Tutorial letter 102/0/2022

Theoretical Computer Science 1
COS1501

Semesters 1 and 2

School of Computing

CONTENTS:
1. TUTORIAL MATTER & CAI TUTORIAL
2. DISCUSSION OF SELF-ASSESSMENT QUESTIONS FOR ASSIGNMENT 02
3. DISCUSSION OF SELF-ASSESSMENT QUESTIONS FOR ASSIGNMENT 03
4. EXTRA QUESTIONS & SOLUTIONS
5. EXTRA QUESTIONS & SOLUTIONS similar to QUESTIONS IN ASSIGNMENTS 01, 02 AND 03

CONTENTS

Dear Students,

This tutorial letter contains information on tutorial matter and the CAI tutorial, solutions to self-assessment questions, assignments 02 and 03, and extra examples and solutions. You are welcome to contact the COS1501 team if you have any academic queries.

If there are any of the tutorial letters listed below that you have not received in good time before the examination, or if you have any administrative queries, consult the brochure *my Studies @ Unisa* for the required procedures that you should follow.

Everything of the best with your studies this year!
Regards,
The COS1501 team

1 TUTORIAL MATTER & CAI TUTORIAL

TUTORIAL MATTER: We provide a list of **tutorial letters** that you should receive during the semester:

COSALLP/301/4/2022 General information about the school and names and telephone numbers of lecturers
COS1501/101/0/2022 General information and assignments
COS1501/102/0/2022 This tutorial letter
COS1501/103/0/2022 Answers to activities in the study guide

CAI TUTORIAL: This is a computer-aided instruction (CAI) tutorial. It is downloadable and you can find the instructions in the Tutorial letter 101. This interactive tutorial with theory, examples and interactive exercises will lead you to understand “sets” and “relations”. There will be no exam questions directly on the content of the CAI.

2 DISCUSSION OF SELF-ASSESSMENT QUESTIONS, ASSIGNMENT 02

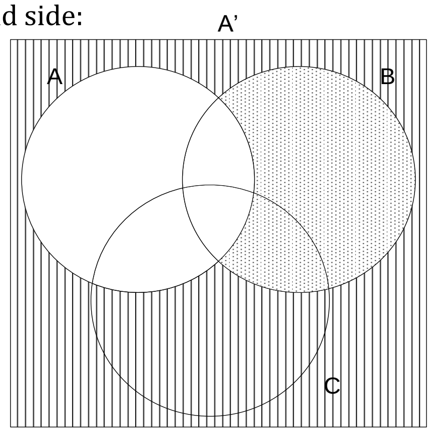
2.1 Assignment 02 Section 1

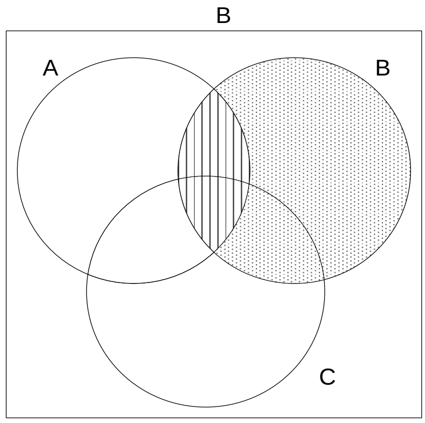
Note: Although the examination paper is an MCQ paper, it is very important to attempt all these self-assessment questions, as you will still have to draw a Venn diagram or truth table etc on rough , or do a proof on rough before you will be able to answer a question in the exam. First attempt the solutions on your own and then compare your solutions to the model solutions to see whether you left out some detail.

Question A

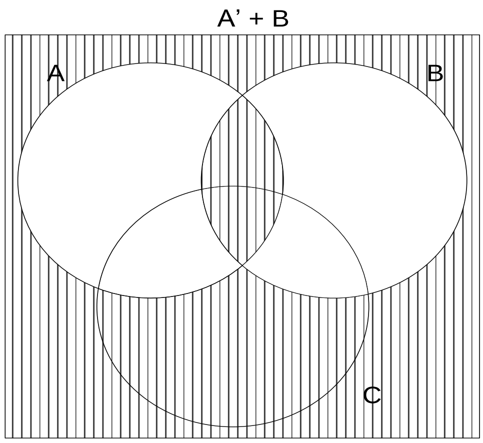
Draw Venn diagrams to show that (A′ + B) = (B ∪ C) ∩ A is not an identity for all subsets A, B and C of U. Draw the diagrams step-by-step.

Left-hand side:





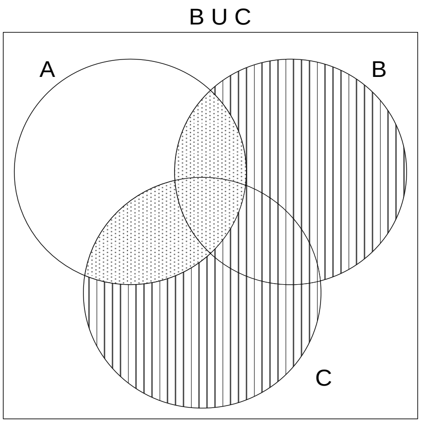
The grey areas are the same for both the above diagrams. In the grey areas the elements live in both A′ and B, i.e. these areas represent A′ ∩ B. For the diagram of A′ + B: Take all the coloured areas in the above diagrams, then remove the grey area (i.e. A′ ∩ B):

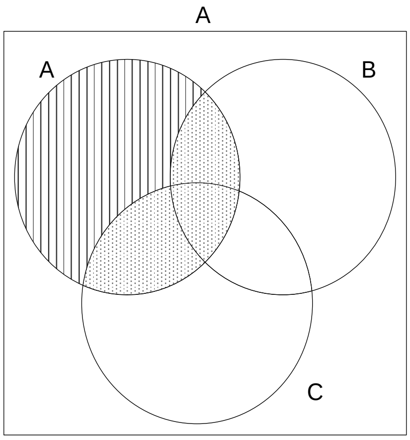


The above should remind you that A′ + B = (A′ ∪ B) − (A′ ∩ B) *(Refer to study guide, activity 4-3).*

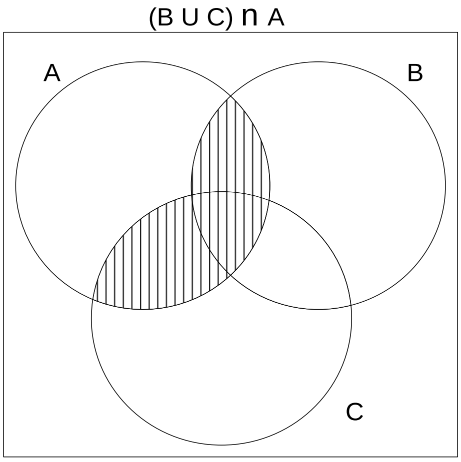
This confirms the definition of symmetric difference: x ∈ A′ + B iff x ∈ A′ or x ∈ B, but not both.

Right-hand side:





The grey areas are the same for both the above diagrams. In the grey areas the elements live in both B ∪ C and A, i.e. these areas represent (B ∪ C) ∩ A.



From the final left-hand and right-hand Venn diagrams it is clear that (A′ + B) and (B ∪ C) ∩ A are not equivalent sets because the coloured areas in the diagrams for (A′ + B) and (B ∪ C) ∩ A are not the same. A counterexample can confirm this result.

(Refer to study guide, pp 48, 51.)

Question B

Provide a counterexample, and then use it to show that (A′ + B) ≠ (B ∪ C) ∩ A.

Solution:

In the solution to question A it is clear that the final diagrams differ. We choose an example in an area where these diagrams differ. It is clear that the two final diagrams differ in the area of set A ∩ C. We name an element living in this area: let **a** be a member of set **A ∩ C only**.

Let U = {a, b, c}, A = {a}, B = {b} and C = {a}. (Note that we use *small* sets in a counterexample.)

We should prove that (A′ + B) ≠ (B ∪ C) ∩ A. We first determine the left-hand side set, then the right-hand side set, then we compare the results to decide whether the statement is an identity.

Left-hand side:

A′ = {b, c} *(b and c live in U but not in A.)*

A′ + B = {b, c} + {b} = {c} *(c lives in A′ or in B, but not in both.)*

Right-hand side:

B ∪ C = {a, b} *(a and b are elements of B or C.)*

(B ∪ C) ∩ A = {a, b} ∩ {a} = {a} *(a is an element of both B ∪ C and A.)*

These results clearly show that (A′ + B) ≠ (B ∪ C) ∩ A.

(Refer to study guide, pp 41 - 43.)

Question C

Using the sets X = {1, 2} and Y = {2, 3}, show that (X ∪ Y) × (X ∩ Y) = (X × (X ∩ Y)) ∪ (Y × (X ∩ Y)).
Does this example show that this is an identity? Justify your answer.

Solution:

Left-hand side:

(X ∪ Y) × (X ∩ Y)
= {1, 2, 3} × {2}
= {(1, 2), (2, 2), (3, 2)} *(Ordered pairs live in a Cartesian product.)*

Right-hand side:

X × (X ∩ Y)
= {1, 2} × {2}
= {(1, 2), (2, 2)}

and

Y × (X ∩ Y)
= {2, 3} × {2}
= {(2, 2), (3, 2)}

Therefore (X × (X ∩ Y)) ∪ (Y × (X ∩ Y)) = {(1, 2), (2, 2), (3, 2)}.

Thus, for the given sets X and Y, (X ∪ Y) × (X ∩ Y) = (X × (X ∩ Y)) ∪ (Y × (X ∩ Y)).

No, one example does *not* show that the expression is an identity for all sets X and Y – there might be other examples that show that it is not an identity. Can you find such an example?

(Refer to study guide, pp 41, 42.)

Question D

Prove that $(X \cup Y) - W' = (X \cap W) \cup (Y \cap W)$ is an identity for all X, Y and $W \subseteq U$.

Solution:

Venn diagrams can confirm that the given sets are equal, but Venn diagrams do not constitute a formal proof.

We provide a proof, starting with the LHS:

$x \in (X \cup Y) - W'$

iff $x \in (X \cup Y)$ and $x \notin W'$

iff ($x \in X$ or $x \in Y$) and $x \in W$ (If x is not an element of W′, then x is an element of W.)

iff ($x \in X$ and $x \in W$) or ($x \in Y$ and $x \in W$)

iff $x \in X \cap W$ or $x \in Y \cap W$

iff $x \in (X \cap W) \cup (Y \cap W)$

Thus $(X \cup Y) - W' = (X \cap W) \cup (Y \cap W)$ for all subsets X, Y and W of U.

(Refer to study guide, pp 41 - 43, comment p 55.)

Question E

Let R be the relation on $\mathbb{Z}$ (the set of integers) defined by $(x, y) \in R$ iff $|y| = |x|$.

a) Provide (i) an ordered pair in R, showing why it belongs to R, and

(ii) an ordered pair not in R, showing why it does not belong to R.

(Use at least one negative integer in one of the pairs.)

b) By doing the appropriate tests, show that R is an equivalence relation.

Solution:

a) We provide two examples:

(i) $(1, -1) \in R$ because $|1| = |-1|$.

$(1, 2) \notin R$ because $|1| \neq |2|$.

b) In order to prove that R is an equivalence relation, we have to prove that **R is reflexive on $\mathbb{Z}$, symmetric and transitive**.

(Refer to study guide, p91.)

Reflexive:

A relation R on A is reflexive on A iff for every $x \in A$, we have $(x, x) \in R$.

For all $x \in \mathbb{Z}$ we have $x = x$,

thus $|x| = |x|$

i.e. $(x, x) \in R$ and

therefore R is reflexive on $\mathbb{Z}$.

(Refer to study guide, p75.)

Symmetry:

A relation R on A is symmetric iff, for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$. This means that for a relation to be symmetric, an ordered pair and its mirror image must both live together in that relation. Goal: to show that whenever $(x, y) \in R$, then (y, x) also belongs to R.

Suppose $(x, y) \in R$,
then $|x| = |y|$
i.e. $|y| = |x|$,
therefore $(y, x) \in R$.
Thus R is symmetric.

(Refer to study guide, p 76.)

Transitivity:

R is transitive iff, for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.
Goal: to show that whenever $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Suppose $(x, y) \in R$,
then $|x| = |y|$ ①
and suppose $(y, z) \in R$,
then $|y| = |z|$. ②
From ① and ② it follows that:
$|x| = |y| = |z|$
i.e. $|x| = |z|$,
therefore $(x, z) \in R$.
Thus R is transitive.

(Refer to study guide, p 77.)

Since R is reflexive, symmetric and transitive it is an equivalence relation.

Question F

Let P and R be relations on A = {1, 2, 3, {1}, {2}, {3}} given by
P = {(1, {1}), ({1}, 1), (1, 2), (2, 1)} and R = {(1, {1}), (3, {3}), (2, {2}), ({2}, {2}), ({3}, 3)}.

a) **Test whether P has the following properties: irreflexive; reflexive; symmetric; antisymmetric; transitive.**

b) **Does R satisfy trichotomy?**

c) **Determine the relations R ○ R and R ○ P (i.e. P; R).**

d) **Give the subset T of R where $(A, B) \in$ T iff $A \subseteq B$.**

e) **Give a partition B of the set A = { 1, 2, 3, {1}, {2}, {3} }.**

Solution:

a)

Irreflexivity:

Is it the case that for all x ∈A, (x, x) ∉ P?

Yes, P is irreflexive: the first and second co-ordinates differ from each other in **each** ordered pair of P.

Reflexivity:

Is it the case that for all x ∈A, (x, x) ∈ P?

No, P is not reflexive, we provide a counterexample: (1,1) ∉ P.

Symmetry:

If (x, y) ∈ P, is it the case that (y, x) ∈ P?

Yes, P is symmetric. The mirror image of each ordered pair in P also lives in P.

Antisymmetry:

If x ≠y and (x, y) ∈ P, is it the case that (y, x) ∉ P?

No, P is not antisymmetric. We provide a counterexample:
1 ≠ {1} and (1, {1}) ∈ P, but it is also the case that ({1}, 1) ∈ P.

Transitivity:

If (x, y) ∈ P and (y, z) ∈ P, is it the case that (x, z) ∈ P?

No, R is not transitive. We provide a counterexample:

(1, {1}) ∈ P, and ({1}, 1) ∈ P, but (1,1) ∉ P.

b) Trichotomy:

Is it the case for all x, y $\in$A, if x ≠ y that (x, y) $\in$ R or (y, x) $\in$ R?

No, R does not satisfy trichotomy, we provide a counterexample:

(2, 3) $\notin$ R and also (3, 2) $\notin$ R.

c) RoR: (x, w) $\in$ R○R iff for some y there exist pairs (x, y) $\in$ R and (y, w) $\in$ R.

R = $\{$(1, {1}), (3, {3}), (2, {2}), ({2}, {2}), ({3}, 3)$\}$

We have that (2, {2}) $\in$ R and ({2}, {2}) $\in$ R, hence (2, {2}) $\in$ R○R.

Also: (3, {3}) $\in$ R and ({3}, 3) $\in$ R, hence (3, 3) $\in$ R○R.

({3}, 3) $\in$ R and (3, {3}) $\in$ R, hence ({3}, {3}) $\in$ R○R.

Thus R○R = { (2, {2}), (3, 3), ({3}, {3}) }.

RoP:

We have that ({1}, 1) $\in$ P and (1, {1}) $\in$ R, hence ({1}, {1}) $\in$ P;R.

Also: (1, 2) $\in$ P and (2, {2}) $\in$ R, hence (1, {2}) $\in$ P;R.

(2, 1) $\in$ P and (1, {1}) $\in$ R, hence (2, {1}) $\in$ P;R.

Thus R○P = P;R = { ({1}, {1}), (1, {2}), (2, {1})}.

d) T: T is a subset of R, so T is also a relation on A. We have (A, B) $\in$ T iff A $\subseteq$ B.

In each ordered pair in T, the first co-ordinate must be a subset of the second co-ordinate.

We have T = { ({2}, {2}) } $\subseteq$ R. (T only has one member because there is only one case in R where the first co-ordinate is a subset of the second co-ordinate: {2} $\subseteq$ {2}.)

(Refer to study guide, pp 75-82.)

e) Give a partition B of the set A = { 1, 2, 3, {1}, {2}, {3} }.

One of the partitions of A is B = {{1, 2}, {3, {1}, {2}, {3}}}.

(The *parts (i.e. elements)* of B are {1, 2} and {3, {1}, {2}, {3}}, and these *parts* of B are *subsets* of A.)

Why is B a partition of A? (Refer to page 94 of the study guide.)

The set B complies with the following:

1. {1, 2} ≠ ∅, and {3, {1}, {2}, {3}} ≠ ∅ (i.e. the parts are *not* empty sets.)
2. {1, 2} $\cap$ {3, {1}, {2}, {3}} = ∅ (i.e. the parts have no common elements.)
3. {1, 2} $\cup$ {3, {1}, {2}, {3}} = {1, 2, 3, {1}, {2}, {3}} = A (i.e. every element of A is in some part.)

Thus B is a partition of A.

Other possible partitions of A:

{{1}, {2, 3}, {{1}, {2}, {3}}}
{{1, 2, 3}, {{1}, {2}, {3}}}
{{1, 2, 3, {1}}, {{2}, {3}}}
{{1, 2, 3, {1}, {2}, {3}}}

Test all the above to make sure that they are partitions of A.

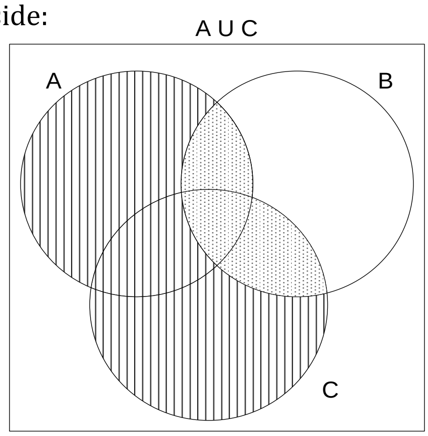
(Refer to study guide, pp 94, 95.)

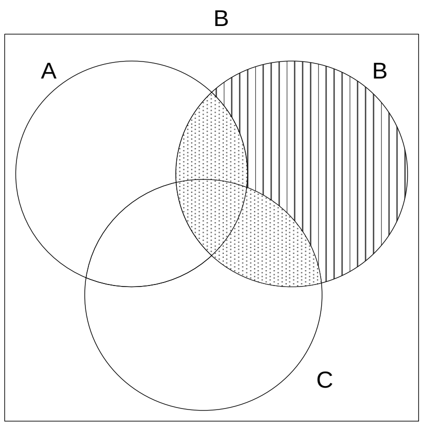
2.2 Assignment 02 Section 2

Question A

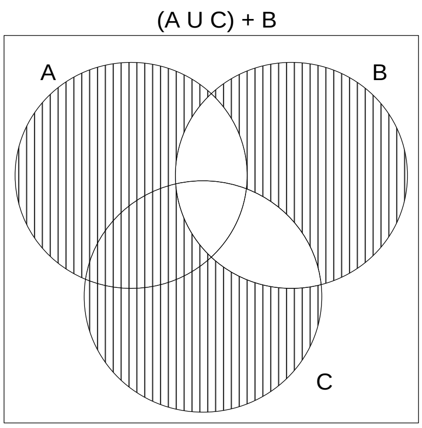
Draw Venn diagrams to show that $(A \cup C) + B = (A \cup B \cup C) - (A \cap B)$ is not an identity for all subsets A, B and C of U. Draw the diagrams step-by-step.

