Department of Mathematical Sciences
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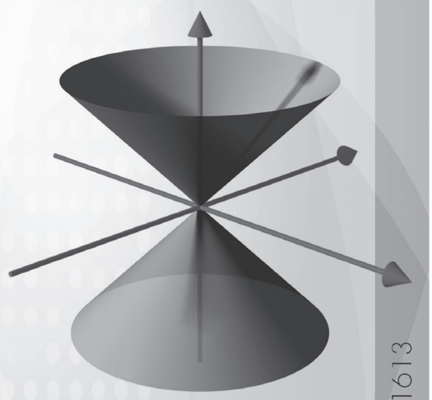

Calculus B

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Contents

About the module

Calculus is the mathematics of *motion and change.* Where there is motion or growth, where variable forces are at work producing acceleration, calculus is the right mathematics to apply. *Differential calculus* deals with the problem of calculating rates of change. *Integral calculus* deals with the problem of determining a function from information about its *rate of change.* It enables people to calculate the future location of a body from its present position and includes the *knowledge of the forces* acting on it, to find the *areas of irregular regions* in the plane, to *measure the lengths* of curves and to *locate the centres of mass of arbitrary solids.*

Today, calculus and its extensions in mathematical analysis are far reaching indeed, and the physicists, mathematicians and astronomers who first invented the subject would surely be amazed and delighted to see what a profusion of problems it solves and what a wide range of fields now use it.

How to study for this module:

- **Prescribed book**

These notes were written in addition to the prescibed textbook.

- **Referenced notes**

References to these notes will be followed by an (S), for instance p. 40(S), means p. 40 of the notes.

The notes must always be used in *conjunction with the prescribed book.* The notes provides extra explanations and examples to some of the prescribed study material.

- **Content**

In the notes we discuss the following two topics namely: Applications of the derivative (Chapters 1, 2 and 6) and Integration methods (applied to specific problems) (Chapters 3, 4 and 5).

- **Exercises**

At the end of each section in your textbook or in the notes some exercises on that section are given. Each time when you finish a section, it is very important that you try to do as many of these exercises as possible. Remember that very little theory is required of you in this module, i.e. you do not need to study many theorems and their proofs. So, please, do as many exercises as possible!!

- **Pocket calculators**

Pocket calculators are not allowed during the examinations. Therefore we expect that you will not use one when doing your assignments. To get used to working without a pocket calculator never use it when working on this module. You need not do any of the exercises which indicate that they are calculator based.

Prescribed Study Material for this module

The sections you must study in the prescribed textbook in conjunction with the notes in the study guide are below.

Topic 1

1.1 Exponential and logarithmic functions and their derivatives and integrals.
1.2 The hyperbolic functions and their derivatives and integrals.
1.3 The inverse trigonometric functions, their definition sets, derivatives and integrals.

Topic 2

2.1 Limits involving infinity.
2.2 Indeterminate forms and L'Hôpital's Rule.

Topic 3

3.1 The mean value theorem and Rolle's theorem.

Topic 4

The sketching of graphs using the following:
4.1 Asymptotes.
4.2 The first derivative to find maximum and minimum values.
4.3 The sign pattern of the first derivative to determine where functions increase and decrease.
4.4 The second derivative and its sign pattern to find the concavity of a function.
4.5 The inflection point(s) of a function (if they exist).
4.6 The second derivative test.

Topic 5

5.1 Optimization.
5.2 Related rates.
5.3 Rates of change in economics and sciences.

Topic 6

6.1 Area between curves.
6.2 Solids of revolution: Finding volumes by using slicing, disks and washers.

Topic 7

Integration
7.1 Review of basic formulas and techniques.
7.2 Integration by substitution.
7.3 Integration by parts.
7.4 Trigonometric substitution and integration using trigonometric identities.
7.5 Integration of rational functions using partial fractions.
7.6 Integration using z-substitution.
7.7 Identification and computing of improper integrals.
7.8 Using the limit comparison test to decide if an integral converges or diverges.

Topic 8

8.1 Finding the limit of a sequence.
8.2 Constructing Taylor polynomials for certain functions. (The remainder is not required.)

Outcomes

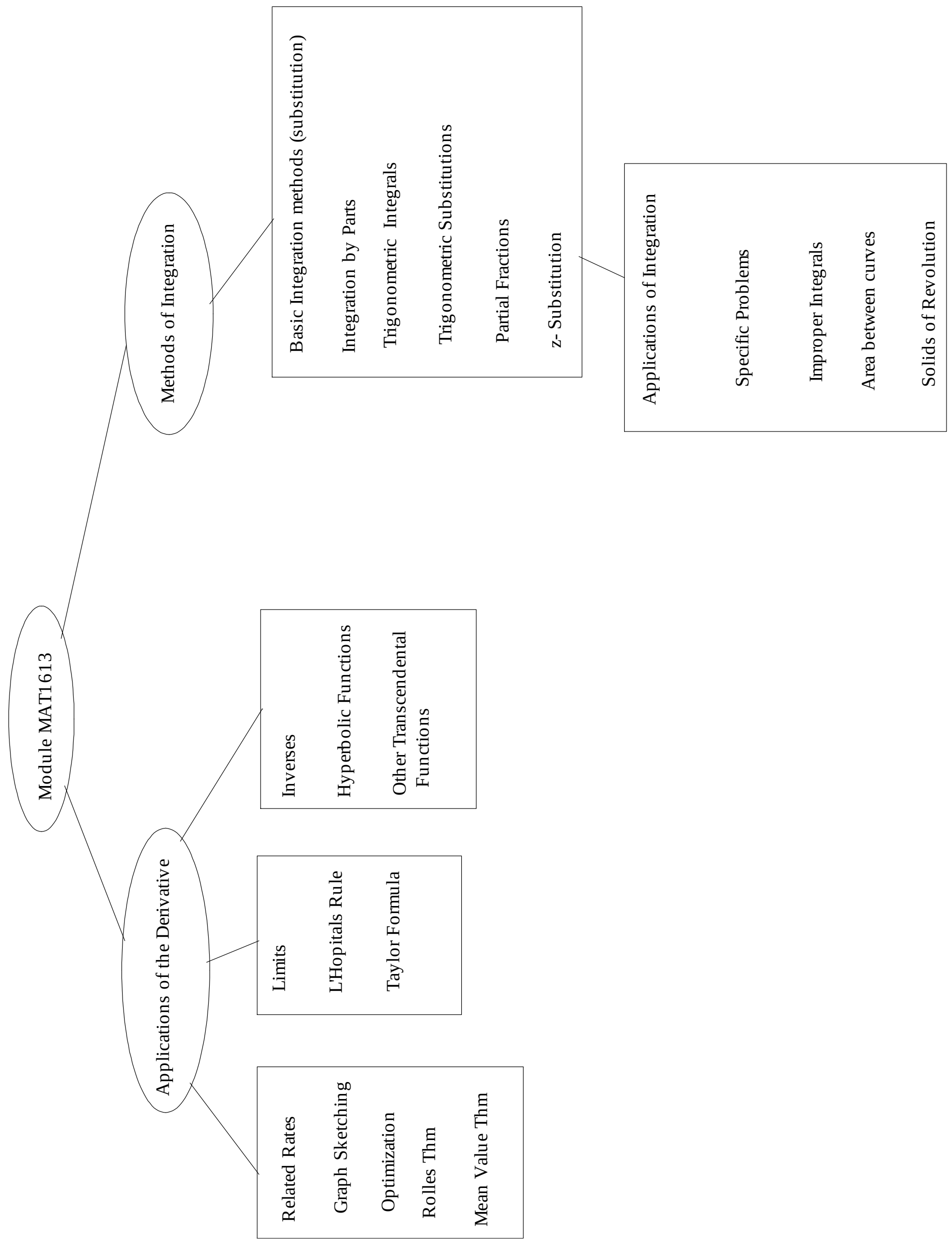
After studying this module, the learner should have acquired skills in

- solving related rates problems, graph sketching (using the derivative) and solving optimization problems.
- solving basic problems using the Mean value theorem and/or Rolle's theorem.
- using L'Hôpital's Rule to solve indeterminate forms.
- obtaining the inverse trigonometric functions, their derivatives and related integrals.
- finding the derivatives and integrals of hyperbolic functions.
- evaluating areas between curves.
- evaluating volumes of solids by revolving plane regions about the co–ordinate axes.

- applying more advanced techniques of integration.
- selecting a suitable method of integration to solve a problem.
- evaluating improper integrals.
- testing for convergence or divergence of improper integrals.
- determining the limit of a sequence if it exists.
- constructing the Taylor polynomial of a certain given function.

We have provided a mind map so that you can see how the topics that we will discuss in this module are related. Since this module is a continuation of MAT1512 some topics are already partially dealt with in MAT1512. In the MAT1613 module, we discuss some advanced methods and applications.

Mind map



Chapter 1

Applications of Derivatives

Introduction

This chapter shows how to *draw conclusions from derivatives*: how to calculate rates of change, how to find a function's extreme values, how to determine the shape of a function's graph, and how to find a function when we know its first derivative and its value at a single point.

The key to *recovering functions from derivatives is the Mean Value Theorem*, a theorem whose corollaries provide the gateway to *integral calculus*.

Outcomes

After studying this chapter the learner should be able to:

- solve basic word problems concerning *related rates*, for example, change in volumes, areas, speeds, etc.
- solve a range of basic mathematical problems using the Mean Value Theorem and/or Rolle's Theorem.
- use first and second derivatives to sketch the graphs of rational functions and trigonometric functions.
- solve basic word problems in optimization, involving, for example, volumes, areas, speed, etc.

In this chapter we will provide some examples on the following topics.You have to work on the relevant topics in your textbook.

1.1 Related Rates.

1.2 Graphs of Functions.

1.3 Optimization.

1.4 Rolle's Theorem and the Mean Value Theorem.

1.1 RELATED RATES

Most quantities encountered in everyday life change with time. This is especially evident in scientific investigations for instance a chemist may be interested in the rate at which a certain substance dissolves in water. An electrical engineer may wish to know the rate of change of current in a part of an electrical circuit. A biologist may be concerned with the rate at which bacteria in a culture decreases or increases.

Let us consider the following general situation which can be applied to all the preceding examples.

Suppose a variable w is a function of time such that at time t, w is given by $\mathbf{w} = \mathbf{g}(\mathbf{t})$, where g is a differentiable function.

We define the **instantaneous rate of change** of w with respect to t as

$$\boxed{\frac{dw}{dt} = g'(t).}$$

Sometimes we are given a problem with a rate of change that we cannot measure directly from the rate we are given. For instance, we may have a function v and we need to find $\frac{dv}{dt}$, but we cannot find $\frac{dv}{dt}$ directly if we have that v is a function of p where p is then a function of t.

In this case we have to use the **Chain rule** and obtain

$$\boxed{v'(t) = \frac{dv}{dt} = \frac{dv}{dp}\frac{dp}{dt}.}$$

In applications it is not unusual to have **two variables** x and y that are differentiable functions of time t, say $x = f(t)$ and $y = g(t)$. In addition, x and y may be related by means of some equation such as

$$\boxed{x^2 - y^3 + 2x + 7y^2 - 4 = 0,\ x = f(t),\ y = g(t).}$$

Differentiating with respect to t and using the chain rule produces an equation involving the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$. The previous equation leads to

$$\boxed{2x\frac{dx}{dt} - 3y^2\frac{dy}{dt} + 2\frac{dx}{dt} + 14y\frac{dy}{dt} = 0.}$$

The derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are called related rates, since they are related by means of the equation.

Worked Examples

1. A spherical balloon is inflated with helium at a rate of $100\pi\ m^3 s^{-1}$. How *fast* is the balloon's *radius increasing* at the instant the radius is $5m$? How *fast* is the *surface area* increasing?

   Solution

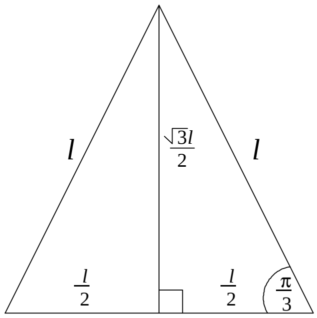
   Let V be the volume (in m^3) of the balloon, r the radius (in m) of the balloon, t the time (in sec) and S the surface area (in m^2) of the balloon.

   We are given that
$$\frac{dV}{dt} = 100\pi,$$
   and need to find
$$\left.\frac{dr}{dt}\right|_{r=5} \text{ and } \left.\frac{dS}{dt}\right|_{r=5}.$$
   We have that
$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2.$$
   Now
$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$$
   i.e.
$$100\pi = 4\pi r^2 \cdot \frac{dr}{dt}$$
   so
$$\frac{dr}{dt} = \frac{100\pi}{4\pi r^2} = \frac{25}{r^2}. \qquad ...(*)$$
   Consequently
$$\left.\frac{dr}{dt}\right|_{r=5} = \frac{25}{5^2} = 1.$$
   Hence the balloon's radius is increasing at the rate of $1ms^{-1}$ at the instant the radius is $5m$. Also,
$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dr}\cdot\frac{dr}{dt}\\ &= 8\pi r\cdot\frac{25}{r^2} \quad \text{(see (*) above)}\\ &= \frac{200\pi}{r}.\end{aligned}$$
   Thus
$$\left.\frac{dS}{dt}\right|_{r=5} = \frac{200\pi}{5} = 40\pi,$$
   and the surface area is increasing at the rate of $40\pi\ m^2 s^{-1}$ when the radius is $5m$.

2. The *area* of an equilateral triangle is *increasing* at a rate of 300 cm^2/min at the instant its edges are each $\sqrt{3}$ cm long. At what rate are the *edges changing* at that instant?

   Solution

Let A denote the area (in cm^2) of an equilateral triangle. Let ℓ denote the side length (in cm) of such a triangle. Let t denote the time in minutes.



From the Pythagoras's Theorem it follows that the height (in cm) of the triangle is

$$\sqrt{\ell^2 - (\ell/2)^2} = \frac{\sqrt{3}}{2}\ell.$$

Hence

$$A = \frac{1}{2} \cdot \ell \cdot \frac{\sqrt{3}}{2}\ell = \frac{\sqrt{3}}{4}\ell^2.$$

It is given that $\left.\frac{dA}{dt}\right|_{\ell=\sqrt{3}} = 300$. We must find $\left.\frac{d\ell}{dt}\right|_{\ell=\sqrt{3}}$. Now

$$\frac{dA}{dt} = \frac{dA}{d\ell} \cdot \frac{d\ell}{dt} = \frac{\sqrt{3}}{4} \cdot 2\ell \cdot \frac{d\ell}{dt} = \frac{\sqrt{3}}{2}\ell \cdot \frac{d\ell}{dt}\ ,$$

so

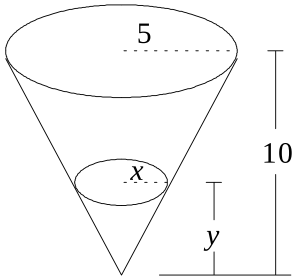
$$\frac{d\ell}{dt} = \frac{2}{\sqrt{3}\ell} \cdot \frac{dA}{dt}\ .$$

Hence

$$\left.\frac{d\ell}{dt}\right|_{\ell=\sqrt{3}} = \frac{2}{\sqrt{3} \cdot \sqrt{3}} \cdot 300 = 200.$$

Thus the edges are each increasing at a rate of 200 cm/min the instant the edges are each $\sqrt{3}$ long.

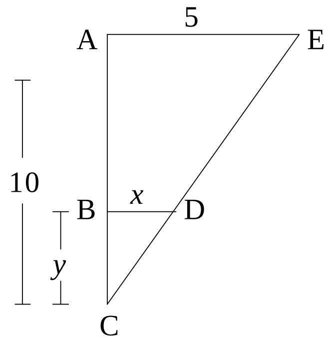
3. Water is running into a conical tank at a rate of $9\text{m}^3/\text{min}$. How *fast* is the water level *rising* when the water is 6 m deep? The dimensions of the tank (in metres) appear in the diagram.



Solution

The volume V of a cone with radius x and height y is:

$$V = \frac{1}{3}\pi x^2 y.$$



To get a relationship between y and x we use a property of similar triangles, namely

$$\frac{BC}{AC} = \frac{BD}{AE}$$

i.e.

$$\frac{y}{10} = \frac{x}{5}$$

i.e.

$$x = \frac{1}{2}y.$$

Substitute this x into the equation for V:

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{1}{12}\pi y^3.$$

Differentiate with respect to time:

$$\frac{dV}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}\ .$$

It is given that $\dfrac{dV}{dt} = 9$ and $y = 6$.

$$\begin{aligned} \therefore\quad 9 &= \frac{\pi}{4}(36)\frac{dy}{dt}\ . \\ \therefore\quad \frac{dy}{dt} &= \frac{1}{\pi}\ . \end{aligned}$$

Therefore, the water level is rising at $\dfrac{1}{\pi}$m/min when the water is 6m deep.

4. According to Poiseuille's law, the volume V of blood flowing through a vein in a unit of time at a fixed pressure is related to the radius r of the vein by the formula

$$V = kr^4,$$

where k is some fixed constant. Now cold water has the effect of contracting the blood vessels in the hands, and the radius of a typical vein might decrease at the rate of 20% /minute. Use Poiseuille's law to calculate the percentage *rate* at which the volume of blood *flowing through* that vein decreases.

Solution

Let t denote the time in minutes. It is given that

$$\frac{dr/dt}{r} = \frac{-20}{100}\,.$$

Now

$$\begin{aligned} V &= kr^4 \\ \Rightarrow \frac{dV}{dt} &= 4kr^3\frac{dr}{dt} \\ \Rightarrow \frac{dV/dt}{V} &= \frac{4kr^3 \cdot dr/dt}{V} \\ &= \frac{4kr^3 \cdot dr/dt}{kr^4} \\ &= 4 \cdot \frac{dr/dt}{r} \\ &= -4 \times \frac{20}{100} = -\frac{80}{100}. \end{aligned}$$

Hence, the volume of blood flowing through the vein decreases at the rate of 80%/minute.

5. Ohm's law for electrical circuits states that

$$V = IR$$

where V is the voltage, I is the current in amperes, and R is the resistance in ohms. Suppose that V is increasing at a rate of 1 volt/s, while I is decreasing at a rate of $\frac{1}{3}$ amperes/s. Find the *rate* at which R is

changing when V is 12 and I is 2. Is R increasing or decreasing?

Solution

Let t denote the time in seconds. It is given that

$$\frac{dV}{dt} = 1, \quad \frac{dI}{dt} = -\frac{1}{3} \quad \text{and} \quad V = IR. \qquad ...(*)$$

Now

$$\begin{aligned} \frac{dV}{dt} &= I\frac{dR}{dt} + R\frac{dI}{dt}. \\ \therefore \quad \frac{dR}{dt} &= \frac{1}{I}\left(\frac{dV}{dt} - R\frac{dI}{dt}\right) \\ &= \frac{1}{I}\left(1 + \frac{1}{3}\frac{V}{I}\right) \qquad \text{(from (*))}. \end{aligned}$$

Hence, when $V = 12$ and $I = 2$, then

$$\frac{dR}{dt} = \frac{1}{2}\left(1 + \frac{1}{3} \times \frac{12}{2}\right) = \frac{3}{2}.$$

Thus R is increasing by $\frac{3}{2}$ohms/second at the instant in question.

1.2 GRAPHS AND FUNCTIONS

Before we do some problems on this topic, we first have to explain the method of using **sign patterns**.

Sign patterns
We use **sign patterns** to determine:

I. over which intervals a given function f is negative and over which intervals f is positive.

II. the intervals where the graph of a given function f rises and falls (we use the first derivative here).

III. the intervals where the graph of a given function f is concave up or concave down (we use the second derivative here).

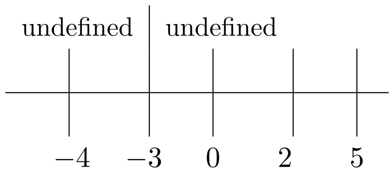
In all of the above cases, make sure that the given functions, first derivative and second derivative are **fully factorize** that means that we have factorized the function into constants, powers of x, linear equations and quadratic equations of the form $x^2 + k,\ k > 0$.

I. Intervals where f is positive and intervals where f is negative:

Suppose

$$f(x) = \frac{(x-2)(x+3)(x-5)}{x(x+4)}.$$

It should be clear that the sign of $f(x)$ changes only at those points where the individual factors change sign. The individual factors change sign at the points where they become zero. Hence, $f(x)$ changes sign at $x = 2$, $x = -3$, $x = 5$, $x = 0$ and $x = -4$. We may mark these points on a number line as follows:



See also that the function $f(x)$ is undefined at $x = -4$ and $x = 0$. (division by zero).

We need to find the sign of $f(x)$ over the intervals $(-\infty, -4)$, $(-4, -3)$, $(-3, 0)$, $(0, 2)$, $(2, 5)$ and $(5, \infty)$.

Say we choose the interval $(5, \infty)$. We have that $y = x - 5$ is a straight line intercepting the x–axis at 5. Thus the sign changes at $x = 5$. When $x > 5$ (since $y = x - 5$), we have that y has positive values and when $x < 5$ we have that y has negative values. For each linear factor in the above problem we now look at the signs. To know where the function $f(x)$ is positive or negative we multiply all the signs in the different intervals.

The *sign pattern* will now look as follows:

		undefined		undefined		
$y = x - 2$	$-$	$-$	$-$	$-$	$+$	$+$
$y = x + 3$	$-$	$-$	$+$	$+$	$+$	$+$
$y = x - 5$	$-$	$-$	$-$	$-$	$-$	$+$
$y = x$	$-$	$-$	$-$	$+$	$+$	$+$
$y = x + 4$	$-$	$+$	$+$	$+$	$+$	$+$
$f(x) = \dfrac{(x-2)(x+3)(x-5)}{x(x+4)}$	$-$	$+$	$-$	$+$	$-$	$+$
	-4	-3	0	2	5	

A word of warning! Not all the factors that occur will change sign, for example:

$$f(x) = \frac{(x^2 + x + 1)(x - 2)}{(x - 3)(x + 1)^2}.$$

$f(x)$ changes sign only at $x = 2$ and at $x = 3$. There is no point at which $x^2 + x + 1$ changes sign – it is always positive. (This follows from the fact that its discriminant is negative.) Also, $(x + 1)^2$ is zero at $x = -1$, but it does not change sign there. $(x + 1)^2$ is always non–negative.

Note also that at $x = -1$ and $x = 3$ the function is undefined.

The **sign pattern** is as follows

		undefined		undefined
$y = x - 2$	$-$	$-$	$+$	$+$
$y = x - 3$	$-$	$-$	$-$	$+$
$f(x) = \dfrac{\left(x^2+x+1\right)(x-2)}{(x-3)(x+1)^2}$	$+$	$+$	$-$	$+$
	-1	2	3	

II. Intervals where the graph of f rises and falls

The *derivative test* to find intervals where the function f is *increasing and decreasing, is as follows:*

> Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
>
> 1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
>
> 2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

So let $f(x) = \dfrac{x^2 - 16}{x + 5}$.

Then we need the **sign pattern** for

$$\begin{aligned} f'(x) &= \frac{2x(x+5) - \left(x^2 - 16\right)}{(x+5)^2} \\ &= \frac{x^2 + 10x + 16}{(x+5)^2} \\ &= \frac{(x+2)(x+8)}{(x+5)^2} \end{aligned}$$

All the rules in I (intervals where f is positive or negative) apply here again.

Thus $x = -5$ is the point where $f(x)$ is undefined. At $x = -2$ and $x = -8$ the sign changes. Since $(x+5)^2$ is always positive, we need not include this in the *sign pattern for* $f'(x)$.

			undefined	
$y = x + 2$	$-$	$-$	$-$	$+$
$y = x + 8$	$-$	$+$	$+$	$+$
$f'(x) = \dfrac{(x+2)(x+8)}{(x+5)^2}$	$+$	$-$	$-$	$+$
	-8	-5	-2	

From the first derivative test, we see that the graph f rises on $(-\infty, -8)$ and $(-2, \infty)$ and falls on $(-8, -5)$ and $(-5, -2)$.

III. Intervals where the graph of f is concave up and concave down.

The test to *determine concavity is as follows*

> Suppose that f'' exists on an interval I.
> If $f''(x) > 0$ on I, then the graph of f is concave up on I and if $f''(x) < 0$ on I, then the graph of f is concave down on I.

So we need the *sign pattern of* $f''(x)$.

Using the same problem as in II, that is

$$f(x) = \frac{x^2 - 16}{x+5}, \quad f'(x) = \frac{(x+2)(x+8)}{(x+5)^2} = \frac{x^2+10x+16}{(x+5)^2}$$

we obtain

$$f''(x) = \frac{18}{(x+5)^3} \text{ (we use the quotient rule for differentiation)}$$

We see again that $f''(x)$ is not defined at $x = -5$. Further $(x+5)^3 = (x+5)^2(x+5)$ the first which is always positive. So we only need to look at $y = x+5$.

The *sign pattern* for $f''(x)$ is as follows:

		undefined	
$x+5$	$-$		$+$
$f''(x)$	$-$		$+$
		-5	

Thus, following the second derivate test, we have that f is concave up on $(-5, \infty)$ and concave down on $(-\infty, -5)$.

Exercises

Determine the sign patterns of the following functions:

(i) $f(x) = \dfrac{(3x+2)(3-x)}{(4+x)(x+1)}$.

(ii) $f(x) = \dfrac{\left(-x^2+3x-4\right)(4+x)}{(3-x)^3 x}$.

(iii) $g'(x) = \dfrac{(x+3)^2 x^2}{(x+4)^4}$.

(iv) $h''(x) = \dfrac{(x-3)(x+1)}{\sqrt{4-x^2}}$.

Solutions

(i)

		undefined		undefined		0		0	
$y = 3x + 2$	$-$		$-$		$-$		$+$		$+$
$y = -x + 3$	$+$		$+$		$+$		$+$		$-$
$y = 4 + x$	$-$		$+$		$+$		$+$		$+$
$y = x + 1$	$-$		$-$		$+$		$+$		$+$
$f(x) = \dfrac{(3x+2)(3-x)}{(4+x)(x+1)}$	$-$		$+$		$-$		$+$		$-$
		-4		-1		$-\frac{2}{3}$		3	

(ii)

				undefined		undefined	
$y = -x^2 - 3x - 4$ (always negative $b^2 - 4ac < 0$)	$-$		$-$		$-$		$-$
$y = 4 + x$	$-$		$+$		$+$		$+$
$y = 3 - x$	$+$		$+$		$+$		$-$
$y = x$	$-$		$-$		$+$		$+$
$f(x) = \dfrac{\left(-x^2 + 3x - 4\right)(4+x)}{(3-x)^3 x}$	$-$		$+$		$-$		$+$
		-4		0		3	

(iii) $g'(x)$ is always positive:

		undefined			
$g'(x)$	$+$		$+$		$+$
		-4		-3	

(iv) $h''(x)$ is only defined on $(-2, 2)$. $h''(x)$ is undefined for $x \geq 2$ and $x \leq -2$

$y = x - 3$		$-$		$-$	
$y = x + 1$		$-$		$+$	
$h''(x)$		$+$		$-$	
	-2		-1		2

Symmetry

We define the concept because it will also help us to draw the graph of a function.

Symmetry
The graph of an equation is **symmetric with respect to the y–axis** if the point $(-x, y)$ is on the graph whenever (x, y) is on the graph.

The graph of an equation is **symmetric with respect to the x–axis** if the point $(x, -y)$ is on the graph whenever (x, y) is on the graph.

The graph of an equation is **symmetric with respect to the origin,** whenever a point (x, y) is on the graph, $(-x, -y)$ is also on the graph.

Symmetry with respect to any other straight line can also be determined but we are not going to discuss this here.

Remarks on the worked problems

In all of the following worked problems you have to determine the following:

(a) Determine the x– and y–intercepts of the graph of f.

(b) Determine the vertical and horizontal asymptotes of the graph of f.

(c) Use the first derivate of f (use the sign pattern of $f(x)$) to

  (i) determine the intervals over which the graph of f rises, and the intervals over which it falls.

  (ii) find the local extrema.

(d) Use the second derivative of f (use the sign pattern of $f''(x)$) to determine the intervals over which the graph of f is concave up, and the intervals over which it is concave down.

(e) (i) Use the sign pattern of $f''(x)$ and determine the inflection points.

  (ii) Obtain the gradient of the inflection points, respectively.

(f) Use the above information to draw the graph of f.

Worked problems

1. Let f be the function defined by

$$\boxed{f(x) = \frac{4x}{x^2 + 4}.}$$

Solution

1. (a) $f(0) = 0$. This is the x- (and y-) intercept.

(b)

$$\lim_{x\to\infty} \frac{4x}{x^2+4} = \lim_{x\to\infty} \frac{\frac{4}{x}}{1+\frac{4}{x^2}} = 0.$$

Similarly,

$$\lim_{x\to-\infty} \frac{4x}{x^2+4} = 0.$$

The horizontal asymptote is the line $y = 0$.

(c)

$$\begin{aligned}
f(x) &= \frac{4x}{x^2+4} = 4\cdot\frac{x}{x^2+4} \\
\Rightarrow f'(x) &= 4\cdot\frac{1\cdot\left(x^2+4\right) - x\cdot 2x}{\left(x^2+4\right)^2} \\
&= \frac{4\left(4-x^2\right)}{\left(x^2+4\right)^2} \\
&= \frac{4(2-x)(2+x)}{\left(x^2+4\right)^2}.
\end{aligned}$$

We have that $f'(x)$ is not undefined at any point. We have that $\left(x^2+4\right)^2$ is always positive. The *sign pattern* of $f'(x)$ is:

$y = 2 - x$	$+$	$+$	$-$
$y = 2 + x$	$-$	$+$	$+$
$f'(x)$	$-$	$+$	$-$
	-2	2	

Hence:

(i) the graph rises on $(-2, 2)$ and falls on $(-\infty, -2)$ and $(2, \infty)$;

(ii) there is a local minimum at $(-2, f(-2)) = (-2, -1)$, and a local maximum at $(2, f(2)) = (2, 1)$.

$$\begin{aligned}
f'(x) &= 4\cdot\frac{4-x^2}{\left(x^2+4\right)^2} \\
\Rightarrow f''(x) &= 4\cdot\frac{-2x\cdot\left(x^2+4\right)^2 - \left(4-x^2\right)\cdot 2\left(x^2+4\right)\cdot 2x}{\left(x^2+4\right)^4} \\
&= 4\cdot\frac{-2x\left(x^2+4\right)\left[\left(x^2+4\right)+2\left(4-x^2\right)\right]}{\left(x^2+4\right)^4} \\
&= \frac{-8x\left(12-x^2\right)}{\left(x^2+4\right)^3} \\
&= \frac{-8x\left[\left(\sqrt{12}-x\right)\left(\sqrt{12}+x\right)\right]}{\left(x^2+4\right)^3}.
\end{aligned}$$

(d) The sign pattern of $f''(x)$ is

$y=\sqrt{12}-x$	$+$	$+$	$+$	$-$
$y=\sqrt{12}+x$	$-$	$+$	$+$	$+$
$y=-8x$	$+$	$+$	$-$	$-$
$f''(x)$	$-$	$+$	$-$	$+$
	$-\sqrt{12}$	0	$\sqrt{12}$	

We have that the graph is concave up on $\left(-\sqrt{12},0\right)$ and $\left(\sqrt{12},\infty\right)$, and concave down on $\left(-\infty,-\sqrt{12}\right)$ and $\left(0,\sqrt{12}\right)$.

(e) (i) The inflection points are where the second derivative becomes zero.Thus there are inflection points at $\left(-\sqrt{12},f\left(-\sqrt{12}\right)\right)=\left(-\sqrt{12},-\sqrt{3}/2\right)$; $(0,f(0))=(0,0)$ and $\left(\sqrt{12},f\left(\sqrt{12}\right)\right)=\left(\sqrt{12},\sqrt{3}/2\right)$.

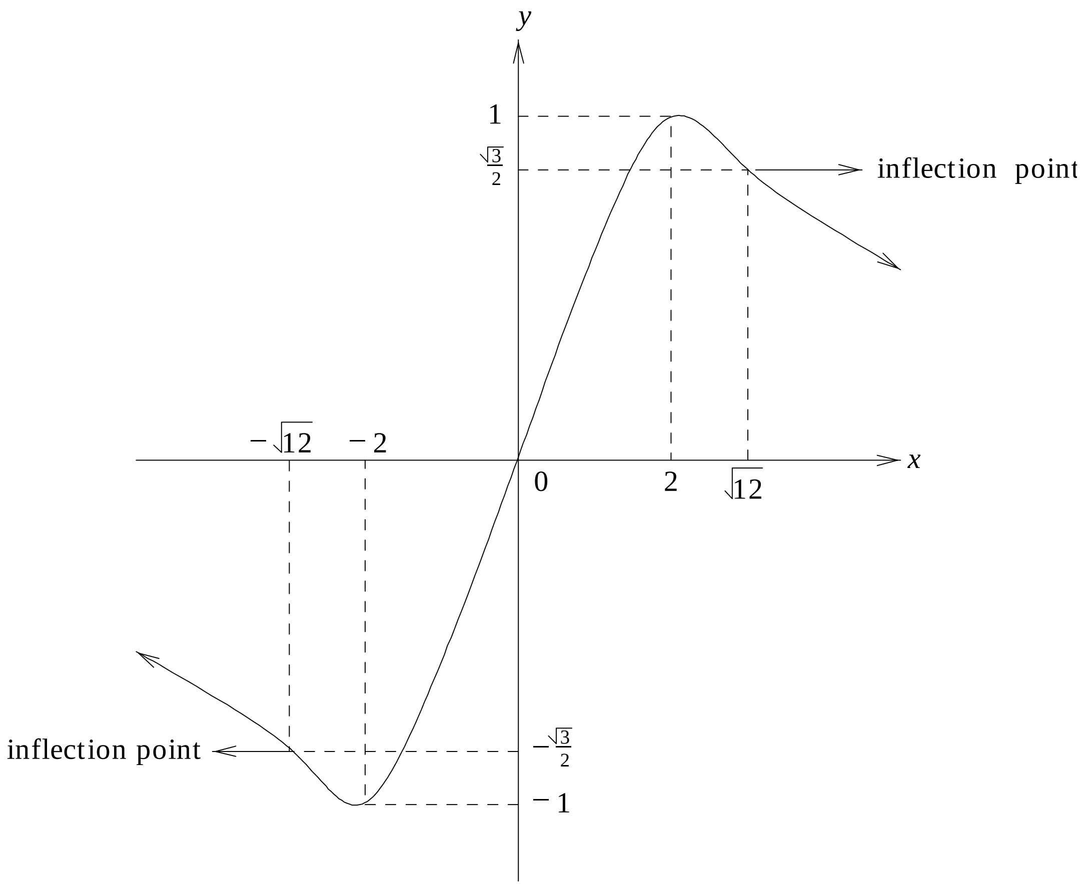
(ii) The gradients at the inflection points are, respectively:

$$\begin{aligned} f'\left(-\sqrt{12}\right) &= -\frac{1}{8}, \\ f'(0) &= 1 \end{aligned}$$

and

$$f'\left(\sqrt{12}\right)=-\frac{1}{8}.$$

(f)



Please see from the graph above that whenever $(x,\ f(x))$ is on the graph, $(-x,f(-x))$ is also on the graph. We thus have symmetry with respect to the origin.

2. Let f be the function defined by

$$\boxed{f(x) = \frac{x}{(x+1)^3}.}$$

Solution

2. (a) The x-intercept is 0 (this is also the y-intercept).

   (b) The vertical asymptote is the line $x = -1$, and the horizontal asymptote is the line $y = 0$.

   (c)

$$\begin{aligned} f(x) &= \frac{x}{(x+1)^3} \\ \Rightarrow f'(x) &= \frac{1 \cdot (x+1)^3 - x \cdot 3(x+1)^2}{(x+1)^6} \\ &= \frac{(x+1)^2 [x + 1 - 3x]}{(x+1)^6} \\ &= \frac{1-2x}{(x+1)^4}. \end{aligned}$$

We see that $f'(x)$ is undefined at $x = -1$ and that $(x+1)^4$ is always positive. The sign pattern for $f'(x)$ is:

		undefined		
$y = 1 - 2x$	$+$		$+$	$-$
$f'(x)$	$+$		$+$	$-$
		-1	$\frac{1}{2}$	

   (i) Hence, the graph of f rises on $(-\infty, -1)$ and on $\left(-1, \frac{1}{2}\right)$, and falls on $\left(\frac{1}{2}, \infty\right)$.

   (ii) There is a local maximum at

$$\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, \frac{4}{27}\right).$$

   (d)

$$\begin{aligned} f'(x) &= \frac{1-2x}{(x+1)^4} \\ \Rightarrow f''(x) &= \frac{-2(x+1)^4 - (1-2x) \cdot 4(x+1)^3}{(x+1)^8} \\ &= \frac{-2(x+1)^3 [x + 1 + 2(1-2x)]}{(x+1)^8} \\ &= \frac{-2(3-3x)}{(x+1)^5} = \frac{6(x-1)}{(x+1)^5}. \end{aligned}$$

We see that the function is undefined at $x = -1$ and $(x+1)^4 (x+1) = (x+1)^5$ is not always positive

		undefined	
$y = x - 1$	$-$	$-$	$+$
$y = x + 1$	$-$	$+$	$+$
$f''(x)$	$+$	$-$	$+$
	-1	1	

Hence, the graph of f is concave up on $(-\infty, -1)$ and on $(1, \infty)$, and concave down on $(-1, 1)$.

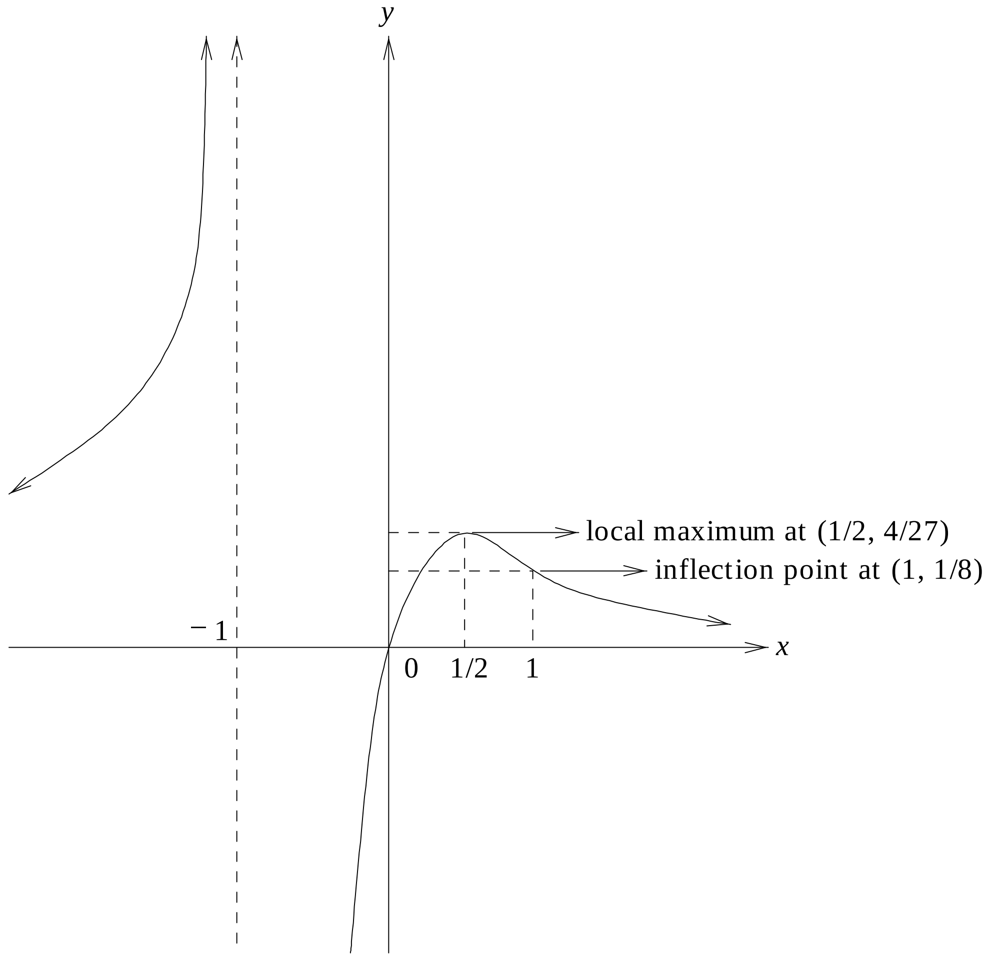
(e) (i) The inflection point is where the second derivative is zero, i.e.

$$(1, f(1)) = \left(1, \frac{1}{8}\right).$$

(ii) The gradient of the inflection point is

$$f'(1) = -\frac{1}{16}.$$

(f)



3. Let f be the function defined by

$$\boxed{f(x) = \frac{x}{(x+1)^2}.}$$

Solution

3. (a) $f(x) = 0 \Leftrightarrow x = 0$. The x-(and y-) intercept is thus 0.

(b) The vertical asymptote is the line $x = -1$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+2x+1} = \lim_{x\to\infty} \frac{1/x}{1+2/x+1/x^2} = 0.$$

Similarly, $\lim\limits_{x\to-\infty} f(x) = 0$.
Hence, the horizontal asymptote is the line $y = 0$.

(c)

$$\begin{aligned} f(x) &= \frac{x}{(x+1)^2} \\ \Rightarrow f'(x) &= \frac{1\cdot(x+1)^2 - x\cdot 2(x+1)}{(x+1)^4} \\ &= \frac{(x+1)[x+1-2x]}{(x+1)^4} \\ &= \frac{1-x}{(x+1)^3} \end{aligned}$$

We see that $f'(x)$ is not defined for $x = -1$ and $(x+1)^3 = (x+1)^2(x+1)$ is not always positive. The sign pattern for $f'(x)$ is:

		undefined	
$y = 1-x$	$+$	$+$	$-$
$y = 1+x$	$-$	$+$	$+$
$f'(x)$	$-$	$+$	$-$
	-1	1	

Hence:

(i) the graph of f rises on $(-1, 1)$ and
the graph of f falls on $(-\infty, -1)$ and $(1, \infty)$.

(ii) there is a local maximum at $(1, f(1)) = \left(1, \frac{1}{4}\right)$.

(d)

$$\begin{aligned} f'(x) &= \frac{1-x}{(x+1)^3} \\ \Rightarrow f''(x) &= \frac{-1(x+1)^3 - (1-x)\,3(x+1)^2}{(x+1)^6} \\ &= \frac{-(x+1)^2[(x+1)+3(1-x)]}{(x+1)^6} \\ &= \frac{2(x-2)}{(x+1)^4}. \end{aligned}$$

The function $y = (x+1)^4$ is always positive so we only need to investigate the linear function. The sign diagram for $f''(x)$ is:

		undefined		
$y = x - 2$	$-$		$-$	$+$
$f''(x)$	$-$		$-$	$+$
		-1		2

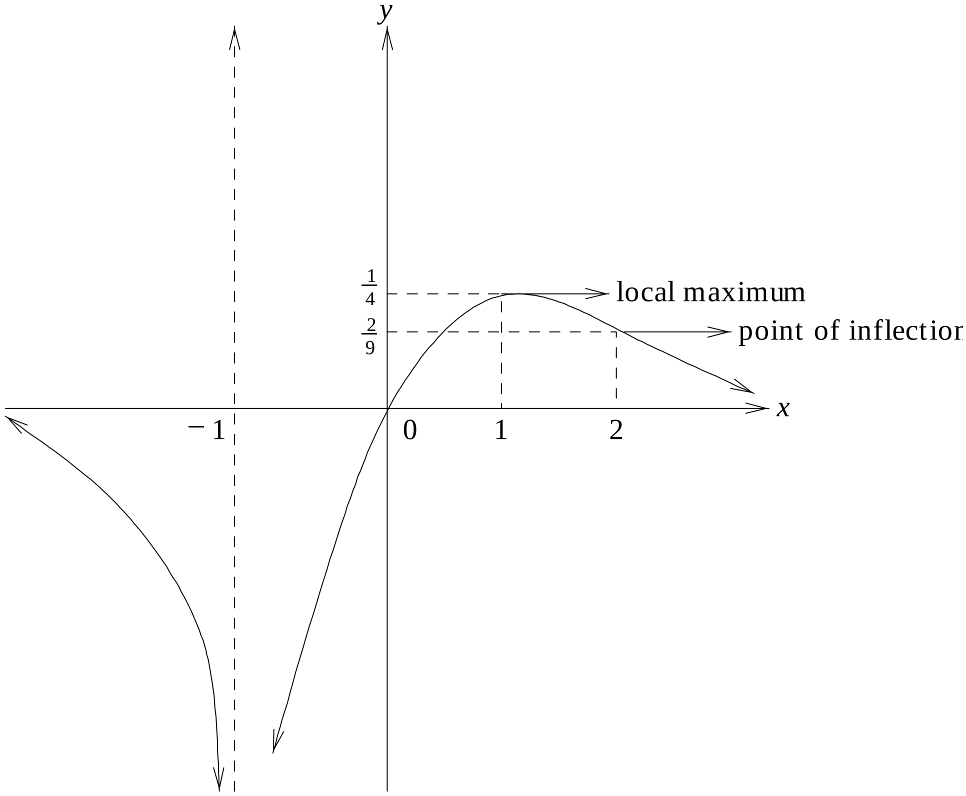
Hence, the graph of f is concave up on $(2, \infty)$ and the graph of f is concave down on $(-\infty, -1)$ and $(-1, 2)$.

(e) (i) There is an inflection point at $(2, f(2)) = \left(2, \frac{2}{9}\right)$.

(ii) The gradient of the inflection point is

$$f'(2) = -\frac{1}{27}.$$

(f)



4. Let f be the function defined by

$$\boxed{f(x) = \frac{x^2}{x^2+3}.}$$

Solution

4. (a) $f(x) = 0 \Leftrightarrow x = 0$. The x- (and y-) intercept is 0.

(b) $\lim\limits_{x\to\pm\infty} \dfrac{x^2}{x^2+3} = \lim\limits_{x\to\pm\infty} \dfrac{1}{1+\frac{3}{x^2}} = 1.$

Hence, the horizontal asymptote is $y = 1$.

(c)

$$\begin{aligned} f(x) &= \frac{x^2}{x^2+3} \\ \Rightarrow f'(x) &= \frac{2x\left(x^2+3\right) - x^2 \cdot 2x}{\left(x^2+3\right)^2} \\ &= \frac{2x^3+6x-2x^3}{\left(x^2+3\right)^2} = \frac{6x}{\left(x^2+3\right)^2}. \end{aligned}$$

We see that $\left(x^2+3\right)^2$ is always positive. The sign pattern of $f'(x)$ is:

$y = x$	$-$	$+$
$f'(x)$	$-$	$+$
	0	

Hence:

(i) the graph rises on $(0, \infty)$ and the graph falls on $(-\infty, 0)$.

(ii) there is a local minimum at $(0, 0)$.

(d)

$$\begin{aligned} f'(x) &= \frac{6x}{\left(x^2+3\right)^2} \\ \Rightarrow f''(x) &= \frac{6\left(x^2+3\right)^2 - 6x \cdot 2\left(x^2+3\right) \cdot 2x}{\left(x^2+3\right)^4} \\ &= \frac{6\left(x^2+3\right)\left[x^2+3-4x^2\right]}{\left(x^2+3\right)^4} \\ &= \frac{6\left(3-3x^2\right)}{\left(x^2+3\right)^3} = \frac{18\,(1-x)\,(1+x)}{\left(x^2+3\right)^3}. \end{aligned}$$

We see that $\left(x^2+3\right)^3$ is always positive. The sign pattern of $f''(x)$ is:

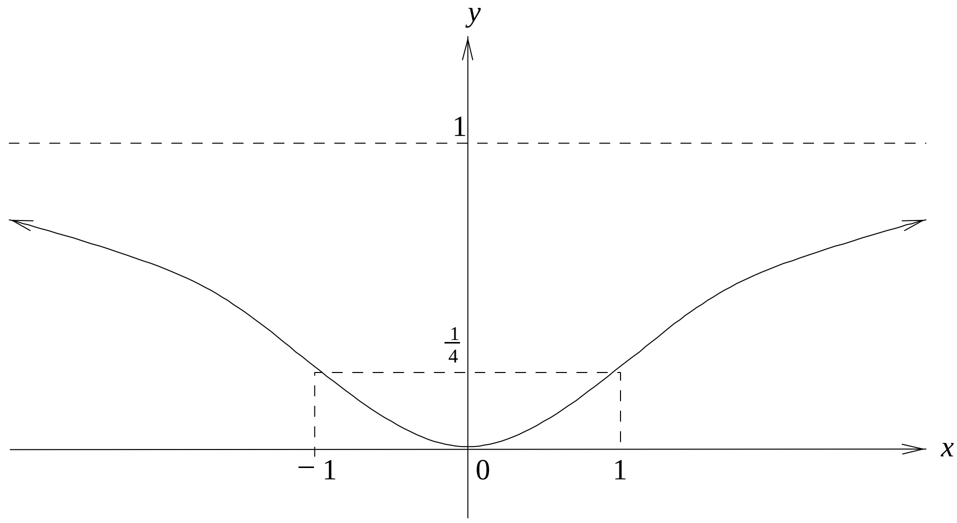
$y = 1 - x$	$+$	$+$	$-$
$y = 1 + x$	$-$	$+$	$+$
$f''(x)$	$-$	$+$	$-$
	-1	1	

Hence, the graph is concave up on $(-1, 1)$ and the graph is concave down on $(-\infty, -1)$ and $(1, \infty)$.

(e) (i) There are inflection points at $(-1, f(-1)) = \left(-1, \frac{1}{4}\right)$ and $(1, f(1)) = \left(1, \frac{1}{4}\right)$.

(ii) The gradients of the inflection points are $f'(1) = \frac{3}{8}$ and $f'(-1) = -\frac{3}{8}$.

(f)



Exercises

1. Suppose that the derivative of a certain function $y = f(x)$ is

$$y' = (x-1)^2 (x-2).$$

Find the intervals over which the graph of f is concave up, and the intervals over which it is concave down. Also, find the values of x at which the graph of f has inflection points.

2. (a) Determine the y-intercept of the curve of

$$f(x) = 2x^3 + 3x^2 - 36x + 10.$$

(b) Use the first derivative of $f(x)$ to find the intervals over which the curve of $f(x)$ is increasing (rising) and the intervals over which it is decreasing (falling). Also find the local extrema.

(c) Use the second derivative of $f(x)$ to determine the intervals over which the curve of $f(x)$ is concave up and the intervals over which it is concave down. Determine the points of inflection.

(d) Determine the slope of the tangent to the curve of $f(x)$ at the point of inflection.

Answers

1.

$$\begin{aligned} f'(x) &= (x-1)^2(x-2) \\ \Rightarrow f''(x) &= 2(x-1)(x-2) + (x-1)^2 \cdot 1 \\ &= (x-1)[2(x-2) + (x-1)] \\ &= (x-1)(3x-5). \end{aligned}$$

The sign pattern for $f''(x)$ is:

		1		$\frac{5}{3}$	
$y = x - 1$	$-$		$+$		$+$
$y = 3x - 5$	$-$		$-$		$+$
$f''(x)$	$+$		$-$		$+$

Hence, the graph of f is concave up on $(-\infty, 1)$ and $(\frac{5}{3}, \infty)$, and concave down on $(1, \frac{5}{3})$. There are inflection points where $x = 1$ and $x = \frac{5}{3}$.

2. (a) One can find the y-intercept by putting x equal to 0. Therefore the y-intercept is

$$f(0) = 10.$$

(b) If $f(x) = 2x^3 + 3x^2 - 36x + 10$, then

$$f'(x) = 6x^2 + 6x - 36.$$

There are possible extremes where $f'(x) = 0$,

i.e.

$$6x^2 + 6x - 36 = 0$$

i.e.

$$x^2 + x - 6 = 0$$

i.e.

$$(x + 3)(x - 2) = 0$$

i.e.

$$x = -3 \quad \text{or} \quad x = 2.$$

The sign pattern for $f'(x)$ is:

		-3		2	
$y = x + 3$	$-$		$+$		$+$
$y = x - 2$	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$

Therefore the curve is increasing over $(-\infty, -3)$ and $(2, \infty)$ and decreasing over $(-3, 2)$. There is a maximum at $(-3, f(-3)) = (-3, 91)$ and a minimum at $(2, f(2)) = (2, -34)$.

(c) If

$$f'(x) = 6x^2 + 6x - 36$$

then

$$f''(x) = 12x + 6 = 6(2x + 1).$$

Possible points of inflection appear where

$$f''(x) = 0$$

i.e.

$$12x + 6 = 0$$

i.e.

$$x = -\frac{1}{2}.$$

The sign pattern for $f''(x)$ is:

$$\begin{array}{r|c|c} y = 2x+1 & - & + \\ \hline f''(x) & - & + \\ & \multicolumn{1}{r}{-\frac{1}{2}} & \end{array}$$

Therefore the curve is concave down over $\left(-\infty, -\frac{1}{2}\right)$ and concave up over $\left(-\frac{1}{2}, \infty\right)$. There is a point of inflection at $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right) = \left(-\frac{1}{2}, \frac{57}{2}\right)$.

(d) The point of inflection is at $x = -\frac{1}{2}$. The gradient of the inflection point is then

$$\begin{aligned} f'\left(-\frac{1}{2}\right) &= 6\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) - 36 \\ &= \frac{6}{4} - 3 - 36 \\ &= -\frac{75}{2}. \end{aligned}$$

1.3 OPTIMIZATION

The theory (used in 1.2(S)) for finding the extrema of functions can be applied to certain practical problems. We see here how the first and second derivative test are used to find the extreme values and to establish whether they are local maximum or a local minimum values.

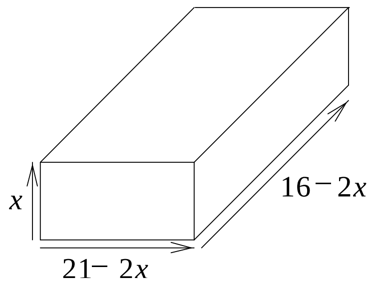
We state the second derivative test:

Second Derivative Test
Suppose a function f is differentiable on an open interval containing a point c and $f'(c) = 0$. If $f''(c) < 0$ then, f has a *local maximum* at c and if $f''(c) > 0$ then, f has a *local minimum* at c.

Say we have the following problem to solve: We have a rectangular piece of cardboard 16 metres wide and 21 metres long. We want to make a box by cutting out a square from each corner and bending the sides. Find the size of the corner which will produce a box having the maximum possible volume.

Guidance: we first have to find the equation for the volume V as a function of x. Thus if we let x be the height of the box, we have

$$\begin{aligned} V &= x(16-2x)(21-2x) \\ &= 2\left(168x - 37x^2 + 2x^3\right). \end{aligned}$$



To find the critical points we first find the derivative

$$\begin{aligned}\frac{dV}{dx} &= 4\left(3x^2 - 37x + 84\right) \\ &= 4\left(3x - 28\right)\left(x - 3\right).\end{aligned}$$

Possible critical points are where $\frac{dV}{dx} = 0$, that is at $x = \frac{28}{3}$ and 3.

$\frac{28}{3}$ is impossible because it is outside the domain of x.

The second derivative test will tell us where the maximum value is

$$\frac{d^2V}{dx^2} = 4\left(6x - 37\right).$$

Then

$$\left.\frac{d^2V}{dx^2}\right]_{x=3} = 4\left(18 - 37\right) = -76 < 0$$

so that V has a relative maximum at $x = 3$.

We also have to check for the endpoints. We see that since $0 \leq x \leq 8$ and since $V = 0$ if either $x = 0$ or $x = 8$, the maximum value of V does not occur at the endpoints of the domain of x.

Thus, a 3-metre square should be cut from each corner to maximise the volume of the resulting box.

Worked Examples

1. Find the largest possible value of $s = 2x + y$ if x and y are side-lengths in a right triangle whose hypotenuse is $\sqrt{5}$ units long.

   Solution

   The Pythagorean Theorem tells us that

   $$x^2 + y^2 = 5$$

   from which follows that

   $$y = \sqrt{5 - x^2}.$$

   Hence,

   $$s = 2x + y = 2x + \sqrt{5 - x^2}.$$

   Observe that $0 < x < \sqrt{5}$. x cannot be 0 or $\sqrt{5}$, for then one of the sides in the triangle would have zero length. If $x > \sqrt{5}$ then $5 - x^2 < 0$, so $\sqrt{5 - x^2}$ is not a real number.

Using the *first derivative* to find the possible critical points, we have

$$\begin{aligned} s &= 2x + \sqrt{5 - x^2} \\ \Rightarrow \frac{ds}{dx} &= 2 + \frac{-2x}{2\sqrt{5 - x^2}} \\ &= \frac{2\sqrt{5 - x^2} - x}{\sqrt{5 - x^2}}. \end{aligned}$$

$\left.\frac{ds}{dx}\right|_{x=\sqrt{5}}$ and $\left.\frac{ds}{dx}\right|_{x=-\sqrt{5}}$ are undefined, and so $\sqrt{5}$ and $-\sqrt{5}$ could be critical values. However $\sqrt{5}$ and $-\sqrt{5}$ lie outside the domain of s. Furthermore

$$\begin{aligned} \frac{ds}{dx} &= 0 \\ \Leftrightarrow 2\sqrt{5 - x^2} - x &= 0 \\ \Leftrightarrow 2\sqrt{5 - x^2} &= x \\ \Rightarrow 4\left(5 - x^2\right) &= x^2 \\ \Rightarrow 5\left(2 - x\right)\left(2 + x\right) &= 0. \end{aligned}$$

The number 2 lies in the domain of s, so 2 is a critical point.

Using the *second derivative* we can investigate if this point is a local maximum or local minimum:

$$\begin{aligned} \frac{ds}{dx} &= 2 - x\left(5 - x^2\right)^{-\frac{1}{2}} \\ \frac{d^2s}{dx^2} &= -\left(5 - x^2\right)^{-\frac{1}{2}} - x \cdot \left(-\frac{1}{2}\left(5 - x^2\right)^{-\frac{3}{2}}\right)(-2x) \\ &= -\frac{1}{\sqrt{5 - x^2}} - \frac{x^2}{\left(5 - x^2\right)^{\frac{3}{2}}}, \end{aligned}$$

and

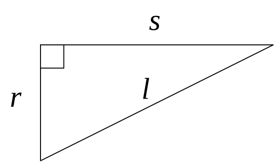
$$\left.\frac{d^2s}{dx^2}\right|_{x=2} = -1 - \frac{4}{1} = -5 < 0.$$

Consequently s has a maximum when $x = 2$. If $x = 2$, then $y = 1$ and the maximum value of s is 5.

2. Suppose that the perimeter of a rectangle is 8 cm. What is the smallest possible length of the diagonal of such a rectangle?

   Solution

   Let r and s denote the side lengths (in cm) of a rectangle.



   It is given that

$$2r + 2s = 8,$$

i.e.

$$r = 4 - s.$$

If ℓ denotes the length (in cm) of the diagonal, then from the Pythagoras's Theorem it follows that

$$\ell = \sqrt{r^2 + s^2} = \sqrt{(4-s)^2 + s^2} = \sqrt{2s^2 - 8s + 16}. \qquad ...(*)$$

Since s is a side length, and since the perimeter is fixed at 8 cm, it follows that $0 < s < 4$. Now

$$\frac{d\ell}{ds} = \frac{4s - 8}{2\sqrt{2s^2 - 8s + 16}} = \frac{2(s-2)}{\sqrt{2s^2 - 8s + 16}}.$$

For $f(s) = 2s^2 - 8s + 16$, the discriminant, $b^2 - 4ac = 64 - 4(2)(16) < 0$, which implies that $f(s)$ is always positive. Thus $\ell > 0$.

The sign pattern for $\dfrac{d\ell}{ds}$ is then only dependent on $y = s - 2$.

$$\begin{array}{r|cc|cc|} y = s-2 & - & - & + & + \\ \hline & 0 & & 2 & & 4 \end{array}$$

(Remember that $\sqrt{}$ denotes only the positive root.) Hence ℓ is decreasing on $(0, 2)$ and increasing on $(2, 4)$, so there is a local (and, in fact, absolute) minimum when $s = 2$. From $(*)$, we get that the smallest possible length of the diagonal is $2\sqrt{2}$cm.

3. Find two positive real numbers with sum 36 and where the product of the square of one number multiplied with the other is a maximum.

   Solution

   Let x be the one number and y the other.

   We have $x + y = 36$, so that $y = 36 - x$.

   Let $T = x^2 y$.

   Thus, $T(x) = x^2(36 - x) = 36x^2 - x^3$.

   Now $\dfrac{dT}{dx} = 72x - 3x^2$.

   Then

   $$\frac{dT}{dx} = 0 \quad \text{when} \quad 72x - 3x^2 = 0;$$

   that is where $3x(24 - x) = 0$.

   The values are then $x = 0$ or $x = 24$.

   Using the second derivative test, we have

   $$\frac{d^2T}{dx^2} = 72 - 6x.$$

   So

   $$\left.\frac{d^2T}{dx^2}\right]_{x=0} = 72 > 0$$

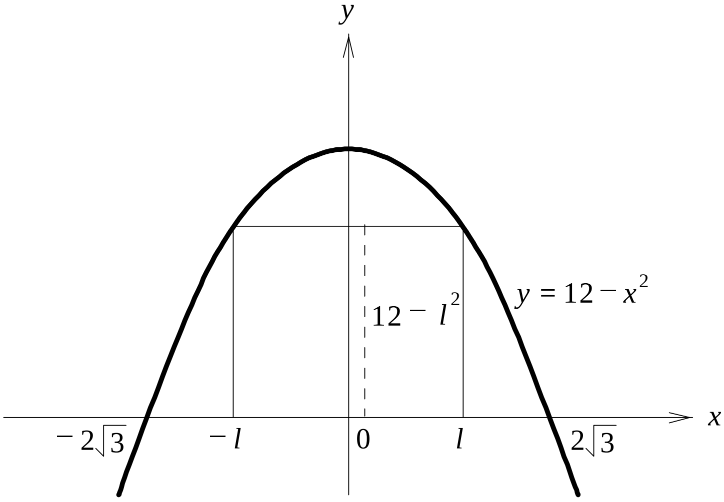
and

$$\left.\frac{d^2T}{dx^2}\right]_{x=24} = -72 < 0 \qquad \text{(maximum when } x = 24\text{)}.$$

Thus, we have that the numbers are 24 and 12.

4. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

   Solution



   Since opposite sides of the rectangle are parallel, it is clear that the rectangle must be symmetric about the y-axis. Let the distance from the lower right–hand corner of the rectangle to the origin be ℓ. If A denotes the area of the rectangle then,

$$A = 2\ell(12 - \ell^2) = 24\ell - 2\ell^3, \quad \text{where} \quad 0 < \ell < 2\sqrt{3}.$$

   Now

$$\frac{dA}{d\ell} = 24 - 6\ell^2 = 6(4 - \ell^2) = 6(2 - \ell)(2 + \ell).$$

   Hence, $\frac{dA}{d\ell} = 0$ if $\ell = 2$. (We cannot have $\ell = -2$ because $0 < \ell < 2\sqrt{3}$.) A possible local extremum thus occurs when $\ell = 2$. Now

$$\frac{d^2A}{d\ell^2} = -12\ell = -24, \quad \text{if} \quad \ell = 2.$$

   This shows that A has a local maximum, and in fact, a maximum value when $\ell = 2$. If $\ell = 2$ then

$$A = 24 \times 2 - 2 \times 8 = 32,$$

   which is the largest area that the rectangle can have.

Exercises

1. A right angle, whose hypotenuse is $\sqrt{3}$ long, is revolved about one of its legs to generate a right circular cone. Find the radius, height and volume of the cone of greatest volume that can be made this way.

2. What is the smallest perimeter possible for a rectangle whose area is 16 square metres?

3. An isosceles triangle has its vertex (the point where the two equal sides meet) at the origin, and its base parallel to the x-axis with its vertices above the x-axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

Answers

1. The volume is given by the formula
$$V = \frac{1}{3}\pi\left(3 - h^2\right)h.$$
(This comes from $V = \frac{1}{3}\pi r^2 h$ and from Pythagoras's Theorem that $r^2 = 3 - h^2$.)
We see that if $\frac{dV}{dh} = 0$, $h = 1$.
The largest volume is $\frac{2\pi}{3}m^3$ and the corresponding radius $\sqrt{2}$m.

2. If we let x and y be the side lengths, then $xy = 16$ and the perimeter $P = 2x + 2y$.
Thus we obtain the function $P(x) = 2x + 32x^{-1}$.
Differentiation gives $\frac{dP}{dx} = 2 - 32x^{-2}$ and if $\frac{dP}{dx} = 0$, then $x = 4$.
The smallest possible perimeter occurs when $x = y = 4$, then $P = 16$ m.
$$A(x) = x\left(27 - x^2\right).$$
At $x = 3$ or $x = -3$, there are local extrema.
The largest area is 54 square units.

1.4 ROLLE'S THEOREM AND THE MEAN VALUE THEOREM

There is strong geometric evidence that between any two points, where a smooth curve crosses the x-axis, there is a point on the curve where the tangent is horizontal.

The theorem stating this is called Rolle's Theorem.

Rolle's Theorem
Suppose $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b) = 0$, there is at least one number c between a and b, with $f'(c) = 0$.

The Mean Value Theorem is a generalised version of Rolle's Theorem and it is stated here.

The Mean Value Theorem
If $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b), then there is at least one number c between a and b at which
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Worked Examples

1. Suppose $f'(x) = 2$ and $f(0) = 5$. Use the Mean Value Theorem to show that $f(x) = 2x + 5$ at every value of x.

   Solution

   We consider the three cases $x > 0$, $x < 0$ and $x = 0$.

   Suppose $x > 0$. The function f is assumed to be differentiable everywhere, so in particular it is differentiable on $(0, x)$ and continuous on $[0, x]$. According to the Mean Value Theorem there exists $c \in (0, x)$ such that
$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$
   i.e.
$$\frac{f(x) - 5}{x} = 2$$
   or
$$f(x) = 2x + 5.$$

   Similarly, if $x < 0$, we can apply the Mean Value Theorem to the function f and the interval $[x, 0]$ to obtain
$$f(x) = 2x + 5.$$

   Finally, if $x = 0$, it clearly holds that
$$f(x) = 2x + 5.$$

2. Prove that the equation
$$x^4 - 4x + c = 0$$
   (where c is an arbitrary fixed constant) does not have two different roots that are less than 1.

   Solution

   Let
$$f(x) = x^4 - 4x + c.$$

   Suppose that the given equation does have two different roots that are less than 1, i.e. there exist numbers α and β such that $\alpha < \beta < 1$, $f(\alpha) = 0$ and $f(\beta) = 0$. The function f, being a polynomial function, is differentiable and continuous everywhere. In particular, f is continuous on $[\alpha, \beta]$ and differentiable on (α, β). According to the Mean Value Theorem there exists $\gamma \in (\alpha, \beta)$ such that

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(\gamma)$$

i.e.

$$0 = f'(\gamma).$$

However,

$$f'(x) = 4x^3 - 4,$$

so

$$\begin{aligned} f'(x) &= 0 \\ \Leftrightarrow 4x^3 - 4 &= 4(x-1)\left(x^2 + x + 1\right) = 0 \\ \Leftrightarrow x &= 1. \end{aligned}$$

Hence $\gamma = 1$. This is a contradiction since $\gamma < \beta < 1$. Thus the given equation does not have two different roots that are less than 1.

3. Given the function

$$f(x) = x^n + px + q$$

where n is a fixed positive even integer, and p and q are arbitrary constants, show that f cannot have more than two zeros.

Solution

Suppose that f has at least three different zeros, say α, β and γ. Also suppose that $\alpha < \beta < \gamma$. Now f is continuous on $[\alpha, \beta]$ and differentiable on (α, β). By assumption $f(\alpha) = f(\beta) = 0$. By Rolle's Theorem there exists $\delta_1 \in (\alpha, \beta)$ such that $f'(\delta_1) = 0$. By applying Rolle's Theorem to the function f and the interval $[\beta, \gamma]$ we also obtain a $\delta_2 \in (\beta, \gamma)$ such that $f'(\delta_2) = 0$. We have thus found two different roots (viz δ_1 and δ_2) of the equation

$$nx^{n-1} + p = 0. \qquad ...(*)$$

But from $(*)$ we obtain

$$x = \left(-\frac{p}{n}\right)^{\frac{1}{n-1}} \quad ,$$

i.e. the equation $(*)$ has only one root because $n - 1$ is odd. This contradiction shows that f cannot have more than 2 zeros.

4. Use the Mean Value Theorem to show that

$$|\sin\ b - \sin\ a| \leq |b - a|$$

for any real numbers a and b.

Solution

If $a = b$ it holds that

$$|\sin\ b - \sin\ a| = 0 \leq |b - a|.$$

Suppose $a < b$. The function $f(x) = \sin\ x$ is continuous over $[a, b]$ and differentiable over (a, b). According to the Mean Value Theorem there exists a $c \in (a, b)$ such that

$$\frac{f\,(b) - f\,(a)}{b - a} = f'(c).$$

Therefore

$$\frac{\sin\,(b) - \sin\,(a)}{b - a} = \cos(c)$$

i.e.

$$\begin{aligned} \left|\frac{\sin\,(b) - \sin\,(a)}{b - a}\right| &= |\cos(c)| \leq 1. \\ \therefore \quad |\sin(b) - \sin(a)| &\leq |b - a| \end{aligned}$$

for all real a and b.

5. Refer to the Mean Value Theorem above. It states that f must be continous at every point of the closed interval $[a, b]$ and also diffferentiable at every point of (a, b) for a c to exists.The converse of the theorem does not hold, i.e. if the conditions do not hold there still may be a c which satisfies the equation. Given $a = -8$, $b = 1$ and $f(x) = x^{1/3}$, show that there exists a value of c so that the equation is satisfied. Which condition of the theorem does not hold?

   Solution

   If

   $$f(x) = x^{1/3}$$

   then

   $$f'(x) = \frac{1}{3}x^{-2/3}.$$

   Since $0 \in (-8, 1)$ and $f'(0)$ does not exist, $f(x)$ is not differentiable over $(a, b) = (-8, 1)$. Therefore the differentiability condition of the Mean Value Theorem does not hold.

   Now,

   $$\begin{aligned} f'\,(c) &= \frac{f\,(b) - f\,(a)}{b - a} \\ \Rightarrow \frac{1}{3c^{\frac{2}{3}}} &= \frac{(1)^{\frac{1}{3}} - (-8)^{\frac{1}{3}}}{1 - (-8)} \\ &= \frac{3}{9} \\ &= \frac{1}{3} \\ \Rightarrow c^{\frac{2}{3}} &= 1 \\ \Rightarrow c &= \pm 1. \end{aligned}$$

   Therefore there exists a $c \in (-8, 1)$, namely $c = -1$, such that

   $$f'(c) = \frac{f\,(b) - f(a)}{b - a}.$$

Exercises

1. Use the Mean Value Theorem to prove that

$$\tan\beta - \beta \geq \tan\alpha - \alpha$$

whenever α and β are any numbers with $\beta \geq \alpha$ which lie in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

2. Use Rolle's Theorem to prove that the equation

$$2x - \cos x = 0$$

has exactly one solution in the interval $[-\pi, \pi]$.

3. Suppose that f is a differentiable odd function with domain the set of all real numbers. Prove that for each positive real number r there exists a number c such that

$$-r < c < r$$

and

$$f(r) = r.f'(c).$$

Answers

1. Let $\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and suppose that $\beta \geq \alpha$. If $\alpha = \beta$ then $\tan\beta - \beta = \tan\ \alpha - \alpha$. Suppose then that $\beta > \alpha$. Now the function $f(x) = \tan x$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). According to the Mean Value Theorem there exists a γ with $\alpha < \gamma < \beta$ such that

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(\gamma),$$

i.e.

$$\frac{\tan\beta - \tan\alpha}{\beta - \alpha} = \sec^2(\gamma). \qquad ...(*)$$

Now for all values of x for which the function sec is defined, we have $|\sec\ x| \geq 1$, so for such x, we have $\sec^2 x \geq 1$. From $(*)$ it then follows that

$$\frac{\tan\beta - \tan\alpha}{\beta - \alpha} \geq 1.$$

Since $\beta - \alpha > 0$, we can multiply through by $\beta - \alpha$ to obtain that

$$\tan\ \beta - \tan\ \alpha \geq \beta - \alpha,$$

and hence that

$$\tan\ \beta - \beta \geq \tan\ \alpha - \alpha.$$

2. Let

$$f(x) = 2x - \cos\ x.$$

Then

$$\begin{aligned} f(-\pi) &= -2\pi - (-1) \\ &= 1 - 2\pi < 0, \end{aligned}$$

while

$$f(\pi) = 2\pi - (-1) = 2\pi + 1 > 0.$$

Because f is continuous, it follows from the Intermediate Value Theorem that $f(c) = 0$ for at least one number c in the interval $(-\pi, \pi)$, i.e. the equation

$$2x - \cos x = 0$$

has at least one solution in the interval $(-\pi, \pi)$.

If there are more than one solution in the interval $(-\pi, \pi)$, then there are numbers a and b in $(-\pi, \pi)$ for which $a < b$ and $f(a) = f(b) = 0$. Now f is continuous on $[a, b]$ and differentiable on (a, b), so it follows from Rolle's Theorem that there is a number d with $a < d < b$ and $f'(d) = 0$, i.e.

$$2 + \sin\ d = 0,$$

or

$$\sin\ d = -2.$$

This is impossible. Consequently, the given equation has exactly one solution in the interval $(-\pi, \pi)$.

3. Let r be a positive real number. By assumption f is differentiable on the interval $(-r,\ r)$, and since f is continuous everywhere (being differentiable), f is continuous on $[-r,\ r]$. According to the Mean-Value Theorem there exists a number $c \in (-r,\ r)$ such that

$$\begin{aligned} f'(c) &= \frac{f(r) - f(-r)}{r - (-r)} \\ &= \frac{f(r) + f(r)}{2r} \quad (-f(-r) = f(r)\,,\ \text{since } f \text{ is odd}) \\ &= \frac{f(r)}{r}. \end{aligned}$$

Hence

$$f(r) = r \cdot f'(c).$$

Summary

We have focussed in this chapter on a large range of applications. The learner should now be able to solve related rates problems, using the Mean Value Theorem and Rolle's Theorem and also optimization problems. The learner should now also be able to sketch graphs of functions using derivatives.

Chapter 2

Transcendental Functions

Introduction

Many of the functions in mathematics and science are inverses of one another. The functions $\ln x$ and e^x are probably the most famous function–inverse pair. The trigonometric functions when suitably restricted, have useful inverses. Less widely known are the hyperbolic functions and their inverses. We also show using L'Hôpital's rule, how a limit of a fraction can be calculated when both the numerator and the denominator approach zero (we call this an indeterminate form).

This chapter consists of the following sections:

2.1 The logarithmic and exponentical functions.

This has already been done in MAT1512. Because of the *importance* of these functions and their properties we do *revision* on these parts.

2.2 Determining indeterminate forms with L'Hôpitals' rule.

2.3 The inverse trigonometric functions.

2.4 Hyperbolic functions.

Outcomes

After studying this chapter the learner should be able to:

- use L'Hôpital's rule to evaluate limits of indeterminate forms.
- use the rules of differentiation and integration to differentiate or integrate f, where f is an inverse trigonometric function or a hyperbolic function.
- obtain integrals and derivatives whenever these functions also appear in products, sums, quotients, or composites.

2.1 THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS

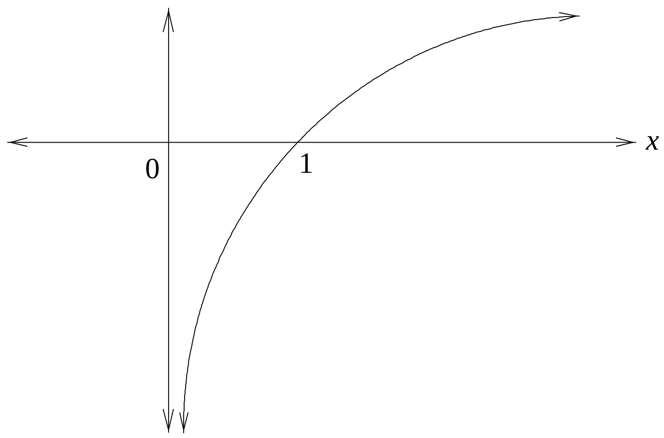
The *inverse* of the exponential function e^x is called the natural logarithm function $y = \ell n\ x$. In this section, we study the natural logarithm both as a differentiable function and as a device for simplifying calculations.

2.1.1 THE LOGARITHMIC FUNCTION AND ITS PROPERTIES

> The natural logarithmic function $y = \ell n\ x$ is the inverse function of the exponential function.

Properties of the ℓn function.

(a) The graph of $\ell n\ x$ looks like this:



(b) ℓn is defined for all $x > 0$, in other words

$$\text{Dom }(\ell n) = \{x|\, x > 0\}\,.$$

(c) $\text{Im }(\ell n) = \mathbb{R}$.

(d) $\ell n\ x < 0$ if $x < 1$,

$\ell n\ 1 = 0$,

$\ell n\ x > 0$ if $x > 1$.

(e) $\dfrac{d}{dx}(\ell n\ x) = \dfrac{1}{x}$, for $x > 0$.

(f) $\ell n(xy) = \ell n\ x + \ell n\ y$, $x, y > 0$.

(g) $\ell n \dfrac{x}{y} = \ell n\ x - \ell n\ y$, $x, y > 0$.

(h) $\ell n\ x^r = r\ \ell n\ x$, $x > 0$.

(i) $\log_b a = \dfrac{\log_c a}{\log_c b}$.

(j) If $x \to \infty$, then $\ell n\ x \to \infty$.

(k) If $x \to 0^+$, then $\ell n\ x \to -\infty$.

The following important mistake is often made when applying rule (h):

$$\ell n\,(x^r) = r\ \ell n\ x \qquad \text{(correct)}$$

but

$$(\ell n\ x)^r \neq r\ \ell n\ x \qquad \text{(incorrect).}$$

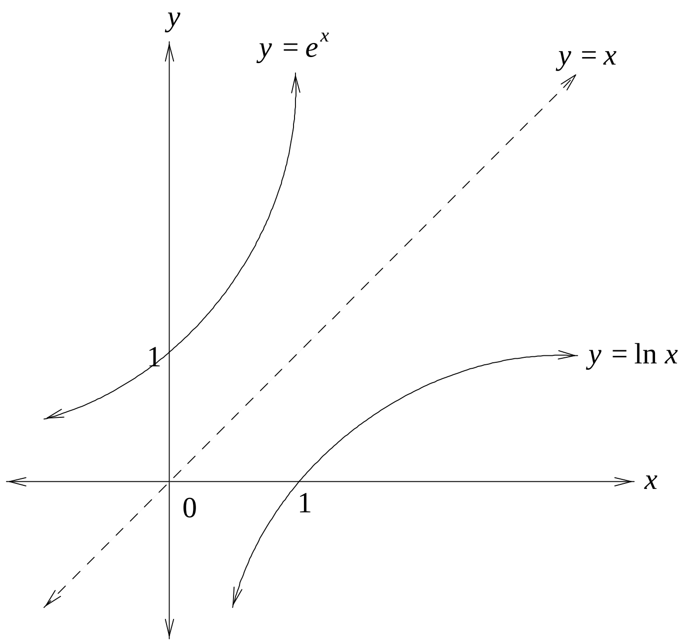
Note that the power r must be a power of x but not of $\ell n\ x$.

2.1.2 THE EXPONENTIAL FUNCTION AND ITS PROPERTIES

The inverse function of the function $\ell n\ x$ is known as the *exponential function* and is indicated by $\exp\ x$ (or e^x).

Properties of the exp function

(a) The graph of e^x is the mirror image of the graph of $\ell n\ x$ in the line $y = x$. Therefore, the graph looks like this:



(b) exp is defined for all $x \in \mathbb{R}$, in other words

$$\text{Dom }(\exp) = \mathbb{R}.$$

(c) Im (exp) $= \{y | y > 0\}$.

(d) $\frac{d}{dx}(e^x) = e^x$.

(e) $e^{x+y} = e^x \cdot e^y$.

(f) $e^{x-y} = e^x/e^y$.

(g) $e^{rx} = (e^x)^r$.

(h) If $x \to \infty$, then $e^x \to \infty$.

(i) If $x \to -\infty$, then $e^x \to 0$.

2.1.3 FUNCTIONS OF THE FORM $f(x) = g(x)^{h(x)}$

We apply logarithmic differentiation to functions when they have an indeterminate form:

$$f(x) = g(x)^{h(x)}.$$

We apply the logarithmic function to both sides and then make use of the properties of the logarithmic function.
Then

$$\ell n\ f(x) = \ell n\ g(x)^{h(x)}.$$

Therefore

$$\ell n\ f(x) = h(x).\ell n\ g(x).$$

Now we have that

$$\begin{aligned} f(x) &= e^{\ln f(x)} \\ &= e^{h(x) \ln f(x)} \end{aligned}$$

This is very important when we use L'Hôpital's rule.

Worked Example
Determine

$$\boxed{f'(x) \text{ if } f(x) = \sqrt{x}^{\sqrt{x}},\ x > 0.}$$

Solution
By applying the logarithmic function on both sides, we obtain

$$\begin{aligned} \ell n\ f(x) &= \ell n \sqrt{x}^{\sqrt{x}} \\ &= \sqrt{x} \cdot \ell n\ \sqrt{x}. \end{aligned}$$

By differentiating, we obtain from the chain rule and the product rule that

$$\begin{aligned} \frac{1}{f(x)} f'(x) &= \sqrt{x} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2} + \ell n\ \sqrt{x} \cdot \frac{1}{2} x^{-1/2} \\ &= \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} \ell n\ \sqrt{x}. \end{aligned}$$

Therefore

$$\begin{aligned} f'(x) &= f(x)\left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x}}\ell n\ \sqrt{x}\right) \\ &= \sqrt{x}^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\left(1+\ell n\ \sqrt{x}\right)\right) \\ &= \frac{\sqrt{x}^{\sqrt{x}}}{2\sqrt{x}}\left(1+\ell n\ \sqrt{x}\right). \end{aligned}$$

2.2 DETERMINING INDETERMINATE FORMS WITH L'HÔPITAL'S RULE

This section is very important and as you will use the methods discussed here regularly in future. Since it is so important, we discuss it in detail.

Limits having one of the following forms

$$\boxed{\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ \pm\infty.0,\ \infty-\infty,\ 0^0,\ 1^\infty,\ \infty^0}$$

are **called limits of indeterminate form**. A very useful rule, which we can apply for determining limits of indeterminate form, is L'Hôpital's rule.

(a) *L'Hôpitals rule for the form* $\frac{0}{0}$

$$\boxed{\text{Let } \mathbf{lim} \text{ indicate one of the limits } \lim_{x\to a},\ \lim_{x\to a^-},\ \lim_{x\to a^+},\ \lim_{x\to\infty},\ \lim_{x\to-\infty},}$$

and suppose $\lim f(x) = 0$ and $\lim g(x) = 0$. If $\lim\ [f'(x)/g'(x)]$ has a finite value L or if this limit is $+\infty$ or $-\infty$, then

$$\boxed{\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}\ .}$$

(b) *L'Hôpitals rule for the form* $\frac{\pm\infty}{\pm\infty}$

Let lim indicate one of the limits in (a) and suppose $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. If $\lim\ [f'(x)/g'(x)]$ has a finite value L or if this limit is $+\infty$ or $-\infty$, then

$$\boxed{\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.}$$

We always try to modify the other forms into one of $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

It is very important to see that we have to find the derivative of the function $f(x)$ ***separately*** *from the derivative of the function $g(x)$. You also have to simplify first before applying the next step, for example applying L'Hôpital's rule again.*

In the following example, we first follow the correct method and then we do the same problem incorrectly.

Worked Example

$$\boxed{\lim_{x\to 1}\frac{-x^2+1}{x-1}}$$

The *correct* method (finding the derivative of the numerator and denominator separately) is:

$$\lim_{x\to 1}\frac{-x^2+1}{x-1}=\lim_{x\to 1}\frac{-2x}{1}=-2.$$

(We first determine the derivative of $-x^2+1$ which is $-2x$ and then separately the derivative of $x-1$ which is 1.)

The *incorrect* method (using the quotient rule and not obtaining the derivatives separately) is:

$$\begin{aligned}
\lim_{x\to 1}\frac{-x^2+1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(-2x)-\left(-x^2+1\right)(1)}{(x-1)^2} \qquad (*)\\
&= \lim_{x\to 1}\frac{-2x^2+2x+x^2-1}{(x-1)^2}\\
& \lim_{x\to 1}\frac{-x^2+2x-1}{(x-1)^2}\\
&= \lim_{x\to 1}-\frac{\left(x^2-2x+1\right)}{(x-1)^2}\\
&= \lim_{x\to 1}-\frac{(x-1)^2}{(x-1)^2}\\
&= -1 \quad \text{which is incorrect.}
\end{aligned}$$

Some examples (indicating how one handles the various limits) follow below:

2.2.1 LIMITS OF THE FORM $\frac{0}{0}$ AND $\frac{\pm\infty}{\pm\infty}$

In these cases we can simply apply L'Hôpital's rule directly.

Worked Examples

1. Determine

$$\boxed{\lim_{x\to\infty}\frac{\ell n\, x}{x}.}$$

Solution

It is an indeterminate form of the type $\frac{\infty}{\infty}$. Thus, we apply L'Hôpital's rule

$$\lim_{x\to\infty}\frac{\ell n\, x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0.$$

2. Determine

$$\boxed{\lim_{x\to\infty}\frac{e^x}{x^k},\ k>0.}$$

Solution

It is an indeterminate form of the type $\frac{\infty}{\infty}$ and therefore

$$\lim_{x\to\infty}\frac{e^x}{x^k}=\lim_{x\to\infty}\frac{e^x}{kx^{k-1}}\ ,$$

which is again of the form $\frac{\infty}{\infty}$. By repeated application of L'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{e^x}{x^k} &= \lim_{x\to\infty}\frac{e^x}{k!} \quad (e^x \text{ is an increasing function})\\ &= \infty.\end{aligned}$$

3. Determine

$$\boxed{\lim_{x\to 0}\frac{x\ e^x}{1-e^x}.}$$

Solution

It is of the type $\frac{0}{0}$ and therefore

$$\begin{aligned}\lim_{x\to 0}\frac{x\ e^x}{1-e^x} &= \lim_{x\to 0}\frac{x\ e^x+e^x}{-e^x}\\ &= \lim_{x\to 0}(-x-1) \quad \text{(simplify)}\\ &= -1.\end{aligned}$$

2.2.2 LIMITS OF THE FORM $\pm\infty\cdot\mathbf{0}$

If $\lim\limits_{x\to\infty}[f(x)\cdot g(x)]$ is an indeterminate form of the type $\pm\infty\cdot 0$, we write it as

$$\boxed{\lim_{x\to\infty}\frac{f(x)}{1/g(x)}}$$

which is again of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Worked Example
Determine

$$\boxed{\lim_{x\to 0^+} x^2 \ell n\ x.}$$

Solution
Since x^2 can be written as $\frac{1}{\frac{1}{x^2}}$ we have

$$\begin{aligned}
\lim_{x\to 0^+} x^2 \ell n\ x &= \lim_{x\to 0^+} \frac{\ell n\ x}{1/x^2} \quad (\text{of the form } \frac{-\infty}{\infty}) \\
&= \lim_{x\to 0^+} \frac{1/x}{-2/x^3} \\
&= \lim_{x\to 0^+} -\frac{1}{2}x^2 \quad (\text{simplify}) \\
&= 0.
\end{aligned}$$

2.2.3 LIMITS OF THE FORM $\infty - \infty$

We try algebraic manipulation (for example by using a common denominator). Then we rewrite the form $\infty - \infty$ into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Worked Example
Determine

$$\boxed{\lim_{x\to 0} \left[\frac{1}{x} - \frac{1}{\ell n\,(1+x)}\right].}$$

Solution
Obtaining a *common denominator* we have

$$\begin{aligned}
\lim_{x\to 0} \left[\frac{1}{x} - \frac{1}{\ell n\,(1+x)}\right] &= \lim_{x\to 0} \left[\frac{\ell n\ (1+x) - x}{x\ \ell n\ (1+x)}\right] \quad (\text{of the form } \frac{0}{0}) \\
&= \lim_{x\to 0} \frac{\frac{1}{1+x} - 1}{\frac{x}{1+x} + \ell n\,(1+x)} \\
&= \lim_{x\to 0} \frac{1-(1+x)}{x + (1+x)\,\ell n\ (1+x)} \quad (\text{simplify}) \\
&= \lim_{x\to 0} \frac{-x}{x + (1+x)\,\ell n\ (1+x)} \quad (\text{of the form } \frac{0}{0}) \\
&= \lim_{x\to 0} \frac{-1}{1 + \frac{1+x}{1+x} + \ell n\,(1+x)} \\
&= -\frac{1}{2}.
\end{aligned}$$

Remark
Note carefully that we simplify the expression after each step before applying the rule again.

2.2.4 LIMITS OF THE FORMS 0^0, 1^∞, ∞^0

We shall once again try to reduce these limits to the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. To do this, we make use of the *logarithmic function.* As the logarithmic function is continuous on $\mathbb{R}^+$ (the positive real numbers),

$$\boxed{\lim_{x\to a} \ell n\ f(x) = \ell n[\lim_{x\to a} f(x)].}$$

Worked Example

1. Determine

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.}$$

(Please see 2.1.3(S).)

Solution

It is a limit of the form 1^∞, therefore we consider

$$\begin{aligned}
\ell n\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right] &= \lim_{x\to\infty}\ \ell n\ \left(1+\frac{1}{x}\right)^x \\
&= \lim_{x\to\infty} x\ \ell n\ \left(1+\frac{1}{x}\right) \quad \text{(of the form } \infty\cdot 0) \\
&= \lim_{x\to\infty} \frac{\ell n\ (1+\frac{1}{x})}{\frac{1}{x}} \quad \text{(of the form } \frac{0}{0}) \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot -\frac{1}{x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\
&= 1.
\end{aligned}$$

Now

$$\ell n\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right] = 1,$$

from which it follows that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e.$

2. Determine

$$\boxed{\lim_{x\to\infty}(1+3x)^{\frac{1}{2x}}.}$$

Solution

It is a limit of the form ∞^0 therefore we consider the method $e^{\ln(f(x))^{g(x)}}$. We have

$$\begin{aligned}
& \ln\left(\lim_{x\to\infty}(1+3x)^{\frac{1}{2x}}\right) \\
= & \lim_{x\to\infty}\ln(1+3x)^{\frac{1}{2x}} \\
= & \lim_{x\to\infty}\frac{1}{2x}\ln(1+3x) \\
= & \lim_{x\to\infty}\frac{\ln(1+3x)}{2x} \qquad (\text{of the form } \frac{\infty}{\infty}) \\
= & \lim_{x\to\infty}\frac{\frac{3}{(1+3x)}}{2} \\
= & \lim_{x\to\infty}\frac{\frac{3}{2}}{(1+3x)} \qquad (\text{Remember } \lim_{x\to\infty}\frac{c}{x}=0, \text{ where } c \text{ is a constant}) \\
= & \ 0
\end{aligned}$$

Consequently

$$\lim_{x\to\infty}(1+3x)^{\frac{1}{2x}} = e^0 = 1.$$

Remarks

We draw your attention to the fact that the following are not *indeterminate forms:*
A limit of the form

(a) $\frac{0}{\infty}$ *has the value 0*

(b) 0^∞ *has the value 0*

(c) $\infty\cdot\infty$ *has the value* ∞

(d) $+\infty+(+\infty)$ *has the value* ∞

(e) $+\infty-(-\infty)$ *has the value* ∞

(f) $-\infty+(-\infty)$ *has the value* $-\infty$

(g) $-\infty-(+\infty)$ *has the value* $-\infty$.

Now you can try to do the following yourself. We again give the answers so that you can make sure that you understand this section thoroughly.

Exercises

Determine the limits:

(1) $\lim_{x\to\infty}\frac{x^k}{e^x}$, $k>0$.

(2) $\lim_{x\to\infty}(\ell n\ x)^{1/x}$.

(3) $\lim_{x\to 0^+} x^x$.

(4) $\lim_{x\to\infty} \frac{x^4+x^2}{e^x+1}$.

(5) $\lim_{x\to\infty} x\ \ell n\left(\frac{x-1}{x+1}\right)$.

(6) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$.

(7) $\lim_{x\to 0}(1+kx)^{1/x}$.

(8) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

(9) $\lim_{x\to 0}\frac{x-\sin x}{e^x-1}$.

Answers

(1) 0

(2) 1

(3) 1

(4) 0

(5) -2

(6) $\frac{1}{2}$

(7) e^k

(8) 0

(9) 0.

2.3 THE INVERSE TRIGONOMETRIC FUNCTIONS

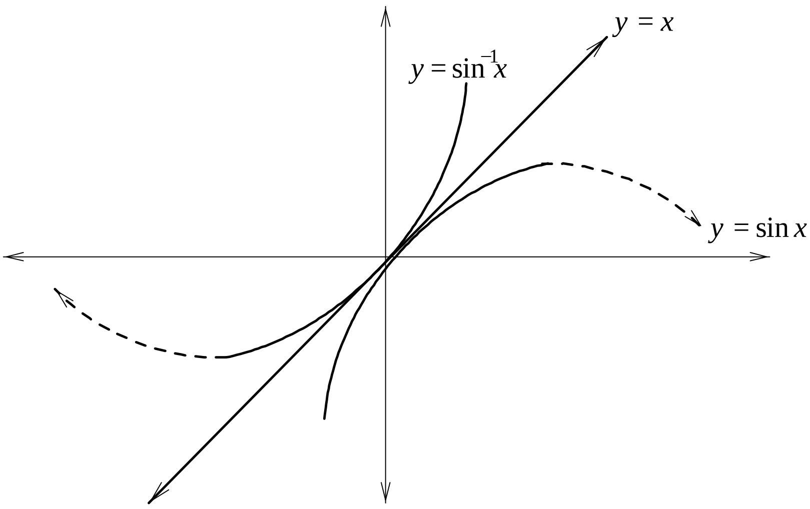
This section is very important since here you will come across further inverse functions that you will use frequently. Also see that the inverse trigonometric function for $\sin x$ can either be denoted by $\arcsin x$ or $\sin^{-1} x$ (similarly for the other inverse trigonometric functions). Look carefully at the following important facts in connection with these functions:

2.3.1 DOMAINS FOR THE INVERSE TRIGONOMETRIC FUNCTIONS

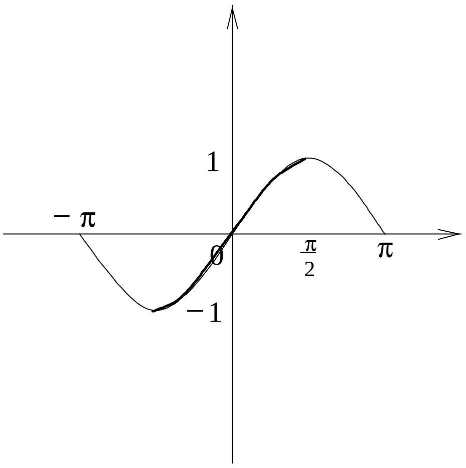
An inverse function f^{-1} of the function f can only be defined if f is one-to-one on its domain, that is for each value y in the range, we have precisely one x value in the domain. The trigonometric functions are not one-to-one. Thus their inverses do not exist. If we *restrict* the domains of the trigonometric functions, then their inverses exist on these specific intervals.

When we **restrict** the domains of the trigonometic functions to specific intervals, their *inverses exist* on these intervals. Let us look at the following example:

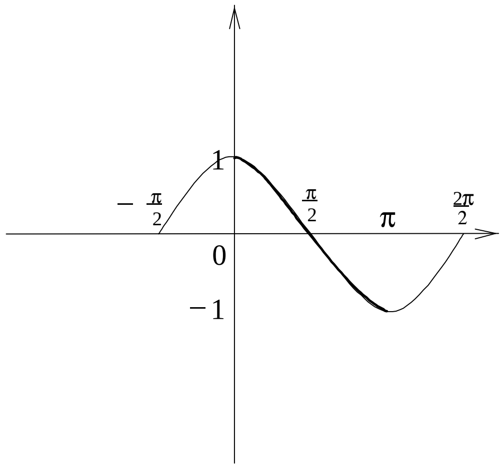
We know that the sine function is strictly increasing on the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and if we restrict $\sin x$ to this interval, the inverse does exist. See the following figure:



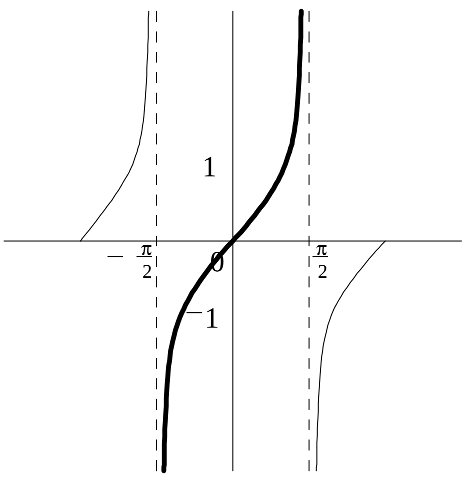
Now consider the graphs of the trigonometric functions:



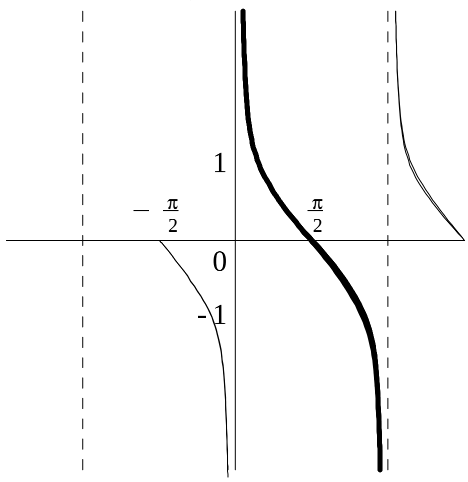
$y = \sin x$



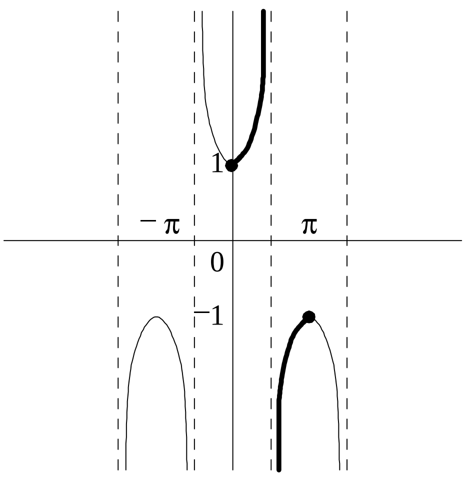
$y = \cos x$



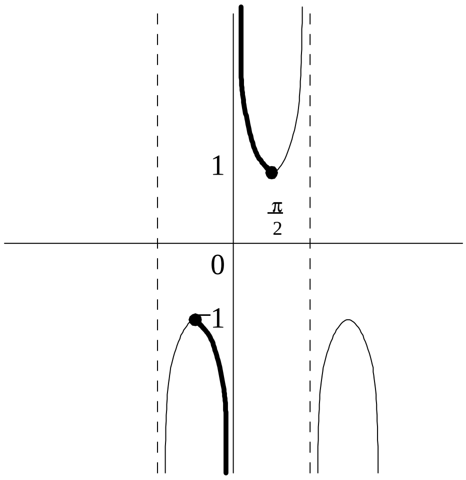
$y = \tan x$



$y = \cot x$



$y = \sec x$



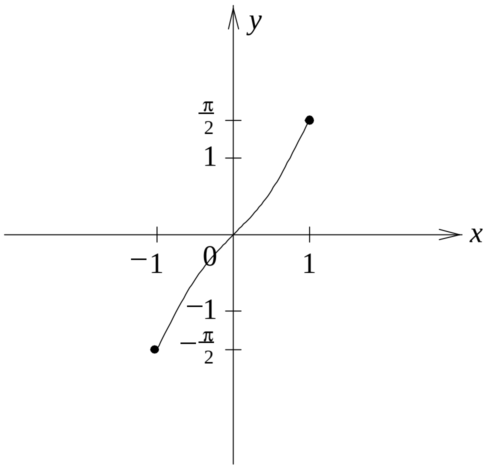
$y = \text{cosec } x$

By restricting the domains of the trigonometric functions as shown in the graphs, it is now possible to define the inverse trigonometric functions.

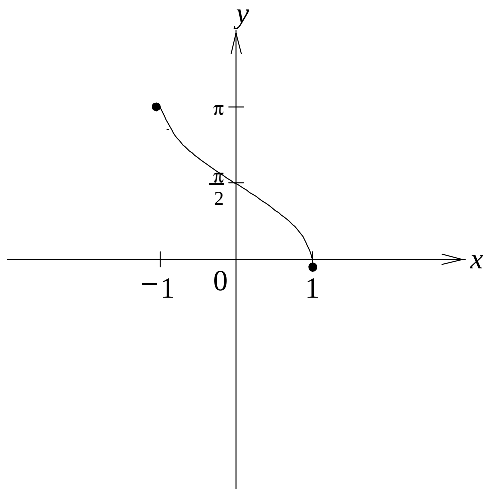
See the following table:(Note that the range of all the inverse functions are NEVER in the third quadrant)

Inverse function	Domain	Range
$y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ quadrants I & IV
$y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$ quadrants I & II
$y = \tan^{-1} x$	$-\infty < x < +\infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$ quadrants I & IV
$y = \sec^{-1} x$	$x \geq 1$ or $x \leq -1$	$0 \leq y \leq \pi \;\; y \neq \frac{\pi}{2}$ quadrants I & II
$y = \text{cosec}^{-1} x$	$x \geq 1$ or $x \leq -1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \;\; y \neq 0$ quadrants I & IV
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$ quadrants I & II

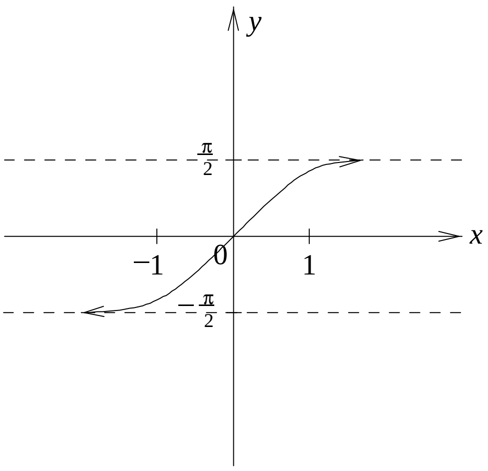
The following diagrams give the graphs of the six inverse trigonometric functions:



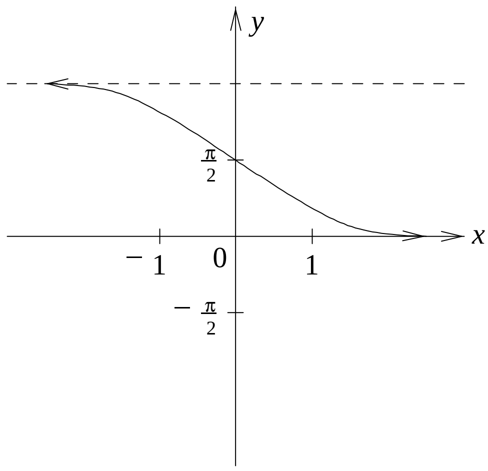
$y = \sin^{-1} x$



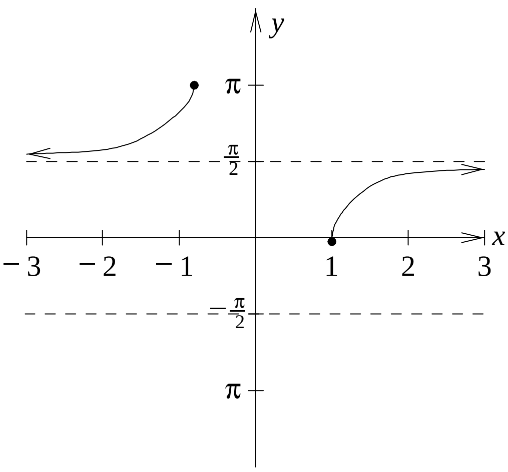
$y = \cos^{-1} x$



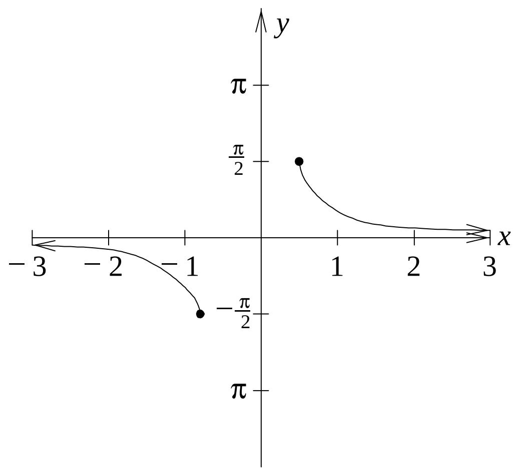
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

As you can see from the above, it is extremely important always to consider the domains of the different *inverse trigonometric functions* when doing proofs or calculations using these functions. Consider the following examples:

Worked Examples

1. Calculate

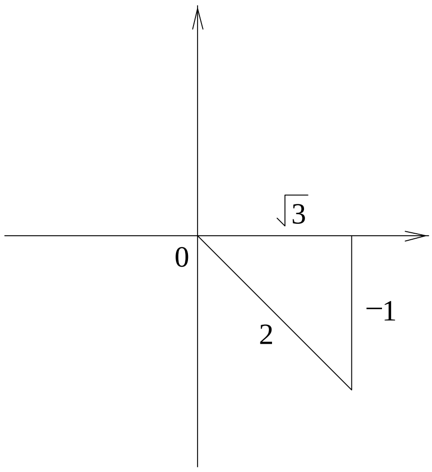
$$\boxed{\tan\left(\sin^{-1}\left(-\tfrac{1}{2}\right)\right).}$$

Solution

Let $\sin^{-1}\left(-\frac{1}{2}\right) = \theta$. Then

$$\sin\theta = -\frac{1}{2} \quad \text{and} \quad \boxed{-\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}}\,.$$

Hence it is obvious that θ must be in the first or fourth quadrant. Since $\sin\theta < 0$, we know that $\sin\theta$ must be in the fourth quadrant. We can represent it as follows:



Hence

$$\tan\left((\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\theta = -\frac{1}{\sqrt{3}}\,.$$

(Note: We do not use a pocket calculator to find an approximate value!)

2. Prove that

$$\boxed{\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}.}$$

Solution

We have $y = \cos^{-1}x$ therefore $x = \cos y$. Because the cosine is $1-1$ and differentiable on the open interval $(0,\pi)$, the inverse of cosine is also differentiable on the open interval $(-1,1)$. We will determine the derivative of $y = \cos^{-1}x$:

$$\begin{aligned} \cos y &= x \\ \frac{d}{dx}(\cos y) &= \frac{d}{dx}(x)\,. \end{aligned}$$

Using the chain rule, we have $-\sin y \frac{dy}{dx} = 1$.

If we now devide by $\sin y > 0$ (where $0 < y < \pi$), we have

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}.$$

The derivative of $y = \cos^{-1} x$ with regard to x is then

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}.$$

2.3.2 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS; RELATED INTEGRALS

Just as important as the differentiation is the simplification of the derived function. Consider the following examples, then attempt the given exercises.

Worked Example

Determine the derivative of

$$\boxed{f(x) = x\ \mathrm{cosec}^{-1}\tfrac{1}{x} + \sqrt{1-x^2}}$$

and simplify your answer.

Solution

$$f'(x) = \mathrm{cosec}^{-1}\frac{1}{x} \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}\left(\mathrm{cosec}^{-1}\frac{1}{x}\right) + \frac{d}{dx}\left(\sqrt{1-x^2}\right).$$

Using the chain rule

$$\frac{d}{dx}\left(\mathrm{cosec}^{-1}\frac{1}{x}\right) = \frac{-1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}} \cdot \frac{d}{dx}\left(\frac{1}{x}\right).$$

Hence

$$\begin{aligned} f'(x) &= \mathrm{cosec}^{-1}\frac{1}{x} \cdot 1 + x\left(\frac{-1}{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x^2}-1}}\right)\left(-\frac{1}{x^2}\right) \\ &\quad + \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) \\ &= \mathrm{cosec}^{-1}\frac{1}{x} + \frac{x}{\left|\frac{1}{x}\right|\sqrt{\frac{1-x^2}{x^2}} \cdot x^2} - \frac{x}{\sqrt{1-x^2}}. \end{aligned}$$

We know that $\sqrt{x^2} = |x|$, with $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$. Also $|x|.|x| = x^2$. Thus

$$\begin{aligned} f'(x) &= \mathrm{cosec}^{-1}\frac{1}{x} + \frac{x}{\frac{1}{|x|} \cdot \frac{1}{|x|}\sqrt{1-x^2} \cdot x^2} - \frac{x}{\sqrt{1-x^2}} \\ &= \mathrm{cosec}^{-1}\frac{1}{x} + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \mathrm{cosec}^{-1}\frac{1}{x}. \end{aligned}$$

Exercises

1. Calculate the exact values of each of the following:

   (a) $\sin\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$.

   (b) $\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$.

2. Determine the derivative of each of the following and simplify your answers:

   (a) $f(x) = (x^2+1)\tan^{-1}\left(\frac{x-1}{x+1}\right)$, $x \neq -1$.

   (b) $f(x) = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}$, $a < 0$.

Answers

1. (a) $4/5$.

   (b) $2/\sqrt{5}$.

2. (a) $2x\tan^{-1}\frac{x-1}{x+1} + 1$.

   (b) $\frac{-2x^2}{\sqrt{a^2-x^2}}$.

2.4 HYPERBOLIC FUNCTIONS

Certain *combinations of the exponential function* appear so often in the application as well as the theory of mathematics, that it is worthwhile to assign special names to them. You will notice that their properties are reminiscent of those of the trigonometric functions which also explains the names of these functions. The hyperbolic functions are defined as follows:

2.4.1

hyperbolic sine function:	$\sinh x = \frac{e^x-e^{-x}}{2}$
hyperbolic cosine function:	$\cosh x = \frac{e^x+e^{-x}}{2}$
hyperbolic tangent function:	$\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$
hyperbolic cotangent function:	$\coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}}$
hyperbolic secant function:	$\operatorname{sech} x = \frac{2}{e^x+e^{-x}}$
hyperbolic cosecant function:	$\operatorname{cosech} x = \frac{2}{e^x-e^{-x}}$

The following identities on the hyperbolic functions can be proved easily.

2.4.2

Identities:

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2\sinh x\cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x+1}{2}$

$\sinh^2 x = \frac{\cosh 2x-1}{2}$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\coth^2 x = 1 + \operatorname{cosech}^2 x$

The derivatives of the hyperbolic function follow easily from the above definitions as well as the facts that $\frac{d}{dx}\left(e^{x}\right)=e^{x}$ and $\frac{d}{dx}\left(e^{-x}\right)=(-1)\,e^{-x}$.
The following table is important:

2.4.3

Derivatives of the hyperbolic functions:
$\frac{d}{dx}(\sinh x)=\cosh x$
$\frac{d}{dx}(\cosh x)=\sinh x$
$\frac{d}{dx}(\tanh x)=\text{sech}^2 x$
$\frac{d}{dx}(\coth x)=-\text{cosech}^2 x$
$\frac{d}{dx}(\text{sech } x)=-\text{sech} x\tanh x$
$\frac{d}{dx}(\text{cosech } x)=-\text{cosech } x\coth x$

The integral formulas for the hyperbolic functions are:

2.4.4

Integrals of the hyperbolic functions:
$\int \sinh x dx=\cosh x+c$
$\int \cosh x dx=\sinh x+c$
$\int \text{sech}^2 x\ dx=\tanh x+c$
$\int \text{cosech}^2 x dx=-\coth x+c$
$\int \text{sech } x\tanh x dx=-\text{sech} x+c$
$\int \text{cosech} x\coth x dx=-\text{cosech} x+c$

Worked Examples

1. Prove that

$$\boxed{\frac{d}{dx}(\sinh x) = \cosh x.}$$

Solution

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right)\\ &= \frac{1}{2}\left(e^{x}-\left(e^{-x}\right)(-1)\right)\\ &= \frac{e^{x}+e^{x}}{2}\\ &= \cosh x.\end{aligned}$$

2. Determine

$$\boxed{\frac{d}{dx}\left\{\ell n(\coth x)\right\}.}$$

Solution

$$\begin{aligned}\frac{d}{dx}\{\ell n\,(\coth\ x)\} &= \frac{1}{\coth\ x}\left(-\text{cosech}^2 x\right)\\ &= \frac{-\sinh\ x}{\cosh\ x}\cdot\frac{1}{\sinh^2 x}\\ &= \frac{-1}{\sinh\ x\cosh\ x}.\end{aligned}$$

3. Determine

$$\boxed{\int \tanh^2 2x\ dx.}$$

Solution

$$\begin{aligned}\int \tanh^2 2x\ dx &= \int \left(1-\text{sech}^2\ 2x\right)dx\\ &= x-\frac{1}{2}\tanh\ 2x+c.\end{aligned}$$

4. Determine

$$\boxed{\int \text{sech}\ x\ dx.}$$

Solution

$$\begin{aligned}\int \text{sech}\ x\ dx &= \int \frac{dx}{\cosh\ x}\\ &= \int \frac{\cosh\ x}{\cosh^2 x}dx\\ &= \int \frac{\cosh\ x}{1+\sinh^2 x}dx\\ &= \arctan(\sinh\ x)+c\end{aligned}$$

(by using the substitution $u = \sinh\ x$).

5. Determine

$$\boxed{\int e^x \cosh\ x\ dx.}$$

Solution

$$\begin{aligned}\int e^x\cosh\ x\ dx &= \int e^x\left(\frac{e^x+e^{-x}}{2}\right)dx\\ &= \frac{1}{2}\int (e^{2x}+1)dx\\ &= \frac{1}{4}e^{2x}+\frac{1}{2}x+c.\end{aligned}$$

6. Determine

$$\boxed{\int_0^{\ell n\ 2} \frac{\sinh}{1+\cosh\ x} dx.}$$

Solution

Consider $\displaystyle\int_0^{\ell n\ 2} \frac{\sinh\ x}{1+\cosh\ x} dx.$

Let $u = 1 + \cosh\ x$. Then $du = \sinh\ x\ dx$. Also, if $x = 0$, then

$$u = 1 + \cosh\ 0 = 1 + \frac{e^0 + e^{-0}}{2} = 1 + \frac{1+1}{2} = 2,$$

and if $x = \ell n\ 2$, then

$$u = 1 + \cosh(\ell n\ 2) = 1 + \frac{e^{\ell n 2} + e^{-\ell n 2}}{2} = 1 + \frac{2 + \frac{1}{2}}{2} = \frac{9}{4}.$$

Hence

$$\begin{aligned} \int_0^{\ell n\ 2} \frac{\sinh\ x}{1+\cosh\ x} dx &= \int_2^{9/4} du/u \\ &= \ell n\ \ u]_2^{9/4} \\ &= \ell n \frac{9}{4} - \ell n\ 2 \\ &= \ell n \frac{9}{8}. \end{aligned}$$

Note how the definition of cosh x is used in Example 6 to calculate the values of cosh 0 and cosh ($\ell n2$).

Again note that $\ell n\frac{9}{8}$ is an exact answer and a pocket calculator is not used to approximate it.

Now try the following:

Exercises

1. Rewrite the expressions in terms of exponentials and simplify the results

   (a) $2\cosh(\ln x)$

   (b) $\cosh 8x + \sinh 8x$

   (c) $\ln(\cosh x - \sinh x) + \ln(\cosh x + \sinh x)$

2. Determine

   (a) $\frac{d}{dx}\left[\cosh\left(x^2 - 1\right)\right].$

   (b) $\frac{d}{dx}\left[\sinh^3(x/2)\right].$

   (c) $\frac{d}{dx}\left[e^{x/2} \tanh\ 2x\right].$

(d) $\frac{d}{dx}\left(\tanh \frac{1}{x}\right)$

3. Determine

(a) $\int \sinh^2 x \ dx$

(b) $\int \text{sech}^4 x \ dx$

(c) $\int x \sinh x \ dx.$

(d) $\int_0^{\ln 2} 4e^{-x} \sinh x dx$

Answers

1. (a) $x + \frac{1}{x}$

   (b) e^{8x}

   (c) 0

2. (a) $2x \sinh (x^2 - 1).$

   (b) $\frac{3}{2} \sinh^2 \frac{x}{2} \cosh \frac{x}{2}.$

   (c) $e^{x/2}(2 \ \text{sech}^2 2x + \frac{1}{2} \tanh 2x).$

   (d) $-\frac{1}{x^2} \text{sech}^2 \left(\frac{1}{x}\right)$

3. (a) $\frac{1}{4} \sinh 2x - \frac{1}{2}x + c.$

   (b) $\tanh x - \frac{1}{3} \tanh^3 x + c.$

   (c) $x \cosh x - \sinh x + c.$

   (d) $\ln 4 - \frac{3}{4}$

Summary

The learner will now be able to obtain limits of indeterminate forms when they appear in the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0.\infty$ and also when they occur in the functions $f(x) = g(x)^{h(x)}$ as 1^0, 1^∞, ∞^0. L'Hôpital's rule can be used in all these forms by applying an adaptation of the original given form. The learner will also be able to demonstrate skills in differentiation and integration of the inverse trigonometric function and hyperbolic function.

Chapter 3

Applications of Integration (areas between curves, solids of revolution)

Introduction

This section shows how to find the areas of regions in the coordinate plane by integrating the functions that define the boundaries of the regions. Solids generated by revolving plane regions about axes are solids of revolution. Threaded spools are solids of revolution and so are billiard balls. These solids have volumes we can find using geometry. But sometimes we want to find the volume of a blimp instead or to predict the weight of a part that will be turned on a lathe. In these cases geometric formulas are of little help and we have to return to calculus for the answers.

Outcomes

After studying this chapter, the learner should be able to:

- find the area between curves of a variety of functions.
- find the volume of a solid of revolution (by rotation about either the x-axis or the y-axis) using the disk method or the washer method.

3.1 AREAS BETWEEN CURVES

Study this whole section carefully and work through Examples 1.1 – 1.6. When studying this section you have to keep the following in mind: It is important to make *sketches* whenever possible, when you have to determine the area underneath the curve, the area between curves or the volume of a solid of revolution. You should always show *all calculations*, even, for example, the calculations to determine points of intersection. If you do not show all your calculations, we cannot identify your problem areas and then we would not be able to help you. Sometimes it could be easier to *integrate with respect to y* than to integrate with respect to x.

3.2 VOLUMES OF SOLIDS OF REVOLUTION

The volume of a solid of revolution about the x–axis (see that the formula is in terms of x): The volume of a solid of revolution between the x–axis and the graph of a continuous function $y = R(x)$, $a \leq x \leq b$ *about the x–axis* is

$$\boxed{V = \int_a^b \pi \, (\text{radius})^2 \, dx = \int_a^b \pi \, (R(x))^2 \, dx}$$

and **the volume of a solid of revolution about the y–axis is (see that the formula is in terms of y):**

$$\boxed{V = \int_c^d \pi \, (\text{radius})^2 \, dy = \int_c^d \pi \, (R(y))^2 \, dy.}$$

Study this whole section thoroughly. Work through all the Examples in this section carefully. Use the following guidelines to calculate the volumes of solids of revolution:

- *Sketch* the given curves.
- Calculating a volume by revolving the region between the two curves about an axis, make sure that you know which of the two curves will be the *greater* over the given interval, otherwise it is possible that you may get a negative answer.
- Make sure *about which axis*, i.e. the x-axis or the y-axis, the area revolves.
- When using the washer method, make sure that you know what $R(x)$ = outer radius and $r(x)$ = inner radius represent.
- When the rotation axis is parallel to the *x-axis* the formula is also in terms of x and similarly, when the rotation axis is parallel to the *y-axis* the formula is also in terms of y.

Now we will do an example to show you once more how to work with the rotations about the different axes:

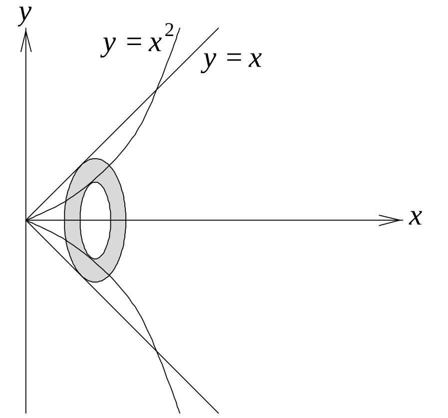
Worked Examples

1. Determine the volume of the solid of revolution which results when the closed region enclosed by $y = x^2$ and the line $y = x$ is rotated about the x-axis;

   Solution

   The graphs of $y = x^2$ and $y = x$ intercept where $x^2 - x = 0$, i.e. where $x = 0$ and $x = 1$. So the points are $(0, 0)$ and $(1, 1)$.

We first sketch the region R as in the following figure:



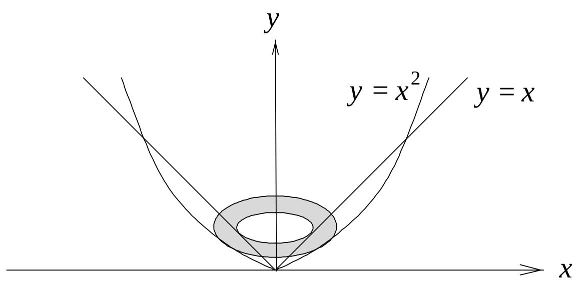
We form the solid by rotating the region R about the x–axis. Note that the line $y = x$ is always above the parabola on the interval $[0, 1]$ so we form a washer to approximate the volume of revolution, the outer radius is $y = R(x) = x$ and the inner radius $y = r(x) = x^2$. Thus the required volume is

$$\begin{aligned} V &= \pi \int_0^1 [R(x)]^2 - [r(x)]^2\, dx \\ &= \pi \int_0^1 x^2 - x^4 dx \\ &= \pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^1 \\ &= \frac{2\pi}{15}. \end{aligned}$$

2. Determine the volume of the solid of revolution which results when the closed region enclosed by $y = x^2$ and the line $y = x$ is rotated about the y-axis.

Solution

Since we are revolving R about the y-axis we use horizontal washers to approximate the solid of revolution. Please see the following figure:



Note that the parabola $x = \sqrt{y}$ is to the right of the line $x = y$ on the interval $[0, 1]$. The washer that we are going to work with has outer radius $R(y) = \sqrt{y}$ and inner radius $r(y) = y$. Then

$$\begin{aligned} V &= \pi \int_0^1 [R(y)]^2 - [r(y)]^2 \, dy \\ &= \pi \int_0^1 (\sqrt{y})^2 - y^2 \, dy \\ &= \pi \int_0^1 y - y^2 \, dy \\ &= \pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 \\ &= \frac{\pi}{6}. \end{aligned}$$

3. Determine the volume of the solid of revolution which results when the closed region enclosed by $y = x^2$ and the line $y = x$ is rotated about the line $y = 2$.

We always first determine the points of intersection of the two curves. