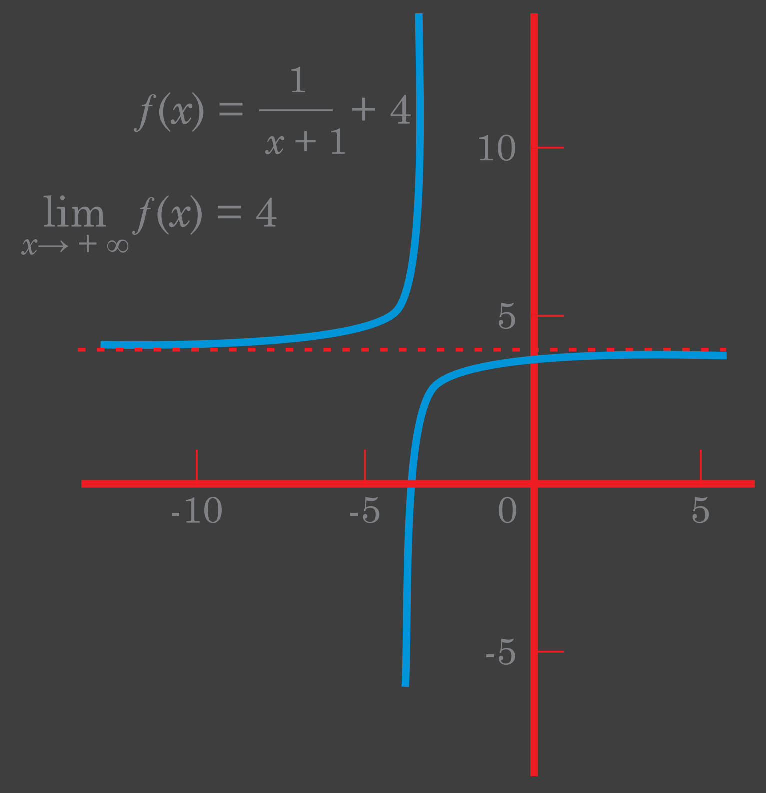

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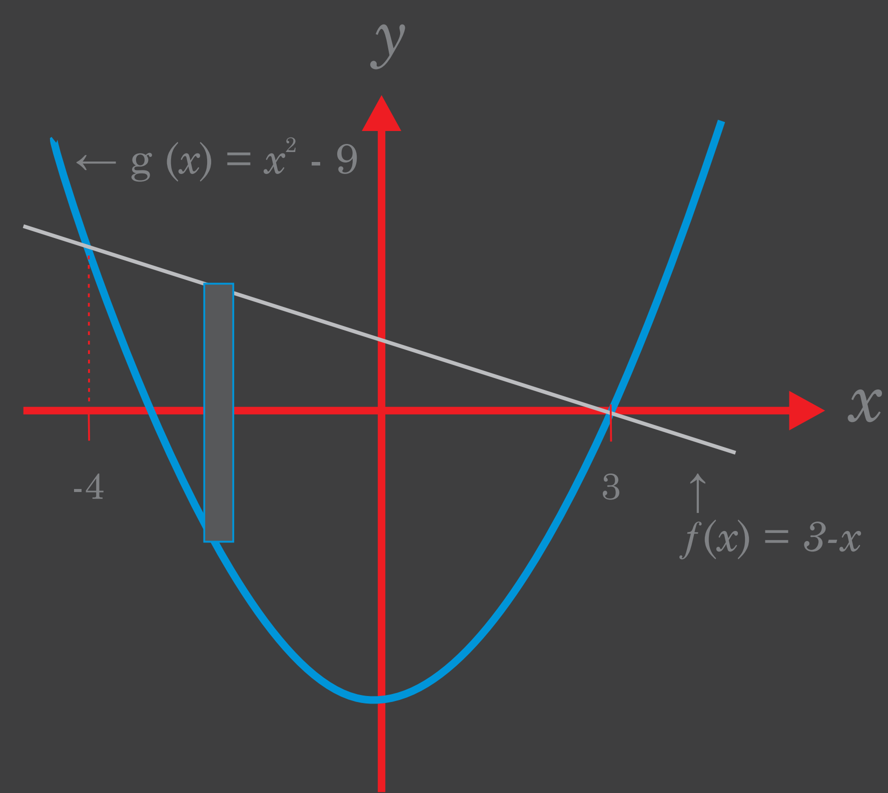

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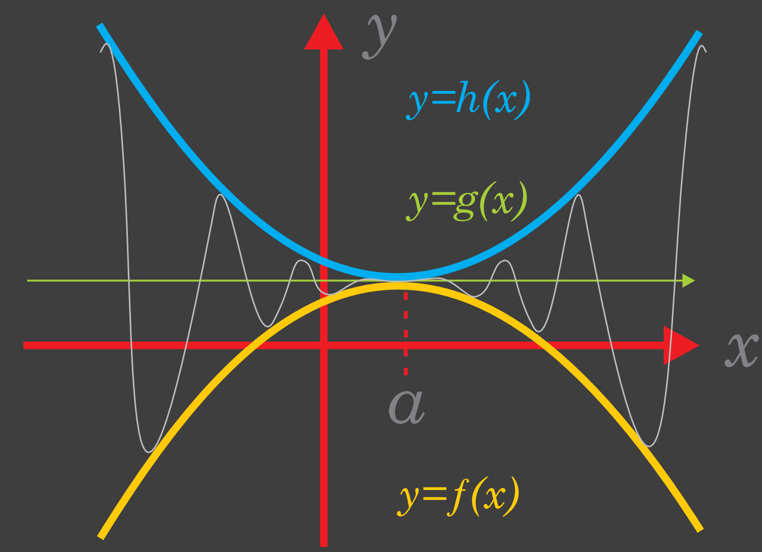

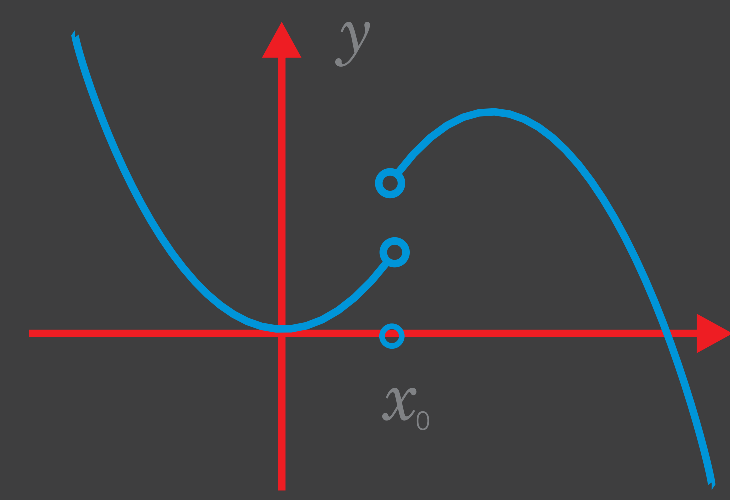

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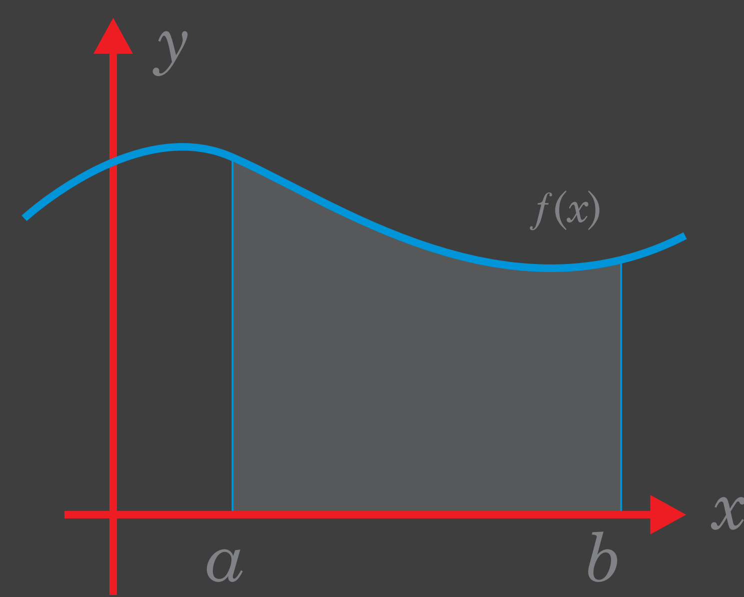

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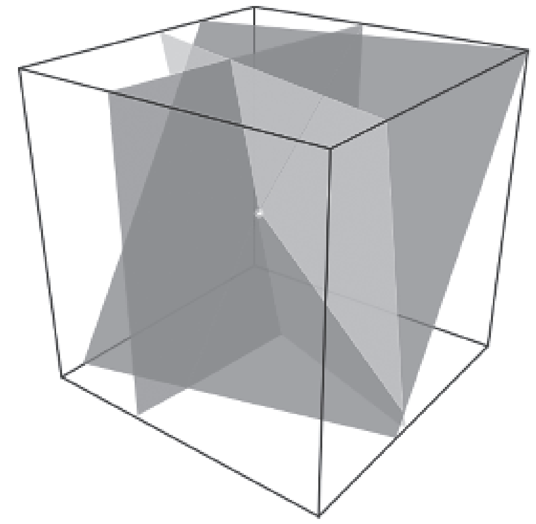


Calculus A

ONLY STUDY GUIDE FOR MAT1512 • XAT1512

SB Mugisha
Department of Mathematical Sciences

UNIVERSITY OF SOUTH AFRICA
PRETORIA

Contents

Study Unit 1

Introduction

Calculus is a set of formal rules and procedures. It gives you the tools you need to measure changes both qualitatively and quantitatively. *Wikipedia* (`www.wikipedia.org`) defines calculus as a branch of mathematics that includes the study of limits, derivatives, integrals, and infinite series, which constitute a major part of modern university education. Calculus has widespread applications in science and engineering and is used to solve complex and expansive problems for which algebra alone is insufficient. It builds on analytical geometry and mathematical analysis and includes two major branches – differential calculus and integral calculus – which are related through the Fundamental Theorem of Calculus. Differential calculus explores and analyses rates of change quantitatively and qualitatively. Integral calculus deals with the analysis (quantitatively and/or qualitatively) of how quantities or measures of values accumulate or diminish over time. The two processes – differentiation and integration – are reciprocal.

The purpose of this module is to equip you, the student, with those basic skills in differential and integral calculus that are essential for the physical, life and economic sciences. Most of the time you will be dealing with functions. Basically, a function is a generalised input-output process that defines the mapping of a set of input values to a set of output values. It is often defined as a rule for obtaining a numerical value from another given numerical value. You are also going to have to develop a very large repertoire of methods for depicting functions graphically/geometrically.

This course is built around your prescribed book. The purpose of this study guide is to guide you through those parts of the prescribed book that you must study for this module, and to provide you with many additional worked examples. The prescribed book is:

James Stewart Calculus
Metric Version 8 Edition
Early Transcendentals
Cengage Learning
ISBN 13:978-1-305-27237-8

1.1 How to Use the Study Guide

From now on we will refer to the prescribed book as "Stewart". The study guide must always be used in conjunction with the prescribed textbook, because it is not a complete set of notes on the book. Chapters 1 to 4 of this study guide contain many additional worked problems, taken from past examination papers and assignments. Before going through them, study the relevant parts in Stewart, and do the examples and some of the exercises in Stewart. Also, before going through our solution to a problem in the study guide, try solving it yourself. **Remember that reading maths often means reading the same thing over and over again.**

We have included numerous worked examples for you. These are designed to stimulate your thinking in such a way that you will come to appreciate and master the delicate beauty and intricacies of the subject. All you have to do is keep going! Follow all the instructions given. Try to write down all the answers to the activities in full. This is extremely important, as a major part of learning mathematics is **thinking** and **writing** down what you think. By writing everything down, you will develop the essential skill of **communicating mathematics effectively**. The other reason for writing down your answers, is to prevent you from losing your train of thought about a concept or mathematical idea. If this happens, it takes a while to get your reasoning back to the same point. If you have everything written down, you can also go back the next day and check your reasoning. Learning mathematics is an activity, and you will only learn by doing. Because you will be thinking about the problems, you will, in most cases, be able to determine by yourself whether your reasoning is right, or not.

Your assignments are included in *Tutorial Letter 101.* Attempt these once you have completed all the related activities in your study guide. Spend a part of your time each day doing some of the questions, and a part studying new material. Being able to do an assignment is proof that you have mastered the work of that particular section. In *Tutorial Letter 101* we have indicated which assignments you should submit for evaluation.

Also remember that statements, theorems and definitions are the building blocks of your mathematical language – you cannot learn anything without knowing the basic facts. Begin by reading the preface of your prescribed book. This should give you a good idea of the importance of the subject you are about to study.

1.2 Keys to Success in Studying Mathematics

Plan	Plan each week according to what will happening in your life that week. Decide how many study hours you need, and where you are going to fit those hours in. Write down what you plan to do in each session. This way you can look forward to the work you have set for yourself and measure your success.
Evaluate	Evaluate each study session. Did you enjoy your studies? If not, why not? What can you change to make things better? Did you achieve what you set out to achieve? Did you use your time constructively? What can you do to improve your time management?
Understand	Understand the way you learn. One can only learn maths through repetition and practice. DO IT, even if you think you know the answer. Understanding comes through repetition, and this understanding eventually brings the JOY of mastering mathematics.
Structure	Structure each study session for best results. A suggestion is to divide your time into four parts: (1) Do five to ten minutes' revision and read through the important points: (2) Go through all the questions of the previous activities. (3) Study new material by following the instructions in the study guide. (4) Attempt the questions in your assignments about the sections you have completed.
Get organised	(1) Make notes of the facts you have to remember. (2) Write down the answers to all your activities. (3) Complete all the ASSIGNMENTS!
Pace yourself	Work out how fast you will have to work. Some sections will take longer than others, but assign about one month for each chapter. This will give you two to four days for each section in each chapter. Bear in mind that this is not the only module you have registered for.

1.3 Preparing for the Examination

You are studying to improve yourself and **NOT** only to pass the examination. The examination is there to **REWARD** you for what you have done and to **CONFIRM** that you have done your work properly. If you have followed all the instructions above, you will already be prepared for the examination and have nothing to worry about. Many students spend their time thinking that they cannot master mathematics. This is not true. If you put your mind to it and work hard, you will eventually master this subject.

GOOD LUCK!

Study Unit 2

Functions and Models

2.1 Background

This chapter of the study guide deals with Chapter 1 in Stewart. It forms a link between school mathematics and calculus. Topics in algebra, trigonometry, analytic geometry and functions which are needed for calculus, are reviewed. Graphs of functions are studied, since they enable us to see and understand the behaviour of functions. In studying this chapter, you should brush up on and revise your school mathematics – especially those parts that are needed for an effective study of calculus. This chapter contains background material. At the same time, some topics which are important to understand calculus are introduced. Therefore, the chapter will contain a few topics which you may have touched on only briefly, or not at all, at school.

2.2 Learning Outcomes

At the end of this chapter, you should be able to

- algebraically manipulate real numbers, solve equations and work comfortably with mathematical concepts such as variables, inequalities and absolute values
- recognize, demonstrate your knowledge of and work with the different types of functions (polynomial, rational, trigonometric, exponential and logarithmic), their properties and their representations

In calculus, the basic mathematical tool at the root of all we do is the function. That is why, in this functions and models chapter, you have to familiarize yourself with mathematical ideas related to functions, such as the different types of functions, their graphs and representations of how they behave. First read the functions and models as given on page 9 in Stewart. You should **study** the whole chapter. Remember that problem solving plays a crucial role in the process of studying calculus.

Note: The way to master calculus is to solve lots of calculus problems!

2.3 Principles of Problem Solving

There are no exact rules for solving problems. However, it is possible to outline some general steps, or to give some hints or principles that may be useful in solving certain problems.

The following steps and principles, which have been adapted from George Polya's book *How to solve it*, are in fact just common sense made explicit.

Step 1: Understand the problem. Read the problem and make sure that you understand what it is about. Ask yourself the following questions: What is the unknown? What are the given quantities? What conditions are given? In many cases it is useful to draw a "given" diagram and identify the given quantities in the diagram. It is usually necessary to introduce suitable notation. In choosing symbols for the unknown quantities, we use letters such as $a, b, c \dots x$ and y, and it also helps to use abbreviations or symbols such as V (volume) and t (time).

Step 2: Think of a plan. Try to find the connection between the data and the unknown. You may have to relate a problem to others you have seen before. Have you seen the same problem in a slightly different form? Do you know a similar/related problem? Do you know a theorem, rule, method or result that could be useful? Could you reformulate the problem? Could you re-state it differently? Go back to your definitions.

If you cannot solve the proposed problem, try to solve some related problem first. Sometimes you will be able to find a similar problem, while in other instances you may have to use components, sections, and methods of other problems to solve a new one. Every problem you encounter becomes a potential method for solving other problems.

Step 3: Carry out the plan. In order to carry out your plan, you have to check each step. Can you see clearly that the step is correct? Can you prove that it is correct?

Step 4: Look back. Examine the solution you have obtained. Can you check the result? Can you check the argument? Can you derive the solution differently? Can you see it at a glance? Can you use result of, or the method to use for, some other problem?

2.4 Summary

Table 2.1 shows a summary of the mathematical ideas you need to master, with references to the relevant examples in your textbook. Try to do as many review exercises as you can.

	Topic(s)	**Sections in Stewart**	**Examples in Stewart**
I.	Four ways to represent a function	Section 1.1	1–11
II.	Mathematical models	Section 1.2	1–6
III.	New function from old function	Section 1.3	1–9
IV.	Exponential functions	Section 1.4	1–4
V.	Inverse functions and logarithms	Section 1.5	1–13

Table 2.1: Sections in Stewart.

Note: Each section (in Stewart) contains examples numbered from 1.

2.5 The Way Forward

The topics in the rest of the study guide are:

Chapter 2: Limits and Derivatives

Chapter 3: Differentiation Rules

Chapter 4: Integrals

Chapter 5: Differential Equations, Growth and Decay, and Partial Derivatives/Chains Rule

Study Unit 3

Limits and Derivatives

3.1 Background

The idea of a **limit** underlies the various branches of calculus. In fact, without limits, calculus simply would not exist. Every single notion of calculus is a limit in one sense or another and the idea of a limit plays a fundamental role in concepts such as instantaneous velocity, the slope of a curve, the length of a curve, the sum of infinite series, etc. The role played by limits in both differential and integral calculus is crucial. In this chapter, three main ideas will be introduced. They are: (a) the basic rules (definitions) of limits and their applications in evaluating limits of algebraic and trigonometric functions; (b) the application of limits to the continuity of functions; and (c) the use of the Squeeze Theorem in determining certain limits. Begin by reading through the preview of this chapter on page 77 in Stewart. You will notice that a mathematical model is used to predict how a function will behave. Graphs are also used to represent function behaviour and will later be used to solve problems.

3.2 Learning Outcomes

At the end of this chapter, you should be able to

- demonstrate an understanding of the concepts relating to limits and how these are applied in evaluating limits of the form $\lim_{x\to c} f(x)$, $\lim_{x\to c^+} f(x)$, $\lim_{x\to c^-} f(x)$, $\lim_{x\to +\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$, where c is a real number and f is an algebraic or trigonometric function
- use the limit definition of continuity
- solve problems based on the Squeeze Theorem

3.3 Prescribed Reading

The prescribed reading in Stewart is Chapter 2, Sections 2.1, 2.2, 2.3, 2.4 and 2.5. Also see Appendix F: A39 (Limit laws) and A42 (The Squeeze Theorem).

Note: The way to master calculus is to solve a lot of calculus problems!

3.4 Limit

3.4.1 Introduction to the Limit Concept

Read carefully through **Section 2.1** in Stewart. You will notice that we have two problems – one dealing with finding the slope of a curve at a specific point and the other dealing with finding the length of a curve. In the first problem, the closer the second point gets to a certain point, the closer the computed value gets to the actual value we are looking for. This calculus problem involves a process called the limit. You can estimate the slope of the curve using a sequence of approximations. The limit allows you to compute the slope exactly. Similarly, in the second problem – in trying to compute the distance along a curved path – you would have to go through better approximations successively until you reach the actual value. The actual arc length is obtained by using the limit. **The limit value is the height or the y-value of a function.**

A left-hand limit is the y-value you obtain by approaching x from the left side.

A right-hand limit is the y-value you obtain by approaching x from the right side.

Let us use two functions to explore the limit concept. The first function, $f(x) = \dfrac{x^2-4}{x-2}$, is undefined at $x = 2$, but its behaviour can be examined in the area close to $x = 2$. From the value in the tables, you can see that the limit of $f(x)$ as x approaches 2 from the left or right is 4.

x	$f(x) = \dfrac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x) = \dfrac{x^2-4}{x-2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

$$\lim_{x\to 2^-} f(x) = 4$$

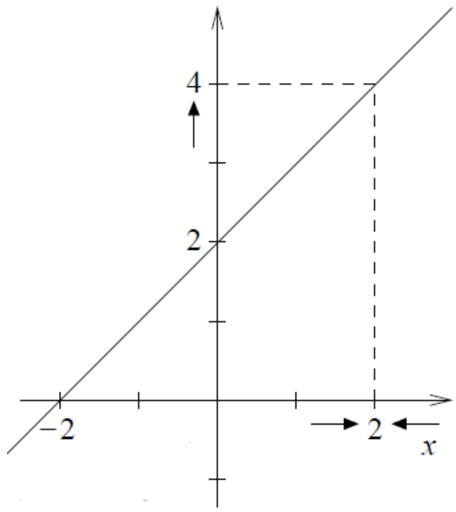


Figure 3.1: Graph of $f(x) = \dfrac{x^2-4}{x-2}$

$\lim_{x\to 2^+} f(x) = 4$
This can be written mathematically as follows:

$$\boxed{\lim_{x\to 2} f(x) = 4.}$$

Consider another function $g(x) = \dfrac{x^2-5}{x-2}$ or $y = \dfrac{x^2-5}{x-2}$.

x	$g(x) = \dfrac{x^2-5}{x-2}$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	10,003.9999

x	$y = \dfrac{x^2-5}{x-2}$
2.1	-5.1
2.01	-95.99
2.001	-995.999
2.0001	-9,995.9999

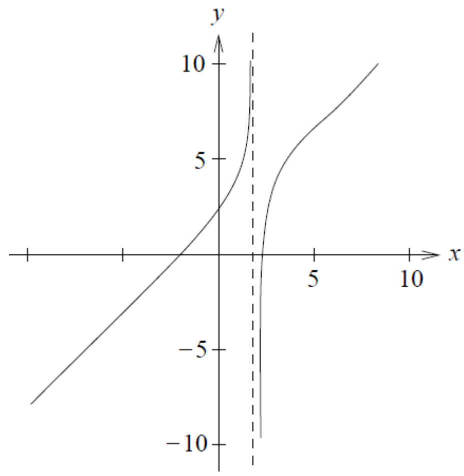


Figure 3.2: Graph of $f\left(x\right)=\dfrac{x^2-5}{x-2}$

Note: As x approaches 2 (from the left or from the right), the value of $g(x)$ gets bigger in absolute value but with opposite signs on opposite sides of 2. The limit of $g(x)$ as x approaches 2 from the left, and the limit of $g(x)$ as x approaches 2 from the right are not equal. Mathematically, this can be written as follows: $\lim\limits_{x\to 2^-} g\left(x\right) \neq \lim\limits_{x\to 2^+} g\left(x\right)$ and this statement means that

$$\boxed{\lim_{x\to 2} g\left(x\right) \text{ does not exist.}}$$

3.4.2 Definition of a Limit: Left- and Right-hand Limits

We say a limit exists if and only if the corresponding one-sided limits are equal. That is, if

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$. In general, $\lim\limits_{x\to a} f\left(x\right) = L$ for some number L if, and only if,

$$\boxed{\lim_{x\to a^-} f\left(x\right) = \lim_{x\to a^+} f\left(x\right) = L.}$$

Let us now consider some worked examples.

3.5 Worked Examples

Our collection of worked examples of the work covered in this study guide can be divided into eight sets (with some overlap), namely:

I. Limits as $x \to c,\ c \in \mathbb{R}$ (cancellations)

II. Limits as $x \to \pm\infty$

III. Limits involving absolute values

IV. Left-hand and right-hand limits

V. Limits involving trigonometric functions

VI. The Squeeze Theorem

VII. The $\varepsilon - \delta$ definition of a limit

VIII. Continuity

Table 3.1 shows which sections of Stewart, and which worked examples in the study guide you must consult.

	Topic(s)	**Sections in Stewart**	**Examples in study guide**
I.	Limits as $x \to c,\ c \in \mathbb{R}$	Sections 2.1, 2.2 & 2.3	1–12
II.	Limits as $x \to \pm\infty$	Section 2.6 (examples 1–5 & 10–11)	13–27
III.	Limits involving absolute values	Section 2.3 (examples 7–9)	22–27
IV.	Left-hand and right-hand limits	Section 2.2 (examples 6, 7 & 9)	28–40
V.	Limits involving trigonometric functions	Section 2.2 (examples 3–5 & 10)	41–52
VI.	The Squeeze Theorem	Section 2.3 page 101 (example 11)	53–57
VII.	The ϵ-δ definition of a limit	Section 2.4 (read only)	58–63
VIII.	Continuity	Section 2.5 (examples 1–9)	64–70

Table 3.1: Sections in Stewart.

In the following sections, we present a number of worked examples based on the mathematical concepts and ideas in each section of this chapter, together with appropriate teaching texts and references to Stewart. The solutions to the problems appear immediately after each set of examples. Unlike the exercises in Stewart, the examples in the study guide are not arranged in order of difficulty. Only attempt them once you have studied the relevant parts and done some of the exercises in Stewart.

3.5.1 Limits as $x \to c$ $(c \in \mathbb{R})$

Determine the following limits:

1. $$\lim_{x\to -1} \frac{x^3+1}{-x^2+x+2}$$

2. $$\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$$

3. $$\lim_{x\to 1} \frac{x^2-2x+1}{x^3-2x^2+x}$$

4. $$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2}$$

5. $$\lim_{x\to -1} \frac{x^2-2x-3}{x^2-1}$$

6. $$\lim_{x\to 3} \frac{x^2-9}{x^3-27}$$

7. $$\lim_{x\to -1} \frac{x^4-1}{x+1}$$

8. $$\lim_{x\to 3} \frac{x^3-27}{x^2-3x}$$

9. $$\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+24}-5}$$

10. $$\lim_{h\to 0} \frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}$$

11. $$\lim_{x\to -2} \frac{\left(x^2-2x\right)^{\frac{1}{3}}-2}{x^2-2x-8}$$

12. $$\lim_{x\to 10} \frac{(x-2)^{\frac{1}{3}}-2}{x-10}$$

Solutions:

1.
$$\begin{aligned}
&\lim_{x\to-1}\frac{x^3+1}{-x^2+x+2}\\
=\ &\lim_{x\to-1}\frac{(x+1)\left(x^2-x+1\right)}{(x+1)(2-x)}\\
=\ &\lim_{x\to-1}\frac{x^2-x+1}{2-x}\\
=\ &\tfrac{3}{3}=1.
\end{aligned}$$

2.
$$\begin{aligned}
&\lim_{x\to-3}\frac{x+3}{x^2+4x+3}\\
=\ &\lim_{x\to-3}\frac{x+3}{(x+3)(x+1)}\\
=\ &\lim_{x\to-3}\frac{1}{x+1}\\
=\ &\frac{1}{-3+1}=-\tfrac{1}{2}.
\end{aligned}$$

3.
$$\begin{aligned}
=\ &\lim_{x\to1}\frac{x^2-2x+1}{x^3-2x^2+x}\\
=\ &\lim_{x\to1}\frac{(x-1)^2}{x(x-1)^2}\\
=\ &\lim_{x\to1}\frac{1}{x}=1.
\end{aligned}$$

4.
$$\begin{aligned}
&\lim_{x\to0}\frac{5x^3+8x^2}{3x^4-16x^2}\\
=\ &\lim_{x\to0}\frac{x^2(5x+8)}{x^2(3x^2-16)}\\
=\ &\lim_{x\to0}\frac{5x+8}{3x^2-16}\\
=\ &=-\tfrac{8}{16}=-\tfrac{1}{2}.
\end{aligned}$$

5.
$$\begin{aligned}
&\lim_{x\to-1}\frac{x^2-2x-3}{x^2-1}\\
=\ &\lim_{x\to-1}\frac{(x+1)(x-3)}{(x+1)(x-1)}\\
=\ &\lim_{x\to-1}\frac{x-3}{x-1}\\
=\ &\frac{-4}{-2}=2.
\end{aligned}$$

6.
$$\begin{aligned}
&\lim_{x\to3}\frac{x^2-9}{x^3-27}\\
=\ &\lim_{x\to3}\frac{(x-3)(x+3)}{(x-3)(x^2+3x+9)}\\
=\ &\lim_{x\to3}\frac{x+3}{x^2+3x+9}\\
=\ &\tfrac{6}{27}=\tfrac{2}{9}.
\end{aligned}$$

7.
$$\begin{aligned}
&\lim_{x\to-1}\frac{x^4-1}{x+1}\\
=\ &\lim_{x\to-1}\frac{\left(x^2-1\right)\left(x^2+1\right)}{x+1}\\
=\ &\lim_{x\to-1}\frac{(x-1)(x+1)\left(x^2+1\right)}{x+1}\\
=\ &\lim_{x\to-1}(x-1)\left(x^2+1\right)\\
=\ &(-1-1)\left[(-1)^2+1\right]\\
=\ &-4.
\end{aligned}$$

8.
$$\begin{aligned}
=\ &\lim_{x\to3}\frac{x^3-27}{x^2-3x}\\
=\ &\lim_{x\to3}\frac{(x-3)\left(x^2+3x+9\right)}{x(x-3)}\\
=\ &\lim_{x\to3}\frac{x^2+3x+9}{x}\\
=\ &9.
\end{aligned}$$

In the following two solutions we make use of (i) the identity and (ii) rationalising:

(i) $A^2-B^2=(A-B)(A+B)$

(ii) Rationalising the denominator, eg

$$\frac{1}{\sqrt{x}}=\frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}=\frac{\sqrt{x}}{x}$$

$$(A-B)(A+B)=A^2-B^2.$$

9.

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+24}-5} &= \lim_{x\to 1}\left(\frac{x-1}{\sqrt{x^2+24}-5}\cdot\frac{\sqrt{x^2+24}+5}{\sqrt{x^2+24}+5}\right)\\
&= \lim_{x\to 1}\frac{(x-1)\left(\sqrt{x^2+24}+5\right)}{x^2+24-25}\\
&= \lim_{x\to 1}\frac{(x-1)\left(\sqrt{x^2+24}+5\right)}{x^2-1}\\
&= \lim_{x\to 1}\frac{(x-1)\left(\sqrt{x^2+24}+5\right)}{(x-1)(x+1)}\\
&= \lim_{x\to 1}\frac{\sqrt{x^2+24}+5}{x+1} = 5.
\end{aligned}$$

10.

$$\begin{aligned}
&\lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}\\
&= \lim_{h\to 0}\frac{\left(\sqrt{(x+h)^2+1}-\sqrt{x^2+1}\right)\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}\\
&= \lim_{h\to 0}\frac{(x+h)^2+1-\left(x^2+1\right)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}\\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}\\
&= \lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}.
\end{aligned}$$

In the following two solutions we make use of the identity

$$(A-B)(A^2+AB+B^2) = A^3-B^3.$$

11.

$$\begin{aligned}
&\lim_{x\to -2}\frac{\left(x^2-2x\right)^{\frac{1}{3}}-2}{x^2-2x-8} = \lim_{x\to -2}\left[\frac{\left(x^2-2x\right)^{\frac{1}{3}}-2}{x^2-2x-8}\cdot\frac{\left(x^2-2x\right)^{\frac{2}{3}}+2\left(x^2-2x\right)^{\frac{1}{3}}+4}{(x^2-2x)^{\frac{2}{3}}+2\,(x^2-2x)^{\frac{1}{3}}+4}\right]\\
&= \lim_{x\to -2}\frac{x^2-2x-8}{(x^2-2x-8)\left[(x^2-2x)^{\frac{2}{3}}+2\,(x^2-2x)^{\frac{1}{3}}+4\right]}\\
&= \lim_{x\to -2}\frac{1}{(x^2-2x)^{\frac{2}{3}}+2\,(x^2-2x)^{\frac{1}{3}}+4}\\
&= \frac{1}{4+4+4} = \frac{1}{12}.
\end{aligned}$$

(Here we put $A = (x^2 - 2x)^{\frac{1}{3}}$ and $B = 2$, and multiplied both numerator and denominator by $A^2 + AB + B^2$.)

12.

$$
\begin{aligned}
& \lim_{x \to 10} \frac{(x-2)^{\frac{1}{3}} - 2}{x - 10} \\
= & \lim_{x \to 10} \left(\frac{(x-2)^{\frac{1}{3}} - 2}{x - 10} \cdot \frac{(x-2)^{\frac{2}{3}} + 2(x-2)^{\frac{1}{3}} + 4}{(x-2)^{\frac{2}{3}} + 2(x-2)^{\frac{1}{3}} + 4} \right) \\
= & \lim_{x \to 10} \frac{(x-2) - 8}{(x-10)\left((x-2)^{\frac{2}{3}} + 2(x-2)^{\frac{1}{3}} + 4\right)} \\
= & \lim_{x \to 10} \frac{1}{(x-2)^{\frac{2}{3}} + 2(x-2)^{\frac{1}{3}} + 4} \\
= & \frac{1}{4+4+4} = \frac{1}{12}.
\end{aligned}
$$

3.5.2 Limits as $x \to \pm\infty$

In this case, if you are required to find the limit of a function which is a polynomial divided by another polynomial, find the highest power of the variable in the denominator and divide each term in both numerator and denominator by that power.

Note:

$$\boxed{\lim_{x \to \pm\infty} \frac{1}{x} = 0.}$$

Example:

Find the following limit: $\displaystyle\lim_{x \to \infty} \frac{6x^2 - 1}{2x^2 + 7x}$

Solution:

$$
\begin{aligned}
\lim_{x \to \infty} \frac{6x^2 - 1}{2x^2 + 7x} &= \lim_{x \to \infty} \frac{\frac{6x^2}{x^2} - \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{7x}{x^2}} \\
&= \lim_{x \to \infty} \frac{6 - \frac{1}{x^2}}{2 + \frac{7}{x^2}} \\
&= \frac{6 - 0}{2 + 0} \\
&= \frac{6}{2} = 3.
\end{aligned}
$$

Notice that the highest power in the denominator is x^2, divide each term in both numerator and denominator by x^2.

Now determine the following limits:

13. $\lim\limits_{x\to\infty} \dfrac{x^4+x^3}{12x^3+128}$

14. $\lim\limits_{x\to\infty} \dfrac{7x^3+1}{x^2+x}$

15. $\lim\limits_{x\to-\infty} \dfrac{7x^3+1}{x^2+x}$

16. $\lim\limits_{x\to\infty} \dfrac{9x^4+x}{2x^4+5x^2-x+6}$

17. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+4x}-2x}{2x}$

18. $\lim\limits_{x\to-\infty} \dfrac{4x+8}{\sqrt{x^2+7x}-2x}$

19. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+2x}+x\right)$

20. $\lim\limits_{x\to-\infty} \dfrac{1}{\sqrt{x^2+5x}+x}$

21. $\lim\limits_{x\to\infty} \left[\left(x^3+x^2\right)^{\frac{1}{3}}-x\right]$

Solutions:

13.

$$\begin{aligned} & \lim_{x\to\infty} \frac{x^4+x^3}{12x^3+128} \\ = & \lim_{x\to\infty} \frac{x+1}{12+(128/x^3)} \\ = & \infty. \end{aligned}$$

14.

$$\begin{aligned} & \lim_{x\to\infty} \frac{7x^3+1}{x^2+x} \\ = & \lim_{x\to\infty} \frac{7x+1/x^2}{1+1/x} \\ = & \infty. \end{aligned}$$

15.

$$\begin{aligned} & \lim_{x\to-\infty} \frac{7x^3+1}{x^2+x} \\ = & \lim_{x\to-\infty} \frac{7x+\frac{1}{x^2}}{1+\frac{1}{x}} \\ = & -\infty. \end{aligned}$$

16.

$$\begin{aligned} & \lim_{x\to\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} \\ = & \lim_{x\to\infty} \frac{9+\left(1/x^3\right)}{2+(5/x^2)-(1/x^3)+(6/x^4)} \\ = & \frac{9+0}{2+0-0+0} \\ = & \frac{9}{2}. \end{aligned}$$

17.

$$
\begin{aligned}
& \lim_{x\to-\infty} \frac{\sqrt{x^2+4x}-2x}{2x} \\
= & \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1+4/x\right)}-2x}{2x} \\
= & \lim_{x\to-\infty} \frac{|x|\sqrt{1+4/x}-2x}{2x} \\
= & \lim_{x\to-\infty} \frac{-x\sqrt{1+4/x}-2x}{2x} \quad (|x|=-x \text{ since } x<0) \\
= & \lim_{x\to-\infty} \frac{-x\left(\sqrt{1+4/x}+2\right)}{2x} \\
= & \lim_{x\to-\infty} \frac{-\left(\sqrt{1+4/x}+2\right)}{2} = -\frac{3}{2}.
\end{aligned}
$$

18.

$$
\begin{aligned}
& \lim_{x\to-\infty} \frac{4x+8}{\sqrt{x^2+7x}-2x} \\
= & \lim_{x\to-\infty} \frac{4x+8}{\sqrt{x^2\left(1+(7/x)\right)}-2x} \\
= & \lim_{x\to-\infty} \frac{4x+8}{|x|\sqrt{1+(7/x)}-2x} \quad (\sqrt{x^2}=|x|) \\
= & \lim_{x\to-\infty} \frac{4x+8}{-x\sqrt{1+(7/x)}-2x} \quad (|x|=-x \;\text{ since } x<0) \\
= & \lim_{x\to-\infty} \frac{4+(8/x)}{-\sqrt{1+(7/x)}-2} \\
= & \frac{4+0}{-\sqrt{1+0}-2} = -\frac{4}{3}.
\end{aligned}
$$

19.

$$
\begin{aligned}
& \lim_{x\to-\infty} \left(\sqrt{x^2+2x}+x\right) \\
= & \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+2x}+x\right)\left(\sqrt{x^2+2x}-x\right)}{\sqrt{x^2+2x}-x} \\
= & \lim_{x\to-\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}-x} \\
= & \lim_{x\to-\infty} \frac{2x}{|x|\sqrt{1+2/x}-x} \\
= & \lim_{x\to-\infty} \frac{2x}{-x\sqrt{1+2/x}-x} \quad (|x|=-x \text{ since } x<0) \\
= & \lim_{x\to-\infty} \frac{2}{-\sqrt{1+2/x}-1} = -1.
\end{aligned}
$$

20.

$$
\begin{aligned}
& \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+5x}+x} \\
= \;& \lim_{x\to-\infty} \left(\frac{1}{\sqrt{x^2+5x}+x} \cdot \frac{\sqrt{x^2+5x}-x}{\sqrt{x^2+5x}-x} \right) \\
= \;& \lim_{x\to-\infty} \frac{\sqrt{x^2+5x}-x}{5x} \\
= \;& \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1+5/x\right)}-x}{5x} \\
= \;& \lim_{x\to-\infty} \frac{|x|\sqrt{1+5/x}-x}{5x} \qquad \left(\sqrt{x^2}=|x|\right) \\
= \;& \lim_{x\to-\infty} \frac{-x\sqrt{1+5/x}-x}{5x} \qquad (|x|=-x \text{ since } x<0) \\
= \;& \lim_{x\to-\infty} \frac{-\sqrt{1+5/x}-1}{5} \\
= \;& \frac{-\sqrt{1+0}-1}{5} = -\frac{2}{5}.
\end{aligned}
$$

21.

$$
\begin{aligned}
& \lim_{x\to\infty} \left[\left(x^3+x^2\right)^{\frac{1}{3}} - x \right] \\
= \;& \lim_{x\to\infty} \frac{\left[\left(x^3+x^2\right)^{\frac{1}{3}} - x \right] \left[\left(x^3+x^2\right)^{\frac{2}{3}} + x\left(x^3+x^2\right)^{\frac{1}{3}} + x^2 \right]}{\left(x^3+x^2\right)^{\frac{2}{3}} + x\left(x^3+x^2\right)^{\frac{1}{3}} + x^2} \\
= \;& \lim_{x\to\infty} \frac{x^3+x^2-x^3}{\left(x^3+x^2\right)^{\frac{2}{3}} + x\left(x^3+x^2\right)^{\frac{1}{3}} + x^2} \\
= \;& \lim_{x\to\infty} \frac{x^2}{x^2\left(1+1/x\right)^{\frac{2}{3}} + x^2\left(1+1/x\right)^{\frac{1}{3}} + x^2} \\
= \;& \lim_{x\to\infty} \frac{1}{\left(1+1/x\right)^{\frac{2}{3}} + \left(1+1/x\right)^{\frac{1}{3}} + 1} = \frac{1}{3}.
\end{aligned}
$$

3.5.3 Limits Involving Absolute Values

Remember from high school that

$$
|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}
$$

The definition of absolute value gives a function in two sections (parts) with corresponding intervals,

ie if $y = |x|$ then

$$
y = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}
$$

Another example:

If $f(x) = |3 - 2x|$, then we can write

$$f(x) = \begin{cases} 3 - 2x & \text{if} \quad 3 - 2x \geq 0 \\ -(3 - 2x) & \text{if} \quad 3 - 2x < 0 \end{cases}$$

But $3 - 2x \geq 0 \Leftrightarrow 3 \geq 2x \Leftrightarrow \frac{3}{2} \geq x \Leftrightarrow x \geq \frac{3}{2}$
Similarly, $3 - 2x < 0 \Leftrightarrow 3 < 2x \Leftrightarrow \frac{3}{2} < x \Leftrightarrow x > \frac{3}{2}$

Thus,

$$f(x) = \begin{cases} 3 - 2x & \text{if} \quad x \leq \frac{3}{2} \\ \\ -3 + 2x & \text{if} \quad x > \frac{3}{2} \end{cases}$$

Let us look at one **example**:

Find the following limit:

$$\lim_{x \to \infty} \frac{x - 5}{|3 - 2x|}$$

Solution:

$$\begin{aligned} \lim_{x \to \infty} \frac{x - 5}{|3 - 2x|} &= \lim_{x \to \infty} \frac{x - 5}{-3 + 2x} = \lim_{x \to \infty} \frac{\frac{x}{x} - \frac{5}{x}}{\frac{-3}{x} + \frac{2x}{x}} \qquad \left(|3 - 2x| = -(3 - 2x) \text{ since } x > \frac{3}{2} \right) \\ &= \lim_{x \to \infty} \frac{1 - \frac{5}{x}}{\frac{-3}{x} + 2} \\ &= \frac{1 - 0}{-0 + 2} \\ &= \frac{1}{2}. \end{aligned}$$

Note: Since we have $x \to \infty$, $x > \frac{3}{2}$ and this is why we have $|3 - 2x| = -3 + 2x$ in our example above.

