Exam Preparation

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XAT1503: LINEAR ALGEBRA

Exercise 1

Question 1.
Assume we are given the system

$$\begin{cases} x & +y & -z & +2t = 2 \\ 2x & & +z & -t = 3 \\ & -y & & +2t = 2. \end{cases}$$

Which of the following elements are solutions of the above system?

(1) (1,-1,0,1)

(2) (1,-1,0,1, 0)

(3) (0,1,3,1)

(4) (2,-3,3,1)

Question 2
Consider the system

$$\begin{cases} x & +2y & & = 1 \\ 3x & +ky & & = 3 \\ x & +ky & +z & = 2 \end{cases}$$

(1) For which value(s) of k is $(1, 0, 1)$ a solution to the above system?

(2) For which value(s) of k is $(1, 0, -1)$ a solution to the above system?

(3) For which value(s) of k is $(3, -1, 5)$ a solution to the above system?

Additional Exercises

Same instructions as Question 2.

(1)

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & k+1 & 3 \\ 0 & 0 & k-3 \end{bmatrix}$$

(2)

$$\begin{bmatrix} \lambda & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & \lambda+3 \end{bmatrix}$$

(3)

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & k-1 & 3 \\ 0 & 0 & (k-1)(k+2) \end{bmatrix}$$

Exercise 2

Suppose A, B and C are 2×2 matrices such that $\det(A) = 3$, $\det(B^{-1}) = -2$ and $\det(C^T) = 4$.

Evaluate

(a) $\det(ABC)$

(b) $\det\left([C^2]^T\right)$

(c) $\det(-4A)$

(d) $\det(-4A^{-1})$

(e) $\det\left([-4A]^{-1}\right)$.

Exercise 3

For which values of k is the coefficient matrix of the system given in Question 2, Exercise 1 invertible?

Exercise 4

Let $A = \begin{bmatrix} -1 & 5 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ -2 & 1 \end{bmatrix}$,

(i) Compute

  (a) A^{-1}, B^{-1} and AB

  (b) AA^{-1}, BB^{-1} and $A^{-1}B^{-1}$

  (c) A^2, B^2 and $A^2 - B^2$.

(ii) Using (a), determine a matrix Z such that $ZA = B$

(iii) Using (a) determine a matrix Y such that $AY = B$.

Exercise 5

(1) Solve the following system with Cramer's Rule

$$\begin{cases} 2x_1 + x_2 + x_3 = 3 \\ x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}$$

(2) Use cofactors to find the determinant of the coefficient matrix of the system.

Exercise 6

Let $\vec{a}$ and $\vec{b}$ be the vectors given by $\vec{a} = \langle 2, 0, -1 \rangle$ and $\vec{b} = \langle 1, -1, 3 \rangle$.

(1) Compute

   (i) $||\vec{a}||$ and $||\vec{b}||$
   (ii) $\vec{a} \cdot \vec{a}$, $\vec{a} \cdot \vec{b}$ in two different ways.
   (iii) $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{b}$

(2) Assume that θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$. Write down $\cos\theta$ in terms of $\vec{a} \cdot \vec{b}$ and $||\vec{a}||$ and $||\vec{b}||$. Deduce $\cos\theta$ and find the angle θ between the two vectors.

(3) Find an equation of the plane P containing $\vec{a}$ and $\vec{b}$

(4) Calculate the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$

(5) Determine the parametric equations of the plane containing $\vec{a}$ and $\vec{b}$

(6) Determine the equation of the plane Q parallel to the plane P in (3) and passing through point

   (i) $(0, 0, 1)$
   (ii) $(0, 1, 0)$
   (iii) $(1, 0, 1)$
   (iv) $(1, 1, 0)$
   (v) $(0, 1, 1)$
   (iv) $(-1, 1, -1)$

(7) Find an equation of the line L perpendicular to P in (3) and passing through the tip (terminal point) of the vector $\vec{u}$ where $\vec{u}$ is

   (i) the unit vector in the direction of $\vec{a}$ with initial point $P_0 = (2, 0, -1)$
   (ii) the unit vector in the direction of $\vec{b}$ with initial point $P_1 = (1, -1, 3)$.

(8) Find a point-normal form of the equation of the plane passing through the point $P(1, -2, -1)$ and having normal $\vec{n} = \langle 1, -1, 4 \rangle$

(9) Determine whether the planes $x + 2y - 7z = 3$ and $-2x + y - 9 = 0$ are perpendicular or parallel.

Exercise 7

(1) Find all the roots (denote them by z_1 and z_2) of $(\sqrt{2} - i\sqrt{2})^{1/2}$ and sketch them as vectors in the complex plane. Write z_1 and z_2 in

   (a) standard form
   (b) polar form

(2) Use your result from 1(a) to find z_1/z_2 and $\overline{z_1/z_2}$

(3) Use de Moivre's Theorem to express $\sin(2\theta)$, $\sin(3\theta)$, $\cos(2\theta)$, $\cos(3\theta)$ and $\cos(4\theta)$ in terms of powers of

(a) $\sin\theta$ and $\cos\theta$.

(b) $\sin\theta$ only

(c) $\cos\theta$ only

(4) Determine the 4^{th} roots of 16 in

(i) standard form

(ii) polar form

(5) Let $z_1 = 2 - 3i$, $z_2 = -1 + 4i$. Find

(i) $\overline{z_1}$, $\overline{z_2}$, $\overline{z_1} - z_1$, $\overline{z_1} + z_1$, $\mathrm{Re}(z_1)$, $\mathrm{Re}(z_2)$ and $\mathrm{Re}(z_1 z_2)$

(ii) $\dfrac{z_1}{z_2}$ and $\overline{\dfrac{z_1}{z_2}}$.

(iii) Deduce the principal arguments of $\dfrac{z_1}{z_2}$ and $\overline{\dfrac{z_1}{z_2}}$.

(6) Use de Moivre's Theorem to determine i^3, $(1+i)^{100}$ in standard and polar form.

Exercise 8

Consider the points $P_0(-1,2)$, $P_1(1,3)$, $P_2(3,-2)$ and $P_3(1,1)$.

Asume that the points P_i, $i = 0,1,2,3$ have position vectors r_i, $i = 0,1,2,3$.

(i) Find the point $-P_0 - P_1 + P_2$

(ii) Solve the system

$$xr_0 + yr_1 + zr_2 = r_3.$$

(iii) Find z whenever $x = y$.

Exercise 9

(a) Let

$A = \begin{bmatrix} 7 & 2 & 1 \\ 3 & 4 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $b = \begin{bmatrix} z-2 \\ 3x - z + 1 \end{bmatrix}$. Verify that $A = -A + 2A$. If we manuaally remove the last column of A, we obtain a matirx B. Write down the matrix B and find x, y and z such that $BX = b$.

Exercise 10

Write your own Summary or Mind Map for each of the following tables

(1)

Linear systems of equations

2

Matrices

3

finding the inverse A^{-1} of a square matrix A

Elementary Matrices

4

Determinant

5

Invertibility of Matrices