Left-hand side:





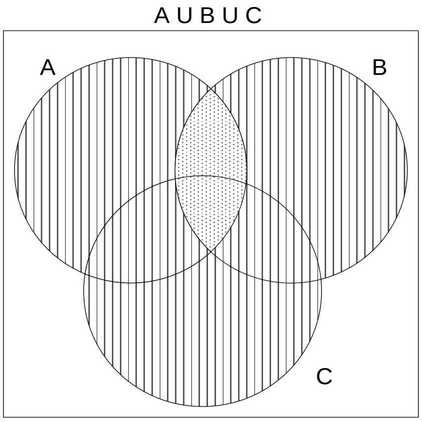
The grey areas are the same for both the above diagrams. In the grey areas the elements live in both $(A \cup C)$ and in B, i.e. these areas represent $(A \cup C) \cap B$. For the diagram of $(A \cup C) + B$: Take all the coloured areas in the above diagrams, then remove the grey area (i.e. $(A \cup C) \cap B$):

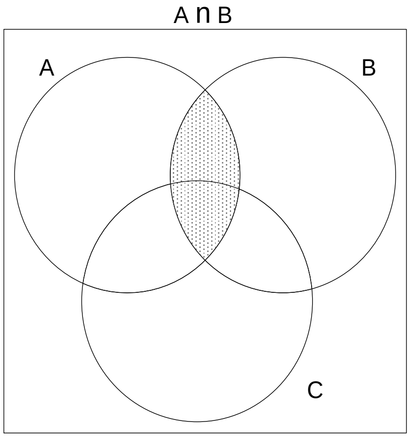


The above should remind you that $(A \cup C) + B = (A \cup C \cup B) - (A \cup C) \cap B$ *(Refer to study guide, activity 4-3)*.

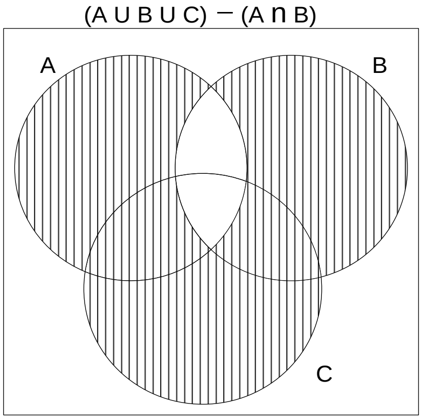
This confirms the definition of symmetric difference: $x \in (A \cup C) + B$ iff $x \in A \cup C$ or $x \in B$, but not both.

Right-hand side:





The grey areas are the same for both the above diagrams. In the grey areas the elements live in both A and B, i.e. these areas represent $A \cap B$. For the diagram of $(A \cup B \cup C) - (A \cap B)$: Take all the coloured areas in the diagram for $A \cup B \cup C$, then remove the grey area (i.e. $(A \cap B)$:



From the final left-hand and right-hand Venn diagrams it is clear that $(A \cup C) + B$ and $(A \cup B \cup C) - (A \cap B)$ are not equivalent sets because the coloured areas in the diagrams for $(A \cup C) + B$ and $(A \cup B \cup C) - (A \cap B)$ are not the same. A counterexample can confirm this result.

(Refer to study guide, pp 48, 51.)

Question B
Provide a counterexample, and then use it to show that (A ∪ C) + B ≠ (A ∪ B ∪ C) − (A ∩ B).

In the solution to A.1 it is clear that the final diagrams differ. We choose an example in an area where these diagrams differ. It is clear that the two final diagrams differ in the area of set B ∩ C. We name an element living in this area: let **3** be a member of set **B ∩ C only**.
(Note that we use *small* sets in a counterexample.)

Let U = {1, 2, 3, 4, 5}, A = {1, 2}, B = {3} and C = {3, 4, 5}

(A ∪ C) + B = {1, 2, 3, 4, 5} + {3}
= {1, 2, 4, 5} *(1, 2, 4, 5 live in A ∪ C or in B, but not in both.)*

(A ∪ B ∪ C) − (A ∩ B) = {1, 2, 3, 4, 5} − { }
= {1, 2, 3, 4, 5} *(1, 2, 3, 4, 5 live in A ∪ B ∪ C, but not in A ∩ B.)*

Clearly (A ∪ C) + B ≠ (A ∪ B ∪ C) − (A ∩ B).

(Refer to study guide, pp 41 - 43.)

Question C
Using the subsets X = {1, 2, 3}, Y = {1} and W = {1, 2} of the universal set U = {1, 2, 3, 4, 5}, show that (X ∪ Y) – W′ = (X ∩ W) ∪ (Y + W).
Does this example show that this is an identity? Justify your answer.

Solution:

Consider (X ∪ Y) – W′ = (X ∩ W) ∪ (Y + W). We first determine the left-hand side set, then the right-hand side set, then we compare the results to decide whether the statement is true for the given sets.

Left-hand side: (X ∪ Y) – W′
= {1, 2, 3} – {3, 4, 5}
= {1, 2} *(1 and 2 live in X ∪ Y, but not in W′.)*

Right-hand side: (X ∩ W) ∪ (Y + W)
= {1, 2} ∪ {2}
= {1, 2} *(1 and 2 live in X ∩ W or in Y + W.)*

so (X ∪ Y) – W′ = (X ∩ W) ∪ (Y + W).

Thus, for the given sets X, Y and W, (X ∪ Y) – W′ = (X ∩ W) ∪ (Y + W).

But one example does *not* show that the expression is an identity for all sets X, Y and W – there might be other examples that show that it is not an identity. Can you find such an example?
(Refer to study guide, pp 41 – 43.)

Question D

Prove that $(X \cup Y) \times W = (X \times W) \cup (Y \times W)$ is an identity for all subsets X, Y and W of a universal set U.

$(a, b) \in (X \cup Y) \times W$
iff $a \in (X \cup Y)$ and $b \in W$
iff ($a \in X$ or $a \in Y$) and $b \in W$
iff ($a \in X$ and $b \in W$) or ($a \in Y$ and $b \in W$)
iff $(a, b) \in (X \times W)$ or $(a, b) \in (Y \times W)$
iff $(a, b) \in (X \times W) \cup (Y \times W)$

(Refer to study guide, pp 41 – 43, 72, 73, comment p 55.)

Question E

Let R be the relation on $\mathbb{Z}$ (the set of integers) defined by
$(x, y) \in R$ iff $y = (k + 1)x$ for some integer $k \geq 0$.

a) Provide (i) an ordered pair in R, showing why it belongs to R, and
(ii) an ordered pair not in R, showing why it does not belong to R.
(Use at least one negative integer in one of the pairs.)

b) By doing the appropriate tests, show that R is a weak partial order.

Solution:

a) Ordered pair in R: (2, 4) (because $4 = (k + 1) \times 2$ with $k = 1$)
Ordered pair not in R: (5, -4) (because $-4 \neq (k + 1) \times 5$ if k is a non-negative integer)

b) *Note: One cannot use a number of examples to show that something is <u>always</u> true. We may, however, show that something is <u>not</u> true by using a counterexample.*

We may call the given relation a **weak partial order** if it is **reflexive** on $\mathbb{Z}$, **antisymmetric**, and **transitive**.

Reflexivity:

Is it the case that for all $x \in \mathbb{Z}$, $(x, x) \in R$?

$x = x$
i.e. $x = (0 + 1)x$
i.e. $x = (k + 1)x$ with $k = 0$.

This means that $(x, x) \in R$ for all $x \in \mathbb{Z}$.

Thus R is reflexive on $\mathbb{Z}$.

Antisymmetry:

If (x, y) $\in$ R and (y, x) $\in$ R, is it the case that x = y?

Assume (x, y) $\in$ R and (y, x) $\in$ R,

then $\quad y = (k + 1)\,x$, ① and

$\quad x = (m + 1)\,y$ ②

Substitute ① in ②:

$$\begin{aligned} x &= (m + 1)\,y \\ &= (m + 1)(k + 1)\,x \quad \text{for some non-negative integers m and k} \\ &= (mk + m + k + 1)\,x \\ &= x \qquad (\text{for } m = k = 0) \end{aligned}$$

Therefore x = y.

Thus R is antisymmetric.

Transitivity:

If (x, y) $\in$ R and (y, z) $\in$ R, is it the case that (x,z) $\in$ R?

Assume (x, y) $\in$ R, i.e. $y = (k + 1)x$ ①

and (y, z) $\in$ R, i.e. $z = (m + 1)y$ ②

Substitute ①in ②:

$$\begin{aligned} z &= (m + 1)\,y \\ &= (m + 1)(k + 1)x \quad \text{for some non-negative integers k and m.} \\ &= (mk + m + k + 1)x \\ &= (t + 1)\,x \qquad \text{for some non-negative integer t } (t = mk + m + k). \end{aligned}$$

Therefore (x, z) $\in$ R.

Thus R is transitive.

R is reflexive on $\mathbb{Z}$, antisymmetric and transitive, thus R is indeed a weak partial order on $\mathbb{Z}$.

(Refer to study guide, pp 84, 85.)

Question F
Let P and R be relations on A = {1, 2, 3, {1}, {2}, {3}} given by
P = {(1, {1}), ({1}, 1), (1, 2), (2, 1)} and R = {(1, {1}), (3, {3}), (2, {2}), ({2}, {2}), ({3}, 3)}.
a) Test whether P has the following properties: irreflexive; reflexive; symmetric; antisymmetric; transitive.
b) Does R satisfy trichotomy?
c) Determine the relations R ○ R and R ○ P (i.e. P; R).
d) Give the subset T of R where $(A, B) \in$ T iff $A \subseteq B$.
e) Give a partition B of the set A = { 1, 2, 3, {1}, {2}, {3} }.

Solution:
a)
Irreflexivity:
Is it the case that for all x $\in$A, (x, x) $\notin$ P?

Yes, P is irreflexive: the first and second co-ordinates differ from each other in **each** ordered pair of P.

Reflexivity:
Is it the case that for all x $\in$A, (x, x) $\in$ P?

No, P is not reflexive, we provide a counterexample: (1,1) $\notin$ P.

Symmetry:
If (x, y) $\in$ P, is it the case that (y, x) $\in$ P?

Yes, P is symmetric. The mirror image of each ordered pair in P also lives in P.

Antisymmetry:
If x ≠y and (x, y) $\in$ P, is it the case that (y, x) $\notin$ P?

No, P is not antisymmetric. We provide a counterexample:
1 ≠ {1} and (1, {1}) $\in$ P, but it is also the case that ({1}, 1) $\in$ P.

Transitivity:
If (x, y) $\in$ P and (y, z) $\in$ P, is it the case that (x, z) $\in$ P?

No, R is not transitive. We provide a counterexample:
(1, {1}) $\in$ P, and ({1}, 1) $\in$ P, but (1,1) $\notin$ P.

b) Trichotomy:
Is it the case for all x, y $\in$A, if x ≠ y that (x, y) $\in$ R or (y, x) $\in$ R?
No, R does not satisfy trichotomy, we provide a counterexample: (2, 3) $\notin$ R and also (3, 2) $\notin$ R.

c) **RoR:** $(x, w) \in R \circ R$ iff for some y there exist pairs $(x, y) \in R$ and $(y, w) \in R$.

$R = \{(1, \{1\}), (3, \{3\}), (2, \{2\}), (\{2\}, \{2\}), (\{3\}, 3)\}$

We have that $(2, \{2\}) \in R$ and $(\{2\}, \{2\}) \in R$, hence $(2, \{2\}) \in R \circ R$.

Also: $(3, \{3\}) \in R$ and $(\{3\},3) \in R$, hence $(3, 3) \in R \circ R$.

$(\{3\}, 3) \in R$ and $(3, \{3\}) \in R$, hence $(\{3\}, \{3\}) \in R \circ R$.

Thus $R \circ R = \{ (2, \{2\}), (3, 3), (\{3\}, \{3\}) \}$.

RoP:
We have that $(\{1\}, 1) \in P$ and $(1, \{1\}) \in R$, hence $(\{1\}, \{1\}) \in P;R$.
Also: $(1, 2) \in P$ and $(2, \{2\}) \in R$, hence $(1, \{2\}) \in P;R$.
$(2, 1) \in P$ and $(1, \{1\}) \in R$, hence $(2, \{1\}) \in P;R$.

Thus $R \circ P = P;R = \{ (\{1\}, \{1\}), (1, \{2\}), (2, \{1\})\}$.

d) **T:** T is a subset of R, so T is also a relation on A. We have $(A, B) \in T$ iff $A \subseteq B$.

In each ordered pair in T, the first co-ordinate must be a subset of the second co-ordinate.

We have $T = \{ (\{2\}, \{2\}) \} \subseteq R$. (T only has one member because there is only one case in R where the first co-ordinate is a subset of the second co-ordinate: $\{2\} \subseteq \{2\}$.)

(Refer to study guide, pp 75-82.)

e) Give a partition B of the set A = { 1, 2, 3, {1}, {2}, {3} }.

One of the partitions of A is B = {{1, 2}, {3, {1}, {2}, {3}}}.
(The *parts (i.e. elements)* of B are {1, 2} and {3, {1}, {2}, {3}}, and these *parts* of B are *subsets* of A.)

Why is B a partition of A? (Refer to page 94 of the study guide.)

The set B complies with the following:
1. {1, 2} ≠ ∅, and {3, {1}, {2}, {3}} ≠ ∅ (i.e. the parts are *not* empty sets.)
2. {1, 2} ∩ {3, {1}, {2}, {3}} = ∅ (i.e. the parts have no common elements.)
3. {1, 2} ∪ {3, {1}, {2}, {3}} = {1, 2, 3, {1}, {2}, {3}} = A (i.e. every element of A is in some part.)

Thus B is a partition of A.

Other possible partitions of A:

{{1}, {2, 3}, {{1}, {2}, {3}}}
{{1, 2, 3}, {{1}, {2}, {3}}}
{{1, 2, 3, {1}}, {{2}, {3}}}
{{1, 2, 3, {1}, {2}, {3}}}

Test all the above to make sure that they are partitions of A.

(Refer to study guide, pp 94, 95.)

3 DISCUSSION OF SELF-ASSESSMENT QUESTIONS, ASSIGNMENT 03

3.1 Assignment 03 – Section A

Question A.1

Let P and R be relations on A = $\{$1, 2, 3, {1}, {2}, {3}$\}$ given by

P = $\{$(1, {1}), ({1}, 1), (1, 2), (2, 1)$\}$ and R = $\{$(1, {1}), (3, {3}), (2, {2}), ({1}, {2}), ({2}, {2}), ({3}, 3)$\}$.

a) Is P functional? Justify your answer.

b) Is P a function from A to A? Justify your answer.

c) Provide the range of R.

d) Is R surjective? Justify your answer.

e) Is R injective? Justify your answer.

Solution:

a) Is P functional? Justify your answer.

A = $\{$1, 2, 3, {1}, {2}, {3}$\}$ and P = $\{$(1, {1}), ({1}, 1), (1, 2), (2, 1)$\}$.

P is not functional since 1 ∈ A appears as first co-ordinate in more than one ordered pair of P:

(**1**, {1}), (**1**, 2) ∈ P.

(Refer to study guide, p 98.)

b) Is P a function from A to A? Justify your answer.

Since P is not functional it is not a function. (Also, dom(P) = {1, {1}, 2} ≠ A.

(Refer to study guide, p 99.)

c) Provide the range of R.

R = $\{$(1, {1}), (3, {3}), (2, {2}), ({1}, {2}), ({2}, {2}), ({3}, 3)$\}$.

The range of R is the **set** containing all the **elements** that appear as second co-ordinates of R:

Ran(R) = {3, {1}, {2}, {3}}.

(Refer to study guide, p 98.)

d) Is R surjective? Justify your answer.

No, ran(R) = {3, {1}, {2}, {3}} ≠ A thus by the definition of surjectivity R is not surjective.

(Refer to study guide, p 105.)

e) Is R injective? Justify your answer.

No, each first co-ordinate does not appear with a unique second co-ordinate in ordered pairs of R:

(2, {2}), ({1}, {2})∈ R which means that {2} appears as second co-ordinate together with two different first co-ordinates 2 and {1} in ordered pairs of R.

(Refer to study guide, p 106.)

Question A.2
Let *f* and *g* be relations on ℤ (the set of integers) defined by

$(x, y) \in f$ iff $y = 30 - x$ and $(x, y) \in g$ iff $y = 31x^2 + 2$.

a) Determine dom(g).
b) Is f injective (one-to-one)? Justify your answer.
c) Is g a surjective function (onto) from Z to Z? Justify your answer.
d) Determine the inverse function f^{-1}.

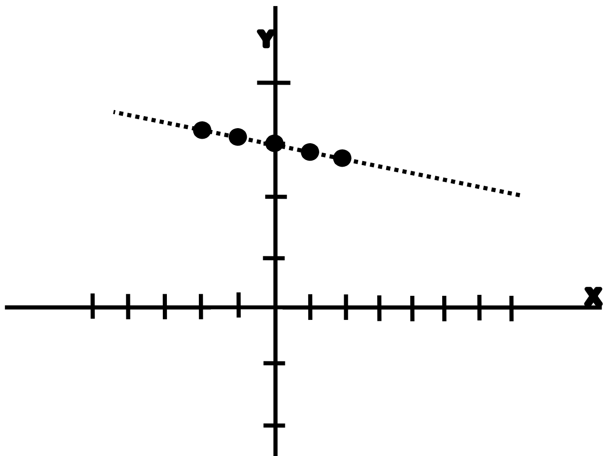
Solution:
Remember, when we speak of a "relation on ℤ", we mean a relation from ℤ to ℤ.
You may wish to make life easier for yourself by roughly sketching the graphs that represent f and g respectively. The graph of f is a broken straight line and g is a broken parabola. These are broken lines because f and g are defined on ℤ. Just by looking at the two graphs should help you to answer some of the questions intuitively. We see that f and g seems to be functional since a vertical line drawn anywhere on the sketches cuts the graphs in just one place. Because any horizontal line cuts the graph of f just once, f appears to be injective. Can you see why g is not injective?

If you don't feel like sketching graphs (or can't remember how to), try the following way of building up a feel for the two relations f and g: Manufacture a few pairs in f and in g according to the instructions that define them. We know that (x, y) belongs to f iff $y = 30 - x$ for some integer x.

Pick your favourite integer x, say $x = 2$ and substitute it in f.
Now $y = 28$, so $(2, 28) \in f$.
And if $x = 0$, then $(0, 30) \in f$.
And if $x = -2$, then $(-2, 32) \in f$.

Similarly, $(x , y) \in g$ iff $y = 31x^2 + 2$.
So if $x = 1$, then $(1, 33) \in g$.
If $x = 0$, then $(0, 2) \in g$,
and if $x = -1$, then $(-1, 33) \in g$. (Draw the graph of g, where $y = 31x^2 + 2$.)

We draw the graph of f, where $y = 30 - x$:



a) Determine dom(g).

Consider g on $\mathbb{Z}$ where $y = 31x^2 + 2$.

$$\begin{aligned}\text{dom}(g) &= \{x \mid \text{for some } y \in \mathbb{Z}, (x, y) \in g\}\\ &= \{x \mid \text{for some } y \in \mathbb{Z}, y = 31x^2 + 2\}\\ &= \{x \mid 31x^2 + 2 \textbf{ is an integer}\}\\ &= \mathbb{Z}\end{aligned}$$

Or we can provide an alternative solution:

dom(g) = $\{x \mid \text{for some } y \in \mathbb{Z}, y = 31x^2 + 2\}$

Because $g \subseteq \mathbb{Z} \times \mathbb{Z}$ by its definition, we know that

dom(g) $\subseteq \mathbb{Z}$. ①

Now take any $x \in \mathbb{Z}$.
For this x there exists a $y \in \mathbb{Z}$, such that $31x^2 + 2$, so we have $(x, 31x^2 + 2) \in g$.
Therefore $x \in$ dom(g).
We proved that if $x \in \mathbb{Z}$ then $x \in$ dom(g), so $\mathbb{Z} \subseteq$ dom (g). ②

From ① and ② it follows that dom(g) = $\mathbb{Z}$.

