars) $\lambda(v + W) = \lambda v + W$.

We have the surjective linear map

$$\pi : V \to V/W, \quad \pi(v) = v + W$$

(called the *quotient map*) with $\operatorname{Ker}(\pi) = W$. If V is finite-dimensional, then

$$\dim(V) = \dim(V/W) + \dim(W).$$

Roughly speaking, *vectors of V/W are vectors of V considered up to addition of vectors of W.*

Suppose that W is spanned by $\{w_i\}_{i\in I}$. Then we can consider vectors of V/W as vectors of V modulo relations

$$\{w_i = 0 \quad \text{for all } i \in I.$$

For a vector space U, we have the bijection

$$\left\{\begin{matrix}\text{Linear maps} \\ \overline{\varphi} : V/W \to U\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Linear maps } \varphi : V \to U \\ \text{such that } \varphi(w_i) = 0 \text{ for all } i \in I\end{matrix}\right\},$$

$$\overline{\varphi} \mapsto \{\varphi, \text{ where } \varphi(v) = \overline{\varphi}(\pi(v))\},$$

$$\{\overline{\varphi}, \text{ where } \overline{\varphi}(v + W) = \varphi(v)\} \leftarrow \varphi.$$

1.4 Change of Basis

Let V be a finite-dimensional vector space, and $(e_i) = (e_1, \ldots, e_n)$ and $(\tilde{e}_i) = (\tilde{e}_1, \ldots, \tilde{e}_n)$ be bases of V.

Decompose vectors of the basis $(\tilde{e}_i)$ in the basis (e_i):

$$\begin{aligned}
\tilde{e}_1 &= A_1^1 e_1 + A_1^2 e_2 + \ldots + A_1^n e_n, \\
\tilde{e}_2 &= A_2^1 e_1 + A_2^2 e_2 + \ldots + A_2^n e_n, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\tilde{e}_n &= A_n^1 e_1 + A_n^2 e_2 + \ldots + A_n^n e_n
\end{aligned}$$

(in one formula, $\tilde{e}_i = A_i^j e_j$) and organize the coefficients into the matrix:

$$A = \begin{pmatrix} A_1^1 & A_2^1 & \ldots & A_n^1 \\ A_1^2 & A_2^2 & \ldots & A_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ A_1^n & A_2^n & \ldots & A_n^n \end{pmatrix}$$

(the ith column of A is formed by the coefficients of the decomposition of $\tilde{e}_i$). The matrix A is called the *transition matrix from the basis (e_i) to the basis $(\tilde{e}_i)$*.

Lemma 1.4.1. *Suppose that $(\tilde{\tilde{e}}_i) = (\tilde{\tilde{e}}_1, \ldots, \tilde{\tilde{e}}_n)$ is another bases of V, then*

$$\begin{pmatrix}\textit{Transition matrix} \\ \textit{from the basis } (e_i) \\ \textit{to the basis } (\tilde{\tilde{e}}_i)\end{pmatrix} = \begin{pmatrix}\textit{Transition matrix} \\ \textit{from the basis } (e_i) \\ \textit{to the basis } (\tilde{e}_i)\end{pmatrix} \cdot \begin{pmatrix}\textit{Transition matrix} \\ \textit{from the basis } (\tilde{e}_i) \\ \textit{to the basis } (\tilde{\tilde{e}}_i)\end{pmatrix}$$

Corollary 1.4.2. *The transition matrix from the basis* $(\tilde{e}_i)$ *to the basis* (e_i) *is* A^{-1}.

Let $(x^i) = (x^1, \dots, x^n)$ be the dual to (e_i) basis and $(\tilde{x}^i) = (\tilde{x}^1, \dots, \tilde{x}^n)$ be the dual to $(\tilde{e}_i)$ basis of the dual space V^*.

Lemma 1.4.3. *The transition matrix from the basis* (x^i) *to the basis* $(\tilde{x}^i)$ *is* $(A^{-1})^\intercal$. *That is,*

$$\begin{aligned}
\tilde{x}^1 &= (A^{-1})^1_1 x^1 + (A^{-1})^1_2 x^2 + \dots + (A^{-1})^1_n x^n,\\
\tilde{x}^2 &= (A^{-1})^2_1 x^1 + (A^{-1})^2_2 x^2 + \dots + (A^{-1})^2_n x^n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\tilde{x}^n &= (A^{-1})^n_1 x^1 + (A^{-1})^n_2 x^2 + \dots + (A^{-1})^n_n x^n
\end{aligned}$$

(in one formula, $\tilde{x}^i = (A^{-1})^i_j x^j$*).*

Let $\psi : V \to V$ be an operator.
Decompose images $\psi(e_i)$ in the basis (e_i):

$$\begin{aligned}
\psi(e_1) &= \Psi^1_1 e_1 + \Psi^2_1 e_2 + \dots + \Psi^n_1 e_n,\\
\psi(e_2) &= \Psi^1_2 e_1 + \Psi^2_2 e_2 + \dots + \Psi^n_2 e_n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\psi(e_n) &= \Psi^1_n e_1 + \Psi^2_n e_2 + \dots + \Psi^n_n e_n
\end{aligned}$$

(in one formula, $\psi(e_i) = \Psi^j_i e_j$) and organize the coefficients into the matrix:

$$\Psi = \begin{pmatrix} \Psi^1_1 & \Psi^1_2 & \dots & \Psi^1_n \\ \Psi^2_1 & \Psi^2_2 & \dots & \Psi^2_n \\ \vdots & \vdots & \ddots & \vdots \\ \Psi^n_1 & \Psi^n_2 & \dots & \Psi^n_n \end{pmatrix}$$

(the ith column of Ψ is formed by the coefficients of the decomposition of $\psi(e_i)$). The matrix Ψ is called the *matrix of* ψ *in the basis* (e_i).

Example 1.4.4. The matrix of the *scalar* operator $\lambda \mathrm{I}_V$ in any basis of V is the scalar matrix $\lambda \mathrm{I}$.

Lemma 1.4.5. *The matrix of* ψ *in the bases* $(\tilde{e}_i)$ *is*

$$\tilde{\Psi} = A^{-1} \Psi A.$$

This formula can be written as

$$\tilde{\Psi}^i_j = (A^{-1})^i_k \Psi^k_l A^l_j.$$

Let U be another finite-dimensional vector space, $(f_i) = (f_1, \dots, f_m)$ be a basis of U, and $\varphi : V \to U$ be a linear map.

Decompose images $\varphi(e_i)$ in the basis (f_i):

$$\begin{gathered}\varphi(e_1) = \Phi_1^1 f_1 + \Phi_1^2 f_2 + \dots + \Phi_1^m f_m,\\ \varphi(e_2) = \Phi_2^1 f_1 + \Phi_2^2 f_2 + \dots + \Phi_2^m f_m,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \varphi(e_n) = \Phi_n^1 f_1 + \Phi_n^2 f_2 + \dots + \Phi_n^m f_m\end{gathered}$$

(in one formula, $\varphi(e_i) = \Phi_i^j f_j$) and organize the coefficients into the matrix:

$$\Phi = \begin{pmatrix} \Phi_1^1 & \Phi_2^1 & \dots & \Phi_n^1 \\ \Phi_1^2 & \Phi_2^2 & \dots & \Phi_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_1^m & \Phi_2^m & \dots & \Phi_n^m \end{pmatrix}$$

(the ith column of Φ is formed by the coefficients of the decomposition of $\varphi(e_i)$). The matrix Φ is called the *matrix of φ in the bases (e_i) and (f_i)*. The rank of Φ ($\overset{def}{=}$ maximal number of linearly independent rows of Φ = maximal number of linearly independent columns of Φ) is equal to the rank of the map φ.

Let $(\tilde{f}_i)$ be another basis of the space U and C be the transition matrix from the basis (f_i) to the basis $(\tilde{f}_i)$.

Lemma 1.4.6. *The matrix of φ in the bases $(\tilde{e}_i)$ and $(\tilde{f}_i)$ is*

$$\tilde{\Phi} = C^{-1}\Phi A.$$

This formula can be written as

$$\tilde{\Phi}_j^i = (C^{-1})_k^i \Phi_l^k A_j^l.$$

Let (x^i) be the dual to (e_i) basis of V^* and (y^i) be the dual to (f_i) basis of U^*.

Lemma 1.4.7. *The matrix of the transpose map $\varphi^t : U^* \to V^*$ (see problem 1.6.6) in the bases (y^i) and (x^i) is $\Phi^\top$. That is,*

$$\begin{gathered}\varphi^t(y^1) = \Phi_1^1 x^1 + \Phi_2^1 x^2 + \dots + \Phi_m^1 x^m,\\ \varphi^t(y^2) = \Phi_1^2 x^1 + \Phi_2^2 x^2 + \dots + \Phi_m^2 x^m,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \varphi^t(y^n) = \Phi_1^n x^1 + \Phi_2^n x^2 + \dots + \Phi_m^n x^m\end{gathered}$$

(in one formula, $\varphi^t(y^i) = \Phi_j^i x^j$).

Consider a linear map

$$\Phi : \mathbf{k}^n \to \mathbf{k}^m, \quad a \mapsto \Phi a,$$

there $\Phi \in \mathrm{Mat}_{m\times n}(\mathbf{k})$. Then, by the previous lemma,

$$\Phi^t : \mathbf{k}^{m*} \to \mathbf{k}^{n*}, \quad s \mapsto s\Phi.$$

1.5 Forms

Bilinear forms.

Let V and U be vector spaces. A map

$$\beta : V \times U \to \mathbf{k}$$

is called a *bilinear form on* $V \times U$ whenever $\beta(v, u)$ linear by v and linear by u.

Examples 1.5.1.

1. Consider $\mathbb{R}^n$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$. Then the map
$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (v, u) \mapsto \langle v, u \rangle$$
is a bilinear form. Note that the map
$$\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \quad (v, u) \mapsto \langle v, u \rangle,$$
where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on $\mathbb{C}^n \times \mathbb{C}^n$, is not a bilinear form because $\langle v, u \rangle$ is antilinear but not linear by u.

2. The map
$$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R},$$
$$\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \mapsto \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$
is a bilinear form.

3. The map
$$\mathrm{Mat}_n(\mathbf{k}) \times \mathrm{Mat}_n(\mathbf{k}) \to \mathbf{k}, \quad (A, B) \mapsto \mathrm{tr}(AB)$$
is a bilinear form.

4. Suppose that $B = (b_{ij}) \in \mathrm{Mat}_{n \times m}(\mathbf{k})$; then the map
$$\mathbf{k}^n \times \mathbf{k}^m \to \mathbf{k}, \quad (v, u) \mapsto v^\top B u$$
is a bilinear form.

The set of bilinear forms on $V \times U$ is a vector space under the following operations:

1. the sum of forms β_1, β_2 is
$$\beta_1 + \beta_2 : V \times U \to \mathbf{k}, \quad (\beta_1 + \beta_2)(v, u) = \beta_1(v, u) + \beta_2(v, u);$$

2. the product of a scalar $\lambda \in \mathbf{k}$ and a form β is
$$\lambda\beta : V \times U \to \mathbf{k}, \quad (\lambda\beta)(v, u) = \lambda\beta(v, u).$$

Suppose that V and U are finite-dimensional, (e_i) is a basis of V, and (f_i) is a basis of U. We have

$$\beta\left(\sum x^i e_i, \sum y^j f_j\right) = \sum_{i,j} \beta(e_i, f_j) x^i y^j = \sum_{i,j} b_{ij} x^i y^j,$$

where $b_{ij} = \beta(e_i, f_j)$. The matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix}.$$

is called the *matrix of the bilinear form β in the bases (e_i) and (f_i)*. In coordinates

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = \sum x^i e_i, \qquad y = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \sum y^j f_j$$

we have

$$\beta(x, y) = x^\top B y.$$

Lemma 1.5.2 (Change of basis)**.** *Suppose that $(\tilde{e}_i)$ is another basis of V, A is the transition matrix from (e_i) to $(\tilde{e}_i)$, $(\tilde{f}_i)$ is another basis of U, C is the transition matrix from (f_i) to $(\tilde{f}_i)$, and $\tilde{B}$ is the matrix of β in the bases $(\tilde{e}_i)$ and $(\tilde{f}_i)$. Then we have*

$$\tilde{B} = A^\top B C.$$

Symmetric bilinear forms.

Let V be a vector space. A map

$$q : V \times V \to \mathbf{k}$$

is called a *symmetric bilinear form on $V \times V$* whenever

1. q is a bilinear form,
2. $q(v, u) = q(u, v)$ for all $v, u \in V$.

Example 1.5.3. Suppose that Q is a symmetric $n \times n$ matrix; then the map

$$\mathbf{k}^n \times \mathbf{k}^n \to \mathbf{k}, \quad (v, u) \mapsto v^\top Q u$$

is a symmetric bilinear form.

Suppose that V is finite-dimensional and (e_i) is a basis of V. Then

$$q\left(\sum x^i e_i, \sum y^j f_j\right) = \sum_{i,j} q(e_i, e_j) x^i y^j = \sum_{i,j} q_{ij} x^i y^j,$$

where $q_{ij} = q(e_i, e_j)$. The matrix

$$Q = \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{pmatrix}.$$

is called the *matrix of q in the basis (e_i)*. In coordinates

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = \sum x^i e_i, \qquad y = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \sum y^j f_j$$

we have

$$\beta(x, y) = x^\top Q y.$$

Exercise 1.5.4. Check that the matrix Q is symmetric.

Lemma 1.5.5 (Change of basis)**.** *Suppose that $(\tilde{e}_i)$ is another bases of V, A is the transition matrix from (e_i) to $(\tilde{e}_i)$, and $\tilde{Q}$ is the matrices of q in the bases $(\tilde{e}_i)$. Then*

$$\tilde{Q} = A^\top Q A.$$

Skew-symmetric bilinear forms.

Let V be a vector space. A map

$$\omega : V \times V \to \mathbf{k}$$

is called a *skew-symmetric bilinear form on $V \times V$* whenever

1. ω is a bilinear form,
2. $\omega(v, u) = -\omega(u, v))$ for all $v, u \in V$.

Example 1.5.6. Suppose that Ω is a skew-symmetric $n \times n$ matrix; then the map

$$\mathbf{k}^n \times \mathbf{k}^n \to \mathbf{k}, \quad (v, u) \mapsto v^\top \Omega u$$

is a skew-symmetric bilinear form.

Suppose that V is finite-dimensional and (e_i) is a basis of V. Then

$$\omega\left(\sum x^i e_i, \sum y^j f_j\right) = \sum_{i,j} \omega(e_i, e_j) x^i y^j = \sum_{i,j} \omega_{ij} x^i y^j,$$

where $\omega_{ij} = \omega(e_i, e_j)$. The matrix

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \dots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \dots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \dots & \omega_{nn} \end{pmatrix}.$$

is called the *matrix of* ω *in the basis* (e_i). In coordinates

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = \sum x^i e_i, \qquad y = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \sum y^j f_j$$

we have

$$\beta(x, y) = x^{\intercal} \Omega y.$$

Exercise 1.5.7. Check that the matrix Ω is skew-symmetric.

Lemma 1.5.8 (Change of basis). *Suppose that* $(\tilde{e}_i)$ *is another bases of* V*,* A *is the transition matrix from* (e_i) *to* $(\tilde{e}_i)$*, and* $\tilde{\Omega}$ *is the matrices of* ω *in the bases* $(\tilde{e}_i)$*. Then*

$$\tilde{\Omega} = A^{\intercal} \Omega A.$$

1.6 Problems

Problem 1.6.1. *Find dimensions and bases of the subspaces*

1.

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{F}_2^5 \mid \begin{cases} x_1 + x_2 + x_4 = 0 \\ x_1 + x_3 + x_5 = 0 \end{cases} \right\} \subset \mathbb{F}_2^5;$$

2.

$$\mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 1 \\ 2 \end{pmatrix} \right) \subset \mathbb{F}_7^4.$$

Problem 1.6.2. *Suppose that* $n \in \mathbb{N}^*$ *and* $x_1, \ldots, x_m \in \mathbf{k}$*. Prove that linear functions*

$$s_i : \mathbf{k}[t]_{\leqslant n} \to \mathbf{k}, \quad s_i(f) = f(x_i), \quad 1 \leqslant i \leqslant m$$

are linearly independent whenever $m \leqslant n + 1$ *and* $x_1, \ldots, x_m$ *are pairwise different.*

Problem 1.6.3. *Prove that the lists of vectors below are bases and find the dual bases*

(1) $\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right) \subset \mathbb{R}^2;$ (2) $\left(\begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 + \sqrt{-1} \\ 1 \end{pmatrix} \right) \subset \mathbb{C}^2;$

(3) $\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \subset \mathbb{F}_3^3.$

Problem 1.6.4. *Suppose that* $\mathbf{k}$ *is a field. Consider the vector space*

$$\mathbf{k}^{\times \mathbb{N}} := \{(x_0, x_1, x_2, \dots) \mid x_i \in \mathbf{k}\}$$

and its subspace

$$\mathbf{k}^{\oplus \mathbb{N}} := \{(x_0, x_1, x_2, \dots) \in \mathbf{k}^{\mathbb{N}} \mid x_i \neq 0 \text{ for finitely many } i\}.$$

Prove that

1. $\mathbf{k}^{\oplus \mathbb{N}} \neq \mathbf{k}^{\times \mathbb{N}}$;
2. $(\mathbf{k}^{\oplus \mathbb{N}})^* \simeq \mathbf{k}^{\times \mathbb{N}}$;
3. $\mathbf{k}^{\oplus \mathbb{N}} \not\simeq \mathbf{k}^{\times \mathbb{N}}$.

Problem 1.6.5. *Find rank, kernel and image of the linear maps*

1. $\begin{pmatrix} 1 & 2 & 1 & 4 \\ 1 & 0 & 4 & 1 \\ 2 & 2 & 0 & 0 \end{pmatrix} : \mathbb{F}_5^4 \to \mathbb{F}_5^3$;
2. $\varphi : \mathbb{F}_3[t]_{\leqslant 2} \to \mathbb{F}_3[t]_{\leqslant 4}$, $f(t) \mapsto t^3 f'(t) - t f''(1) + f(t+1)$.

Problem 1.6.6. *Suppose that* V, U *are finite-dimensional vector spaces and* $\varphi : V \to U$ *is a linear map. Prove that the map*

$$\varphi^t : U^* \to V^*, \quad r \mapsto \varphi^t(r), \text{ where } \varphi^t(r)(v) = r(\varphi(v))$$

is a well-defined linear map (it is called transpose *of* φ*);*

Problem 1.6.7. *Suppose that* V *is a finite-dimensional vector space. Prove that the map*

$$\alpha : V \to (V^*)^*, \quad v \mapsto \alpha(v) : V^* \to \mathbf{k}, \text{ where } \alpha(v)(s) = s(v),$$

is a natural isomorphism between vector spaces ("natural" means that, for any operator $\varphi : V \to V$*, we have* $\alpha \circ \varphi = (\varphi^t)^t \circ \alpha$*).*

Using the isomorphism α from the previous problem, we identify a space V and its doudle dual $(V^*)^*$.

Remark. For a vector space V there is no isomorphism $\alpha : V \to V^*$ such that, for any operator $\varphi : V \to V$, we have $\alpha \circ \varphi = \varphi^t \circ \alpha$. In other words, there is no natural isomorphism $V \simeq V^*$.

Problem 1.6.8. *Suppose that* V, U *are finite-dimensional vector spaces and* $\varphi : V \to U$ *is a linear map. Prove that* $(\varphi^t)^t = \varphi$.

Problem 1.6.9. *Consider a vector space* V *over a field* $\mathbf{k}$ *with a basis* $(e_1, \dots, e_5)$*, its subspace*

$$W = \mathrm{Span}(e_1 + e_3 - e_5, e_2 - e_4 + e_5),$$

and the quotient space V/W*. Construct an isomorphism* $\varphi : V/W \to \mathbf{k}^3$.

An operator $\varphi : V \to V$ is called a *projection* onto a subspace $V_1 \subset V$ parallel to a subspace $V_2 \subset V$ whenever

1. $V = V_1 \oplus V_2$,

2. $\varphi(v_1 + v_2) = v_1$ for all $v_1 \in V_1, v_2 \in V_2$.

Thus, a projection φ is a projection onto $\mathrm{Im}(\varphi)$ parallel to $\mathrm{Ker}(\varphi)$.

Problem 1.6.10. *Consider an operator $\varphi : V \to V$. Prove that*

$$\varphi \textit{ is a projection} \quad \Longleftrightarrow \quad \varphi^2 = \varphi \quad \Longleftrightarrow \quad \varphi|_{\mathrm{Im}(\varphi)} = \mathrm{I}_{\mathrm{Im}(\varphi)}.$$

Chapter 2

Tensors

In this section we define tensor products of vector spaces and prove some of their properties. Vector spaces we consider in this chapter are assumed defined over $\mathbf{k} = \mathbb{Q}$, or over $\mathbf{k} = \mathbb{R}$ (real vector spaces), or over $\mathbf{k} = \mathbb{C}$ (complex vector spaces).

2.1 Tensor Product of Vector Spaces

Tensor product of two vector spaces

Let V and U be vector spaces.