Now determine the rest of the limits below.

22. $\displaystyle \lim_{x \to \infty} \frac{3x + 1}{|2 - x|}$

23. $\displaystyle \lim_{x \to -\infty} \frac{|x|}{|x| + 1}$

24. $\displaystyle \lim_{x \to -\infty} \frac{x}{|x|}$

25. $\displaystyle \lim_{x \to -\infty} \frac{|x + 1|}{x + 1}$

26. $\displaystyle \lim_{x \to -\infty} \frac{|x + 1|}{2x + 3}$

27. $\displaystyle \lim_{x \to -\infty} \frac{2 + |2x + 1|}{2x + 1}$

Solutions:

22.

$$\begin{aligned}|2-x| &= \begin{cases} 2-x & \text{if} \quad 2-x \geq 0 \\ -(2-x) & \text{if} \quad 2-x < 0 \end{cases} \\ &= \begin{cases} 2-x & \text{if} \quad x \leq 2 \\ x-2 & \text{if} \quad x > 2 \end{cases}\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{x\to\infty} \frac{3x+1}{|2-x|} \\ =& \lim_{x\to\infty} \frac{3x+1}{x-2} \qquad (\,|2-x| = x-2 \quad \text{since} \quad x>2) \\ =& \lim_{x\to\infty} \frac{3+1/x}{1-2/x} \\ =& \frac{3+0}{1-0} = 3.\end{aligned}$$

23.

$$\begin{aligned}&\lim_{x\to-\infty} \frac{|x|}{|x|+1} \\ =& \lim_{x\to-\infty} \frac{-x}{-x+1} \quad (|x| = -x \text{ since } x<0) \\ =& \lim_{x\to-\infty} \frac{-1}{-1+\frac{1}{x}} = 1.\end{aligned}$$

24.

$$\begin{aligned}&\lim_{x\to-\infty} \frac{x}{|x|} \\ =& \lim_{x\to-\infty} \frac{x}{-x} \quad (|x| = -x \text{ since } x<0) \\ =& \lim_{x\to-\infty} (-1) = -1.\end{aligned}$$

25.

$$\begin{aligned}&\lim_{x\to-\infty} \frac{|x+1|}{x+1} \\ =& \lim_{x\to-\infty} \frac{-(x+1)}{x+1} \quad (|x+1| = -(x+1) \text{ since } x<-1) \\ =& \lim_{x\to-\infty} (-1) = -1.\end{aligned}$$

26.

$$\begin{aligned}&\lim_{x\to-\infty} \frac{|x+1|}{2x+3} \\ =& \lim_{x\to-\infty} \frac{-(x+1)}{2x+3} \quad (\text{since } x<-1) \\ =& \lim_{x\to-\infty} \frac{-1-1/x}{2+3/x} = -\frac{1}{2}.\end{aligned}$$

27.

$$\begin{aligned}
& \lim_{x\to-\infty} \frac{2+|2x+1|}{2x+1} \\
= & \lim_{x\to-\infty} \frac{2-(2x+1)}{2x+1} \quad (|2x+1| = -(2x+1) \text{ since } x < -\frac{1}{2}) \\
= & \lim_{x\to-\infty} \frac{1-2x}{2x+1} \\
= & \lim_{x\to-\infty} \frac{\frac{1}{x}-2}{2+\frac{1}{x}} = -1.
\end{aligned}$$

3.5.4 Left-hand and Right-hand Limits

A left-hand limit is the y-value you obtain by approaching x from the left side.

A right-hand limit is the y-value you obtain by approaching x from the right side.

Let us try out one **example**:

Suppose

$$h(x) = \begin{cases} x & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 < x \leq 2 \\ 8-x & \text{if} \quad x > 2. \end{cases}$$

Evaluate the following limits:

28. $\lim\limits_{x\to 0^-} h(x)$

29. $\lim\limits_{x\to 0+} h(x)$

30. $\lim\limits_{x\to 2^-} h(x)$

31. $\lim\limits_{x\to 2^+} h(x)$

32. $\lim\limits_{x\to 2} h(x)$

Solutions:

28. $\lim\limits_{x\to 0^-} h(x) = \lim\limits_{x\to 0^-} x = 0.$

29. $\lim\limits_{x\to 0+} h(x) = \lim\limits_{x\to 0^+} x^2 = 0^2 = 0.$

30. $\lim\limits_{x\to 2^-} h(x) = \lim\limits_{x\to 2^-} x^2 = 2^2 = 4.$

31. $\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} (8-x) = 8-2 = 6.$

32. $\lim_{x\to 2} h(x)$ does not exist, since $\lim_{x\to 2^-} h(x) \neq \lim_{x\to 2^+} h(x)$.

Now determine the following limits:

33. $\lim_{x\to 2^-} \frac{1}{x-2}$

34. $\lim_{x\to 2^-} f(x)$, where

$$f(x) = \begin{cases} x+2 & \text{if} \quad x > 2 \\ x^2 & \text{if} \quad x < 2 \\ 0 & \text{if} \quad x = 2 \end{cases}$$

35. $\lim_{x\to 0^-} \frac{1}{|x|}$

36. $\lim_{x\to 0^-} \frac{x}{|x|}$

37. Let

$$f(x) = \begin{cases} 2 & \text{if} \quad x > 1 \\ 1-x^2 & \text{if} \quad x < 1 \end{cases}$$

(a) Draw the graph of f.

(b) Determine $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(c) Does $\lim_{x\to 1} f(x)$ exist? Give a reason for your answer.

38. Let

$$f(x) = \begin{cases} x^2+1 & \text{if} \quad x > 2 \\ 0 & \text{if} \quad x = 2 \quad \text{or} \quad x = 0 \\ \frac{1}{x} & \text{if} \quad x < 2 \quad \text{and} \quad x \neq 0 \end{cases}$$

Determine $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2} f(x)$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$, if these limits exist.

39. Let

$$f(x) = \begin{cases} 3-x & \text{if} \quad x < 2 \\ 2 & \text{if} \quad x = 2 \\ \frac{x}{2} & \text{if} \quad x > 2 \end{cases}$$

Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it?

Solutions:

33. $\lim_{x \to 2^-} \dfrac{1}{x-2}$ (because $x - 2 \to 0$ and $\frac{1}{x-2} < 0$)

$= -\infty.$

34. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2$ (since $x < 2$)

$= 4.$

35. $\lim_{x \to 0^-} \dfrac{1}{|x|}$

$= \lim_{x \to 0^-} \dfrac{1}{-x}$ ($|x| = -x$ since $x < 0$)

$= \infty$.

36. If x tends to 0^-, then x is negative. According to the definition of the absolute value, we then have that $|x| = -x$.

Consequently,

$\lim_{x \to 0^-} \dfrac{x}{|x|}$

$= \lim_{x \to 0^-} \dfrac{x}{-x} = \lim_{x \to 0^-} (-1) = -1.$

37. (a)

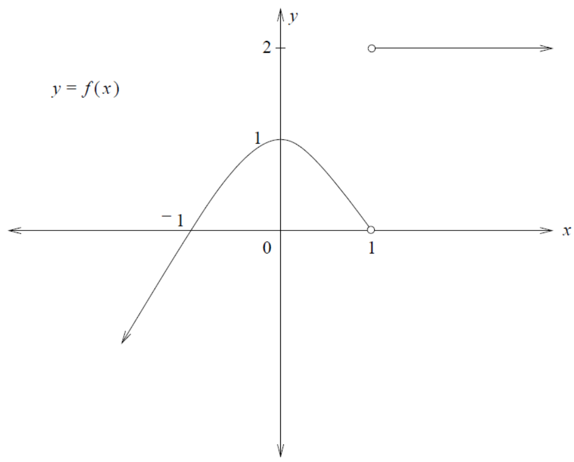


Figure 3.3: Graph of $f(x)$

(b) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - x^2) = 1 - 1 = 0.$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2) = 2.$

(c) $\lim_{x \to 1} f(x)$ does not exist, because

$$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x).$$

38. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 5.$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{1}{x}\right) = \frac{1}{2}.$$

$\lim_{x \to 2} f(x)$ does not exist, since $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (x^2 + 1) = \infty.$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0.$$

39. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} = \frac{2}{2} = 1$

while

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 3 - 2 = 1.$$

$\lim_{x \to 2} f(x)$ exists, since $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$.

We have $\lim_{x \to 2} f(x) = 1$.

3.5.5 Limits Involving Trigonometric Functions

In order to compute the derivatives of the trigonometric functions (for example, by using the first principle of differentiation), we first have to know how to evaluate the limits of trigonometric functions.

Below are some of the theorems you need to know:

(i) $\lim_{x \to 0} \sin x = 0,$

(ii) $\lim_{x \to 0} \cos x = 1,$

(iii) $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = 1,$

(iv) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0.$

Examples:

Find the following limit:

$$\lim_{\theta\to 0} \frac{\sin(5\theta)}{3\theta}$$

Solution:

In order to apply theorem (iii) above, we first have to re-write the function as follows:

$$\frac{\sin 5\theta}{3\theta} = \frac{\sin 5\theta}{3\theta}.\frac{5}{5} = \frac{5\sin 5\theta}{3.5\theta} = \frac{5}{3}\left(\frac{\sin 5\theta}{5\theta}\right)$$

Notice that as $\theta \to 0$ we have $5\theta \to 0$. So by theorem (iii) above, with $x = 5\theta$,

we have $\lim\limits_{\theta\to 0} \dfrac{\sin 5\theta}{5\theta} = \lim\limits_{5\theta\to 0} \dfrac{\sin(5\theta)}{5\theta} = 1.$

Therefore,

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin 5\theta}{3\theta} &= \lim_{\theta\to 0} \frac{5}{3}\left(\frac{\sin(5\theta)}{5\theta}\right) \\
&= \frac{5}{3}\lim_{\theta\to 0} \frac{\sin 5\theta}{5\theta} \\
&= \frac{5}{3}.1 \\
&= \frac{5}{3}.
\end{aligned}$$

Also find the following limit

$$\lim_{x\to 0} x\cot x.$$

Solution:

$$\begin{aligned}\lim_{x\to 0} x\cot x &= \lim_{x\to 0} x\frac{\cos x}{\sin x}\\ &= \lim_{x\to 0}\frac{\cos x}{\frac{\sin x}{x}}\\ &= \frac{\lim\limits_{x\to 0}\cos x}{\lim\limits_{x\to 0}\frac{\sin x}{x}}\\ &= \frac{1}{1} = 1.\end{aligned}$$

Also, determine the following limits:

40. $\displaystyle\lim_{x\to 0^-}\frac{x}{\tan x}$

41. $\displaystyle\lim_{\theta\to 0}\frac{\theta^2}{\tan\theta}$

42. $\displaystyle\lim_{\theta\to 0}\frac{\sin(5\theta)}{2\theta}$

43. $\displaystyle\lim_{x\to 0}\frac{x+\sin x}{x}$

44. $\displaystyle\lim_{t\to 0}\frac{\tan(2t)}{t}$

45. $\displaystyle\lim_{\theta\to 0}\frac{\sin(5\theta)}{\sin(2\theta)}$

46. $\displaystyle\lim_{x\to 0}(\tan(2x)\cdot\csc(4x))$

47. $\displaystyle\lim_{\theta\to 5}\frac{\sin(2\theta-10)}{\sin(\theta-5)}$

48. $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$

49. $\displaystyle\lim_{x\to -\infty}\left(x\sin\tfrac{1}{x}\right)$

50. $\displaystyle\lim_{x\to\infty}\left(x\sin\tfrac{1}{x}\right)$

51. $\displaystyle\lim_{x\to -\infty}\left(x^2\sin\tfrac{1}{x}\right)$

52. $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{1-\cos(3\theta)}$

Solutions:

40. $\lim\limits_{x\to 0^-}\frac{x}{\tan x} = \lim\limits_{x\to 0^-}\left(\frac{x}{\sin x}\cdot\cos x\right)$

$= \lim\limits_{x\to 0^-}\frac{x}{\sin x}\cdot\lim\limits_{x\to 0^-}\cos x$

$= \frac{1}{\lim\limits_{x\to 0^-}\frac{\sin x}{x}}\cdot\lim\limits_{x\to 0^-}\cos x$

$= \frac{1}{1}\cdot 1 = 1.$

41.
$$\begin{aligned}
&\lim_{\theta\to 0}\frac{\theta^2}{\tan\theta} = \lim_{\theta\to 0}\tfrac{\theta^2\cos\theta}{\sin\theta}\\
=\;&\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\theta\cos\theta\right)\\
=\;&\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\lim_{\theta\to 0}(\theta\cos\theta)\\
=\;&\lim_{\theta\to 0}\tfrac{\theta}{\sin\theta}\cdot\lim_{\theta\to 0}\theta\cdot\lim_{\theta\to 0}\cos\theta\\
=\;&1\times 0\times 1 = 0.
\end{aligned}$$

42.
$$\begin{aligned}
&\lim_{\theta\to 0}\frac{\sin(5\theta)}{2\theta}\\
=\;&\lim_{\theta\to 0}\frac{5\sin(5\theta)}{2.5\theta}\\
=\;&\frac{5}{2}\times 1 = \tfrac{5}{2}.
\end{aligned}$$

43.
$$\begin{aligned}
&\lim_{x\to 0}\frac{x+\sin x}{x}\\
=\;&\lim_{x\to 0}\left(1+\frac{\sin x}{x}\right)\\
=\;&\lim_{x\to 0}1+\lim_{x\to 0}\frac{\sin x}{x}\\
=\;&1+1 = 2.
\end{aligned}$$

44.
$$\begin{aligned}
&\lim_{t\to 0}\frac{\tan(2t)}{t}\\
=\;&\lim_{t\to 0}\frac{\sin(2t)}{t\cdot\cos(2t)}\\
=\;&\lim_{t\to 0}\left(\frac{\sin(2t)}{2t}\cdot\frac{2}{\cos(2t)}\right)\\
=\;&1\times\frac{2}{1} = 2.
\end{aligned}$$

45.
$$\begin{aligned}
&\lim_{\theta\to 0}\frac{\sin(5\theta)}{\sin(2\theta)}\\
=\;&\lim_{\theta\to 0}\left(\frac{\sin(5\theta)}{5\theta}\cdot\frac{5\theta}{2\theta}\cdot\frac{2\theta}{\sin(2\theta)}\right)\\
=\;&\lim_{\theta\to 0}\frac{\sin(5\theta)}{5\theta}\times\frac{5}{2}\times\lim_{\theta\to 0}\frac{2\theta}{\sin(2\theta)}\\
=\;&1\times\frac{5}{2}\times 1 = \frac{5}{2}.
\end{aligned}$$

46.
$$\begin{aligned}
&\lim_{x\to 0}(\tan(2x)\cdot\csc(4x))\\
=\;&\lim_{x\to 0}\frac{\sin(2x)}{\cos(2x)\cdot\sin(4x)}\\
=\;&\lim_{x\to 0}\left(\frac{\sin(2x)}{\cos(2x)}\cdot\frac{1}{2\sin(2x)\cos(2x)}\right)\\
=\;&\lim_{x\to 0}\left(\frac{1}{2}\cdot\frac{1}{\cos^2(2x)}\right) = \frac{1}{2}.
\end{aligned}$$

47. Put $x=\theta-5$. Then $2\theta-10=2x$ and $x\to 0\Leftrightarrow\theta\to 5$. Now,

$$\begin{aligned}
\lim_{\theta\to 5}\frac{\sin(2\theta-10)}{\sin(\theta-5)} &= \lim_{x\to 0}\frac{\sin(2x)}{\sin x}\\
&= \lim_{x\to 0}\left(\frac{\sin(2x)}{2x}\cdot\frac{2x}{x}\cdot\frac{x}{\sin x}\right)\\
&= 1\times 2\times 1 = 2.
\end{aligned}$$

48.

$$\begin{aligned}
&\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{(1-\cos\theta)(1+\cos\theta)}{\theta(1+\cos\theta)}\\
=\;&\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\\
=\;&\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\\
=\;&\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\\
=\;&1.0 = 0.
\end{aligned}$$

We have used property (iv) on page 28.

49. Put $\theta = \dfrac{1}{x}$. Then $x = \dfrac{1}{\theta}$ and $x \to -\infty \Leftrightarrow \theta \to 0^-$. Hence,

$$\begin{aligned} & \lim_{x\to-\infty}\left(x\sin\frac{1}{x}\right) \\ = & \lim_{\theta\to 0^-}\left(\frac{\sin\theta}{\theta}\right) \\ = & 1. \end{aligned}$$

50.

$$\begin{aligned} & \lim_{x\to\infty}\left(x\sin\frac{1}{x}\right) \\ = & \lim_{\theta\to 0^+}\left(\frac{\sin\theta}{\theta}\right) \quad (\text{where } \theta = \frac{1}{x}) \\ = & 1. \end{aligned}$$

51. Put $\theta = \frac{1}{x}$. Then $x \to -\infty \Leftrightarrow \theta \to 0^-$. Thus,

$$\begin{aligned} & \lim_{x\to-\infty}\left(x^2\sin\frac{1}{x}\right) \\ = & \lim_{\theta\to 0^-}\left(\frac{\sin\theta}{\theta}\cdot\frac{1}{\theta}\right) \\ = & -\infty. \end{aligned}$$

52.

$$\begin{aligned} & \lim_{\theta\to 0}\frac{1-\cos\theta}{1-\cos(3\theta)} \\ = & \lim_{\theta\to 0}\left[\frac{1-\cos\theta}{1-\cos(3\theta)}\cdot\frac{1+\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos(3\theta)}{1+\cos(3\theta)}\right] \\ = & \lim_{\theta\to 0}\left[\frac{\left(1-\cos^2\theta\right)(1+\cos(3\theta))}{\left(1-\cos^2(3\theta)\right)(1+\cos\theta)}\right] \\ = & \lim_{\theta\to 0}\frac{\sin^2\theta\cdot(1+\cos(3\theta))}{\sin^2(3\theta)\cdot(1+\cos\theta)} \\ = & \lim_{\theta\to 0}\left[\left(\frac{\sin\theta}{\theta}\right)^2\cdot\frac{\theta^2}{9\theta^2}\cdot\left(\frac{3\theta}{\sin(3\theta)}\right)^2\cdot\frac{1+\cos(3\theta)}{1+\cos\theta}\right] \\ = & 1^2\cdot\frac{1}{9}\cdot 1^2\cdot\frac{2}{2} = \frac{1}{9}. \end{aligned}$$

3.5.6 The Squeeze Theorem

Note: The Squeeze Theorem is also known as the Sandwich Theorem or Pinching Theorem.

The Squeeze Theorem is used to evaluate the limit values of a complicated function, or the limit values of an unknown function. The Squeeze Theorem is stated as follows:

If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

The Sandwich Theorem is illustrated by the sketches below.

Imagine you have a sandwich as shown in Figure 3.4:

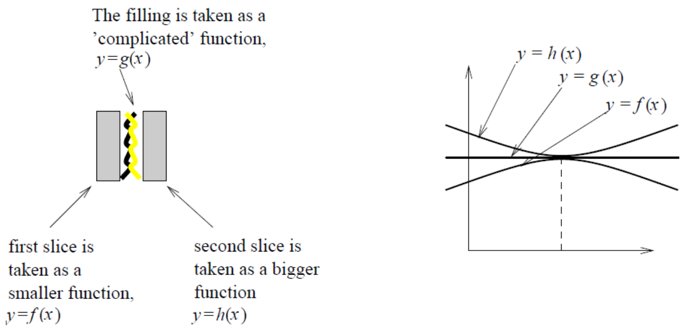


Figure 3.4: Illustrating Squeeze Theorem

The theorem says that if $g(x)$ is squeezed or pinched between $f(x)$ and $h(x)$ near a, and if f and h have the same limit value L at a, then g is forced to have the same limit value L at a.

Suppose you have to find the following limit:

$$\lim_{x \to 0} x^2 \sin \frac{1}{x}$$

This is a complicated function, in other words, it is difficult to determine the limit of the function. In such a case we can use the Squeeze (Sandwich) Theorem.

Squeeze Theorem

A:Functions only eg polynomials

If we have $f(x) \leq g(x) \leq h(x)$
and for some value L,

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$,

then, by the Sandwich (Squeeze) Theorem

$\lim_{x \to c} g(x) = L$

Squeeze Theorem
B:Trigonometric functions only

We apply
$-1 \leq \sin x \leq 1$
OR
$-1 \leq \cos x \leq 1$
$f(x) \leq g(x) \leq h(x)$
and for some value L,
$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$,
then, by the Sandwich (Squeeze) Theorem
$\lim_{x \to c} g(x) = L$

Now let us find the limit of

$\lim_{x\to 0} x^2 \sin \frac{1}{x}$.   (This is an example of a complicated function.)

Solution:

To use the Squeeze Theorem, we first have to find functions f and h such that

$$f(x) \leq x^2 \sin \frac{1}{x} \leq h(x)$$

where

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x).$$

Recall that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \quad \text{for all } x \neq 0.$$

Now we multiply by x^2 (note that for $x \neq 0,\ x^2 > 0$).

This means that the inequality signs will not change in direction, and we get

$x^2(-1) \leq x^2.\sin\left(\frac{1}{x}\right) \leq x^2.1$, which means that $-x^2 \leq x^2.\sin\left(\frac{1}{x}\right) \leq x^2$ for all $x \neq 0$.

Then $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$.

So from the Sandwich Theorem, it follows that $\lim_{x\to 0} x^2.\sin\left(\frac{1}{x}\right) = 0$.

Go through the given worked examples below.

53. Find $\lim_{x\to\infty} f(x)$ if

$$\frac{2x^2}{x^2+1} < f(x) < \frac{2x^2+5}{x^2}$$

54. Suppose that

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

for x close to zero. Determine

$$\lim_{x\to 0} \frac{x \sin x}{2 - 2\cos x}$$

55. Use the Squeeze Theorem to determine

$$\lim_{x\to -\infty} \frac{\sin x}{x}$$

56. Use the Squeeze Theorem to determine

$$\lim_{x\to\infty} (\lfloor x \rfloor / x) \quad \text{and} \quad \lim_{x\to-\infty} (\lfloor x \rfloor / x)$$

57. Use the Squeeze Theorem to determine

$$\lim_{x\to\infty} \frac{1-\cos x^2}{1+x^3}$$

Solutions:

53. $\displaystyle\lim_{x\to\infty} \frac{2x^2}{x^2+1} = \lim_{x\to\infty} \frac{2}{1+1/x^2} = 2$ and

$$\lim_{x\to\infty} \frac{2x^2+5}{x^2} = \lim_{x\to\infty} \frac{2+5/x^2}{1} = 2.$$

By the Squeeze Theorem, it follows that $\displaystyle\lim_{x\to\infty} f(x) = 2$.

54.
$$\lim_{x\to 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{0}{6} = 1,$$

while

$$\lim_{x\to 0} 1 = 1,$$

so, by the Squeeze Theorem

$$\lim_{x\to 0} \frac{x \sin x}{2 - 2\cos x} = 1.$$

55. For all x it holds that

$$-1 \leq \sin\ x \leq 1.$$

If $x < 0$, we can divide by x to obtain

$$\frac{1}{x} \geq \frac{\sin x}{x} \geq -\frac{1}{x} \Rightarrow \frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

Now $\displaystyle\lim_{x\to-\infty} \frac{1}{x} = 0$, and $\displaystyle\lim_{x\to-\infty} \left(-\frac{1}{x}\right) = -\lim_{x\to-\infty} \frac{1}{x} = 0$. It follows by the Squeeze Theorem that

$$\lim_{x\to-\infty} \frac{\sin x}{x} = 0.$$

56. For all x it holds that

$$x - 1 < \lfloor x \rfloor \le x. \tag{3.1}$$

Where $\lfloor x \rfloor$ is known as the floor function. If $x > 0$ then

$$\frac{x-1}{x} < \lfloor x \rfloor / x \le 1.$$

Now

$$\lim_{x\to\infty} \frac{x-1}{x} = \lim_{x\to\infty} \frac{1 - \frac{1}{x}}{1} = 1,$$

while $\lim_{x\to\infty} 1 = 1$. From the Squeeze Theorem it follows that

$$\lim_{x\to\infty} (\lfloor x \rfloor / x) = 1.$$

If $x < 0$, then we can divide (3.1) by x to obtain

$$\frac{x-1}{x} > \lfloor x \rfloor / x \ge 1.$$

Similarly, as above, it follows that

$$\lim_{x\to-\infty} (\lfloor x \rfloor / x) = 1.$$

57. For all x it holds that

$$-1 \le -\cos x^2 \le 1$$

Add 1:

$$0 \le 1 - \cos x^2 \le 2$$

For $x > 0$ (because $x \to \infty$), we divide by $1 + x^3 > 0$ to obtain

$$\frac{0}{1+x^3} \le \frac{1-\cos x^2}{1+x^3} \le \frac{2}{1+x^3}.$$

Now $\lim_{x\to\infty} 0 = 0$ and $\lim_{x\to\infty} \frac{2}{1+x^3} = 0$. It follows by the Squeeze Theorem that

$$\lim_{x\to\infty} \frac{1-\cos x^2}{1+x^3} = 0.$$

3.5.7 The ε-δ Definition of a Limit (Read only for other modules eg MAT2615)

This section is not for examination purposes, therefore you should not spend too much time on it. However, those of you who intend to continue with pure mathematics should be familiar with the contents of this section.

58. Determine $\delta > 0$ so that

$$0 < \left| x - \frac{1}{4} \right| < \delta \Rightarrow \left| \frac{1}{4x} - 1 \right| < \frac{1}{20} .$$

59. Prove from the ε-δ definition that $\lim_{x \to 2} (7x - 1) = 13$.

60. Prove from the ε-δ definition that $\lim_{x \to -1} f(x) = 10$, where $f(x) = -3x + 7$.

61. Prove from the ε-δ definition that $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$.

62. Prove from the ε-δ definition that $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$.

63. Prove from the ε-δ definition that $\lim_{x \to -3} \dfrac{x^2 - 9}{x + 3} = -6$.

Solutions:

58. $\left| \dfrac{1}{4x} - 1 \right| < \dfrac{1}{20}$

$$\Leftrightarrow -\frac{1}{20} < \frac{1}{4x} - 1 < \frac{1}{20}$$

$$\Leftrightarrow 1 - \frac{1}{20} < \frac{1}{4x} - 1 + 1 < \frac{1}{20} + 1$$

$$\Leftrightarrow \frac{19}{20} < \frac{1}{4x} < \frac{21}{20}$$

$$\Leftrightarrow \frac{20}{19} > 4x > \frac{20}{21}$$

$$\Leftrightarrow \frac{5}{21} < x < \frac{5}{19}$$

$$\Leftrightarrow \frac{5}{21} - \frac{1}{4} < x - \frac{1}{4} < \frac{5}{19} - \frac{1}{4}$$

$$\Leftrightarrow -\frac{1}{84} < x - \frac{1}{4} < \frac{1}{76} .$$

If we choose $\delta = \dfrac{1}{84}$, we should have that

$$0 < \left| x - \frac{1}{4} \right| < \delta \Rightarrow \left| \frac{1}{4x} - 1 \right| < \frac{1}{20}\ .$$

59. According to the definition of a limit, let $f(x) = 7x - 1$, $x_o = 2$ and $L = 13$. Given $\epsilon > 0$, we need to find $\delta > 0$ so that for all x

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 13| < \epsilon.$$

Now

$$\begin{aligned} |f(x) - 13| &= |7x - 1 - 13| \\ &= |7x - 14| \\ &= 7\,|x - 2| \end{aligned}$$

So let $\delta = \dfrac{\epsilon}{7}$. Then

$$\begin{aligned} 0 < |x - 2| < \delta \Rightarrow |f(x) - 13| &= 7\,|x - 2| \ \text{(from above)} \\ &< 7\delta \\ &= 7 \cdot \frac{\epsilon}{7} = \epsilon. \end{aligned}$$

60. We have that

$$\begin{aligned} |f(x) - 10| = |-3x + 7 - 10| = |-3x - 3| &= |-3(x + 1)| \\ &= 3|x + 1| \\ &= 3\,|x - (-1)|\,. \end{aligned}$$

If $\epsilon > 0$ is given, put $\delta = \dfrac{\epsilon}{3}$. Then

$$\begin{aligned} 0 < |x - (-1)| < \delta \Rightarrow |f(x) - 10| &= 3\,|x - (-1)| \\ &< 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that

$$\lim_{x \to -1} f(x) = 10.$$

61. Let $\epsilon > 0$ be given. We have that

$$\begin{aligned} & \left| \frac{x^2 - 4}{x - 2} - 4 \right| \\ = & \left| \frac{(x + 2)\,(x - 2)}{x - 2} - 4 \right| \\ = & |x + 2 - 4| \ \text{(if } \ x \neq 2) \\ = & |x - 2|. \end{aligned}$$

Put $\delta = \epsilon$. Then, if $0 < |x-2| < \delta$, it follows that $x \neq 2$, and so

$$\left|\frac{x^2-4}{x-2} - 4\right| = |x-2| < \delta = \epsilon.$$

62. Let $\epsilon > 0$ be given. Then

$$\begin{aligned}\left|\frac{x^2-1}{x-1} - 2\right| &= \left|\frac{(x-1)(x+1)}{x-1} - 2\right| \\ &= |x+1-2| \text{ (if } x \neq 1) \\ &= |x-1|.\end{aligned}$$

Put $\delta = \epsilon$. Then

$$0 < |x-1| < \delta \Rightarrow \left|\frac{x^2-1}{x-1} - 2\right| < \epsilon.$$

Hence,

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

63. In the definition of a limit, we put $f(x) = (x^2-9)/(x+3)$, $x_o = -3$ and $L = -6$. Let $\epsilon > 0$ be given. Then

$$\begin{aligned}\left|\frac{x^2-9}{x+3} - (-6)\right| &= \left|\frac{(x+3)(x-3)}{x+3} + 6\right| \\ &= |x-3+6| \text{ (if } x \neq -3) \\ &= |x+3| \\ &= |x-(-3)|.\end{aligned}$$

Let $\delta = \epsilon$. If

$$0 < |x-(-3)| < \delta$$

then $x \neq -3$, and then it follows from the above that

$$\left|\frac{x^2-9}{x+3} - (-6)\right| = |x-(-3)| < \epsilon.$$

Hence,

$$\lim_{x \to -3} \frac{x^2-9}{x+3} = -6.$$

3.5.8 Continuity

In the previous topics, notice that the limit of a function as x approaches a can sometimes be found by just calculating the value of the function (y) at $x = a$. The functions with this property are said to be continuous at a. **A continuous process is one that takes place gradually, without interruption or abrupt changes.**

Definition 1

A function f is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$.

Definition 2

A function f is continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$ and f is continuous from the left at a number a if $\lim_{x \to a^-} f(x) = f(a)$.

If f is not continuous at a, we say that f is discontinuous at a, or that f has a discontinuity at a. When we consider the two definitions above, we notice that three things (or conditions) are required for f to be continuous at a.

A function f is continuous at $x = a$ when the following conditions are satisfied:

(i) if $f(a)$ is defined (that is, a is in the domain of f)

(ii) if $\lim_{x \to a} f(x)$ exists [that is, f must be defined in an open interval that contains a] and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

(iii) if $\lim_{x \to a} f(x) = f(a)$.

If one or more of the three conditions above are not satisfied, then f is said to be discontinuous at $x = a$. [Try to think of the process of finding a limit as making a prediction as to what the y-value will be when x finally gets to a.]

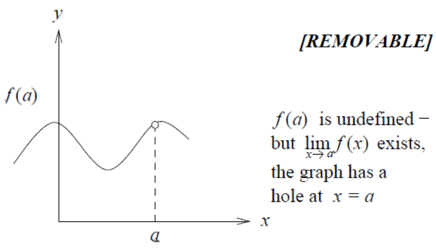
In figures (a), (b), (c) and (d) below, three main types of discontinuities – **removable**, **jump** and **non-removable** are shown. Are you able to explain why the function f is discontinuous at $x = a$ in all four diagrams?

Let us now work through some examples.

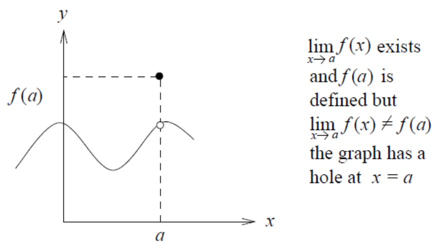
Example 1:

(a) Determine whether the function $f(x) = \begin{cases} 1 - x, & \text{if} \quad x \leq 2 \\ x^2 - 2x, & \text{if} \quad x > 2 \end{cases}$ is continuous or discontinuous at $a = 2$, and explain why.

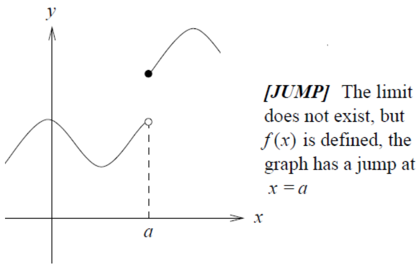
(b) Sketch the graph of the function.



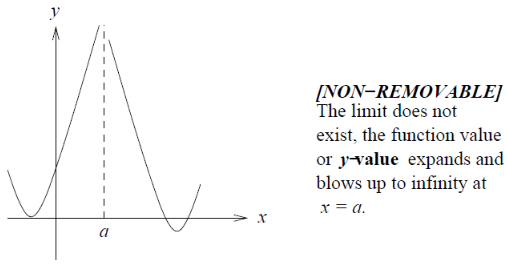
(a)



(b)



(c)



(d)

Solution:

(a)

$$\begin{aligned}
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (1 - x) = 1 - 2 = -1 \\
\lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} \left(x^2 - 2x\right) = (2)^2 - 2(2) = 4 - 4 = 0
\end{aligned}$$

Since

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

$\lim_{x \to 2} f(x)$ does not exist [that is, condition (ii) has not been met].

Therefore the function is discontinuous at $x = 2$.

(b) The key points on the graph are:

(i) the vertical intercept(s) (where $x = 0$)

(ii) the horizontal intercept(s) (where $f(x) = 0$)

(iii) the turning points

$f(x) = 1 - x$ is a straight line with intercepts at $(0, 1)$ and $(1, 0)$ for the interval where $x \leq 2$.

$f(x) = x^2 - 2x$ is a curve with intercepts where $x(x - 2) = 0$ that is $(0, 0)$ and $(2, 0)$ are the x-intercepts, but they do not fall in the interval $x > 2$.

Example 2:

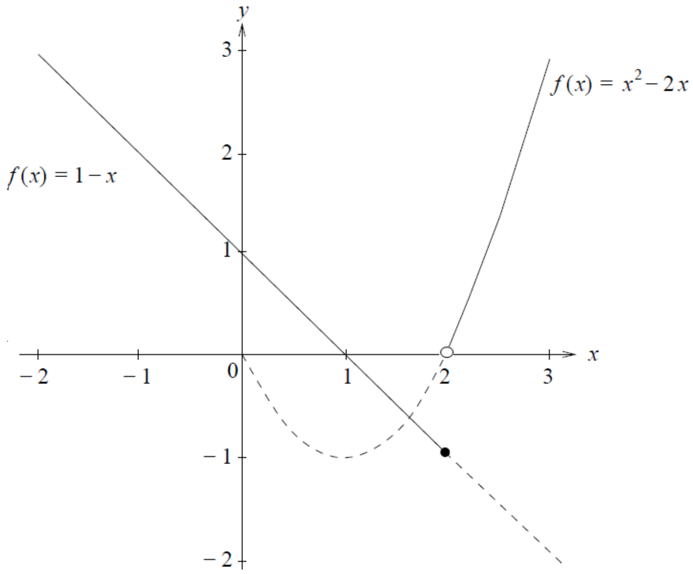


Figure 3.5: Graph of $f(x) = x^2 - 2x$ and $f(x) = 1 - x$

If

$$f\left(x\right) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1, & \text{if} \quad x = 2, \end{cases}$$

determine if the function $f(x)$ is continuous at $x = 2$.