(Refer to study guide, pp 98, 99.)

b) Is f injective (one-to-one)? Justify your answer.

The easiest test for injectivity is often to test whether: if $f(u) = f(v)$, is it the case that $u = v$?

Consider f on $\mathbb{Z}$ where $y = 30 - x$.

Suppose $f(u) = f(v)$
then $30 - u = 30 - v$
i.e. $u = v$.

Therefore f is injective.

(Refer to study guide, pp 106, 107.)

c) Is g a surjective function (onto) from $\mathbb{Z}$ to $\mathbb{Z}$? Justify your answer.

A function is surjective iff the range of the function is exactly the same set as its codomain.

Consider g on $\mathbb{Z}$ where $y = 31x^2 + 2$.

ran(g) = $\{y \mid \text{ for some } x \in \mathbb{Z}, y = 31x^2 + 2.\}$

Does the range use up the whole codomain? Maybe not ... look at your graph.

Provide a counterexample:
Choose $y = -3$.
There is no $x \in \mathbb{Z}$ such that $31x^2 + 2 = -3$ (i.e. $x^2 = -5/31$)
and so $-3 \notin$ ran(g).

Thus ran(g) $\neq \mathbb{Z}$, and therefore g is **not** surjective.

(Refer to study guide, pp 98, 104 - 106.)

d) Determine the inverse function f^{-1}.

Recall that $f: \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = 30 - x$.

$(y, x) \in f^{-1}$ iff $(x, y) \in f$
iff $y = 30 - x$
iff $x = 30 - y$

So $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is the function defined by $f^{-1}(y) = 30 - y$.

***Note**: f^{-1} is the **function** from $\mathbb{Z}$ to $\mathbb{Z}$ and $f^{-1}(y)$ is the **image** of y under f^{-1}.*

(Refer to study guide, pp 112, 113.)

Question A.3
Use a truth table to determine whether the following proposition is a tautology, a contradiction or neither: $((p \to q) \land (q \to r)) \to (p \lor r)$

Solution :

p	**q**	**r**	**p → q**	**(q → r)**	**(p → q) ∧ (q → r)**	**→**	**(p ∨ r)**
T	T	T	T	T	T	**T**	T
T	T	F	T	F	F	**T**	T
T	F	T	F	T	F	**T**	T
T	F	F	F	T	F	**T**	T

F	T	T	T	T	T	**T**	T
F	T	F	T	F	F	**T**	F
F	F	T	T	T	T	**T**	T
F	F	F	T	T	T	**F**	F

↑

One entry in the final column is F, and the others are all T, thus the proposition is neither a tautology nor a contradiction.

(Refer to study guide, pp 138-144)

Question A.4

Give a counterexample to indicate that the following statement is not true for all $x \in \mathbb{Z}^+$:

If x is even or divisible by 3, then $x^2 + 2x - 3$ is odd.

Solution:

Consider the statement 'If x is even or divisible by 3, then $x^2 + 2x - 3$ is odd'.

We do not substitute x = 1 in the given expression because it is not an even number and is not divisible by 3.

Choose $x = 2$. This means that x is even and $x^2 + 2x - 3 = 4 + 4 - 3 = 5$ is odd.
Choose $x = 3$. In this case, x is divisible by 3 and $x^2 + 2x - 3 = 9 + 6 - 3 = 12$ is not an odd number.

This counterexample proves that the statement is not true for all $x \in \mathbb{Z}^+$.

(Refer to study guide, p 163)

Question A.5
Prove by contradiction (*Reductio ad absurdum*) that for all $n \in \mathbb{Z}$ if $n^2 + 6n$ is odd, then n is odd.

Solution:

Suppose $n^2 + 6n$ is **odd**.
There are two possibilities for n: either n is odd or n is even.
The former is the good possibility, so we try to eliminate the latter.

Suppose n is even, i.e. $n = 2k$ for some integer k. (bad possibility)
Determine $n^2 + 6n$:

$$\begin{aligned} n^2 + 6n &= (2k)^2 + 6(2k) \\ &= 4k^2 + 12k \\ &= 2(2k^2 + 6k) \\ &= 2t \text{ for some integer } t. \end{aligned}$$

This means that $n^2 + 6n$ is an **even** number which is a **contradiction** to our initial supposition, so the bad possibility can be discarded.

Thus we conclude that for all $n \in \mathbb{Z}$, if $n^2 + 6n$ is odd, then n is odd.

(Refer to study guide, p 160)

Question A.6
Prove by contrapositive that for all $n \in \mathbb{Z}$ if $n^2 + 6n$ is odd, then n is odd.

Solution:

Suppose n is **not odd**, i.e. n is even.
(We have to prove that $n^2 + 6n$ is even.)

We supposed n is even, say $n = 2k$ for some integer k.

Determine $n^2 + 6n$:

$$\begin{aligned} n^2 + 6n &= (2k)^2 + 6(2k) \\ &= 4k^2 + 12k \\ &= 2(2k^2 + 6k) \\ &= 2t \text{ for some integer } t. \end{aligned}$$

This means that $n^2 + 6n$ is an **even** number which is what we needed to prove.

Thus we conclude that for all $n \in \mathbb{Z}$, if $n^2 + 6n$ is odd, then n is odd.

(Refer to study guide, pp 160, 161)
Question A.7
Consider the following statement:

If $x^3 = x$ then $x^2 = 1$, for all $x \in \mathbb{Z}$.
If it appears to be true, give a direct proof; if not, give a counterexample.

Solution:

We provide a counterexample to show that the given statement is not true for all $x \in \mathbb{Z}$.
Let x = 0, then
$x^2 = 0 \neq 1$.

(Refer to study guide, p 159)

Question A.8
Give a direct proof to show that the following statement holds for all $n \in \mathbb{Z}$.
If *n* is even, then the product of *n* and its successor is even.

(*Hint:* The successor of an integer *n* is the integer *n* +1.)

Solution:

The successor of n is $n + 1$.
We thus need to show that $n(n + 1)$ is even.

Assume n is even, then it can be written as $n = 2k$ for some $k \in \mathbb{Z}$.

$$
\begin{aligned}
n(n+1) &= 2k(2k+1) \\
&= 2.2k^2 + 2k \\
&= 2(2k^2 + k)
\end{aligned}
$$

Thus $n(n + 1)$ is even.

(Refer to study guide, p 159)

Question A.9
Consider the following statement:
If 3*n* is odd, then *n* is odd.
(i) Write the contrapositive statement.
(ii) Prove the original statement by proving its contrapositive.

Solution:
(i) If n is not odd, then $3n$ is not odd, i.e. if n is even, then $3n$ is even.

(ii) Suppose n is not odd, then it means that n is even.

Thus n can be written in the form $n = 2k$ for some $k \in \mathbb{Z}$.
So $3n = 3(2k)$
$= 6k$
$= 2(3k)$, which means that $3n$ is even, i.e. $3n$ is not odd.

(Refer to study guide, p 161)

Question A.10
Prove by contradiction that, for all $n \in \mathbb{Z}$, if n^2 is odd, then n is odd.

Solution:

Suppose n^2 is odd.

Now there are just two possibilities for n: either n is odd or n is even. The former is the good possibility, so let's eliminate the latter.

Suppose just for the moment that n is not odd, i.e. n is even.
Then $n = 2k$ for some $k \in \mathbb{Z}$.

i.e. $n^2 = 4k^2$
$= 2(2k^2)$.

This means that n^2 is even, contradicting our starting supposition that n^2 is odd.

Thus, if n^2 is odd, then n is odd.

(Refer to study guide, p 160)

3.2 Assignment 03 – Section B

Note : You are welcome to work through the section on vectors, but this section is not examinable. We do give some questions below for students who would like to work through them.

Question B.1 **Alternative 3**

Given vector $u = (1, 2, 3, 4, 5)$ and vector $v = (5, 4, 3, 2, 1)$. According to the definition the sum $u + v$ is determined as follows:

$u + v = (1+5, 2+4, 3+3, 4+2, 5+1) = (6, 6, 6, 6, 6)$.

(Refer to study guide, p 124)

Question B.2 **Alternative 1**

Given vector $u = (1, -3, 5)$ and vector $v = (0, -1, 1)$. According to the definition the dot product $u \cdot v$ is obtained as follows:

$u \cdot v = (1 \cdot 0) + (-3 \cdot -1) + (5 \cdot 1) = 0 + 3 + 5 = 8$.

(Refer to study guide, pp 124, 125)

Question B.3 **Alternative 4**

Given matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 3 & -3 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. The multiplication AB can be computed only if the number of columns in A is equal to the number of rows in B. In this case A has 2 columns and B has 3 rows so the AB multiplication is not possible.

(Refer to study guide, p 131)

Question B.4 **Alternative 4**

Provide the truth values for the compound expression $[(\neg p) \rightarrow (q \vee r)] \rightarrow p$ in the provided table:

Solution:

p	**q**	**r**	**¬ p**	**q ∨ r**	**(¬ p) → (q ∨ r)**	**[(¬ p) → (q ∨ r)] → p**
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	F
F	T	F	T	T	T	F
F	F	T	T	T	T	F
F	F	F	T	F	F	T

(Refer to study guide, pp 138-145)

Question B.5 **Alternative 1**

Draw a truth table for the following expression:

$(p \vee q) \vee \neg [p \vee (q \wedge r)]$

p	**q**	**r**	**$p \vee q$**	**$q \wedge r$**	**$p \vee (q \wedge r)$**	**$\neg[p \vee (q \wedge r)]$**	**$(p \vee q) \vee \neg [p \vee (q \wedge r)]$**
T	T	T	T	T	T	F	**T**
T	T	F	T	F	T	F	**T**
T	F	T	T	F	T	F	**T**
T	F	F	T	F	T	F	**T**
F	T	T	T	T	T	F	**T**
F	T	F	T	F	F	T	**T**
F	F	T	F	F	F	T	**T**
F	F	F	F	F	F	T	**T**

From the truth table it is clear that there are only Ts in the final column, thus the expression is a tautology.

(Refer to study guide, pp 146-148)

Question B.6 **Alternative 2**

Use the property of double negation and De Morgan's properties to rewrite $\neg [(\neg p \vee q) \wedge \neg (q \wedge \neg r)]$ as an equivalent statement that does **not** have the not symbol ($\neg$) outside parentheses.

$\neg [(\neg p \vee q) \wedge \neg (q \wedge \neg r)]$
$\equiv \neg (\neg p \vee q) \vee \neg\neg (q \wedge \neg r)$ De Morgan
$\equiv (\neg\neg p \wedge \neg q) \vee \neg\neg (q \wedge \neg r)$ De Morgan
$\equiv (p \wedge \neg q) \vee (q \wedge \neg r)$ Double negation

(Refer to study guide, pp 146 - 148)

Question B.7 **Alternative 2**

Let *p* denote the statement (proposition) '$x^2 - 9 < 0$' and *q* denote the statement (proposition) '$y^2 - 4 < 0$'. Use *p* and *q* along with connectives $\wedge$, $\vee$ or $\neg$ to write down the following statement symbolically (in propositional logic):

'$x^2 - 9$ is negative if and only if $y^2 - 4$ is negative.'

$p \leftrightarrow q$
$\equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$
$\equiv (\neg p \vee q) \wedge (p \vee \neg q)$

(Refer to study guide, pp 146 - 148)

Question B.8 **Alternative 2**

Which one of the following quantified statements is NOT TRUE?

1.	There exists an integer x such that $x > 1$ and $x < 4$ if $x^2 - 5x + 6 = 0$.	TRUE
2.	There exists an integer x such that $x < 0$ and $x^2 + 4 = 0$.	FALSE
3.	There exists an integer $x \in \{10, 100, 1000\}$ such that $x^2 \in \{10, 100, 1000\}$.	TRUE
4.	For all integers $x \in \{10, 100, 1000\}$, x^2 is an even number.	TRUE

Alternative 1:
If $x^2 - 5x + 6 = 0$
then $(x - 3)(x - 2) = 0$
i.e. $x = 3$ or $x = 2$
Thus it is true that the value of x lies between 1 and 4, i.e. $1 < x < 4$.

Alternative 2 is false since there exists no integer x such that $x^2 + 4 = 0$. It is impossible that $x^2 = -4$ since x^2 must be a positive integer.

Alternative 3:
If $x = 10$ then $x^2 = 100$.

Alternative 4:
$(10)^2 = 100$; $(100)^2 = 10000$; $(1000)^2 = 1000000$. All these numbers are even.

(Refer to study guide, pp 152-158)

Question B.9 **Alternative 3**

Consider the following expression:
$[(p \wedge q) \vee \neg r] \rightarrow \neg [(q \vee \neg r) \rightarrow (\neg p \wedge r)]$

The converse of the expression above is obtained as follows, where $[(p \wedge q) \vee \neg r]$ is the hypothesis and $\neg [(q \vee \neg r) \rightarrow (\neg p \wedge r)]$ is the conclusion : $\neg [(q \vee \neg r) \rightarrow (\neg p \wedge r)] \rightarrow [(p \wedge q) \vee \neg r]$.

(Refer to study guide, p 161)

Question B.10 **Alternative 1**

Consider the following expression where $x \in \mathbb{Z}$:

If $x^2 + 2x - 3 = 0$, then $x < 2$.

The contrapositive of the expression above is obtained as follows, where $P(x)$ is the hypothesis and stands for "$x^2 + 2x - 3 = 0$", and $Q(x)$ is the conclusion and stands for "$x < 2$".

If $\neg Q(x)$ then $\neg P(x)$,
i.e. if $x \geq 2$, then $x^2 + 2x - 3 \neq 0$.

(Refer to study guide, p 160)

3.3 Assignment 03 – Section C

Question C.1 **Alternative 4**

Given vector $u = (3, 0, -2, 1, 4)$ and vector $v = (1, -3, 2, 5)$.
The sum $u + v$ can be computed only if u and v have the same number of co-ordinates. In this case u has 5 co-ordinates whereas v has only 4 co-ordinates, thus the sum cannot be computed.

(Refer to study guide, p 124)

Question C.2 **Alternative 3**

Given vector $u = (-3, 4, 2)$ and vector $v = (2, -1, 5)$. According to the definition the dot product $u \cdot v$ is obtained as follows:

$u \cdot v = (-3 \cdot 2) + (4 \cdot -1) + (2 \cdot 5) = (-6) + (-4) + 10 = 0.$

(Refer to study guide, pp 124, 125)

Question C.3 **Alternative 1**

Given matrix $A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -5 & 1 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$. According to the definition the multiplication $A \cdot B$ is obtained as follows:

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (1 \cdot 4) + (0 \cdot 1) + (-3 \cdot 0) \\ (2 \cdot 4) + (-5 \cdot 1) + (1 \cdot 0) \end{bmatrix} = \begin{bmatrix} 4 + 0 + 0 \\ 8 - 5 + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

(Refer to study guide, p 131)

Question C.4 **Alternative 2**

Give the truth values of the last column of the following truth table for the given compound expression:

p	q	r	$p \vee [(\neg p) \wedge (q \rightarrow r)]$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Solution:

p	q	r	$\neg p$	$q \rightarrow r$	$(\neg p) \wedge (q \rightarrow r)$	$p \vee [(\neg p) \wedge (q \rightarrow r)]$
T	T	T	F	T	F	T
T	T	F	F	F	F	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	T	T

(Refer to study guide, pp 138-145)

Question C.5 **Alternative 3**

Compile a truth table for the following expression:

$[(p \rightarrow q) \wedge r] \leftrightarrow [(q \rightarrow p) \wedge r]$

p	q	r	$p \rightarrow q$	$(p \rightarrow q) \wedge r$	$\leftrightarrow$	$q \rightarrow p$	$(q \rightarrow p) \wedge r$
T	T	T	T	T	**T**	T	T
T	T	F	T	F	**T**	T	F
T	F	T	F	F	**F**	T	T
T	F	F	F	F	**T**	T	F
F	T	T	T	T	**F**	F	F
F	T	F	T	F	**T**	F	F
F	F	T	T	T	**T**	T	T
F	F	F	T	F	**T**	T	F

Considering the final column containing Ts and Fs, it is clear that the expression is neither a tautology nor a contradiction.

(Refer to study guide, pp 138-145)

Question C.6 **Alternative 4**

Use the law of double negation and De Morgan's laws to rewrite the following expression as an equivalent statement that does **not** have the not symbol ($\neg$) outside parentheses.

$\neg [(p \wedge \neg q \wedge \neg r) \vee (\neg p \vee \neg r)]$

$\equiv \neg (p \wedge \neg q \wedge \neg r) \wedge \neg (\neg p \vee \neg r)$	De Morgan's law
$\equiv (\neg p \vee \neg\neg q \vee \neg\neg r) \wedge (\neg\neg p \wedge \neg\neg r)$	De Morgan's law
$\equiv (\neg p \vee q \vee r) \wedge (p \wedge r)$	Double negation

(Refer to study guide, pp 146 - 148)

Question C.7 **Alternative 2**

Which of the following pairs of propositions are equivalent?

A. $\neg [(p \vee q) \wedge \neg r]$ and $(\neg p \wedge \neg q) \vee r$.

B. $(p \vee q) \rightarrow r$ and $\neg (\neg p \wedge \neg q) \vee r$.

C. $\neg [(p \vee q) \wedge \neg r]$ and $(\neg p \vee r) \wedge (\neg q \vee r)$.

We test these as follows:

A. $\neg [(p \vee q) \wedge \neg r]$ and $[(\neg p \wedge \neg q) \vee r]$ are equivalent.

$\neg [(p \vee q) \wedge \neg r]$

$\equiv \neg (p \vee q) \vee \neg\neg r$	De Morgan's law
$\equiv (\neg p \wedge \neg q) \vee \neg\neg r$	De Morgan's law
$\equiv (\neg p \wedge \neg q) \vee r$	Double negation

B. $[(p \vee q) \rightarrow r]$ and $[\neg (\neg p \wedge \neg q) \vee r]$ are not equivalent.

$\neg (\neg p \wedge \neg q) \vee r$

$\equiv (\neg p \wedge \neg q) \rightarrow r$	Refer to comment p 158.
$\equiv \neg (p \vee q) \rightarrow r$	De Morgan's law

C. $\neg [(p \vee q) \wedge \neg r]$ and $(\neg p \vee r) \wedge (\neg q \vee r)$ are equivalent.

$\neg [(p \vee q) \wedge \neg r]$

$\equiv \neg (p \vee q) \vee \neg\neg r$	De Morgan's law
$\equiv (\neg p \wedge \neg q) \vee \neg\neg r$	De Morgan's law
$\equiv (\neg p \wedge \neg q) \vee r$	Double negation
$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$	Distributive law

(Refer to study guide, pp 146 - 148)

Question C.8 **Alternative 3**

Let $P(x)$ denote '$x^2 - 3x + 2 < 0$'. Write the following statement using some quantifiers and predicates, as well as a negation: 'It is not the case that $x^2 - 3x + 2 < 0$ for all integers x.'

$\neg [\forall x \in \mathbb{Z}, P(x)]$

$\equiv \exists x \in \mathbb{Z}, \neg P(x)$

(Refer to study guide, p 157)

Question C.9 **Alternative 1**

Consider the following expression:

$\neg [(p \wedge q) \rightarrow (q \vee \neg r)] \rightarrow [(\neg p \wedge r) \vee \neg q]$

The contrapositive of the expression above is obtained as follows, where $\neg [(p \wedge q) \rightarrow (q \vee \neg r)]$ is the hypothesis and $[(\neg p \wedge r) \vee \neg q]$ is the conclusion:

$\neg [(\neg p \wedge r) \vee \neg q] \rightarrow \neg \neg [(p \wedge q) \rightarrow (q \vee \neg r)]$

$\equiv \neg [(\neg p \wedge r) \vee \neg q] \rightarrow [(p \wedge q) \rightarrow (q \vee \neg r)]$ Double negation

(Refer to study guide, p 160)

Question C.10 **Alternative 2**

Consider the following expression where $x \in \mathbb{Z}$:

If $x > 0$, then $x^2 - 4x - 5 \leq 0$.