Consider an infinite dimensional vector space with the basis $(v \otimes u \mid v \in V, u \in U)$. Vectors of this space are formal expressions of the form

$$\lambda_1 v_1 \otimes u_1 + \ldots + \lambda_l v_l \otimes u_l, \quad \text{where } l \in \mathbb{N}^*, \lambda_i \in \mathbf{k}, v_i \in V, u_i \in U,$$

with the natural addition and multiplication by scalars; the zero vector is the expression $0 \otimes 0$. By definition, the tensor product $V \otimes U$ is the quotient of this space modulo relations

$$\begin{aligned} &(1) \quad (v_1 + v_2) \otimes u = v_1 \otimes u + v_2 \otimes u, \\ &(2) \quad v \otimes (u_1 + u_2) = v \otimes u_1 + v \otimes u_2, \\ &(3) \quad \lambda v \otimes u = (\lambda v) \otimes u = v \otimes (\lambda u). \end{aligned} \tag{2.1.1}$$

Elements of $V \otimes U$ are called *tensors*. The standard notation of the *class* of an expression $\lambda_1 v_1 \otimes u_1 + \ldots + \lambda_l v_l \otimes u_l$ in $V \otimes U$ is not

$$\overline{\lambda_1 v_1 \otimes u_1 + \ldots + \lambda_l v_l \otimes u_l} \quad \text{or} \quad [\lambda_1 v_1 \otimes u_1 + \ldots + \lambda_l v_l \otimes u_l]$$

but

$$\lambda_1 v_1 \otimes u_1 + \ldots + \lambda_l v_l \otimes u_l$$

(the same as of the expression). Thus, every tensor $T \in V \otimes U$ can be written as

$$T = \lambda_1 v_1 \otimes u_1 + \ldots + \lambda_l v_l \otimes u_l.$$

Two tensors are equal if one of them can be transformed into the other by using (2.1.1).

Example 2.1.1.

$$(v_1+v_2)\otimes(2u_1-u_2)+v_2\otimes u_2 = v_1\otimes(2u_1-u_2)+v_2\otimes(2u_1-u_2)+v_2\otimes u_2 =$$
$$2v_1\otimes u_1 - v_1\otimes u_2 + 2v_2\otimes u_1 - v_2\otimes u_2 + v_2\otimes u_2 =$$
$$2v_1\otimes u_1 - v_1\otimes u_2 + 2v_2\otimes u_1.$$

Exercise 2.1.2. Prove that if one of a factor of a tensor product is the zero space, then the tensor product is the zero space.

A tensor $v\otimes u$ is called the *tensor product* of vectors $v\in V$, $u\in U$. A tensor is called *decomposable* whenever it is a tensor product of some vectors.

From definition it follows that tensor product $V\otimes U$ is spanned by decomposable tensors. Thus, to define a linear map

$$\varphi : V\otimes U \to W,$$

where W is a vector space, we have to define the images of decomposable tensors such that these images are consistent with relations (2.1.1). That is, we have to define $\varphi(v\otimes u)$ for all $v\in V$, $u\in U$ such that

$$\begin{aligned}
&\varphi((v_1+v_2)\otimes u) = \varphi(v_1\otimes u)+\varphi(v_2\otimes u),\\
&\varphi(v\otimes(u_1+u_2)) = \varphi(v\otimes u_1)+\varphi(v\otimes u_2),\\
&\lambda\varphi(v\otimes u) = \varphi((\lambda v)\otimes u) = \varphi(v\otimes(\lambda u)).
\end{aligned} \tag{2.1.2}$$

Roughly speaking, (2.1.2) means that $\varphi(v\otimes u)$ is linear by v and linear by u.

Example 2.1.3. If $s:V\to\mathbf{k}$ and $r:U\to\mathbf{k}$ are linear functions, then

$$V\otimes U\to\mathbf{k},\quad v\otimes u\mapsto s(v)r(u)$$

is a well-defined linear function.

Suppose that the spaces V and U are finite-dimensional, $(e_1,\dots,e_n)$ is a basis of V and $(f_1,\dots,f_m)$ is a basis of U.

Lemma 2.1.4. *$(e_i\otimes f_j)$ is a basis of $V\otimes U$.*

Proof. First, let us prove that every tensor of the tensor product is a linear combination of tensors $e_i\otimes f_j$. Note that every tensor is a finite sum of decomposable tensors. Therefore, it is sufficient to present every decomposable tensor as a linear combination of tensors $e_i\otimes f_j$. Suppose that $T = v\otimes u$, where $v = x^ie_i\in V$, $u = y^if_i\in U$. Using (2.1.1), we remove the parentheses:

$$(x^1e_1+\ldots+x^ne_n)\otimes(y^1f_1+\ldots+y^mf_m) \overset{(1)}{=}$$
$$(x^1e_1)\otimes(y^1f_1+\ldots+y^mf_m)+\ldots+(x^ne_n)\otimes(y^1f_1+\ldots+y^mf_m) \overset{(2)}{=}$$
$$(x^1e_1)\otimes(y^1f_1)+\ldots+(x^1e_1)\otimes(y^mf_m)+\ldots+$$
$$(x^ne_n)\otimes(y^1f_1)+\ldots+(x^ne_n)\otimes(y^mf_m) \overset{(3)}{=}$$
$$(x^1y^1)e_1\otimes f_1+\ldots+(x^1y^m)e_1\otimes f_m+\ldots+$$
$$(x^ny^1)e_n\otimes f_1+\ldots+(x^ny^m)e_n\otimes f_m.$$

Now, let us prove that the tensors $e_i \otimes f_j$ are linearly independent. Let $(x^1, \dots, x^n)$ be the dual to (e_i) basis of the space V^* and $(y^1, \dots, y^m)$ be the dual to (f_i) basis of the space U^*. For $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant m$ define the linear function

$$F_{kl} : V \otimes U \to \mathbf{k}, \quad v \otimes u \mapsto x^k(v) y^l(u).$$

From example 2.1.3 it follows that these functions are well-defined. Now suppose that $c^{ij} e_i \otimes f_j = 0$. Applying the linear function F_{kl} to both sides of this relation we obtain $c^{kl} = 0$ for all k, l. This proves that tensors $e_i \otimes f_j$ are linearly independent. $\square$

By lemma 2.1.4, a tensor $T \in V \otimes U$ can be uniquely decomposed as

$$T = T^{ij} e_i \otimes f_j.$$

The coefficients (T^{ij}) are called the *coordinates of the tensor T in the basis $(e_i \otimes f_j)$*. We can organize the coefficients of the tensor $T = T^{ij} e_i \otimes f_j$ into the matrix:

$$(T^{ij}) = \begin{pmatrix} T^{11} & T^{12} & \cdots & T^{1m} \\ T^{21} & T^{22} & \cdots & T^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ T^{n1} & T^{n2} & \cdots & T^{nm} \end{pmatrix}.$$

The matrix (T^{ij}) is called the *matrix of the tensor T in the bases (e_i) and (f_i)*.

Remark 2.1.5. Sometimes tensors are identified by their coordinates. In this situation, the corresponding basis must be in mind. For example, the phrase "Let T^{ij} be a tensor ..." means "Let $T^{ij} e_i \otimes f_j$ be a tensor ...," where (e_i) is some basis of V and (f_i) is some basis of U.

Lemma 2.1.6 (Change of Basis)**.** *Suppose that $(\tilde{e}_i)$ is another basis of V, A is the transition matrix from (e_i) to $(\tilde{e}_i)$, $(\tilde{f}_j)$ is another basis of U, B is the transition matrix from (f_j) to $(\tilde{f}_j)$, and $(\tilde{T}^{ij})$ is the matrix of the tensor T in the bases $(\tilde{e}_i)$ and $(\tilde{f}_i)$. Then*

$$(T^{ij}) = A(\tilde{T}^{ij})B^\top.$$

This formula can be written as

$$T^{ij} = A^i_l B^j_k \tilde{T}^{lk}.$$

The *rank* of a tensor $T \in V \otimes U$ is defined to be the rank of its matrix in some bases.

Exercise 2.1.7. Prove that rank of tensor is well-defined.

Lemma 2.1.8. *Suppose that V and U are finite-dimensional vector spaces. Then we have the isomorphism*

$$V^* \otimes U^* \to \{\text{bilinear forms on } V \times U\},$$
$$s \otimes r \mapsto \left\{ \begin{matrix} V \times U \to \mathbf{k}, \\ (v, u) \mapsto s(v) r(u) \end{matrix} \right\}.$$

There are *canonical* isomorphisms between some tensor products. Some of them are discribed in the following theorem.

Theorem 2.1.9. *For vector spaces $V, U, V_1, \dots, V_k$ we have the isomorphisms*

$$V \otimes U \to U \otimes V, \quad v \otimes u \mapsto u \otimes v; \tag{2.1.3}$$

$$\begin{gathered}(V_1 \oplus \dots \oplus V_k) \otimes U \to (V_1 \otimes U) \oplus \dots \oplus (V_k \otimes U), \\ (v_1, \dots, v_k) \otimes u \mapsto (v_1 \otimes u, \dots, v_k \otimes u).\end{gathered} \tag{2.1.4}$$

$$V^* \otimes U \to \mathrm{Hom}(V, U), \quad s \otimes u \mapsto \varphi, \ \textit{where } \varphi(v) = s(v)u; \tag{2.1.5}$$

$$V^* \otimes U^* \to (V \otimes U)^*, \quad s \otimes r \mapsto \varphi, \ \textit{where } \varphi(v \otimes u) = s(v)r(u). \tag{2.1.6}$$

Tensor product of finitely many vector spaces

Let $V_1, \dots, V_k$ be vector spaces.

Consider the set of formal expressions of the form

$$\lambda_1 v_1^{(1)} \otimes \dots \otimes v_k^{(1)} + \dots + \lambda_l v_1^{(l)} \otimes \dots \otimes v_k^{(l)}, \quad \text{where} \quad l \in \mathbb{N}^*, \lambda_i \in \mathbf{k}, v_j^{(i)} \in V_j,$$

as a vector space with the natural addition and multiplication by scalars; the zero vector is the expression $0 \otimes 0$.[1] By definition, the tensor product $V_1 \otimes \dots \otimes V_k$ is the quotient of this space modulo relations

$$\begin{gathered}v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v_i') \otimes v_{i+1} \otimes \dots \otimes v_k = \\ v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \dots \otimes v_k + v_1 \otimes \dots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \dots \otimes v_k, \\ \lambda v_1 \otimes \dots \otimes v_k = v_1 \otimes \dots \otimes v_{i-1} \otimes (\lambda v_i) \otimes v_{i+1} \otimes \dots \otimes v_k.\end{gathered} \tag{2.1.7}$$

Elements of $V_1 \otimes \dots \otimes V_k$ are called *tensors*. The standard notation of the *class* of an expression

$$\lambda_1 v_1^{(1)} \otimes \dots \otimes v_k^{(1)} + \dots + \lambda_l v_1^{(l)} \otimes \dots \otimes v_k^{(l)}$$

in $V_1 \otimes \dots \otimes V_k$ is

$$\lambda_1 v_1^{(1)} \otimes \dots \otimes v_k^{(1)} + \dots + \lambda_l v_1^{(l)} \otimes \dots \otimes v_k^{(l)}$$

(the same as of the expression). Thus every tensor $T \in V_1 \otimes \dots \otimes V_k$ can be written as

$$T = \lambda_1 v_1^{(1)} \otimes \dots \otimes v_k^{(1)} + \dots + \lambda_l v_1^{(l)} \otimes \dots \otimes v_k^{(l)}.$$

Two tensors are equal if one of them can be transformed into the other by using (2.1.7).

---

[1] This vector space can be defined formally as

$$\bigoplus_{v_i \in V_i} \mathbf{k} v_1 \otimes \dots \oplus v_k$$

Example 2.1.10.

$$\begin{gathered}(v_1 + v_2) \otimes (-u) \otimes (w_1 - w_2) + v_2 \otimes u \otimes w_1 = \\ -v_1 \otimes u \otimes (w_1 - w_2) - v_2 \otimes u \otimes (w_1 - w_2) + v_2 \otimes u \otimes w_1 = \\ -v_1 \otimes u \otimes w_1 + v_1 \otimes u \otimes w_2 - v_2 \otimes u \otimes w_1 + v_2 \otimes u \otimes w_2 + v_2 \otimes u \otimes w_1 = \\ -v_1 \otimes u \otimes w_1 + v_1 \otimes u \otimes w_2 + v_2 \otimes u \otimes w_2.\end{gathered}$$

Exercise 2.1.11. Prove that if one of a factor of a tensor product is the zero space, then the tensor product is the zero space.

A tensor $v_1 \otimes \ldots \otimes v_k$ is called the *tensor product* of vectors $v_i \in V_i$. A tensor is called *decomposable* whenever it is a tensor product of some vectors.

From definition it follows that tensor product $V_1 \otimes \ldots \otimes V_k$ is spanned by decomposable tensors. Thus to define a linear map

$$\varphi : V_1 \otimes \ldots \otimes V_k \to W,$$

where W is a vector space, we have to define the images of decomposable tensors such that these images are consistent with relations (2.1.7). That is, we have to define $\varphi(v_1 \otimes \ldots \otimes v_k)$ for all $v_i \in V_i$ such that

$$\begin{gathered}\varphi(v_1 \otimes \ldots \otimes v_{i-1} \otimes (v_i + v_i') \otimes v_{i+1} \otimes \ldots \otimes v_k) = \\ \varphi(v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \ldots \otimes v_k) + \varphi(v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i' \otimes v_{i+1} \otimes \ldots \otimes v_k), \\ \lambda\varphi(v_1 \otimes \ldots \otimes v_k) = \varphi(v_1 \otimes \ldots \otimes v_{i-1} \otimes (\lambda v_i) \otimes v_{i+1} \otimes \ldots \otimes v_k).\end{gathered} \tag{2.1.8}$$

Roughly speaking, (2.1.8) means that $\varphi(v_1 \otimes \ldots \otimes v_k)$ is linear by every v_i.

The properties of tensor products of finitely many spaces are similar to the corresponding properties of tensor products of two spaces.

Suppose that $V_1, \ldots, V_k$ are finite-dimensional vector spaces and $(e_j^{(i)})$ is a basis of V_i, $1 \leqslant i \leqslant k$.

Lemma 2.1.12. $(e_{j_1}^{(1)} \otimes \ldots \otimes e_{j_k}^{(k)})$ *is a basis of the tensor product* $V_1 \otimes \ldots \otimes V_k$.

Thus, a tensor $T \in V_1 \otimes \ldots \otimes V_k$ can be decomposed as

$$T = T^{j_1,\ldots,j_k} e_{j_1}^{(1)} \otimes \ldots \otimes e_{j_k}^{(k)}.$$

The coefficients $(T^{j_1,\ldots,j_k})$ are called the *coordinates of* T *in the basis* $(e_{j_1}^{(1)} \otimes \ldots \otimes e_{j_k}^{(k)})$.

Lemma 2.1.13 (Change of basis)**.** *Suppose that* $(\tilde{e}_j^{(i)})$ *is another basis of the space* V_i*,* $A[i]$ *is the transition matrix from the basis* $(e_j^{(i)})$ *to the basis* $(\tilde{e}_j^{(i)})$*. Then*

$$T^{j_1,\ldots,j_k} = A[1]^{j_1}_{l_1} \ldots A[k]^{j_k}_{l_k} \tilde{T}^{l_1,\ldots,l_k},$$

where $(\tilde{T}^{j_1,\ldots,j_k})$ *are the coordinates of the tensor* T *in the basis* $(\tilde{e}_{j_1}^{(1)} \otimes \ldots \otimes \tilde{e}_{j_k}^{(k)})$*.*

Theorem 2.1.14. *For vector spaces $V_1, \dots, V_k$ we have the isomorphisms*

$$V_1 \otimes \dots \otimes V_k \to V_{\sigma(1)} \otimes \dots \otimes V_{\sigma(k)}, \quad v_1 \otimes \dots \otimes v_k \mapsto v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)} \tag{2.1.9}$$

where $\sigma \in \mathbf{S}_k$;

$$(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3), \quad (v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3); \tag{2.1.10}$$

$$\begin{gathered} V_1^* \otimes \dots \otimes V_k^* \to (V_1 \otimes \dots \otimes V_k)^*, \\ s_1 \otimes \dots \otimes s_k \mapsto \left\{ \begin{array}{l} V_1 \otimes \dots \otimes V_k \to \mathbf{k}, \\ v_1 \otimes \dots \otimes v_k \mapsto s_1(v_1) \dots s_k(v_k) \end{array} \right\}. \end{gathered} \tag{2.1.11}$$

2.2 Symmetric Powers of Vector Spaces

Let V be a vector space.

By definition, the *k-th tensor power of V* is

$$V^{\otimes k} := \begin{cases} \mathbf{k} & \text{if } k = 0, \\ \underbrace{V \otimes \dots \otimes V}_{k} & \text{if } k > 0. \end{cases}$$

An element $\sigma \in \mathbf{S}_k$ defines the isomorphism (we denote it by the same symbol)

$$\sigma : V^{\otimes k} \to V^{\otimes k}, \quad v_1 \otimes \dots \otimes v_k \mapsto v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}.$$

For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} : V^{\otimes 4} \to V^{\otimes 4},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} : v_1 \otimes v_2 \otimes v_3 \otimes v_4 \mapsto v_4 \otimes v_1 \otimes v_3 \otimes v_2.$$

Clearly,

$$\sigma(\tau(T)) = (\sigma \circ \tau)(T)$$

for all $\sigma, \tau \in \mathbf{S}_k$, $T \in V^{\otimes k}$.

A tensor $T \in V^{\otimes k}$ is called *symmetric* whenever

$$\sigma(T) = T \quad \text{for all } \sigma \in \mathbf{S}_k.$$

Example 2.2.1. The simplest symmetric tensor is the zero tensor. The following tensors are symmetric

$$e_1 \otimes e_1 - 2e_1 \otimes e_3 - 2e_3 \otimes e_1 \in (\mathbb{R}^6)^{\otimes 2},$$

$$e_2 \otimes e_2 \otimes e_5 + e_2 \otimes e_5 \otimes e_2 + e_5 \otimes e_2 \otimes e_2 \in (\mathbb{C}^5)^{\otimes 3}.$$

The tensor

$$T = e_2 \otimes e_2 + e_3 \otimes e_1 - e_1 \otimes e_3 \in (\mathbb{R}^3)^{\otimes 2}$$

is not symmetric because

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}(T) = e_2 \otimes e_2 + e_1 \otimes e_3 - e_3 \otimes e_1 \neq T.$$

Consider the subset

$$\mathrm{S}^k(V) := \{T \in V^{\otimes k} \mid \sigma(T) = T \text{ for all } \sigma \in \mathbf{S}_k\}$$

of symmetric tensors. Clearly, the subset $\mathrm{S}^k(V)$ is a subspace of $V^{\otimes k}$. Consider the map

$$\mathrm{sym} : V^{\otimes k} \to V^{\otimes k}, \quad T \mapsto \frac{1}{k!} \sum_{\sigma \in \mathbf{S}_k} \sigma(T).$$

Lemma 2.2.2. *For a tensor $T \in V^{\otimes k}$ and an element $\tau \in \mathbf{S}_k$ we have*

$$\tau(\mathrm{sym}(T)) = \mathrm{sym}(\tau(T)) = \mathrm{sym}(T).$$

Corollary 2.2.3. *Map* sym *is a projection onto* $\mathrm{S}^k(V)$.

For vectors $v_1, \ldots, v_k \in V$ define their *symmetric product*

$$v_1 \cdot \ldots \cdot v_k := \mathrm{sym}(v_1 \otimes \ldots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in \mathbf{S}_k} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)} \in \mathrm{S}^k(V)$$

A symmetric tensor T is called *decomposable* whenever it is a symmetric product of some vectors.

Lemma 2.2.4. *We have*

$$\begin{gathered} v_1 \cdot \ldots \cdot v_{i-1} \cdot (\lambda v_i + \lambda' v_i') \cdot v_{i+1} \cdot \ldots \cdot v_k = \\ \lambda v_1 \cdot \ldots \cdot v_{i-1} \cdot v_i \cdot v_{i+1} \cdot \ldots \cdot v_k + \lambda' v_1 \cdot \ldots \cdot v_{i-1} \cdot v_i' \cdot v_{i+1} \cdot \ldots \cdot v_k, \\ v_{\sigma(1)} \cdot \ldots \cdot v_{\sigma(k)} = v_1 \cdot \ldots \cdot v_k \end{gathered} \tag{2.2.1}$$

for all $i = 1, \ldots, k$; $\lambda, \lambda' \in \mathbf{k}$; $v_j, v_j' \in V$; $\sigma \in \mathbf{S}_k$.

Suppose that V is finite-dimensional and $(e_1, \ldots, e_n)$ is a basis of V.

Lemma 2.2.5. *The list $(e_{i_1} \cdot \ldots \cdot e_{i_k} \mid i_1 \leqslant \ldots \leqslant i_k)$ is a basis of* $\mathrm{S}^k(V)$.

From this lemma it follows that

$$\dim(\mathrm{S}^k(V)) = \binom{\dim(V) + k - 1}{k}$$

and a tensor $q \in \mathrm{S}^k(V)$ can be decomposed:

$$q = \sum_{i_1 \leqslant \ldots \leqslant i_k} f^{i_1, \ldots, i_k} e_{i_1} \cdot \ldots \cdot e_{i_k}.$$

Remark 2.2.6. We can define the symmetric powers of V without using tensors. Namely, we define the kth symmetric power $\mathrm{S}^k(V)$ as the space of formal expressions of the form

$$f_1 v_{11} \cdot \ldots \cdot v_{1k} + \ldots + f_l v_{l1} \cdot \ldots \cdot v_{lk}, \quad \text{where } l \in \mathbb{N}, f_i \in \mathbf{k}, v_{ij} \in V$$

modulo relations from (2.2.1). One can prove that this definition is consistent with the definition with tensors.

If $V = \mathbf{k}^{n*}$ and $(x^1, \ldots, x^n)$ is a basis of V^*, then $\mathrm{S}^k(\mathbf{k}^{n*})$ is the "usual" space of homogenuous polynomials of degree k in the variables $x^1, \ldots, x^n$.

2.3 Exterior Powers of Vector Spaces

Let V be a vector space.