Solution:

For $\lim\limits_{x \to 2} f\left(x\right)$ to exist, we must first find $\lim\limits_{x \to 2^-} f\left(x\right)$ and $\lim\limits_{x \to 2^+} f\left(x\right)$. Since

$$\begin{aligned} \lim_{x \to 2^-} f\left(x\right) &= \lim_{x \to 2^+} f\left(x\right) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} \\ &= \lim_{x \to 2} \frac{\left(x-2\right)\left(x+1\right)}{x-2} = \lim_{x \to 2} \left(x+1\right) = 3, \end{aligned}$$

the limit exists.

But $f(2) = 1$ is also defined and $\lim\limits_{x \to 2} f(x) \neq f(2)$.

So $f\left(x\right)$ is not continuous at $x = 2$ (this is the “removable” type of discontinuity).

Example 3:

Find the value of the constant c that would make the function f continuous at $x = 3$, where:

$$f\left(x\right) = \begin{cases} cx + 1, & \text{if} \quad x \leq 3 \\ cx^2 - 1, & \text{if} \quad x > 3. \end{cases}$$

Solution:

We determine the left- and right-hand limits as follows:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (cx+1) = 3c+1$$

and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \left(cx^2-1\right) = 9c-1.$$

The condition for f to be continuous at $x=3$ is that $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$,

that is

$$\begin{array}{rrcl} & 3c+1 & = & 9c-1 \\ \Leftrightarrow & 9c-3c & = & 1+1 \\ \Leftrightarrow & 6c & = & 2 \\ \Leftrightarrow & c & = & \frac{1}{3}. \end{array}$$

Thus $f(x)$ is continuous at $x=3$, for

$$c = \frac{1}{3}.$$

You can now work through the last examples in this chapter.

64. Let

$$f(x) = \begin{cases} x & \text{if} \quad x \leq 1 \\ (x-1)^2 & \text{if} \quad x > 1. \end{cases}$$

(a) Draw the graph of $f(x)$.

(b) Determine $\lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1^-} f(x)$. Does $\lim\limits_{x\to 1} f(x)$ exist? Give a reason for your answer.

(c) Is $f(x)$ continuous at $x=1$? Give a reason for your answer.

65. Let

$$f(x) = \begin{cases} 4-x^2 & \text{if} \quad x > 1 \\ 3x^2 & \text{if} \quad x < 1. \end{cases}$$

(a) Draw the graph of $f(x)$.

(b) Calculate $\lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1^-} f(x)$.

(c) What value, if any, must be assigned to $f(1)$ to make $f(x)$ continuous at $x=1$? Give reasons for your answer.

66. (a) At which points (if any) is the function

$$f(x) = \frac{x+a}{x^2 + (a+b)\,x + ab}$$

discontinuous? (a and b are constants.)

(b) At which points (if any) can

$$f(x) = \frac{x+a}{x^2 + (a+b)\,x + ab}$$

be made continuous by assigning a certain value to $f(x)$, and what is the value?

67. Let

$$g(x) = \begin{cases} \frac{x^2+2x-15}{x-3} & \text{if} \quad x \neq 3 \\ k & \text{if} \quad x = 3. \end{cases}$$

What value (if any) must k be for g to be continuous at $x = 3$? Give reasons for your answer.

68. Let

$$g(x) = \begin{cases} bx^2 & \text{if} \quad x \geq \frac{1}{2} \\ x^3 & \text{if} \quad x < \frac{1}{2}. \end{cases}$$

What value must b be for g to be continuous at $x = \frac{1}{2}$? Give reasons for your answer.

69. Let

$$g(x) = \begin{cases} x^3 + 4 & \text{if} \quad x \leq 0 \\ \dfrac{b+1}{x+2} & \text{if} \quad x > 0. \end{cases}$$

What value must b be for g to be continuous at $x = 0$? Give reasons for your answer.

70. Let

$$g(x) = \begin{cases} \alpha x^2 & \text{if} \quad x \geq 2 \\ x+1 & \text{if} \quad x < 2. \end{cases}$$

What value must α be for g to be continuous at $x = 2$? Give reasons for your answer.

Solutions:

64. (a)

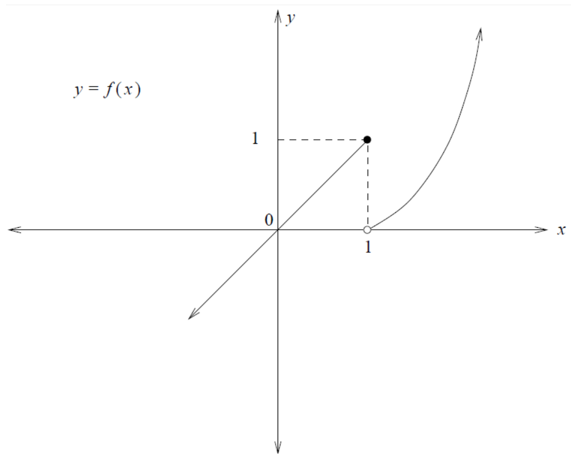


Figure 3.6: Graph of $f(x)$

b(i) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = 0.$

b(ii) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1.$

b(iii) $\lim_{x\to 1} f(x)$ does not exist, because $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$.

(c) The function $f(x)$ is thus not continuous at $x = 1$, because $\lim_{x\to 1} f(x)$ does not exist.

65. (a)

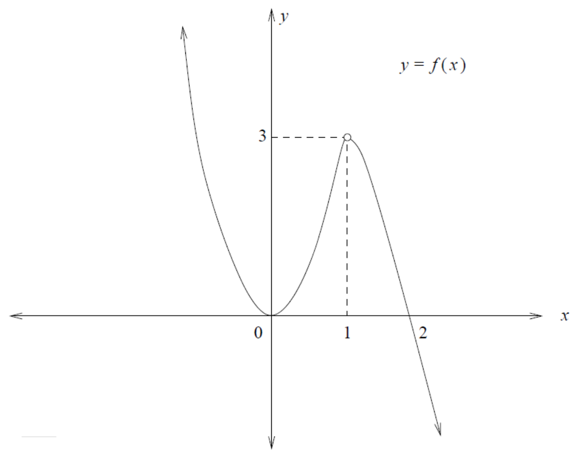


Figure 3.7: Graph of $f(x)$

(b) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4-x^2) = 4-1 = 3,$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x^2) = 3.$$

(c) Let $f(1) = 3$, since $\lim_{x \to 1} f(x) = 3$.

66. (a) $f(x) = \dfrac{x+a}{x^2 + (a+b)\,x + ab}$

$$= \frac{x+a}{(x+a)\,(x+b)}.$$

$f(x)$ is discontinuous where

$$(x+a)(x+b) = 0$$

that is at $x = -a$ and $x = -b$.

b(i) The case $a \neq b$

We calculate the limits of $f(x)$ when x tends to $-a$ and to $-b$.

$$\begin{aligned} & \lim_{x \to -b^+} \frac{x+a}{(x+a)\,(x+b)} \\ = \; & \lim_{x \to -b^+} \frac{1}{x+b} \\ = \; & \infty. \end{aligned}$$

The discontinuity at $x = -b$ cannot be removed. Now,

$$\begin{aligned} & \lim_{x \to -a} \frac{x+a}{(x+a)\,(x+b)} \\ = \; & \lim_{x \to -a} \frac{1}{x+b} \\ = \; & \frac{1}{b-a}. \end{aligned}$$

The discontinuity at $x = -a$ can be removed. We can make $f(x)$ continuous at $x = -a$ by putting

$$f(-a) = \frac{1}{b-a}\ .$$

b(ii) The case $a = b$

In this case there is a non-removable discontinuity at $x = -b$. **Note**: Removable discontinuity is when a discontinuity can be removed by redefining the function at that point. In the case of a non-removable discontinuity, the limits do not exist, so there is no way to redefine the function at that point of discontinuity. See pages 115–116 of Stewart.

67. $\displaystyle\lim_{x\to 3}\frac{x^2+2x-15}{x-3}$

$$= \lim_{x\to 3}\frac{(x-3)(x+5)}{x-3}$$

$$= \lim_{x\to 3}(x+5)$$

$$= 8.$$

From the definition of continuity it follows that k must be equal to 8 to make g continuous at $x=3$.

68. For $g(x)$ to be continuous at $x=\frac{1}{2}$, $\lim_{x\to\frac{1}{2}} g(x)$ must exist (and be equal to $g\left(\frac{1}{2}\right)$). For this limit to exist, $\lim_{x\to\frac{1}{2}^+} g(x)$ and $\lim_{x\to\frac{1}{2}^-} g(x)$ must exist and be equal. Now

$$\lim_{x\to\frac{1}{2}^+} g(x) = \lim_{x\to\frac{1}{2}^+}(bx^2) = \frac{b}{4}$$

and

$$\lim_{x\to\frac{1}{2}^-} g(x) = \lim_{x\to\frac{1}{2}^-}(x^3) = \frac{1}{8}.$$

Thus we must have that

$$\frac{b}{4} = \frac{1}{8}$$

or

$$b = \frac{1}{2}.$$

69. The function $g(x)$ will be continuous at $x=0$ if

$$g(0) = \lim_{x\to 0} g(x).$$

Also, $\lim_{x\to 0} g(x)$ will exist provided that $\lim_{x\to 0^-} g(x)$ and $\lim_{x\to 0^+} g(x)$ exist and are equal. Hence we must determine the latter two limits, set them equal, and solve for b. Now

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(x^3+4) = 0+4 = 4,$$

while

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}\left(\frac{b+1}{x+2}\right) = \frac{b+1}{0+2} = \frac{b+1}{2}.$$

Let
$$\frac{b+1}{2} = 4.$$

Then
$$b = 7.$$

Hence if $b = 7$, then $g\left(x\right)$ is continuous at $x = 0$.

70. If $g\left(x\right)$ is to be continuous at $x = 2$, then it must hold that
$$g(2) = \lim_{x \to 2} g(x),$$

that is
$$4\alpha = \lim_{x \to 2^+} g(x). \tag{3.2}$$

Now $\lim_{x \to 2} g(x)$ will exist provided $\lim_{x \to 2^+} g(x)$ and $\lim_{x \to 2^-} g(x)$ exist, and
$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^-} g(x). \tag{3.3}$$

We have that
$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (\alpha x^2) = 4\alpha,$$

while
$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (x+1) = 3.$$

Hence both (3.2) and (3.3) will be satisfied if $4\alpha = 3$. Thus if $\alpha = \frac{3}{4}$, then $g\left(x\right)$ is continuous at $x = 2$.

Key points

You should now be comfortable with the notion of the limit of a function and its use. This includes one-sided limits, infinite limits (limits resulting in infinity) and limits at infinity (limits of functions as x approaches infinity).

At this stage you should be able to

- evaluate limits of the form $\lim_{x \to c} f\left(x\right)$, $\lim_{x \to c^-} f\left(x\right)$, $\lim_{x \to c^+} f\left(x\right)$ and $\lim_{x \to \pm\infty} f\left(x\right)$, where c is a real number and $f\left(x\right)$ is an algebraic or trigonometric function
- provide a formal definition of the limit, using the correct mathematical notation

- apply the laws governing limits
- determine and evaluate the limits of sums, products, quotients and compositions of functions
- evaluate the limits of functions analytically, graphically and numerically
- use the limit definition of continuity to determine whether a function is continuous or discontinuous at a point
- use the Squeeze Theorem to determine certain limits

Continue practising solving problems until you have mastered the basic techniques! Go through the section "For your review"at the end of each chapter to consolidate what you have learnt and also use other calculus textbooks.

Study Unit 4

Differentiation Rules

4.1 Background

Differentiation is widely applied in engineering, chemistry, physics, biology and many other disciplines in science and technology. It is basically a concept that deals with rates of change. Among other things, it is used to define the slopes of curves, to calculate the velocities and accelerations of moving objects, to find the firing angle that gives a cannon its greatest range, and to predict the times when planets will be closest to each other or farthest apart. In this chapter we begin the study of **differential calculus**. The central concept of differential calculus is the **derivative**. After learning how to calculate derivatives, you will use them to solve problems involving rates of change. This chapter is devoted to (a) the definition of the derivative, or differentiation from first principles; (b) the rules used to find derivatives of algebraic and trigonometric functions; (c) problems involving tangent and normal lines; (d) techniques such as logarithmic and implicit differentiation; and (e) the application of the Mean Value Theorem and Rolle's Theorem.

4.2 Learning Outcomes

At the end of this chapter, you should be able to

- demonstrate an understanding of the first principles of differentiation
- apply basic differential formulas such as the power rule, product rule, quotient rule, chain rule, and combinations of these rules to differentiate between a variety of algebraic and trigonometric functions
- compute the derivatives of trigonometric functions and inverse trigonometric functions, and of exponential and logarithmic functions
- use the methods of logarithmic and implicit differentiation to find derivatives
- solve problems involving tangent and normal lines and the Mean Value Theorem

Note: The way to master calculus is to solve lots of calculus problems!

4.3 Prescribed Reading

The prescribed readings in Stewart are:

- Chapter 2 sections 2.1–2.3 and 2.5–2.7.
- Chapter 3 sections 3.1–3.7 and
- Chapter 4 section 4.2 only.

4.4 The Derivative

4.4.1 Introducing the Derivative

The definition of the derivative can be approached in two different ways. One is **geometrical** (as the slope of a curve) and the other is **physical** (as a rate of change). Both of these definitions can be used, but the emphasis should be on using the **derivative as a tool for solving calculus problems**.

Remember, in high school you used the first principle of **differentiation** to find the slope of a tangent line at a particular point.

Definition: The slope $m_{\tan}$ of the tangent line $y = f(x)$ at $x = a$ is:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example:

Find the slope of the tangent line to the curve $y = x^2 + 1$ at $x = 1$.

Solution:

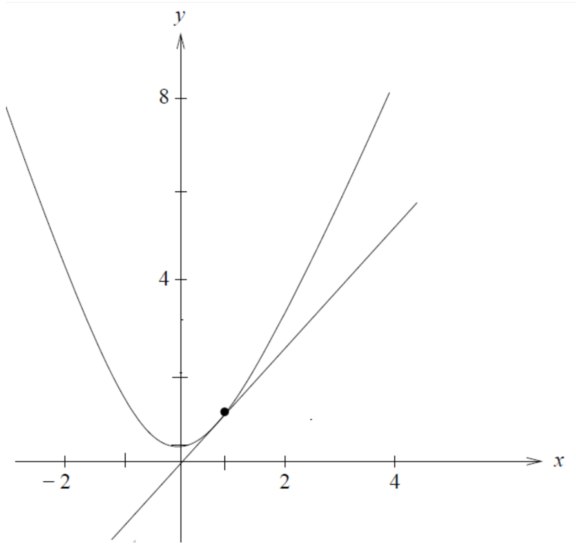


Figure 4.1: Graph of $y = x^2 + 1$ and $x = 1$

$$m_{\tan} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{\left[(1+h)^2 + 1\right] - (1+1)}{h}$$
$$= \lim_{h\to 0} \frac{1 + 2h + h^2 + 1 - 2}{h}$$
$$= \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} \frac{h(2+h)}{h}$$
$$= \lim_{h\to 0} (2+h) = 2$$

Thus at $x = 1,\ m_{\tan} = 2$.

You should also be familiar with the approach used in mechanics when dealing with problems **involving average and instantaneous velocity** (rate of change).

Average velocity:

$$v_{ave} = \frac{\text{total distance}}{\text{total time}}$$

Instantaneous velocity:

$$v = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

Example:

Consider an object that is dropped from a height of 64 m. Its position is given as $f(t) = 64 - 16t^2$. Find its average velocity between

(a) $t = 1$ and $t = 2$

(b) $t = 1.5$ and $t = 2$

(c) $t = 1.9$ and $t = 2$

(d) Also find its instantaneous velocity at $t = 2$.

Solution:

(a) $v_{ave} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{64 - 16(2)^2 - \left[64 - 16(1)^2\right]}{1} = -48$ m/s.

(b) $v_{ave} = \dfrac{f(2) - f(1.5)}{2 - 1.5} = \dfrac{64 - 16(2)^2 - \left[64 - 16(1.5)^2\right]}{0.5} = -56$ m/s.

(c) $v_{ave} = \dfrac{f(2) - f(1.9)}{2 - 1.9} = \dfrac{64 - 16(2)^2 - \left[64 - 16(1.9)^2\right]}{0.1} = -62.4$ m/s.

(d) The instantaneous velocity is given by:

$$\begin{aligned}
v_{\text{instantaneous}} &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{(2+h) - 2} \\
&= \lim_{h\to 0} \frac{\left[64 - 16(2+h)^2\right] - \left[64 - -16(2)^2\right]}{h} \\
&= \lim_{h\to 0} \frac{\left[64 - 16\left(4 + 4h + h^2\right)\right] - \left[64 - -16(2)^2\right]}{h} \\
&= \lim_{h\to 0} \frac{-64h - 16h^2}{h} = \lim_{h\to 0} \frac{-16h(h+4)}{h} \\
&= \lim_{h\to 0} -16(h+4) = -64 \text{ m/s.}
\end{aligned}$$

4.4.2 Definition of the Derivative

The derivative of the function $f(x)$ (at a particular point where $x = a$) is defined as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided that the limit exists. We say that f is differentiable at $x = a$ when the above limit exists.

In general, the derivative of $f(x)$ is the function $f'(x)$ given by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

provided that this limit exists. Finding the first derivative by using the above method is called the **First Principle of Differentiation**.

Note: If we write $x = a + h$, then $h = x - a$ and h will approach 0 if and only if x approaches a. Therefore another way of defining the derivative is as follows:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Let us try an **example**:

Use the definition of the derivative to find $g'(x)$ if $g(x) = x^4$.

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h \to 0} \frac{\left[(x+h)^2 - x^2\right]\left[(x+h)^2 + x^2\right]}{h} \\
&= \lim_{h \to 0} \frac{h\,[2x+h]\left[(x+h)^2 + x^2\right]}{h} \\
&= \lim_{h \to 0} [2x+h]\left[(x+h)^2 + x^2\right] \\
&= (2x)\left(x^2 + x^2\right) \\
&= 4x^3.
\end{aligned}$$

4.5 Worked Examples

In some instances, we do compute derivatives directly from the definition. However, such computations are tedious and very time-consuming. Fortunately, several differential rules or basic differentiation formulas have been developed to find derivatives without using the definition directly. In order to introduce you to the basic differentiation techniques, these rules are presented together with worked examples. You should go and read about these rules in Stewart and work through the examples, following each rule. You will find the rules on the following pages in Stewart:

Our collection of worked examples relating to Chapters 2, 3 and 4 of Stewart divides naturally into eight sets, with some overlap, namely:

I. Differentiation from first principles (derivative as a function)

Derivative of a constant	p. 172
The **power rule** for positive integer powers of x	p. 172
The constant multiple rule	p. 175
The sum and difference rule	p. 176
The power rule for negative integer powers of x	p. 174
The **product rule**	p. 183
The **quotient rule**	p. 185
The **chain rule**	p. 197

Table 4.1: Pages in Stewart for different rules of differentiation.

II. Basic differentiation formulas (use of the power rule, product rule, quotient rule, chain rule, and combinations of these rules to differentiate between a variety of functions)

III. Derivatives of trigonometric functions and inverse trigonometric functions

IV. Derivatives of exponential and logarithmic functions

V. Logarithmic differentiation

VI. Implicit differentiation

VII. Tangent and normal lines

VIII. The Mean Value Theorem

Attempt the problems appearing immediately after each section, once you have studied the relevant parts of Stewart and done some of the exercises there.

Table 4.2 shows how these worked examples and their solutions are organised.

4.5.1 Differentiation from First Principles (derivative as a Function)

1. Find the derivative of $3x^2 + 2x - 1$ at $x = 1$.
2. If $f(x) = \frac{1}{x}\,(x \neq 0)$, find $f'(x)$.
3. If $f(x) = \sqrt{x}$ for $(x \geq 0)$, find $f'(x)$.

	Topic(s)	**Section(s) in Stewart**	**Examples in study guide**
I.	Differentiation from first principle (derivative as a function)	Section 2.8 (examples in 2.1–2.7	1–5
II.	Basic rules of differentiation Power rule Product rule Quotient rule Chain rule Combinations of rules (derivative of varieties of) functions)	 Section 3.1 Section 3.2 Section 3.4 Section 3.2 Section 3.4	6–31
III.	Derivatives of trigonometric functions and inverse trigonometric functions	Section 3.3 page 190	32–49 50–52
IV.	Derivatives of exponential functions	Section 3.1 page 172	53–63
V.	Derivative of logarithmic functions	Section 3.6 page 218	64–74
VI.	Implicit differentiation	Section 3.5: page 208	75–82
VII.	Tangent and normal lines	Sections 3.1 page 172 and 2.7 page 140	83–92
VIII.	The Mean Value Theorem	Section 4.2 page 287	93–95

Table 4.2: Sections in Stewart.

Solutions:

1.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 + 2(x+h) - 1 - \left(3x^2 + 2x - 1\right)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 + 2x + 2h - 1 - 3x^2 - 2x + 1}{h} \\
&= \lim_{h \to 0} \frac{6xh + h^2 + 2h}{h} \\
&= \lim_{h \to 0} \frac{h(6x + h + 2)}{h} \\
&= \lim_{h \to 0} 6x + h + 2 = 6x + 2.
\end{aligned}
$$

At $x = 1,\ f'(1) = 6(1) + 2 = 8.$

2.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\left(\frac{1}{x+h}\right)-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{x-(x+h)}{x(x+h)}\cdot\frac{1}{h} \\
&= \lim_{h\to 0}\frac{-h}{x^2+xh}\cdot\frac{1}{h} = \lim_{h\to 0}\frac{-1}{x^2+xh} = -\frac{1}{x^2}.
\end{aligned}
$$

3.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

4. Use the definition of the derivative to prove that $g'(0) = 0$ if $g(x) = x^2|x|$.

Solution:

By definition,

$$g'(0) = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h},$$

provided this limit exists. Now

$$
\begin{aligned}
& \lim_{h\to 0}\frac{g(0+h)-h(0)}{h} \\
=\ & \lim_{h\to 0}\frac{h^2|h|-0}{h} \\
=\ & \lim_{h\to 0}(h|h|) \\
=\ & 0.
\end{aligned}
$$

5. Use the definition of the derivative to show that $f'(2)$ does not exist when $f(x) = |x-2|$.

(a)

$$\begin{aligned}
\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+}\frac{|(2+h)-2|-|2-2|}{h} \\
&= \lim_{h\to 0^+}\frac{|h|}{h} \\
&= \lim_{h\to 0^+}\frac{h}{h} \qquad (|h| = h \text{ since } \ h>0) \\
&= 1,
\end{aligned}$$

(b) while similarly

$$\begin{aligned}
\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-}\frac{|h|}{h} \\
&= \lim_{h\to 0}\frac{-h}{h} \qquad (|h| = -h \text{ since } \ h<0) \\
&= -1.
\end{aligned}$$

Since the left-hand and right-hand limits are not equal, it follows that

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

does not exist, that is, f is not differentiable at 2.

4.5.2 Basic Differentiation Formulas

(a) The Power Rule

If $\quad y = x^n$, then $\dfrac{d}{dx}(x^n) = nx^{n-1}$ i.e. $\dfrac{dy}{dx} = nx^{n-1} \qquad for \quad n \in \mathbb{R}$

Examples:

Find the derivatives of the following:

6. $\quad f(x) = x^8$ 7. $\quad \dfrac{d}{dx}\sqrt[3]{x^2}$ 8. $\quad m(x) = x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

Solutions:

6. $f'(x) = 8x^{8-1} = 8x^7$. 7. $\dfrac{d}{dx}\sqrt[3]{x^2} = \dfrac{d}{dx}x^{\frac{2}{3}} = \dfrac{2}{3}x^{\frac{2}{3}-1} = \dfrac{2}{3}x^{-\frac{1}{3}} = \dfrac{2}{3x^{\frac{1}{3}}}$.

8. $m(x) = x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} \Rightarrow m'(x) = \dfrac{1}{2x^{\frac{1}{2}}} - \dfrac{3}{2x^{\frac{3}{2}}}$.

Examples involving absolute values:

Find the derivatives of the following:

9. $k(a) = \dfrac{1}{|a|}$ 10. $k(s) = |s|$

Solutions:

9. $k(a) = \dfrac{1}{|a|}$

$$= \begin{cases} \dfrac{1}{a} & \text{if} \quad a > 0 \\ -\dfrac{1}{a} & \text{if} \quad a < 0. \end{cases}$$

Hence,

$$k'(a) = \begin{cases} -\dfrac{1}{a^2} & \text{if} \quad a > 0 \\ \dfrac{1}{a^2} & \text{if} \quad a < 0. \end{cases}$$

In fact, $k'(a)$ can be written as a single expression, namely $k'(a) = -\dfrac{1}{a\,|a|}$.

10.

$$\begin{aligned} k(s) &= |s| \\ \Rightarrow \quad k'(s) &= \begin{cases} 1 & \text{if} \quad s > 0 \\ -1 & \text{if} \quad s < 0. \end{cases} \end{aligned}$$

Note: $k(s) = |s|$ is not differentiable at $s = 0$.

(b) The Product Rule

$$\boxed{\text{If } y = f.g, \text{ then } y' = f'.g + f.g' \quad \text{or} \quad \frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + f(x)\,g'(x)}$$

Examples:

11. Find $f'(x)$ if $f(x) = \left(2x^4 - 3x + 5\right)\left(x^2 - \sqrt{x} + \frac{2}{x}\right)$

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[2x^4 - 3x + 5\right]\left(x^2 - \sqrt{x} + \frac{2}{x}\right) + \left(2x^4 - 3x + 5\right)\frac{d}{dx}\left(x^2 - \sqrt{x} + \frac{2}{x}\right) \\ &= \left(8x^3 - 3\right)\left(x^2 - \sqrt{x} + \frac{2}{x}\right) + \left(2x^4 - 3x + 5\right)(2x - \frac{1}{2\sqrt{x}} - \frac{2}{x^2}). \end{aligned}$$

12. Differentiate the following function:

$$y = (x+1)^{10}\,(2x+3)^{11}\,(4-x)^{12}$$

Solution:

$$\begin{aligned} y &= (x+1)^{10}(2x+3)^{11}(4-x)^{12} \\ \Rightarrow \quad \frac{dy}{dx} &= (x+1)^{10}(2x+3)^{11}\frac{d}{dx}(4-x)^{12} + (x+1)^{10}(4-x)^{12}\frac{d}{dx}(2x+3)^{11} \\ &\quad + (2x+3)^{11}(4-x)^{12}\frac{d}{dx}(x+1)^{10} \\ &= (x+1)^{10}(2x+3)^{11} \cdot 12(4-x)^{11}\frac{d}{dx}(4-x) + \\ &\quad +(x+1)^{10}(4-x)^{12} \cdot 11(2x+3)^{10} \cdot \frac{d}{dx}(2x+3) \\ &\quad + (2x+3)^{11}(4-x)^{12}.10(x+1)^9 \\ &= -12(x+1)^{10}(2x+3)^{11}(4-x)^{11} + 22(x+1)^{10}(2x+3)^{10}(4-x)^{12} \\ &\quad + 10(x+1)^9(2x+3)^{11}(4-x)^{12}. \end{aligned}$$

(c) The Quotient Rule:

$$\boxed{\begin{aligned} &\text{If } y = \frac{f}{g}, \text{ then } \frac{f'.g - f.g'}{g^2} = y' \\ &\text{or } \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{\left[g(x)\right]^2} \end{aligned}}$$

13. Compute the derivative of $\dfrac{x^2-2}{x^2+1}$.

Solution:

$$\begin{aligned} f'(x) &= \frac{\left[\frac{d}{dx}\left(x^2-2\right)\right]\left(x^2+1\right)-\left(x^2-2\right)\left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^2} \\ &= \frac{2x\left(x^2+1\right)-\left(x^2-2\right)(2x)}{\left(x^2+1\right)^2} \\ &= \frac{6x}{\left(x^2+1\right)^2}. \end{aligned}$$

More examples:

Differentiate the following functions:

14. $g(t) = \dfrac{\sqrt{2t+1}}{\sqrt{2t+1}+1}$ 15. $g(x) = \dfrac{x+1}{(x+2)^{10}}$ 16. $y = \dfrac{2x+5}{3x-2}$

Solutions:

14.

$$\begin{aligned} g(t) &= \frac{\sqrt{2t+1}}{\sqrt{2t+1}+1} = \frac{(2t+1)^{\frac{1}{2}}}{(2t+1)^{\frac{1}{2}}+1} \\ \Rightarrow \quad g'(t) &= \left[\frac{1}{2}(2t+1)^{-\frac{1}{2}}\cdot 2\cdot\left((2t+1)^{\frac{1}{2}}+1\right)- \right. \\ & \left. (2t+1)^{\frac{1}{2}}\cdot\frac{1}{2}(2t+1)^{-\frac{1}{2}}\cdot 2\right] / \left((2t+1)^{\frac{1}{2}}+1\right)^2 = \frac{1}{\sqrt{2t+1}\left(\sqrt{2t+1}+1\right)^2}. \end{aligned}$$

15.

$$\begin{aligned} g(x) &= \frac{x+1}{(x+2)^{10}} \\ \Rightarrow \quad g'(x) &= \frac{1.(x+2)^{10}-(x+1)(x+2)^9 .10.1}{(x+2)^{20}} = \frac{-9x-8}{(x+2)^{11}}. \end{aligned}$$

16.

$$\begin{aligned} y &= \frac{2x+5}{3x-2} \\ \Rightarrow \quad \frac{dy}{dx} &= \frac{(3x-2)\,2-(2x+5)\,3}{(3x-2)^2} = \frac{-19}{(3x-2)^2}. \end{aligned}$$

(d) The Chain Rule

$$\boxed{\begin{array}{l}\text{If } y = f \circ g, \text{ then } y' = f'(g).g' \\[2ex] \text{or } \dfrac{d}{dx}\left[f\left(g\left(x\right)\right)\right] = f'\left(g\left(x\right)\right)g'\left(x\right)\end{array}}$$

Examples:

17. Differentiate $y = \left(x^3 + x + 1\right)^5$

18. Find $\dfrac{d}{dt}\left(\sqrt{100 + 8t}\right)$

19. Find $\dfrac{dy}{dx}$ if $y = (4 - 3x)^9$

Solutions:

17. Let $u = x^3 + x + 1$, then $y = u^5$, and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}\left(u^5\right)\frac{du}{dx} \\ &= 5u^4\frac{d}{dx}\left(x^3 + x + 1\right) = 5\left(x^3 + x + 1\right)^4\left(3x^2 + 1\right).\end{aligned}$$

18. Let $u = 100 + 8t$, then $\sqrt{100 + 8t} = (100 + 8t)^{\frac{1}{2}} = u^{\frac{1}{2}}$, and

$$\frac{d}{du}(u)^{\frac{1}{2}} = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dt} = \frac{1}{2\sqrt{100 + 8t}}\frac{d}{dt}(100 + 8t) = \frac{4}{\sqrt{100 + 8t}}.$$

19. $y = (4 - 3x)^9 \Rightarrow \dfrac{dy}{dx} = 9\left(4 - 3x\right)^8(-3)$.

(e) Combinations of rules

In the following worked examples, combinations of rules are used to determine the required derivatives.

Find the first derivatives of the following. Do not simplify.

20. $f(x) = (5x+11)^{20}(4-x)^{30}$

21. $f(w) = (3w^2+2)^{\frac{2}{3}}(9w-1)^{-\frac{1}{3}}$

22. $f(x) = \sqrt{\dfrac{1-x}{1+x^2}}$

23. $f(x) = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^{100}$

24. $\theta(y) = \dfrac{1}{3}(2y^2+5y)^{\frac{3}{2}}$

25. $f(x) = (2x^3+3)^{10}(3x^2+2)^{20}$

26. $h(\theta) = (\theta^{\frac{1}{3}}+7)^{100}(2\theta^{-\frac{1}{3}}+7)^{200}$

27. $y = \left(\dfrac{x}{5} + \dfrac{1}{5x}\right)^5$

28. $y = (1-6x)^{\frac{2}{3}}$

Solutions:

20. $f(x) = (5x+11)^{20}(4-x)^{30}$

$$\Rightarrow f'(x) = 20(5x+11)^{19}5(4-x)^{30} + (5x+11)^{20}30(4-x)^{29}(-1).$$

21. $f(w) = (3w^2+2)^{\frac{2}{3}}(9w-1)^{-\frac{1}{3}}$

$$\Rightarrow f'(w) = \tfrac{2}{3}(3w^2+2)^{-\frac{1}{3}}(6w)(9w-1)^{-\frac{1}{3}} + (3w^2+2)^{\frac{2}{3}}\left(-\tfrac{1}{3}\right)(9w-1)^{-\frac{4}{3}}9.$$

22. $f(x) = \sqrt{\dfrac{1-x}{1+x^2}} = \left(\dfrac{1-x}{1+x^2}\right)^{\frac{1}{2}}$

$$\Rightarrow f'(x) = \frac{1}{2}\left(\frac{1-x}{1+x^2}\right)^{-\frac{1}{2}} \cdot \frac{(-1)\left(1+x^2\right) - (1-x)\cdot 2x}{\left(1+x^2\right)^2}.$$

23. $f(x) = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^{100}$

$$\Rightarrow f'(x) = 100\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{99} \cdot \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x^{\frac{3}{2}}}\right).$$

24. $\theta(y) = \dfrac{1}{3}(2y^2+5y)^{\frac{3}{2}}$

$$\Rightarrow \theta'(y) = \frac{1}{3}\cdot\frac{3}{2}(2y^2+5y)^{\frac{1}{2}}(4y+5).$$

25. $f(x) = (2x^3+3)^{10}(3x^2+2)^{20}$

$$\Rightarrow f'(x) = 10(2x^3+3)^9 \cdot 6x^2 \cdot (3x^2+2)^{20} + 20(3x^2+2)^{19} \cdot 6x \cdot (2x^3+3)^{10}.$$

26. $h(\theta) = (\theta^{\frac{1}{3}} + 7)^{100}(2\theta^{-\frac{1}{3}} + 7)^{200}$

$$\Rightarrow h'(\theta) = 100(\theta^{\frac{1}{3}}+7)^{99}\cdot\frac{1}{3}\theta^{-\frac{2}{3}}\cdot(2\theta^{-\frac{1}{3}}+7)^{200}+(\theta^{\frac{1}{3}}+7)^{100}\cdot 200(2\theta^{-\frac{1}{3}}+7)^{199}\cdot(-\frac{2}{3}\theta^{-\frac{4}{3}}).$$

27. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$

$$\Rightarrow \frac{dy}{dx} = 5\left(\frac{x}{5} + \frac{1}{5x}\right)^4 \left(\frac{1}{5} - \frac{1}{5}x^{-2}\right).$$

28. $y = (1 - 6x)^{\frac{2}{3}}$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{3}\frac{1}{(1-6x)^{\frac{1}{3}}}(-6).$$

Also do the following exercises:

29. Show that $y' = 2\,|x|$ if $y = x|x|$.

30. Let $g(x), h(x)$ and $k(x)$ be differentiable functions such that

$$f(x) = g(x)\,h(x)\,k(x)$$

for all x. Use the product rule for the derivative of the product of two functions to write down the derivative of f in terms of $g(x)$, $h(x)$ and $k(x)$ and their derivatives.

31. Use the chain rule and product rule to derive the quotient rule for derivatives.

Solutions:

29. **Case I** Suppose $x > 0$. Then $|x| = x$, so

$$y = x^2$$

and

$$y' = 2x = 2|x|.$$

Case II Suppose $x < 0$. Then $|x| = -x$, so

$$y = -x^2$$

and

$$y' = -2x = 2(-x) = 2|x|.$$

Case III Suppose $x = 0$. If $f(x) = x|x|$, then

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0}\frac{h|h|-0}{h}\\ &= \lim_{h\to 0}|h|\\ &= 0 = 2|0|.\end{aligned}$$

Hence, for all values of x we have that

$$y' = 2|x|.$$

30. Let

$$f(x) = \ell(x)k(x),$$

where

$$\ell(x) = g(x)h(x).$$

Then

$$\begin{aligned}f'(x) &= \ell'(x)k(x) + \ell(x)k'(x)\\ &= (g'(x)h(x) + g(x)h'(x))k(x) + \ell(x)k'(x)\\ &= g'(x)h(x)k(x) + g(x)h'(x)k(x) + g(x)h(x)k'(x).\end{aligned}$$

31. Let $f(x) = \dfrac{g(x)}{h(x)}$, where g and h are differentiable functions. Now

$$f(x) = g(x)\cdot(h(x))^{-1}$$

so

$$\begin{aligned}f'(x) &= g'(x)\cdot(h(x))^{-1} + g(x)\cdot\left(-\frac{1}{(h(x))^2}\cdot h'(x)\right)\\ &\quad\text{(by the product and chain rules)}\\ &= \frac{g'(x)}{h(x)} - \frac{g(x)\cdot h'(x)}{(h(x))^2}\\ &= \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}.\end{aligned}$$

4.5.3 Derivatives of Trigonometric Functions and Inverse Trigonometric Functions

(a) Derivatives of trigonometric functions

Below are limits which will help you find some of the trigonometric derivatives.

$$\lim_{\theta\to 0}\cos\theta = 1, \qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$$

$$\lim_{\theta\to 0}\sin\theta = 0, \qquad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1$$

A very important table

You should know these derivative formulae by heart!

Note: These derivative formulae can also be derived from first principles.

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$

Here is an example worked out from first principles:

Prove that $\frac{d}{dx}\sin x = \cos x$.

Solution:

$$\begin{aligned}
\text{If} \quad f(x) &= \sin x, \quad \text{then} \\
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right] \\
&= \lim_{h\to 0}\sin x.\lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x.\lim_{h\to 0}\frac{\sin h}{h} \\
&= \sin x.0 + \cos x.1 \\
&= \cos x.
\end{aligned}$$

Now determine the derivatives of the following functions:

32. $f(x) = x^5\cos x$

33. $f(x) = \sin\left(\frac{2x}{x+1}\right)$

Solutions:

32.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^5 \cos x\right) = \left[\frac{d}{dx}\left(x^5\right)\right]\cos x + x^5 \frac{d}{dx}(\cos x) \qquad \textbf{\{product rule\}} \\ &= 5x^4 \cos x - x^5 \sin x. \end{aligned}$$

33. The derivative of $f(x) = \sin\left(\dfrac{2x}{x+1}\right)$ is

$$\begin{aligned} f'(x) &= \cos\left(\frac{2x}{x+1}\right)\frac{d}{dx}\left(\frac{2x}{x+1}\right) && \textbf{\{chain rule\}} \\ &= \cos\left(\frac{2x}{x+1}\right)\frac{2(x+1) - 2x(1)}{(x+1)^2} && \textbf{\{quotient rule\}} \\ &= \cos\left(\frac{2x}{x+1}\right)\frac{2}{(x+1)^2}. \end{aligned}$$

Now go through the rest of the worked examples. Find the first derivative. Do not simplify.