The converse of the expression above is obtained as follows, where '$x > 0$' is the hypothesis and '$x^2 - 4x - 5 \leq 0$' is the conclusion.

If $x^2 - 4x - 5 \leq 0$, then $x > 0$.

(Refer to study guide, p 161)

4 EXTRA EXAMPLES & SOLUTIONS

Question 1 *(Refer to study guide, p 139)*

Write the English sentence 'If the wind is blowing it will bring wind or rain' in symbolic logic notation.
Use the letter *t* to represent the statement 'the wind is blowing', the letter *d* for 'it will bring wind' and the letter *h* for 'it will bring rain'.

The sentence is written in symbolic logic notation as follows:

$t \rightarrow (d \vee h)$

Question 2 *(Refer to study guide, pp 147, 148)*

Use the double negation property and De Morgan's laws to rewrite the following expression as an equivalent statement that does not have the not symbol (¬) outside parentheses.

$\neg (q \wedge (\neg q \vee p))$

The not symbol ¬ is taken into the parenthesis step by step:

$\neg (q \wedge (\neg q \vee p))$	
$= \neg q \vee \neg (\neg q \vee p)$	De Morgan's law
$= \neg q \vee (\neg \neg q \wedge \neg p)$	De Morgan's law
$= \neg q \vee (q \wedge \neg p)$	Double negation

Question 3 *(Refer to study guide, study unit 9)*

Use a truth table to determine whether the compound statement $(\neg p \vee q) \wedge [\neg(p \rightarrow q)]$ is a tautology, a contradiction or neither.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$(\neg p \vee q) \wedge [\neg(p \rightarrow q)]$
T	T	F	T	T	F	**F**
T	F	F	F	F	T	**F**
F	T	T	T	T	F	**F**
F	F	T	T	T	F	**F**

↑

The last column represents the sentence. Since **all** its values are **false**, the sentence is a **contradiction**.

Question 4 *(Refer to study guide, pp 157, 158)*

Let D = {1, 2, 4}. Write the negation of the following statement then decide which is true, the original statement or the negation.

$$\forall x \in D, 4x + 1 \leq 16$$

The negation of the given statement is: $\exists x \in D$, $4x + 1 > 16$.

x	$4x + 1$
1	5
2	9
4	17

The element x = 4 is the only element of D for which $4x + 1 > 16$ is true.
Thus there exists an $x \in D$ such that $4x + 1 > 16$.
Hence the negation is true.

(The original statement is not true since $4x + 1 \leq 16$ is not true for all $x \in D$.)

Question 5 *(Refer to study guide, pp 157, 158)*

Let $P(x)$ denote '$x^2 = 9$'. Write the following statement using $P(x)$, a quantifier and a negation: 'There exists an integer x, such that $x^2 \neq 9$.'

Let $P(x)$ denote '$x^2 = 9$'.
There exists an integer x such that $x^2 \neq 9$, i.e. $\exists x \in \mathcal{Z}, \neg P(x)$.

Question 6 *(Refer to study guide, pp 147, 148, 152, 153, 157, 158)*

Write the negation of the following statement:

$$\forall x \in \mathcal{Z}, \text{ if } 2x^2 = 8 \text{ then } x \geq 0$$

Let $P(x)$ denote $2x^2 = 8$ and $Q(x)$ denote $x \geq 0$.
The statement $\forall x \in \mathcal{Z}$, if $2x^2 = 8$, then $x \geq 0$ can be written as $\forall x \in \mathcal{Z}, P(x) \longrightarrow Q(x)$.

The negation of the above statement is written as follows:

$\neg (\forall x \in \mathcal{Z}, P(x) \longrightarrow Q(x))$	
$\equiv \neg(\forall x \in \mathcal{Z}, \neg P(x) \vee Q(x))$	study guide, middle p 158
$\equiv \exists x \in \mathcal{Z}, \neg(\neg P(x) \vee Q(x))$	
$\equiv \exists x \in \mathcal{Z}, \neg\neg P(x) \wedge \neg Q(x)$	De Morgan's law
$\equiv \exists x \in \mathcal{Z}, P(x) \wedge \neg Q(x)$	Double negation
$\equiv \exists x \in \mathcal{Z}, (2x^2 = 8) \wedge (\neg (x \geq 0))$	
$\equiv \exists x \in \mathcal{Z}, (2x^2 = 8) \wedge (x < 0)$	This statement is true but the original statement is not true.

Question 7 *(Refer to study guide, pp 160 - 161)*

a) Give a proof by contrapositive that for any integer *n*, if 5n^2 is odd then *n* is odd. (p → q)

We have to prove that the contrapositive statement is true,
i.e. if *n* is not odd, then 5n^2 is not odd. ($\neg q \rightarrow \neg p$)

Suppose *n* is even, then *n* can be written as $n = 2k$, where k is some integer.
Then $5n^2 = 5(2k)^2 = 5(4k^2)$.
This can be written as $2(10k^2)$, i.e. $5n^2$ is even.

Thus $5n^2$ is not odd, and we are done.

b) Prove by contradiction that for any integer *n*, if $n^2 + 2n$ is even then *n* is even.

Suppose $n^2 + 2n$ is even.
Now there are just two possibilities for n: either *n* is odd or *n* is even.
The latter is the good possibility, so we eliminate it.
Suppose *n* is odd. This is our **questionable supposition**.
Then *n* can be written as $n = 2k + 1$, for some integer k.

$$\begin{aligned} \text{So } n^2 + 2n &= (2k+1)^2 + 2(2k+1) \\ &= 4k^2 + 4k + 1 + 4k + 2 \\ &= 4k^2 + 8k + 3 \\ &= (4k^2 + 8k + 2) + 1 \\ &= 2(2k^2 + 4k + 1) + 1 \end{aligned}$$

This means that $n^2 + 2n$ is odd, but this contradicts our initial supposition.

Our **questionable supposition** is thus **wrong**, meaning that n must be even if $n^2 + 2n$ is even.

When you solve this type of problem, remember that there are convenient notations to express a description in mathematical terms. **For example, an even number can be expressed as 2k, an odd number as 2k + 1, a multiple of three as 3k, where k ∈ ℤ. Two consecutive numbers can be written as k and k + 1 where k ∈ ℤ.**

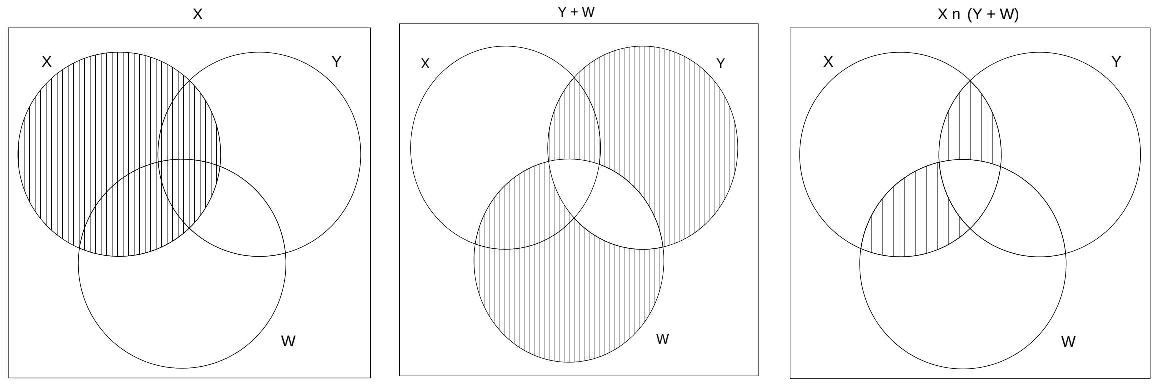
Question 8 *(Refer to study guide, pp 50 - 61)*

a) Use Venn diagrams to investigate whether, for all subsets *X*, *Y* and *W* of *U*,

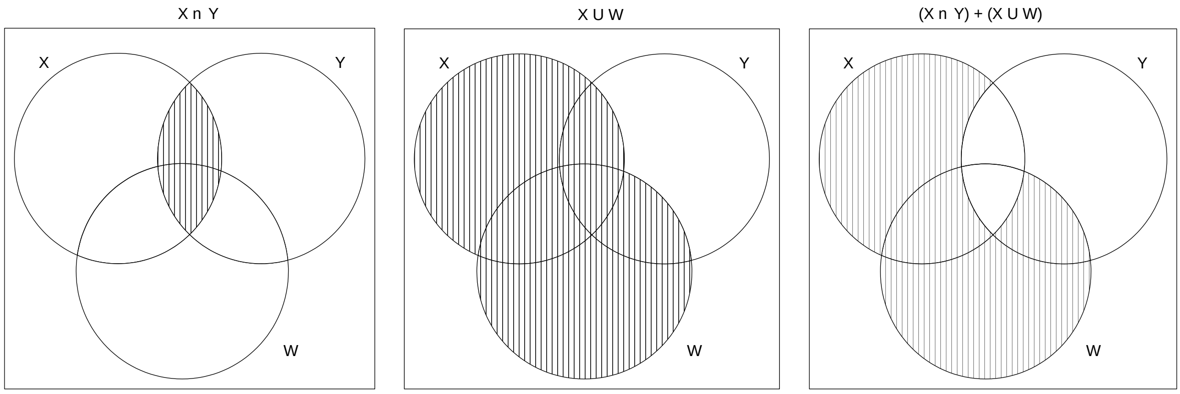
$$\boldsymbol{X \cap (Y + W) = (X \cap Y) + (X \cup W)}$$

If it appears to be an identity, give a proof; if not, give a counterexample.

Left-hand side:



Right-hand side:



The final diagrams differ, so $X \cap (Y + W) = (X \cap Y) + (X \cup W)$ is not an identity.

We need a counterexample. That is, we want an example with sets that make the left-hand side different from the right-hand side. Choose (a) set(s) that has/have some member(s) in the region(s) where the final diagrams differ. We choose 0 as a member of <u>W only</u>.

Let $X = \{1, 2, 3\}$, $Y = \{2\}$ and $W = \{0, 2\}$.

Left-hand side:

$X \cap (Y + W) = \{1, 2, 3\} \cap \{0\} = \{\}$

Right-hand side:

$(X \cap Y) + (X \cup W) = \{2\} + \{0, 1, 2, 3\} = \{0, 1, 3\}$

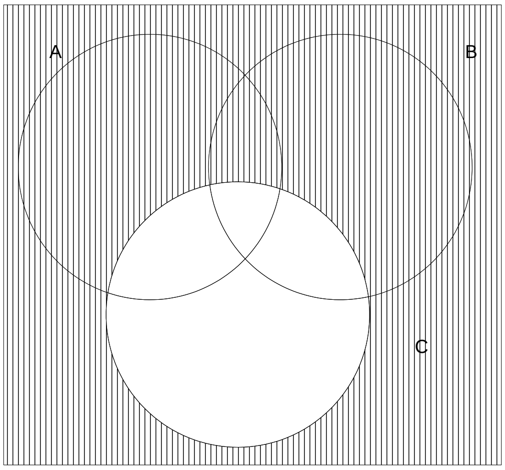
b) Use Venn diagrams to investigate whether or not, for all subsets A, B and C of a universal set U

$$A \cup (B \cap C') = (A + B) - C.$$

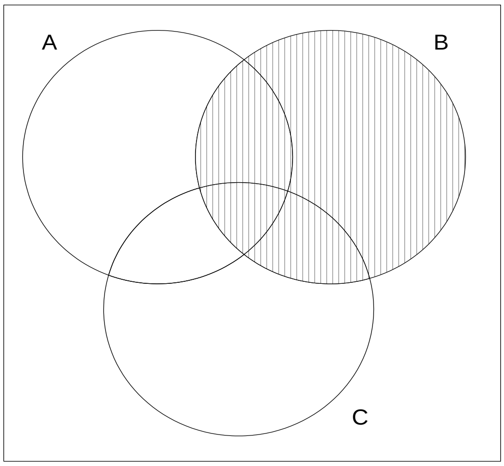
If the statement appears to hold, give a proof; if not, give a counterexample.

Left-hand side:

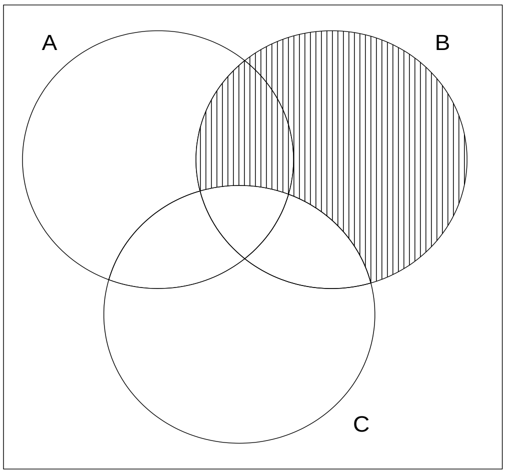
C'



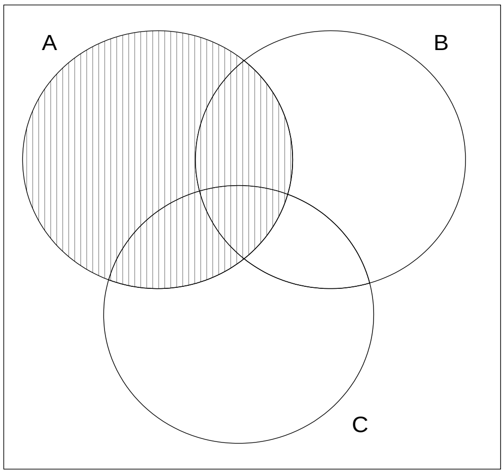
B



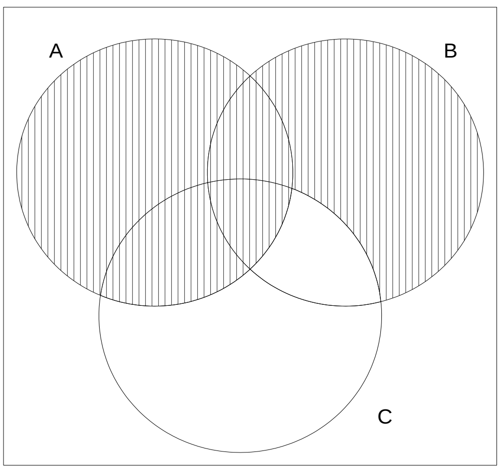
B n C'



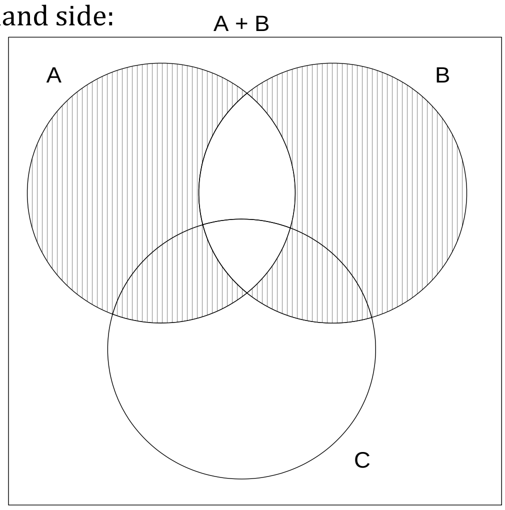
A

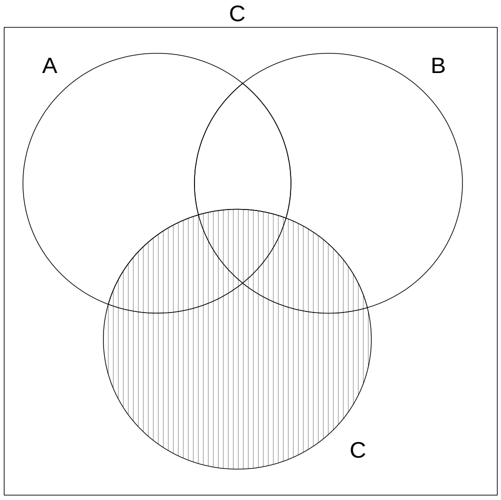


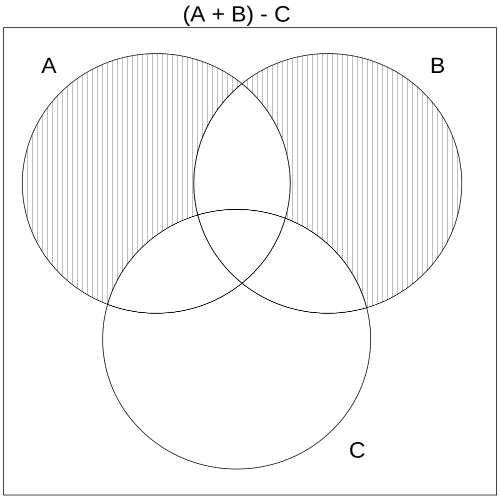
A U (B n C')



Right-hand side:







From the Venn diagrams it is clear that A ∪ (B ∩ C′) = (A + B) − C is not an identity. Hence we provide a counterexample. We choose an element in an area where the final two diagrams differ. For example, the area A ∩ B ∩ C differs in the diagrams for A ∪ (B ∩ C′) and (A + B) – C, so we choose an element to live in this area:

Choose $a \in$ A and $a \in$ B and $a \in$ C. Let U = {a, b, c}, A = {a}, B = {a, b} and C = {a}.

Consider A ∪ (B ∩ C′) = (A + B) – C:

Left-hand side:

C′ = {b, c} *(b and c are members of U but not of C.)*

B ∩ C′ = {a, b} ∩ {b, c}= {b} *(b is a member of B and C′.)*

A ∪ (B ∩ C′) = {a} ∪ {b} = {a, b} *(a and b are members of A or (B ∩ C′).)*

Right-hand side:

A + B = {b} *(b is a member of A or B, but not both.)*

(A + B) − C = {b} − {a} = {b} *(b is a member of* A + B *but not of C.)*

The example supports the result of the Venn diagrams.

Question 9 *(Refer to study guide, pp 42, 43, 55, 56, 73)*

a) Prove that for all subsets X, Y and W of U, $X \cap (Y + W) = (X \cap Y) + (X \cap W)$.

We do a single bi-directional proof with 'iff':

x $\in X \cap (Y + W)$

iff x $\in X$ and x $\in (Y + W)$

iff x $\in X$ and (x $\in Y$ or x $\in W$, but not both)

iff (x $\in X$ and x $\in Y$) or (x $\in X$ and x $\in W$), but not both

iff x $\in (X \cap Y)$ or x $\in (X \cap W)$, but not both

iff x $\in (X \cap Y) + (X \cap W)$

b) Determine whether, for all sets X, Y and W, $(X - Y) \times W \subseteq (X \times W) - (Y \times W)$.

Suppose $(p, q) \in (X - Y) \times W$,

then $p \in (X - Y)$ and $q \in W$

i.e. ($p \in X$ and $p \notin Y$) and $q \in W$

i.e. ($p \in X$ and $q \in W$) and ($p \notin Y$ and $q \in W$)

i.e. $(p, q) \in (X \times W)$ and $(p, q) \notin (Y \times W)$

Thus $(p, q) \in (X \times W) - (Y \times W)$.

We proved that $(X - Y) \times W \subseteq (X \times W) - (Y \times W)$.

Question 10 *(Refer to study guide, pp 63 - 66)*

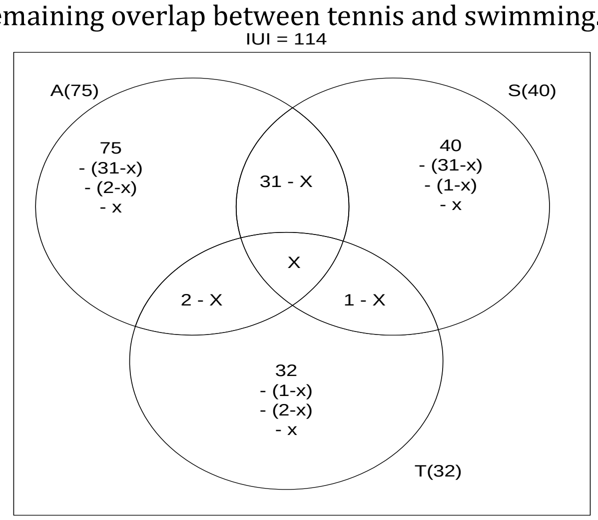
Pofadder High is setting up its summer sports programme, comprising tennis, swimming and athletics. Of the 114 pupils, 75 are involved in athletics, 40 swim, and 32 play tennis; 1 plays tennis and swims, 31 combine swimming with athletics and 2 do both tennis and athletics.

a) How many pupils take part in all three sports?

Let A be the set of athletes, T the set of tennis players, and S the set of swimmers.

$|U| = 114$; $|A| = 75$; $|S| = 40$; $|T| = 32$; $|T \cap S| = 1$; $|S \cap A| = 31$, $|T \cap A| = 2$, $|T \cap S \cap A| = x$.

Start filling in from the triple overlap in the centre: x do/does all three. Now 1 plays tennis and swims, but x of them is/are already positioned in the triple overlap. So $1 - x$ must be placed in the remaining overlap between tennis and swimming. And so on.



How many of the 114 are placed in the middle?

$(75 - (31 - x) - (2 - x) - x) + (31 - x) + (40 - (31 - x) - (1 - x) - x) + (1 - x) + (32 - (1 - x) - (2 - x) - x) + x + (2 - x) = 114$
i.e. $113 + x = 11$, i.e. $x = 1$
So 1 takes part in all three sports.

b) How many take part in tennis, but not athletics?

$(1 - x) + (32 - (1 - x) - (2 - x) – x) = 30$
So 30 take part in tennis but not athletics.

Question 11 *(Refer to study guide, pp 98 – 99, 105 – 106, 108, 112)*
Let f be the relation on the set of integers $\mathbb{Z}$ such that (x, y) ∈ f iff y = 3x - 1
and let g be the relation on $\mathbb{Z}$ such that (x, y) ∈ g iff y = 3 – x.

a) Prove that f is a function from $\mathbb{Z}$ to $\mathbb{Z}$, i.e prove that f is functional and that dom(f) = $\mathbb{Z}$.

<u>Is f functional</u>?
Suppose $(x, y) \in f$ and $(x, z) \in f$,
then $y = 3x – 1$ and $z = 3x – 1$
i.e. $y = 3x – 1 = z$
i.e. $y = z$.
Thus f is functional.

<u>Is dom (f) = $\mathbb{Z}$</u>?
dom (f) = {x| for some y ∈ $\mathbb{Z}$, (x, y) ∈ f }
= {x| for some y ∈ $\mathbb{Z}$, y = 3x – 1}
= {x| 3x – 1 is an integer}
= $\mathbb{Z}$

Because f is functional and dom (f) = $\mathbb{Z}$, f is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

b) Although f and g are both functions from $\mathbb{Z}$ to $\mathbb{Z}$, only one of them is bijective. Name the function that is not bijective and show your reasoning by doing two tests on it.

We show that f is not bijective because it is injective (one-to-one), but not surjective (onto).
<u>Is f injective?</u>

Suppose f(u) = f(v) for some u, v ∈ $\mathbb{Z}$,
then $3u – 1 = 3v – 1$
i.e. $u = v$.
Thus f is injective

Is f surjective?

Consider the following counterexample:
Choose y = 3.
There is no $x \in \mathbb{Z}$ such that f(x) = y,
i.e. such that 3x – 1 = 3. (In this case, $x = 4/3 \notin \mathbb{Z}$)
Hence $3 \notin$ ran(f) and thus ran(f) $\neq \mathbb{Z}$.
Thus f is not surjective.

c) Determine the inverse of the bijective function.

g is bijective. We determine the inverse function g^{-1}.

$(y, x) \in g^{-1}$ iff $(x, y) \in g$
iff $y = 3 - x$
iff $x = 3 - y$

Hence g^{-1}: $\mathbb{Z} \times \mathbb{Z}$ is defined by $g^{-1}(y) = 3 - y = g(y)$ ($g^{-1}(y)$, interestingly, is the same as g(y)).

d) Determine the composition function f o g.

f o g(x) = f (g(x))
= f(3 – x)
= 3 (3 – x) – 1
= 8 – 3x

Hence f o g: $\mathbb{Z} \times \mathbb{Z}$ is defined by f o g(x) = 8 – 3x.

Question 12 *(Refer to study guide, pp 70, 84 – 85)*

R is the relation on $\mathbb{Z}$ (the set of all integers) such that

$(x, y) \in$ R iff *m*x = y for some positive integer m.

a) Write down two ordered pairs in R followed by two ordered pairs not in R.

$(3, 12) \in R$ and $(5, 10) \in R$; $(3, 4) \notin R$ and $(-6, -2) \notin R$.

b) Test whether R is a weak partial order on $\mathbb{Z}$.

If $(x, y) \in R$, it means that y = *m*x, i.e. y is a multiple of x.
To be a weak partial order, R should be **reflexive on $\mathbb{Z}$**, **antisymmetric**, and **transitive**.

Reflexivity: *Is it true that for all $x \in \mathbb{Z}$, $(x, x) \in R$?*
Well, x = x
i.e. $1 \cdot x = x$.
Hence $(x, x) \in R$.

Thus R is reflexive on $\mathbb{Z}$.

Antisymmetry: *Suppose $(x, y) \in R$ and $x \neq y$, does it follow that $(y, x) \notin R$?*
Suppose $(x, y) \in R$ and $x \neq y$
then $m \cdot x = y$ for some $m \in \mathbb{Z}^+$.
i.e. $x = (1/m) \cdot y$, but $1/m \notin \mathbb{Z}^+$.
Hence $(y, x) \notin R$.

Thus R is antisymmetric.

Transitivity: *If $(x, y) \in R$ and $(y, z) \in R$, does it follow that $(x, z) \in R$?*
Suppose $(x, y) \in R$ and $(y, z) \in R$,
then $m \cdot x = y$ for some $m \in \mathbb{Z}^+$ and $k \cdot y = z$ for some $k \in \mathbb{Z}^+$.
Now we substitute y into $k \cdot y = z$ to get $k \cdot (m \cdot x) = z$.
Then $(k \cdot m) \cdot x = z$, with $k \cdot m \in \mathbb{Z}^+$.
Hence $(x, z) \in R$.

Thus R is transitive.

It follows that R is a weak partial order because R is reflexive, antisymmetric and transitive.
(Note: We do not have investigate whether this relation is trichotomous, but we include it for interest's sake.)

Does R satisfy trichotomy? *Is it true that for all $x, y \in \mathbb{Z}^+$ such that $x \neq y$, $(x, y) \in R$ or $(y, x) \in R$?*
No, R does not satisfy trichotomy.
Counterexample:
$(3, 4) \notin R$ and $(4, 3) \notin R$.

So R does not satisfy trichotomy.

Question 13: *(Refer to study guide, pp 90 - 92)*
R is the relation on $\mathbb{Z}$ defined by
$(x, y) \in R$ iff $y - x$ is even.
Prove that R is an equivalence relation and show all its equivalence classes.

In order to be an equivalence relation, R will need to be reflexive on $\mathbb{Z}$, symmetric and transitive.

Because $y - x$ is even, i.e. a multiple of two, we can describe R by the formula
$y - x = 2k$ where $k \in \mathbb{Z}$.

Reflexivity:
What is our goal? For every $x \in \mathbb{Z}$, it is true that $(x, x) \in R$? i.e. Is $x - x = 2k$ for all $x \in \mathbb{Z}$?
For all $x \in \mathbb{Z}$,
$x - x = 0$
$= 2(0)$, $0 \in \mathbb{Z}$.
Hence $(x, x) \in R$.

Thus R is reflexive on $\mathbb{Z}$.

<u>Symmetry</u>:

R is symmetric iff, for all x, y $\in$ X, if (x, y) $\in$ R then (y, x) $\in$ R.

Goal: to show that whenever (x, y) $\in$ R, we also have that (y, x) $\in$ R.

Suppose (x, y) $\in$ R, then y – x = 2k, for some $k \in \mathbb{Z}$ (1)

Now consider x - y

$$\begin{aligned} x - y &= -y + x \\ &= -(y - x) \\ &= -(2k) && \text{from (1)} \\ &= 2(-k) && \text{with } -k \in \mathbb{Z} \end{aligned}$$

This means that (y, x) $\in$ R.

Thus R is symmetric.

<u>Transitivity</u>:

Goal: to show that whenever (x, y) $\in$ R and (y, z) $\in$ R, we also have that (x, z) $\in$ R.

Suppose (x, y) $\in$ R and (y, z) $\in$ R then

$$\begin{aligned} y - x &= 2k, \text{ and} && (1) \\ z - y &= 2m. && (2) \end{aligned}$$

Add (2) to (1) then

$(y - x) + (z - y) = 2k + 2m$

i.e. $z - x = 2(k + m)$ with $(k + m) \in \mathbb{Z}$

Hence (x, z) $\in$ R.

Thus R is transitive.

It follows that R is an equivalence relation.

Next we calculate the *equivalence classes* of R:

$[x] = \{ y \mid (x, y) \in R \}$

In our case we have:

$$\begin{aligned} [x] &= \{y \mid y - x = 2k \text{ for some } k \in \mathbb{Z}\} \\ &= \{y \mid y = 2k + x \text{ for some } k \in \mathbb{Z}\} \end{aligned}$$

Let us substitute values for x until we no longer encounter new equivalence classes:

Let x = 0, then we determine the y-values:

$$\begin{aligned} [0] &= \{y \mid y = 2k, \text{ for some } k \in \mathbb{Z}\} \\ &= \{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\} \end{aligned}$$

This is the set of even integers.

Let x = 1, then we have:

[1] $= \{y \mid y = 2k + 1$, for some $k \in \mathbb{Z}\}$
$= \{..., -7, -5, -3, -1, 1, 3, 5, 7, ...\}$

This is the set of odd integers.

Try x = 2: We find that [2] = [0].

Try x = 3: Similarly, [3] = [1].

We find that … = [–4] = [–2] = [0] = [2] = [4] =… (all the even integers included in the classes) and also have that

… = [–5] = [–3] = [–1] = [1] = [3] = [5] =… (all the odd integers included in the classes).

We have only two equivalence classes: $[0] \cup [1] = \mathbb{Z}$.

S = {[0], [1]} is a partition of $\mathbb{Z}$.

How can we be sure that S a partition of $\mathbb{Z}$?

[0] and [1] are two **non-empty** subsets of $\mathbb{Z}$, and

$[0] \cap [1] = \emptyset$, and

$[0] \cup [1] = \mathbb{Z}$.

Because S has the above properties, it is a partition of $\mathbb{Z}$.

Question 14 *(Refer to study guide, pp 119 - 122)*

Let X = {b, c, d}

a i) Use a table to give an example of a binary operation on X which is commutative and has an identity element.

*	**b**	**c**	**d**
b	b	c	d
c	c	c	c
d	d	c	d

a ii) Show that your operation has both these properties, and name the identity element.

* is commutative:

b * b = b = b * b; c * c = c = c * c;
c * d = c = d * c; b * c = c = c * b;
d * d = d = d * d; b * d = d = d * b.

We can also verify commutativity by checking that the top half of the table is symmetric to the bottom, by drawing a diagonal from the top left corner to the bottom right corner.

The identity element is *b*:

b * *b* = *b* * b = b
c * *b* = *b* * c = c, and
d * *b* = *b* * d = d.

a iii) Using your table, show briefly how you would test for associativity, giving any single example (one or two lines only). Using your example, is it possible to decide whether your binary operation has the property of associativity? Give a reason for your answer.

(b * c) * d = c * d = c and b * (c * d) = b * c = c

This example shows associativity, but <u>one</u> example does not show that it is true for all possible combinations.

b i) Let Y = $\{\emptyset, \{\emptyset\}\}$. Complete the following table for the binary operation $\cap$ on Y:

(Note: $\cap$ is the existing intersection operation. It is not an operation that you can arbitrarily define.)

$\cap$	$\emptyset$	$\{\emptyset\}$
$\emptyset$		
$\{\emptyset\}$		

Solution:

$\cap$	$\emptyset$	$\{\emptyset\}$
$\emptyset$	$\emptyset$	$\emptyset$
$\{\emptyset\}$	$\emptyset$	$\{\emptyset\}$

Consider the operation $\cap$ in (b i) above:

b ii) Is the operation commutative? Justify your answer.

Yes, $\cap$ is commutative:

$\emptyset \cap \emptyset = \emptyset = \emptyset \cap \emptyset$

$\emptyset \cap \{\emptyset\} = \emptyset = \{\emptyset\} \cap \emptyset$

$\{\emptyset\} \cap \{\emptyset\} = \{\emptyset\} = \{\emptyset\} \cap \{\emptyset\}$

b iii) Does the operation have an identity element? Justify your answer fully.

The identity element is $\{\emptyset\}$, because:

$\{\emptyset\} \cap \emptyset = \emptyset = \emptyset \cap \{\emptyset\}$

$\{\emptyset\} \cap \{\emptyset\} = \{\emptyset\} = \{\emptyset\} \cap \{\emptyset\}$

b iv) Write the operation in list notation.

$\{((\emptyset, \emptyset), \emptyset), ((\emptyset, \{\emptyset\}), \emptyset), ((\{\emptyset\}, \emptyset), \emptyset), ((\{\emptyset\}, \{\emptyset\}), \{\emptyset\})\}$

b v) Give an example of a relation R on Y that satisfies trichotomy.

Y = $\{\emptyset, \{\emptyset\}\}$

The two elements in Y are not equal, therefore they must be combined in an ordered pair belonging to a relation R. i.e. $(\emptyset, \{\emptyset\}) \in$ R or $(\emptyset, \{\emptyset\}) \in$ R. It is also possible that both these pairs can belong to a relation R.

R = $\{(\emptyset, \{\emptyset\})\}$ or R = $\{(\{\emptyset\}, \emptyset)\}$ or R = $\{(\{\emptyset\}, \emptyset), (\emptyset, \{\emptyset\})\}$

c i) Provide the truth table for the connective $\rightarrow$ with two atomic sentences *p* and *q*.

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

c ii) Rewrite the truth table in c i) as a binary operation $\to$ on the set of truth values. Is the operation commutative? Give a reason for your answer.

$\to$	**T**	**F**
T	*T*	*F*
F	*T*	*T*

$\to$ is not commutative: T $\to$ *F* = F but F $\to$ T = *T*
(List notation: { ((T, T), *T*), ((T, F), *F*), ((F, T), *T*), ((F, F), *T*) })

Question 15 *(Refer to study guide, pp 41 - 44, 91 – 92 , 104 – 107)*
Let A = {1, 2, 3}, B = {0, 1} and C = {∅}.

a) Write down A + B and give an equivalence relation on A + B which is not the identity relation.

A + B = {0, 2, 3} (The elements 0, 2 and 3 belong to A or B, but not to both A and B.)

Equivalence relation on A + B:

{(0, 0), (2, 2), (3, 3), (0, 2), (2, 0)} This relation is reflexive on A + B, symmetric and transitive.

b) Determine the values of the sets
i) $\mathcal{P}$(B) ∩ $\mathcal{P}$(C), ii) $\mathcal{P}$(B ∩ C), iii) $\mathcal{P}$(B) – $\mathcal{P}$(C), and iv) $\mathcal{P}$(B) + $\mathcal{P}$(C).

i) $\mathcal{P}$(B) = {∅, {0}, {1}, {0, 1} }
$\mathcal{P}$(C) = {∅, {∅ }

So $\mathcal{P}$(B) ∩ $\mathcal{P}$(C) = {∅}.

ii) B ∩ C = ∅

So $\mathcal{P}$(B ∩ C) = {∅}.

iii) $\mathcal{P}$(B) – $\mathcal{P}$(C) = {{0}, {1}, {0, 1}}

iv) $\mathcal{P}$(B) + $\mathcal{P}$(C) = {{∅}, {0}, {1}, {0, 1}}

c) Give an injective function on B × B.

B × B = $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$

The identity relation on B × B is an injective function:

$\{((0, 0), (0, 0)), ((0, 1), (0, 1)), ((1, 0), (1, 0)), ((1, 1), (1, 1))\}$

d) Give an example of a surjective function from B × B to B ∪ C.

In order to be surjective, it must be the case that the range of the function is equal to B ∪ C. This means that each member of B ∪ C must appear as second co-ordinate. It is a function, so each member of B × B must appear only once as first co-ordinate.

B × B = $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and

B ∪ C = {0, 1, ∅}

An example of a surjective function:

$\{((0, 0), 0), ((0, 1), 1), ((1, 0), 1), ((1, 1), \emptyset)\}$

e) Give the simplest equivalence relation on $\mathcal{P}$(C).

The relation must be reflexive on $\mathcal{P}$(C), symmetric and transitive.

$\mathcal{P}(C) = \{\emptyset, \{\emptyset\}\}$

The simplest equivalence relation on $\mathcal{P}$(C) is the identity relation:

$\{(\emptyset, \emptyset), (\{\emptyset\}, \{\emptyset\})\}$

f) Give an example of a partition of B = {0, 1}.

{{0, 1}} or {{0}, {1}}

How can we be sure that {{0}, {1}} is a partition of B?

{0} and {1} are two **non-empty** subsets of B, and

{0} ∩ {1} = ∅, and

{0} ∪ {1} = B.

Because {{0}, {1}} has the above properties, it is a partition of B.

5 EXTRA EXAMPLES & SOLUTIONS similar to QUESTIONS IN ASSIGNMENTS 01, 02 AND 03

This section contains example assignment questions and solutions. We hope these example questions will give you a better idea of how to approach multiple choice questions. All alternatives in a question should be considered before you make a choice.

Although the exam is an MCQ exam, you will still need to write down proofs, draw Venn diagrams, do calaculations for matrices etc. It is therefore a good idea to work through the additional examples in the tutorial letter.

Questions similar to assignment 01 questions.

Consider the following sets, where U represents a universal set:

U = {∅, a, b, c, {b}, {c}, {b, c}} A = {∅, {b, c}} B = {a, {b}, {c}} C = {∅, a, b, c}.

Questions 1 to 10 are based on the sets defined above.

NOTE: The Venn diagrams in study unit 4 will help you to understand the definitions in study unit 3.

Question 1 **Alternative 1**

B′ is the set:

1. {∅, b, c, {b, c}}
2. {∅, b, c, {b}, {c}}
3. {b, c, {b, c}}
4. {b, c}

B′ = U − B = {∅, a, b, c, {b}, {c}, {b, c}} − {a, {b}, {c}} = {∅, b, c, {b, c}}

Set complement: The elements ∅, b, c and {b, c} belong to U but not to B.

Refer to study guide, pp 42, 43.

Question 2 **Alternative 1**

A – C is the set:

1. {{b, c}}
2. {a, b, {b, c}}
3. {a, {b, c}}
4. {b, c}

A – C

= {∅, {b, c}} − {∅, a, b, c} = {{b, c}}.

Set difference: Only the element {b, c} belongs to A but not to C.

Refer to study guide, pp 42, 43.

Question 3 **Alternative 3**

$A \cup B$ is the set:

1. $\{\emptyset, a, \{b, c\}\}$
2. $\{a, \{b, c\}, \{b\}, \{c\}\}$
3. $\{\emptyset, a, \{b, c\}, \{b\}, \{c\}\}$
4. $\{\emptyset, a, \{b\}, \{c\}\}$

$A \cup B$

$= \{\emptyset, \{b, c\}\} \cup \{a, \{b\}, \{c\}\}$

$= \{\emptyset, a, \{b, c\}, \{b\}, \{c\}\}$.

Set union: The elements ∅, a, {b, c}, {b} and {c} belong to A or B.
Refer to study guide, p 41.

Question 4 **Alternative 4**

$A + C$ is the set:

1. $\{\emptyset, a, b, c, \{b, c\}\}$
2. $\{\{b, c\}\}$
3. $\{a, \{b, c\}\}$
4. $\{a, b, c, \{b, c\}\}$

$A + C$

$= \{\emptyset, \{b, c\}\} + \{\emptyset, a, b, c\}$

$= \{a, b, c, \{b, c\}\}$.

Set symmetric difference: The elements a, b, c and {b, c} belong to A or to C, but not both.

It is also the case that $A + C = (A \cup C) - (A \cap C)$, so

$A + C = (A \cup C) - (A \cap C)$ *Include elements belonging to* $A \cup C$ *but not to* $A \cap C$.

$= (\{\emptyset, \{b, c\}\} \cup \{\emptyset, a, b, c\}) - (\{\emptyset, \{b, c\}\} \cap \{\emptyset, a, b, c\})$

$= \{\emptyset, a, b, c, \{b, c\}\} - \{\emptyset\}$

$= \{a, b, c, \{b, c\}\}$

Refer to study guide, pp 43, 44.

Question 5 **Alternative 1**

$(A \cap B) \cap C$ is the set:

1. $\emptyset$
2. $\{\{b, c\}\}$
3. $\{\{b\}, \{c\}\}$
4. $\{\emptyset, a, b, c, \{b\}, \{c\}, \{b, c\}\}$

$(A \cap B) \cap C = (\{\emptyset, \{b, c\}\} \cap \{a, \{b\}, \{c\}\}) \cap \{\emptyset, a, b, c\}$
$= \{\ \} \cap \{\emptyset, a, b, c\}$
$= \{\ \}$

No element belongs to A, B and C.

Refer to study guide, p 42.

Question 6 **Alternative 2**

$\mathcal{P}(A)$ is the set:

1. $\{\emptyset, \{\{b, c\}\}, \{\emptyset, \{b, c\}\}\}$
2. $\{\emptyset, \{\emptyset\}, \{\{b, c\}\}, \{\emptyset, \{b, c\}\}\}$
3. $\{\emptyset, \{b, c\}, \{\emptyset, \{b, c\}\}\}$
4. $\{\{\emptyset\}, \{b, c\}\}$

The number of elements in a set is called the cardinality of the set.

We first determine the cardinality of $\mathcal{P}(A)$:

$|\mathcal{P}(A)| = 2^n$ (n is the number of elements in A.)
$= 2^2$
$= 4$

The elements of $\mathcal{P}(A)$ are all the subsets of A, so we have to determine the subsets of A.

Let's look at the definition of a subset:
*For all sets F and G, F is a subset of G if and only if every element of F is also an element of G. Subsets of G can be formed by **keeping the outside brackets** of G and then throwing away **none**, **one** or **more** elements of G.*

The elements of $A = \{\emptyset, \{b, c\}\}$ *are* $\emptyset$ *and* $\{b, c\}$.
We form the subsets of A:

Throw away no element of set A, then $\{\emptyset, \{b, c\}\} \subseteq A$;
throw away the element $\emptyset$ *of set A, then* $\{\{b, c\}\} \subseteq A$;
throw away the element $\{b, c\}$ *of set A, then* $\{\emptyset\} \subseteq A$; *and*
throw away the elements $\emptyset$ *and* $\{b, c\}$ *of set A, then* $\{\ \} \subseteq A$.

If $A = \{\emptyset, \{b, c\}\}$ then the subsets of A, namely $\emptyset$, $\{\emptyset\}$, $\{\{b, c\}\}$ and $\{\emptyset, \{b, c\}\}$ are the elements of $\mathcal{P}(A)$, thus

$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{b, c\}\}, \{\emptyset, \{b, c\}\}\}$.

The set provided in alternative 2 is the set $\mathcal{P}(A)$.

The set $\{\emptyset, \{\{b, c\}\}, \{\emptyset, \{b, c\}\}\}$ is a **subset** of $\mathcal{P}(A)$ but **not equal** to $\mathcal{P}(A)$.

The sets $\{\emptyset, \{b, c\}, \{\emptyset, \{b, c\}\}\}$ and $\{\{\emptyset\}, \{b, c\}\}$ provided in alternatives 3 and 4 contain the element $\{b, c\}$ which is not an element of $\mathcal{P}(A)$ thus these sets are not subsets of $\mathcal{P}(A)$ nor are they equal to $\mathcal{P}(A)$.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, pp 40, 44, 45.

Question 7 **Alternative 3**

What is the cardinality of the set $\mathcal{P}(U)$?

1. 32
2. 64
3. 128
4. 256

The number of elements in a set is called the cardinality of the set. The set $U = \{\emptyset, a, b, c, \{b\}, \{c\}, \{b, c\}\}$ has 7 elements namely $\emptyset$, a, b, c, $\{b\}$, $\{c\}$ and $\{b, c\}$, i.e. $|U| = 7$.

We determine the cardinality of $\mathcal{P}(U)$:

$|\mathcal{P}(U)| = 2^n$ (n is the cardinality of U.)

$= 2^7$

$= 128$

Thus the cardinality of $\mathcal{P}(U)$ is 128.

Refer to study guide, pp 44, 45.

Question 8 **Alternative 2**

Which one of the following sets is a **proper subset** of U?

1. $\{\{\emptyset\}, a, b, c\}$
2. $\{\emptyset\}$
3. $\{\{a, b, c\}\}$
4. $\{\emptyset, a, b, c, \{b\}, \{c\}, \{b, c\}\}$

*Let's first discuss **proper subsets**:*

*If we throw away **one** or **more** elements of some set G (**keeping the outside brackets** of G), then the resulting set (let's call it F) is a proper subset of G. So every element of F is also an element of G, but set F has less elements than set G. We may write $F \subset G$.*

We consider the sets provided in the different alternatives:

Alternative 1: $\{\{\emptyset\}, a, b, c\}$ is **not** a proper subset of U. We provide a counterexample:

$\{\emptyset\} \in \{\{\emptyset\}, a, b, c\}$ but $\{\emptyset\} \notin U = \{\emptyset, a, b, c, \{b\}, \{c\}, \{b, c\}\}$.

Alternative 2: $\{\emptyset\}$ is a proper subset of U. If we keep the outside brackets of U and throw away the elements a, b, c, {b}, {c} and {b, c} of U, we form the proper subset $\{\emptyset\}$ of U, i.e. $\{\emptyset\} \subset$ U.

Alternative 3: $\{\{a, b, c\}\}$ is **not** a proper subset of U. We provide a counterexample:
$\{a, b, c\} \in \{\{a, b, c\}\}$ but $\{a, b, c\} \notin$ U.
The only element of $\{\{a, b, c\}\}$ namely {a, b, c} is **not** an element of U, thus $\{\{a, b, c\}\} \not\subset$ U.

Alternative 4: The set $\{\emptyset, a, b, c, \{b\}, \{c\}, \{b, c\}\}$ provided in this alternative is the set U but U $\not\subset$ U. (We must throw away **one or more** elements from U to form a proper subset of U.)

From the arguments provided we can deduce that alternative 2 should be selected.
Refer to study guide, p 41.

Question 9 **Alternative 4**

Which one of the following sets is NOT a subset of U?

1. $\{\emptyset, a, b, c\}$
2. $\{b, c, \{b, c\}\}$
3. $\{\{b\}, \{c\}, \{b, c\}\}$
4. $\{\{\emptyset\}, \{b, c\}\}$

U = $\{\emptyset, a, b, c, \{b\}, \{c\}, \{b, c\}\}$**.** We consider the sets in the different alternatives:

Alternative 1: If we throw away elements {b}, {c} and {b, c} of set U, we are left with elements $\emptyset$, a, b and c that belong to the subset $\{\emptyset, a, b, c\}$ of U, thus $\{\emptyset, a, b, c\} \subseteq$ U.

Alternative 2: If we throw away elements $\emptyset$, a, {b} and {c} of set U, we are left with elements b, c and {b, c} that belong to the subset $\{b, c, \{b, c\}\}$ of U, thus $\{b, c, \{b, c\}\} \subseteq$ U.

Alternative 3: If we throw away elements $\emptyset$, a, b and c of set U, we are left with elements {b}, {c} and {b, c} that belong to the subset $\{\{b\}, \{c\}, \{b, c\}\}$ of U, thus $\{\{b\}, \{c\}, \{b, c\}\} \subseteq$ U.

Alternative 4: $\{\{\emptyset\}, \{b, c\}\}$ is not a subset of U. We provide a counterexample:
$\{\emptyset\} \in \{\{\emptyset\}, \{b, c\}\}$ but $\{\emptyset\} \notin$ U, therefore $\{\{\emptyset\}, \{b, c\}\} \nsubseteq$ U.

From the arguments provided we can deduce that alternative 4 should be selected.
Refer to study guide, pp 40, 41.

Question 10 **Alternative 2**

Which one of the following is a set D such that D + U = $\{\emptyset, b, \{b\}\}$?

1. $\{\emptyset, a, b, c, \{b\}, \{c\}, \{b, c\}\}$
2. $\{a, c, \{c\}, \{b, c\}\}$
3. $\{a, c, \{b, c\}\}$

4. {a, c, {b}}

U = {∅, a, b, c, {b}, {c}, {b, c}}. We consider the sets in the different alternatives:

Alternative 1: D + U = {∅, a, b, c, {b}, {c}, {b, c}} + {∅, a, b, c, {b}, {c}, {b, c}} = { } ≠ {∅, b, {b}}.
Thus alternative 1 does not provide the required result.

Alternative 2: By the definition of symmetric difference, the elements of D + U = {∅, b, {b}}, namely ∅, b and {b}, belong to D or to U but not both.
In the question it is stated that D + U = D + {∅, a, b, c, {b}, {c}, {b, c}} = {∅, b, {b}}.
Let D = {a, c, {c}, {b, c}},
then {**a**, **c**, **{c}**, **{b, c}**} + {∅, **a**, b, **c**, {b}, **{c}**, **{b, c}**} = {∅, b, {b}}, i.e. ∅, b and {b}, belong to D or to U but not both.

It is also the case that D + U = (D ∪ U) − (D ∩ U), so

D + U = (D ∪ U) − (D ∩ U)   *Include elements belonging to D ∪ U but not to D ∩ U.*
= ({a, c, {c}, {b, c}} ∪ {∅, a, b, c, {b}, {c}, {b, c}}) −
({a, c, {c}, {b, c}} ∩ {∅, a, b, c, {b}, {c}, {b, c}})
= {∅, a, b, c, {b}, {c}, {b, c}} − {a, c, {c}, {b, c}}
= {∅, b, {b}}

Alternative 3: D + U = {**a**, **c**, **{b, c}**} + {∅, **a**, b, **c**, {b}, {c}, **{b, c}**} = {∅, b, {b}, {c}} ≠ {∅, b, {b}}.
Thus alternative 3 does not provide the required result.

Alternative 4: D + U = {**a**, **c**, **{b}**} + {∅, **a**, b, **c**, **{b}**, {c}, {b, c}} = {∅, b, {c}, {b, c}} ≠ {∅, b, {b}}.
Thus alternative 4 does not provide the required result.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, p 43.

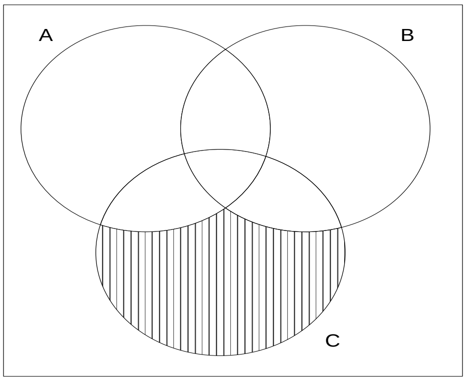
Questions similar to assignment 02 questions.

Question 1 **Alternative 1**

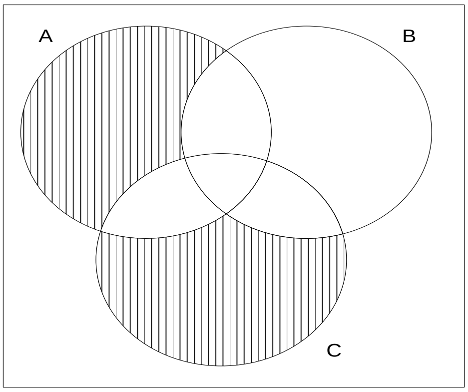
Let A, B and C be subsets of a universal set U.

Which one of the following four Venn diagrams presents the set (A ∪ B)′ ∩ (C + A)?

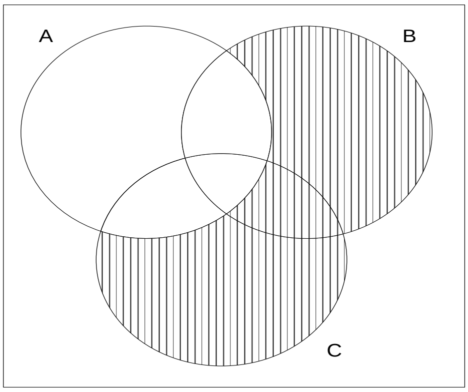
1.



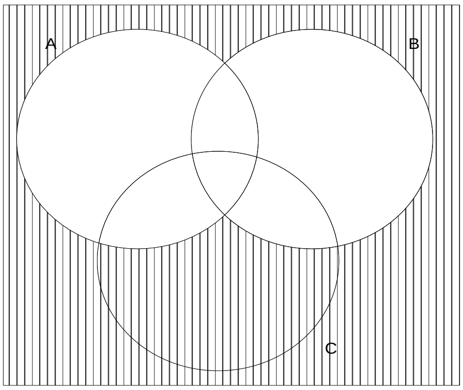
2.



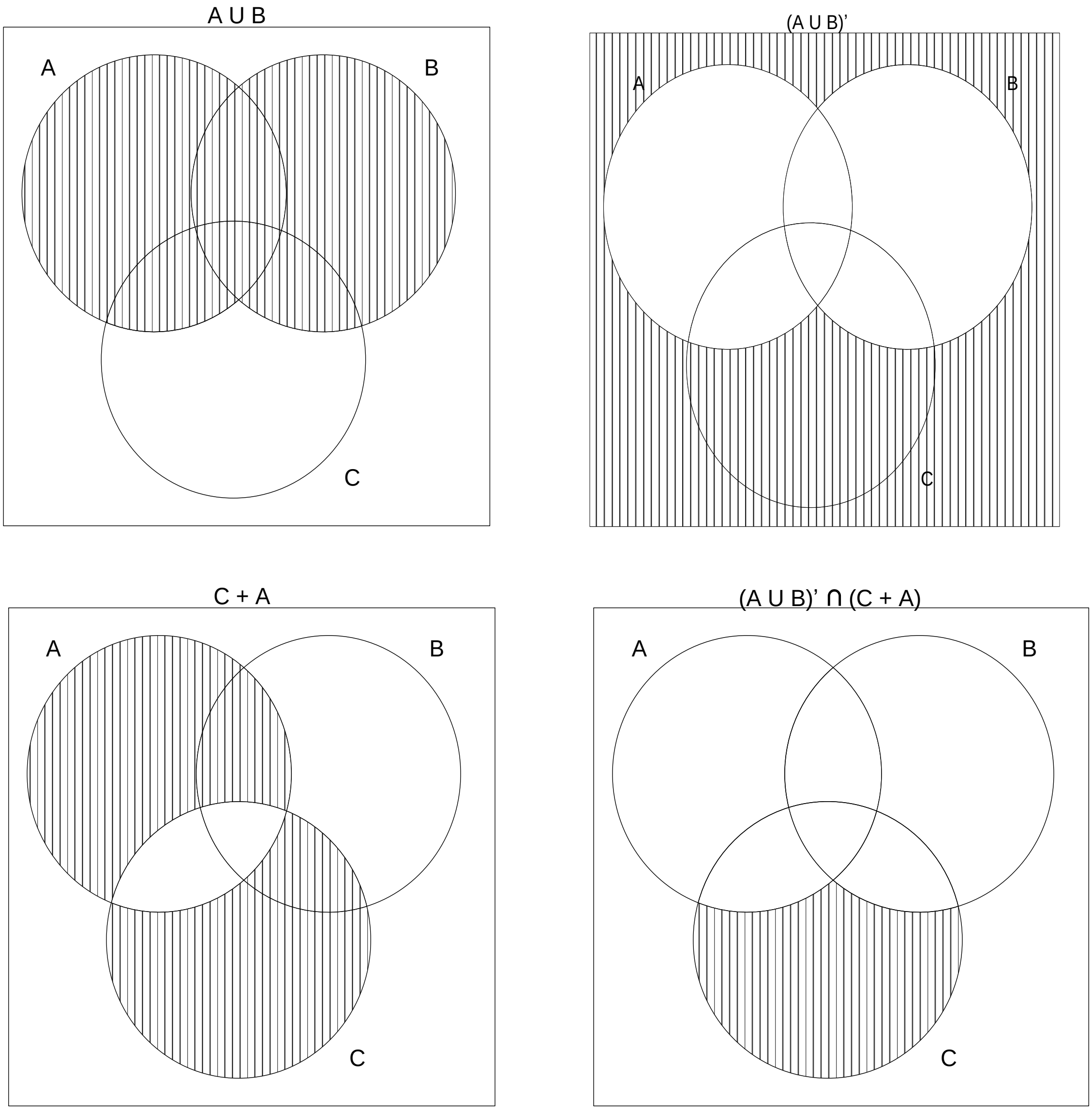
3.



4.



We determine the Venn diagram for the set $(A \cup B)' \cap (C + A)$ step by step:



Refer to study guide, pp 50, 51.

Question 2 **Alternative 3**

Let A, B and C be subsets of a universal set U. The statement $(A \cap B) - C' = A \cap (B - C)$ is NOT an identity. Which of the following subsets A, B and C of a universal $U = \{1, 2, 3, 4\}$ provides a *counterexample* that can be used to show that the given statement is not an identity?

1. A = {2}, B = {3} & C = {4}
2. A = {1}, B = {3} & C = {3}
3. A = {1}, B = {1} & C = {1}
4. A = {1, 2}, B = {3, 4} & C = {4}

Discussion

Given: A, B, C $\subseteq$ U with U = {1, 2, 3, 4}, and (A $\cap$ B) – C′ = A $\cap$ (B – C) is not an identity.

*We do **not** start our counterexample solution with (A $\cap$ B) – C′ ≠ A $\cap$ (B – C).*

***First** we **determine (A $\cap$ B) – C′, then** we **determine A $\cap$ (B – C)** by using the sets provided in the different alternatives, then we **compare the answers** and come to a conclusion.*

We consider the different alternatives:

1. We use the sets A = {2}, B = {3} & C = {4} to determine (A $\cap$ B) – C′ and A $\cap$ (B – C), then we compare the answers. Note that we use **curly brackets** for **sets**.

   (A $\cap$ B) – C′ = { } – {1, 2, 3}
   = { }
   A $\cap$ (B – C) = {2} $\cap$ {3}
   = { }

   Thus (A $\cap$ B) – C′ = A $\cap$ (B – C).

2. We use the sets A = {1}, B = {3} & C = {3} to determine (A $\cap$ B) – C′ and A $\cap$ (B – C), then we compare the answers.

   (A $\cap$ B) – C′ = { } – {1, 2, 4}
   = { }
   A $\cap$ (B – C) = {1} $\cap$ { }
   = { }

   Thus (A $\cap$ B) – C′ = A $\cap$ (B – C).

3. We use the sets A = {1}, B = {1} & C = {1} to determine (A $\cap$ B) – C′ and A $\cap$ (B – C), then we compare the answers.

   (A $\cap$ B) – C′ = {1} – {2, 3, 4}
   = {1}
   A $\cap$ (B – C) = {1} $\cap$ { }
   = { }

   Clearly {1} ≠ { }, so (A $\cap$ B) – C′ ≠ A $\cap$ (B – C).

4. We use the sets $A = \{1, 2\}$, $B = \{3, 4\}$ & $C = \{4\}$ to determine $(A \cap B) - C'$ and $A \cap (B - C)$, then we compare the answers.

$$\begin{aligned} (A \cap B) - C' &= \{\ \} - \{1, 2, 3\} \\ &= \{\ \} \\ A \cap (B - C) &= \{1, 2\} \cap \{3\} \\ &= \{\ \} \end{aligned}$$

Thus $(A \cap B) - C' = A \cap (B - C)$.

Alternatives 1, 2 and 4 do not provide counterexamples, but a counterexample is provided in alternative 3, thus this alternative should be selected.
Refer to study guide, pp 60, 61.

Question 3 **Alternative 2**

We want to prove that for all $X, Y \subseteq U$,

$$X \times (Y \cup W)' = (X \times Y') \cap (X \times W') \text{ is an identity.}$$

Consider the following proof:

$(u, v) \in X \times (Y \cup W)'$

iff $u \in X$ and $v \in (Y \cup W)'$

iff $u \in X$ and $v \notin (Y \cup W)$

iff Step 4

iff $(u \in X$ and $v \notin Y)$ and $(u \in X$ and $v \notin W)$

iff Step 6

iff $(u, v) \in (X \times Y')$ and $(u, v) \in (X \times W')$

iff $(u, v) \in (X \times Y') \cap (X \times W')$

*Note: The given proof involves **Cartesian products**, i.e. ordered pairs, so the proof starts with: "(u, v) ∈ ...".*

Which one of the following alternatives provides valid steps 4 and 6 to complete the given proof?

1. Step 4: iff $u \in X$ and $(v \notin Y$ or $v \notin W)$
   Step 6: iff $(u \in X$ or $v \in Y')$ and $(u \in X$ or $v \in W')$

2. Step 4: iff $u \in X$ and $(v \notin Y$ and $v \notin W)$
   Step 6: iff $(u \in X$ and $v \in Y')$ and $(u \in X$ and $v \in W')$

3. Step 4: iff $u \in X$ and $(u \notin Y$ and $u \notin W)$
   Step 6: iff $(u, v) \notin (X \times Y')$ and $(u, v) \notin (X \times W')$

4. Step 4: iff $u \in X$ or $(v \notin Y$ and $v \notin W)$
   Step 6: iff $(u \in X$ and $v \in Y')$ and $(u \in X$ and $v \in W')$

Discussion

In the proof we apply the definitions of Cartesian product, union and complement of sets. The notation should be correct and all the necessary steps should appear in the proof.

Proof:

$(u, v) \in X \times (Y \cup W)'$

iff $u \in X$ and $v \in (Y \cup W)'$

iff $u \in X$ and $v \notin (Y \cup W)$ *$u \in X$, and v is not an element of Y or W.*

iff $u \in X$ and ($v \notin Y$ and $v \notin W$) *$u \in X$, and v is neither an element of Y nor of W.*

iff ($u \in X$ and $v \notin Y$) and ($u \in X$ and $v \notin W$)

iff ($u \in X$ and $v \in Y'$) and ($u \in X$ and $v \in W'$)

iff $(u, v) \in (X \times Y')$ and $(u, v) \in (X \times W')$

iff $(u, v) \in (X \times Y') \cap (X \times W')$

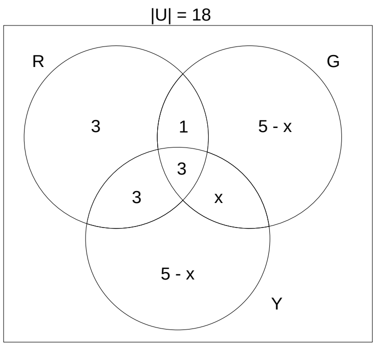
Refer to study guide, pp 41, 43, 55-57, 73.

Question 4 **Alternative 4**

Painters paint 18 toys. Of these toys 10 are painted red, 9 are painted green, and 11 are painted yellow. (Toys are not necessarily painted only one colour.)
Furthermore, some toys are painted as follows:
4 are painted red & green, 6 are painted red & yellow, and 3 are painted red, green & yellow.
How many toys are painted only green and yellow?

1. 1
2. 4
3. 5
4. 2

Solution:
$|U| = 18$, $|R| = 10$, $|G| = 9$, $|Y| = 11$, (U = universal set; R = red; G = green; Y = yellow)
$|R \cap G| = 4$, $|R \cap Y| = 6$, $|R \cap G \cap Y| = 3$, and x is the unknown we want to solve, i.e. the number of toys painted only green and yellow. *Fill in the various regions in a Venn diagram. Firstly fill in x for the number of toys painted only green and yellow*.



$|R \cup G \cup Y| = 3 + 3 + 3 + 1 + (5 - x) + (5 - x) + x = 18$
i.e. $20 - x = 18$
i.e. $x = 2$, i.e. 2 toys are painted only green and yellow.
From the argument provided we can deduce that alternative 4 should be selected.

Refer to study guide, pp 63 – 66.

Question 5 **Alternative 2**

Let T be a relation from A = {0, 1, 2, 3} to B = {0, 1, 2, 3, 4} such that

(a, b) $\in$T iff b – a is a multiple of 2. (A, B $\subseteq$ U = $\mathbb{Z}$.)

Which one of the following alternatives provides elements belonging to T?

1. (0, 4), (2, 3), (2, 4)
2. (1, 1), (1, 3), (0, 2)
3. (0, 4), (1, 4), (2, 4)
4. (0, 2), (2, 8), (4, 0)

We consider the ordered pairs provided in the different alternatives:

1. Is (0, 4) $\in$ T? Yes, 4 – 0 = 4 and 4 is a multiple of 2.
   Is (2, 3) $\in$ T? No, 3 – 2 = 1 and 1 is *not* a multiple of 2.
   Is (2, 4) $\in$ T? Yes, 4 – 2 = 2 and 2 is a multiple of 2.
   (2, 3) $\notin$ T, thus this alternative does not provide only elements belonging to T.

2. Is (1, 1) $\in$T? Yes, 1 – 1 = 0 and 0 is a multiple of 2.
   Is (1, 3) $\in$T? Yes, 3 – 1 = 2 and 2 is a multiple of 2.
   Is (0, 2) $\in$T? Yes, 2 – 0 = 2 and 2 is a multiple of 2.
   Thus all the given ordered pairs in the alternative are elements of T.

3. Is (0, 4) $\in$ T? Yes, 4 – 0 = 4 and 4 is a multiple of 2.
   Is (1, 4) $\in$ T? No, 4 – 1 = 3 and 3 is *not* a multiple of 2.
   Is (2, 4) $\in$ T? Yes, 4 – 2 = 2 and 2 is a multiple of 2.
   (1, 4) $\notin$ T, thus this alternative does not provide only elements belonging to T.

4. Is (0, 2) $\in$ T? Yes, 2 – 0 = 2 and 2 is a multiple of 2.
   Is (2, 8) $\in$ T? No, T is defined from A to B but 8 $\notin$ B = {0, 1, 2, 3, 4} thus (2, 8) $\notin$ T.
   Is (4, 0) $\in$ T? Yes, 0 – 4 = – 4 and – 4 is a multiple of 2.
   (2, 8) $\notin$ T, thus this alternative does not provide only elements belonging to T.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, p 73.

Consider the following relation P on the set B = {a, b, {a, b}}:

P = {(a, a), (a, b), (b, {a, b}), ({a, b}, a)}.

Answer questions 6 to 8 by using the given relation P.

Question 6 **Alternative 4**

Which one of the following alternatives represents the domain of P (dom(P))?

1. {a, b}
2. {{a, b}}
3. a, b, {a, b}
4. {a, b, {a, b}}

Discussion

First we look at the definition of the domain of the given relation P:
dom(P) = {x / for some y ∈ B, (x, y) ∈ P}. (Note: The domain is a ***set*** *with all the* ***first co-ordinates*** *of the ordered pairs of P as its* ***elements****.)*

Dom(P) is the **set** of elements appearing as first co-ordinates in the ordered pairs of P. The elements a, b and {a, b} appear as first co-ordinates in the ordered pairs of P, thus dom(P) = {a, b, {a, b}}. From this discussion one can conclude that alternative 4 should be selected. Clearly no other alternative is suitable.

Refer to study guide, p 74.

Question 7 **Alternative 2**

Which one of the following relations represents the composition relation P ○ P?

1. {(a, a), (a, b), (a, {a, b}), (b, a)}
2. {(a, a), (a, b), (a, {a, b}), (b, a), ({a, b}, a), ({a, b}, b)}
3. {(a, a), (a, b), (b, a)}
4. {(a, a), (a, b), (b, a), ({a, b}, {a, b})}

Discussion
We first look at the definition of a composition relation:

Given relation ***P*** *from B to B and P from B to B, the composition of* ***P*** *followed by P (P ○* ***P*** *or* ***P****; P) is the relation from B to B defined by*

P ○ **P** = **P**; *P = {(m, o) | there is some n ∈ B such that (m, n) ∈* ***P*** *and (n, o) ∈ P}.*

(P and ***P*** *is exactly the same relation, but for the purpose of our explanations we make the subtle distinction.)*

P **=** {(a, a), (a, b), (b, {a, b}), ({a, b}, a)} is defined on B = {a, b, {a, b}}.

To determine **P**; *P* we start with the pair (a, **a**) of **P**, and then we look for a pair in *P* that has as first co ordinate an **a**, and then see where it takes us.

Link (a, **a**) of **P** with (**a**, a) of *P*, then (a, a) ∈ **P**; *P*.
Link (a, **a**) of **P** with (**a**, b) of *P*, then (a, b) ∈ **P**; *P*.
Link (a, **b**) of **P** with (**b**, {a, b}) of *P*, then (a, {a, b}) ∈ **P**; *P*.
Link (b, **{a, b}**) of **P** with (**{a, b}**, a} of *P*, then (b, a) ∈ **P**; *P*.
Link ({a, b}, **a**) of **P** with (**a**, a) of *P*, then ({a, b}, a) ∈ **P**; *P*.
Link ({a, b}, **a**) of **P** with (**a**, b) of *P*, then ({a, b}, b) ∈ **P**; *P*.
No other pairs can be linked, so P ○ P = {(a, a), (a, b), (a, {a, b}), (b, a), ({a, b}, a), ({a, b}, b)}.

From the above one can conclude that alternative 2 should be selected. Clearly no other alternative is suitable.

Refer to study guide, pp 79, 108, 109.

Question 8 **Alternative 3**
The relation P is not transitive. Which ordered pairs can be included in P so that P would satisfy transitivity?

1. (a, {a, b}) & (b, a)
2. (b, b) & ({a, b}, {a, b})
3. (b, a), (b, b), (a, {a, b}), ({a, b}, b) & ({a, b}, {a, b})
4. (b, a), (a, {a, b}) & ({a, b}, b)

Discussion
We first provide the definition for transitivity:
A relation $R \subseteq A \times A$ is transitive iff R has the property that for all x, y, z $\in$ A,
whenever (x, y) $\in$ R and (y, z) $\in$ R, then (x, z) $\in$ R. Let's say the ordered pairs (x, y) and (y, z) y plays the role of an "intermediary".
P = {(a, a), (a, b), (b, {a, b}), ({a, b}, a)} is defined on B = {a, b, {a, b}}. We determine which ordered pairs have an "intermediary".

For transitivity:
(b, {a, b}), ({a, b}, a) $\in$ P thus (b, a) should also be an element of P;
(a, b), (b, {a, b}) $\in$ P thus (a, {a, b}) should also be an element of P; and
({a, b}, a), (a, b) $\in$ P thus ({a, b}, b) should also be an element of P.
Since (b, a), (a, b) $\in$ P then (b, b) should also be an element of P; and
since ({a, b}, b), (b, {a, b}) and also ({a, b}, a), (a, {a, b}) $\in$ P then ({a, b}, {a, b}) should also be an element of P.
Thus, if we add the ordered pairs (b, a), (b, b), (a, {a, b}), ({a, b}, b) & ({a, b}, {a, b}) to the relation P, we actually have a new relation, say P_1, such that
P_1 = {(a, a), (b, b), (a, b), (b, a), ({a, b}, b), (b, {a, b}), ({a, b}, a), (a, {a, b}), ({a, b}, {a, b})}
is a transitive relation. Thus alternative 3 should be selected.

Refer to study guide, p 77.

Suppose U = {1, 2, 3, a, b, c} is a universal set with the subset A = {a, c, 2, 3}.
Let R = { (a, c), (3, c), (3, a) } be a relation on A.
Answer questions 9 to 12 by using the given sets A, U and relation R.

Question 9 **Alternative 4**
Which one of the following relations on **A** is irreflexive and antisymmetric?
1. { (a, a), (a, c), (3, c), (3, a) }
2. { (a, c), (c, a), (3, a), (2, 3) }
3. { (a, c), (3, c), (3, a), (3, 3) }
4. { (a, c), (3, c), (3, a), (2, 3) }

Discussion
Irreflexive: Ask: Is it true that for all x $\in$ A that we have (x, x) $\notin$ R?
Antisymmetric: Ask: Is it true that for all x, y $\in$ A, if x ≠ y and (x, y) $\in$ R then (y, x) $\notin$ R?

We consider the relations provided in the different alternatives:
1. (a, a) $\in$ { (a, a), (a, c), (3, c), (3, a) } thus this relation is not irreflexive.
2. (a, c), (c, a) $\in$ { (a, c), (c, a), (3, a), (2, 3) } thus this relation is not antisymmetric.
3. (3, 3) $\in$ { (a, c), (3, c), (3, a), (3, 3) } thus this relation is not irreflexive.
4. Let S = { (a, c), (3, c), (3, a), (2, 3) } (say). S is both irreflexive and antisymmetric:

In S we have no ordered pair such that (x, x) $\in$ S (with x $\in$ A), thus S is irreflexive.
Antisymmetry:
(a, c) but (c, a) $\notin$ S;
(3, c) but (c, 3) $\notin$ S;

(3, a) but (a, 3) $\notin$ S; and
(2, 3) but (3, 2) $\notin$ S.
Thus S is antisymmetric.

From the arguments provided we can deduce that alternative 4 should be selected.

Refer to study guide, pp 75, 76.

Question 10 **Alternative 1**

Which one of the following sets is a partition of **U** = {1, 2, 3, a, b, c}?

1. { {1}, {2, 3, a, b, c} }
2. {1, 2, 3, a, b, c}
3. { {1, 2}, {a, b} }
4. { {1, 2, a, c}, {1, 3, b} }

Discussion

First we look at the definition of a partition:

For a nonempty set A, a partition of A is a set S = {S_1, S_2, S_3, …}. The members of S are subsets of A (each set S_i is called a part of S) such that

a. for all i, $S_i \neq \emptyset$ (that is, each part is nonempty),

b. for all i and j, if $S_i \neq S_j$, then $S_i \cap S_j = \emptyset$ (that is, different parts have nothing in common), and

c. $S_1 \cup S_2 \cup S_3 \cup \ldots = A$ (that is, every element in A is in some part S_i).

We consider the sets provided in the different alternatives:

1. Let P = { {1}, {2, 3, a, b, c} } (say).

   We test whether P is a partition of U = {1, 2, 3, a, b, c}:

   a. {1} and {2, 3, a, b, c} are two **non-empty** subsets of U,
   b. {1} $\cap$ {2, 3, a, b, c} = $\emptyset$, and
   c. {1} $\cup$ {2, 3, a, b, c} = U.

   Since P has all the necessary properties, it is a partition of U.

2. The elements of {1, 2, 3, a, b, c} are not subsets of U thus the given set is not a partition of U.

3. Let T = { {1, 2}, {a, b} } (say).
   We test whether T is a partition of U = {1, 2, 3, a, b, c}:

   a. {1, 2} and {a, b} are two **non-empty** subsets of U,
   b. {1, 2} $\cap$ {a, b} = $\emptyset$, but
   c. {1, 2} $\cup$ {a, b} = {1, 2, a, b} $\neq$ U.

   Since T does not have all the necessary properties, it is not partition of U.

4. Let M = $\{$ {1, 2, a, c}, {1, 3, b} $\}$ (say).
   We test whether M is a partition of U = {1, 2, 3, a, b, c}:

   a. {1, 2, a, c} and {1, 3, b} are two **non-empty** subsets of U, but
   b. {1, 2, a, c} ∩ {1, 3, b} = {1} ≠ ∅.
   c. We do have that {1, 2, a, c} ∪ {1, 3, b} = {1, 2, 3, a, b, c} = U but
   since M does not have all the necessary properties, it is not partition of U.

From the arguments provided we can deduce that alternative 1 should be selected.

Refer to study guide, pp 94, 95.

Question 11 **Alternative 1**

Which one of the following statements regarding the relation R is TRUE?

1. R is a strict partial order.
2. R is strict total order.
3. R is weak partial order.
4. R is weak total order.

Discussion

In addition to the definitions for irreflexivity, antisymmetry and transitivity which we looked at in other question solutions, we provide the definition for trichotomy.

A relation P on B satisfies the requirement for trichotomy iff, for every x and y chosen from B such that x ≠ y, we have that x and y are comparable,
i.e. for all x, y ∈ B such that x ≠ y, x P y or y P x (i.e. (x, y) ∈ P or (y, x) ∈ P).

Furthermore, a <u>weak partial order</u> is reflexive on A, antisymmetric and transitive,
a <u>strict partial order</u> is irreflexive, antisymmetric and transitive,
a <u>weak total (or linear) order</u> is reflexive on A, antisymmetric and transitive, and satisfies trichotomy, and
a <u>strict total (or linear) order</u> is irreflexive, antisymmetric and transitive, and satisfies trichotomy.

R = { (a, c), (3, c), (3, a) } is a relation on A = {a, c, 2, 3}. Let's determine which properties the relation R has:

Irreflexivity:
There is no ordered pair in R such that (x, x) ∈ R (with x ∈ A), thus R is irreflexive.

Antisymmetry:
(a, c) but (c, a) ∉ *R;*
(3, c) but (c, 3) ∉ *R;*
(3, a) but (a, 3) ∉ *R.*
Thus R is antisymmetric.

Transitivity:
(3, a), (a, c) ∈ R then also (3, c) ∈ R,
thus R is transitive.

R does not satisfy trichotomy.
We provide a counterexample:

$2 \neq 3$ but $(2, 3) \notin R$ and $(3, 2) \notin R$.

R is irreflexive, antisymmetric and transitive, thus R is a strict partial order and alternative 1 should be chosen.

Refer to study guide, pp 75 – 78, 88.

Question 12 **Alternative 1**

The relation R does not satisfy trichotomy. Which ordered pair(s) can be included in R so that R would satisfy trichotomy?

1. (a, 2), (c, 2) & (3, 2)
2. (c, a), (c, 3) & (a, 3)
3. (c, a) & (c, 3)
4. (2, 3)

Discussion

For trichotomy we ask the question: Which ordered pairs can be included in R so that it will be true that for all $x, y \in A$ with $x \neq y$, we have $(x, y) \in R$ or $(y, x) \in R$?

We compare the elements of A: $a \neq 2$; $c \neq 2$ and $2 \neq 3$ but these elements are not grouped in ordered pairs of R. We only have $(a, c), (3, c), (3, a) \in R$. We can include the ordered pairs
(a, 2), (c, 2) & (3, 2) in R then R_1 (say) will satisfy trichotomy.

Let $R_1 = \{(a, c), (3, c), (3, a), (a, 2), (c, 2), (3, 2)\}$ then R_1 satisfies trichotomy.

From the arguments provided we can deduce that alternative 1 should be selected.

Refer to study guide, p 78.

Let R be the relation on $\mathbb{Z}$ (the set of integers) defined by $(x, y) \in R$ iff $y^2 = x^2$.
Answer questions 13 to 15 by using the given relation R.

Question 13 **Alternative 3**

Which one of the following is an ordered pair in R?

1. $(1, 3)$
2. $(2, 1)$
3. $(-1, -1)$
4. $(6, 4)$

Relation R on $\mathbb{Z}$ is defined by $(x, y) \in R$ iff $y^2 = x^2$.
We consider the ordered pairs provided in the different alternatives:

1. Let $x = 1$ and $y = 3$ then
   $y^2 = 3^2 = 9$ and
   $x^2 = 1^2 = 1 \neq 9$
   thus $(1, 3) \notin R$.

2. Let $x = 2$ and $y = 1$ then
   $y^2 = 1^2 = 1$ and
   $x^2 = 2^2 = 4 \neq 1$
   thus $(2, 1) \notin R$.

3. Let $x = -1$ and $y = -1$, then
   $y^2 = (-1)^2 = 1$ and
   $x^2 = (-1)^2 = 1$
   Thus $y^2 = x^2$, therefore $(-1, -1) \in R$.

4. Let $x = 6$ and $y = 4$ then
   $y^2 = 4^2 = 16$ and
   $x^2 = 6^2 = 36 \neq 16$
   thus $(6, 4) \notin R$.

From the arguments provided we can deduce that alternative 3 should be selected.

Refer to study guide, pp 71, 72.

Question 14 **Alternative 3**

R is an equivalence relation since R is reflexive, symmetric and transitive. Which one of the following is a valid proof to show that R is reflexive on A?

1. Let $x, y \in \mathbb{Z}$ be given.
   then $y^2 - x^2 = 0$
   i.e. $x^2 = y^2$.
   Thus $(x, x) \in R$.
2. Let $x, y \in \mathbb{Z}$ be given.
   $y^2 - x^2 = 0$
   thus $x^2 = x^2$.
   Thus $(x, x) \in R$.
3. Let $x \in \mathbb{Z}$ be given.
   $x = x$
   i.e. $x^2 = x^2$.
   Thus $(x, x) \in R$.
4. Let $x, y \in \mathbb{Z}$ be given.
   $x^2 = x^2$
   i.e. $y = \pm x^2$
   Thus $(x, x) \in R$.

Discussion

For reflexivity we should prove that for all $x \in A$, $(x, x) \in R$. Note that only one variable (namely x) should play a role in a valid proof. This condition rules out alternatives 1, 2 & 4 since two variables namely x and y play roles in the attempted proofs.

Alternative 3 provides a valid proof for reflexivity of R on A:

Let $x \in \mathbb{Z}$ be given.
$x = x$
i.e. $x^2 = x^2$.
Thus $(x, x) \in R$.

Refer to study guide, p 75.

Question 15 **Alternative 4**

Which one of the following is an equivalence class of R?

1. $[2] = \{y \mid y^2 = 2\}$
2. $[1] = \{1\}$
3. $[0] = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$
4. $\{-3, 3\}$

Discussion
We know that for each $x \in \mathbb{Z}$ we define the equivalence class $[x]$ by
$[x] = \{y \mid y \in \mathbb{Z} \text{ and } x\, R\, y\} = \{y \mid y \in \mathbb{Z} \text{ and } y^2 = x^2\}$.

We consider the possible equivalence class provided in the different alternatives:

1. By the definition provided for [x],

$$\begin{aligned} [2] &= \{y \mid y^2 = 2^2\} \\ &\neq \{y \mid y^2 = 2\} \end{aligned}$$

Thus alternative 1 does not provide a valid equivalence class for R.

2. By the definition provided for [x],

$$\begin{aligned} [1] &= \{y \mid y^2 = 1^2\} \\ &= \{y \mid y^2 = 1\} \\ &= \{y \mid y = \pm\sqrt{1}\} \\ &= \{-1, 1\} \\ &\neq \{1\} \end{aligned}$$

Thus alternative 2 does not provide a valid equivalence class for R.

3. By the definition provided for [x],

$$\begin{aligned} [0] &= \{y \mid y^2 = 0^2\} \\ &= \{y \mid y^2 = 0\} \\ &= \{y \mid y = 0\} \\ &= \{0\} \\ &\neq \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\} \end{aligned}$$

Thus alternative 3 does not provide a valid equivalence class for R.

4. Let x = 3, then we determine the y-values in the set:

[3] $= \{y \mid y^2 = 3^2\}$
$= \{y \mid y^2 = 9\}$
$= \{y \mid y = \pm \sqrt{9}\}$
$= \{-3, 3\}$

Thus alternative 4 provides a valid equivalence class for R and therefore alternative 4 should be selected as the correct one.

Refer to study guide, p 91.

Questions similar to assignment 03 questions.

Suppose U = {1, 2, 3, a, b, c} is a universal set with the subset A = {a, c, 2, 3}.
Answer questions 1 and 2 by using the given sets U and A.

Question 1 **Alternative 4**

Which one of the following relations is a function on **A**?

1. {(1, 3), (b, 3)}
2. {(a, c), (b, c), (c, b), (1, 3), (2, 3), (3, a)}
3. {(a, a), (c, c), (2, 2)}
4. {(a, c), (c, c), (2, 3), (3, 3)}

Discussion

First we look at the definition of a function:

> *Suppose R $\subseteq$ B × C is a binary relation from a set B to a set C. We may call R a function from B to C if every element of B appears exactly once as the first co-ordinate of an ordered pair in R (i.e. f is functional), and the domain of R is exactly the set B, i.e. dom(R) = B.*

We consider the relations provided in the different alternatives:

1. Let L = {(1, 3), (b, 3)}. L is not a relation on A. The first co-ordinates 1 and b are not elements of A and furthermore dom(L) ≠ A. Thus L is not a function on A.

2. Let M = {(a, c), (b, c), (c, b), (1, 3), (2, 3), (3, a)}. M is not a relation A. The first co-ordinates b and 1 are not elements of A. Thus M is not a function on A.

3. Let N = {(a, a), (c, c), (2, 2)}. {a, c, 2} = dom(N) ≠ A. Thus N is not a function on A.

4. Let S = {(a, c), (c, c), (2, 3), (3, 3)}. S is a function on A.
   Reason: for each first co-ordinate there is only one corresponding second co-ordinate (S is functional) and dom(S) = A. (We note that {c, 3} = ran(S) $\subseteq$ A.)

From the arguments provided we can deduce that alternative 4 should be selected.

Refer to study guide, pp 98, 99.

Question 2 **Alternative 1**

Which one of the following alternatives represents an injective function from **A** to **U** with the range {1, 3, a, c}?

1. {(a, 1), (3, a), (c, c), (2, 3)}
2. {(1, 1), (2, 2), (3, 3), (a, a)}
3. {(a, 2), (c, a), (2, 1), (3, d)}
4. {(a, 1), (a, c), (3, a), (2, 2), (2, 3), (3, a)}

Discussion

First we look at the definition of an injective function:

A function f: B $\rightarrow$ C is injective iff f has the property that *whenever $f(a_1) = f(a_2)$ then $a_1 = a_2$ (or whenever $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$) .*

We consider the relations provided in the different alternatives:

1. Let L = {(a, 1), (3, a), (c, c), (2, 3)}. For each first co-ordinate there is only one second co-ordinate in an ordered pair of L, thus L is injective. (Different elements of the domain clearly go to different elements of the range.)

2. Let M = {(1, 1), (2, 2), (3, 3), (a, a)}. M is not a relation from A to U.
   The ordered pair (1, 1) is an element of M but the first co-ordinate 1 is not an element of A. Thus M is not a function from A to U and thus M cannot be an injective function.

3. Let N = {(a, 2), (c, a), (2, 1), (3, d)}. N is not a relation from A to U.
   (3, d) is an element of N but the second co-ordinate d is not an element of the range nor of U. Thus N is not a function with range {1, 3, a, c} and thus N cannot be the injective function.

4. Let S = {(a, 1), (a, c), (3, a), (2, 2), (2, 3), (3, a)}. The elements a and 2 appear twice as first co-ordinates thus S is not an injective function. (Actually, S is not a function.)

From the arguments provided we can deduce that alternative 1 should be selected.

Refer to study guide, p 106.

Let g be a function from $\mathbb{Z}^+$ (the set of positive integers) to $\mathbb{Q}$ (the set of rational numbers) defined by $(x, y) \in g$ iff $y = \frac{2}{5}x - 3$ ($g \subseteq \mathbb{Z}^+ \times \mathbb{Q}$) and let f be a function on $\mathbb{Z}^+$ defined by $(x, y) \in f$ iff $y = 5x + 1$ ($f \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$).

Answer questions 3 to 7 by using the given functions g and f.

Question 3 **Alternative 3**

Which one of the following is an ordered pair belonging to g?

1. (5, 1)
2. $(1, 2\frac{3}{5})$
3. $(3, -1\frac{4}{5})$
4. $(\frac{5}{2}, -2)$

Discussion

The first co-ordinates of elements of g are elements of $\mathbb{Z}^+$ and the second co-ordinates are elements of $\mathbb{Q}$.

We consider the ordered pairs provided in the different alternatives:

1. $(x, y) \in g$ iff $y = \frac{2}{5}x - 3$. Is $(5, 1) \in g$?
   Let x = 5 then
   $$y = \frac{2}{5}x - 3$$
   $$= \frac{2}{5} \cdot 5 - 3$$
   $$= 2 - 3$$
   $$= -1$$

Thus $(5, -1) \in g$ but $(5, 1) \notin g$.

2. Is $(1, 2\frac{3}{5}) \in g$?
Let x = 1 then
$y = \frac{2}{5}x - 3$
$= \frac{2}{5} \cdot 1 - 3$
$= \frac{2}{5} - 3$
$= -2\frac{3}{5}$
Thus $(1, -2\frac{3}{5}) \in g$ but $(1, 2\frac{3}{5}) \notin g$.

3. Is $(3, -1\frac{4}{5}) \in g$?
Let x = 3 then
$y = \frac{2}{5}x - 3$
$= \frac{2}{5} \cdot 3 - 3$
$= \frac{6}{5} - 3$
$= -1\frac{\mathbf{4}}{\mathbf{5}}$
Thus $(3, -1\frac{4}{5}) \in g$.

4. Is $(\frac{5}{2}, -2) \in g$?
Let $x = \frac{5}{2}$, then
$y = \frac{2}{5}x - 3$
$= \frac{2}{5} \cdot \frac{5}{2} - 3$
$= 1 - 3$
$= -2$

It seems as if $(\frac{5}{2}, -2) \in g$ but $(\frac{5}{2}, -2) \notin g$ since $\frac{5}{2} \notin \mathbb{Z}^+$ and thus $\frac{5}{2}$ is not an element of the domain of g.

From the arguments provided we can deduce that alternative 3 should be selected.

Refer to study guide, p 98.

Question 4 **Alternative 4**

Which one of the following alternatives represents the domain of g (i.e. dom(g))?

1. $x = \frac{5}{2}(y + 3)$
2. {x | for some $x \in \mathbb{Z}^+$, $y = \frac{\mathbf{2}}{\mathbf{5}}\boldsymbol{x} - \mathbf{3}$}
3. {x | $\frac{\mathbf{2}}{\mathbf{5}}\boldsymbol{x} - \mathbf{3}$ is not a rational number}
4. {x | $\frac{\mathbf{2}}{\mathbf{5}}\boldsymbol{x} - \mathbf{3}$ is a rational number}

Discussion

Let's first look at the definition for the domain of a function:
Given a function T from X to Y, the domain of T is defined by
dom(T) = {x | for some y ∈ Y, (x, y) ∈ T}, i.e. the set of all elements that appear as first co-ordinates in the ordered

pairs of T (each first co-ordinate appears only once as first co-ordinate). Furthermore, each first co-ordinate is

associated with a second co-ordinate which is an element of Y. Also, dom(T) = X.

By the definition of a function
dom(g) = {x | for some y ∈ $\mathbb{Q}$, (x, y) ∈ g}
= {x | for some y ∈ $\mathbb{Q}$, y = $\frac{2}{5}x - 3$ }
= {x | $\frac{2}{5}x - 3$ is a rational number} (y ∈ $\mathbb{Q}$ thus y = $\frac{2}{5}x - 3$ is a rational number)

From the above we can deduce that alternative 4 should be selected.

Refer to study guide, p 98.

Question 5 **Alternative 2**

The function f is NOT surjective. Which of one of the following values for y provides a counterexample that can be used to prove that f is not surjective?

1. y = 6
2. y = 7
3. y = 11
4. y = 16

Discussion

The function f is NOT surjective thus ran(f) ≠ $\mathbb{Z}^+$. A counterexample provides a value y for which there is ***no*** *element x ∈ $\mathbb{Z}^+$ such that y = 5x + 1 ∈ $\mathbb{Z}^+$ i.e. x = (y − 1)/5 ∈ $\mathbb{Z}^+$.*

We consider the ordered pairs provided in the different alternatives:

1. Let y = 6 then
   x = (y − 1)/5
   = (6 − 1)/5
   = 5/5 = 1 ∈ $\mathbb{Z}^+$ thus 6 ∈ ran(f).

2. Let y = 7 then
   x = (7 − 1)/5
   = 6/5 ∉ $\mathbb{Z}^+$ thus 7 ∉ ran(f).

3. Let y = 11 then
   x = (11 − 1)/5
   = 10/5 = 2 ∈ $\mathbb{Z}^+$ thus 11 ∈ ran(f).

4. Let y = 16 then
   x = (16 − 1)/5
   = 15/5 = 3 ∈ $\mathbb{Z}^+$ thus 16 ∈ ran(f).

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, p 89.

Question 6 **Alternative 1**

Which one of the following alternatives represents the image of x under g ○ f (i.e. g ○ f(x))?

1. $2x - 2\frac{3}{5}$
2. $2x - 14$
3. $10x - 3\frac{1}{5}$
4. $2x + 3\frac{2}{5}$

Discussion

Given the functions f: A → B and g: B → C the composite function g ○f: A → C is defined by g ○f(x) = g(f(x)).

g ○ f(x)

= g(f(x))

= $\frac{2}{5}$ f(x) – 3

= $\frac{2}{5}(5x + 1) - 3$

= $2x + \frac{2}{5} - 3$

= $2x - 2\frac{3}{5}$

From the above we can deduce that alternative 1 should be selected.

Refer to study guide, p 110.

Question 7 **Alternative 1**

Which one of the following statements regarding the relation f is TRUE?

1. The composition f ○ f is a function.
2. The composition f ○ f is surjective.
3. f is invertible.
4. f is a bijective function.

We consider the statements provided in the different alternatives:

1. By theorem 7.1 the composition of two functions is a function thus f ○ f is a function.

2. In question 5 we proved that f is not surjective thus by theorem 7.2 f ○ f is not surjective.

3. f is not surjective thus ran(f) ≠ $\mathbb{Z}^+$.

The inverse relation f^{-1} is defined by {(y, x) | (x, y) ∈ f}.

However, f^{-1} is not a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ since dom(f^{-1}) ≠ $\mathbb{Z}^+$. (Remember, ran(f) ≠ $\mathbb{Z}^+$ thus dom(f^{-1}) ≠ $\mathbb{Z}^+$.

By the definition of invertible functions f^{-1} is not invertible.

Alternatively we can argue that f is not bijective (surjective and injective) then by theorem 7.4 f is not invertible.

4. Since f is not surjective and injective then by definition it is not a bijective function.

From the above arguments we can deduce that alternative 1 should be selected.
Refer to study guide, pp 105, 110 - 113.

Let A = {□, ◊, ☼, ⌂} and let # be a binary operation from A × A to A presented by the following table:

#	□	◊	☼	⌂
□	⌂	◊	□	◊
◊	◊	⌂	◊	□
☼	□	◊	☼	⌂
⌂	◊	□	⌂	⌂

Question 8 **Alternative 3**

Which one of the following symbols is the identity element for #?

1. □
2. ◊
3. ☼
4. ⌂

Discussion
Definition of an identity element of a binary operation:
An element e of X is an identity element in respect of the binary operation
◊ : $X \times X \rightarrow X$ iff e ◊ x = x ◊ e = x for all $x \in X$.
Is it possible to identify an element *e* in A such that *e* # x = x # *e* = x for all x ∈ A?
Yes, ☼ is such an element of A:

☼ # □ = □ # ☼ = □,
☼ # ◊ = ◊ # ☼ = ◊, **and**
☼ # ⌂ = ⌂ # ☼ = ⌂**.**

So ☼ acts as an identity element for #.

From the above we can deduce that alternative 3 should be selected.

Refer to study guide, pp 119, 120.

Question 9 **Alternative 2**

can be written in list notation. Which one of the following ordered pairs is NOT an element of the list notation set representing #?

1. ((□, ⌂), ◊)
2. ((⌂, ☼), ◊)
3. ((☼, □), □)
4. ((◊, ⌂), □)

We consider the ordered pairs provided in the different alternatives:

1. From the table □ # ⌂ = ◊

thus ((□, ⌂), ◊) ∈ #.

2. ⌂ # ☼ = ⌂

thus ((⌂, ☼), ⌂) ∈ # but ((⌂, ☼), ◊) ∉ #.

3. From the table ☼ # □ = □

thus ((☼, □), □) ∈ #.

4. From the table ◊ # ⌂ = □

thus ((◊, ⌂), □) ∈ #.

From the above arguments we can deduce that alternative 2 should be selected.

Refer to study guide, pp 116, 117.

Question 10 **Alternative 4**

Perform the following matrix multiplication operation:

$$\begin{bmatrix} 11 & 3 & 0 \\ 2 & 4 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ 5 & -2 \end{bmatrix}$$

Which one of the following alternatives represents the correct answer to the above operation?

1. The operation is not possible.

2. $\begin{bmatrix} -11 & -3 & 0 \\ 4 & 4 & 2 \\ 9 & 0 & 6 \end{bmatrix}$

3. $\begin{bmatrix} 20 & 12 \\ 13 & 5 \\ 11 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -8 \\ 9 & 0 \\ 10 & -7 \end{bmatrix}$

We perform the following matrix multiplication operation:

$$\begin{bmatrix} 11 & 3 & 0 \\ 2 & 4 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ 5 & -2 \end{bmatrix}$$

Discussion

A $\mathbf{3} \times 3$ *matrix multiplied by a* $3 \times \mathbf{2}$ *matrix gives a* $\mathbf{3} \times \mathbf{2}$ *matrix.*

We determine a_{ij}:

$a_{11} = 11 \cdot 0 + 3 \cdot 1 + 0 \cdot 5 = 3$

$a_{12} = 11 \cdot -1 + 3 \cdot 1 + 0 \cdot -2 = -8$

$a_{21} = 2 \cdot 0 + 4 \cdot 1 + 1 \cdot 5 = 9$

$a_{22} = 2 \cdot -1 + 4 . 1 + 1 \cdot -2 = 0$

$a_{31} = 3 \cdot 0 + 0 \cdot 1 + 2 \cdot 5 = 10$

$a_{32} = 3 \cdot -1 + 0 \cdot 1 + 2 \cdot -2 = -7$

Thus the answer to the multiplication of the matrices is

$$\begin{bmatrix} 3 & -8 \\ 9 & 0 \\ 10 & -7 \end{bmatrix}$$

From the above we can deduce that alternative 4 should be selected.

Refer to study guide, p 131, 132.