A tensor $T \in V^{\otimes k}$ is called *skew-symmetric* whenever

$$\sigma(T) = \mathrm{sgn}(\sigma) T \quad \text{for all } \sigma \in \mathbf{S}_k.$$

Example 2.3.1. The simplest skew-symmetric tensor is the zero tensor. The following tensors are skew-symmetric

$$e_1 \otimes e_3 - e_3 \otimes e_1 + 5 e_2 \otimes e_3 - 5 e_3 \otimes e_2 \in (\mathbb{R}^7)^{\otimes 2},$$

$$e_2 \otimes e_3 \otimes e_5 + e_3 \otimes e_5 \otimes e_2 + e_5 \otimes e_2 \otimes e_3 - e_2 \otimes e_5 \otimes e_3 - e_3 \otimes e_2 \otimes e_5 - e_5 \otimes e_3 \otimes e_2 \in (\mathbb{C}^5)^{\otimes 3}.$$

The tensor

$$T = e_1 \otimes e_1 + e_2 \otimes e_1 - e_1 \otimes e_2 \in (\mathbb{R}^2)^{\otimes 2},$$

is not skew-symmetric because

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}(T) = e_1 \otimes e_1 + e_1 \otimes e_2 - e_2 \otimes e_1 \neq \mathrm{sgn} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} T = -T.$$

Consider the subset

$$\Lambda^k(V) := \{T \in V^{\otimes k} \mid \sigma(T) = \mathrm{sgn}(\sigma) T \text{ for all } \sigma \in \mathbf{S}_k\}$$

of skew-symmetric tensors. Clearly, the subset $\Lambda^k(V)$ is a subspace of $V^{\otimes k}$. Consider the map

$$\mathrm{alt} : V^{\otimes k} \to V^{\otimes k}, \quad T \mapsto \frac{1}{k!} \sum_{\sigma \in \mathbf{S}_k} \mathrm{sgn}(\sigma) \sigma(T).$$

Lemma 2.3.2. *For a tensor $T \in V^{\otimes k}$ and an element $\tau \in \mathbf{S}_k$ we have*

$$\tau(\mathrm{alt}(T)) = \mathrm{alt}(\tau(T)) = \mathrm{sgn}(\tau) \mathrm{alt}(T).$$

Corollary 2.3.3. *Map* alt *is a projection onto $\Lambda^k(V)$.*

For vectors $v_1, \ldots, v_k \in V$ define their *external product*

$$v_1 \wedge \ldots \wedge v_k := \operatorname{alt}(v_1 \otimes \ldots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in \mathbf{S}_k} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)} \in \Lambda^k(V).$$

A skew-symmetric tensor T is called *decomposable* whenever it is an external product of some vectors.

Lemma 2.3.4. *We have*

$$\begin{gathered} v_1 \wedge \ldots \wedge v_{i-1} \wedge (\lambda v_i + \lambda' v_i') \wedge v_{i+1} \wedge \ldots \wedge v_k = \\ \lambda v_1 \wedge \ldots \wedge v_{i-1} \wedge v_i \wedge v_{i+1} \wedge \ldots \wedge v_k + \lambda' v_1 \wedge \ldots \wedge v_{i-1} \wedge v_i' \wedge v_{i+1} \wedge \ldots \wedge v_k, \\ v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(k)} = \operatorname{sgn}(\sigma) v_1 \wedge \ldots \wedge v_k \end{gathered} \tag{2.3.1}$$

for all $i = 1, \ldots, k$*;* $\lambda, \lambda' \in \mathbf{k}$*;* $v_j, v_j' \in V$*;* $\sigma \in \mathbf{S}_k$*.*

Suppose that V is finite-dimensional and $(e_1, \ldots, e_n)$ is a basis of V.

Lemma 2.3.5. *The list of vectors* $(e_{i_1} \wedge \ldots \wedge e_{i_k} \mid i_1 < \ldots < i_k)$ *is a basis of* $\Lambda^k(V)$*.*

Corollary 2.3.6. *1.* $\dim(\Lambda^k(V)) = \binom{\dim(V)}{k}$ *for* $0 \leqslant k \leqslant \dim(V)$*;*

2. $\Lambda^k(V) = 0$ *for* $k > \dim(V)$*.*

Remark 2.3.7. We can define the exterior powers of V without using tensors. Namely, we define the kth exterior power $\Lambda^k(V)$ as the space of formal expressions of the form

$$\lambda_1 v_{11} \wedge \ldots \wedge v_{1k} + \ldots + \lambda_l v_{l1} \wedge \ldots \wedge v_{lk}, \quad \text{where } l \in \mathbb{N}, \lambda_i \in \mathbf{k}, v_{ij} \in V,$$

modulo relations (2.3.1). One can prove that this definition is consistent with the definition with tensors.

Remark 2.3.8. In analysis, when we consider an open subset $U \subset \mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$, elements of $\Lambda^k(T^*_{x_0}(U))$, where $T_{x_0}(U)$ is the tangent space of U at $x_0 \in U$, are called *k-forms*. We have the basis of $\Lambda^k(T^*_{x_0}(U))$:

$$(dx^{i_1} \wedge \ldots \wedge dx^{i_k} \mid i_1 < \ldots < i_k).$$

So, a k-form ω can be decomposed:

$$\omega = \sum_{i_1 < \ldots < i_k} \omega_{i_1, \ldots, i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

2.4 Tensor Product of Linear maps

Let $V_1, \ldots, V_k, U_1, \ldots, U_k$ be vector spaces and $\varphi_i : V_i \to U_i$, $i = 1, \ldots, k$ be linear maps. Define the linear map

$$\begin{aligned} \varphi_1 \otimes \ldots \otimes \varphi_k : V_1 \otimes \ldots \otimes V_k &\to U_1 \otimes \ldots \otimes U_k, \\ v_1 \otimes \ldots \otimes v_k &\mapsto \varphi_1(v_1) \otimes \ldots \otimes \varphi_k(v_k). \end{aligned} \tag{2.4.1}$$

It is easy to see that $\varphi_1 \otimes \ldots \otimes \varphi_k$ is well-defined; it is called the *tensor product of the maps* $\varphi_1, \ldots, \varphi_k$.

Let V be a vector space and $\varphi : V \to V$ be an operator. We have the endomorphism

$$\varphi^{\otimes k} := \underbrace{\varphi \otimes \ldots \otimes \varphi}_{k} : V^{\otimes k} \to V^{\otimes k}.$$

Lemma 2.4.1. *We have the inclusions*

1. $\varphi^{\otimes k}(\Lambda^k V) \subset \Lambda^k V$;
2. $\varphi^{\otimes k}(\mathrm{S}^k(V)) \subset \mathrm{S}^k(V)$.

From this lemma it follows that the operators

$$\Lambda^k(\varphi) : \Lambda^k V \to \Lambda^k V, \qquad T \mapsto \varphi^{\otimes k}(T);$$
$$\mathrm{S}^k(\varphi) : \mathrm{S}^k(V) \to \mathrm{S}^k(V), \qquad T \mapsto \varphi^{\otimes k}(T).$$

are well-defined.

2.5 Canonical Forms of Tensors

In this section we consider finite-dimensional vector spaces.

There is the following

General problem. *Given a tensor T, find a "simplest" presentation of T in coordinates (the* canonical form *of T).*

Here are some special cases of this problem.

1. $q \in \mathrm{S}^2(V)$, $\mathbf{k} = \mathbb{C}$. In this case, the canonical form is

   $$q = e_1^2 + \ldots + e_k^2,$$

   where $k \geqslant 0$, (e_i) is some basis of V. Thus k is the rank of q and the matrix of q in the basis (e_i) is

   $$\begin{pmatrix} \mathrm{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

   If $k = \dim(V)$, then q is called *nondegenerate.*

2. $q \in \mathrm{S}^2(V)$, $\mathbf{k} = \mathbb{R}$. In this case, the canonical form is

   $$q = e_1^2 + \ldots + e_k^2 - e_{k+1}^2 - \ldots - e_{k+l}^2,$$

   where $k, l \geqslant 0$, (e_i) is some basis of V. Thus $k + l$ is the rank of q and the matrix of q in the basis (e_i) is

   $$\begin{pmatrix} \mathrm{I}_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_l & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The pair of integers (k, l) is called the *signature of* q. If $k + l = \dim(V)$, then q is called *nondegenerate*.

3. $\omega \in \Lambda^2 V$, $\mathbf{k}$ is an arbitrary field. In this case, the canonical form is

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{2k-1} \wedge e_{2k},$$

where $k \geqslant 0$, (e_i) is some basis of V.

4. $\varphi \in V \otimes V^* = \mathrm{Hom}(V, V)$, $\mathbf{k} = \mathbb{C}$. In this case, the canonical form is

$$\varphi = \Phi_i^j e_j \otimes x^i,$$

where (e_i) is a basis of V, (x^i) is the dual basis of the dual space V^*, and (Φ_i^j) is a Jordan matrix.

5. $f \in \mathrm{S}^3(V)$, where $\dim(V) = 3$, $\mathbf{k} = \mathbb{C}$. In this case, if f is "in general position," then there exists a basis (e_1, e_2, e_3) of V such that

$$Tf = \lambda_1(e_1^3 + e_2^3 + e_3^3) + \lambda_2 e_1 e_2 e_3,$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$.

Remark 2.5.1. In some cases canoniacal form is uniquely determined (examples (1), (2), and (3) above), and in some cases it is not uniquely determined (examples (4) and (5) above).

Remark 2.5.2. For tensors of a "general" space, there are no "good" canonical forms. For example, there are no "good" canonical forms for tensors of the space $\mathrm{S}^4(V)$, where $\dim(V) = 3$, $\mathbf{k} = \mathbb{C}$.

Exercise 2.5.3. Find the canonical form of the tensor

$$x^1 x^4 - (x^3)^2 \in \mathrm{S}^2(\mathbb{C}^{4*}),$$

where (x^i) is the standard basis of $\mathbb{C}^{4*}$.

Exercise 2.5.4. Find the canonical form of the tensor

$$x^1 x^5 - x^2 x^4 + (x^3)^2 \ \in \ \mathrm{S}^2(\mathbb{R}^{5*}),$$

where (x^i) is the standard basis of $\mathbb{R}^{5*}$.

Exercise 2.5.5. Find the canonical form of the tensor

$$x^1 \wedge x^2 + x^1 \wedge x^5 - x^2 \wedge x^4 + x^3 \wedge x^3 + x^3 \wedge x^6 \ \in \ \Lambda^2(\mathbb{F}_3^{6*}),$$

where (x^i) is the standard basis of $\mathbb{F}_3^{6*}$.

2.6 *Multilinear Maps

Definition 2.6.1. If $V_1, \ldots, V_k, W$ are vector spaces, then

1. A map $\varphi : V_1 \times \ldots \times V_k \to W$ is called *multilinear* whenever $\varphi(v_1, \ldots, v_k)$ is linear by v_i for all $1 \leqslant i \leqslant k$.

2. A multilinear map of the form $\varphi : V_1 \times \ldots \times V_k \to \mathbf{k}$ is called a *multilinear form*.

The concept of multilinear form is a generalization of the concept bilinear form.

Example 2.6.2. The map

$$\mathrm{Mat}_{k\times l} \times \mathrm{Mat}_{l\times m} \times \mathrm{Mat}_{m\times n} \to \mathrm{Mat}_{k\times n}, \quad (A, B, C) \mapsto ABC.$$

is trilinear.

Lemma 2.6.3. *Consider the map*

$$\alpha : V_1 \times \ldots \times V_k \to V_1 \otimes \ldots \otimes V_k, \quad (v_1, \ldots, v_k) \mapsto v_1 \otimes \ldots \otimes v_k.$$

Then we have:

1. *α is a multilinear map.*

2. $\mathrm{Span}(\alpha(V_1 \times \ldots \times V_k)) = V_1 \otimes \ldots \otimes V_k$.

3. *Suppose that $\varphi : V_1 \times \ldots \times V_k \to W$ is a multilinear map; then there exists a unique linear map $\varphi_\alpha : V_1 \otimes \ldots \otimes V_k \to W$ such that $\varphi = \varphi_\alpha \circ \alpha$. The following commutative diagram illustarates this statement*

$$\begin{tikzcd}
V_1 \times \ldots \times V_k \arrow[rr, "\alpha"] \arrow[dr, "\varphi"'] & & V_1 \otimes \ldots \otimes V_k \arrow[dl, dotted, "\varphi_\alpha"] \\
& W &
\end{tikzcd}$$

The map α from this lemma is called the *universal multilinear map*.
Clearly, the set of multilinear maps $\varphi : V_1 \times \ldots \times V_k \to W$ is a vector space.

Lemma 2.6.4. *We have the isomorphisms*

$$\begin{aligned} V_1^* \otimes \ldots \otimes V_k^* &\to \{\textit{multilinear forms on } V_1 \times \ldots \times V_k\}, \\ s_1 \otimes \ldots \otimes s_k &\mapsto \left\{ \begin{matrix} V_1 \times \ldots \times V_k \to \mathbf{k}, \\ (v_1, \ldots, v_k) \mapsto s_1(v_1) \ldots s_k(v_k) \end{matrix} \right\}. \end{aligned}$$

Definition 2.6.5. If $k \in \mathbb{N}^*$, W is a vector space, and V is a vector space. Then

1. A multilinear map $\varphi : V^{\times k} \to W$ is called *symmetric* whenever

$$\varphi(\sigma(v_1, \dots, v_k)) = \varphi(v_1, \dots, v_k) \quad \text{for all } \sigma \in \mathbf{S}_k.$$

2. A symmetric map of the form $\varphi : V^{\times k} \to \mathbf{k}$ is called a *symmetric form on* $V^{\times k}$.

The concept of symmetric multilinear form is a generalization of the concept bilinear symmetric form.

Example 2.6.6. The following maps are symmetric:

1. the Euclidean dot product

$$\cdot \,:\, \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (v, u) \mapsto v \cdot u;$$

2. the map

$$\mathrm{Mat}_n(\mathbf{k}) \times \mathrm{Mat}_n(\mathbf{k}) \to \mathrm{Mat}_n(\mathbf{k}), \quad (A, B) \mapsto AB + BA.$$

Lemma 2.6.7. *Consider the map*

$$\gamma := \mathrm{sym} \circ \alpha : V^{\times k} \to \mathrm{S}^k(V),$$

where α is the universal multilinear map. Then we have:

1. *γ is a symmetric map.*
2. $\mathrm{Span}(\gamma(V^{\times k})) = \mathrm{S}^k(V)$.
3. *Suppose that $\varphi : V^{\times k} \to W$ is a symmetric map; then there exists a unique linear map $\varphi_\gamma : \mathrm{S}^k V \to W$ such that $\varphi = \varphi_\gamma \circ \gamma$. The following commutative diagram illustrates this statement:*

$$\begin{tikzcd} V^{\times k} \arrow[r, "\gamma"] \arrow[dr, "\varphi"'] & \mathrm{S}^k V \arrow[d, dotted, "\varphi_\gamma"] \\ & W \end{tikzcd}$$

The map γ from this lemma is called the *universal symmetric map.*
Clearly, the set of symmetric maps of the form $\varphi : V^{\times k} \to W$ is a vector space.

Lemma 2.6.8. *We have the isomorphisms*

$$\mathrm{S}^k(V^*) \to \{\textit{symmetric forms on } V^{\times k}\},$$
$$s_1 \cdot \ldots \cdot s_k \mapsto \left\{ \begin{array}{c} V^{\times k} \to \mathbf{k}, \\ (v_1, \dots, v_k) \mapsto \dfrac{1}{k!} \sum_{\sigma \in \mathbf{S}_k} s_{\sigma(1)}(v_1) \dots s_{\sigma(k)}(v_k) \end{array} \right\}$$

Suppose that V is a finite-dimensional vector space and $k \in \mathbb{N}^*$. A map $p : V \to \mathbf{k}$ is called a *homogeneous form of degree* k on V whenever

$$p(v) = q(\underbrace{v, \dots, v}_{k}),$$

where $q : V^{\times k} \to \mathbf{k}$ is a symmetric form. Clearly, the set of homogeneous forms $p : V \to \mathbf{k}$ of degree k is a vector space. The concept of homogeneous form is a generalization of the concept of quadratic form.

Lemma 2.6.9. *Suppose that V is a finite-dimensional vector space and $k \in \mathbb{N}^*$. Then the map*

$$\{\text{symmetric forms on } V^{\times k}\} \to \left\{ \begin{matrix} \text{homogeneous forms} \\ p : V \to \mathbf{k} \text{ of degree } k \end{matrix} \right\},$$
$$q(v_1, \dots, v_k) \mapsto p(v) = q(v, \dots, v)$$

is an isomorphism between vector spaces.

Definition 2.6.10. If $k \in \mathbb{N}^*$, W is a vector spaces, and V is a vector spaces. Then

1. A multilinear map $\varphi : V^{\times k} \to W$ is called *skew-symmetric* whenever

$$\varphi(\sigma(v_1, \dots, v_k)) = \operatorname{sgn}(\sigma)\varphi(v_1, \dots, v_k) \quad \text{for all } \sigma \in \mathbf{S}_k.$$

2. A skew-symmetric map of the form $\varphi : V^{\times k} \to \mathbf{k}$ is called a *skew-symmetric form.*

The concept of skew-symmetric form is a generalization of the concept of skew-symmetric bilinear form.

Example 2.6.11. The following maps are skew-symmetric:

1. the cross product

$$\cdot \times \cdot \ : \ \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (v, u) \mapsto v \times u;$$

2. the Lie bracket

$$[\cdot,\cdot] : \mathrm{Mat}_n(\mathbf{k}) \times \mathrm{Mat}_n(\mathbf{k}) \to \mathrm{Mat}_n(\mathbf{k}), \quad (A, B) \mapsto AB - BA.$$

3. the map

$$(\mathbf{k}^n)^{\times n} \to \mathbf{k}, \quad (\begin{pmatrix} a_1^1 \\ \vdots \\ a_1^n \end{pmatrix}, \dots, \begin{pmatrix} a_n^1 \\ \vdots \\ a_n^n \end{pmatrix}) \mapsto \det \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \dots & a_n^n \end{pmatrix}.$$

Lemma 2.6.12. *Consider the map*

$$\beta := \mathrm{alt} \circ \alpha : V^{\times k} \to \Lambda^k(V),$$

where α is the universal multilinear map. Then we have:

1. *β is a skew-symmetric map.*
2. $\mathrm{Span}(\beta(V^{\times k})) = \Lambda^k(V)$.
3. *Suppose that $\varphi : V^{\times k} \to W$ is a skew-symmetric map; then there exists a unique linear map $\varphi_\beta : \Lambda^k(V) \to W$ such that $\varphi = \varphi_\beta \circ \beta$. The following commutative diagram illustrates this statement:*

$$\begin{tikzcd} V^{\times k} \arrow[rr, "\beta"] \arrow[dr, "\varphi"'] & & \Lambda^k(V) \arrow[dl, dotted, "\varphi_\beta"] \\ & W & \end{tikzcd}$$

The map β from this lemma is called the *universal skew-symmetric map.*

Clearly, the set of skew-symmetric maps of the form $\varphi : V^{\times k} \to W$ is a vector space.

Lemma 2.6.13. *We have the isomorphism*

$$\Lambda^k(V^*) \to \left\{\textit{skew-symmetric forms on } V^{\times k}\right\},$$
$$s_1 \wedge \ldots \wedge s_k \mapsto \left\{\begin{array}{c} V^{\times k} \to \mathbf{k}, \\ (v_1, \ldots, v_k) \mapsto \dfrac{1}{k!} \displaystyle\sum_{\sigma \in \mathbf{S}_k} \mathrm{sgn}(\sigma) s_{\sigma(1)}(v_1) \ldots s_{\sigma(k)}(v_k) \end{array}\right\}$$

2.7 *Tensor Fields

Let U be an open subset of $\mathbb{R}^n$. A *scalar field on U* is a map $v : U \to \mathbb{R}$. A *vector field on U* is a map $v : U \to \mathbb{R}^n$. For example, electric and magnetic fields. The natural generalization of scalar and vector fields are tensor fields. A *tensor field* is a map of the form $v : U \to (\mathbb{R}^n)^{\otimes p} \otimes (\mathbb{R}^{n*})^{\otimes q}$, where $p, q \in \mathbb{N}$. For example, a Riemannian metric on U is a positive-definite tensor field $g : U \to \mathrm{S}^2\mathbb{R}^{n*}$. The point is that many physical laws are of the form $D(f) = 0$, where f is a tensor field which describes a physical object and D is some *universal differential operation.* Maxwell's equations in electrodynamics and Einstein's equation in gravitational theory are of this form. Here is an example of a universal differential operation.

Example 2.7.1. Consider the space $\mathbb{R}^n$ with the coordinates $(x^1, \ldots, x^n)$, the dual space $\mathbb{R}^{n*}$ with the dual basis $dx^1, \ldots, dx^n$ of . Consider the space $\Omega^k(U)$ of tensor fields of the form

$$\omega : U \to \Lambda^k(\mathbb{R}^{n*}).$$

A tensor field $\omega \in \Omega^k(U)$ can be written in the form

$$w_{i_1,\dots,i_k}(x)\ dx^{i_1} \wedge \ldots \wedge dx^{i_k}, \quad \text{where } w_{i_1,\dots,i_k}(x) \in C^\infty(U).$$

The *exterior derivative* is a universal differential operation. It is defined as

$$d : \Omega^k(U) \to \Omega^{k+1}(U),$$
$$w_{i_1,\dots,i_k}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_k} \mapsto \frac{\partial w_{i_1,\dots,i_k}(x)}{\partial x^l} dx^l \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Maxwell's equations in electrodynamics.