34. $h(t) = \sec\ t + \tan\ t$

35. $y = x\ \csc\ x$

36. $g(\theta) = \dfrac{\sin\theta}{\theta}$

37. $h(t) = \dfrac{\sin t}{\tan t + 1}$

38. $f(t) = \dfrac{\sqrt{3t + \cos t}}{t+3}$

39. $y = \dfrac{\cos x}{1 + \sin x}$

40. $h(w) = \sin^3(\cot w)$

41. $k(x) = \cot^2(3x)$

42. $g(x) = \sqrt{\cos cx + \cos c\alpha}$, where α is a constant

43. $g(\theta) = \sec^{\frac{3}{2}}(1-\theta)$

44. $g(x) = \tan^4(\tan\ x)$

45. $h(t) = (t^{-\frac{1}{3}} + 1).\tan\ t$

46. $y = \dfrac{2x+5}{3x-2}$

47. $y = \cos(3\pi x/2)$

48. $y = \cos^3 x$

49. $y = [\sin(x+5)]^{\frac{5}{4}}$

Solutions:

34. $h(t) = \sec\ t + \tan\ t$

$\Rightarrow h'(t) = (\sec\ t)(\tan\ t) + \sec^2 t.$

35. $y = x \ \csc x$

$\Rightarrow y' = x.(-\csc x \ \cot \ x) + (\csc x).1.$

36. $g(\theta) = \dfrac{\sin\theta}{\theta}$

$\Rightarrow g'(\theta) = \dfrac{\theta\cos\theta - (\sin\theta)\,(1)}{\theta^2}$.

37. $h(t) = \dfrac{\sin t}{\tan t + 1}$

$\Rightarrow h'(t) = \dfrac{\cos t\,(\tan t + 1) - \sin t\left(\sec^2 t\right)}{(\tan t + 1)^2}.$

38. $f(t) = \dfrac{\sqrt{3t + \cos t}}{t+3}$

$\Rightarrow f'(t) = \dfrac{\frac{1}{2\sqrt{3t+\cos t}}\,(3 - \sin t)\,(t+3) - \sqrt{3t + \cos t}\cdot 1}{(t+3)^2}.$

39. $y = \dfrac{\cos x}{1 + \sin x}$

$\Rightarrow y' = \dfrac{(1+\sin x)\,(-\sin x) - \cos x\,(0 + \cos x)}{(1 + \sin x)^2}$.

40. $h(w) = \sin^3(\cot \ w)$

$\Rightarrow h'(w) = 3\sin^2(\cot \ w).\cos(\cot \ w).(-\csc^2 w).$

41. $k(x) = \cot^2(3x)$

$\Rightarrow k'(x) = 2\cot(3x).(-\csc^2(3x)).3.$

42. $g(x) = \sqrt{\cos cx + \cos c\alpha}$ where α is a constant

$\Rightarrow g'(x) = \dfrac{1}{2\sqrt{\csc x + \csc\alpha}}\cdot(-\csc x.\cot x).$

43. $g(\theta) = \sec^{\frac{3}{2}}(1-\theta)$

$\Rightarrow g'(\theta) = \dfrac{3}{2}\sec^{\frac{1}{2}}(1-\theta)\cdot\sec(1-\theta)\cdot\tan(1-\theta)\cdot(-1).$

44. $g(x) = \tan^4(\tan\ x)$

$\Rightarrow g'(x) = 4\tan^3(\tan\ x).\sec^2(\tan\ x).\sec^2 x.$

45. $h(t) = (t^{-\frac{1}{3}} + 1) \cdot \tan t$

$\Rightarrow h'(t) = -\dfrac{1}{3}t^{-\frac{4}{3}} \cdot \tan t + (t^{-\frac{1}{3}} + 1) \cdot \sec^2 t.$

46. $g(\theta) = \cos^2(\theta^2)$

$\Rightarrow g'(\theta) = 2\cos(\theta^2) \cdot (-\sin(\theta^2)) \cdot 2\theta.$

47. $y = \cos(3\pi x/2)$

$\Rightarrow \dfrac{dy}{dx} = -\sin(3\pi x/2)\dfrac{d}{dx}(3\pi x/2)$

$= -(3\pi/2)\sin(3\pi x/2).$

48. $y = \cos^3 x$

$\Rightarrow \dfrac{dy}{dx} = 3(\cos^2 x)(-\sin\ x).$

49. $y = [\sin(x+5)]^{\frac{5}{4}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{5}{4}[\sin(x+5)]^{\frac{1}{4}}[\cos(x+5)].$

(b) Derivatives of inverse trigonometric functions

These are useful in applications and are essential for solving problems. The rules or definitions are as follows:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

Examples:

Compute the derivatives of the following functions:

50. $\cos^{-1}\left(3x^2\right)$ 51. $\left(\sec^{-1}x\right)^2$ 52. $\tan^{-1}\left(x^3\right)$

Solutions:

50. $\dfrac{d}{dx}\cos^{-1}\left(3x^2\right) = \dfrac{-1}{\sqrt{1-\left(3x^2\right)^2}}\dfrac{d}{dx}\left(3x^2\right) = \dfrac{-6x}{\sqrt{1-9x^4}}.$

51. $\dfrac{d}{dx}\left(\sec^{-1}x\right)^2 = 2\left(\sec^{-1}x\right)\dfrac{d}{dx}\sec^{-1}x = 2\left(\sec^{-1}x\right)\dfrac{1}{|x|\sqrt{x^2-1}}.$

52. $\dfrac{d}{dx}\tan^{-1}\left(x^3\right) = \dfrac{1}{1+\left(x^3\right)^2}\dfrac{d}{dx}x^3 = \dfrac{3x^2}{1+x^6}.$

4.5.4 Derivatives of Exponential and Logarithmic Functions

(a) The exponential function

Definition:

The inverse function of the function $\ln x$ is known as the *exponential function* and is indicated by exp x (or e^x).

Properties of the exponential function

(i) The graph of e^x is the mirror image of the graph of $\ln x$ in the line $y = x$. Therefore the graph looks like this: Figure 4.2

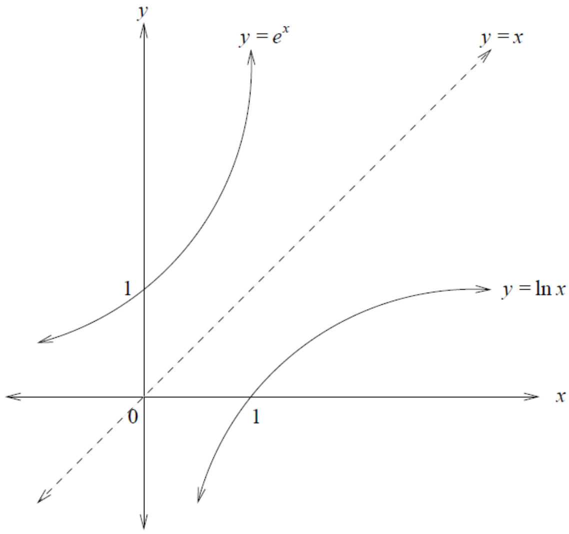


Figure 4.2: Graph of $y = e^x$ and $y = \ln x$

(ii) $\exp x$ is defined for all $x \in \mathbb{R}$, in other words

$$\text{Dom } (\exp x) = \mathbb{R}.$$

(iii) $\text{Im } (\exp x) = \{y|y > 0\}$.

(iv) $\dfrac{d}{dx}(e^x) = e^x$.

(v) $e^{x+y} = e^x \cdot e^y$.

(vi) $e^{x-y} = e^x/e^y$.

(vii) $e^{rx} = (e^x)^r$.

(viii) If $x \to \infty$, then $e^x \to \infty$.

(ix) If $x \to -\infty$, then $e^x \to 0$.

(b) The logarithmic function

Definition:

The natural logarithmic function $y = \ln x$ is the inverse function of the exponential function.

Properties of the $\ln$ function

(i) The graph of $\ln x$ looks like this:

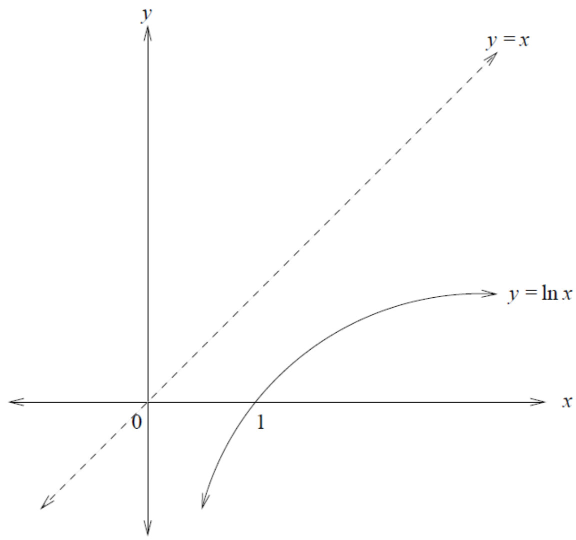


Figure 4.3: Graph of $y = x$ and $y = \ln x$

(ii) $\ln x$ is defined for all $x > 0$, in other words

$$\text{Dom } (\ln x) = \{x | \, x > 0\}.$$

(iii) $\text{Im } (\ln x) = \mathbb{R}$.

(iv) $\ln x < 0$ if $0 < x < 1$,

$\ln 1 = 0$,

$\ln x > 0$ if $x > 1$.

(v) $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

(vi) $\ln(xy) = \ln x + \ln y$, if $x, y > 0$.

(vii) $\ln \dfrac{x}{y} = \ln x - \ln y$, if $x, y > 0$.

(viii) $\ln x^r = r \ \ln x$, if $x > 0$, r rational

(ix) $\log_b a = \dfrac{\log_c a}{\log_c b}$

(x) If $x \to \infty$, then $\ln x \to \infty$.

(xi) If $x \to 0^+$, then $\ln x \to -\infty$.

Remark:

The following mistake is often made when applying rule (h):

$$\ln(x^r) = r\ \ln x \quad \text{(this is correct)}$$

but

$$(\ln x)^r \neq r\ \ln x \quad \text{(incorrect)}.$$

The index r must be an index of x, but not of $\ln x$.

The *inverse* of the exponential function e^x is called the natural logarithmic function $y = \ln x$. In this section, we study the natural logarithm both as a differentiable function and as a device for simplifying calculations.

(c) Examples of the exponential function

Find the derivatives of the following:

53. $f(x) = e^{\sqrt{x}}$
54. $y = e^{x\cos x}$
55. $y = \tan\left(e^{3x-2}\right)$
56. $y = \dfrac{e^{3x}}{1+e^x}$
57. $y = e^{x+e^x}$

Solutions:

53. $f(x) = e^{\sqrt{x}}$

$$f'(x) = e^{\sqrt{x}} \cdot \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

54. $y = e^{x\cos x}$

$$y' = e^{x\cos x}\left(1.\cos x + x(-\sin x)\right) = e^{x\cos x}\left(\cos x - x\sin x\right).$$

55. $y = \tan\left(e^{3x-2}\right)$

$$y' = \sec^2\left(e^{3x-2}\right)\left(e^{3x-2}.3\right) = 3e^{3x-2}\sec^2\left(e^{3x-2}\right).$$

56. $y = \dfrac{e^{3x}}{1+e^x}$

$$y' = (\frac{e^{3x}}{1+e^x})' = \frac{(1+e^x)\left(e^{3x}.3\right) - e^{3x}\left(0+e^x(1)\right)}{(1+e^x)^2} = \frac{3e^{3x}+3e^{4x}-e^{4x}}{(1+e^x)^2} = \frac{3e^{3x}+2e^{4x}}{(1+e^x)^2}.$$

57. $y = e^{x+e^x}$

$$y' = e^{x+e^x}\left(1+e^x(1)\right) = e^{x+e^x}(1+e^x).$$

(d) Examples of the logarithmic function

Find the derivatives of the following:

58. $f(x) = x^2 \ln\left(1-x^2\right)$
59. $f(x) = \log_3\left(x^2-4\right)$
60. $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right)$
61. $y = \ln\left(x\sqrt{1-x^2}\sin x\right)$
62. $y = x^{\sin x}$
63. $y = (\sin x)^x$

Solutions:

58. $f(x) = x^2 \ln\left(1-x^2\right)$

$$f'(x) = 2x\ln\left(1-x^2\right) + x^2 \cdot \frac{1}{1-x^2}(0-2x) = 2x\ln\left(1-x^2\right) - \frac{2x^3}{1-x^2}.$$

59. $f(x) = \log_3\left(x^2-4\right)$

Note that (or recall that) $\log_a x = \dfrac{\ln x}{\ln a}$,

so $f(x) = \log_3\left(x^2-4\right) = \dfrac{\ln\left(x^2-4\right)}{\ln 3}$

$$\therefore \quad f'(x) = \frac{1}{\ln 3}\left[\frac{1}{x^2-4}(2x-0)\right] = \frac{1}{\ln 3}\left[\frac{2x}{x^2-4}\right] = \frac{2x}{(x^2-4)\ln 3}.$$

60. $f(x) = \log_{10}\left(\frac{x}{x-1}\right)$

Recall: $\log_a x = \frac{\ln x}{\ln a}$,

$$\text{so } f(x) = \log_{10}\left(\frac{x}{x-1}\right) = \frac{\ln\left(\frac{x}{x-1}\right)}{\ln 10} = \frac{1}{\ln 10}\left[\ln\left(\frac{x}{x-1}\right)\right] = \frac{1}{\ln 10}\left[\ln x - \ln(x-1)\right]$$

$$\therefore \quad f'(x) = \frac{1}{\ln 10}\left[\frac{1(x-1)-1.x}{x(x-1)}\right] = \frac{1}{\ln 10}\left[\frac{x-1-x}{x(x-1)}\right] = \frac{-1}{x(x-1)\ln 10}$$

61. $y = \ln\left(x\sqrt{1-x^2}\sin x\right)$

$$y = \ln x + \frac{1}{2}\ln\left(1-x^2\right) + \ln\sin x$$

$$\text{so } y' = \frac{1}{x} + \frac{1}{2}\left[\frac{1}{1-x^2}(0-2x)\right] + \frac{1}{\sin x}(\cos x\,(1))$$

$$\text{that is} \quad y' = \frac{1}{x} + \frac{1}{2}\left(\frac{-2x}{1-x^2}\right) + \frac{\cos x}{\sin x}$$

$$= \frac{1}{x} + \frac{-x}{1-x^2} + \cot x.$$

62. $y = x^{\sin x}$

You should take natural logs on both sides first and get:

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

Now take the derivatives of both sides:

$$\frac{1}{y}y' = \cos x.\ln x + \sin x.\frac{1}{x}$$

$$y' = y\left[\cos x\ln x + \frac{\sin x}{x}\right] = x^{\sin x}\left[\cos x.\ln x + \frac{\sin x}{x}\right].$$

63. $y = (\sin x)^x$

Take logs on both sides and get:

$$\ln y = x\ln(\sin x)$$

and $\frac{1}{y}y' = 1.\ln(\sin x) + x(\frac{1}{\sin x}.\cos x)$,

so $y' = y\left[\ln(\sin x) + x.\frac{\cos x}{\sin x}\right]$

$$= (\sin x)^x \left[\ln(\sin x) + x.\cot x\right].$$

4.5.5 Logarithmic Functions

(a) The simplification of functions

As a result of its properties, a function involving the logarithmic function can be simplified before it is differentiated, as becomes clear in the following example:

Example:

Determine $f'(x)$ if $f(x) = \ln\left(\sqrt[3]{\frac{x-1}{x^2}}\right)$.

Solution:

Now using properties (f), (g) and (h) of logarithmic functions described earlier on

$$\begin{aligned} f(x) &= \ln\left(\frac{x-1}{x^2}\right)^{1/3} \\ &= \frac{1}{3}\left(\ln(x-1) - \ln\ x^2\right) \\ &= \frac{1}{3}\ln(x-1) - \frac{2}{3}\ln\ x. \end{aligned}$$

Therefore

$$\begin{aligned} f'(x) &= \frac{1}{3}\left(\frac{1}{(x-1)}\right) - \frac{2}{3}\left(\frac{1}{x}\right) \\ &= \frac{-x+2}{3x(x-1)}. \end{aligned}$$

Even if the logarithmic function itself does not occur in the expression, it can be used for the simplification of functions. We apply the logarithmic function to both sides of the equation and then differentiate (see below).

Remark:

Always remember that when applying the chain rule to $\frac{d}{dx}(\ln\ f(x))$, it becomes

$$\boxed{\frac{d}{dx}(\ln\ f(x)) = \frac{1}{f(x)} f'(x).}$$

For example:

$$\begin{aligned} & \frac{d}{dx}(\ln\ \sin x) \\ = & \frac{1}{\sin x} \cdot \cos x. \end{aligned}$$

Examples:

64. Determine y' if $y = x^2 e^{2x} \cos\ x^3$.

65. Determine $f'(x)$ if $f(x) = \frac{x(1-x^2)^2}{\sqrt{1+x^2}}$.

66. Determine $f'(x)$ if $f(x) = \log_3 \frac{\sqrt{x^2+4}}{(x-1)^{\frac{3}{2}}}$.

Solutions:

64. We have that

$$y = x^2 e^{2x} \cos\ x^3$$

By applying the logarithmic function on both sides, we find that

$$\begin{aligned} \ln\ y &= \ln\left(x^2 e^{2x} \cos x^3\right) \\ &= 2\ \ln\ x + 2x + \ln\ \cos x^3. \end{aligned}$$

By differentiating we obtain

$$\frac{1}{y} \cdot y' = \frac{2}{x} + 2 + \frac{1}{\cos x^3} \cdot \left(-\sin x^3\right) \cdot 3x^2.$$

Therefore

$$\begin{aligned} y' &= y\left(\frac{2}{x} + 2 - 3x^2 \tan x^3\right) \\ &= x^2 e^{2x} \cos x^3 \left(\frac{2}{x} + 2 - 3x^2 \tan x^3\right). \end{aligned}$$

65.

$$f(x) = \frac{x\left(1-x^2\right)^2}{\sqrt{1+x^2}}$$

Once again we apply the logarithmic function on both sides and thus obtain

$$\begin{aligned}\ln\ f(x) &= \ln \frac{x\left(1-x^2\right)^2}{\sqrt{1+x^2}} \\ &= \ln\ x + 2\ \ln(1-x^2) - \frac{1}{2}\ln(1+x^2).\end{aligned}$$

By differentiation we therefore obtain that

$$\frac{1}{f(x)} \cdot f'(x) = \frac{1}{x} + \frac{-4x}{1-x^2} - \frac{x}{1+x^2}.$$

Therefore

$$\begin{aligned}f'(x) &= \frac{x\left(1-x^2\right)^2}{\sqrt{1+x^2}}\left(\frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}\right) \\ &= \frac{\left(1-x^2\right)^2}{\sqrt{1+x^2}} - \frac{4x^2\left(1-x^2\right)}{\sqrt{1+x^2}} - \frac{x^2\left(1-x^2\right)^2}{(1+x^2)^{\frac{3}{2}}} \\ &= \frac{\left(1-x^2\right)^2\left(1+x^2\right) - 4x^2\left(1-x^2\right)\left(1+x^2\right) - x^2\left(1-x^2\right)^2}{(1+x^2)^{\frac{3}{2}}} \\ &= \frac{\left(1-x^2\right)\left[\left(1-x^2\right)\left(1+x^2\right) - 4x^2\left(1+x^2\right) - x^2\left(1-x^2\right)\right]}{(1+x^2)^{\frac{3}{2}}} \\ &= \frac{\left(1-5x^2-4x^4\right)\left(1-x^2\right)}{(1+x^2)^{\frac{3}{2}}}.\end{aligned}$$

66.

$$\begin{aligned}f(x) &= \log_3 \frac{\sqrt{x^2+4}}{(x-1)^{\frac{3}{2}}} \\ &= \frac{1}{2}\log_3(x^2+4) - \frac{3}{2}\log_3(x-1)\end{aligned}$$

Now using the fact that

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{[Formula 10 on page 62 in Stewart]}$$

we find that

$$\begin{aligned}f(x) &= \frac{1}{2}\frac{\log_e\left(x^2+4\right)}{\log_e 3} - \frac{3}{2}\frac{\log_e(x-1)}{\log_e 3} \\ &= \frac{1}{2}\frac{\ln\left(x^2+4\right)}{\ln\ 3} - \frac{3}{2}\frac{\ln(x-1)}{\ln\ 3}.\end{aligned}$$

If we keep in mind that $\ln\ 3$ is a constant, it follows that

$$\begin{aligned}f'(x) &= \frac{1}{2\ \ln\ 3} \cdot \frac{1}{x^2+4} \cdot 2x - \frac{3}{2\ \ln\ 3} \cdot \frac{1}{x-1} \\ &= \frac{x}{(x^2+4)\ln\ 3} - \frac{3}{2(x-1)\ln\ 3}.\end{aligned}$$

(b) Functions of the form $f(x) = g(x)^{h(x)}$

We employ logarithmic differentiation to differentiate functions of the form

$$f(x) = g(x)^{h(x)}$$

Just as in (a), we do this by applying the logarithmic function on both sides and then making use of the properties of the logarithmic function.

Then

$$\ln\ f(x) = \ln\ g(x)^{h(x)}.$$

Therefore

$$\ln\ f(x) = h(x).\ln\ g(x).$$

Example:

Determine $f'(x)$ if $f(x) = \sqrt{x}^{\sqrt{x}}$.

Solution:

By applying the logarithmic function on both sides, we obtain

$$\begin{aligned} \ln\ f(x) &= \ln \sqrt{x}^{\sqrt{x}} \\ &= \sqrt{x} \cdot \ln\ \sqrt{x} \end{aligned}$$

By differentiating, we obtain from the chain rule and the product rule that

$$\begin{aligned} \frac{1}{f(x)} f'(x) &= \sqrt{x} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} + \ln\ \sqrt{x} \cdot \frac{1}{2} x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} \ln\ \sqrt{x}. \end{aligned}$$

Therefore

$$\begin{aligned} f'(x) &= f(x) \left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} \ln\ \sqrt{x} \right) \\ &= \sqrt{x}^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \left(1 + \ln\ \sqrt{x}\right) \right) \\ &= \frac{\sqrt{x}^{\sqrt{x}}}{2\sqrt{x}} (1 + \ln\ \sqrt{x}). \end{aligned}$$

[See the remark earlier on under (a) The simplification of functions]. Now attempt the following yourself. (The answers are given at the end, so that you can make sure that you understand this important section well.)

Determine $f'(x)$ below:

67. $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ (be careful with this exercise!)

68. $f(x) = (\ln\ x)^{\sqrt{x}}$

69. $f(x) = \ln\sqrt[3]{\dfrac{x^2-1}{x^2+1}}$

70. $f(x) = x^{\ln\ x} + 7^{x^{\frac{2}{3}}}$

71. $f(x) = \dfrac{x\,(x-1)^{\frac{1}{3}}}{(x+1)^{\frac{2}{3}}}$

72. $f(x) = a^{3x^2}$

73. $f(x) = x^{e^{-x^2}}$

74. $f(x) = \log_{10}(\log_{10} x)$

Solutions:

67. $\dfrac{1}{\sqrt{x^2+1}}$. (Use the chain rule on the right-hand side and simplify.)

68. $\dfrac{(\ln\ x)^{\sqrt{x}}}{\sqrt{x}}\left(\dfrac{1}{\ln\ x} + \dfrac{\ln\ \ln\ x}{2}\right)$.

69. $\dfrac{4x}{3\,(x^4-1)}$.

70. $\dfrac{2\ln\ x}{x} \times x^{\ln\ x} + \dfrac{2\ln\ 7}{3x^{\frac{1}{3}}} \times 7^{x^{\frac{2}{3}}}$.

71. $\dfrac{2x^2 + 3x - 3}{3\,(x-1)^{\frac{2}{3}}\,(x+1)^{\frac{5}{3}}}$.

72. $6x\; a^{3x^2} \ln\, a$.

73. $x^{e^{-x^2}} \cdot e^{-x^2}(\frac{1}{x} - 2x\;\; \ln\;\; x)$.

74. $\dfrac{1}{x\;\; \ln\;\; x\;\; \ln\;\; 10}$.

4.5.6 Implicit Differentiation

This technique is used when the function contains variables that cannot be separated from each other. This process of differentiating both sides of the equation with respect to x and then solving for $y'(x)$ is called **implicit differentiation**.

Example:

Find $y'(x)$ for $x^2 + y^3 - 2y = 3$. Then find the slope of the tangent line at the point $(2, 1)$.

Solutions:

Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(x^2 + y^3 - 2y\right) = \frac{d}{dx}(3)$$

$$2x + 3y^2 y'(x) - 2y'(x) = 0$$

Solve for $y'(x)$:

$$3y^2 y'(x) - 2y'(x) = -2x$$

$$y'(x) = \frac{-2x}{3y^2 - 2}$$

Find the slope of the tangent line at $(2, 1)$:

$$\boxed{y'(2) = \frac{-4}{3-2} = -4}$$

Therefore the equation of the tangent line is

$$y - 1 = -4(x - 2).$$

Now try the following exercises:

75. Find $y'(x)$ for $x^2y^2 - 2x = 4 - 4y$. Then find the slope of the tangent line at the point $(2, -2)$.

76. Determine the derivative of $\tan^{-1} x$.

77. Van der Waal's equation for a specific gas is $\left(P + \dfrac{5}{V^2}\right)(V - 0.03) = 9.7$.

    Using the volume V as a function of pressure, use implicit differentiation to find the derivative $\dfrac{dV}{dP}$ at the point $(2, -2)$.

78. Find $\dfrac{dy}{dx}$ if $\tan\left(xy^2\right) + 3y = 2xy$.

79. Find $\dfrac{dy}{dx}$ if $y^2x^2 + y + \cos(xy) = 0$.

80. Find $\dfrac{dy}{dx}$ if $\sin(x+y) = y^2 \cos x$.

Solutions:

75.

$$\begin{aligned}
& \frac{d}{dx}\left(x^2y^2 - 2x\right) &=& \frac{d}{dx}(4 - 4y) \\
\Rightarrow\quad & 2xy^2 + x^2\left(2y\frac{dy}{dx}\right) - 2 &=& 0 - 4\frac{dy}{dx} \\
\Rightarrow\quad & \left(2x^2y + 4\right)\frac{dy}{dx} &=& 2 - 2xy^2 \\
\Rightarrow\quad & \frac{dy}{dx} &=& \frac{2 - 2xy^2}{2x^2y + 4} \\
\therefore\quad & \left.\frac{dy}{dx}\right|_{(2,-2)} = m_t &=& \frac{2 - 2(2)(-2)^2}{2(-2)(2)^2 + 4} = \frac{-14}{-12} = \frac{7}{6}
\end{aligned}$$

where m_t represents slope of the tangent line.

The equation of the tangent line is

$$\begin{aligned}
y - (-2) &= \frac{7}{6}(x - 2) \\
\text{that is} \qquad y + 2 &= \frac{7}{6}x - \frac{14}{6} \\
\text{that is} \qquad y &= \frac{7}{6}x - \frac{14}{6} - 2 \\
&= \frac{7}{6}x - \frac{14}{6} - \frac{12}{6} \\
&= \frac{7}{6}x - \frac{26}{6} \\
&= \frac{7}{6}x - \frac{13}{3}.
\end{aligned}$$

76. The derivative of $\tan^{-1} x$:

Note: Since $\tan x$ is differentiable, $\tan^{-1} x$ is also differentiable. To find its derivative, let $y = \tan^{-1} x$. Then
$\tan y = \tan\left(\tan^{-1} x\right) \Rightarrow \tan y = x.$

Differentiate this equation implicitly with respect to x to get:

$\sec^2 y \dfrac{dy}{dx} = 1.$

Then $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}.$

Therefore $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}.$

77.

$$
\begin{aligned}
& \left(P + \tfrac{5}{V^2}\right)(V - 0.03) = 9.7 \\
\Rightarrow\quad & PV - 0.03P + \tfrac{5}{V} - \frac{0.15}{V^2} = 9.7 \\
\Rightarrow\quad & \frac{d}{dP}\left(PV - 0.03P + 5V^{-1} - 0.15V^{-2}\right) = 0 \\
\Rightarrow\quad & \left(1V + P\frac{dV}{dP}\right) - 0.03 + 5\,(-1)\,V^{-2}\frac{dV}{dP} + 2\cdot(0.15)\,V^{-3}\frac{dV}{dP} = 0 \\
\Rightarrow\quad & \left(P - \frac{5}{V^2} + \frac{0.3}{V^3}\right)\frac{dV}{dP} = 0.03 - V \\
\Rightarrow\quad & \frac{dV}{dP} = \frac{0.03 - V}{P - \dfrac{5}{V^2} + \dfrac{0.3}{V^3}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\quad \left.\frac{dV}{dP}\right|_{(2,-2)} &= \frac{0.03-(-2)}{2 - \dfrac{5}{(-2)^2} + \dfrac{0.3}{(-2)^3}} \\
&= \frac{0.03+2}{2 - \dfrac{5}{4} - \dfrac{0.3}{8}} \\
&= \frac{203/100}{57/80} = \frac{812}{150}.
\end{aligned}
$$

78. $\tan\left(xy^2\right) + 3y = 2xy$

Differentiate on both sides:

$$
\begin{aligned}
\sec^2(xy^2)[y^2 + 2xy\frac{dy}{dx}] + 3\frac{dy}{dx} &= 2y + 2x\tfrac{dy}{dx} \\
\Rightarrow\quad \left[2xy\sec^2(xy^2) - 2x + 3\right]\frac{dy}{dx} &= 2y - y^2\sec^2(xy^2) \\
\Rightarrow\quad \frac{dy}{dx} &= \frac{2y - y^2\sec^2(xy^2)}{2xy\sec^2(xy^2) - 2x + 3}.
\end{aligned}
$$

79. $y^2x^2 + y + \cos(xy) = 0$

Differentiate on both sides:

$$\begin{aligned}
2y\frac{dy}{dx}x^2 + 2xy^2 + \frac{dy}{dx} - \sin(xy)\left[y + x\frac{dy}{dx}\right] &= 0 \\
\Rightarrow \qquad \left[2yx^2 + 1 - x\sin(xy)\right]\frac{dy}{dx} &= -2xy^2 + y\sin(xy) \\
\Rightarrow \qquad \frac{dy}{dx} &= \frac{y\sin(xy) - 2xy^2}{2x^2y + 1 - x\sin(xy)}.
\end{aligned}$$

80. $\sin(x+y) = y^2\cos x$

Differentiate on both sides:

$$\begin{aligned}
\cos(x+y)\cdot\left(1 + \frac{dy}{dx}\right) &= 2y\tfrac{dy}{dx}\cos x + y^2(-\sin x) \\
\Rightarrow \quad \cos(x+y) + \cos(x+y)\frac{dy}{dx} &= 2y\tfrac{dy}{dx}\cos x - y^2\sin x \\
\Rightarrow \quad \frac{dy}{dx}\left[\cos(x+y) - 2y\cos x\right] &= -y^2\sin x - \cos(x+y) \\
\Rightarrow \quad \frac{dy}{dx} &= \frac{-y^2\sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x}.
\end{aligned}$$

4.5.7 Tangents and Normal Lines

From your high-school mathematics, you should know that differentiating the equation of a curve gives you a formula for the gradient (m) of the curve. The gradient of a curve at a point is equal to the gradient of the tangent at that point. For example, in order to find the equation of the tangent to the curve $y = x^3$ at the point $(2, 8)$, you first have to determine the derivative: $m = y' = 3x^2$.

The gradient of the tangent at the point $(2, 8)$ when $x = 2$ is $3\,(2)^2 = 12$

The equation for the tangent line is: $y = 12x - 16$ since $\left\{\dfrac{\Delta y}{\Delta x} = \dfrac{8-y}{2-x} = 12\right\}$.

The normal to the curve is the line which is perpendicular (at right angles) to the tangent to the curve at that point. If two lines are perpendicular, then the product of their gradients is -1. So if the gradient of the tangent at the point $(2, 8)$ of the curve $y = x^3$ is 12, then the gradient of the normal is $-\dfrac{1}{12}$, since $-\dfrac{1}{12} \times 12 = -1$. Now try to work through the following problems involving tangents and normal lines. Note, we have used the fact that $\boxed{m_t m_n = -1}$ where m_t and m_n are the slopes of tangent and normal lines respectively.

Examples:

81. Two curves are said to be *orthogonal* at a point where the curves intersect if their tangents are perpendicular to each other at the point of intersection. Show that the curves $y = \sin(2x)$ and $y = -\sin(\frac{x}{2})$ are orthogonal at the origin.

82. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

83. Does the graph of the function

$$y = 2x + \sin\ x$$

have any horizontal tangents in the interval $0 \leq x \leq 2\pi$? If so, where? If not, why not?

84. Find equations for the lines that are tangent and normal to the curve $f(x) = 1 + \cos x$ at the point $(\pi/2, 1)$.

85. (a) Find the equation for the tangent to the graph of

$$h(x) = \frac{4}{(1+x)^2}$$

which makes an angle of 45° with the x-axis.

[**NB:** Assume that the scales along the x- and y-axes are the same. Angles are measured counterclockwise from the positive x-axis.]

(b) What is the equation of the normal to the curve of h at the point of tangency for the tangent considered in (a) above?

86. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis. What are the equations for these tangents?

87. Determine the equation for the normal to the curve of

$$f(x) = \frac{1}{x^2+1}$$

at the point $x = 1$.

88. Determine the equations for the tangents to the curve of $y = x^3 - 6x + 2$ which are parallel to the line $y = 6x - 2$.

89. The line normal to the curve $y = x^2 + 2x - 3$ at $(1, 0)$ intersects the curve at what other point?

90. What are the equations for the tangents to the curve $f(x) = x^2 + 5x + 9$ which passes through the origin?

91. Find the equations of the tangent and the normal lines to the curve of

$$2xy + \pi \sin y = 2\pi \quad \text{at the point } \left(1, \frac{\pi}{2}\right).$$

92. Find the equations for the tangent and the normal lines to the curve of

$$x \sin(2y) = y \cos(2x)$$

at the point $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Solutions:

81.

$$\begin{aligned} y &= \sin(2x) \\ \Rightarrow \frac{dy}{dx} &= (\cos(2x))\frac{d}{dx}(2x) \\ &= 2\cos(2x), \end{aligned}$$

so

$$\left.\frac{dy}{dx}\right|_{x=0} = 2 \times 1 = 2.$$

Also,

$$\begin{aligned} y &= -\sin(\frac{x}{2}) \\ \Rightarrow \frac{dy}{dx} &= -\cos(\frac{x}{2})\frac{d}{dx}\left(\frac{x}{2}\right) \\ &= -\frac{1}{2}\cos(\frac{x}{2}). \end{aligned}$$

so

$$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{1}{2} \times 1 = -\frac{1}{2}.$$

Since $\left(-\frac{1}{2}\right)(2) = -1$, the given curves are orthogonal at the origin.

82. Let

$$f(x) = x^3.$$

Then

$$f'(x) = 3x^2,$$

so the gradient at the point $(-2, -8)$ is $f'(-2)$, that is 12. The equation of the tangent at the point $(-2, -8)$ is therefore given by

$$y - (-8) = 12(x - (-2))$$

or

$$\begin{aligned} y &= 12x + 16 \\ &= 4(3x + 4). \end{aligned}$$

The x-intercept of this tangent is $-\frac{4}{3}$ and its y-intercept is 16.

83. Let
$$f(x) = 2x + \sin\ x.$$

Then
$$f'(x) = 2 + \cos x.$$

Now the gradient of a horizontal line is 0, so if the graph of f is to have a horizontal tangent at the point $(x, f(x))$, then the value of $f'(x)$ must be 0. We must therefore solve the equation
$$f'(x) = 0$$

for all values of x occurring in the interval $0 \leq x \leq 2\pi$. But if
$$2 + \cos\ x = 0$$

then
$$\cos\ x = -2,$$

which is impossible. Hence the graph of f has no horizontal tangents.

84.
$$\begin{aligned} f(x) &= 1 + \cos x \\ \Rightarrow f'(x) &= -\sin\ x \\ m_t &= -1 \end{aligned}$$

Hence the gradient at the point $\left(\frac{\pi}{2}, 1\right)$ is $f'\left(\frac{\pi}{2}\right)$ that is -1. The equation of the tangent at the point $\left(\frac{\pi}{2}, 1\right)$ is therefore
$$\begin{aligned} y - y_1 &= m_t (x - x_1) \\ y - 1 &= -1\left(x - \frac{\pi}{2}\right) \\ y &= -x + \frac{\pi}{2} + 1. \end{aligned}$$

Since $m_t m_n = -1 \Rightarrow m_n = \frac{-1}{m_t} = \frac{-1}{-1} = 1$. The gradient of the normal at the point $\left(\frac{\pi}{2}, 1\right)$ is 1. The equation for the normal at the point $\left(\frac{\pi}{2}, 1\right)$ is therefore
$$\begin{aligned} y - y_1 &= m_n (x - x_1) \\ y - 1 &= 1\left(x - \frac{\pi}{2}\right) \\ y &= x - \frac{\pi}{2} + 1. \end{aligned}$$

85. (a) The tangent which makes an angle of 45° with the x-axis has a gradient of 1. To find the x-coordinate of the corresponding point of tangency, we let $m = \tan\theta$ be the slope
$$h'(x) = 1$$

and solve for x. Now

$$\begin{aligned} h(x) &= 4(1+x)^{-2} \\ \Rightarrow h'(x) &= -8(1+x)^{-3}, \end{aligned}$$

and

$$\begin{aligned} -\frac{8}{(1+x)^3} &= 1 \\ \Leftrightarrow \quad x &= -3. \end{aligned}$$

We have that

$$h(-3) = \frac{4}{(1-3)^2} = 1.$$

The tangent thus passes through the point $(-3, 1)$ and has a gradient of 1. Its equation is

$$y = 1.(x+3) + 1$$

that is

$$y = x + 4.$$

(b) We obtain the equation for the required normal:

$$y = -1.(x+3) + 1$$

that is

$$y = -x - 2.$$

86.

$$\begin{aligned} y &= 2x^3 - 3x^2 - 12x + 20 \\ \Rightarrow y' &= 6x^2 - 6x - 12. \end{aligned}$$

Now

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ \Rightarrow 6(x^2 - x - 2) &= 0 \\ \Rightarrow 6(x-2)(x+1) &= 0 \\ \Rightarrow x = 2 \ \text{ or } \ x &= -1. \end{aligned}$$

If $x = 2$ then $y = 2 \times 8 - 3 \times 4 - 24 + 20 = 0$.

If $x = -1$ then $y = -2 - 3 + 12 + 20 = 27$.

The required points are $(2, 0)$ and $(-1, 27)$. The equations for the tangents are $y = 0$ and $y = 27$ respectively.

87.

$$\begin{aligned} f(x) &= \frac{1}{x^2+1} = \left(x^2+1\right)^{-1} \\ \Rightarrow \quad f'(x) &= -(x^2+1)^{-2} \cdot 2x \\ \therefore \quad f'(1) &= -\frac{1}{2}, \end{aligned}$$

so the gradient of the normal at $x = 1$ is 2. Now

$$f(1) = \frac{1}{2},$$

so the equation for the normal at $x = 1$ is

$$y - \frac{1}{2} = 2(x - 1).$$

88. The slope of the tangent to the curve of $y = x^3 - 6x + 2$ is given by $y' = \dfrac{dy}{dx}$. Now

$$y' = 3x^2 - 6.$$

Since the tangent must be parallel to $y = 6x - 2$, we must have that y' is equal to the slope of $y = 6x - 2$, that is

$$\begin{aligned} y' &= 6 \\ \Leftrightarrow \quad 3x^2 - 6 &= 6 \\ \Leftrightarrow \quad 3x^2 - 12 &= 0 \\ \Leftrightarrow \quad x^2 - 4 &= 0 \\ \Leftrightarrow \quad x &= \pm 2. \end{aligned}$$

The corresponding y-values are

$$\begin{aligned} y &= (-2)^3 - 6(-2) + 2 \\ &= -8 + 12 + 2 = 6 \end{aligned}$$

for $x = -2$, and

$$\begin{aligned} y &= (2)^3 - 6(2) + 2 \\ &= 8 - 12 + 2 = -2 \end{aligned}$$

for $x = 2$.