Consider the 4-dimensional space-time $\mathbb{R}^4$ with coordinates (t, x, y, z) and the corresponding basis dt, dx, dy, dz of $\mathbb{R}^{4*}$. Consider an open subset $U \subset \mathbb{R}^4$. For Maxwell's equations we have the following data.

(ρ) The density of charge

$$\rho : U \to \mathbb{R},$$

where $\rho(t, x, y, z)$ is the density of charge at the point (x, y, z) at the time t.

(J) The current density

$$\mathrm{J} : U \to \mathbb{R}^3,$$
$$\mathrm{J}(t, x, y, z) = (\mathrm{J}_x(t, x, y, z), \mathrm{J}_y(t, x, y, z), \mathrm{J}_z(t, x, y, z)),$$

where $\mathrm{J}(t, x, y, z)$ is the current density at the point (x, y, z) at the time t.

(E) The electric field

$$\mathrm{E} : U \to \mathbb{R}^3,$$
$$\mathrm{E}(t, x, y, z) = (\mathrm{E}_x(t, x, y, z), \mathrm{E}_y(t, x, y, z), \mathrm{E}_z(t, x, y, z)),$$

where $\mathrm{E}(t, x, y, z)$ is the electric field at the point (x, y, z) at the time t.

(B) The magnetic field

$$\mathrm{B} : U \to \mathbb{R}^3,$$
$$\mathrm{B}(t, x, y, z) = (\mathrm{B}_x(t, x, y, z), \mathrm{B}_y(t, x, y, z), \mathrm{B}_z(t, x, y, z)),$$

where $\mathrm{B}(t, x, y, z)$ is the magnetic field at the point (x, y, z) at the time t.

For Maxwell's equations we need the following tensor fields or, simply, tensors:

$$\mathbb{F} := -\mathrm{E}_x dt \wedge dx - \mathrm{E}_y dt \wedge dy - \mathrm{E}_z dt \wedge dz + +\mathrm{B}_x dy \wedge dz + \mathrm{B}_y dz \wedge dx + \mathrm{B}_z dx \wedge dy \in \Omega^2(U)$$

– Faraday's tensor,

$$\star\mathbb{F} := \mathrm{B}_x dt \wedge dx + \mathrm{B}_y dt \wedge dy + \mathrm{B}_z dt \wedge dz + +\mathrm{E}_x dy \wedge dz + \mathrm{E}_y dz \wedge dx + \mathrm{E}_z dx \wedge dy \in \Omega^2(U)$$

– the dual to Faraday's tensor, and

$$\varphi = \rho dx \wedge dy \wedge dz - \mathrm{J}_x dt \wedge dy \wedge dz - \mathrm{J}_y dt \wedge dz \wedge dx - \mathrm{J}_z dt \wedge dx \wedge dy \in \Omega^3(U).$$

Exercise 2.7.2. Check that the matrix of Faraday's tensor in the basis (dt, dx, dy, dz) of the space $\mathbb{R}^{4*}$ is

$$\begin{pmatrix} 0 & -\mathrm{E}_x & -\mathrm{E}_y & -\mathrm{E}_z \\ \mathrm{E}_x & 0 & \mathrm{B}_z & -\mathrm{B}_y \\ \mathrm{E}_y & -\mathrm{B}_z & 0 & \mathrm{B}_x \\ \mathrm{E}_z & \mathrm{B}_y & -\mathrm{B}_x & 0 \end{pmatrix}$$

Maxwell's equations are

$$d(\mathbb{F}) = 0, \qquad d(\star\mathbb{F}) = \varphi.$$

Einstein's equation in gravitational theory.

Consider $\mathbb{R}^4$ with coordinates $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$, the standard basis

$$\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$$

of $\mathbb{R}^4$ and the dual basis

$$(dx^0, dx^1, dx^2, dx^3)$$

of the dual space $\mathbb{R}^{4*}$. Consider an open subset $U \subset \mathbb{R}^4$. For Einstein's equation we have the following data.

(g) The metric tensor

$$g : U \to \mathrm{S}^2(R^{4*}), \quad g(x) = g_{ij}(x) dx^i \otimes dx^j$$

and its dual

$$g^{-1} : U \to \mathrm{S}^2(R^4), \quad g^{-1}(x) = g^{ij}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$

where (g^{ij}) is the inverse to (g_{ij}) matrix.

(T) The stress-energy tensor

$$T : U \to \mathrm{S}^2(R^{4*}), \quad T(x) = T_{ij}(x) dx^i \otimes dx^j.$$

Einstein's equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where

- Λ is the cosmological constant;
- G is Newton's gravitational constant;
- c is the speed of light;
- $$R_{\mu\nu} = \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\rho} - \frac{\partial \Gamma^\rho_{\mu\rho}}{\partial x^\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\nu\sigma},$$
  where
  $$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\nu\rho}}{\partial x^\sigma} + \frac{\partial g_{\nu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\nu}\right);$$
- $R = g^{\mu\nu} R_{\mu\nu}$.

2.8 Problems

Problem 2.8.1. *Prove that a tensor $T \in V \otimes U$ can be represented in the form*

$$T = \sum_{i=1}^{k} v_i \otimes u_i,$$

where $v_1, \dots, v_k \in V$ and $u_1, \dots, u_k \in U$ are linearly independent vectors.

Problem 2.8.2. *Consider the 3-dimensional space $\mathbb{R}^3$. Prove that the following linear maps are well-defined:*

1. $\varphi : \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}$, $\varphi(v \otimes u) = v \cdot u$, *(the dot product of v and u);*
2. $\varphi : \mathbb{R}^3 \otimes \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(v \otimes u) = v \times u$ *(the cross-product of v and u).*

Problem 2.8.3. *Prove that the following "linear maps" are not well-defined:*

1. $\mathbb{R}^1 \otimes \mathbb{R}^1 \to \mathbb{R}^1, \quad (a_1) \otimes (b_1) \mapsto (-a_1 + 3b_1);$
2. $\mathbb{R}^2 \otimes \mathbb{R}^1 \to \mathbb{R}^1, \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes (b_1) \mapsto (a_1 a_2 b_1);$
3. $\mathbb{R}^2 \otimes \mathbb{R}^2 \to \mathbb{R}^1, \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \mapsto (a_1 b_2 + a_2 b_1 + 1).$
4. $\mathbb{R}^2 \otimes \mathbb{R}^1 \to \mathbb{R}^2, \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes (b_1) \mapsto \begin{pmatrix} 2a_1 \\ -a_2 \end{pmatrix}.$

Problem 2.8.4. *Suppose that $v_1, v_2, v_3 \in V$ are linearly independent vectors. Prove that the tensor $v_1 \otimes v_2 + v_2 \otimes v_3 \in V \otimes V$ is not decomposable.*

Problem 2.8.5. *Find decomposable tensors on the list*

1. $T^{ij} = i + j;$
2. $T^{ij} = i - j;$
3. $T^{ij} = ij.$

Problem 2.8.6. *Suppose that V and U are vector spaces.*

1. *For $0 \neq v \in V$ prove that*
   (a) *the map $\varphi : U \to V \otimes U$, $\varphi(u) = v \otimes u$ is linear;*
   (b) $\mathrm{Ker}(\varphi) = 0$*;*

(c) *the image of φ is a subspace of $V \otimes U$ of dimension $\dim(U)$ (it is denoted by $\mathrm{Span}(v) \otimes U$).*

2. *For linearly independent $v_1, \dots v_d \in V$ prove that the sum*

$$\mathrm{Span}(v_1) \otimes U + \dots + \mathrm{Span}(v_d) \otimes U$$

of subspaces is direct.

Problem 2.8.7. *Suppose that V is a finite-dimensional vector space. Define the tensor*

$$\delta := x^i \otimes e_i \in V^* \otimes V,$$

where (e_i) is a basis of V, (x^i) is the dual basis of the dual space V^*. Prove that δ is well-defined.

Consider a tensor product $V_1 \otimes \dots \otimes V_k$ such that $V_j = V_i^*$, where $i < j$. Then we have the well-defined linear map

$$\begin{gathered} V_1 \otimes \dots \otimes V_k \to V_1 \otimes \dots \otimes \hat{V}_i \otimes \dots \otimes \hat{V}_j \otimes \dots \otimes V_k, \\ v_1 \otimes \dots \otimes v_k \mapsto v_j(v_i) v_1 \otimes \dots \otimes \hat{v}_i \otimes \dots \otimes \hat{v}_j \otimes \dots \otimes v_k; \end{gathered}$$

it is called a *contruction.*

Problem 2.8.8. *Find all possible contractions of the tensor*

$$(x^1 + x^2) \otimes x^3 \otimes e_1 \otimes (e_1 - e_2) \in \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^n.$$

Problem 2.8.9. *Prove lemma 2.2.2 and its corollary.*

Problem 2.8.10. *Prove lemma 2.2.5.*

Problem 2.8.11. *Prove lemma 2.3.2 and its corollary.*

Problem 2.8.12. *Prove lemma 2.3.5.*

Problem 2.8.13. *Suppose that V is a finite-dimensional vector space. Prove that*

1. $V^{\otimes 2} = \Lambda^2(V) \oplus \mathrm{S}^2(V)$;

2. $V^{\otimes k} \neq \Lambda^k(V) \oplus \mathrm{S}^k(V)$ *for* $\dim(V) \geqslant 2$, $k \geqslant 3$.

Problem 2.8.14. *Suppose that V is a finite-dimensional vector space. Construct explicitly the isomorphisms*

1. $\Lambda^2(V \otimes U) \simeq \Lambda^2(V) \otimes \mathrm{S}^2(U) \oplus \mathrm{S}^2(V) \otimes \Lambda^2(U)$;

2. $\mathrm{S}^2(V \otimes U) \simeq \mathrm{S}^2(V) \otimes \mathrm{S}^2(U) \oplus \Lambda^2(V) \otimes \Lambda^2(U)$.

Chapter 3

Operators

Vector spaces we consider in this chapter are assumed finite-dimensional real or complex. We describe operators in general vector spaces, real inner product spaces, and Hermitian spaces.

3.1 Realification and Complexification

A complex vector space U, considered as a real vector space, is called the *realification of* U and denoted by $U^{\mathbb{R}}$.

If $(e_1, \dots, e_n)$ is a basis of U, then $(e_1, \sqrt{-1}e_1, \dots, e_n, \sqrt{-1}e_n)$ is a basis of $U^{\mathbb{R}}$ (problem 3.8.1). From this it follows that

$$\dim_{\mathbb{R}}(U^{\mathbb{R}}) = 2\dim_{\mathbb{C}}(U).$$

Definition 3.1.1. Suppose that V is a real vector space. An operator $J : V \to V$ such that $J^2 = -\mathrm{I}_V$ is called a *complex structure* on V.

Example 3.1.2. Consider a complex vector space U and an operator

$$U \to U, \quad u \mapsto \sqrt{-1}u.$$

In a natural way, this operator defines the complex structure on $U^{\mathbb{R}}$.

A real vector space V with a complex structure J can be considered as a complex vector space with the multiplication by complex numbers

$$(a + b\sqrt{-1}) \cdot v \;:=\; av + bJ(v), \quad \text{where } v \in V,\ a + b\sqrt{-1} \in \mathbb{C}.$$

In this situation, the space V considered as a complex vector space is denoted by V^J.

Lemma 3.1.3. *Suppose that V is a real vector space. Then*

1. $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ *with the multiplication by complex numbers*

$$\lambda \cdot (v \otimes \mu) := v \otimes (\lambda\mu), \quad \textit{where } v \in V \textit{ and } \lambda, \mu \in \mathbb{C},$$

is a complex vector space (it is called the complexification *of V).*

2. *If (e_i) is a basis of V, then $(e_i \otimes 1)$ is a basis of $V^{\mathbb{C}}$.*

3. $\dim_{\mathbb{C}}(V^{\mathbb{C}}) = \dim_{\mathbb{R}}(V)$.

Consider $\mathbb{R}^n$ and its complexification $(\mathbb{R}^n)^{\mathbb{C}}$. We have the isomorphism of complex vector spaces

$$(\mathbb{R}^n)^{\mathbb{C}} \to \mathbb{C}^n, \qquad \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} \otimes \lambda \mapsto \begin{pmatrix} \lambda a^1 \\ \vdots \\ \lambda a^n \end{pmatrix}$$

Using this isomorphism, we identify $(\mathbb{R}^n)^{\mathbb{C}}$ and $\mathbb{C}^n$.

Lemma 3.1.4. *Suppose that V, U are real vector spaces and $\varphi : V \to U$ is a linear map. Then*

1. *The map*
$$\varphi^{\mathbb{C}} : V^{\mathbb{C}} \to U^{\mathbb{C}}, \quad v \otimes \lambda \mapsto \varphi(v) \otimes \lambda$$
between complex vector spaces is linear.

2. *If V and U are finite-dimensional, (e_i) is a basis of V, and (f_i) is a basis of U, then the matrix of $\varphi^{\mathbb{C}}$ in the bases $(e_i \otimes 1)$ and $(f_i \otimes 1)$ is equal to the matrix of φ in the bases (e_i) and (f_i).*

Remark 3.1.5. For a complex vector space U, there is no natural complex conjugation map $\bar{\cdot} : U \to U$ with natural properties

1. $\overline{u_1 + u_2} = \overline{u_1} + \overline{u_2}$,

2. $\overline{\lambda u} = \overline{\lambda}\overline{u}$.

But if U is the complexification of a real vector space V, that is, $U = V^{\mathbb{C}}$, then we have the natural complex conjugation map

$$\bar{\cdot} : V^{\mathbb{C}} \to V^{\mathbb{C}}, \quad v \otimes \lambda \mapsto v \otimes \overline{\lambda}.$$

3.2 Jordan Normal Form

Suppose that $\varphi : V \to V$ is an operator. A subspace $U \subset V$ is called *φ-invariant* or, simply, *invariant* whenever $\varphi(U) \subset U$. For example, $\{0\}$, V, $\mathrm{Im}(\varphi)$ and $\mathrm{Ker}(\varphi)$ are invariant subspaces.

If $\varphi : V \to V$ is an operator and $U \subset V$ is its invariant subspace, then we have the operator

$$\varphi_U : U \to U, \quad u \mapsto \varphi(u);$$

it is called the *restriction of φ to U*.

Lemma 3.2.1. *Suppose that $\varphi : V \to V$ is an operator and U, W are its invariant subspaces such that $V = U \oplus W$. Consider basis $(e_1, \ldots, e_l)$ of U and basis $(f_1, \ldots, f_m)$ of W. Then the matrix of φ in the basis $(e_1, \ldots, e_l, f_1, \ldots, f_m)$ of V is the matrix*

$$\begin{pmatrix} \Phi_U & 0 \\ 0 & \Phi_W \end{pmatrix},$$

where Φ_U is the matrix of φ_U in the basis $(e_1, \ldots, e_l)$ and Φ_W is the matrix of φ_W in the basis $(f_1, \ldots, f_m)$.

Definition 3.2.2.

- The *characteristic polynomial* of a matrix $\Phi \in \mathrm{Mat}_{n\times n}(\mathbf{k})$ is the polynomial
$$p_\Phi(t) := \det(t\mathrm{I} - \Phi).$$
Clearly, the characteristic polynomial of an operator $\varphi : V \to V$ is a monic polynomial of dergee $\dim(V)$.

- The *characteristic polynomial* of an operator $\varphi : V \to V$ is the characteristic polynomial of its matrix in some basis (see problem 3.8.6).

- The roots of the characteristic polynomial of an operator φ are called *eigenvalues* of φ. The multiset[1] of eigenvalues of φ is called the *spectrum of* φ and denoted by $\mathrm{Spec}(\varphi)$.

- Suppose that λ is an eigenvalue of an operator $\varphi : V \to V$. Then

  1. the subspace
  $$V_\lambda := \mathrm{Ker}(\varphi - \lambda \mathrm{I}_V) = \{v \in V \mid \varphi(v) = \lambda v\}$$
  is called the *eigenspace* of φ corresponding to the eigenvalue λ;
  2. a nonzero vector $v \in V_\lambda$ is called an *eigenvector* of φ corresponding to the eigenvalue λ.

Exercise 3.2.3. Suppose that $\varphi : V \to V$ is an operator and $\lambda \in \mathrm{Spec}(\varphi)$. Prove that the eigenspace V_λ is a nonempty invariant subspace of V.

Exercise 3.2.4. Suppose that $\Phi \in \mathrm{Mat}_{n\times m}(\mathbf{k})$ and $\Psi \in \mathrm{Mat}_{m\times n}(\mathbf{k})$. Prove that $t^m p_{\Phi\Psi}(t) = t^n p_{\Psi\Phi}(t)$.

Example 3.2.5. Consider an operator

$$\Phi = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

[1] By definition, a *multiset* is a set with multiplicities of its elements.

Then

$$p_\Phi(t) = \det\begin{pmatrix} t-1 & -2 \\ -4 & t+1 \end{pmatrix} = t^2 - 9, \quad \mathrm{Spec}(\Phi) = \{3, -3\}.$$

The vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are eigenvectors of φ corresponding to the eigenvalues 3 and -3 respectively and thus

$$(\mathbb{R}^2)_3 = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle, \quad (\mathbb{R}^2)_{-3} = \langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rangle.$$

The matrix of Φ in the basis $(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix})$ is $\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$

Jordan Normal Form

Definition 3.2.6. 1. An $n \times n$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & 0 & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & 0 & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix}$$

is called a *Jordan block*.

2. A block matrix of the form

$$\begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{pmatrix},$$

where $J_1, \dots, J_k$ are Jordan blocks, is called a *Jordan* matrix.

3. A *Jordan basis of an operator* $\varphi : V \to V$ is a basis of V such that the matrix of φ in this basis is a Jordan matrix.

4. The *Jordan form of an operator* is the Jordan matrix of the operator in a Jordan basis.

Exercise 3.2.7. Prove that for the operator

$$\mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix},$$

where $\alpha \neq \pi k$ with $k \in \mathbb{Z}$, there is no a Jordan basis.

Exercise 3.2.8. Suppose that V is a vector space and $\varphi : V \to V$ is an operator such that there exists a Jordan basis of φ. Prove that

1. A Jordan basis of φ is not uniquely determined.
2. The Jordan form of φ is uniquely (up to permutation of Jordan blocks) determined.

Exercise 3.2.9. Find Jordan form of the operator

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : \mathbf{k}^7 \to \mathbf{k}^7.$$

Theorem 3.2.10 (Jordan Normal Form Theorem)**.** *Suppose that $\varphi : V \to V$ is an operator. Then*

$$\left\{ \begin{matrix} \textit{there exists} \\ \textit{a Jordan basis of } \varphi \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} \textit{the characteristic polynomial of } \varphi \\ \textit{is completely decomposable} \end{matrix} \right\}.$$

Corollary 3.2.11. *Suppose that V is a complex vector space; then for every operator $\varphi : V \to V$ there exists a Jordan basis.*

Also, there is the Normal Form Theorem for operators in real vector spaces. Here is its special case:

Theorem 3.2.12. *Suppose that V is a real vector space and $\varphi : V \to V$ is an operator such that the Jordan form of the complexification $\varphi^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ is a diagonal matrix. Then there exist a basis of V such that the matrix of φ in this basis is a matrix of the form*

$$\begin{pmatrix} R_1 & & & & & \\ & \ddots & & & 0 & \\ & & R_n & & & \\ & & & \lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix},$$

where

$$R_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}), \quad \lambda_j \in \mathbb{R},$$

3.3 Operators in Real Inner Product Spaces

In this section we consider real vector spaces.

Let V be a real vector space. An *inner product* on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that

1. $\langle v, v \rangle \in \mathbb{R}_{\geqslant 0}$ and $\langle v, v \rangle = 0$ whenever $v = 0$ (positive-definetness);
2. $\langle \lambda v + \lambda' v', u \rangle = \lambda \langle v, u \rangle + \lambda' \langle v', u \rangle$ (linearity by the first argument);
3. $\langle v, \lambda u + \lambda' u' \rangle = \lambda \langle v, u \rangle + \lambda' \langle v, u' \rangle$ (linearity by the second argument);
4. $\langle v, u \rangle = \langle u, v \rangle$ (symmetry).

A real vector space with a fixed inner product is called a *real inner product space.*

Remark 3.3.1. In the definition above, properties 2–4 mean that $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form on $V \times V$, and property 1 means that the corresponding quadratic form is positive-definite.

A basis (e_i) of a real inner product space V is called *orthonormal* whenever

$$\langle e_i, e_j \rangle = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Example 3.3.2. The *Euclidean space* is defined to be the coordinate vector space $\mathbb{R}^n$ with the *Euclidean inner product* (also called the *dot product*)

$$\langle x, y \rangle = x^\top y = x_1 y_1 + \ldots + x_n y_n, \quad \text{where} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

The standard basis of the Euclidean space $\mathbb{R}^n$ is orthonormal.

Example 3.3.3 (A generalization of the previous example)**.** Reacall that an $n \times n$ symmetric matrix Q is called *positive definite* iff determinants of their upper left $k \times k$ submatrices are positive. Now, a positive define real symmetric $n \times n$ matrix Q defines the inner product

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x, y) \mapsto x^\top Q y.$$

Example 3.3.4 (Frobenius inner product)**.** For $V = \mathrm{Mat}_{n \times m}(\mathbb{R})$ we have the Frobenius inner product $\langle \cdot, \cdot \rangle_F$, where

$$\langle \Phi, \Psi \rangle_F = \mathrm{tr}(\Phi \Psi^\top).$$

Lemma 3.3.5. *Suppose that V is a real inner product space and U is a subspace of V. Then the map*

$$U \times U \to \mathbb{R}, \quad (v,u) \mapsto \langle v,u \rangle$$

is an inner product on U. It is called the restriction *of the inner product $\langle \cdot,\cdot \rangle$ to the subspace U.*

Lemma 3.3.6. *Suppose that V is a real inner product space and U is a subspace of V. Then*

1. *$U^\perp := \{v \in V \mid \langle v,u \rangle = 0$ for all $u \in U\}$ is a subspace of V;*
2. *$V = U \oplus U^\perp$*

Lemma 3.3.7. *Suppose that V is a real inner product space. Then the maps*

$$V \to V^*, \quad v \mapsto \langle v,\cdot \rangle, \text{ where } \langle v,\cdot \rangle(u) = \langle v,u \rangle, \tag{3.3.1}$$

$$\begin{aligned} &\mathrm{Hom}(V,V) \to \{\text{Bilinear forms on } V \times V\},\\ &\qquad \varphi \mapsto \beta_\varphi, \text{ where } \beta_\varphi(v,u) = \langle \varphi(v),u \rangle \end{aligned} \tag{3.3.2}$$

are isomorphisms between vector spaces.

Using isomorphisms (3.3.1) **and** (3.3.2)**, we identify a real inner product space V and its dual V^*, and operators on a real inner product space V and bilinear forms on $V \times V$.**

Exercise 3.3.8. Suppose that V is a real inner product space and $(e_1,\dots,e_n)$ is an orthonormal basis. Check that

1. The dual to $(e_1,\dots,e_n)$ basis of V^* coincides with the corresponding under the identification (3.3.1) basis of V^*, that is, coincides with $(\langle e_1,\cdot \rangle,\dots,\langle e_1,\cdot \rangle)$.
2. If φ is an operator in V and β_φ is the corresponding under the identification (3.3.2) bilinear form, then the matrix Φ of φ in $(e_1,\dots,e_n)$ coincides with the transpose of the matrix B_Φ of β_φ in $(e_1,\dots,e_n)$:

$$\Phi = \mathrm{B}_\Phi^\top.$$

Let V,U be a real inner product spaces and $\varphi : V \to U$ be a linear map. A linear map $\varphi^* : U \to V$ is called the *adjoint* of φ whenever

$$\langle \varphi(v),u \rangle_U = \langle v,\varphi^*(u) \rangle_V \quad \text{for all } v \in V, u \in U.$$

Lemma 3.3.9. *Suppose that V,U are real inner product spaces and $\varphi : V \to U$ is a linear map. Then*

1. *there exist a unique adjoint of* φ.

2. *the matrix* Φ^* *of* φ^* *in an orthonormal basis of* V *and in an orthonormal basis of* U *is the transpose of the matrix* Φ *of* φ *in that bases:*

$$\Phi^* = \Phi^\top.$$

Exercise 3.3.10. Consider $\mathbb{R}^n$ with the inner product defined by a positive define symmetric matrix Q and $\mathbb{R}^m$ with the inner product defined by a positive define symmetric matrix S (Example 3.3.3). Prove that the adjoint of an operator $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is th operator $Q^{-1}\Phi^\top S : \mathbb{R}^m \to \mathbb{R}^n$.

Remark 3.3.11. Let V, U be a real inner product spaces and $\varphi : V \to U$ be a linear map. The following commutative diagram explains the connection between the adjoint and the transpose of φ:

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[d, "\varphi^*"'] & U^* \arrow[d, "\varphi^t"] \\ V \arrow[r, "\cong"] & V^* \end{tikzcd}$$

So that, under the identifications $V \cong V^*$ and $U \cong U^*$,

$$\boxed{\textbf{adjoint of } \varphi = \textbf{transpose of } \varphi}$$

Exercise 3.3.12. Suppose that V, U are real inner product spaces. Prove that

1. $0^* = 0$;

2. $(\mathrm{I}_V)^* = \mathrm{I}_V$;

3. $(\lambda\varphi)^* = \lambda\varphi^*$ for every $\lambda \in \mathbb{R}$ and every operator $\varphi : V \to U$;

4. $(\varphi + \psi)^* = \varphi^* + \psi^*$ for operators $\varphi, \psi : V \to U$.

Definition 3.3.13. Let V be a real inner product space. An operator $\varphi : V \to V$ is called

- *normal* whenever $\varphi\varphi^* = \varphi^*\varphi$.

- *orthogonal* whenever $\varphi\varphi^* = \varphi^*\varphi = \mathrm{I}_V$ or, equivalently,

$$\langle\varphi(v), \varphi(u)\rangle = \langle v, u\rangle \quad \text{for all } v, u \in V.$$

- *symmetric* (also called *sel-fadjoint*) whenever $\varphi^* = \varphi$.

- *skew-symmetric* whenever $\varphi^* = -\varphi$.

Exercise 3.3.14. Prove that orthogonal, symmetric, and skew-symmetric operators are normal.

Exercise 3.3.15. Suppose that V is a real inner product space, $\varphi : V \to V$ is an operator, Φ is the matrix of φ in an orthonormal basis. Check that

- φ is normal whenever $\Phi\Phi^\top = \Phi^\top\Phi$;
- φ is orthogonal whenever $\Phi\Phi^\top = \Phi^\top\Phi = \mathrm{I}$;
- φ is symmetric whenever $\Phi^\top = \Phi$;
- φ is skew-symmetric whenever $\Phi^\top = -\Phi$.

Exercise 3.3.16. Suppose that V is a real inner product space, $\varphi : V \to V$ is an operator, and β_φ is the corresponding bilinear form. Check that

1. φ^* is symmetric whenever β_φ is symmetric;
2. φ^* is skew-symmetric whenever β_φ is skew-symmetric.

Theorem 3.3.17. *Suppose that V is a real inner product space and $\varphi : V \to V$ is a normal operator. Then there exists an orthonormal basis of V such that the matrix of φ in that basis is a matrix of the form*

$$\begin{pmatrix} R_1 & & & & & \\ & \ddots & & & 0 & \\ & & R_n & & & \\ & & & \lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix},$$

where $\lambda_j \in \mathbb{R}$,

$$R_i = r_i \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}, \qquad r_i > 0 \text{ and } \alpha_i \neq \pi k \text{ with } k \in \mathbb{Z}$$

Proof. We prove by induction.

Case $\dim(V) = 1$ is evident, case $\dim(V) = 2$ is in Problem 3.8.28.

Suppose that $\dim(V) \geqslant 3$. Consider the complexification $\varphi^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ and its eigenvalue λ. By statement of Problem 3.8.5, we have 2-dimensional φ-invariant subspace $U \subset V$. By statement of Problem 3.8.26, the subspace $U^\perp$ is φ-invariant. Note that $V = U \oplus U^\perp$. Now we use Lemma 3.2.1, induction, and statement of Problem 3.8.27. □

Corollary 3.3.18. *Suppose that V is a real inner product space and $\varphi : V \to V$ is an operator. Then*

1. *If φ is orthogonal, then there is an orthonormal basis of V such that the matrix of φ in that basis is a matrix of the form*

$$\begin{pmatrix} R_1 & & & & & \\ & \ddots & & & 0 & \\ & & R_n & & & \\ & & & \lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_m \end{pmatrix},$$

   where $\lambda_j = \pm 1$,

$$R_i = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}.$$

2. *If φ is symmetric, then there is an orthonormal basis of V such that the matrix of φ in that basis is a diagonal matrix.*

3. *If φ is skew-symmetric, then there is an orthonormal basis of V such that the matrix of φ in that basis is a matrix of the form*

$$\begin{pmatrix} \Omega_1 & & & & & \\ & \ddots & & & 0 & \\ & & \Omega_n & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

   where

$$\Omega_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}, \quad \omega_i > 0.$$

Corollary 3.3.19. *Suppose that V is a real inner product space and $\beta : V \times V \to \mathbb{R}$ is a symmetric bilinear form. Then there exists an orthonormal basis of V such that the matrix of β in that basis is diagonal.*

3.4 Operators in Hermitian Spaces

In this section we consider complex vector spaces.

Let V be a complex vector space. A function $s : V \to \mathbb{C}$ is called *conjugate linear* (also called *antilinear*) whenever

$$s(\lambda_1 v_1 + \ldots + \lambda_k v_k) = \overline{\lambda_1} s(v_1) + \ldots + \overline{\lambda_k} s(v_k) \text{ for all } \lambda_i \in \mathbb{C}, v_i \in V.$$

Exercise 3.4.1. Prove that conjugate linear functions on $\mathbb{C}^n$ are functions of the form

$$\mathbb{C}^n \to \mathbb{C}, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \lambda_1 \overline{x_1} + \ldots + \lambda_n \overline{x_n},$$

where $\lambda_i \in \mathbb{C}$.

The set of conjugate linear functions on a complex vector space V is a complex vector space under the following compositions:

1. the sum of conjugate linear functions s_1, s_2 is the function

$$s_1 + s_2 : V \to \mathbb{C}, \quad (s_1 + s_2)(v) = s_1(v) + s_2(v);$$

2. the product of a scalar $\lambda \in \mathbb{C}$ and a conjugate linear function s is the function

$$\lambda s : V \to \mathbb{C}, \quad (\lambda s)(v) = \lambda s(v).$$

The space of conjugate linear functions on V is called *conjugate dual* of V; it is denoted by $\overline{V}^*$.

Let V, U be complex vector spaces and $\varphi : V \to U$ be a linear map. It can be easily checked that the map

$$\overline{\varphi}^t : \overline{U}^* \to \overline{V}^*, \quad s \mapsto s \circ \varphi.$$

is linear; it is called the *conjugate transpose* of φ.

Let V be a complex vector space. A map

$$\beta(\cdot,\cdot) : V \times V \to \mathbb{C}$$

is called *sesquilinear* iff

1. $\beta(\lambda v + \lambda' v', u) = \lambda\beta(v, u) + \lambda'\beta(v', u)$ (linearity by the first argument);
2. $\beta(v, \lambda u + \lambda' u') = \overline{\lambda}\beta(v, u) + \overline{\lambda'}\beta(v, u')$ (conjugate linearity by the second argument).

A sesquilinear form $\beta(\cdot,\cdot)$ on $V \times V$ is called *conjugate symmetric* iff $\beta(v, u) = \overline{\beta(u, v)}$.

Exercise 3.4.2. Prove that a sesquilinear form $\beta(\cdot,\cdot)$ on $V \times V$ is conjugate symmetric iff $\beta(v, v) \in \mathbb{R}$ for all $v \in V$.

An *Hermitian inner product* on $V \times V$ is a conjugate symmetric sesquilinear map

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$$

such that

$$\langle v, v\rangle \geqslant 0 \text{ and } \langle v, v\rangle = 0 \text{ whenever } v = 0 \text{ (positive-definetness).}$$

A complex vector space with a fixed Hermitian inner product is called an *Hermitian space* (also called a *unitary space*). A basis (e_i) of an Hermitian space is called *orthonormal* whenever

$$\langle e_i, e_j\rangle = \delta_{ij}$$

Example 3.4.3. The n-dimensional coordinate vector space $\mathbb{C}^n$ is an Hermitian inner product space with respect to the *standard Hermitian inner product*

$$\langle x, y\rangle = x^\top \overline{y} = x_1\overline{y_1} + \ldots + x_n\overline{y_n}, \quad \text{where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

The standard basis of $\mathbb{C}^n$ is orthonormal. It is easy to note that the restriction of the standard Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ on $\mathbb{R}^n \times \mathbb{R}^n$ is the standard inner product on $\mathbb{R}^n$.

Example 3.4.4 (A generalization of the previous example)**.** Recall that

- the *conjugate transpose* of a complex matrix H is defined to be $\overline{H}^\top$;
- a complex matrix H is called *Hermitian* if $\overline{H}^\top = H$;
- an Hermitian $n\times n$ matrix H is called *positive definite* iff determinants of their upper left $k \times k$ submatrices are positive.

Now, a complex positive define Hermitian $n \times n$ matrix H defines the Hermitian inner product

$$\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x, y) \mapsto x^\top H\overline{y}.$$

Example 3.4.5 (Frobenius inner product)**.** For $V = \mathrm{Mat}_{n\times m}(\mathbb{C})$ we have the Frobenius inner product $\langle\cdot,\cdot\rangle_F$, where

$$\langle\Phi, \Psi\rangle_F = \mathrm{tr}(\Phi\overline{\Psi}^\top).$$

Lemma 3.4.6. *Suppose that V is an Hermitian space and U is a subspace of V. Then the map*

$$U \times U \to \mathbb{C}, \quad (v, u) \mapsto \langle v, u\rangle$$

is an Hermitian inner product on $U\times U$. It is called the restriction *of the Hermitian inner product $\langle\cdot,\cdot\rangle$ to the subspace U.*

Lemma 3.4.7. *Suppose that V is an Hermitian space and U is a subspace of V. Then*

1. *The subset*
$$U^\perp := \{v \in V \mid \langle v, u\rangle = 0 \text{ for all } u \in U\}$$
is a subspace of V. It is called the orthogonal *supplement of U.*
2. $V = U \oplus U^\perp$.

Lemma 3.4.8. *Suppose that V is an Hermitian space. Then the maps*

$$V \to \overline{V}^*, \quad v \mapsto \langle v, \cdot \rangle, \text{ where } \langle v, \cdot \rangle(u) = \langle v, u \rangle, \tag{3.4.1}$$

$$\begin{gathered} \operatorname{Hom}(V, V) \to \{\text{Sesquilinear forms on } V \times V\}, \\ \varphi \mapsto \beta_\varphi, \text{ where } \beta_\varphi(v, u) = \langle \varphi(v), u \rangle \end{gathered} \tag{3.4.2}$$

are isomorphisms between vector spaces.

Using isomorphisms (3.4.1) **and** (3.3.2)**, we identify an Hermitian space V and its conjugate dual $\overline{V}^*$, and operators on an Hermitian space V and sesquilinear forms on $V \times V$.**

Let V, U be Hermitian spaces and $\varphi : V \to U$ be an operator. An operator $\varphi^* : U \to V$ is called the *Hermitian adjoint* of φ whenever

$$\langle \varphi(v), u \rangle_U = \langle v, \varphi^*(u) \rangle_V \quad \text{for all } v \in V, u \in U.$$

Lemma 3.4.9. *Suppose that V, U are Hermitian spaces and $\varphi : V \to U$ is an operator. Then*

1. *there exist a unique Hermitian adjoint of φ;*
2. *the matrix Φ^* of φ^* in an orthonormal basis of V and in an orthonormal basis of U is the cojugate transpose of the matrix Φ of φ in that bases:*

$$\Phi^* = \overline{\Phi}^\top$$

Exercise 3.4.10. Consider $\mathbb{C}^n$ with the Hermitian inner product defined by an Hermitian matrix H and $\mathbb{C}^m$ with the Hermitian inner product defined by an Hermitian matrix S (Example 3.4.4). Prove that the adjoint of an operator $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ is the operator $\overline{H}^{-1} \overline{\Phi}^\top \overline{S} : \mathbb{C}^m \to \mathbb{C}^n$.

Remark 3.4.11. Let V, U be Hermitian spaces and $\varphi : V \to U$ be an operator. The following commutative diagram explains the connection between Hermitian adjoint and conjugate transpose of φ:

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[d, "\varphi^*"'] & \overline{U}^* \arrow[d, "\overline{\varphi}^t"] \\ V \arrow[r, "\cong"] & \overline{V}^* \end{tikzcd}$$

So that, under the identification $V \cong \overline{V}^*$,

$$\boxed{\textbf{Hermitian adjoint of } \varphi = \textbf{conjugate transpose of } \varphi}$$

Exercise 3.4.12. Suppose that V, U are Hermitian spaces. Prove that

1. $0^* = 0$;
2. $(\mathrm{I}_V)^* = \mathrm{I}_V$;
3. $(\lambda\varphi)^* = \overline{\lambda}\varphi^*$ for every $\lambda \in \mathbb{C}$ and every linear map $\varphi : V \to U$;
4. $(\varphi + \psi)^* = \varphi^* + \psi^*$ for every linear maps $\varphi, \psi : V \to U$.

Definition 3.4.13. Let V be an Hermitian space and $\varphi : V \to V$ be an operator.

1. φ is called *normal* whenever $\varphi\varphi^* = \varphi^*\varphi$.
2. φ is called *unitary* whenever $\varphi\varphi^* = \varphi^*\varphi = \mathrm{I}_V$ or, equivalently,
$$\langle \varphi(v), \varphi(u) \rangle = \langle v, u \rangle \quad \text{for all } v, u \in V.$$
3. φ is called *Hermitian* or *self-adjoint* whenever $\varphi^* = \varphi$.
4. φ is called *skew-Hermitian* whenever $\varphi^* = -\varphi$.

Exercise 3.4.14. Prove that unitary, Hermitian, and skew-Hermitian operators are normal.

Exercise 3.4.15. Suppose that V is an Hermitian space, $\varphi : V \to V$ is an operator, Φ is the matrix of φ in an orthonormal basis. Check that

- φ is normal whenever $\Phi\overline{\Phi}^\top = \overline{\Phi}^\top\Phi$;
- φ is unitary whenever $\Phi\overline{\Phi}^\top = \overline{\Phi}^\top\Phi = \mathrm{I}$;
- φ is Hermitian whenever $\overline{\Phi}^\top = \Phi$;
- φ is skew-Hermitian whenever $\overline{\Phi}^\top = -\Phi$.

Exercise 3.4.16. Suppose that V is an Hermitian space, $\varphi : V \to V$ is an operator, and β_φ is the corresponding sesquilinear form. Check that

1. φ^* is Hermitian whenever β_φ is conjugate symmetric;
2. φ^* is skew-Hermitian whenever $\beta_\varphi(v, u) = -\overline{\beta_\varphi(u, v)}$ for all $v, u \in V$, that is, β_φ is conjugate skew-symmetric.

Theorem 3.4.17. *Suppose that V is an Hermitian space and $\varphi : V \to V$ is a normal operator. Then there exists an orthonormal basis of V such that the matrix of φ in that basis is a diagonal matrix.*

Proof. We prove by induction.

Case $\dim(V) = 1$ is evident.

Suppose that $\dim(V) \geqslant 2$. Consider an eigenvalue λ of φ, a corresponding eigenvector v_λ of φ, and the φ-invariant subspave $U = \mathrm{Span}(v_\lambda)$. By statement of Problem 3.8.26, the subspace $U^\perp$ is φ-invariant. Note that $V = U \oplus U^\perp$. Now we use Lemma 3.2.1, induction, and statement of Problem 3.8.27. □

Corollary 3.4.18. *Suppose that V is an Hermitian space and $\varphi : V \to V$ is an operator. Then*

1. *If φ is unitary, then there is an orthonormal basis of V such that the matrix of φ in that basis is a diagonal matrix with unit complex numbers on diagonal.*

2. *If φ is Hermitian, then there is an orthonormal basis of V such that the matrix of φ in that basis is a diagonal matrix with real numbers on diagonal.*

3. *If φ is skew-Hermitian, then there is an orthonormal basis of V such that the matrix of φ in that basis is a diagonal matrix with purely imaginary numbers on diagonal.*

3.5 Operators Decompositions

In this section we consider LU, QR, and SVD decompositions. From the SVD we deduce properties of Moore-Penrose inverse. To make the layout more transparent, we consider only simplest version of the theorems for real vector spaces.

⧫ **LU decompisition**.

LU decompisition is a corollary of Gaussian elimination algorithm.

Theorem 3.5.1. *Suppose that*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{R}),$$

where

$$\det(a_{1,1}) \neq 0, \ \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \neq 0, \ \dots \ , \det\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \neq 0.$$

Then we have the decomposition $A = LU$ with uniquely determined lower triangular matrix L with identity elements on the diagonal and invertible upper triangular matrix U.

Example 3.5.2. Here are matrices and their LU decompositions:

$$\begin{pmatrix}2\end{pmatrix} = \begin{pmatrix}1\end{pmatrix}\begin{pmatrix}2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 2 & 1\\ -1 & 1 & 0\\ 0 & 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 2 & 1\\ 0 & -1 & -1\\ 0 & 0 & -2\end{pmatrix}$$

⧫ **QR decompisition**.

Theorem 3.5.3 (QR decomposition)**.** *Suppose that A is an $n \times n$ invertible real matrix. Then there exist a unique $n \times n$ orthogonal matrix Q and a unique upper triangular matrix R with positive diagonal elements such that $A = QR$.*

QR decompisition is a corollary of Gram-Schmidt orthonormalization algorithm.

Example 3.5.4. Here are matrices and their QR decompositions:

$$\begin{pmatrix}2\end{pmatrix} = \begin{pmatrix}1\end{pmatrix}\begin{pmatrix}2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} & \frac{3\sqrt{2}}{2}\\ 0 & \frac{\sqrt{2}}{2}\end{pmatrix}$$

$$\begin{pmatrix}-1 & 2 & 1\\ -1 & 1 & 0\\ 0 & 1 & -1\end{pmatrix} = \begin{pmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3}\\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3}\\ 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3}\end{pmatrix}\begin{pmatrix}\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{6}\\ 0 & 0 & \frac{2\sqrt{2}}{3}\end{pmatrix}$$

⧫ SVD **- singular value decompisition**.
We start from the geometric version of SVD.

Theorem 3.5.5. *Suppose that V and U are real inner product spaces and $\varphi : V \to U$ is a linear map. Then there are orthonormal bases $(e_1, e_2, \ldots)$ of V and orthonormal bases $(f_1, f_2, \ldots)$ of U such that*

$$\varphi(e_i) = \begin{cases}\lambda_i f_i & \text{if } 1 \leqslant i \leqslant m,\\ 0 & \text{if } i \geqslant m+1,\end{cases}$$

where $m \leqslant \min\{\dim V, \dim U\}$, $\lambda_1 \geqslant \ldots \geqslant \lambda_m \geqslant 0$.

Elements λ_i in this theorem are called *singular values*.

Proof. Firstly, it is enough to find an orthonormal basis $(e_1, e_2, \ldots)$ of V such that

$$\varphi(e_i) = \begin{cases} h_i \neq 0, & \text{for } 1 \leqslant i \leqslant m, \\ 0, & \text{for } m+1 \leqslant i, \end{cases}$$

where $0 \leqslant m \leqslant n$, $(h_1, \ldots, h_m)$ is an orthogonal list of vectors.

Secondly, consider the preimage under φ of the inner product on U, that is, consider the symmetric bilinear form

$$\beta : V \times V \to \mathbb{R}, \quad \beta(v_1, v_2) = \langle \varphi(v_1), \varphi(v_2) \rangle$$

And, by Corollary 3.3.19, we take an orthonormal basis $(e_1, e_2, \ldots)$ of V such that

$$\beta(e_i, e_j) = 0 \text{ for all } i \neq j.$$

$\square$

The standard matrix version of SVD is the following:

Corollary 3.5.6. *Suppose that Φ is $n \times k$ real matrix. Then there exist an orthogonal $n \times n$ matrix P, an orthogonal $k \times k$ matrix R, and $n \times k$ matrix*

$$\Sigma = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & 0 \\ & & \lambda_m & \\ & 0 & & 0 \end{pmatrix},$$

where $m \leqslant \min\{n, k\}$, $\lambda_1 \geqslant \ldots \geqslant \lambda_m \geqslant 0$, such that $\Phi = P\Sigma R$.

Example 3.5.7. Consider $\mathbb{R}^3$ and $\mathbb{R}^2$ with the standard bases (e_1, e_2, e_3) and $f_1, f_2)$ and the standard inner products. Consider the linear map

$$\Phi = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^2.$$

The preimage of the inner product on $\mathbb{R}^2$ is the symmetric bilinear form on $\mathbb{R}^3$ with the matrix

$$\Phi^\top \Phi := \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 3 \\ 4 & 3 & 5 \end{pmatrix}$$

The eigenvalues of $\Phi^\top \Phi$ are $10, 5, 0$ with the corresponding eigenvectors

$$\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -3 \\ 5 \end{pmatrix}.$$

We have the corresponding orthonormal basis

$$(\tilde{e}_1 = \begin{pmatrix} \frac{2\sqrt{2}}{5} \\ \frac{3\sqrt{2}}{10} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \tilde{e}_2 = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}, \tilde{e}_3 = \begin{pmatrix} -\frac{4\sqrt{2}}{10} \\ -\frac{3\sqrt{2}}{10} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

of $\mathbb{R}^3$. Now we calculate:

$$\Phi\tilde{e}_1 = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix} = \lambda_1 \tilde{f}_1, \text{ where } \lambda_1 = \sqrt{10}, \ \tilde{f}_1 = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{pmatrix},$$

$$\Phi\tilde{e}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \lambda_2 \tilde{f}_2 \text{ where } \lambda_2 = \sqrt{5}, \ \tilde{f}_2 = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{pmatrix},$$

$$\Phi\tilde{e}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix of Φ in the bases $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ and $(\tilde{f}_1, \tilde{f}_2)$ is

$$\tilde{\Phi} = \begin{pmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{pmatrix}$$

The matrix SVD deconposition is

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{10} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{2\sqrt{2}}{5} & \frac{3\sqrt{2}}{10} & \frac{\sqrt{2}}{2} \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4\sqrt{2}}{10} & -\frac{3\sqrt{2}}{10} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

⧫ **Moore-Penrose inverse**.

Let V, U be real inner product spaces and $\varphi : V \to U$ be a linear map. A linear map $\psi : V \to U$ is called *Moore–Penrose inverse* of φ iff

1. $\varphi\psi\varphi = \varphi$,
2. $\psi\varphi\psi = \psi$,
3. $(\varphi\psi)^* = \varphi\psi$,
4. $(\psi\varphi)^* = \psi\varphi$.

Theorem 3.5.8. *Suppose that V, U are real inner product spaces and $\varphi : V \to U$ is a linear map. Then there is a unique Moore–Penrose inverse of φ.*

This theorem is a corollary of SVD.

Example 3.5.9. Consider $\mathbb{R}^4$ and $\mathbb{R}^3$ with the standard bases $(e_1, e_2, e_3, e_4,)$ and (f_1, f_2, f_3) and the standard inner products. Consider the linear map

$$\Phi = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^3.$$

The Moore-Penrose inverse of Φ is

$$\Psi = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^4.$$

3.6 *Normed Spaces

Let V be a finite-dimensional vector space defined over $\mathbf{k}$, where $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$.

Recall that a *norm* on V is a map

$$\|\cdot\| \ : \ V \to \mathbb{R}, \quad v \mapsto \|v\|$$

such that for all $v, u \in V$, $\lambda \in \mathbf{k}$, we have

1. $\|v\| \geqslant 0$ and $\|v\| = 0 \Leftrightarrow v = 0$;
2. $\|\lambda v\| = |\lambda| \|v\|$;
3. $\|v + u\| \leqslant \|v\| + \|u\|$.

Examples 3.6.1. 1. Manhattan norm $\|\cdot\|_1 : \mathbf{k}^n \to \mathbb{R}$, where

$$\|x\|_1 = \sum_{1 \leqslant i \leqslant n} |x_i|.$$

2. Euclidean norm $\|\cdot\|_2 : \mathbf{k}^n \to \mathbb{R}$, where

$$\|x\|_2 = \left(\sum_{1 \leqslant i \leqslant n} |x_i|^2 \right)^{1/2}.$$

3. Maximum norm $\|\cdot\|_\infty : \mathbf{k}^n \to \mathbb{R}$, where

$$\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|.$$

4. (a generalization of the previous examples) $\|\cdot\|_p : \mathbf{k}^n \to \mathbb{R}$, where $p \geqslant 1$,

$$\|x\|_p = \left(\sum_{1 \leqslant i \leqslant n} |x_i|^p \right)^{1/p}.$$

Examples 3.6.2. Consider a nonempty interval $[a,b] \subset \mathbb{R}$. We have the following norms

1. $\|\cdot\|_1 : \mathbb{R}[t]_{\leqslant n} \to \mathbb{R}$, where
$$\|f\|_1 = \int_a^b |f(t)|dt.$$

2. $\|\cdot\|_2 : \mathbb{R}[t]_{\leqslant n} \to \mathbb{R}$, where
$$\|f\|_2 = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}.$$

3. Uniform norm $\|\cdot\|_\infty : \mathbb{R}[t]_{\leqslant n} \to \mathbb{R}$, where
$$\|f\|_\infty = \max_{t\in[a,b]} |f(t)|.$$

4. (a generalization of the previous examples) $\|\cdot\|_p : \mathbb{R}[t]_{\leqslant n} \to \mathbb{R}$, where $p \geqslant 1$,
$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p}.$$

Examples 3.6.3. 1. $\|\cdot\| : \mathrm{Hom}(U,W) \to R$, where U, W are normed spaces,
$$\|\varphi\| = \sup_{0\neq u\in U} \frac{\|\varphi(u)\|_W}{\|u\|_U}.$$

2. (Frobenius norm) $\|\cdot\|_F : \mathrm{Mat}_{n\times m}(\mathbf{k}) \to \mathbb{R}$, where
$$\|\Phi\|_F = \begin{cases} \mathrm{tr}(\Phi\Phi^\top), & \text{if } \mathbf{k} = \mathbb{R}, \\ \mathrm{tr}(\Phi\overline{\Phi}^\top), & \text{if } \mathbf{k} = \mathbb{C}. \end{cases}$$

Note that $\|(\Phi_{i,j})\|_F = \sum_{i,j} |\Phi_{i,j}|^2$.

Exercise 3.6.4. Find maximum of the function
$$f : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{R}, \quad f(\Phi) = |\det(\Phi)|$$
on the ball $B = \{\Phi \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \|\Phi\|_F \leqslant 1\}$.

Lemma 3.6.5. *Suppose that $\langle\cdot,\cdot\rangle$ is an inner product on V, then*
$$\|\cdot\| \ : \ V \to \mathbb{R}, \quad \|v\| := \sqrt{\langle v, v\rangle}$$
is a norm.

What kind of norms can be obtained from inner products? The answer is in the lemma below.

Lemma 3.6.6. *Suppose that $\|\cdot\|$ is norm on V. Then the following conditions are equivalent.*

1. *$\|\cdot\|$ is defined by some inner product on V.*
2. *$\|\cdot\|$ satisfies the* parallerogram identity*:*

$$\|v+u\|^2 + \|v-u\|^2 = 2\|v\|^2 + 2\|u\|^2 \quad \textit{for all } v, u \in V.$$

Theorem 3.6.7. *Every two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a finite-dimensional vector space V are equivalent; that is, there are real numbers $c_1, c_2 > 0$ such that such that*

$$\|v\|_2 \leqslant c_1\|v\|_1 \quad \textit{and} \quad \|v\|_1 \leqslant c_2\|v\|_2 \quad \textit{for all } v \in V.$$

3.7 *Functions of Operators

In this section we discribe some type of functions of operators.

Invariants of Operators.

A function

$$f : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}$$

is called an *invariant* iff $f(A^{-1}\Phi A) = f(\Phi)$ for all $\Phi \in \mathrm{Mat}_{n\times n}(\mathbf{k})$, $A \in \mathrm{GL}_n(\mathbf{k})$.

Example 3.7.1. The functions

$$\mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}, \quad \Phi \mapsto \mathrm{tr}(\Phi^m), \quad m = 1, 2, \dots$$

are polynomial invariants.

Example 3.7.2. Consider the decomposition

$$p_\Phi(t) = \det(t\mathrm{I} - \Phi) = t^n - p_1(\Phi)t^{n-1} + p_2(\Phi)t^{n-2} - \ldots + (-1)^n p_n(\Phi),$$

where

$$p_m : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}, \quad m = 1, \dots, n$$

are polynomial functions. Then the functions $p_1, \dots, p_n$ are polynomial invariants.

Theorem 3.7.3. *Suppose that $f : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}$ is a polynomial invariant. Then there is a polynomial $s(t_1, \dots, t_n)$ in $t_1, \dots, t_n$ such that*

$$f(\Phi) = s(p_1(\Phi), \dots, p_n(\Phi)).$$

Remark 3.7.4. The set of polynomial invariant functions of the form $f : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}$ is a ring with natural additions and multiplication. Theorem 3.7.3 claims that $p_1, \dots, p_n$ are generators of this ring.

Let V be a vector space over $\mathbf{k}$. An invariant function $f : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}$, where $n = \dim(V)$, well-define the function (we denote it by the same symbol)

$$\begin{aligned} &f : \mathrm{Hom}(V,V) \to \mathbf{k}, \\ &\qquad \varphi \mapsto f(\Phi), \text{ where } \Phi \text{ is the matrix of } \varphi \text{ in a basis } (e_i) \text{ of } V. \end{aligned}$$

For an operator $\varphi \in \mathrm{Hom}(V,V)$, the value $f(\varphi)$, where $f : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}$ is an invariant function, is called an *invariant* of the operator φ. Roughly speaking, Theorem 3.7.3 claims that $(p_1(\varphi), \ldots, p_n(\varphi))$ is a complete list of polynomial invariants of an operator $\varphi : V \to V$.

Covariants of Operators.

A map
$$F : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathrm{Mat}_{n\times n}(\mathbf{k})$$
is called a *covariant* iff $F(A^{-1}\Phi A) = A^{-1}F(\Phi)A$ for all $\Phi \in \mathrm{Mat}_{n\times n}(\mathbf{k})$, $A \in \mathrm{GL}_n(\mathbf{k})$.

Theorem 3.7.5. *Suppose that $F : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}$ is a polynomial covariant. Then*

$$F(\Phi) = f_0(\Phi)\mathrm{I} + f_1(\Phi)\Phi + \ldots + f_{n-1}(\Phi)\Phi^{n-1},$$

where $f_i : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathbf{k}$, $i = 0, 1, \ldots, n-1$ are polynomial invariants.

Let V be a vector space over $\mathbf{k}$. A covariant $F : \mathrm{Mat}_{n\times n}(\mathbf{k}) \to \mathrm{Mat}_{n\times n}(\mathbf{k})$, where $n = \dim(V)$, well-defines the map (we denote it by the same symbol)

$$\begin{aligned} &F : \mathrm{Hom}(V,V) \to \mathrm{Hom}(V,V), \\ &\qquad \Phi \mapsto F(\Phi), \text{ where } \Phi \text{ and } F(\Phi) \text{ are the matrices} \\ &\qquad\qquad \text{of } \varphi \text{ and } F(\Phi) \text{ resp. in a basis } (e_i) \text{ of } V. \end{aligned}$$

The map F is called *covariant*.

Analytical Functions of Operators.

Let U be an open subset of $\mathbb{C}$. Consider

– $\mathcal{O}(U)$ – the ring of analytical functions on U, and

– $\mathrm{Mat}_{n\times n}(\mathbb{C})_U$ – the set of $n \times n$ complex matrices Φ with $\mathrm{Spec}(\Phi) \subset U$.

We have the map

$$\mathcal{O}(U) \times \mathrm{Mat}_{n\times n}(\mathbb{C})_U \to \mathrm{Mat}_{n\times n}(\mathbb{C}), \quad (f(z), \Phi) \mapsto f(\Phi), \tag{3.7.1}$$

where $f(\Phi)$ (substitution of Φ into $f(z)$) is defined in the following way: find a polynomial $r(z)$ of degree $\leqslant n-1$ such that $f(z) - r(z)$ is divisible by $p_\Phi(z)$ in $\mathcal{O}(U)$ and put
$$f(\Phi) = r(\Phi).$$

Theorem 3.7.6. *The map* (3.7.1) *satisfies the following properties.*

- *If $f(z) \in \mathcal{O}(U)$, $\Phi \in \mathrm{Mat}_{n\times n}(\mathbb{C})_U$, and $A \in \mathrm{GL}_n(\mathbb{C})$, then*
$$f(A^{-1}\Phi A) = A^{-1}f(\Phi)A.$$
- *If $f(z), g(z) \in \mathcal{O}(U)$, $\lambda, \mu \in \mathbb{C}$, and $\Phi \in \mathrm{Mat}_{n\times n}(\mathbb{C})_U$, then*
$$(\lambda f + \mu g)(\Phi) = \lambda f(\Phi) + \mu g(\Phi)$$
and a similar statement holds for convergent series of functions.
- *If $f(z), g(z) \in \mathcal{O}(U)$ and $\Phi \in \mathrm{Mat}_{n\times n}(\mathbb{C})_U$, then*
$$(fg)(\Phi) = f(\Phi)g(\Phi)$$
and a similar statement holds for convergent infinite products of functions.

Example 3.7.7. Consider

$$U = \mathbb{C} \setminus 0, \quad \Phi = \begin{pmatrix} 3 & 4 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 4 \end{pmatrix}.$$

Then $\mathrm{Spec}(\Phi) = \{4, 5, -1\} \subset U$. Let us take $f(z) = z^{-1}$ and calculate $f(\Phi) = \Phi^{-1}$. To find $r(z)$, we have the conditions

$$r(4) = f(4) = \frac{1}{4}, \quad r(5) = f(5) = \frac{1}{5}, \quad r(-1) = f(-1) = -1.$$

From this we find $r(z) = \frac{1}{20}(-z^2 + 8z - 11)$ and thus

$$f(\Phi) = r(\Phi) = \frac{1}{20}(-\Phi^2 + 8\Phi - 11\mathrm{I}) = \frac{1}{20}\begin{pmatrix} -4 & 16 & 0 \\ 8 & -12 & 0 \\ -3 & 7 & 5 \end{pmatrix}.$$

From theorem 3.7.6 it follows that the matrix $f(\Phi)$ is the "usual" inverse of Φ.

Exercise 3.7.8. Calculate $\Gamma(\begin{pmatrix} 12 & 4 \\ -25 & -8 \end{pmatrix})$, where $\Gamma(z)$ is the gamma function.

3.8 Problems

Problem 3.8.1. *Suppose that $(e_1, \dots, e_n)$ is a basis of a complex vector space U.*

1. *Prove that $(e_1, \sqrt{-1}e_1, \dots, e_n, \sqrt{-1}e_n)$ is a basis of the realification $U^{\mathbb{R}}$.*
2. *Find the matrix of the corresponding complex structure (example 3.1.2) in that basis.*

Problem 3.8.2. *Suppose that V is a real vector space. Prove that every vector of its complexification $V^{\mathbb{C}}$ can be uniquelly represented as $v \otimes 1 + u \otimes \sqrt{-1}$, where $v, u \in V$.*

Problem 3.8.3. *Prove lemma 3.1.3.*

Problem 3.8.4. *Prove lemma 3.1.4.*

Problem 3.8.5. *Suppose that V is a real vector space, $\varphi : V \to V$ is an operator, $\varphi^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ is its complexification, and $\lambda = a + b\sqrt{-1} \in \mathbb{C}$ is an eigenvalue of $\varphi^{\mathbb{C}}$. Prove that*

1. *If $\lambda \in \mathbb{R}$, then λ is an eigenvalue of φ;*
2. *If $\lambda \notin \mathbb{R}$ and $v \otimes 1 + u \otimes \sqrt{-1} \in V^{\mathbb{C}}$ is an eigenvector corresponding to λ, then*
   (a) *v and u are linearly independent;*
   (b)
   $$\varphi(v) = av - bu, \quad \varphi(u) = bv + au.$$
   In particular, $\mathrm{Span}(v, u)$ *is a 2-dimensional φ-invariant subspace of V.*

Problem 3.8.6. *Prove that the characteristic polynomial of a linear operator is well-defined. That is, if $\varphi : V \to V$ is an operator, Φ and $\tilde{\Phi}$ are matrices of φ in some bases of V, then*

$$\det(t\mathrm{I} - \Phi) = \det(t\mathrm{I} - \tilde{\Phi}).$$

Problem 3.8.7. *Suppose that $\Phi \in \mathrm{Mat}_{n\times n}(\mathbf{k})$, $p_\Phi(t) = t^n + p_{n-1}t^{n-1} + \ldots + p_0$. Prove that*

1. $p_{n-1} = -\mathrm{tr}(\Phi)$;
2. $p_{n-2} = \frac{1}{2}((\mathrm{tr}(\Phi))^2 - \mathrm{tr}(\Phi^2))$;
3. $p_0 = (-1)^n \det(\Phi)$.

Problem 3.8.8. *Find invariant subspaces of the operator*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^4.$$

Problem 3.8.9. *Find invariant subspaces of the operator*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^4.$$

Problem 3.8.10. *Suppose that* $\varphi : V \to V$ *is an operator and*

$$\operatorname{Spec}(\varphi) = \{\lambda_1, \dots, \lambda_n\},$$

where $n = \dim(V)$*,* $\lambda_i \neq \lambda_j$ *for* $i \neq j$*. Suppose that* e_i *is an eigenvector of* φ *corresponding to* λ_i*,* $1 \leqslant i \leqslant n$*. Prove that*

1. (e_i) *is a Jordan basis for* φ*;*

2. *the Jordan matrix of* φ *is a diagonal matrix with the diagonal entries* $\lambda_1, \dots, \lambda_n$*.*

Problem 3.8.11. *Consider the operator*

$$\Phi = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

1. *Find a Jordan basis and the Jordan matrix of* Φ*.*

2. *Calculate* Φ^n*, where* $n \in \mathbb{N}$*.*

Problem 3.8.12. *Find a Jordan basis and the Jordan form of the operator*

$$\Phi = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2.$$

Problem 3.8.13. *Find a Jordan basis and the Jordan form of the operator*

$$\Phi = \begin{pmatrix} 3 & 4 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 4 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3.$$

Problem 3.8.14. *Find a Jordan basis and the Jordan form of the operator*

$$\Phi = \begin{pmatrix} 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 9 & 2 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} : \mathbb{R}^5 \to \mathbb{R}^5.$$

The *minimal polynomial* of an operator φ is a nonzero polynomial $f(t) \in \mathbf{k}[t]$ of minimal degree such that $f(\varphi) = 0$. For example, the minimal polynomial of the identity operator is $t - 1$ and the minimal polynomial of a nontrivial projection is $t^2 - t$.

Problem 3.8.15. *Suppose that* V *is a finite-dimensional vector space and* $\varphi : V \to V$ *is an operator. Prove the following statements.*

1. *There exists a unique (up to a scalar factor) minimal polynomial of* φ*.*

2. *If $f(t)$ is a polynomial such that $f(\varphi) = 0$, then the minimal polynomial divides $f(t)$; in particular, the minimal polynomial divides the characteristic polynomial of φ.*

Problem 3.8.16. *Prove that the minimal polynomial of $n \times n$ diagonal matrix with different diagonal elements is equal to its characteristic polynomial.*

Problem 3.8.17. *Consider a complex vector space V with a basis $(e_1, \dots, e_n)$ and the operator*

$$\varphi : V \to V, \quad e_i \mapsto \begin{cases} e_{i+1}, & \text{if } i \neq n, \\ e_1, & \text{if } i = n. \end{cases}$$

1. *Find eigenvalues and corresponding eigenvectors of φ.*

2. *Prove that the minimal polynomial of φ and its characteristic polynomial are both equal to $t^n - 1$.*

Problem 3.8.18. *Consider an $n \times n$ Jordan block J with λ on the diagonal.*

1. *Find eigenvalues and corresponding eigenvectors of J.*

2. *Prove that the minimal polynomial of J and its characteristic polynomial are both equal to $(t - \lambda)^n$.*

Problem 3.8.19. *Consider a Jordan matrix Φ with $p_\Phi(t) = \prod\limits_{1 \leqslant i \leqslant m} (t - \lambda_i)^{n_i}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$ Prove that*

1. *n_i is the sum of sizes of Jordan blocks with λ_i on the diagonal.*

2. *the minimal polynomial of Φ is $\prod\limits_{1 \leqslant i \leqslant m} (t - \lambda_i)^{l_i}$, where l_i is the size of a maximal Jordan block with λ_i on the diagonal.*

An operator is called *diagonalizable* whenever its matrix in some basis is diagonal.

Problem 3.8.20. *Suppose that for an operator φ there is a Jordan basis. Prove that*

1. *φ is diaginalizable whenever its minimal polynomial has no multiple roots;*

2. *if $\varphi^m = \mathrm{I}$, then φ is diaginalizable.*

Problem 3.8.21. *Suppose that $\Phi \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ is such that the sequence (Φ^n) is bounded. Prove that* $\mathrm{Spec}(\Phi) \subset \{z \mid |z| \leqslant 1\}$.

A matrix $P \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ is a called *right stochastic*, iff

(a) $P_{i,j} \geqslant 0$ for all $1 \leqslant i, j \leqslant n$,

(b) $\sum_{1 \leqslant j \leqslant n} P_{i,j} = 1$ for all $1 \leqslant i \leqslant n$.

Problem 3.8.22. *Suppose that $P \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ is a right stochastic matrix. Consider P as a complex matrix. Prove that*

1. $1 \in \mathrm{Spec}(P) \subset \{z \mid |z| \leqslant 1\}$;
2. *for the Jordan normal form of P, every Jordan block corresponding to $\lambda \in \mathrm{Spec}(P)$, where $|\lambda| = 1$, is a block of size 1×1 .*

Problem 3.8.23. *Prove lemma 3.3.7.*

Problem 3.8.24. *Suppose that V is a finite-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle_V$. Prove that*

1. $$\langle \cdot, \cdot \rangle_{V^{\mathbb{C}}} : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}, \quad \langle v \otimes \lambda, u \otimes \mu \rangle_{V^{\mathbb{C}}} = \lambda \overline{\mu} \langle v, u \rangle_V$$
   is an Hermitian inner product on $V^{\mathbb{C}} \times V^{\mathbb{C}}$.
2. *For an operator $\varphi : V \to V$ we have $(\varphi^{\mathbb{C}})^* = (\varphi^*)^{\mathbb{C}}$.*

Problem 3.8.25. *Suppose that V is an inner product space. Prove that a projection $\varphi : V \to V$ is a normal operator whenever φ is an orthogonal projection.*

Problem 3.8.26. *Suppose that V is an inner product space, $\varphi : V \to V$ is a normal operator, and U is a φ-invariant subspace. Prove that $U^{\perp}$ is φ-invariant subspace.*

Problem 3.8.27. *Suppose that V is an inner product space, $\varphi : V \to V$ is a normal operator, and U is a φ-invariant subspace. Prove that φ_U is a normal operator.*

Problem 3.8.28. *Suppose that V is a 2-dimensional real inner product space and $\varphi : V \to V$ is a normal operator. Prove that*

1. *if φ have no real eigenvalue, then there is an orthonormal basis of V such that the matrix of φ in this basis is*
   $$r \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \quad \text{where } r > 0 \text{ and } \alpha \neq \pi k \text{ with } k \in \mathbb{Z};$$
2. *if φ have a real eigenvalue, then there is an orthonormal basis of V such that the matrix of φ in this basis is*
   $$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \text{where } \lambda_i \in \mathbb{R}.$$

Problem 3.8.29. *Prove corollary 3.3.19.*

Problem 3.8.30. *Prove* LU *decomposition theorem (theorem 3.5.1).*

Problem 3.8.31. *Prove* QR *decomposition theorem (theorem 3.5.3).*

Problem 3.8.32. *Prove* SVD *decomposition theorem (theorem 3.5.5).*

Problem 3.8.33. *Prove theorem 3.5.8.*

Problem 3.8.34. *Prove lemma 3.6.5.*

Problem 3.8.35. *Suppose that V is a real or complex vector space and $\|\cdot\|$ is a norm on V. Prove that the first condition in lemma 3.6.6 implies the second condition.*

Problem 3.8.36. *Prove that in examples 3.6.1, norms 2 and 5 can be obtained from inner products, and norms 1, 2, and 4 cannot be obtained from inner products.*

Problem 3.8.37. *Suppose that V and U are finite-dimensional vector spaces and $\varphi : V \to V$ and $\psi : U \to U$ are operators. Prove that*

1. $\mathrm{rk}(\varphi \otimes \psi) = \mathrm{rk}(\varphi)\mathrm{rk}(\psi)$.

2. *If the characteristic polynomials of φ and ψ are completely decomposable, than the spectrum of $\varphi \otimes \psi$ is $\{\mu_i \lambda_j\}$, where $\{\mu_i\}$ and $\{\lambda_i\}$ are the spectrums of φ and ψ respectively.*

3. $\mathrm{tr}(\varphi \otimes \psi) = \mathrm{tr}(\varphi)\mathrm{tr}(\psi)$.

4. $\det(\varphi \otimes \psi) = \det(\varphi)^{\dim(U)} \det(\psi)^{\dim(V)}$.

Problem 3.8.38. *Suppose that $\varphi : V \to V$ and $\psi : U \to U$ are operators with the Jordan forms*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively. Find the Jordan form of the operator $\varphi \otimes \psi$.

Problem 3.8.39. *Suppose that $\varphi : V \to V$ is a operator with the Jordan form*

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Find the Jordan form of operators $\mathrm{S}^2(\varphi)$ and $\Lambda^2(\varphi)$.

Problem 3.8.40. *Suppose that V is an n-dimensional vector space, $\varphi : V \to V$ is an operator with a completely decomposable characteristic polynomial, and $\{\mu_i\}$ is the spectrum of φ. Prove that*

1. *The spectrum of $\mathrm{S}^k(\varphi)$ is $\{\mu_{i_1} \dots \mu_{i_k}\}_{1 \leqslant i_1, \dots, i_k \leqslant n}$,*

2. *The spectrum of $\Lambda^k(\varphi)$ is $\{\mu_{i_1} \dots \mu_{i_k}\}_{1 \leqslant i_1 < \dots < i_k \leqslant n}$.*

Problem 3.8.41. *Suppose that V is a finite-dimensional vector space and $\varphi : V \to V$ is an operator such that $\mathrm{tr}(\Lambda^q(\varphi)) = 0$ for all q. Prove that φ is a nilpotent operator.*

Chapter 4

Rings and Modules

The natural generalization of fields are rings and the natural generalization of vector spaces over fields are modules over rings. In this section we present an introduction to rings and modules over rings.

4.1 Rings

Definition 4.1.1. A *ring* is a set R with a fixed element $0 \in R$ (the zero) and two compositions: an addition

$$+ : R \times R \to R, \quad (r,s) \mapsto r+s$$

and a multiplication

$$R \times R \to R, \quad (r,s) \mapsto rs$$

satisfying the following axioms:

1. $(r+s)+t = r+(s+t)$ for all $r,s,t \in R$;
2. $r+s = s+r$ for all $r,s \in R$;
3. $r+0 = r$ for all $r \in R$;
4. for every $r \in R$ there exists $-r \in R$ such that $r+(-r) = 0$.
5. $r(s+t) = rs+rt$ and $(s+t)r = sr+tr$ for all $r,s,t \in R$.

Examples 4.1.2.

- Every field is a ring. For example, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{F}_p$ are rings.
- $\mathbb{Z}$ – the ring of integers.
- $\mathbb{Z}/n\mathbb{Z}$ – the ring of integers modulo n.
- $R[t]$ – the polynomial ring in the variable t over a ring R. It is defined to be the set of expressions of the form

$$r_0 + r_1 t + \ldots + r_n t^n, \quad \text{where } n \in \mathbb{N}, r_i \in R$$

with the natural addition

$$(r_0 + r_1 t + \ldots + r_n t^n) + (s_0 + s_1 t + \ldots + s_m t^m) =$$
$$(r_0 + s_0) + (r_1 + s_1)t + (r_2 + s_2)t^2 + \ldots$$

and the multiplication

$$(r_0 + r_1 t + \ldots + r_n t^n)(s_0 + s_1 t + \ldots + s_m t^m) =$$
$$(r_0 s_0) + (r_0 s_1 + r_1 s_0)t + (r_0 s_2 + r_1 s_1 + r_2 s_0)t^2 + \ldots$$

of polynomials. In particular, we have the polynomial ring $\mathbf{k}[t]$ in the variable t over a field $\mathbf{k}$.

- $R[[t]]$ – the ring of *formal power series* in the variable t over a ring R. It is defined to be the set of series of the form

$$r_0 + r_1 t + r_2 t^2 + \ldots = \sum_{i \geqslant 0} r_i t^i, \quad \text{where } r_i \in R$$

with the natural addition

$$(r_0 + r_1 t + r_2 t^2 + \ldots) + (s_0 + s_1 t + s_2 t^2 + \ldots) =$$
$$(r_0 + s_0) + (r_1 + s_1)t + (r_2 + s_2)t^2 + \ldots$$

and the multiplication

$$(r_0 + r_1 t + r_2 t^2 + \ldots)(s_0 + s_1 t + s_2 t^2 + \ldots) =$$
$$(r_0 s_0) + (r_0 s_1 + r_1 s_0)t + (r_0 s_2 + r_1 s_1 + r_2 s_0)t^2 + \ldots$$

of power series.

- $R((t))$ – the ring of *formal Laurent series* in the variable t over a ring R. It is defined to be the set of series of the form

$$r_m t^m + r_{m+1} t^{m+1} + r_{m+2} t^{m+2} + \ldots = \sum_{i \geqslant m} r_i t^i, \quad \text{where } m \in \mathbb{Z}, r_i \in R$$

with the natural addition and the multiplication of Laurent series.

- $\mathrm{Mat}_n(R)$ – the *matrix ring* of $n \times n$ matrices with entries in a ring R with the addition and the multiplication of matrices.

Definition 4.1.3. • A ring R is called *associative* whenever

$$r(st) = (rs)t \quad \text{for all } r, s, t \in R.$$

- A ring R is called *commutative* whenever

$$rs = sr \quad \text{for all } r, s \in R.$$

- A ring R is called a *ring with identity* whenever there exists an identity elemenet $1 \in R$ such that

$$1r = r1 = r \quad \text{for all } r \in R.$$

- An element r of a ring R with identity is called *invertible* whenever there exists $r^{-1} \in R$ such that

$$rr^{-1} = r^{-1}r = 1.$$

  The set of invertible elements of R is denoted by R^*.

Remark 4.1.4. There are associative and commutative, associative and noncommutative, nonassociative and commutative, nonassociative and noncommutative rings with and without identity.

Definition 4.1.5. A ring R is called an *algebra over a field* $\mathbf{k}$ whenever R is a vector space over $\mathbf{k}$ and $\lambda(rs) = (\lambda r)s = r(\lambda s)$ for all $\lambda \in \mathbf{k}$, $r, s \in R$.

For example, $\mathbf{k}[t]$, $\mathbf{k}[[t]]$, $\mathbf{k}((t))$, and $\mathrm{Mat}_n(\mathbf{k})$, where $\mathbf{k}$ is a field, are $\mathbf{k}$-algebras.

Tensor algebra of a vector space.

Let V be a vector space over a field $\mathbf{k}$. Consider the space

$$\mathrm{T}(V) = \mathbf{k} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \ldots$$

as an algebra with the multiplication $\otimes$, where the product of

$$v_1 \otimes \ldots \otimes v_k \in \mathrm{T}^k(V) \quad \text{and} \quad u_1 \otimes \ldots \otimes u_l \in \mathrm{T}^l(V)$$

is

$$v_1 \otimes \ldots \otimes v_k \otimes u_1 \otimes \ldots \otimes u_l \in \mathrm{T}^{k+l}(V).$$

Lemma 4.1.6. *The algebra* $\mathrm{T}(V)$ *is well-defined.*

The algebra $\mathrm{T}(V)$ is called the *tensor algebra of the space* V.

Clearly, the tensor algebra of the 0-dimensional vector space $V = \{0\}$ is $\mathbf{k}$.

Consider an 1-dimensional vector space V. Fix a basis $(e_1 = e)$ of V. Then $V^{\otimes k}$ is an 1-dimensional vector space with the basis $(e^{\otimes k})$, where

$$e^{\otimes k} := \underbrace{e \otimes \ldots \otimes e}_{k}.$$

We have the isomorphism

$$\mathbf{k}[t] \to T(V),$$
$$f_0 + f_1 t + f_2 t^2 + \ldots \mapsto f_0 + f_1 e + f_2 e^{\otimes 2} + \ldots$$

Symmetric algebra of a vector space

Let V be a vector space. Consider the space

$$\mathrm{S}(V) = \mathbf{k} \oplus \mathrm{S}^1(V) \oplus \mathrm{S}^2(V) \oplus \dots .$$

The space $\mathrm{S}(V)$ is an algebra with the multiplication $\cdot$, where the product of

$$f \in \mathrm{S}^k(V) \quad \text{and} \quad g \in \mathrm{S}^l(V)$$

is

$$f \cdot g \coloneqq \mathrm{sym}(f \otimes g) \in \mathrm{S}^{k+l}(V).$$

Lemma 4.1.7. *The algebra* $\mathrm{S}(V)$ *is well-defined.*

The algebra $\mathrm{S}(V)$ is called the *symmetric algebra of* V; it is associative, commutative, and with identity. The algebra $\mathrm{S}(V)$ is not a subalgebra of $\mathrm{T}(V)$; while the map $\mathrm{sym} : \mathrm{T}(V) \to \mathrm{S}(V)$ is an algebra homomorphism.

Exterior algebra of a vector space

Let V be a vector space. Consider the vector space

$$\Lambda(V) = \mathbf{k} \oplus \Lambda^1 V \oplus \dots \oplus \Lambda^{\dim(V)} V.$$

The space $\Lambda(V)$ is an algebra over $\mathbf{k}$ with the multiplication $\wedge$, where the product of

$$\omega_1 \in \Lambda^k(V) \quad \text{and} \quad \omega_2 \in \Lambda^l(V)$$

is

$$\omega_1 \wedge \omega_2 \coloneqq \mathrm{alt}(\omega_2 \otimes \omega_2) \in \Lambda^{k+l}(V).$$

Lemma 4.1.8. *The algebra* $\Lambda(V)$ *is well-defined.*

The algebra $\Lambda(V)$ is called the *exterior algebra of* V; it is associative, noncommutative for $\dim(V) \geqslant 2$, and with identuty. The algebra $\Lambda(V)$ is not a subalgebra of $\mathrm{T}(V)$; while the map $\mathrm{alt} : \mathrm{T}(V) \to \Lambda(V)$ is an algebra homomorphism.

We have

$$\dim(\Lambda(V)) = 2^{\dim(V)}.$$

A form $\omega \in \Lambda(V)$ is called *homogeneous* whenever $\omega \in \Lambda^k(V)$ for some k; in this case k is called the *degree* of w and is denoted by $\deg(\omega)$. We have

$$\omega_1 \wedge \omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)} \omega_1 \wedge \omega_2$$

for homogeneous forms $\omega_1, \omega_2 \in \Lambda(V)$.

4.2 Modules

Let R be an associative, commutative ring with identuty.

Definition 4.2.1. An *R-module* is defined to be a set M with a fixed element $0 \in M$ (the *zero*) and two compositions:

$$+ : M \times M \to M, \quad (m, l) \mapsto m + l \qquad \text{(an addition)}$$

and

$$R \times M \to M, \quad (r, m) \mapsto r \cdot m \qquad \text{(a multiplication by elements of } R\text{)}$$

satisfying the following axioms:

1. $m + l = l + m$;
2. $(m + l) + n = m + (l + n)$;
3. $m + 0 = m$;
4. for every $m \in M$ there exists $-m \in M$ such that $m + (-m) = 0$.
5. $r \cdot (m + l) = r \cdot m + r \cdot l$;
6. $(r + s) \cdot m = r \cdot m + s \cdot m$;
7. $(rs) \cdot m = r \cdot (s \cdot m)$;
8. $1 \cdot m = m$

for all $r, s \in R$ and $m, l, n \in M$.

The first four properties mean that M is an abelian group with the neutral element 0 and the composition $+$.

Examples 4.2.2. 1. An abelian group M can be considered as a $\mathbb{Z}$-module with the multiplication

$$r \cdot m = \operatorname{sgn}(r)(\underbrace{m + \ldots + m}_{|r|}), \quad \text{where } r \in \mathbb{Z}, m \in M.$$

2. A vector space over a field $\mathbf{k}$ is a $\mathbf{k}$-module.

3. the set $\{0\}$ (consists of one element – the zero) with the multiplication

$$r \cdot 0 = 0, \quad \text{where } r \in R$$

is an R-module; it is called the *trivial R-module*.

4. the ring R itself with the multiplication

$$r \cdot m = rm, \quad \text{where } r \in R, m \in R$$

is an R-module; it is denoted by R^1.

5. The set

$$R^k := \{\begin{pmatrix} r^1 \\ \vdots \\ r^k \end{pmatrix} \mid r^i \in R\}$$

with the addition

$$\begin{pmatrix} r^1 \\ \vdots \\ r^k \end{pmatrix} + \begin{pmatrix} s^1 \\ \vdots \\ s^k \end{pmatrix} := \begin{pmatrix} r^1 + s^1 \\ \vdots \\ r^k + s^k \end{pmatrix}$$

and the multiplication

$$r \cdot \begin{pmatrix} r^1 \\ \vdots \\ r^k \end{pmatrix} = \begin{pmatrix} rr^1 \\ \vdots \\ rr^k \end{pmatrix}$$

is an R-module.

Example 4.2.3. Suppose that V is a vector space over a field $\mathbf{k}$ and $\varphi \in \mathrm{End}(V)$. Then we have the $\mathbf{k}[t]$-module V_φ, where

- $V_\varphi = V$ as a set,
- the multiplication is $f(t) \cdot v = f(\varphi)(v)$.

Exercise 4.2.4. Consider the ring $\mathbf{k}[t]$, where $\mathbf{k}$ is a field, and $\mathbf{k}[t]$-module V. Suppose that V, considered as a vector space over $\mathbf{k}$, is finite-dimensional. Prove that $V = V_\varphi$ for some $\varphi \in \mathrm{End}(V)$.

Example 4.2.5. Suppose that $M_1, \ldots, M_k$ are R-modules. Then their direct product is an R-module with respect to the addition

$$(m_1, \ldots, m_k) + (l_1, \ldots, l_k) = (m_1 + l_1, \ldots, m_k + l_k)$$

and the multiplication

$$r \cdot (m_1, \ldots, m_k) = (r \cdot m_1, \ldots, r \cdot m_k), \quad \text{where } r \in R, m_i \in M_i;$$

it is called the *direct sum* of R-modules $M_1, \ldots, M_k$ and denoted by $M_1 \oplus \ldots \oplus M_k$. We have

$$\underbrace{R \oplus \ldots \oplus R}_{n} \simeq R^n.$$

Definition 4.2.6. Suppose that M is an R-module. A subset $L \subset M$ is called an *R-submodule* whenever

$$m_1, \ldots, m_k \in L \text{ and } r_1, \ldots, r_k \in R \quad \text{imply} \quad r_1 \cdot m_1 + \ldots + r_k \cdot m_k \in L.$$

Example 4.2.7. Suppose that M is an R-module. Then

1. $\{0\}$ is an R-submodule; it is called the *trivial* submodule.

2. M itself is an R-submodule.

3. An element $m \in M$ defines the submodule

$$Rm := \{r \cdot m \mid r \in R\};$$

it is called the *cyclic submodule generated by m*.

4. (A generalization of the previous example) A subset $X \subset M$ generates the submodule
$$\{r_1 \cdot x_1 + \ldots + r_k x_k \mid k \in \mathbb{N}, r_i \in R, x_i \in X\};$$
it is called the *submodule generated by X*.

Definition 4.2.8. Suppose that M and L are R-modules. A map $\varphi : M \to L$ is called an *R-module homomorphism* or, simply, homomorphism if

$$\varphi(r_1 \cdot m_1 + \ldots + r_k \cdot m_k) = r_1 \cdot \varphi(m_1) + \ldots + r_k \cdot \varphi(m_k) \quad \text{for all } r_i \in R, m_i \in M.$$

An R-module homomorphism is called an *R-module isomorphism* whenever it is a bijective map and the set-theoretic inverse map is an R-module homomorphism; in this case M is called isomorphic to L and we write $M \simeq L$.

Lemma 4.2.9. *1. The map*

$$\Phi : R^n \to R^k, \quad \begin{pmatrix} r^1 \\ r^2 \\ \vdots \\ r^n \end{pmatrix} \mapsto \begin{pmatrix} \Phi_1^1 & \Phi_2^1 & \ldots & \Phi_n^1 \\ \Phi_1^2 & \Phi_2^2 & \ldots & \Phi_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_1^k & \Phi_2^k & \ldots & \Phi_n^k \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \\ \vdots \\ r^n \end{pmatrix},$$

where
$$\Phi = \begin{pmatrix} \Phi_1^1 & \Phi_2^1 & \ldots & \Phi_n^1 \\ \Phi_1^2 & \Phi_2^2 & \ldots & \Phi_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_1^k & \Phi_2^k & \ldots & \Phi_n^k \end{pmatrix} \in \mathrm{Mat}_{k \times n}(R),$$
is an R-module homomorphism;

2. An R-module homomorphism of the form $R^n \to R^k$ is a left multiplication by some $\Phi \in \mathrm{Mat}_{k\times n}(R)$.

Suppose that M, N are R-modules. Consider the set $\mathrm{Hom}(M, N)$ of R-modules homomorphisms $\varphi : M \to N$. Then $\mathrm{Hom}(M, N)$ is an R-module under the following compositions:

1. the sum of homomorphisms $\varphi_1, \varphi_2 \in \mathrm{Hom}(M,N)$ is the homomorphisms

$$\varphi_1 + \varphi_2 : M \to N, \quad (\varphi_1 + \varphi_2)(m) = \varphi_1(m) + \varphi_2(m);$$

2. the product of $\lambda \in R$ and an homomorphism $\varphi \in \mathrm{Hom}(M,N)$ is the homomorphism

$$\lambda\varphi : M \to N, \quad (\lambda\varphi)(m) = \lambda\varphi(m).$$

In particular, for an R-module M we have the R-module $M^* := \mathrm{Hom}(M,R)$.

Exercise 4.2.10. Calculate $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/18\mathbb{Z}, \mathbb{Z}/24\mathbb{Z})$

Lemma 4.2.11. *Suppose that M is an R-module, and L is a submodule of M. Then*

1. *the set*

$$M/L := \{m + L \mid m \in M\}$$

with the zero element is $0 + L$, with the addition

$$(m + L) + (l + L) = m + l + L,$$

and the multiplication

$$r \cdot (m + L) = r \cdot m + L$$

is an R-module; it is called the quotient *of M by L.*

2. *The map*

$$\pi : M \to M/L, \quad \varphi(m) = m + L$$

is a surjective R-module homomorphism; it is called the quotient *homomorphism.*

4.3 Tensor Products of Modules

Let R be an associative and commutative ring, M and L be R-modules.

Consider the set of formal expressions of the form

$$r_1 m_1 \otimes l_1 + \ldots + r_k m_k \otimes l_k, \quad \text{where } k \in \mathbb{N}^*, r_i \in R, m_i \in M, l_i \in L,$$

as an R-module with the natural addition and multiplication by elements of R; the zero element is the expression $0 \otimes 0$.[1] By definition, the tensor product $V \otimes U$ is the quotient of this R-module modulo relations

$$\begin{aligned} &(1) \quad (m_1 + m_2) \otimes l = m_1 \otimes l + m_2 \otimes l, \\ &(2) \quad m \otimes (l_1 + l_2) = m \otimes l_1 + m \otimes l_2, \\ &(3) \quad rm \otimes l = (r \cdot m) \otimes l = m \otimes (r \cdot l). \end{aligned} \tag{4.3.1}$$

[1] This R-module can be defined formally by

$$\bigoplus_{m \in M, l \in L} Rm \otimes l$$

The notation of the *class* of an expression $r_1m_1 \otimes l_1 + \ldots + r_km_k \otimes l_k$ in $M \otimes L$ is

$$r_1m_1 \otimes l_1 + \ldots + r_km_k \otimes l_k$$

(the same as of the element). Two element of $M \otimes L$ are equal if one of them can be transformed into the other by using (4.3.1). From definition it follows that

$$M \otimes \{0\} = \{0\} \otimes L = \{0\}.$$

Example 4.3.1. In $\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ we have

$$m \otimes l = (21 \cdot m) \otimes l = m \otimes (21 \cdot l) = m \otimes 0 = 0$$

for every $m \in \mathbb{Z}/10\mathbb{Z}$, $l \in \mathbb{Z}/3\mathbb{Z}$. Thus $\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = \{0\}$.

From definition it follows that elements of the form $m \otimes l$, where $m \in M$, $l \in L$ are generators of $M \otimes L$. Thus to define an R-module homomorphism

$$\varphi : M \otimes L \to W,$$

where W is an R-module, we have to define the images $\varphi(m \otimes l) \in W$, where $m \in M$, $l \in L$ such that these images are consistent with (4.3.1), that is,

$$\begin{array}{ll} (1) & \varphi((m_1 + m_2) \otimes l) = \varphi(m_1 \otimes l) + \varphi(m_2 \otimes l), \\ (2) & \varphi(m \otimes (l_1 + l_2)) = \varphi(m \otimes l_1) + \varphi(m \otimes l_2), \\ (3) & r\varphi(m \otimes l) = \varphi((r \cdot m) \otimes l) = \varphi(m \otimes (r \cdot l)). \end{array} \tag{4.3.2}$$

Roughly speaking, (4.3.2) means that $\varphi(m \otimes l)$ is linear by m and linear by l.

4.4 *A Proof of Cayley-Hamilton's Theorem

Consider the ring $\mathrm{Mat}_n(\mathbf{k})$ of $n \times n$ matrices with entries in a field $\mathbf{k}$. Recall that

- The *determinant* is the function of the form

  $$\det : \mathrm{Mat}_n(\mathbf{k}) \to \mathbf{k}$$

  that defined inductively by the formulas:

  1. for 1×1 matrix $A = (a_{1,1})$, $\det(A) = a_{1,1}$;
  2. if $A = (a_{i,j})$ is a $n \times n$ matrix, where $n > 1$, then

     $$\det(A) = \sum_{1 \leqslant j \leqslant n} (-1)^{1+j} a_{1,j} M_{1,j},$$

     where $M_{i,j}$ is the determinant of the submatrix of A formed by deleting the ith row and the jth column.

- The determinant satisfies the following properties:
  1. $\det(\mathrm{I}) = 1$.
  2. $\det(AB) = \det(A)\det(B)$ for every $A, B \in \mathrm{Mat}_n(\mathbf{k})$.
  3. $\det(A)$ is a skew-symmetric form of rows of $A \in \mathrm{Mat}_n(\mathbf{k})$.
- The *characterictic polynomial* of a matrix $A \in \mathrm{Mat}_n(\mathbf{k})$ is

$$p_A(t) = \det(t\mathrm{I} - A) \in \mathbf{k}[t].$$

- The *adjugate* of a matrix $A \in \mathrm{Mat}_n(\mathbf{k})$ is the matrix

$$\mathrm{adj}(A) := \begin{pmatrix} M_{1,1} & -M_{2,1} & M_{3,1} & \dots \\ -M_{1,2} & M_{2,2} & -M_{3,2} & \dots \\ M_{1,3} & -M_{2,3} & M_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathrm{Mat}_n(\mathbf{k}).$$

Lemma 4.4.1. *Suppose that* $\mathbf{k}$ *is a field and* $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathbf{k})$*. Then*

1. *for any* m, $\det(A) = \sum_{1\leqslant i\leqslant n}(-1)^{m+i}a_{m,i}M_{m,i} = \sum_{1\leqslant i\leqslant n}(-1)^{m+i}a_{i,m}M_{i,m}$*;*
2. $A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A)\mathrm{I}$*.*

Let R be an associative, commutative ring with identity. Consider the set $\mathrm{Mat}_n(R)$ of $n \times n$ matrices with entries in R. The set $\mathrm{Mat}_n(R)$ is an associative ring with identity. Similarly to the matrices with entries in a field, we have the following definitions and facts.

- The *determinant* is the function of the form

$$\det : \mathrm{Mat}_n(R) \to R$$

  that defined inductively by the formulas:
  1. for 1×1 matrix $A = (a_{1,1})$, $\det(A) = a_{1,1}$.
  2. if $A = (a_{i,j})$ is a $n \times n$ matrix, where $n > 1$, then

$$\det(A) = \sum_{1\leqslant j\leqslant n} (-1)^{1+j} a_{1,j} M_{1,j},$$

     where $M_{i,j}$ is the determinant of the submatrix of A formed by deleting the ith row and the jth column.
- The determinant satisfies the following properties:
  1. $\det(\mathrm{I}) = 1$.

2. $\det(AB) = \det(A)\det(B)$ for every $A, B \in \mathrm{Mat}_n(R)$.
3. $\det(A)$ is a skew-symmetric form of rows of $A \in \mathrm{Mat}_n(R)$.

♦ The *characterictic polynomial* of a matrix $A \in \mathrm{Mat}_n(R)$ is

$$p_A(t) = \det(t\mathrm{I} - A) \in R[t].$$

♦ The *adjugate* of a matrix $A \in \mathrm{Mat}_n(R)$ is the matrix

$$\mathrm{adj}(A) := \begin{pmatrix} M_{1,1} & -M_{2,1} & M_{3,1} & \dots \\ -M_{1,2} & M_{2,2} & -M_{3,2} & \dots \\ M_{1,3} & -M_{2,3} & M_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathrm{Mat}_n(R).$$

Lemma 4.4.2. *Suppose that R is an associative, commutative ring with identity and $A = (a_{i,j}) \in \mathrm{Mat}_n(R)$. Then*

1. *for any m, $\det(A) = \sum_{1\leqslant i\leqslant n}(-1)^{m+i}a_{m,i}M_{m,i} = \sum_{1\leqslant i\leqslant n}(-1)^{m+i}a_{i,m}M_{i,m}$;*

2. *$A \cdot \mathrm{adj}(A) = \mathrm{adj}(A)\cdot A = \det(A)\mathrm{I}$.*

Exercise 4.4.3. Suppose that R is an associative and commutative ring with identity and $A \in \mathrm{Mat}_n(R)$. Prove that $\mathrm{tr}(\mathrm{adj}(A)) = p_{n-1}(A)$.

Cayley-Hamilton's theorem. *Suppose that R is an associative, commutative ring with identity, $A \in \mathrm{Mat}_n(R)$, and*

$$p_A(t) = t^n + (-1)p_1(A)t^{n-1} + \ldots + (-1)^n p_n(A)$$

is the characterictic polynomial of A. Then

$$p_A(A) = A^n + (-1)p_1(A)A^{n-1} + \ldots + (-1)^n p_n(A) = 0.$$

Proof. Consider the ring $R((t^{-1}))$ of formal Laurent series in the variable t^{-1}. Elements of $R((t^{-1}))$ are formal series

$$\sum_{i\leqslant m} r_i t^i, \quad \text{where } m \in \mathbb{Z}, r_i \in R.$$

The matrix $t\mathrm{I} - A$ is an invertible element of $\mathrm{Mat}_n(R((t^{-1})))$:

$$(t\mathrm{I} - A)^{-1} = t^{-1}\mathrm{I} + t^{-2}A + t^{-3}A^2 + \ldots \in \mathrm{Mat}_n(R((t^{-1}))).$$

By lemma 4.4.2(2),

$$\mathrm{adj}(t\mathrm{I} - A)(t\mathrm{I} - A) = \det(t\mathrm{I} - A)\mathrm{I} = p_A(t)\mathrm{I}.$$

Thus,

$$\operatorname{adj}(t\mathrm{I} - A) = p_A(t)(t\mathrm{I} - A)^{-1}$$

or

$$\begin{gathered}\operatorname{adj}(t\mathrm{I} - A) = \\ (t^n + (-1)p_1(A)t^{n-1} + \ldots + (-1)^n p_n(A))(t^{-1}\mathrm{I} + t^{-2}A + t^{-3}A^2 + \ldots).\end{gathered} \tag{4.4.1}$$

Now we have

$$\begin{gathered}0 = \left\{\begin{matrix}\text{the coefficient of } t^{-1} \\ \text{of the left side of (4.4.1)}\end{matrix}\right\} = \left\{\begin{matrix}\text{the coefficient of } t^{-1} \\ \text{of the right side of (4.4.1)}\end{matrix}\right\} = \\ A^n + (-1)p_1(A)A^{n-1} + \ldots + (-1)^n p_n(A)\mathrm{I} = p_A(A).\end{gathered}$$

□

Remark 4.4.4. (In notations of Caley-Hamilton's theorem and its proof) We have

$$\begin{gathered}p_{t\mathrm{I}-A}(\lambda) = \det(\lambda\mathrm{I} - (t\mathrm{I} - A)) = \det(-((t-\lambda)\mathrm{I} - A)) = (-1)^n p_A(t-\lambda) = \\ (-1)^n(t-\lambda)^n + (-1)^{n-1}p_1(A)(t-\lambda)^{n-1} + \ldots + p_n(A)\end{gathered}$$

and thus

$$\begin{gathered}\{\text{trace of the left side of (4.4.1)}\} \overset{\text{exercise 4.4.3}}{=} p_{n-1}(t\mathrm{I} - A) = \\ (-1)^{n-1} \cdot \{\text{coefficient of } \lambda \text{ of } p_{t\mathrm{I}-A}(\lambda)\} = \\ nt^{n-1} - (n-1)p_1(A)t^{n-1} + (n-2)p_2(A)t^{n-2} + \ldots + (-1)^{n-1}p_{n-1}(A) = \\ \{\text{trace of the right side of (4.4.1)}\} = \\ \operatorname{tr}((t^n + (-1)p_1(A)t^{n-1} + \ldots + (-1)^n p_n(A))(t^{-1}\mathrm{I} + t^{-2}A + t^{-3}A^2 + \ldots)).\end{gathered}$$

Equating coefficients of the monomials $t^{n-2}, t^{n-3}, \ldots$ of the middle expression with the coefficients of these monomials of the last expression we obtain the matrix version of the *Newton's identities*

$$\begin{aligned}p_1(A) &= \operatorname{tr}(A), \\ 2p_2(A) &= p_1(A)\operatorname{tr}(A) - \operatorname{tr}(A^2), \\ 3p_3(A) &= p_2(A)\operatorname{tr}(A) - p_1(A)\operatorname{tr}(A^2) + \operatorname{tr}(A^3), \\ 4p_4(A) &= p_3(A)\operatorname{tr}(A) - p_2(A)\operatorname{tr}(A^2) + p_1(A)\operatorname{tr}(A^3) - \operatorname{tr}(A^4), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\end{aligned}$$

where $p_k(A) = 0$ for $k > n$.

4.5 *Tor and Ext

Let R be a commutative associative ring with identity and M, N be R-modules. Consider a *resolution*

$$\ldots \xrightarrow{\varphi_3} R^{\oplus d_2} \xrightarrow{\varphi_2} R^{\oplus d_1} \xrightarrow{\varphi_1} R^{\oplus d_0} \xrightarrow{\varphi_0} M \longrightarrow 0 \tag{4.5.1}$$

of M. It generates the sequence

$$\dots \xrightarrow{\hat{\varphi}_3} R^{\oplus d_2} \otimes N \xrightarrow{\hat{\varphi}_2} R^{\oplus d_1} \otimes N \xrightarrow{\hat{\varphi}_1} R^{\oplus d_0} \otimes N \xrightarrow{\hat{\varphi}_0} 0,$$

where $\hat{\varphi}_0 = 0$ and $\hat{\varphi}_n = \varphi_n \otimes \mathrm{I}_N$ for $n \geqslant 1$, and the sequence

$$\dots \xleftarrow{\tilde{\varphi}_3} \mathrm{Hom}(R^{\oplus d_2}, N) \xleftarrow{\tilde{\varphi}_2} \mathrm{Hom}(R^{\oplus d_1}, N) \xleftarrow{\tilde{\varphi}_1} \mathrm{Hom}(R^{\oplus d_0}, N) \xleftarrow{\tilde{\varphi}_0} 0,$$

where $\tilde{\varphi}_0 = 0$ and $(\hat{\varphi}_n(h))(a) = h(\varphi_n(a))$ for $n \geqslant 1$. It is not hard to check that for $n \geqslant 0$ we have

$$\mathrm{Im}(\hat{\varphi}_{n+1}) \subset \mathrm{Ker}(\hat{\varphi}_n)$$

and

$$\mathrm{Im}(\tilde{\varphi}_n) \subset \mathrm{Ker}(\tilde{\varphi}_{n+1}).$$

Thus, for $n \geqslant 0$ we can define

$$\mathrm{Tor}_n^R(M, N) := \mathrm{Ker}(\hat{\varphi}_n)/\mathrm{Im}(\hat{\varphi}_{n+1})$$

and

$$\mathrm{Ext}_R^n(M, N) := \mathrm{Ker}(\tilde{\varphi}_{n+1})/\mathrm{Im}(\tilde{\varphi}_n).$$

It is not hard to check that R-modules $\mathrm{Tor}_n^R(M, N)$ and $\mathrm{Ext}_R^n(M, N)$ are well-defined (do not depend on a choice of resolution (4.5.1)).

Exercise 4.5.1. Check that for vector spaces V, U over a field $\mathbf{k}$ we have

$$\mathrm{Tor}_n^{\mathbf{k}}(V, U) = 0 \quad \text{and} \quad \mathrm{Ext}_{\mathbf{k}}^n(V, U) = 0 \quad \text{for} \quad n \geqslant 1.$$

Exercise 4.5.2. Suppose that R is a commutative and associative ring with identity and M, N are R-modules. Prove that

$$\mathrm{Tor}_0^R(M, N) = M \otimes N \text{ and } \mathrm{Ext}_R^0(M, N) = \mathrm{Hom}(M, N).$$

Exercise 4.5.3. Calculate $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/18\mathbb{Z})$ for all $n \geqslant 0$.

Exercise 4.5.4. Calculate $\mathrm{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/16\mathbb{Z})$ for all $n \geqslant 0$.

Exercise 4.5.5. Calculate $\mathrm{Tor}_n^{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$ for all $n \geqslant 0$.

Exercise 4.5.6. Calculate $\mathrm{Ext}_{\mathbb{Z}/24\mathbb{Z}}^n(\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$ for all $n \geqslant 0$.

Exercise 4.5.7. Calculate $\mathrm{Tor}_n^{\mathbb{R}[t]}(\mathbb{R}_A^2, \mathbb{R}_B^2)$ for all $n \geqslant 0$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Exercise 4.5.8. Calculate $\mathrm{Ext}_{\mathbb{R}[t]}^n(\mathbb{R}_A^2, \mathbb{R}_B^3)$ for all $n \geqslant 0$, where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

4.6 Problems

Problem 4.6.1. *Prove that $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are not algebras over any fields.*

Problem 4.6.2. *Suppose that V is a finite-dimensional vector space, $0 \neq v \in V$, and $\omega \in \Lambda^k(V)$. Prove that*

$$v \wedge \omega = 0 \;\Leftrightarrow\; \omega = v \wedge \omega' \text{ for some } \omega' \in \Lambda^{k-1}(V).$$

Problem 4.6.3. *Suppose that V is a finite-dimensional vector space, $0 \neq \omega \in \Lambda^k(V)$, and $U = \{u \in V \mid u \wedge \omega = 0\}$. Prove that*

1. $\dim(U) \leqslant k$;
2. *for every $0 \leqslant m \leqslant k$ we have: $\dim(U) \geqslant m$ iff $\omega = u_1 \wedge \ldots \wedge u_m \wedge \omega'$ for some $u_1, \ldots, u_m \in V$, $\omega' \in \Omega^{k-m}(V)$.*

Problem 4.6.4. *Suppose that V is a finite-dimensional vector space and $\omega \in \Lambda^2(V)$. Prove that $\omega \wedge \omega = 0$ whenever $\omega = v_1 \wedge v_2$ for some $v_1, v_2 \in V$.*

Problem 4.6.5. *Suppose that R is a commutative and associative ring with identity. Prove that $\sum_{i \geqslant 0} r_i t^i \in R[[t]]$ is invertible in $R[[t]]$ whenever r_0 is invertible in R.*

Problem 4.6.6. *Find the first 4 terms of the series $(5 + t + 4t^2)^{-1} \in \mathbb{Z}/6\mathbb{Z}[[t]]$.*

Problem 4.6.7. *Find the coefficient of t^{100} of the series $(2 + t + 4t^2)^{-1} \in \mathbb{F}_5[[t]]$.*

Problem 4.6.8. *Suppose that R is a commutative and associative ring with identity. Consider a formal Laurent series $r = \sum_{i \geqslant m} r_i t^i \in R((t))$. Prove that*

1. *if r_m is invertible in R, then r is invertible in $R((t))$;*
2. *$2 + t \in \mathbb{Z}/6\mathbb{Z}((t))$ is invertible in $\mathbb{Z}/6\mathbb{Z}((t))$ (while 2 is not invertible in $\mathbb{Z}/6\mathbb{Z}$).*

Problem 4.6.9. *Suppose that R is an associative and commutative ring with identity, and $A, B \in \mathrm{Mat}_n(R)$. Prove that $p_{AB}(t) = p_{BA}(t)$.*

Problem 4.6.10. *Prove lemma 4.2.9.*

Problem 4.6.11. *Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ for a finite abelian group A.*

Problem 4.6.12. *Suppose that R is a commutative and associative ring with identity and M is an R-module. Prove that the map*

$$M \to R \otimes_R M, \quad m \mapsto 1 \otimes m$$

is an isomorphism between R-modules.

Problem 4.6.13. *Suppose that $n, m \in \mathbb{N}^*$; prove that*

$$(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

Problem 4.6.14. *Consider* $\mathbf{k}[t]$*-modules* V_φ *and* U_ψ*, where* $\gcd(p_\varphi(t), p_\psi(t)) = 1$*. Prove that* $\operatorname{Hom}_{\mathbf{k}[t]}(V_\varphi, U_\psi) = 0$ *and* $V_\varphi \otimes_{\mathbf{k}[t]} U_\psi = 0$*.*

Problem 4.6.15. *Calculate* $\operatorname{Hom}_{\mathbb{R}[t]}(\mathbb{R}^2_A, \mathbb{R}^3_B)$ *and* $\operatorname{Hom}_{\mathbb{R}[t]}(\mathbb{R}^3_B, \mathbb{R}^2_A)$*, where*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 4.6.16. *Calculate* $\operatorname{Hom}_{\mathbb{R}[t]}(\mathbb{R}^5_A, \mathbb{R}^6_B)$*, where*

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4.6.17. *Calculate* $\mathbb{R}^2_A \otimes_{\mathbb{R}[t]} \mathbb{R}^3_B$ *and* $\mathbb{R}^3_B \otimes_{\mathbb{R}[t]} \mathbb{R}^2_A$*, where*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 4.6.18. *Calculate* $\mathbb{R}^5_A \otimes_{\mathbb{R}[t]} \mathbb{R}^6_B$*, where*

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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