Therefore the tangent points are $(-2, 6)$ and $(2, -2)$.

The equation for a line with slope m which passes through (x_o, y_o) is given by

$$y - y_o = m(x - x_o).$$

The tangent through $(-2, 6)$ therefore has the equation

$$\begin{aligned} y - 6 &= 6(x - (-2)) \\ &= 6x + 12 \end{aligned}$$

that is

$$y = 6x + 18.$$

The tangent through $(2, -2)$ has the equation

$$y - (-2) = 6(x - 2)$$

that is

$$y + 2 = 6x - 12$$

that is

$$y = 6x - 14.$$

89. The normal to the curve $y = x^2 + 2x - 3$ at the point $(1, 0)$ is perpendicular to the tangent at that point. Hence, if we know the gradient of the tangent we can calculate the gradient of the normal. Now if

$$f(x) = x^2 + 2x - 3$$

then

$$f'(x) = 2x + 2.$$

Thus

$$f'(1) = 4,$$

which is the gradient of the tangent. Hence the gradient of the normal is $-\frac{1}{4}$. Since the normal passes through the point $(1, 0)$, its equation is

$$y - 0 = -\frac{1}{4}(x - 1)$$

that is

$$y = -\frac{1}{4}x + \frac{1}{4}.$$

To find the points of intersection of the curve $y = x^2+2x-3$ and the line $y = -\frac{1}{4}x+\frac{1}{4}$, let

$$x^2 + 2x - 3 = -\frac{1}{4}x + \frac{1}{4}.$$

Then

$$4x^2 + 8x - 12 = -x + 1$$

so

$$4x^2 + 9x - 13 = 0,$$

that is

$$(x-1)(4x+13) = 0.$$

Hence $x = 1$ or $x = -\dfrac{13}{4}$. If $x = -\dfrac{13}{4}$ then

$$-\frac{1}{4}x + \frac{1}{4} = \frac{13}{16} + \frac{1}{4} = \frac{17}{16}.$$

The other point of intersection of the normal and the curve is therefore $\left(-\dfrac{13}{4}, \dfrac{17}{16}\right)$.

90. The equation of the tangent at a point $(\alpha, f(\alpha))$ is

$$y - f(\alpha) = f'(\alpha)(x - \alpha),$$

that is

$$y = f'(\alpha) \cdot x - \alpha \cdot f'(\alpha) + f(\alpha).$$

The tangent will pass through the origin only if the y-intercept is 0, that is

$$f(\alpha) - \alpha \cdot f'(\alpha) = 0,$$

and then the equation of the tangent becomes

$$y = f'(\alpha) \cdot x. \tag{4.1}$$

Hence the solutions of the equation

$$f(x) - x \cdot f'(x) = 0$$

will give the x-coordinates α of the points of contact between the tangents and the curve, and then the equations of the corresponding tangents can be determined from $(*)$.

Now

$$\begin{aligned} f(x) &= x^2 + 5x + 9 \\ \Rightarrow f'(x) &= 2x + 5, \end{aligned}$$

and

$$\begin{aligned} f(x) - x \cdot f'(x) &= 0 \\ \Rightarrow x^2 + 5x + 9 - x(2x+5) &= 9 - x^2 = 0 \\ \Rightarrow x = 3 \quad \text{or} \quad x &= -3. \end{aligned}$$

We have that $f'(3) = 11$ and $f'(-3) = -1$. Hence the required tangents have the formulae $y = 11x$ and $y = -x$.

91. $2xy + \pi \sin y = 2\pi$

Differentiate both sides:

$$\begin{aligned} 2y + 2x\tfrac{dy}{dx} + \pi \cos y \frac{dy}{dx} &= 0 \\ \Rightarrow \qquad (2x + \pi \cos y)\frac{dy}{dx} &= -2y \\ \Rightarrow \qquad \frac{dy}{dx} &= \frac{-2y}{2x + \pi \cos y}. \end{aligned}$$

At $(1, \frac{\pi}{2})$:

$$\frac{dy}{dx} = \frac{-2 \cdot \frac{\pi}{2}}{2 + \pi \cos \frac{\pi}{2}} = -\frac{\pi}{2}.$$

Equation of the tangent line:

$y - \frac{\pi}{2} = -\frac{\pi}{2}(x - 1) \Rightarrow y = -\frac{\pi}{2}x + \pi.$

Equation of the normal line:

$y - \frac{\pi}{2} = \frac{2}{\pi}(x - 1) \Rightarrow y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}.$

92. $x \sin(2y) = y \cos(2x)$

Differentiate both sides:

$$\begin{aligned} \sin 2y + x \cos 2y \frac{dy}{dx} &= \frac{dy}{dx} \cos 2x - 2y \sin 2x \\ \Rightarrow \quad \frac{dy}{dx}\left[2x \cos 2y - \cos 2x\right] &= -2y \sin 2x - \sin 2y \\ \Rightarrow \qquad \frac{dy}{dx} &= \frac{2y \sin 2x + \sin 2y}{\cos 2x - 2x \cos 2y}. \end{aligned}$$

At the point $(\frac{\pi}{4}, \frac{\pi}{2})$ the slope is:

$$\frac{dy}{dx} = \frac{\pi \sin(\frac{\pi}{2}) + \sin \pi}{\cos(\frac{\pi}{2}) - \frac{2\pi}{4} \cos \pi} = \frac{\pi}{\frac{2\pi}{4}} = 2.$$

Therefore the equation of the tangent at the point $(\frac{\pi}{4}, \frac{\pi}{2})$ is:

$$\begin{aligned} y - \frac{\pi}{2} &= 2(x - \frac{\pi}{4}) \\ \Rightarrow y &= 2x, \end{aligned}$$

and the equation of the normal line is:

$$y = -\frac{1}{2}x + \frac{5\pi}{8}.$$

4.5.8 The Mean Value Theorem

In this section we discuss the Mean Value Theorem. Please read carefully through Sections 4.2 of Chapter 4 in Stewart. We would advise you to work through the examples in order to become familiar with how the Mean Value Theorem is applied. In section 4.2 you have to understand theorems 1 and 3. You have to be able to apply these theorems. Please work through examples 3 to 5 in this section.

The Mean Value Theorem is one of the most important tools in calculus. It states that if $f(x)$ is defined and continuous on the interval $[a, b]$ and differentiable on (a, b), then there is at least one number c in the interval (a, b) (that is $a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The special case when $f(a) = f(b)$ is known as Rolle's Theorem. In that case, we have $f'(c) = 0$. In other words, there exists a point in the interval (a, b) which has a horizontal tangent. In fact, the Mean Value Theorem can be stated in terms of slopes.

The number

$$\frac{f(b) - f(a)}{b - a}$$

is the slope of the line passing through $(a, f(a))$ and $(b, f(b))$, see Figure 4.4.

Example:

Let $f(x) = \frac{1}{x}$, $a = -1$ and $b = 1$. We have

$$\frac{f(b) - f(a)}{b - a} = \frac{1-(-1)}{1-(-1)} = \frac{2}{2} = 1.$$

On the other hand, for any $c \in [-1, 1]$, not equal to 0, we have $f'(c) = -\frac{1}{c^2} \neq 1$.

So the equation $f'(c) = \frac{f(b) - f(a)}{b - a}$ does not have a solution for c. This does not contradict the Mean Value Theorem, since $f(x)$ is not continuous on $[-1, 1]$.

Note: The derivative of a constant function is 0. You may wonder whether a function with a derivative of zero is constant – the answer is yes. To show this, we let $f(x)$ be a

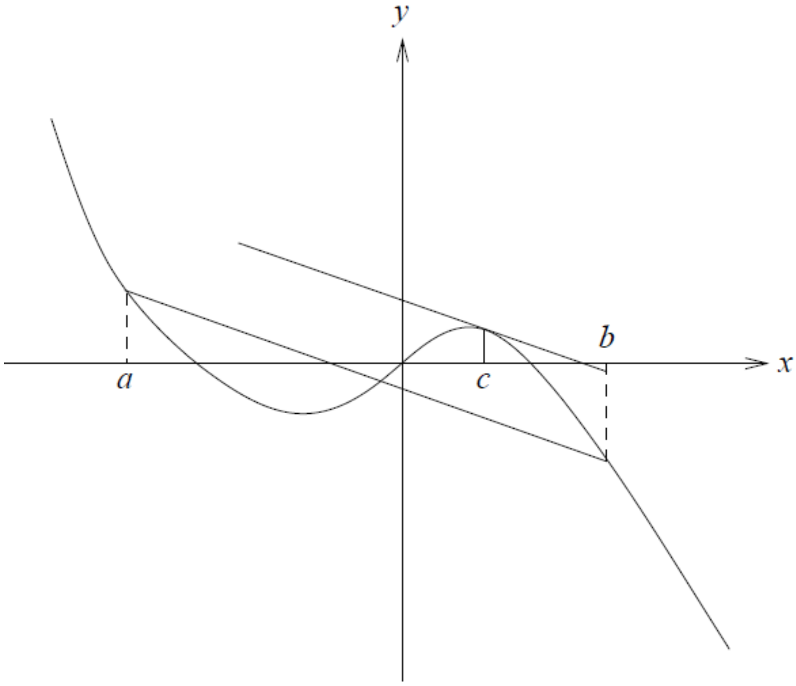


Figure 4.4: Illustration of the Mean Value Theorem

differentiable function on an interval I, with $f'(x) = 0$, for every $x \in I$. Then for a and b in I, the Mean Value Theorem implies that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ for some point c between a and b. So our assumption implies that $f(b) - f(a) = 0.(b - a)$. Thus $f(b) = f(a)$ for any a and b in I, which means that $f(x)$ is a constant function.

Worked Examples:

For the following functions, find a value of c which complies with the Mean Value Theorem.

93. $f(x) = x^2 + 1$, on the interval $[-2, 2]$.

94. $f(x) = x^3 + x^2$, on the interval $[0, 1]$.

95. $f(x) = \sin x$, on the interval $[0, \frac{\pi}{2}]$.

Solutions:

93. Since $f(x) = x^2 + 1$ is a polynomial, $f(x)$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. According to the Mean Value Theorem, this means that there is a number c in $(-2, 2)$ for which

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{(2^2 + 1) - \left((-2)^2\right) + 1}{4} = \frac{5 - 5}{4} = \frac{0}{4} = 0.$$

To find this number c, we obtain $f'(c)$ and then set $f'(c) = 0$.

$f'(c) = 2c$ and $2c = 0$, so $c = 0$

Notice that $c = 0 \in (-2, 2) \quad \therefore \quad c = 0$.

94. $f(x) = x^3 + x^2$ is a polynomial, continuous on $[0, 1]$ and differentiable on $(0, 1)$. According to the Mean Value Theorem, there is a number c in $(0, 1)$ for which

$$f'(c) = \frac{f(1) - f(0)}{1 - (0)} = \frac{2 - 0}{1} = 2.$$

But $f'(c) = 3c^2 + 2c$.

So, $f'(c) = 3c^2 + 2c = 2 \Rightarrow 3c^2 + 2c - 2 = 0$.

Using the quadratic formula we get

$$\begin{aligned} c &= \frac{-2 \pm \sqrt{2^2 - 4(3)(-2)}}{2(3)} = \frac{-2 \pm \sqrt{28}}{6} = \frac{-2 \pm 2\sqrt{7}}{6} \\ c &= \frac{-1 \pm \sqrt{7}}{3} \Rightarrow c = \frac{-1 - \sqrt{7}}{3} \quad \text{or} \quad c = \frac{-1 + \sqrt{7}}{3} \\ &\Rightarrow c \approx -1.22 \quad \text{or} \quad c \approx 0.55. \end{aligned}$$

But since $c \approx -1.22 \notin [0, 1]$, we only accept the alternative $c \approx 0.55$ that is

$$c = \frac{-1 + \sqrt{7}}{3}.$$

95. $f(x) = \sin x$, on the interval $\left[0, \frac{\pi}{2}\right]$.

As $f(x) = \sin x$ is a trigonometric function, it means that $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$ and differentiable on $\left(0, \frac{\pi}{2}\right)$. The conditions for the Mean Value Theorem hold, and so there exists $c \in \left(0, \frac{\pi}{2}\right)$such that

$$f'(c) = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} = \frac{\sin\left(\frac{\pi}{2}\right)\sin(0)}{\frac{\pi}{2}} = \frac{1 - 0}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

But $f'(x) = \cos x$, $f'(c) = \cos c$ and c has to be in the first quadrant. Thus $c = \cos^{-1}\left(\frac{2}{\pi}\right) \approx 0.88$, and so $c = 0.88 \in \left(0, \frac{\pi}{2}\right)$. Therefore the number c we are looking for is $c = \cos^{-1}\left(\frac{2}{\pi}\right) \approx 0.88$.

Key points

You should by now have an understanding of the notion of differentiability and should also be able to compute the derivatives of different functions using basic differential formulas such as the power, product, quotient and chain rules. Think of differentiation as a process during which a function is "processed" to produce another function - the derivative. This process (differentiation) is carried out using different formulas and rules. We have introduced the basic ones in this chapter. We have also demonstrated some techniques you can use to compute the derivatives of different algebraic and trigonometric functions. In addition to this, we have demonstrated how some theorems, such as the Mean Value Theorem, can be applied.

At this stage you should be able to

- compute the derivative of a function from first principles (using the definition of the derivative)
- distinguish between the continuity and differentiability of a function
- use the basic differentiation formulas to compute derivatives of algebraic and trigonometric functions
- compute the derivatives of trigonometric functions and inverse trigonometric functions, as well as exponential and logarithmic functions
- use the methods of logarithmic differentiation and implicit differentiation to find derivatives
- solve problems involving tangents and normal lines and apply the Mean Value Theorem

Continue practising solving problems until you have mastered the basic techniques! Go through the section "For your review" at the end of each chapter to consolidate what you have learnt and also use other calculus textbooks.

Study Unit 5

Integrals

5.1 Background

In principle, calculus deals with two geometric problems: the one is to find the tangent line to a curve and the other is to find the area of a region under a curve. Both these problems are limiting processes. The first is called *differentiation* and the second process is *integration*. In this chapter we turn to the latter process.

To find the area of the region under the graph of a positive continuous function f defined on an interval $[a, b]$ we subdivide the interval $[a, b]$ into a finite number of subintervals, say n, the k-th subinterval having length Δx_k, and then we consider the sum of the form $\sum_{k=1}^{n} f(t_k)\Delta x_k$, where t_k is some point in the k-th subinterval. This sum is an approximation of the area that is arrived at by adding up n rectangles. Making the subdivisions finer and finer, this sum will tend to a limit as we let $n \to \infty$ and, roughly speaking, this limit is the definition of the definite integral $\int_a^b f(x)dx$ (Riemann's definition).

The two concepts, "derivative" and "integral", arise in entirely different ways and it is a remarkable fact indeed that the two are intimately connected. We will show that differentiation and integration are, in a sense, inverse operations.

There is a connection between differential calculus and integral calculus. This connection is called the Fundamental Theorem of Calculus and we shall see in this chapter that it greatly simplifies the solving of many problems.

5.2 Learning Outcomes

At the end of this chapter, you should be able to

- show that you understand the notion of the antiderivative by finding the antiderivative of basic algebraic, trigonometric, exponential and logarithmic functions

- use the Fundamental Theorem of Calculus to find the derivatives of functions of the form $F(x) = \int_a^{g(x)} f(t)\,dt$

- evaluate definite integrals and use them to determine the area between a curve and the x-axis, and the area between curves

- use substitution or term-by-term integration techniques to integrate basic algebraic, trigonometric, exponential and logarithmic functions

- solve problems involving the Mean Value Theorem for definite integrals

Note: The way to master calculus is to solve lots of calculus problems!

5.3 Prescribed Reading

The prescribed sections of Stewart are 4.9, 5.1, 5.2, 5.3, 5.4, 5.5 and in Chapter 6 only Section 6.1.

5.4 Worked Examples

Our collection of worked examples pertains to the work covered in Section 4.9, Chapter 5 and Section 6.1 of Chapter 6 of Stewart. It can be divided into 11 sets (with some overlap), namely:

I. Antiderivative

II. The definite integral and the Fundamental Theorem of Calculus – Part II

III. The definite integral and the area between the curve and the x-axis

IV. The definite integral and the area under the curve

V. The Mean Value Theorem for definite integrals

VI. The Fundamental Theorem of Calculus – Part I

VII. Integration in general

VIII. Indefinite integrals

IX. The substitution rule

X. Integration of exponential and logarithmic functions

XI. Review of formulas and techniques of integration

Attempt the problems appearing immediately after each set once you have studied the relevant parts of Stewart and done some of the exercises there.

Table 5.1 shows how these worked examples and their solutions are organised.

	Topic(s)	**Sections in Stewart**	**Study guide examples**
I.	Antiderivatives	Section 4.9	1–10
II.	The definite integral and the Fundamental Theorem of Calculus – Part II	Section 5.3 pages 396–399	11–14
III.	The definite integral and the area between the curve and the x-axis	Sections 5.2 and 6.1	15–17
IV.	The definite integral and the area under the curve	Section 6.1	18–21
V.	The Mean Value Theorem for definite integrals	Section 6.5 page 461–462	22 & 23
VI.	The Fundamental Theorem of Calculus – Part I	Section 5.3 pages 394–398	24–30
VII.	Integration in general	Chapters 5 & 6, Section 4.9	11–52
VIII.	Indefinite integrals	Section 5.4 pages 402–405	31–34
IX.	The substitution rule	Section 5.5 pages 412–418	35–37
X.	Integration of exponential and logarithmic functions	Appendix G A50–A53	38–39, 43, 45 and 51
XI.	Review of formulas and techniques of integration	page 471	40–50

Table 5.1: Sections in Stewart.

5.4.1 Antiderivatives

If we have been given a function, we can find its derivative. But many mathematical problems and their applications require us to solve the inverse (the reverse) of the derivative problem. This means that, given a function f, you may be required to find a function F of which the derivative is f. If such a function F exists, it is called an antiderivative of f.

The process of undoing differentiation (in other words, the reverse of differentiation) is known as integration. The antiderivative is known as an integral.

Find the general antiderivative of the following functions:

1. $x^{-3} + x^2$

2. $\frac{1}{2\sqrt[6]{x}}$

3. $x^{\frac{2}{3}} + 2x^{-\frac{1}{3}}$

4. $\theta + \sec\theta\tan\theta$

5. $-\dfrac{3}{2}\csc^2\left(\dfrac{3\theta}{2}\right)$

6. $\cos\frac{\pi x}{2} + \pi\cos x$

   Find the function f if f' is given as follows:

7. $f'(x) = \sin x - \sqrt[5]{x^2}$

8. $f'(x) = 3\cos x + 5\sin x$

9. $f'(t) = \sin 4t - 2\sqrt{t}$

10. $f'(t) = \sqrt[7]{t^4} + t^{-6}$

Solutions:

1. $x^{-3} + x^2$

   General antiderivative:
   $$-\frac{1}{2}x^{-2} + \frac{x^3}{3} + c$$
   by the power rule (see Stewart pages 173, 175, 200 and 221).

2. $\frac{1}{2\sqrt[6]{x}} = \frac{1}{2}x^{-\frac{1}{6}}$

   General antiderivative:
   $$\begin{aligned} & \frac{1}{2}\cdot\frac{6}{5}x^{\frac{5}{6}} + c \text{ (power rule)} \\ = & \frac{3}{5}x^{\frac{5}{6}} + c. \end{aligned}$$

3. $x^{\frac{2}{3}} + 2x^{-\frac{1}{3}}$

   General antiderivative:
   $$\begin{aligned} & \frac{3}{5}x^{\frac{5}{3}} + 2.\frac{3}{2}x^{\frac{2}{3}} + c \text{ (power rule)} \\ = & \frac{3}{5}x^{\frac{5}{3}} + 3x^{\frac{2}{3}} + c. \end{aligned}$$

4. $\theta + \sec\theta\tan\theta$

   General antiderivative:
   $$\frac{1}{2}\theta^2 + \sec\theta + c.$$

5. $-\frac{3}{2}\csc^2\frac{3}{2}\theta$

   General antiderivative:
   $$\cot\frac{3}{2}\theta + c.$$

6. $\cos\frac{\pi}{2}x + \pi\cos x$

   General antiderivative:
   $$\frac{2}{\pi}\sin\frac{\pi}{2}x + \pi\sin x + c.$$

7.
$$\begin{aligned} f'(x) &= \sin x - \sqrt[5]{x^2} \\ &= \sin x - x^{\frac{2}{5}} \\ \Rightarrow f(x) &= -\cos x - \frac{5}{7}x^{\frac{7}{5}} + c. \end{aligned}$$

8.
$$\begin{aligned} f'(x) &= 3\cos x + 5\sin x \\ \Rightarrow f(x) &= 3\sin x - 5\cos x + c. \end{aligned}$$

9.
$$\begin{aligned} f'(t) &= \sin 4t - 2\sqrt{t} = \sin 4t - 2t^{\frac{1}{2}} \\ \Rightarrow f(t) &= -\frac{1}{4}\cos 4t - 2\cdot\frac{2}{3}t^{\frac{3}{2}} + c \\ &= -\frac{1}{4}\cos 4t - \frac{4}{3}t^{\frac{3}{2}} + c. \end{aligned}$$

10.
$$\begin{aligned} f'(t) &= \sqrt[7]{t^4} + t^{-6} = t^{\frac{4}{7}} + t^{-6} \\ \Rightarrow f(t) &= \frac{7}{11}t^{\frac{11}{7}} - \frac{1}{5}t^{-5} + c. \end{aligned}$$

5.4.2 The Definite Integral and the Fundamental Theorem of Calculus – Part II

The Fundamental Theorem of Calculus, Part II is stated as follows:

> If f is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

Examples:

Compute the following integrals using the Fundamental Theorem of Calculus:

11.

$$\begin{aligned}\int_0^2 \left(x^2 - 2x\right) dx &= \left(\frac{1}{3}x^3 - x^2\right)\Big|_0^2 \\ &= \left(\frac{8}{3} - 4\right) = -\frac{4}{3}.\end{aligned}$$

12.

$$\begin{aligned}\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx &= \int_1^4 x^{\frac{1}{2}} - x^{-2} dx \\ &= \left(\frac{2}{3}x^{\frac{3}{2}} + x^{-1}\right)\Big|_1^4 = \left[\frac{2}{3}(4)^{\frac{3}{2}} + 4^{-1}\right] - \left(\frac{2}{3} + 1^{-1}\right) \\ &= \frac{16}{3} + \frac{1}{4} - \frac{2}{3} - 1 = \frac{47}{12}.\end{aligned}$$

13.

$$\begin{aligned}&\int_0^4 e^{-2x} dx \\ &= \left(-\frac{1}{2}e^{-2x}\right)\Big|_0^4 = -\frac{1}{2}e^{8} - \left(-\frac{1}{2}e^{0}\right) \\ &= -\frac{1}{2}e^{-8} + \frac{1}{2}.\end{aligned}$$

Note that in the example above, the integral of an exponential function is just the exponential function with its index divided by the derivative of its index.

14.

$$\begin{aligned}&\int_{-3}^{-1} \frac{2}{x} dx \\ &= 2\ln|x|_{-3}^{-1} = 2\left(\ln|-1| - \ln|-3|\right) \\ &= 2\left(\ln 1 - \ln 3\right) = -2\ln 3.\end{aligned}$$

5.4.3 The Definite Integral and the Area Between the Curve and the x-axis

It is important to understand that the definite integral of a continuous function f evaluates the area between the graph of this function f and the x-axis. See the following examples.

Examples:

15. Find the area under the curve of $f(x) = \sin x$ on the interval $[0, \pi]$, always true for $x \in [0, \pi]$.

Solution:

$$\text{Area} = \int_0^\pi \sin x dx = (-\cos \pi) + (\cos 0) = (1) - (-1) = 2.$$

16. *[The connection between area and calculus]*

    Calculate the value of $\int_2^5 (2x - 4)\, dx$ geometrically by using a suitable sketch. Then test your answer with the aid of integration.

Solution:

(a) Geometrically:

The graph of $y = 2x - 4$ looks like this:

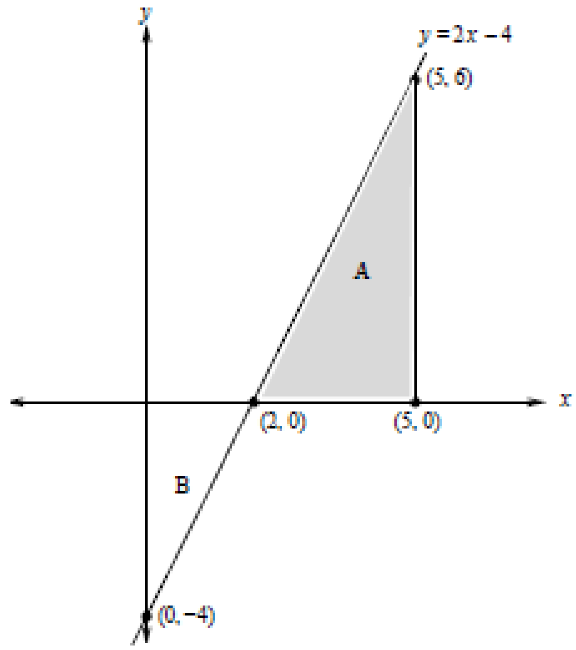


Figure 5.1: Graph of $y = 2x - 4$

Therefore

$$\begin{aligned}\int_2^5 (2x-4)dx &= \text{area of triangle A} \\ &= 1/2 \times \text{ base } \times \text{ height} \\ &= 1/2 \times 3 \times 6 \\ &= 9.\end{aligned}$$

(b) Test with the aid of integration:

$F(x) = x^2 - 4x$ is an antiderivative of $y = 2x - 4$ and hence

$$\begin{aligned}\int_2^5 (2x-4)dx &= F(5) - F(2) \\ &= (25-20) - (4-8) \\ &= 9,\end{aligned}$$

which is the same as in (a) above.

17. Find the area of the region between the x-axis and the graph of $f(x) = 2x + x^2 - x^3, \quad -1 \leq x \leq 2$.

Solution:

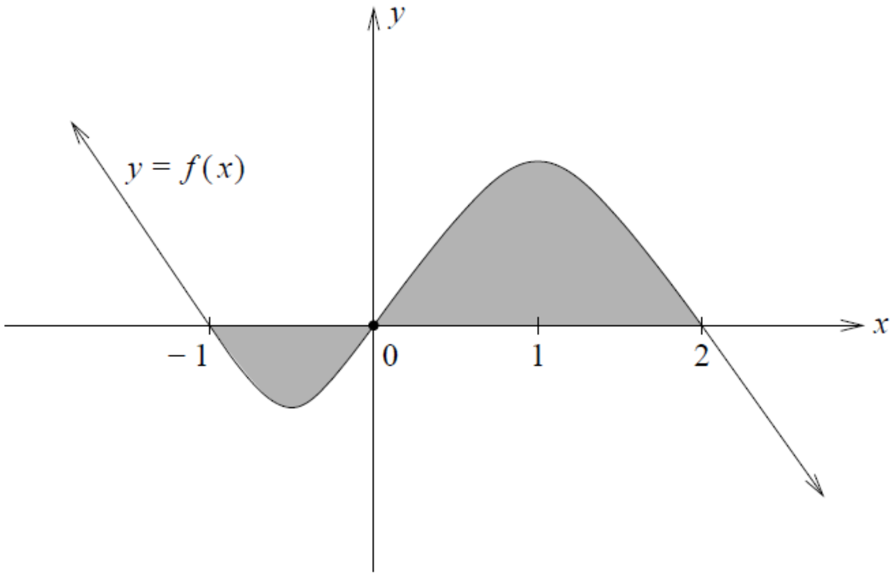


Figure 5.2: Graph of $f(x) = 2x + x^2 - x^3$

First find the zeros of f. Since

$$f(x) = 2x + x^2 - x^3 = -x(-2 - x + x^2) = -x(x+1)(x-2),$$

the zeros are $x = 0, -1$, and 2. It is clear that the zeros partition the interval $[-1, 2]$ into two subintervals:$[-1, 0]$ and $[0, 2]$ with $f(x) \leq 0$ for all $x \in [-1, 0]$ and $f(x) \geq 0$ for all $x \in [0, 2]$. We integrate over each subinterval and add the absolute values of the calculated values.

Integral over $[-1, 0]$:

$$\begin{aligned}
\int_{-1}^{0} (2x + x^2 - x^3)dx &= \left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_{-1}^{0} \\
&= 0 - \left[(-1)^2 + \frac{(-1)^3}{3} - \frac{(-1)^4}{4}\right] \\
&= -\frac{5}{12}.
\end{aligned}$$

Integral over $[0, 2]$:

$$\begin{aligned}\int_0^2 (2x + x^2 - x^3)dx &= \left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 \\ &= \left[(2)^2 + \frac{(2)^3}{3} - \frac{(2)^4}{4}\right] - 0 \\ &= [4 + \frac{8}{3} - 4] \\ &= \frac{8}{3}.\end{aligned}$$

Enclosed area:

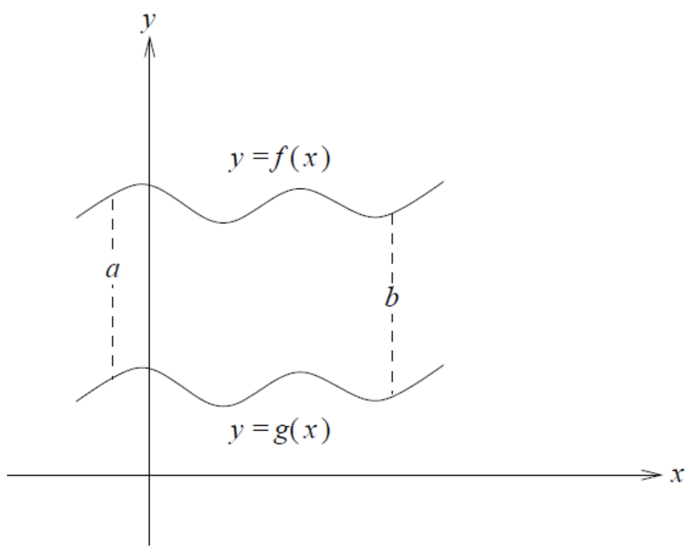
Total enclosed area $= \left|\frac{-5}{12}\right| + \left|\frac{8}{3}\right| = \frac{37}{12}$.

5.4.4 The Definite Integral and Area Under the Curve

If f and g are continuous functions with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves of f and g from a to b is the integral of $[f - g]$ from a to b. The area bounded by the two curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ where f and g are continuous, is written as follows:

$$\boxed{A = \int_a^b [f(x) - g(x)]\, dx.}$$

This is shown in the sketch below:



Steps To Find The Area Under a Curve

(i) Sketch the graphs of the given functions.

(ii) Determine the boundaries of integration, which are the points of intersection of the graphs by equating both functions.

(iii) Apply the formula for determining the area under a curve

$$\boxed{A = \int_a^b [f(x) - g(x)]\, dx.}$$

where $f(x)$ bounds or is above $g(x)$, ie $f(x) \geq g(x)$ on $[a, b]$, a and b are the boundaries of integration determined in (ii) above.

(iv) Then integrate to evaluate the area.

Examples:

18. Evaluate the area between the curves $f(x) = 5 - x^2$ and $g(x) = |x + 1|$.

Solution:

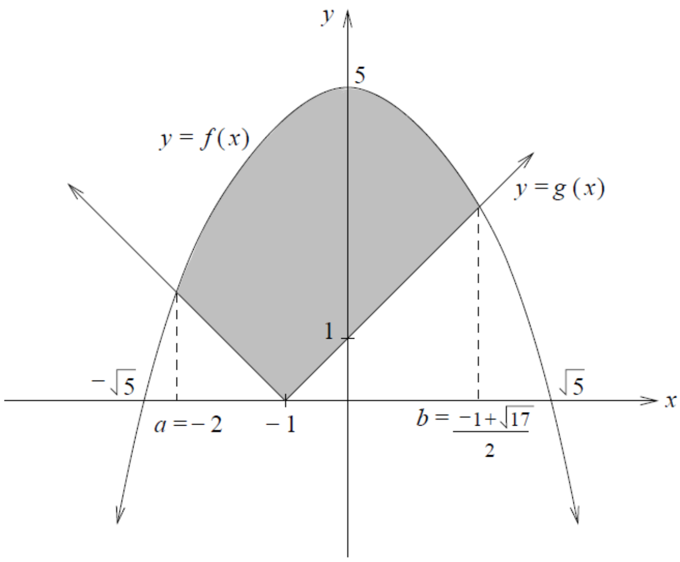


Figure 5.3: Graph of $f(x) = 5 - x^2$ and $g(x) = |x + 1|$

Points of intersection:

$$\begin{aligned} -(x+1) &= 5 - x^2 \\ \Rightarrow \quad x^2 - x - 6 &= 0 \\ \Rightarrow (x-3)(x+2) &= 0 \Rightarrow x = 3 \quad \text{or} \quad x = -2, \end{aligned}$$

and

$$\begin{aligned} x + 1 &= 5 - x^2 \\ \Rightarrow x^2 + x - 4 &= 0 \\ \Rightarrow \qquad x &= \frac{-1 \pm \sqrt{17}}{2}. \end{aligned}$$

It is clear that the limits of integration are the x-values of the points of intersection of the two curves: $a = -2$ and $b = \frac{-1+\sqrt{17}}{2}$. Now we evaluate the area of the region by integrating:

$$
\begin{aligned}
A &= \int_{-2}^{\frac{-1+\sqrt{17}}{2}} [5 - x^2 - |x+1|]dx \\
&= \int_{-2}^{-1} [5 - x^2 - (-x-1)]dx + \int_{-1}^{\frac{-1+\sqrt{17}}{2}} [5 - x^2 - (x+1)]dx \\
&= \int_{-2}^{-1} [-x^2 + x + 6]dx + \int_{-1}^{\frac{-1+\sqrt{17}}{2}} [-x^2 - x + 4]dx \\
&= \Big[-\frac{x^3}{3} + \frac{x^2}{2} + 6x\Big]_{-2}^{-1} + \Big[-\frac{x^3}{3} - \frac{x^2}{2} + 4x\Big]_{-1}^{\frac{-1+\sqrt{17}}{2}} \\
&= \frac{1}{3} + \frac{1}{2} - 6 - \left[\frac{8}{3} + 2 - 12\right] + \frac{-\left(-1+\sqrt{17}\right)^3}{24} + \frac{-\left(-1+\sqrt{17}\right)^2}{8} \\
&\quad + 4\left(\frac{-\left(-1+\sqrt{17}\right)}{2}\right) - \left[\frac{1}{3} - \frac{1}{2} - 4\right] \\
&= \frac{-8}{3} + 9 - \frac{(-1+\sqrt{17})^3}{24} - \frac{(-1+\sqrt{17})^2}{8} + 2(-1+\sqrt{17})
\end{aligned}
$$

We would like to encourage you to leave your answers unsimplified, as you are not allowed to use a calculator in your examinations!

19. Find the area bounded by the graphs of $y = 3 - x$ and $y = x^2 - 9$.

Solution:

Find the points of intersection by equating the two functions:

$$
\begin{aligned}
3 - x &= x^2 - 9 \\
\Rightarrow \quad 0 &= x^2 + x - 12 = (x-3)(x+4) \\
\text{that is} \quad x &= 3 \quad \text{or} \quad x = -4.
\end{aligned}
$$

The two curves intersect at $x = 3$ and $x = -4$ (see Figure 5.4).

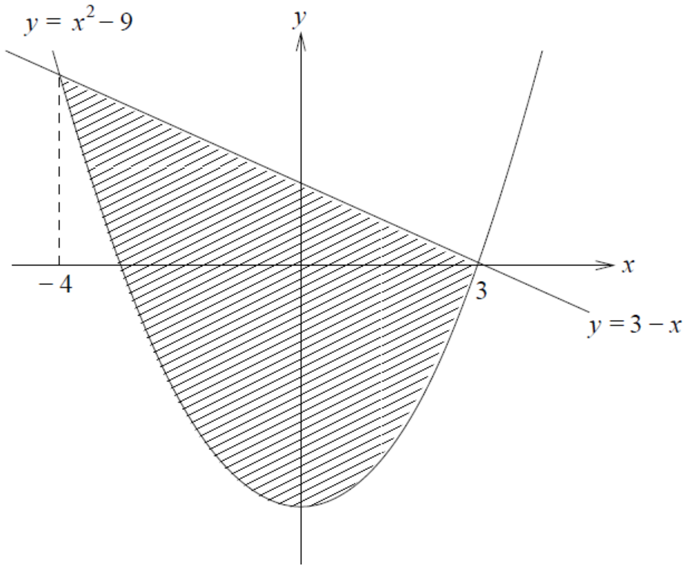


Figure 5.4: Graph of $y = 3 - x$ and $y = x^2 - 9$

The height of the bounded area is $h(x) = (3 - x) - \left(x^2 - 9\right)$.

Therefore

$$\begin{aligned}
A &= \int_{-4}^{3} \left[(3 - x) - \left(x^2 - 9\right)\right] dx \\
&= \int_{-4}^{3} \left(-x^2 - x + 12\right) dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 12x\right]_{-4}^{3} \\
&= \left[-\frac{3^3}{3} - \frac{3^2}{2} + 12(3)\right] - \left[-\frac{(-4)^3}{3} - \frac{(-4)^2}{2} + 12(-4)\right] \\
&= \frac{343}{6}.
\end{aligned}$$

20. Find the area bounded by the graphs of $y = x^2$ and $y = 2 - x^2$ for $0 \leq x \leq 2$.

Solution:

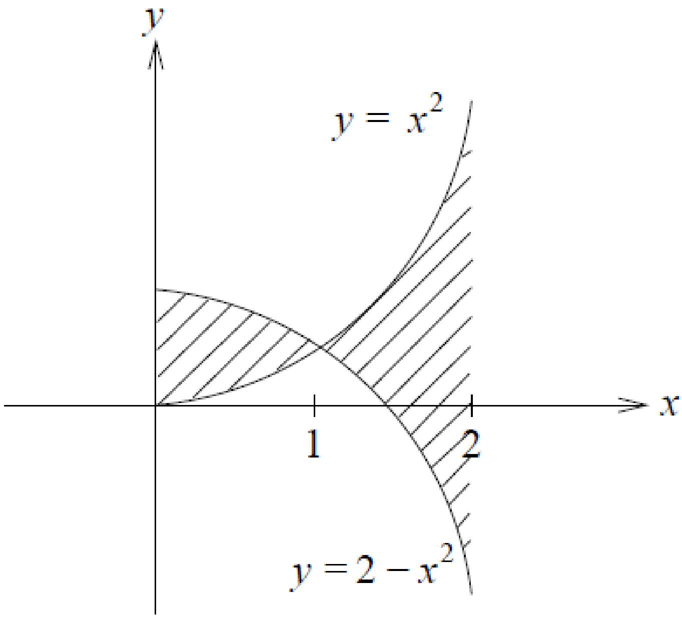


Figure 5.5: Graph of $y = x^2$ and $y = 2 - x^2$

Since the curves intersect in the middle of the interval, you have to compute two integrals, namely for $2 - x^2 \geq x^2$ and for $x^2 \geq 2 - x^2$.

To find the point of intersection, solve for x :

$$x^2 = 2 - x^2 \Rightarrow 2x^2 = 2 \Rightarrow x = \pm 1.$$

Note that only $x = +1$ is relevant since $x = -1$ is outside the given interval.

$$\begin{aligned}
A &= \int_0^1 \left[\left(2 - x^2\right) - x^2\right] dx + \int_1^2 \left[x^2 - \left(2 - x^2\right)\right] dx \\
&= \int_0^1 \left(2 - 2x^2\right) dx + \int_1^2 \left(2x^2 - 2\right) dx \\
&= \left[2x - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - 2x\right]_1^2 = \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) \\
&= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4.
\end{aligned}$$

21. Find the area bounded by the graphs of $x = y^2$ and $x = 2 - y^2$.

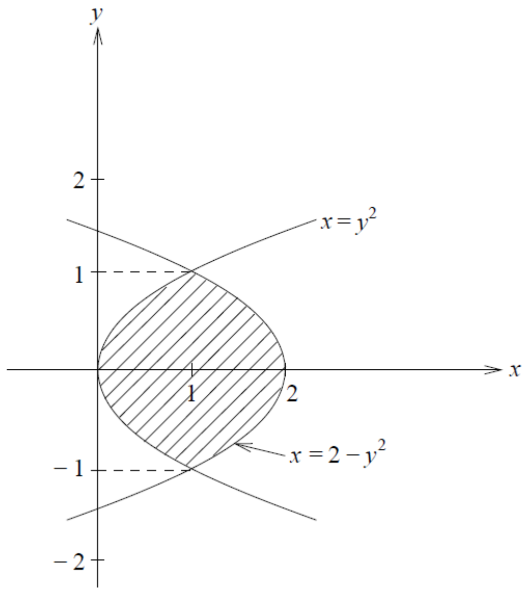


Figure 5.6: Graph of $x = y^2$ and $x = 2 - y^2$

Solution:

It is easier to compute this area by integrating with respect to y. Therefore we find the points of intersection by solving for y.

$$y^2 = 2 - y^2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$$

In the interval $[-1, 1]$,

$$\begin{aligned} A &= \int_{-1}^{1} \left[\left(2 - y^2\right) - y^2\right] dy = \int_{-1}^{1} \left(2 - 2y^2\right) dy \\ &= \left[2y - \frac{2}{3}y^3\right]_{-1}^{1} = \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) = \frac{8}{3}. \end{aligned}$$

5.5 The Mean Value Theorem for Definite Integrals

Now we state the **Mean Value Theorem for definite integrals**

> If f is continuous on the interval $[a, b]$ there is at least one number c between a and b such that
>
> the area under the curve between a and $b = \int_a^b f(x)\,dx = (b - a) f(c)$
>
> or $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$.

22. *[Finding c using the Mean Value Theorem for integrals]*

Find a value c guaranteed by the Mean Value Theorem for integrals for $f(x) = x^2 + 2x + 3$ on $[0, 2]$.

Solution:

$$\begin{aligned} \int_0^2 x^2 + 2x + 3 \, dx &= \left. \frac{1}{3}x^3 + x^2 + 3x \right]_0^2 \\ &= \frac{8}{3} + 4 + 6 \\ &= \frac{8 + 12 + 18}{3} \\ &= \frac{38}{3}. \end{aligned}$$

The region bounded by the graph of f and the x-axis on $[0, 2]$ is shaded in Figure 5.7.

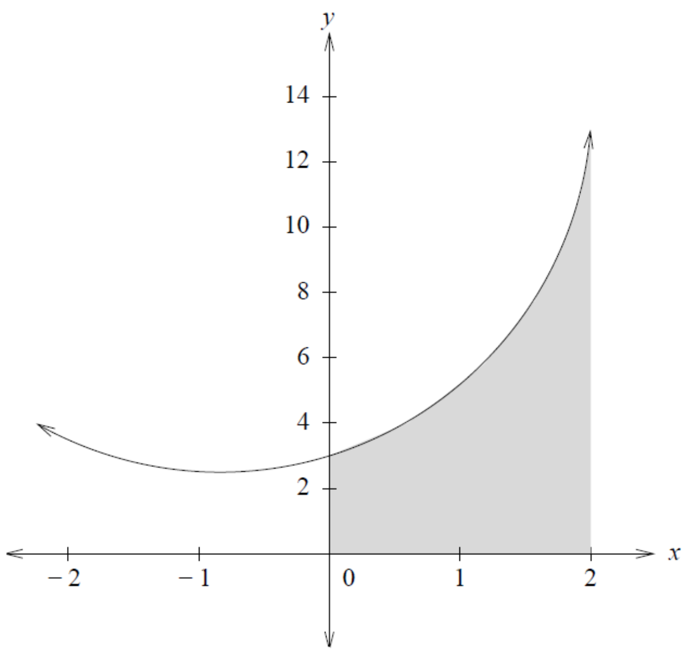


Figure 5.7: Graph of $f(x) = x^2 + 2x + 3$ and $x - axis$ on [0, 2]

The Mean Value Theorem for integrals asserts the existence of a number c on $[0, 2]$ such that $f(c)(b - a) = \frac{38}{3}$. We solve this equation to find c:

$$\begin{aligned} & f(c)(b - a) &= \frac{38}{3} \\ \Rightarrow \quad & \left(c^2 + 2c + 3\right)(2 - 0) &= \frac{38}{3} \\ \Rightarrow \quad & 2c^2 + 4c + 6 &= \frac{38}{3} \\ \Rightarrow \quad & 6c^2 + 12c - 20 &= 0 \\ \therefore \quad & 3c^2 + 6c - 10 &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad c &= \frac{-6 \pm \sqrt{36 + 4(3)(10)}}{6} \\ &= \frac{-6 \pm \sqrt{156}}{6} \\ &= -1 \pm \frac{\sqrt{39}}{3}. \end{aligned}$$

Since only one of these points, that is $c = -1 + \frac{\sqrt{39}}{3} \approx 1,08$, lies within the interval $[0, 2]$, this value of c satisfies the Mean Value Theorem for integrals.

The following is **the definition of the *average value* of a function:**

If f is integrable on $[a, b]$, then the average value or mean value of f on $[a, b]$ denoted by $av(f)$ is $av(f) = \frac{1}{b-a} \int_a^b f(x)\, dx$

23. *[Finding the average value of a function on an interval]*

Find the average value of $f(x) = 2x$ on the interval $[1, 5]$.

Solution:

The average value is

$$\begin{aligned} av(f) &= \frac{1}{5-1} \int_1^5 2x dx = \frac{1}{4}\left(x^2\right)\Big]_1^5 \\ &= \frac{1}{4}(25 - 1) \\ &= 6. \end{aligned}$$

Obviously this average value occurs when $f(x) = 6$ and when $x = 3$.

5.6 The Fundamental Theorem of Calculus – Part I

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$ on $[a, b]$.

In other words, if $F(x) = \int_a^x f(t)\, dt$ where a is any real number, then

$$\frac{d}{dx}(F(x)) = \frac{d}{dx} \int_a^x f(t)\, dt$$

that is $F'(x) = f(x) \frac{d}{dx}(x) = f(x)(1)$; therefore $F'(x) = f(x)$.

To apply Part I of the Fundamental Theorem of Calculus, just leave the function $f(t)$ as it is, but replace the variable t with the variable x times the derivative of x.

Let us write down the two parts of the Fundamental Theorem of Calculus again:

Part II

If $G(x)$ is any antiderivative of $f(x)$ on $[a, b]$ so that $G'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = G(b) - G(a).$$

Part I (The Fundamental Theorem of Calculus stated in another way)

Suppose that the function f is continuous on the interval $[a, b]$.

Let the function F be defined on $[a, b]$ by $F(x) = \int_a^x f(t)\, dt$.

Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$, which means that F is an antiderivative of f on $[a, b]$.

Expressed in symbols:

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x) = F'(x).$$

Remarks:

(1) You should remember that both conclusions of the Fundamental Theorem are useful. Part I concerns the *derivative* of an integral – it tells you how to *differentiate a definite integral* with respect to its upper limit. Part II concerns the *integral* of a derivative – it tells you how to *evaluate a definite integral* if you can find an antiderivative of the integrand.

(2) Remember that differentiation and integration are inverse operations and cancel each other to a certain extent.

(3) Notice that the lower limit of the integral may be *any* constant in the interval $[a, b]$ and it does not affect the answer after differentiation. Thus

$$\frac{d}{dx} \int_0^x f(t)dt = \frac{d}{dx} \int_{-10}^x f(t)dt = \frac{d}{dx} \int_{513}^x f(t)dt$$

etc. This is so because the derivative of a constant is always zero.

(4) Note that the upper limit must be the same as the variable with respect to which we differentiate, in other words, if we determine $\dfrac{d}{dx}$ (the integral) then the upper limit must be x (or a function of x).

(5) The dt in the integral shows you that integration is done with respect to the variable t. Hence you must change all ts in the given function $f(t)$ to xs to obtain $f(x)$, or dF/dx.

(6) If the upper limit of the integral (which you must differentiate) does not consist of the variable of interest only, but is in fact a function, then you must remember to use the chain rule for differentiation. Thus:

$$\boxed{\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x)).g'(x).}$$

In this case you replace all ts by $g(x)$ and then multiply by the derivative of $g(x)$ (look at examples 25 and 26 below).

Keeping the above-mentioned in mind, consider the following examples:

Examples:

Determine the derivatives of $F(x)$ in each of the following:

24. $F(x) = \int_a^x \dfrac{1}{1+\sin^2 t}dt$

25. $F(x) = \int_a^{x^3} \dfrac{1}{1+\sin^2 t}dt$

26. $F(x) = \int_{\sin x}^{\cos x} t(5-t)dt$

Solutions:

24. $F(x) = \int_a^x \dfrac{1}{1+\sin^2 t}dt$

It follows from Fundamental Theorem of Calculus Part I that

$$F'(x) = \frac{1}{1+\sin^2 x}.$$

25. $F(x) = \int_a^{x^3} \dfrac{1}{1+\sin^2 t}dt$

In this case we must apply the chain rule, because the upper limit is a function of x. Let $f(t) = \frac{1}{1+\sin^2 t}$ and $g(x) = x^3$. Then

$$\begin{aligned} F'(x) &= f(g(x)) \cdot g'(x) \\ &= f\left(x^3\right) \cdot \frac{d}{dx} x^3 \\ &= \frac{1}{1+\sin^2\left(x^3\right)} \cdot 3x^2. \end{aligned}$$

(In other words, substitute t in the function $f(t) = \frac{1}{1+\sin^2 t}$ with x^3, but also remember to multiply the derivative of x^3 with this answer.)

26. In this example see that both the upper and lower limit are functions of x, so that it follows from remark 3 above that we may introduce any constant in this case. We call it a.

$$\begin{aligned} F(x) &= \int_{\sin x}^{\cos x} t(5-t)dt \\ &= \int_{\sin x}^{a} t(5-t)\,dt + \int_{a}^{\cos x} t(5-t)dt \\ &= -\int_{a}^{\sin x} t(5-t)\,dt + \int_{a}^{\cos x} t(5-t)dt. \end{aligned}$$

By again applying the chain rule as in example 25, we obtain

$$\begin{aligned} F'(x) &= -\sin x\,(5-\sin x) \cdot \frac{d}{dx}\sin x + \cos x\,(5-\cos x) \cdot \frac{d}{dx}\cos x \\ &= -\sin x \cos x\,(5-\sin x) - \cos x \sin x(5-\cos x). \end{aligned}$$

Now attempt the following yourself.

Find the derivatives of F in each of the following cases:

27. $F(x) = \int_x^{10} \left(t + \frac{1}{t}\right) dt$

28. $F(x) = \int_{2x}^{x^3} \cos t\; dt$

29. $F(x) = \int_3^{x^3} \sin^3 t\; dt$

30. $F(x) = \int_0^{\sin x} \sqrt{1-t^2}dt$

Solution:

27. $-x - \dfrac{1}{x}$.

    (This is plainly Part II of the Fundamental Theorem of Calculus with $\int_x^{10} f(x)\,dx = -\int_{10}^{x} f(x)\,dx$.)

28. $3x^2 \cos x^3 - 2\cos 2x$.

    (Again we must introduce constants for the lower limits, for example $\int_{2x}^{0} \ldots + \int_0^{x^3} \ldots = -\int_0^{2x} \ldots + \int_0^{x^3} \ldots$.)

29. $3x^2 \sin^3 x^3$.

    (Use the chain rule, in other words first substitute t with x^3 and then multiply the answer with the derivative of x^3.)

30. $|\cos x| \cos x$.

5.7 Integration in General

The process of computing an integral (integration) is written as follows:

If $F'(x) = f(x)$, then $\int F'(x)\,dx = \int f(x)\,dx$,

that is $F(x) = \int f(x)\,dx$.

$f(x)$ is the integrand and the term dx identifies x as the variable of integration.

Given a function f, find, if possible, a function F such that:

$$F'(x) = f(x)$$

A function F defined on an interval which has the property that $F'(x) = f(x)$ is called an indefinite integral or antiderivative of f on the same interval for all x, and we write:

$$\begin{aligned} F(x) &= \int f(x)\,dx \quad \text{(for indefinite integrals)} \\ \text{or} \qquad F(x) &= \int_a^b f(x)\,dx \quad \text{(for definite integrals)} \end{aligned}$$

The term "dx" is part of the notation. It represents the statement "We integrate with respect to x".

5.8 Indefinite Integrals

For indefinite integrals, if F and G are both antiderivatives of f on the interval I, then $G(x) = F(x) + c$ for some constant c.

If F is any antiderivative of f, the **indefinite integral** of $f(x)$ with respect to x is defined by $\int f(x)\,dx = F(x) + c$ (where c is the constant of integration).

Example:

27. Evaluate $\int t^5 dt$.

Solution:

We know that $\dfrac{d}{dt}t^6 = 6t^5$, so $\dfrac{d}{dt}\left(\dfrac{1}{6}t^6\right) = t^5$. Therefore $\int t^5 dt = \dfrac{1}{6}t^6 + c$.

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$ since $\dfrac{d}{dx}\left[\dfrac{1}{n+1}x^{n+1} + c\right] = x^n$.

However, we could also have $\int x^n dx = \dfrac{x^{n+1}}{n+1} + 5$, because $\dfrac{d}{dx}\left[\dfrac{x^{n+1}}{n+1} + 5\right] = x^n$.

So, in general, $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c$, where c is the constant of integration.

Note: This rule does not work for $n = -1$ since it would produce a division by 0.

Examples of indefinite integrals:

31.

$$\begin{aligned} &\int x^{17} dx \\ = \;& \frac{x^{17+1}}{17+1} + c = \frac{x^{18}}{18} + c. \end{aligned}$$

32.

$$\begin{aligned} &\int \frac{1}{\sqrt[3]{x}} dx \\ = \;& \int x^{-\frac{1}{3}} dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c \\ = \;& \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2}x^{\frac{2}{3}} + c. \end{aligned}$$

33.

$$\begin{aligned} & \int \left(3\cos x + 4x^8\right) dx \\ = \; & 3\int \cos x dx + 4\int x^8 dx \\ = \; & 3\sin x + 4\frac{x^9}{9} + c = 3\sin x + \frac{4}{9}x^9 + c. \end{aligned}$$

34.

$$\begin{aligned} & \int \left(3e^x - \frac{2}{1+x^2}\right) dx \\ = \; & 3\int e^x dx - 2\int \frac{1}{1+x^2} dx \\ = \; & 3e^x - 2\tan^{-1} x + c. \end{aligned}$$

5.9 The Substitution Rule

(a) Indefinite integrals

The method we are going to discuss now is the principal method by which integrals are evaluated.

When the integrand is not in standard form, *first of all* determine whether the integrand does not consist of two factors, *one of which is the derivative of the other.* The method of substitution is the integration version of the chain rule.

Remember that, according to the chain rule, the derivative of $\left(3x^2 - 5x + 2\right)^4$ is

$$\frac{d}{dx}\left(3x^2 - 5x + 2\right)^4 = 4\left(3x^2 - 5x + 2\right)^3 (6x - 5).$$

On the right-hand side we now have two expressions, namely

$$\begin{aligned} & \left(3x^2 - 5x + 2\right) \text{ and } 6x - 5, \text{ which are such that} \\ & \frac{d}{dx}\left(3x^2 - 5x + 2\right) = 6x - 5. \end{aligned}$$

Since we know that differentiation and integration are inverse operations, we see that

$$\int \left[4\left(3x^2 - 5x + 2\right)^3 \cdot (6x - 5)\right] dx = \left(3x^2 - 5x + 2\right)^4 + c.$$

Example:

Integration of an indefinite integral using substitution

Find

$$\int 7\left(x^2+3x+5\right)^6(2x+3)\,dx.$$

Solution:

Look at the problem and make the observation as shown in the boxes:

$$7\int \boxed{\left(x^2+3x+5\right)^6}\quad \boxed{(2x+3)}\ dx$$

The derivative of the function in the left-hand box which is $(2x+3)\,dx$ is sitting in the right-hand box.

So let $u = x^2+3x+5$.

(NB: Note that the function u appears *without the index* 6.)

Then $\dfrac{du}{dx} = 2x+3$, that is $(2x+3)\,dx = du$.

(This is the form in the right-hand box.)

Now substitute everything under the integral sign to express the integral in terms of u. We have

$$7\int u^6 du = 7\left(\frac{1}{7}u^7\right)+c = u^7 + c.$$

Then back-substitute to express u in terms of x:

$$7\int u^6 du = \left(x^2+3x+5\right)^7 + c.$$

Remarks:

1. Sometimes the two functions under the integral sign are linked, but a *constant might be missing.*

   If, however, you follow the procedure of substitution correctly, the end product will be correct.

Example:

Find

$$\int \boxed{\left(x^2 + 2x + 5\right)^4} \quad \boxed{(x+1)\,dx}$$

Solution:

Let

$$u = x^2 + 2x + 5.$$

Then

$$\frac{du}{dx} = 2x + 2 = 2\,(x+1)\,.$$

Thus $(x+1)\,dx = \dfrac{du}{2}$.

Substitution then gives

$$\begin{aligned} \int u^4 \frac{du}{2} &= \frac{1}{2}\int u^4 du \\ &= \frac{1}{10}u^5 + c \\ &= \frac{1}{10}\left(x^2 + 2x + 5\right)^5 + c. \end{aligned}$$

If you differentiate this using the chain rule, you will obtain the function under the integral sign.

2. After you have done your substitution and simplified, there should be *no left-over x-values* in the integrand.

Example:

[*Substitution with leftover x-values*]

Find

$$\int x\,(3x-5)^3\,dx$$

Solution:

In this case it is so that the derivative of the function with index 3, that is $(3x-5)$, is not linked to the other function in the ordinary way. However, we follow the same method.

Let $u = 3x - 5$, so $du = 3dx$ and $dx = \frac{du}{3}$.

Now substitute:

$$\int x\,(3x-5)^3\,dx = \int xu^3\frac{du}{3} = \frac{1}{3}\int xu^3du$$

It is clear that there is a left-over x-value.

We now follow the next procedure to eliminate the left-over x-term:

Because $u = 3x - 5$, we can solve for x :

$$x = \frac{u+5}{3},$$

so

$$\begin{aligned}
\frac{1}{3}\int xu^3du &= \frac{1}{3}\int\left(\frac{u+5}{3}\right)u^3du \\
&= \frac{1}{9}\int\left(u^4+5u^3\right)du \\
&= \frac{1}{9}\left(\frac{u^5}{5}+\frac{5}{4}u^4\right)+c \\
&= \frac{1}{45}(3x-5)^5+\frac{5}{36}(3x-5)^4+c.
\end{aligned}$$

Examples:

Determine the following integrals:

35. $\int 3x\sqrt{1-2x^2}\;dx$

36. $\int \cos^3 x\;\sin^5 x\;dx$

37. $\int \dfrac{dx}{x^2\left(1+\frac{1}{x}\right)^3}$.

Solutions:

35. We see that x is the derivative (except for a constant factor) of $1-2x^2$. Therefore set

$$u = 1 - 2x^2.$$

Then

$$du = -4x\;dx.$$

In other words

$$-\frac{1}{4}du = x\;dx.$$

Now

$$\begin{aligned}\int 3x\sqrt{1-2x^2}dx &= 3\int \sqrt{1-2x^2}\ x\ dx \\ &= 3\int \sqrt{u}\left(-\frac{1}{4}\right) du \\ &= -\frac{3}{4}\int u^{\frac{1}{2}}\ du \\ &= -\frac{3}{4}\frac{u^{\frac{3}{2}}}{3/2}+c \\ &= -\frac{1}{2}u^{\frac{3}{2}}+c \\ &= -\frac{1}{2}\left(1-2x^2\right)^{\frac{3}{2}}+c.\end{aligned}$$

Note: You must not give your answer in terms of u. You *must give your answer in terms of* x, since that was the original variable!

36. In this case we use trigonometric identities. We do this with two different methods.

Method 1

$$\begin{aligned} & \int \cos^3 x \sin^5 x\ dx \\ = & \int \cos x \cos^2 x \ \boxed{\sin^5 x}\ dx.\end{aligned}$$

The part in the block, without its power, has the derivative $\cos x$.

Now we must also rewrite the other extra term in terms of $\sin x$, that is $\cos^2 x = 1-\sin^2 x$.

The integral then becomes

$$\begin{aligned} & \int \cos x\left(1-\sin^2 x\right)\sin^5 x\ dx \\ = & \int \cos x \sin^5 x\ dx - \int \cos x \sin^7 x\ dx.\end{aligned}$$

Since $\cos x$ is the derivative of $\sin x$ we let $u = \sin x$.

Then

$$du = \cos x\ dx$$

and so it follows that

$$\begin{aligned}\int \cos^3 x \sin^5 x\ dx &= \int \left(u^5-u^7\right) du \\ &= \frac{u^6}{6}-\frac{u^8}{8}+c \\ &= \frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+c.\end{aligned}$$

Method 2

$$\int \cos^3 x \sin^5 x \; dx$$

$$= \int \boxed{(\cos x)^3} \; \sin^4 x \sin x \; dx.$$

The part in the block, without the power, has the derivative $(-\sin x)$ (thus the constant (-1) is missing). Now rewrite $\sin^4 x$ in terms of $\cos x$:

$$\sin^4 x = \left(1 - \cos^2 x\right)^2 = 1 - 2\cos^2 x + \cos^4 x.$$

Thus

$$\begin{aligned}
\int \cos^3 x \sin^5 x \; dx &= \int (\cos x)^3 \left(1 - 2\cos^2 x + \cos^4 x\right) \sin x \; dx \\
&= \int \cos^3 x \sin x \; dx - 2 \int \cos^5 x \sin x \; dx + \int \cos^7 x \; \sin x \; dx.
\end{aligned}$$

Put $u = \cos x$, so that $du = (-\sin x)\, dx$.

Then

$$\begin{aligned}
\int \cos^3 x \sin^5 x \; dx &= -\int u^3 du + 2 \int u^5 du - \int u^7 du \\
&= -\frac{1}{4} u^4 + \frac{2}{6} u^6 - \frac{1}{8} u^8 + c \\
&= -\frac{1}{4} (\cos x)^4 + \frac{1}{3} (\cos x)^6 - \frac{1}{8} (\cos x)^8 + c.
\end{aligned}$$

37. In this case we see that $-\frac{1}{x^2}$ is the derivative of $1 + \frac{1}{x}$ and hence we set

$$u = 1 + \frac{1}{x},$$

and therefore

$$du = -\frac{1}{x^2} dx$$

and

$$\begin{aligned}
\int \frac{dx}{x^2 \left(1 + \frac{1}{x}\right)^3} &= -\int \frac{du}{u^3} \\
&= -\frac{u^{-2}}{-2} + c \\
&= \frac{1}{2} \frac{1}{\left(1 + \frac{1}{x}\right)^2} + c \\
&= \frac{1}{2} \left(\frac{x}{x+1}\right)^2 + c.
\end{aligned}$$

(b) Definite integrals

The section on changing the limits of integration on pages 416–418 in Stewart is very important and students tend to have problems with this. We shall discuss it in detail and show you three different methods you may use. Work thoroughly through all three and make very sure that you understand each step.

Example:

Calculate

$$\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}}\,dx.$$

Solution:

Method 1

When you have done substitution with the variable u, also change the limits of integration in terms of the variable u, and *do not* change back to the variable x.

$$\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}}\,dx$$

Let

$$u = (x^2+1)^{\frac{1}{2}}.$$

Then

$$u^2 = x^2 + 1$$

and

$$2udu = 2xdx, \text{ that is } u\,du = x\,dx.$$

Now we also change the limits of integration:

If

$$x = 0 \text{ then } \; u = (0^2+1)^{\frac{1}{2}} = 1$$

and if

$$x = 1 \text{ then } \; u = (1^2+1)^{\frac{1}{2}} = \sqrt{2}.$$

Hence,

$$\begin{aligned}
\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}} dx &= \int_0^1 \frac{x^2 x \; dx}{(x^2+1)^{\frac{3}{2}}} \\
&= \int_1^{\sqrt{2}} \frac{(u^2-1)}{u^3} u \; du \\
&= \int_1^{\sqrt{2}} \left(1 - \frac{1}{u^2}\right) du \\
&= u + \frac{1}{u}\Big]_1^{\sqrt{2}}.
\end{aligned}$$

Now substitute the values of the limits directly and do not change the variable back to x:

$$\begin{aligned}
\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}} dx &= \sqrt{2} + \frac{1}{\sqrt{2}} - (1+1) \\
&= \sqrt{2} + \frac{\sqrt{2}}{2} - 2 \\
&= \frac{3}{2}\sqrt{2} - 2.
\end{aligned}$$

Method 2

With this method we first determine the indefinite integral, change the answer back to the variable x after substitution, and then put in the values of the limits of integration in terms of the variable x.

We first determine

$$\int \frac{x^3}{(x^2+1)^{\frac{3}{2}}} dx.$$

As before, we let $u = (x^2+1)^{\frac{1}{2}}$. Then $u \; du = x \; dx$ and

$$\begin{aligned}
\int \frac{x^3}{(x^2+1)^{\frac{3}{2}}} dx &= u + \frac{1}{u} + c \\
&= \left(x^2+1\right)^{\frac{1}{2}} + \frac{1}{(x^2+1)^{\frac{1}{2}}} + c.
\end{aligned}$$

Now we calculate the definite integral in terms of x:

$$\begin{aligned}
\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}} dx &= \left(x^2+1\right)^{\frac{1}{2}} + \frac{1}{(x^2+1)^{\frac{1}{2}}}\Bigg]_0^1 \\
&= \sqrt{2} + \frac{1}{\sqrt{2}} - (1+1) \\
&= \frac{3}{2}\sqrt{2} - 2.
\end{aligned}$$

Method 3

With this method we do not change the limits when we make the substitution, but we show clearly in terms of which variable the limits are given:

$$\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}}\,dx.$$

As before, we let $u = (x^2+1)^{\frac{1}{2}}$. Then $u\ du = x\ dx$ and

$$\begin{aligned}
\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}}\,dx &= \int_{x=0}^{x=1} \frac{(u^2-1)}{u^3}\,u\ du \\
&= u + \frac{1}{u}\Big]_{x=0}^{x=1} \\
&= \left(x^2+1\right)^{\frac{1}{2}} + \frac{1}{(x^2+1)^{\frac{1}{2}}}\Bigg]_0^1 \\
&= \frac{3}{2}\sqrt{2} - 2.
\end{aligned}$$

Remarks:

1. You must decide for yourself which of these three methods you want to use. Please note the error of writing the following:

$$\int_0^1 \frac{x^3}{(x^2+1)^{\frac{3}{2}}}\,dx = \int_0^1 \left(1 - \frac{1}{u^2}\right) du$$

   This is wrong, because the limits are not given in terms of the variable u.

   The answer you obtain may be correct, but the method is totally wrong. In the examinations or an assignment you will get **no** marks for an answer like this.

2. There are only two general methods of integration: integration by parts (which is not part of this module) and integration by substitution. All other methods boil down to algebraic manipulation, followed by the application of one of the above-mentioned two methods. Keeping this in mind, you will realise the importance of this section.

(c) Steps for integration by substitution

1. Choose a new variable u (usually the innermost part of the integrand).
2. Compute $\dfrac{du}{dx}$.
3. Replace all terms in the original integrand with an expression involving u and du.

4. Evaluate the resulting (u) integral. (You may have to try a different u if the first one does not work.)

5. Replace each occurrence of u in the antiderivative with the corresponding expression in x.

5.10 Integration of Exponential and Logarithmic Functions

We state the following integration formulas again:

1.	$\int e^x dx$	$=$	$e^x + c$
2.	$\int a^x dx$	$=$	$\dfrac{a^x}{\ln\ a} + c$
3.	$\int \dfrac{1}{x} dx$	$=$	$\ln\|x\| + c$

which have the following forms when we work with functions $f(x)$.

Following normal substitution, we have

$1'$	$\int e^{f(x)} f'(x)\, dx$	$=$	$e^{f(x)} + c$
$2'$	$\int a^{f(x)} f'(x)\, dx$	$=$	$\dfrac{a^{f(x)}}{\ln\ a} + c$
$3'$	$\int \dfrac{1}{f(x)} f'(x)\, dx$	$=$	$\ln\|f(x)\| + c.$

Now we shall do one more example of integration by substitution to show you how the logarithmic function is used in this case:

Example:

Determine the following integral:

$$\int \frac{x^2}{1-2x^3} dx.$$

Note that this is of the form ($3'$) above, where $f(x) = 1 - 2x^3$ and $f'(x) = -6x^2$. (Remember that the constant (-6) will only make a difference to the answer as seen in Remark 1 on page 124 of the study guide.)

Solution:

In this case we let
$$u = 1 - 2x^3.$$

Then
$$du = -6x^2 dx.$$

In other words
$$-\frac{1}{6} du = x^2 dx.$$

Then
$$\begin{aligned}\int \frac{x^2}{1-2x^3}dx &= -\frac{1}{6}\int \frac{du}{u} \\ &= -\frac{1}{6}\ln|u| + c \\ &= -\frac{1}{6}\ln|1-2x^3| + c.\end{aligned}$$

Find the following integrals:

38. $\int \dfrac{\sec^2 x}{\tan x}dx = \ln|\tan x| + c.$

39. $\int \dfrac{2x}{x^2+1}dx = \ln\left|x^2+1\right| + c.$ Note that: $\left\{\dfrac{d}{dx}\left(x^2+1\right) = 2x\right\}$

5.11 Review of Formulas and Techniques of Integration

Go through the rest of the examples carefully. Evaluate:

40. $\int 3\cos 4x dx$

41. $\int \sec 2x \tan 2x dx$

42. $\int e^{3-2x} dx$

43. $\int \dfrac{4}{x^{\frac{1}{3}}\left(1+x^{\frac{2}{3}}\right)}dx$

44. $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}dx$

45. $\int_0^{\pi} \cos x e^{\sin x} dx$

46. $\int_{-\frac{\pi}{4}}^{0} \frac{\sin x}{\cos^2 x} dx$

47. $\int \frac{3}{16 + x^2} dx$

48. $\int \frac{x}{\sqrt{1 - x^4}} dx$

49. $\int \frac{1 + x}{1 + x^2} dx$

50.
$$\int_0^2 f(x)dx, \qquad where \qquad f(x) = \begin{cases} \frac{x}{x^2+1} & if \qquad x \leq 1 \\ \frac{x}{x^2+1} & if \qquad x > 1 \end{cases}$$

51.
$$f(x) = \begin{cases} xe^{x^2} & if \qquad x < 0 \\ x^2 e^{x^3} & if \qquad x > 0 \end{cases}$$

52. $\int \frac{2 + \sqrt{x}}{3 - \sqrt{x}} dx$

Solutions:

40. $\int 3 \cos 4x dx = 3 \sin 4x \frac{1}{4} + c = \frac{3}{4} \sin 4x + c.$

41. $\int \sec 2x \tan 2x dx = \sec 2x \frac{1}{2} + c = \frac{1}{2} \sec 2x + c.$

42. $\int e^{3-2x} dx = e^{3-2x} \left(-\frac{1}{2} \right) + c = -\frac{1}{2} e^{3-2x} + c.$

43. $\int \frac{4}{x^{\frac{1}{3}} \left(1 + x^{\frac{2}{3}} \right)} dx :$

Let $u = 1 + x^{\frac{2}{3}}$ then $du = \frac{2}{3} x^{-\frac{1}{3}} dx = \frac{2}{3x^{\frac{1}{3}}} dx \Rightarrow dx = \frac{3x^{\frac{1}{3}}}{2} du.$

So $\int \frac{4}{x^{\frac{1}{3}} \left(1 + x^{\frac{2}{3}} \right)} dx = 4 \int \frac{1}{x^{\frac{1}{3}}.u} . \frac{3x^{\frac{1}{3}}}{2} du = 4 \left(\frac{3}{2} \right) \int \frac{1}{u} du$

$= 6 \ln |u| + c = 6 \ln \left| 1 + x^{\frac{2}{3}} \right| + c.$

44. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx :$

Let $u = \sqrt{x}$ then $du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2\sqrt{x} du = 2u du = dx.$

So $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int \frac{\sin u}{u}2udu = 2\int \sin udu$

$= 2\left(-\cos u\right) + c = -2\cos\sqrt{x} + c.$

45. $\int_0^{\pi} \cos xe^{\sin x}dx$:

Let $u = \sin x$ then $du = \cos xdx \Rightarrow dx = \dfrac{du}{\cos x}$.

We have to change the limits, that is, if $x = 0$, $u = \sin 0 = 0$ and if $x = \pi$, $u = \sin\pi = 0$.

So $\int\limits_0^{\pi} \cos xe^{\sin x}dx = \int\limits_0^{0} \cos xe^{u}.\dfrac{du}{\cos x} = \int\limits_0^{0} e^{u}du = \left[e^{u}\right]_0^0$

$= e^0 - e^0 = 1 - 1 = 0.$

46. $\int_{-\frac{\pi}{4}}^{0} \dfrac{\sin x}{\cos^2 x}dx$

$= \int\limits_{-\frac{\pi}{4}}^{0} \sec x\tan xdx = \sec x\Big]_{-\frac{\pi}{4}}^{0} = \sec\left(0\right) - \sec\left(-\frac{\pi}{4}\right)$

$= \dfrac{1}{\cos\left(0\right)} - \dfrac{1}{\cos\left(-\frac{\pi}{4}\right)} = \dfrac{1}{1} - \left(\dfrac{1}{\frac{1}{\sqrt{2}}}\right) = 1 - \sqrt{2}.$

47. $\int \dfrac{3}{16 + x^2}dx = 3\int \dfrac{1}{16 + x^2}dx$:

Recall: $\int \dfrac{1}{a^2 + x^2}dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + c$

$\therefore \quad 3\int \dfrac{1}{4^2 + x^2}dx = 3.\dfrac{1}{4}\tan^{-1}\left(\dfrac{x}{4}\right) + c$

$$= \frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right) + c.$$

48. $\int \dfrac{x}{\sqrt{1 - x^4}}dx = \int \dfrac{x}{\sqrt{1 - \left(x^2\right)^2}}dx$:

Let $u = x^2$ then $du = 2xdx \Rightarrow dx = \dfrac{du}{2x}$.

So $\int \dfrac{x}{\sqrt{1 - u^2}}.\dfrac{du}{2x} = \dfrac{1}{2}\int \dfrac{x}{\sqrt{1 - u^2}}\dfrac{du}{x} = \dfrac{1}{2}\int \dfrac{du}{\sqrt{1 - u^2}}$

$= \dfrac{1}{2}\sin^{-1}\left(u\right) + c = \dfrac{1}{2}\sin^{-1}\left(x^2\right) + c.$

49. $\int \dfrac{1 + x}{1 + x^2}dx = \int \dfrac{1}{1 + x^2}dx + \int \dfrac{x}{1 + x^2}dx$

$$= \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}.\frac{2}{2}dx \qquad \left(\text{Recall that } \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c\right)$$

$$= \int \frac{1}{1+x^2}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= \tan^{-1} x + \frac{1}{2}\ln\left|1+x^2\right| + c.$$

50. $\int_0^2 f(x)dx,$ *where* $f(x)dx = \begin{cases} \frac{x}{x^2+1} & if \quad x \le 1 \\ \frac{x^2}{x^2+1} & if \quad x > 1 \end{cases}$

$$\int_0^2 f(x)\,dx = \int_0^1 \frac{x}{x^2+1}dx + \int_1^2 \frac{x^2}{x^2+1}dx.$$

Note: Rationalise the first integral by $\frac{2}{2}$ and use long division on the second integral to obtain:

$$\frac{1}{2}\int_0^1 \frac{2x}{x^2+1}dx + \int_1^2 \left(1 - \frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}dx + \int_1^2 1dx - \int_1^2 \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\ln\left(x^2+1\right)\Big]_0^1 + x\Big]_1^2 - \arctan x\Big]_1^2$$

$$= \frac{1}{2}\ln 2 + 1 + \frac{\pi}{4} - \arctan 2.$$

51. $f(x) = \begin{cases} xe^{x^2} & if \quad x < 0 \\ x^2e^{x^3} & if \quad x > 0 \end{cases}.$

$$\int_{-2}^2 f(x)\,dx = \int_{-2}^0 xe^{x^2}dx + \int_0^2 x^2e^{x^3}dx.$$

For the first integral, $\int_{-2}^0 xe^{x^2}dx$:

Let $u_1 = x^2$. Then

$du_1 = 2xdx$ and

$dx = \frac{du_1}{2x}$.

For the second integral, $\int_0^2 x^2e^{x^3}dx$:

Let $u_2 = x^3$.

Then $du_2 = 3x^2dx$ and

$$dx = \frac{du_2}{3x^2}.$$

So we have

$\int xe^{x^2}dx + \int x^2e^{x^3}dx$

$= \int xe^{u_1} \cdot \frac{du_1}{2x} + \int x^2e^{u_2} \cdot \frac{du_2}{3x^2} = \frac{1}{2}\int e^{u_1}du_1 + \frac{1}{3}\int e^{u_2}du_2$

$= \frac{1}{2}e^{u_1} + c_1 + \frac{1}{3}e^{u_2} + c_2.$

When considering the given definite integral, we ignore the constants c_1 and c_2 and get:

$\int_{-2}^{0} xe^{x^2}dx + \int_0^2 x^2e^{x^3}dx$

$= \left.\frac{e^{x^3}}{2}\right|_{-2}^{0} + \left.\frac{e^{x^3}}{3}\right|_{0}^{2}$

$= \left.\frac{1}{2}e^{x^2}\right|_{-2}^{0} + \left.\frac{1}{3}e^{x^3}\right|_{0}^{2} = \frac{1}{2}e^0 - \frac{1}{2}e^{(-2)^2} + \frac{1}{3}e^{(2)^3} - \frac{1}{3}e^{(0)^3}$

$= \frac{1}{2}(1) - \frac{1}{2}e^4 + \frac{1}{3}e^8 - \frac{1}{3}e^0 = \frac{1}{2} - \frac{e^4}{2} + \frac{e^8}{3} - \frac{1}{3}(1)$

$= \frac{1-e^4}{2} + \frac{e^8-1}{3}.$

52. $\int \frac{2+\sqrt{x}}{3-\sqrt{x}}dx$

$\int \frac{2+\sqrt{x}}{3-\sqrt{x}}dx = \int \frac{\sqrt{x}+2}{-\sqrt{x}+3}dx.$

This expression is an improper fraction, so we apply long division first.

$$\left[\begin{array}{r|r} & -1 \\ \hline -\sqrt{x}+3 & \sqrt{x}+2 \\ & +\sqrt{x}-3 \\ \cline{2-2} & 5 \end{array}\right]$$

So $\int \frac{2+\sqrt{x}}{3-\sqrt{x}}dx = \int \left(-1 + \frac{5}{-\sqrt{x}+3}\right)dx$

$$= \int -1dx + 5\int \frac{1}{-\sqrt{x}+3}dx$$

Now find $\int \frac{1}{-\sqrt{x}+3}dx$:

Let $u = -\sqrt{x} + 3 \Rightarrow \sqrt{x} = 3 - u$.

Then $\dfrac{du}{dx} = -\dfrac{1}{2\sqrt{x}}$ and

$du = \frac{-1}{2\sqrt{x}} dx \Rightarrow dx = -2\sqrt{x} du$.

So $dx = -2(3-u)\,du$,

that is $dx = (-6 + 2u)\,du$.

$$\text{So } \int \frac{1}{-\sqrt{x}+3} dx = \int \frac{1}{u}(-6+2u)\,du$$

$$= -6\int \frac{1}{u} du + \int \frac{2u}{u} du$$

$$= -6\int \frac{1}{u} du + \int 2du$$

$$= -6\ln|u| + 2u + c.$$

Therefore,

$$\int \frac{\sqrt{x}+2}{-\sqrt{x}+3} dx = \int -1dx + 5\int \frac{1}{-\sqrt{x}+3} dx$$

$$= -x + 5(-6\ln|u| + 2u + c)$$

$$= -x - 30\ln|u| + 10u + 5c$$

$$= -x - 30\ln|(3-\sqrt{x})| + 10(3-\sqrt{x}) + k, \text{ where } C = 5c = k$$

$$= x + 10(3-\sqrt{x}) - 30\ln|(3-\sqrt{x})| + k.$$

Key points

You should by now have an understanding of the notion of integration as the reverse of differentiation. You should particularly be able to link it to determining areas between different curves and the x-axis, and the area between curves. You should now be comfortable with calculating the integrals of several basic algebraic, trigonometric, exponential and logarithmic functions.

At this stage you should be able to

- show that you understand the notion of the antiderivative by finding the antiderivative of basic algebraic, trigonometric, exponential and logarithmic functions
- use the Fundamental Theorem of Calculus to find the derivatives of functions of the form $F(x) = \int_a^{g(x)} f(t)\,dt$

- evaluate definite integrals and use them to determine the areas between a curve and the x-axis, and the area between curves
- use substitution or term-by-term integration techniques to integrate basic algebraic, trigonometric, exponential and logarithmic functions
- solve problems involving the Mean Value Theorem for integrals

Continue practising solving problems until you have mastered the basic techniques! Go through the section "For your review"at the end of each chapter to consolidate what you have learnt and also use other calculus textbooks.

Study Unit 6

Differential Equation, Growth and Decay and Partial Derivatives/Chain Rule

6.1 Background

In this chapter we start by looking at first-order differential equations and then we move on to growth and decay problems, followed by partial derivatives. It has applications in chemistry, physics, biology, economics and engineering. In real life, physical quantities usually depend on two or more variables, so we shall also turn our attention to functions with several variables and extend the basic ideas of differential calculus to these functions.

6.2 Learning Outcomes

At the end of this chapter, you should be able to

- solve first-order differential equations with initial values
- solve basic real-life problems involving exponential growth and decay
- determine the partial derivatives of functions with several variables

Note: The way to master calculus is to solve lots of calculus problems!

6.3 Prescribed Reading

In Chapter 9 we cover Sections 9.1, 2.3, 9.4 and Chapter 14 Sections 14.3 and 14.5. Please also study some Sections in Chapter 3 (Section 3.8).

6.4 Worked Examples

The greater part of the collection of worked examples is taken from Chapters 3, 9 and 14 of the prescribed textbook by Stewart. They can be divided into three sets (with some overlap), namely,

I. differential equations

II. growth and decay

III. partial derivatives/chain rule

Attempt the problems appearing immediately after each set, once you have studied the relevant parts of Stewart and done some of the exercises there. Table 6.1 shows how these worked examples and their solutions are organised.

	Topic(s)	**Sections in Stewart**	**Study guide examples**
I.	Differential equations	9.1 & 9.3	1–9
II.	Growth and decay	3.8 & 9.4	10–20
III.	Partial derivatives/chain rule	14.1, 14.3 & 14.5	21–40

Table 6.1: Sections in Stewart.

6.4.1 Differential Equations

An equation that contains an unknown function and some of its derivatives is called a differential equation (DE). Differential equations are of the type:

$$\begin{aligned} \frac{dy}{dx} &= f(x) \\ \text{or} \quad \frac{dy}{dx} &= y. \end{aligned}$$

A first-order differential equation is an equation involving the first derivative.

The general **first-order differential equation** is of the type:

$$\frac{dy}{dx} = g\left(x, y\right),$$

where $g\left(x, y\right)$ is a function with two variables. We refer to "first-order" because of the presence of the first derivative only. In general, a differential equation is one that contains an unknown function and one or more of its derivatives. The **order** of the differential equation is the order of the highest derivative that occurs in the equation.

(a) The solution of differential equations

A function f is called a solution for a differential equation if the equation is satisfied when $y = f\left(x\right)$ and its derivatives are substituted into the equation. When solving a differential equation, we are expected to find all possible solutions for the equation.

Example:

1. Solve the differential equation $y' = \dfrac{x^2 + 7x + 3}{y^2}$.

   Solution:

   (i) Separate the variables: $y^2 y' = x^2 + 7x + 3$.

   (ii) Integrate both sides with respect to x:

   $$\begin{aligned} \int y^2 y'\left(x\right) dx &= \int \left(x^2 + 7x + 3\right) dx \quad \text{or} \\ \int y^2 dy &= \int \left(x^2 + 7x + 3\right) dx. \end{aligned}$$

   Result of integration: $\dfrac{y^3}{3} = \dfrac{x^3}{3} + 7\dfrac{x^2}{2} + 3x + c.$

   (iii) Solve for y: $y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 3c}.$

(b) Separable differential equations

A differential equation is separable if you are able to group the ys with dy and xs with dx. For example, $y' = xy^2 - 2xy$ is a separable DE because you can rewrite it as $y' = x\left(y^2 - 2y\right)$ or $\dfrac{dy}{dx} = x\left(y^2 - 2y\right)$.

Divide both sides by $\left(y^2 - 2y\right)$ and multiply both sides by dx and obtain:

$$\frac{1}{y^2 - 2y} dy = x dx.$$

On the other hand, an equation like $y' = xy^2 - 2x^2y$ is not separable.

A separable equation is a first-order differential equation in which the expression for $\dfrac{dy}{dx}$ can be factored as a function of x times a function of y. That is, it can be written in the form

$$\frac{dy}{dx} = g\left(x\right) f(y).$$

The term **separable** comes from the fact that the expression on the right side can be "separated" into a function of x and a function of y. If $f\left(y\right) \neq 0$, we could re-write the equation as

$$\frac{dy}{dx} = \frac{g\left(x\right)}{h\left(y\right)} = \frac{g\left(x\right)}{h\left(y\left(x\right)\right)} \tag{6.1}$$

where $h\left(y\right) = \dfrac{1}{f\left(y\right)}$.

To solve the equation $\dfrac{dy}{dx} = \dfrac{g\left(x\right)}{h\left(y\right)}$, we rewrite it in the differential form as

$$h\left(y\right) dy = g\left(x\right) dx,$$

so that all ys are on one side of the equation with dy and all xs are on the other side of the equation with dx. Then we integrate both sides of the equation:

$$\int h\left(y\right) dy = \int g\left(x\right) dx.$$

This equation defines y as a function of x. In some cases we may be able to solve for y in terms of x.

Note: The justification for the equation above, $\int h\left(y\right) dy = \int g\left(x\right) dx$, comes from the substitution rule, as follows:

$$\begin{aligned} \text{L.H.S.} &= \int h\left(y\right) dy = \int h\left(y\left(x\right)\right) \frac{dy}{dx}\, dx \\ &= \int h\left(y\left(x\right)\right) \frac{g\left(x\right)}{h\left(y\left(x\right)\right)} dx \quad \text{(substituting (1))} \\ &= \int g\left(x\right) dx = \text{R.H.S.} \end{aligned}$$

In general, the first-order differential equation is

$$\frac{dy}{dx} = h(x, y).$$

We consider the special case in which $h\left(x, y\right) = f\left(y\right) g(x)$, the product of a function of y alone and a function of x alone.

Examples are:

$$\begin{aligned} \frac{dy}{dx} &= x^2y^3, \\ \frac{dy}{dx} &= e^x \cos y, \\ \text{and} \quad \frac{dy}{dx} &= \frac{y}{\sqrt{1+x^2}}. \end{aligned}$$

The examples below are not included in this special case:

$$\frac{dy}{dx} = x + y : \qquad \frac{dy}{dx} = \sin(xy), \qquad \frac{dy}{dx} = \sqrt{1+x^2+y^2}$$

since in each of these equations the right-hand side is not of the form $f(y)\,g(x)$.

We solve $\frac{dy}{dx} = f(y)\,g(x)$ by first separating the variables, putting all the ys on one side and all the xs on the other side.

Write

$$\frac{dy}{dx} = f(y)\,g(x)$$

as

$$\frac{dy}{f(y)} = g(x)\,dx.$$

Then integrate both sides (if you can) to get:

$$F(y) = G(x) + c,$$

where $F(y)$ is an antiderivative of $\frac{1}{f(y)}$ and $G(x)$ is an antiderivative of $g(x)$.

If this equation can be solved for y in terms of x, the resulting function is a solution of the differential equation.

Note that in separating the variables, you must assume that $f(y) \neq 0$.

If $f(y) = 0$ for $y = y_0$, it can be checked that the constant function $y(x) = y_0$ is a solution for the equation.

Examples of general solutions:

2. Solve $\dfrac{dy}{dx} = xy$.

Solution:

It can be noted that $y(x) = 0$ is a solution. Now assume $y(x) \neq 0$ and separate the variables:

$$\begin{aligned}
\frac{dy}{y} &= x\ dx \\
\Rightarrow \quad \int \frac{dy}{y} &= \int x\ dx \\
\Rightarrow \quad \ln|y| &= \frac{1}{2}x^2 + c \\
\Rightarrow \quad e^{\ln|y|} &= e^{\frac{x^2}{2}+c} \\
\Rightarrow \quad |y| &= e^{\frac{x^2}{2}} \cdot e^c \\
\text{or} \quad |y| &= ke^{\frac{x^2}{2}} \qquad \text{where } k = e^c \text{ is a positive constant.}
\end{aligned}$$

If $y \geq 0$, then $|y| = y = ke^{\frac{x^2}{2}}$.

If $y < 0$, then $|y| = -y = +ke^{\frac{x^2}{2}}$.

Therefore,

$$y = \begin{cases} ke^{\frac{x^2}{2}} & \text{if} \quad y \geq 0, \\ -ke^{\frac{x^2}{2}} & \text{if} \quad y < 0. \end{cases}$$

This is equivalent to:

$$y = ae^{\frac{x^2}{2}}$$

where a is any constant.

Note that the solution $y(x) = 0$ is included.

To check if the answer obtained is the solution to the original differential equation, just differentiate this answer and you should obtain the original differential equation, namely:

$$\begin{aligned}
y &= ae^{\frac{x^2}{2}} \\
\Rightarrow \quad \frac{dy}{dx} &= \frac{d}{dx}\left(ae^{\frac{x^2}{2}}\right) = ae^{\frac{x^2}{2}} \cdot \frac{1}{2} \cdot 2x \\
&= axe^{\frac{x^2}{2}} \\
&= x \cdot ae^{\frac{x^2}{2}} \\
&= xy \qquad \text{(proved).}
\end{aligned}$$

3. Solve $\dfrac{dy}{dx} = \dfrac{2x(y-1)}{x^2+1}$.

Solution:

Note that $y\left(x\right)=1$ is a solution.

Now assume that $y\left(x\right)\neq 1$ and separate the variables:

$$\begin{aligned}
\frac{dy}{y-1} &= \frac{2x}{x^2+1}dx \\
\Rightarrow \quad \int \frac{1}{y-1}dy &= \int \frac{2x}{x^2+1}dx \\
\Rightarrow \quad \ln\left|y-1\right| &= \ln\left(x^2+1\right)+c \\
\Rightarrow \quad e^{\ln\left|y-1\right|} &= e^{\ln\left(x^2+1\right)+c} \\
\Rightarrow \quad \left|y-1\right| &= \left(x^2+1\right)e^c = k\left(x^2+1\right), \text{ where } k=e^c.
\end{aligned}$$

Therefore

$$y-1=\pm k\left(x^2+1\right)$$

where k is a positive constant.

The answer may be written as:

$$y=1+a\left(x^2+1\right)=1+ax^2+a=ax^2+\left(a+1\right)$$

where a is an arbitrary constant and $a=\pm k$. This answer includes the special solution $y\left(x\right)=1$.

Check:

$$\begin{aligned}
\frac{dy}{dx} &= 2ax+0 \\
\Rightarrow \frac{dy}{dx} &= 2x\left(\frac{y-1}{x^2+1}\right) \qquad \text{where } a=\frac{y-1}{x^2+1} \qquad \text{since} \qquad y=1+a(x^2+1).
\end{aligned}$$

(c) Initial-value problems (IVPs)

In many physical (real-world) problems, we need to find the particular solution that satisfies a condition of the form $y\left(t_0\right)=y_0$. This is called the initial condition. The problem of finding a solution for the differential equation that satisfies the initial condition is called the **initial-value problem**. In such problems, you are asked to find the solution $y=y\left(x\right)$ or $y=f\left(x\right)$ for the differential equation

$$\frac{dy}{dx}=g(x,y),$$

the graph of which passes through a given point (a,b), that is, which satisfies the initial condition

$$y\left(a\right)=b.$$

Steps For Solving IVPs

(i) Separate the given differential equation into its components.

(ii) Integrate both sides of (i) above.

(iii) Solve for c (integration constant) by substituting the initial conditions.

(iv) Re-write your final answer by substituting the value for c.

Examples:

4. Find a solution for the initial-value problem $\dfrac{dy}{dx} = x^2$, with initial condition $y(-3) = +1$.

Solution:

There are two steps to follow:

(i) Solve the differential equation by integrating both sides:

$$\begin{aligned} \frac{dy}{dx} &= x^2 \\ \Rightarrow \quad dy &= x^2 dx \\ \Rightarrow \quad \int dy &= \int x^2 dx \\ \Rightarrow \quad y &= \frac{1}{3}x^3 + c \\ \text{or} \quad y(x) &= \frac{1}{3}x^3 + c. \quad \text{This is the general solution for the DE.} \end{aligned}$$

(ii) We find the constant c by substituting the initial condition $y(-3) = 1$:

$$\begin{aligned} 1 &= \frac{1}{3}(-3)^3 + c \\ \Rightarrow \quad 1 &= -9 + c \\ \Rightarrow \quad c &= 10. \end{aligned}$$

Therefore, the particular solution to the given initial-value problem is

$$y(x) = \frac{1}{3}x^3 + 10.$$

5. Find a solution for the differential equation $\dfrac{dy}{dx} = 8x^3 + 1$ with initial condition $y(1) = -1$.

Solution:

(i)

$$\begin{aligned}
\frac{dy}{dx} &= 8x^3+1 \quad \text{(separate } dx \text{ from } dy \text{ and integrate)} \\
\Rightarrow \quad dy &= \left(8x^3+1\right)dx \\
\Rightarrow \quad \int dy &= \int \left(8x^3+1\right)dx \\
\Rightarrow \quad y &= \frac{8x^4}{4}+x+c \\
\text{or } y(x) &= 2x^4+x+c\cdot
\end{aligned}$$

(ii) Evaluate the value of c:

$$\begin{aligned}
y(x) &= 2x^4+x+c \\
\Rightarrow \quad -1 &= 2(1)^4+1+c \quad \text{(initial condition } y(1)=-1\text{)} \\
\Rightarrow \quad -1 &= 2+1+c \\
\Rightarrow \quad c &= -1-2-1 \\
\Rightarrow \quad c &= -4.
\end{aligned}$$

The particular solution to the given initial-value problem is

$$y = 2x^4+x-4.$$

Other initial-value problems:

6. Solve the initial-value problem

$$\frac{dy}{dx} = \frac{2x}{3y-1}, \quad y(1)=0.$$

Solution:

$$\begin{aligned}
\int (3y-1)\,dy &= \int 2x\ dx \\
\Rightarrow \quad \frac{3}{2}y^2-y &= \frac{2x^2}{2}+c.
\end{aligned}$$

Now substitute the given initial values and obtain:

$$\begin{aligned}
\frac{3}{2}(0)^2-0 &= (1)^2+c \\
\Rightarrow \quad 0 &= 1+c \\
\therefore \quad c &= -1. \\
\text{Thus} \quad \frac{3}{2}y^2-y &= x^2-1 \\
\text{or} \quad 3y^2-2y &= 2x^2-2 \text{ is the particular solution.}
\end{aligned}$$

7. Solve the initial-value problem

$$\frac{dy}{dx} = x^2y^2, \qquad y(0) = b.$$

Solution:

Note that $y(x) = 0$ is the solution if $b = 0$.

Assume $y \neq 0$ and separate the variables:

$$\begin{aligned}
\frac{dy}{dx} &= x^2y^2, \qquad y(0) = b \\
\Rightarrow \quad \frac{dy}{y^2} &= x^2dx \\
\Rightarrow \quad y^{-2}dy &= x^2dx \\
\Rightarrow \quad \int y^{-2}dy &= \int x^2dx \\
\Rightarrow \quad \frac{y^{-1}}{-1} &= \frac{x^3}{3} + c \\
\Rightarrow \quad -\frac{1}{y} &= \frac{1}{3}x^3 + c.
\end{aligned}$$

Substitute the initial values:

$$\begin{aligned}
-\frac{1}{y} &= \frac{1}{3}x^3 + c \\
\Rightarrow \quad -\frac{1}{b} &= \frac{1}{3}(0)^3 + c \\
\Rightarrow \quad -\frac{1}{b} &= 0 + c \\
\therefore \quad c &= -\frac{1}{b}.
\end{aligned}$$

Therefore

$$\begin{aligned}
-\frac{1}{y} &= \frac{x^3}{3} - \frac{1}{b} \\
\Rightarrow \quad -\frac{1}{y} &= \frac{bx^3 - 3}{3b} \\
\Rightarrow \quad -1(3b) &= y\left(bx^3 - 3\right) \\
\Rightarrow \quad 3b &= -y\left(bx^3 - 3\right) = y\left(-bx^3 + 3\right) \\
\therefore \quad y &= \frac{3b}{3 - bx^3}.
\end{aligned}$$

The solution includes $y(x) = 0$.

Note: In initial-value problems (IVPs), you are given initial conditions (in other words, the x and y values), and you use them in order to evaluate a particular value of a constant c.

More examples:

8. Solving a logistic growth problem: Given a maximum sustainable population of $M = 1\ 000$ and a growth rate of $k = 0.007$, find an expression for the population at any time t, given an initial population of $y(0) = 350$ and assuming logistic growth.

Solution:

Using the solution for a logistic equation, we have $kM = 7$ and

$$y = \frac{AMe^{kMt}}{1 + Ae^{kMt}}, \text{ where } \begin{cases} A = \text{constant} \\ M = \text{carrying capacity} \\ k = \text{growth rate and} \\ t = \text{time} \end{cases}$$

From the initial conditions, we have $350 = y\,(0) = \dfrac{1\ 000A}{1 + A}$

and solving for A, we obtain $A = \dfrac{35}{65}$

which gives the solution of the IVP as $y = \dfrac{35\ 000e^{7t}}{65 + 35e^{7t}}$.

9. Investment strategies for making a million: Money is invested at 8% interest, compounded continuously. If deposits are made at a rate of R2 000 per year, find the size of the initial investment needed to reach R1 million in 20 years.

Solution:

It would be easier to recall the general solution (or the equation) first. The constant of integration c is obtained by setting $t = 0$ and taking $A\,(0) = x$,

that is $12.5 \ln |0.08x + 2000| = c$

so that $12.5 \ln |0.08A + 2000| = t + 12.5 \ln |0.08x + 2000|$.

Now find the value of x such that $A\,(20) = 1\ 000\ 000$.

$12.5 \ln |0.08\,(1\ 000\ 000) + 2000| = 20 + 12.5 \ln |0.08x + 2000|$

or $\dfrac{12.5 \ln |82\ 000| - 20}{12.5} = \ln |0.08x + 2000|$

Solve for x by taking the exponential of both sides:

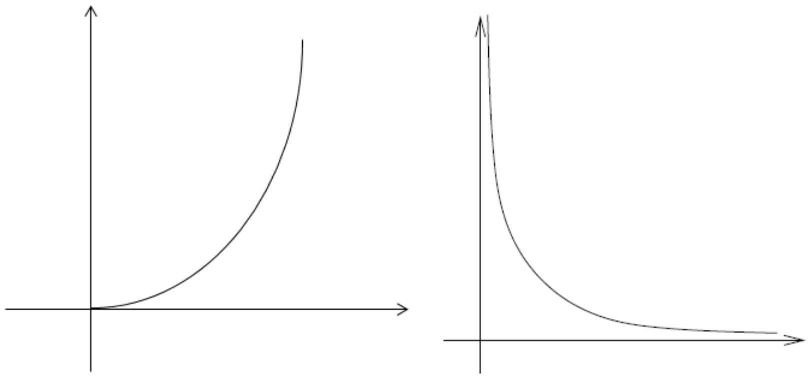
$$e^{(12.5\ln 82\ 000-20)/12.5} = 0.08x + 2000$$

$$\Rightarrow \quad x = \frac{e^{\ln 82000-1.6} - 2000}{0.08} \approx 181,943.93.$$

6.4.2 Growth and Decay

$y\,(t) = Ae^{kt}$ is the general solution for the differential equation $y'\,(t) = \dfrac{dy}{dt} = ky(t)$.

For $k > 0$ the equation reflects an **exponential growth** law and for $k < 0$, the equation reflects an **exponential decay** law.



(a) (b)

Figure 6.1: (a) $k > 0$ is the growth curve (b) $k < 0$ is the decay curve

Exponential growth or decay means that:

$$y'\,(t) = \frac{dy}{dt} = ky\,(t) \tag{6.2}$$

Since equation (6.2) involves the derivative of an unknown function, we call it a differential equation (DE). The aim is to solve this DE, that is, to find the function $y(t)$.

Recall how to solve separable differential equations. We have

$$\frac{dy}{dt} = ky\,(t) \tag{6.3}$$

This is a separable DE, that is

$$\frac{dy}{y\,(t)} = kdt. \tag{6.4}$$

Integrating both sides of equation (6.4) with respect to t, we obtain

$$\int \frac{1}{y\,(t)}dy = \int kdt$$

$$\begin{aligned} \Rightarrow \quad \ln|y| + c_1 &= kt + c_2 \\ \text{or} \quad \ln|y| &= kt + C \text{ (where } C = c_2 - c_1\text{)}. \end{aligned}$$

Since $y(t) > 0$, we have $\ln y(t) = kt + C$

and taking exponentials of both sides , we get $e^{\ln y(t)} = e^{kt+C}$,

that is, $y(t) = e^{kt}.e^{C}$.

Therefore $y(t) = Ae^{kt}$ (where $A = e^{C}$).

Examples:

10. A bacterial culture contains 100 cells at a certain point in time. Sixty minutes later, there are 450 cells. Assuming **exponential growth**, determine the number of cells present at time t and find the doubling time.

Solution:

Exponential growth means that $y'(t) = ky(t)$ and after solving this equation, we have $y(t) = Ae^{kt}$.

At $t = 0$, $y(0) = 100$ (initial conditions), we have $100 = y(0) = Ae^0 = A$.

Then $y(t) = 100e^{kt}$.

At $t = 60$ minutes, we have $450 = y(60) = 100e^{60k}$.

Solve for k by taking natural logarithms of both sides and obtain

$$\begin{aligned} \ln 4.5 &= \ln e^{60k} = 60k, \\ \text{so that } k &= \frac{\ln 4.5}{60}. \end{aligned}$$

Therefore, at any time t we have $y(t) = 100e^{kt} = 100\exp\left(\frac{\ln 4.5}{60}t\right)$.

At doubling time, $y(t) = 2y(0) = 200$,

so that $200 = y(t) = 100\exp\left(\frac{\ln 4.5}{60}t\right)$.

Solve for it and get $\ln 2 = \frac{\ln 4.5}{60}t$.

11. A cup of coffee is 180°F when poured. After 2 minutes in a room at 70°F, the coffee has cooled to 165°F. Find the temperature at time t and find the time at which the coffee will have cooled to 120°F.

Solution:

The differential equation is $y'(t) = k[y(t) + 70]$.

Solving this differential equation, we obtain $y(t) = Ae^{kt} + 70$.

Initial conditions are : $180 = y(0) = Ae^0 + 70 = A + 70$.

This gives $y(0) = 180$,

so that $A = 110$ and $y(t) = 110e^{kt} + 70$.

We use the second measured temperature (165°F) to solve for k :

$165 = y(2) = 110e^{2k} + 70$.

Solve for k by subtracting 70 from both sides and dividing by 110 to obtain

$$e^{2k} = \frac{165 - 70}{110} = \frac{95}{110}.$$

Taking natural logarithms of both sides, we get $2k = \ln\left(\frac{95}{110}\right)$ and, therefore $k = \frac{1}{2}\ln\left(\frac{95}{110}\right)$.

When the temperature $T = 120$, we have $120 = y(t) = 110e^{kt} + 70$.

Solve for t and obtain $t = \frac{1}{k}\ln\frac{5}{11}$.

12. Compound interest and an example of decay. Suppose the value of a R10 000 asset decreases continuously at a constant rate of 24% per year. Find its worth after

    (a) 10 years

    (b) 20 years

Solution:

The value $v(t)$ of any quantity that is changing at a constant rate r satisfies the equation: $v' = rv$.

Here $r = -0.24$, so that $v(t) = Ae^{-0.24t}$

Since the initial asset value is R10 000, we have $v(t) = 10\ 000e^{-0.24t}$.

We now have $10\ 000 = v(0) = Ae^0 = A$.

At time $t = 10$, the asset value is R10 000$e^{-0.24(10)} \approx$ R907.18 and

at time $t = 20$, the asset value is R10 000$e^{-0.24(20)} \approx$ R82.30.

More Worked Examples:

13. A bacterial culture starts with 1 000 bacteria and after 2 hours there are 2 500 bacteria. Assuming that the culture grows at a rate proportional to its size, find the population after 6 hours.

14. The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1 013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.

    (a) Solve the initial value problem –

    Differential equation: $dp/dh = kp$ (k is a constant).

    Initial condition: $p = p_0$ when $h = 0$,

    Express p in terms of h. Determine the values of p_0 and k from the given altitude and pressure data.

    (b) What is the atmospheric pressure at $h = 50$ km?

    (c) At what altitude does the pressure equal 900 millibars?

15. In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-gluconolactone into gluconic acid, for example,
$$\frac{dy}{dt} = -0.6y$$
when t is measured in hours. If there are 100 grams of δ-gluconolactone present when $t = 0$, how many grams will be left after the first hour?

16. The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation
$$\frac{dL}{dx} = -kL.$$
As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. Up to about how deep can you expect to work without artificial light?

17. If R1 000 is invested for 4 years at 12% interest compounded continuously, what is the value of the investment at the end of the 4 years?

18. You have just placed A_0 rand in a bank account that pays 14% interest, compounded continuously.

    (a) How much money will you have in the account in 6 years' time?

    (b) How long will it take for your money to double and to triple?

19. Suppose an object takes 40 min to cool down from 30°C to 24°C in a room that is kept at 20°C.

    (a) What would the temperature of the object be 15 min after it has been 30°C?

    (b) How long will it take for the object to cool down to 21°C?

20. Suppose that a cup of soup cooled down from 90°C to 60°C within 10 minutes in a room of which the temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.

    (a) How much longer would it take for the soup to cool to 35°C?

    (b) Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer in which the temperature is −15°C. How long will it take for the soup to cool down from 90°C to 35°C?

Solutions:

13. Let $y(t)$ be the number of bacteria after t hours. Then $y_0 = y(0) = 1000$ and $y(2) = 2500$. Since we are assuming $\frac{dy}{dt} = ky$ (Law of Exponential Growth on page 237 in Stewart),

$$\begin{aligned} y(t) &= y_0e^{kt} = 1000e^{kt} \\ \Rightarrow \quad y(2) &= 1000e^{2k} = 2500. \end{aligned}$$

Therefore $e^{2k} = 2.5$ and $2k = \ln 2.5$.

Thus $k = \dfrac{\ln 2.5}{2}$.

Substituting the value of $k = \frac{1}{2}\ln 2.5$ back into the expression for $y(t)$, we have

$$\begin{aligned} y(t) &= 1000e^{\frac{\ln 2,5}{2}t} \\ &= 1000\left(e^{\ln 2.5}\right)^{\frac{t}{2}} \\ &= 1000\,(2.5)^{\frac{t}{2}} \quad \text{(since } e^{\ln 2.5} = 2.5\text{)}. \end{aligned}$$

Therefore the population after 6 hours is

$$y(6) = 1000\,(2.5)^3 = 15\ 625.$$

14. (a) Differential equation: $\dfrac{dp}{dh} = kp$ (k is a constant).

Initial condition: $p = p_0$ when $h = 0$.

We separate the variables by dividing the differential equation by p to have

$$\begin{aligned}
\frac{1}{p}\frac{dp}{dh} &= k \\
\Rightarrow \ln|p| &= kh + c \ \text{ (by integrating the R.H.S. with respect to } h) \\
\Rightarrow |p| &= e^{kh} + c \ \text{ (by exponentiating)} \\
\Rightarrow |p| &= e^c e^{kh} \\
\Rightarrow p &= \pm e^c e^{kh} \\
\Rightarrow p &= Ae^{kh} \ \text{ where } A = \pm e^c.
\end{aligned}$$

But the initial condition $p = p_0$ when $h = 0$. Therefore $p_0 = Ae^{k0} = A$ and thus

$$p = p_0 e^{kh}$$

Then

$$p(0) = p_0 = 1013 \text{ and } p(20) = 90 = 1013e^{20k}.$$

Thus

$$\begin{aligned}
e^{20k} &= \frac{90}{1013} \\
\Rightarrow 20k &= \ln\frac{90}{1013} \\
\Rightarrow k &= \frac{1}{20}\ln\frac{90}{1013}.
\end{aligned}$$

Therefore $p(h)$ is given by

$$p(h) = 1013e\ln\left(\frac{90}{1013}\right)\frac{h}{20} = 1013\left(\frac{90}{1013}\right)^{\frac{h}{20}}.$$

(b) If $h = 50$ km, then

$$\begin{aligned}
p(50) &= 1013e^{\ln\left(\frac{90}{1013}\right)\cdot\frac{5}{2}} \\
&= 1013\left(e^{\ln\left(\frac{90}{1013}\right)}\right)^{\frac{5}{2}} \\
&= 1013\left(\frac{90}{1013}\right)^{\frac{5}{2}} \ \left(\text{since } e^{\ln\left(\frac{90}{1013}\right)} = \frac{90}{1013}\right) \\
&= \frac{90^{2.5}}{1013^{1.5}}.
\end{aligned}$$

(c) If $p = 900$, then

$$\begin{aligned}
900 &= 1013\left(\frac{90}{1013}\right)^{\frac{h}{20}} \\
\Rightarrow \left(\frac{90}{1013}\right)^{\frac{h}{20}} &= \frac{900}{1013} \\
\Rightarrow \frac{h}{20}\ln\left(\frac{90}{1013}\right) &= \ln\left(\frac{900}{1013}\right) \\
\Rightarrow \quad h &= 20\cdot\frac{1}{\ln\left(\frac{90}{1013}\right)}\cdot\ln\left(\frac{900}{1013}\right).
\end{aligned}$$

15. We have the following initial-value problem:

Differential equation: $\dfrac{dy}{dt} = -0.6y$

Initial condition: $y_0 = 100$ when $t = 0$.

We use the Radioactive decay equation on page 239 in Stewart:

$$y = y_0e^{-kt}.$$

Since $y_0 = 100$ and $k = -0.6$ we have

$$y = 100e^{-0.6t}.$$

After the first hour the number of grams will be

$$\begin{aligned}
y &= 100e^{(-0{,}6)(1)} \\
&= 100e^{-0{,}6}.
\end{aligned}$$

16. We have the following initial-value problem:

Differential equation: $\dfrac{dL}{dx} = -kL$

Initial condition: $L = L_0$ when $x = 0$. We are given that $L = \frac{1}{2}L_0$ when $x = 18$. We want to find x when $L = \frac{1}{10}L_0$.

$$L(x) = L_0e^{-kx}$$

where

$$\begin{aligned}
L(x) &= y(t), \\
L_0 &= A.
\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}L_0 &= L_0e^{-18k}\\ \therefore \quad e^{-18k} &= \frac{1}{2}\\ \Rightarrow e^{18k} &= 2\\ \Rightarrow 18k &= \ln 2\\ \Rightarrow k &= \frac{1}{18}\ln 2.\end{aligned}$$

When $L = \frac{1}{10}L_0$ we have

$$\begin{aligned}\frac{1}{10}L_0 &= L_0e^{-kx}\\ \Rightarrow e^{-kx} &= \frac{1}{10}\\ \Rightarrow -kx &= \ln\left(\frac{1}{10}\right)\\ \Rightarrow -kx &= -\ln 10\\ \Rightarrow x &= \frac{1}{k}\ln 10\\ &= \frac{18}{\ln 2}\ln 10 \text{ (since } k = \frac{1}{18}\ln 2).\end{aligned}$$

17. We use the Continuously compound interest formula on page 242 in Stewart:

$$A(t) = A_0e^{rt}.$$

It is given that A_0 =R1000 and $r = 12\% = 0.12$. So the value of the investment in 4 years will be

$$\begin{aligned}A(4) &= 1000e^{(0.12)4}\\ &= 1000e^{0.48}.\end{aligned}$$

18. We use the continuous compound interest formula:

$$A(t) = A_0e^{rt}.$$

(a) We are given that $r = 14\% = 0.14$. In 6 years the money will be

$$\begin{aligned}A(6) &= A_0e^{(0.14)6}\\ &= A_0e^{0.84}.\end{aligned}$$

(b) We want to find t when $A = 2A_0$, and when $A = 3A_0$. We have $A(t) = A_0e^{0.14t}$.

When $A = 2A_0$ we have

$$\begin{aligned}2A_0 &= A_0e^{0.14t}\\ \Rightarrow e^{0.14t} &= 2\\ \Rightarrow 0.14t &= \ln 2\\ \Rightarrow t &= \frac{\ln 2}{0.14} = \frac{100}{14}\ln 2.\end{aligned}$$

When $A = 3A_0$ we have

$$\begin{aligned}
3A_0 &= A_0 e^{0.14t} \\
\Rightarrow e^{0.14t} &= 3 \\
\Rightarrow 0.14t &= \ln 3 \\
\Rightarrow t &= \frac{\ln 3}{0.14} = \frac{100}{14} \ln 3.
\end{aligned}$$

19. We use Newton's Law of Cooling on page 240 in Stewart:

$$\begin{aligned}
T &= T_s + (T_0 - T_s)\, e^{-kt} \\
\text{where} \quad T &= y(t)\,, \; T_s = T_\alpha \\
\text{and} \quad T_0 - T_s &= A
\end{aligned}$$

where T is the temperature at any given time t, T_s is the surrounding temperature and T_0 is the initial temperature (in other words, T_0 is the value of T at $t = 0$).

Now, we are given that $T_s = 20°\text{C}$ and $T_0 = 30°\text{C}$.

Then the object's temperature after t minutes is

$$\begin{aligned}
T &= 20 + (30 - 20)\, e^{-kt} \\
&= 20 + 10e^{-kt}.
\end{aligned}$$

(a) We first find k, by using the information that $T = 24$ when $t = 40$:

$$\begin{aligned}
24 &= 20 + 10e^{-40k} \\
\Rightarrow 10e^{-40k} &= 4 \\
\Rightarrow e^{-40k} &= \frac{4}{10} \\
\Rightarrow -40k &= \ln\left(\frac{4}{10}\right) \\
\Rightarrow k &= -\frac{1}{40} \ln\left(\frac{4}{10}\right).
\end{aligned}$$

Now the object's temperature after time t is

$$T = 20 + 10e^{\frac{1}{40}\ln\frac{4}{10}\cdot t}.$$

We now find T when $t = 15$:

$$\begin{aligned}
T &= 20 + 10e^{\frac{1}{40}\ln\frac{4}{10}\cdot 15} \\
&= 20 + 10\left(e^{\ln\frac{4}{10}}\right)^{\frac{15}{40}} \\
&= 20 + 10\left(\frac{4}{10}\right)^{\frac{15}{40}}.
\end{aligned}$$

(b) We now find the time t when $T = 21°$C:

$$\begin{aligned} 21 &= 20 + 10\left(e^{\ln\frac{4}{10}}\right)^{\frac{t}{40}} \\ \Rightarrow 10\left(\frac{4}{10}\right)^{\frac{t}{40}} &= 1 \\ \Rightarrow \left(\frac{4}{10}\right)^{\frac{t}{40}} &= \frac{1}{10} \\ \Rightarrow \frac{t}{40}\ln\left(\frac{4}{10}\right) &= \ln\left(\frac{1}{10}\right) \\ \Rightarrow t &= \frac{40}{\ln\left(\frac{4}{10}\right)} \cdot \ln\left(\frac{1}{10}\right) \\ &= -\frac{40\ln 10}{\ln(0,4)}. \end{aligned}$$

20. As in the solution of the question above,

$$T = T_s + (T_0 - T_s)\, e^{-kt}.$$

It is given that $T_s = 20°$C and $T_0 = 90°$C. The soup's temperature after t minutes is

$$\begin{aligned} T &= 20 + (90 - 20)\, e^{-kt} \\ &= 20 + 70e^{-kt}. \end{aligned}$$

(a) We find k by using the information that $T = 60°$C when $t = 10$ minutes:

$$\begin{aligned} 60 &= 20 + 70e^{-10k} \\ \Rightarrow 40 &= 70e^{-10k} \\ \Rightarrow e^{-10k} &= \frac{4}{7} \\ \Rightarrow -10k &= \ln\left(\frac{4}{7}\right) \\ \Rightarrow k &= -\frac{1}{10}\ln\left(\frac{4}{7}\right). \end{aligned}$$

The soup's temperature at time t is

$$T = 20 + 70e^{\left(\frac{1}{10}\ln\left(\frac{4}{7}\right)\right)\cdot t}$$

We now find the time t when $T = 35°$C:

$$\begin{aligned} 35 &= 20 + 70\left(e^{\ln\frac{4}{7}}\right)^{\frac{t}{10}} \\ \Rightarrow 15 &= 70\left(\frac{4}{7}\right)^{\frac{t}{10}} \\ \Rightarrow \left(\frac{4}{7}\right)^{\frac{t}{10}} &= \frac{15}{70} \\ \Rightarrow \frac{t}{10}\ln\left(\frac{4}{7}\right) &= \ln\left(\frac{15}{70}\right) \\ \Rightarrow t &= 10\frac{1}{\ln\left(\frac{4}{7}\right)}\ln\frac{15}{70}. \end{aligned}$$

(b) With $T = -15°\text{C}$ and $T_0 = 90°\text{C}$, we have

$$\begin{aligned} T &= -15 + (90 - (-15))\, e^{-kt} \\ &= -15 + 105e^{-kt}. \end{aligned}$$

So we find t when $T = 35°\text{C}$:

$$\begin{aligned} 35 &= -15 + 105\left(e^{\ln\frac{4}{7}}\right)^{\frac{t}{10}} \\ \Rightarrow 50 &= 105\left(\frac{4}{7}\right)^{\frac{t}{10}} \\ \Rightarrow \left(\frac{4}{7}\right)^{\frac{t}{10}} &= \frac{50}{105} \\ \Rightarrow \frac{t}{10}\ln\left(\frac{4}{7}\right) &= \ln\left(\frac{50}{105}\right) \\ \Rightarrow t &= 10\frac{1}{\ln\left(\frac{4}{7}\right)}\ln\left(\frac{50}{105}\right). \end{aligned}$$

6.4.3 Partial Derivatives/Chain Rule

Functions with more than one variable appear more often in science than functions with one variable. The reason for this is that in real life the mathematical application in respect of any dynamic phenomenon depends on more than one variable. One example of a dynamic phenomenon is a moving train, where the motion will depend on variables such as time, distance, friction, mass and so forth. The applied mathematician uses functions with several variables, their derivatives and their integrals to study continuum mechanics, probability, statistics, fluid mechanics and electricity – to mention but a few examples.

If f is a function of two independent variables, we usually call the independent variables x and y and picture the domain of $f(x, y)$ as a region in the XY-plane. If f is a function of three independent variables, we call the variables x, y, and z and picture the domain of $f(x, y, z)$ as a region in space.

Now, to talk about the derivative of f does not make sense if we do not specify with respect to which independent variable the derivative is required. When we keep all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a partial derivative. We would like to encourage you to work through examples 1–8 in Section 14.3 and to study Section 14.5 of Chapter 14 in Stewart carefully, in conjunction with the examples in the study guide.

(a) The partial derivatives technique

This technique is used when functions have more than one variable.

Consider a function f of two variables, x and y. Suppose we let only x vary, while keeping y fixed and $y = b$ where b is a constant.

Then this is a function of a single variable x : $g(x) = f(x, b)$.

If g has a derivative at a, this is called the partial derivative of f with respect to x at (a, b), denoted by
$f_x(a, b) = g'(x)$, that is $g'(a) = \lim\limits_{h \to 0} \dfrac{g(a+h) - g(a)}{h}$ or $f_x(a, b) = \lim\limits_{h \to 0} \dfrac{f(a+h, b) - f(a, b)}{h}$.

Similarly, the partial derivative of f with respect to y at (a, b), denoted by $f_y(a, b)$ is obtained by keeping x fixed (that is $x = a$) and finding the derivative at b of the function $G(y) = f(a, y)$.

So $f_x(a, b) = \lim\limits_{h \to 0} \dfrac{f(a+h, b) - f(a, b)}{h}$ and $f_y(a, b) = \lim\limits_{h \to 0} \dfrac{f(a, b+h) - f(a, b)}{h}$.

If we now let the point (a, b) vary, f_x and f_y become functions of two variables.

Definition
If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined as:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

(b) Notations for partial derivatives

If $z = f(x, y)$, we write:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Note: f_1 or $D_1 f$ indicates differentiation with respect to the first variable and f_2 or $D_2 f$ indicates differentiation with respect to the second variable.

(c) Rules for finding the partial derivatives of $\mathbf{z = f(x, y)}$

1. To find f_x, regard y as a constant and differentiate with respect to x.
2. To find f_y, regard x as a constant and differentiate with respect to y.

Examples:

21. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.
22. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$.

Solutions:

21. If $f(x, y) = x^3 + x^2y^3 - 2y^2$,

    then holding y constant and differentiating with respect to x, we get

$$\begin{aligned} f_x(x, y) &= 3x^2 + 2xy^3 - 0 \\ &= 3x^2 + 2xy^3 \end{aligned}$$

and

$$\begin{aligned} f_x(2, 1) &= 3(2)^2 + 2(2)\left(1^3\right) \\ &= 12 + 4 \\ &= 16. \end{aligned}$$

Holding x constant and differentiating with respect to y, we get

$$\begin{aligned} f_y(x, y) &= 0 + 3x^2y^2 - 4y \\ &= 3x^2y^2 - 4y \end{aligned}$$

and

$$\begin{aligned} f_y(2, 1) &= 3\left(2^2\right)\left(1^2\right) - 4(1) \\ &= 12 - 4 \\ &= 8. \end{aligned}$$

22. Given that $x^3 + y^3 + z^3 + 6xyz = 1$.

Then, holding y constant and differentiating implicitly with respect to x, we get

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0.$$

Solving this equation for $\frac{\partial z}{\partial x}$, we get

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{-3x^2-6yz}{3z^2+6xy} = \frac{-3\left(x^2+2yz\right)}{3\left(z^2+2xy\right)} \\ &= -\frac{x^2+2yz}{z^2+2yz}.\end{aligned}$$

Similarly, holding x constant and differentiating implicitly with respect to y gives

$$\begin{aligned}0+3y^2+3z^2\frac{\partial z}{\partial y}+6xz+6xy\frac{\partial z}{\partial y} &= 0 \\ \Rightarrow \quad \left(3z^2+6xy\right)\frac{\partial z}{\partial y} &= -3y^2-6xz \\ \Rightarrow \quad \frac{\partial z}{\partial y} &= \frac{-3y^2-6xz}{3z^2+6xy} \\ &= \frac{-3\left(y^2+2xz\right)}{3\left(z^2+2xy\right)} \\ &= -\frac{y^2+2xz}{z^2+2xy}.\end{aligned}$$

(d) Partial differentiation

If $z = f(x, y)$ where f is a differentiable function of x and y where $x = x(s, t)$ and $y = y(s, t)$, and both have partial derivatives, then the **chain rule** below holds:

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \text{ and} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.\end{aligned}$$

Example:

23. Suppose $f(x, y) = e^{xy},\ x(u, v) = 3u\sin v$ and $y(u, v) = 4v^2u$.

    For $g(u, v) = f(x(u, v), y(u, v))$, find the partial derivatives $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$.

Solution:

$f(x, y) = e^{xy},\ x(u, v) = 3u\sin v,\ y(u, v) = 4uv^2$ and

$$g(u, v) = f(x(u, v), y(u, v)).$$

Then

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

So

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{xy} \cdot y = ye^{xy}, \\
\frac{\partial f}{\partial y} &= e^{xy} \cdot x = xe^{xy}, \\
\frac{\partial x}{\partial u} &= 3\sin v, \\
\frac{\partial y}{\partial u} &= 4v^2, \\
\frac{\partial x}{\partial v} &= 3u\left(-\cos u\,(1)\right) = -3u\cos v \text{ and} \\
\frac{\partial y}{\partial v} &= 8uv.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial g}{\partial u} &= ye^{xy}\left(3\sin v\right) + xe^{xy}\left(4v^2\right) \\
&= 4uv^2 e^{3u\sin v\left(4uv^2\right)}\left(3\sin v\right) + 3u\sin v e^{3u\sin v\left(4uv^2\right)}\left(4v^2\right) \\
&= 12uv^2 e^{12u^2v^2\sin v}\sin v + 12uv^2 e^{12u^2u^2\sin v}\sin v \\
&= 24uv^2 e^{12u^2v^2\sin v}\sin v \text{ and}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g}{\partial v} &= ye^{xy}\left(3u\cos v\right) + xe^{xy}\left(8uv\right) = 3yue^{xy}\cos v + 8xuve^{xy} \\
&= 3\left(4uv^2\right)ue^{3u\sin v\left(4uv^2\right)}\cos v + 8\left(3u\sin v\right)uve^{3u\sin v\left(4uv^2\right)} \\
&= 12u^2v^2 e^{12u^2v^2\sin v}\cos v + 24u^2 v e^{12u^2v^2\sin v}\sin v.
\end{aligned}$$

(e) Functions of more than two variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables, x, y and z, then its partial derivative with respect to x is defined as:

$$f_x\left(x, y, z\right) = \lim_{h\to 0}\frac{f\left(x+h, y, z\right) - f\left(x, y, z\right)}{h}$$

and is found by, for example, regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x.

Example:

24. Find $f_x,\ f_y$ and f_z if $f\left(x, y, z\right) = e^{xy}\ln z$.

Solution:

If $f(x,y,z) = e^{xy} \ln z$,

then to compute the partial derivative with respect to x, we treat y and z as constants and obtain

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(e^{xy} \ln z\right) \\ &= e^{xy}(y) \ln z \\ &= ye^{xy} \ln z. \end{aligned}$$

To compute the partial derivative with respect to y, we treat x and z as constants and obtain

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}\left(e^{xy} \ln z\right) \\ &= e^{xy}(x) \ln z \\ &= xe^{xy} \ln z. \end{aligned}$$

To compute the partial derivative with respect to z, we treat x and y as constants and obtain

$$\begin{aligned} f_z &= \frac{\partial}{\partial z}\left(e^{xy} \ln z\right) \\ &= e^{xy} \cdot \frac{1}{z}(1) \\ &= \frac{e^{xy}}{z}. \end{aligned}$$

(f) Higher-order derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$ and $(f_y)_y$, which are called the second partial derivatives of f.

If $z = f(x,y)$, we use the following notation:

$$\begin{aligned} (f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}, \\ (f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}, \\ (f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \text{ and} \\ (f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}. \end{aligned}$$

Note: The notation f_{xy} or $\left(\dfrac{\partial^2 f}{\partial y \partial x}\right)$ means we first differentiate with respect to x and then with respect to y, and in computing f_{yx}, the order of differentiation is reversed.

Example:

25. Find the second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$.

Solution:

Given $f(x, y) = x^3 + x^2y^3 - 2y$.

In example 21 we found that

$f_x(x, y) = 3x^2 + 2xy^3$ and

$f_y(x, y) = 3x^2y^2 - 4y$.

Therefore

$f_{xx}(x, y) = \dfrac{\partial}{\partial x}\left(3x^2 + 2xy^3\right) = 6x + 2y^3$,

$f_{xy}(x, y) = \dfrac{\partial}{\partial y}\left(3x^2 + 2xy^3\right) = 0 + 6xy^2 = 6xy^2$,

$f_{yx}(x, y) = \dfrac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2 - 0 = 6xy^2$ and

$f_{yy}(x, y) = \dfrac{\partial}{\partial y}\left(3x^2y^2 - 4y\right) = 6x^2y - 4$.

Now work through the rest of the examples provided.

Worked Examples:

26. Determine $\partial f/\partial x$ and $\partial f/\partial y$ if:

    (a) $f(x, y) = \sin(x + y)$ and

    (b) $f(x, y) = \dfrac{x}{x^2 + y^2}$.

27. Use the chain rule for partial derivatives to determine the value of dw/dt when $t = 0$ if $w = x^2 + y^2$, $x = \cos t + \sin t$ and $y = \cos t - \sin t$.

28. Use the chain rule for partial derivatives to find dw/dt when $w = -\sin(xy)$,

$$x = 1 + t \quad \text{and} \quad y = t^2.$$

29. Suppose

$$w = \frac{1}{xy} + \frac{1}{y}, \quad x = t^2 + 1 \text{ and } y = t^4 + 3.$$

Use the chain rule for partial derivatives to find dw/dt.

30. Assume that the equation

$$yx^{\frac{4}{3}} + y^{\frac{4}{3}} = 144$$

defines y as a differentiable function of x. Use partial derivatives to find the value of dy/dx at the point $(-8, 8)$.

31. Determine $\partial z/\partial u$ when $u = 0$ and $v = 1$, if $z = \sin(xy) + x \sin y$, $x = u^2 + v^2$ and $y = uv$.

32. Let g be a function of x and y, and suppose that x and y are both functions of u and v. Which one of the following is $\partial g/\partial v$?

(a) $\dfrac{\partial g}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial g}{\partial u} \cdot \dfrac{\partial u}{\partial y}$

(b) $\dfrac{dg}{dx} \cdot \dfrac{\partial x}{\partial v} + \dfrac{dg}{dy} \cdot \dfrac{\partial y}{\partial v}$

(c) $\dfrac{\partial g}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial g}{\partial y} \cdot \dfrac{\partial y}{\partial v}$

(d) $\dfrac{dg}{dx} \cdot \dfrac{dx}{dv} + \dfrac{dg}{dy} \cdot \dfrac{dy}{dv}$

33. Determine $\partial^2 f/\partial y^2$ if $f(x, y) = \tan(xy) - \cot x$.

34. Suppose that the equation

$$y = \sin(xy)$$

defines y as a differentiable function of x. Use partial derivatives to find dy/dx.

35. Suppose $w = (-2x^2 + 2y^2 - 2)^2$, $x = 2u - 2v + 2$ and $y = -u + 4v + 2$. Find the value of $\partial w/\partial v$ when $u = 0$ and $v = -1$.

36. Use the chain rule for partial derivatives to find dw/dt when $w = -\sin(xy)$, $x = 1+t$ and $y = t^2$.

37. Assume that the equation

$$xy + y^2 - 3x - 3 = 0$$

defines y as a differentiable function of x. Find the value of dy/dx at the point $(-1, 1)$ by using partial derivatives.

38. Use the chain rule for partial derivatives to find $\partial w/\partial r$ when $r = \pi$ and $s = 0$, if $w = \sin(2x - y)$, $x = r + \sin s$ and $y = rs$.

39. Suppose that the equation

$$7y^4 + x^3 y + x = 4$$

defines y as a differentiable function of x. Find the value of dy/dx at the point $(4, 0)$.

40. The dimensions a, b and c of a rectangular solid vary with time (t). At the instant in question

$$a = 13 \text{ cm},\ b = 9 \text{ cm},\ c = 5 \text{ cm},\ \frac{da}{dt} = \frac{dc}{dt} = 2 \text{ cm/sec}$$

and

$$\frac{db}{dt} = -5 \text{ cm/sec}.$$

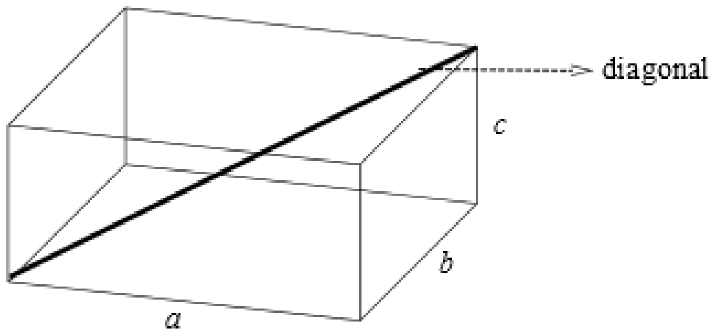


Figure 6.2: Rectangular solid

(a) How fast is the volume V changing at the given instant? Is V increasing or decreasing?

(b) How fast is the surface area S changing at that moment? Is the area increasing or decreasing?

(c) How fast is the length D of the diagonal changing at that instant? Does the length increase or decrease?

Solutions:

26. (a)

$$\begin{aligned} f(x,y) &= \sin(x+y) \\ \Rightarrow \frac{\partial f}{\partial x} &= \cos(x+y) \quad \text{and} \quad \frac{\partial f}{\partial y} = \cos(x+y). \end{aligned}$$

(b)

$$\begin{aligned} f(x,y) &= \frac{x}{x^2+y^2} \\ \Rightarrow \frac{\partial f}{\partial x} &= \frac{1.\left(x^2+y^2\right) - x\,(2x+0)}{\left(x^2+y^2\right)^2} \\ &= \frac{y^2-x^2}{\left(x^2+y^2\right)^2}. \end{aligned}$$

Also,

$$\begin{aligned} f(x,y) &= \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1} \\ \Rightarrow \frac{\partial f}{\partial y} &= x(-(x^2+y^2)^{-2}).(0+2y) \\ &= -\frac{2xy}{(x^2+y^2)^2}. \end{aligned}$$

27.

$$\begin{aligned} w &= x^2+y^2,\ x=\cos t+\sin\ t \text{ and } y=\cos t-\sin\ t \\ \Rightarrow \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= (2x+0)(-\sin\ t+\cos\ t)+(0+2y)(-\sin\ t-\cos\ t) \\ &= 2x\,(\cos t-\sin t)-2y\,(\sin t+\cos t) \\ & \quad \text{by substituting for } x \text{ and } y \\ &= 2(\cos\ t+\sin\ t)(\cos\ t-\sin\ t)-2(\cos\ t-\sin\ t)(\cos\ t+\sin\ t) \\ &= 0. \end{aligned}$$

Hence,

$$\left.\frac{dw}{dt}\right|_{t=0} = 0.$$

28.

$$w = -\sin(xy)$$

$$\Rightarrow \quad \frac{\partial w}{\partial x} = -\cos(xy)\cdot y$$

and

$$\frac{\partial w}{\partial y} = -\cos(xy)\cdot x.$$

Also,

$$\begin{aligned} x &= 1+t \\ \Rightarrow \quad \frac{dx}{dt} &= 1, \end{aligned}$$

and

$$\begin{aligned} y &= t^2 \\ \Rightarrow \quad \frac{dy}{dt} &= 2t. \end{aligned}$$

We have

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt} \\ &= -\cos(xy)\cdot y\,(1)-\cos(xy)\cdot x\cdot 2t \\ &= -\cos((1+y)\,(+2))\cdot y^2-\cos\left((1+t)\,t^2\right)(1+t)\cdot 2t. \end{aligned}$$

29.

$$\begin{aligned} w &= \frac{1}{xy} + \frac{1}{y} \\ \Rightarrow \frac{\partial w}{\partial x} &= -\frac{1}{yx^2} \end{aligned}$$

and

$$\frac{\partial w}{\partial y} = \frac{-1}{xy^2} - \frac{1}{y^2}.$$

Also,

$$\begin{aligned} x &= t^2 + 1 \\ \Rightarrow \frac{dx}{dt} &= 2t \end{aligned}$$

and

$$\begin{aligned} y &= t^4 + 3 \\ \Rightarrow \frac{dy}{dt} &= 4t^3. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \\ &= -\frac{1}{yx^2} \cdot 2t + \left(-\frac{1}{xy^2} - \frac{1}{y^2} \right) \cdot 4t^3 \\ &= -\frac{2t}{(t^4+3)(t^2+1)^2} - \frac{4t^3}{(t^2+1)(t^4+3)^2} - \frac{4t^3}{(t^4+3)^2}. \end{aligned}$$

30. Let

$$F(x, y) = yx^{\frac{4}{3}} + y^{\frac{4}{3}} - 144.$$

Then

$$\begin{aligned} \frac{dy}{dx} &= -\frac{F_x}{F_y} \\ &= -\frac{\frac{4}{3} yx^{\frac{1}{3}}}{x^{4/3} + \frac{4}{3} y^{\frac{1}{3}}}, \end{aligned}$$

so

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{(x,y)=(-8,8)} &= -\frac{\frac{4}{3} \cdot 8 \cdot (-2)}{16 + \frac{4}{3} \cdot 2} \\ &= \frac{8}{7}. \end{aligned}$$

31.

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ &= (y\cos(xy)+\sin y)2u+(x\cos(xy)+x\cos\ y)v.\end{aligned}$$

If $u=0$ and $v=1$, then $x=u^2+v^2=1$ and $y=uv=0$. Then

$$\begin{aligned}\frac{\partial z}{\partial u} &= (0\cdot\cos 0+\sin 0)\cdot 0+(1\cdot\cos 0+1\cdot\cos 0)\cdot 1\\ &= 2.\end{aligned}$$

32.

$$\frac{\partial g}{\partial v}=\frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

33.

$$\begin{aligned}f(x,y) &= \tan(xy)-\cot\ x\\ \Rightarrow \frac{\partial f}{\partial y} &= \sec^2(xy).x\\ \Rightarrow \frac{\partial^2 f}{\partial y^2} &= 2\sec(xy).\sec(xy)\tan(xy).x.x\\ &= 2x^2.\sec^2(xy).\tan(xy).\end{aligned}$$

34.

$$\begin{aligned}y &= \sin(xy)\\ \Rightarrow y-\sin(xy) &= 0.\end{aligned}$$

Let

$$F(x,y)=y-\sin(xy).$$

Then

$$\begin{aligned}\frac{dy}{dx} &= -\frac{F_x}{F_y}\\ &= -\frac{-\cos(xy)\cdot y}{1-\cos(xy)\cdot x}\\ &= \frac{y\cos(xy)}{1-x\cos(xy)}.\end{aligned}$$

35.

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}\\ &= 2(-2x^2+2y^2-2)(-4x)(-2)+2(-2x^2+2y^2-2)(4y)(4).\end{aligned}$$

If $u=0$ and $v=-1$, then

$$x=0+2+2=4$$

and

$$y = 0 - 4 + 2 = -2$$

so that

$$\begin{aligned} \frac{\partial w}{\partial v} &= 2\left(-32+8-2\right)\left(-16\right)\left(-2\right)+2\left(-32+8-2\right)\left(-8\right)\left(4\right) \\ &= 0. \end{aligned}$$

36.

$$\begin{aligned} w &= -\sin(xy) \\ \Rightarrow \frac{\partial w}{\partial x} &= -\cos(xy).y \end{aligned}$$

and

$$\frac{\partial w}{\partial y} = -\cos(xy).x.$$

Also,

$$\begin{aligned} x &= 1+t \\ \Rightarrow \frac{dx}{dt} &= 1 \end{aligned}$$

and

$$\begin{aligned} y &= t^2 \\ \Rightarrow \frac{dy}{dt} &= 2t. \end{aligned}$$

We have

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt} \\ &= -\cos(xy)\cdot y\cdot 1-\cos(xy)\cdot x\cdot 2t \\ &= -\cos((1+t)t^2)\cdot t^2-\cos((1+t)t^2)\cdot(1+t)2t. \end{aligned}$$

37. Put

$$F(x,y) = xy + y^2 - 3x - 3.$$

Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}.$$

Hence,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-1,1)} = -\frac{1-3}{-1+2} = 2.$$

38.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} \\ &= 2\cdot\cos(2x-y)\cdot 1+(-\cos(2x-y))\cdot s.\end{aligned}$$

If $r=\pi$ and $s=0$, then $x=\pi$ and $y=0$, so that

$$\frac{\partial w}{\partial r}=2\cdot\cos(2\pi)\cdot 1-\cos(2\pi).0=2.$$

39. Let

$$F(x,y)=7y^4+x^3y+x-4.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= -\frac{F_x}{F_y} \\ &= -\frac{3x^2y+1}{28y^3+x^3}.\end{aligned}$$

Hence

$$\left.\frac{dy}{dx}\right|_{(x,y)=(4,0)} = -\frac{1}{64}.$$

40. (a) Denote derivatives with respect to time by a dot. If

$$V=abc,$$

then

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial a}\dot{a}+\frac{\partial V}{\partial b}\dot{b}+\frac{\partial V}{\partial c}\dot{c} \\ &= \dot{a}bc+a\dot{b}c+ab\dot{c}.\end{aligned}$$

Substitute the given values:

$$\begin{aligned}\dot{V} &= (2)(9)(5)+(13)(-5)(5)+(13)(9)(2) \\ &= 90-325+234 \\ &= -1.\end{aligned}$$

Therefore the volume decreases at 1 cm^3/sec.

(b) Differentiate with respect to time. If

$$S=2ac+2bc+2ab,$$

then

$$\begin{aligned}\dot{S} &= \frac{\partial S}{\partial a}\dot{a} + \frac{\partial S}{\partial b}\dot{b} + \frac{\partial S}{\partial c}\dot{c} \\ &= (2c+2b)\dot{a} + (2c+2a)\dot{b} + (2a+2b)\dot{c}.\end{aligned}$$

Substitute the given values:

$$\begin{aligned}\dot{S} &= (2\times 5 + 2\times 9)(2) + (2\times 5 + 2\times 13)(-5) + (2\times 13 + 2\times 9)(2) \\ &= -36.\end{aligned}$$

The area decreases at a rate of 36 cm^2/sec.

(c) According to the Theorem of Pythagoras

$$D = (a^2 + b^2 + c^2)^{\frac{1}{2}}.$$

Differentiate with respect to time:

$$\begin{aligned}\dot{D} &= \frac{1}{2}(a^2+b^2+c^2)^{-\frac{1}{2}}(2a\dot{a} + 2b\dot{b} + 2c\dot{c}) \\ &= \frac{a\dot{a} + b\dot{b} + c\dot{c}}{\sqrt{a^2+b^2+c^2}}. \\ \text{Thus}\quad \dot{D}(13,\ 9,\ 5) &= \frac{(13)(2) + (9)(-5) + (5)(2)}{\sqrt{(13)^2 + (9)^2 + (5)^2}} \quad \text{(from the given values)} \\ &= \frac{26 - 45 + 10}{\sqrt{169+81+25}} \\ &= \frac{-9}{\sqrt{275}} \\ &= \frac{-9}{5\sqrt{11}}.\end{aligned}$$

Therefore the diagonal decreases at $\frac{9}{5\sqrt{11}}$cm/sec.

Key points

In this chapter we have introduced you to specific solutions of simple, first-order differential equations. You should now be comfortable with solving some basic growth and decay problems. We have also introduced the notion of partial derivatives, where we have seen how to approach scientific problems involving functions of several variables, by changing one of the variables at a time, and keeping the remaining variable(s) fixed.

At this stage you should be able to:

- solve first-order differential equations with initial values

- solve basic real-life problems involving exponential growth and decay
- determine the partial derivatives of the functions of several variables

Continue practising solving problems until you have mastered the basic techniques! Go through the section "For your review" at the end of each chapter to consolidate what you have learnt and also use other calculus textbooks.

Appendix A

Sequence and Summation Notation

The purpose of the Appendix is to help you to make the transition from high school to the more advanced first-year calculus module MAT1512.

A sequence is a list of numbers written in a **specific order**. Sequences have many applications.

To read: Stewart Appendix E A34–A37

Outcomes:

After studying this section you should

(a) know the definition of a sequence

(b) know what is meant by

- a recursive sequence
- a partial sum of a sequence
- sigma notation

(c) be able to

- find the terms of a sequence
- find the n^{th} term of a sequence
- find a partial sum of a sequence

- use sigma notation

Examples:

A. 1.1 Write the first four terms of the sequence given by the formula in each case.

(a) $c_j = 3\left(\frac{1}{10}\right)^{j-1}$

Solution:

$$\begin{aligned} c_1 &= 3\left(\frac{1}{10}\right)^{1-1} = 3\left(\frac{1}{10}\right)^{0} = 3, \\ c_2 &= 3\left(\frac{1}{10}\right)^{2-1} = 3\left(\frac{1}{10}\right)^{1} = \frac{3}{10}, \\ c_3 &= 3\left(\frac{1}{10}\right)^{3-1} = 3\left(\frac{1}{10}\right)^{2} = \frac{3}{100} \\ c_4 &= 3\left(\frac{1}{10}\right)^{4-1} = 3\left(\frac{1}{10}\right)^{3} = \frac{3}{1000}. \end{aligned}$$

(b) $a_n = \frac{(-1)^{n+1}}{n+3}$

Solution:

$$\begin{aligned} a_1 &= \frac{(-1)^{1+1}}{1+3} = \frac{1}{4}, \quad a_2 = \frac{(-1)^{2+1}}{2+3} = -\frac{1}{5} \\ a_3 &= \frac{(-1)^{3+1}}{3+3} = \frac{1}{6}, \quad a_4 = \frac{(-1)^{4+1}}{4+3} = -\frac{1}{7}. \end{aligned}$$

(c) $x_k = \frac{k}{k+1} - \frac{k+1}{k}$

Solution:

$$\begin{aligned} x_1 &= \frac{1}{1+1} - \frac{1+1}{1} = \frac{1}{2} - 2 = -\frac{3}{2}, \\ x_2 &= \frac{2}{2+1} - \frac{2+1}{2} = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}, \\ x_3 &= \frac{3}{3+1} - \frac{3+1}{3} = \frac{3}{4} - \frac{4}{3} = -\frac{7}{12}, \\ x_4 &= \frac{4}{4+1} - \frac{4+1}{4} = \frac{4}{5} - \frac{5}{4} = -\frac{9}{20}. \end{aligned}$$

A. 1.2 Find the ninth and tenth terms of: $0,\ 4,\ 0, \ldots, \dfrac{2^n + (-2)^n}{n}, \ldots$

Solution:

$$
\begin{aligned}
a_n &= \frac{2^n + (-2)^n}{n} \\
\text{Then } \ a_9 &= \frac{2^9 + (-2)^9}{9} = \frac{2^9 - 2^9}{9} = \frac{0}{9} = 0 \text{ and} \\
a_{10} &= \frac{2^{10} + (-2)^{10}}{10} = \frac{2^{10} + 2^{10}}{10} = \frac{2 \cdot 2^{10}}{10} = \frac{2048}{10} = \frac{1024}{5}.
\end{aligned}
$$

A. 1.3 Find the first six terms of the sequence defined by: $a_1 = 6$ and $a_n = \dfrac{3}{a_{n-1}}$.

Solution:

$$
\begin{aligned}
a_1 &= 6, \quad a_2 = \frac{3}{a_{2-1}} = \frac{3}{6} = \frac{1}{2}, \\
a_3 &= \frac{3}{a_{3-1}} = \frac{3}{a_2} = \frac{3}{\left(\frac{1}{2}\right)} = 6, \\
a_4 &= \frac{3}{a_{4-1}} = \frac{3}{a_3} = \frac{3}{6} = \frac{1}{2}, \\
a_5 &= \frac{3}{a_{5-1}} = \frac{3}{a_4} = \frac{3}{\left(\frac{1}{2}\right)} = 6, \\
a_6 &= \frac{3}{a_{6-1}} = \frac{3}{a_5} = \frac{3}{6} = \frac{1}{2}.
\end{aligned}
$$

A. 1.4 Find the sum of the first five terms of the sequence given by the formula in each case.

(a) $a_k = (-1)^k \dfrac{1}{k}$

Solution:

$$
\begin{aligned}
\sum_{k=1}^{5} a_k &= a_1 + a_2 + a_3 + a_4 + a_5 \\
&= (-1)^1 \frac{1}{1} + (-1)^2 \frac{1}{2} + (-1)^3 \frac{1}{3} + (-1)^4 \frac{1}{4} + (-1)^5 \frac{1}{5} \\
&= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \\
&= -\frac{47}{60}.
\end{aligned}
$$

(b) $b_i = i^3$

Solution:

$$\begin{aligned}\sum_{i=1}^{5} i^3 &= b_1 + b_2 + b_3 + b_4 + b_5 \\ &= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \\ &= 1 + 8 + 27 + 64 + 125 \\ &= 225.\end{aligned}$$

(c) Find $\sum\limits_{n=0}^{8} x_n$ where $x_n = \dfrac{1}{2^n}$.

Solution:

$$\begin{aligned}\sum_{n=0}^{8} x_n &= x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \\ &= \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} \\ &= \frac{511}{256}.\end{aligned}$$

A. 1.5 Evaluate each of the following:

(a) $\sum\limits_{k=1}^{6} (5k)$

Solution:

$$\begin{aligned}\sum_{k=1}^{6} (5k) &= 5.1 + 5.2 + 5.3 + 5.4 + 5.5 + 5.6 \\ &= 5 + 10 + 15 + 20 + 25 + 30 \\ &= 105.\end{aligned}$$

(b) $\sum\limits_{k=1}^{7} (-1)^k$

Solution:

$$\begin{aligned}\sum_{k=1}^{7}(-1)^k &= (-1)^1+(-1)^2+(-1)^3+(-1)^4+(-1)^5+(-1)^6+(-1)^7\\ &= -1+1-1+1-1+1-1\\ &= -1.\end{aligned}$$

(c) $\sum_{n=1}^{3}\left(\frac{n+1}{n}-\frac{n}{n+1}\right)$

Solution:

$$\begin{aligned}\sum_{n=1}^{3}\left(\frac{n+1}{n}-\frac{n}{n+1}\right) &= \left(\frac{1+1}{1}-\frac{1}{1+1}\right)+\left(\frac{2+1}{2}-\frac{2}{2+1}\right)+\left(\frac{3+1}{3}-\frac{3}{3+1}\right)\\ &= 2-\frac{1}{2}+\frac{3}{2}-\frac{2}{3}+\frac{4}{3}-\frac{3}{4}\\ &= \frac{35}{12}.\end{aligned}$$

A. 1.6 Rewrite each series using sigma notation.

(a) $4+8+12+16+20+24$

Solution:

$$\begin{aligned}4+8+12+16+20+24 &= 4\cdot 1+4\cdot 2+4\cdot 3+4\cdot 4+4\cdot 5+4\cdot 6\\ &= \sum_{k=1}^{6}(4k).\end{aligned}$$

(b) $-4-2+0+2+4+6+8$

Solution:

$$\begin{aligned}-4-2+0+2+4+6+8 &= 2(-2)+2(-1)+2\cdot 0+2\cdot 1+2\cdot 2+2\cdot 3+2\cdot 4\\ &= \sum_{k=-2}^{4}(2k).\end{aligned}$$

Appendix B

Mathematical Induction

We consider here a special kind of proof method called <u>mathematical induction</u>.

To read: Stewart pages 72, 74 and Appendix E A36.

Outcomes:

After studying this section you should

(a) know what the principle of mathematical induction is

(b) be able to prove using mathematical induction that a statement $P(n)$ is true for all natural numbers n

Examples:

B. 2.1 Use mathematical induction to prove the following statement for all positive integers n :

$$1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2}. \tag{B.1}$$

Solution:

Let $P(n)$ be the statement

$$1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2} \tag{B.2}$$

for any positive integer n.

We first show that $P(1)$ is true.

$$\begin{aligned} \text{LHS of } (B.2) &= 3 \cdot 1 - 2 \\ &= 1. \end{aligned}$$

$$\begin{aligned} \text{RHS of } (B.2) &= \frac{1(3 \cdot 1 - 1)}{2} \\ &= \frac{2}{2} \\ &= 1. \end{aligned}$$

Thus $P(1)$ is true.

Suppose that $P(k)$ is true for any k, that is, we assume

$$1 + 4 + 7 + \ldots + (3k - 2) = \frac{k(3k-1)}{2}.$$

We want to prove that $P(k+1)$ is true, in other words:

$$\begin{aligned} & 1 + 4 + 7 + \ldots + (3k-2) + (3(k+1) - 2) \\ = & \frac{(k+1)(3(k+1)-1)}{2} \\ = & \frac{(k+1)(3k+2)}{2}. \end{aligned}$$

Now

$$\begin{aligned} & 1 + 4 + 7 + \ldots + (3k-2) + (3(k+1) - 2) \\ = & \frac{k(3k-1)}{2} + (3k+1) \\ = & \frac{1}{2}\left[k(3k-1) + 2(3k+1)\right] \\ = & \frac{1}{2}\left(3k^2 - k + 6k + 2\right) \\ = & \frac{1}{2}(3k^2 + 5k + 2) \\ = & \frac{(k+1)(3k+2)}{2}. \end{aligned}$$

Since both conditions of the principle of mathematical induction have been satisfied, it follows that

$$1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2}$$

is true for all integers $n \geq 1$.

B. 2.2 Prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3} \text{ for all } n \in \mathbb{N}.$$

Solution:

We must verify that the equation is true for $n = 1$.

The LHS is $1(1+1) = 2$.

The RHS is $\dfrac{1(1+1)(1+2)}{3} = 2$.

So the equation is true for $n = 1$.

Suppose that the equation is true for $n = k$, that is

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3} \qquad (*)$$

We now use the assumption $(*)$ to deduce that the equation is true for $n = k+1$, in other words:

$$\begin{aligned} & 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) + (k+1)\left[(k+1)+1\right] \\ = \quad & \frac{(k+1)\left[(k+1)+1\right]\left[(k+1)+2\right]}{3}. \end{aligned}$$

Now,

$$\begin{aligned}
\text{LHS} &= 1\cdot 2+2\cdot 3+3\cdot 4+\ldots+k(k+1)+(k+1)\left[(k+1)+1\right]\\
&= \frac{k(k+1)(k+2)}{3}+(k+1)\left[(k+1)+1\right]\\
&= (k+1)\left[\frac{k(k+2)}{3}+(k+1)+1\right]\\
&= (k+1)\left(\frac{k^2+2k+3k+6}{3}\right)\\
&= (k+1)\left(\frac{k^2+5k+6}{3}\right)\\
&= \frac{(k+1)(k+2)(k+3)}{3}\\
&= \frac{(k+1)\left[(k+1)+1\right]\left[(k+1)+2\right]}{3}\\
&= \text{RHS.}
\end{aligned}$$

Since the equation is true for $n = 1$, and if it is true for $n = k$, then it is true for $n = k+1$, we conclude by mathematical induction that the equation is true for all integers $n \geq 1$.

B. 2.3 Prove that

$$3+3^2+3^3+\ldots+3^n = \frac{3^{n+1}-3}{2} \quad \text{for all integers } n \geq 1.$$

Solution:

We must verify that the equation is true for $n = 1$.

The LHS is $3^1 = 3$.

The RHS is $\dfrac{3^{1+1}-3}{2} = 3$

So the equation is true for $n = 1$.

Suppose that the equation is true for $n = k$, that is

$$3+3^2+3^3+\ldots 3^k = \frac{3^{k+1}-3}{2} \tag{B.3}$$

We now use the assumption $(*)$ to deduce that the equation is true for $n = k+1$, in other words:

$$3+3^2+3^3+\ldots+3^k+3^{k+1} = \frac{3^{(k+1)+1}-3}{2}.$$

Now,

$$\begin{aligned}
\text{LHS} &= 3 + 3^2 + 3^3 + \ldots + 3^k + 3^{k+1} \\
&= \frac{3^{k+1} - 3}{2} + 3^{k+1} \\
&= \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2} \\
&= \frac{3^{k+1}(1+2) - 3}{2} \\
&= \frac{3^{k+1} \cdot 3 - 3}{2} \\
&= \frac{3^{(k+1)+1} - 3}{2} \\
&= \text{RHS.}
\end{aligned}$$

Since the equation is true for $n = 1$, and if it is true for $n = k$, then it is true for $n = k+1$, we conclude by mathematical induction that the equation is true for all integers $n \geq 1$.

B. 2.4 Use mathematical induction to prove that

$$\left(\frac{3}{4}\right)^n < \frac{3}{4} \text{ for } n \geq 2.$$

Solution:

Our starting point is $n = 2$, so we first verify that the inequality is true for $n = 2$.

LHS is $\left(\frac{3}{4}\right)^2 = \frac{3}{4} \cdot \frac{3}{4} < \frac{3}{4} = \text{RHS}$ since $0 < \frac{3}{4} < 1$.

So the inequality is true for $n = 2$.

Suppose that the inequality is true for $n = k$, that is

$$\left(\frac{3}{4}\right)^k < \frac{3}{4} \tag{B.4}$$

We now use the assumption $(B.4)$ to deduce that the inequality is true for $n = k+1$, in other words

$$\left(\frac{3}{4}\right)^{k+1} < \frac{3}{4}.$$

Now,

$$\begin{aligned}
\text{LHS} &= \left(\frac{3}{4}\right)^{k+1} \\
&= \left(\frac{3}{4}\right)^{k} \cdot \frac{3}{4} \\
&< \frac{3}{4} \cdot \frac{3}{4} && \text{by } (B.4) \\
&< \frac{3}{4} && \text{by the case } n = 2 \\
&= \text{RHS.}
\end{aligned}$$

Therefore if the inequality is true for $n = k$, it is true for $n = k + 1$. By mathematical induction we conclude that it is true for all positive integers $n \geq 2$.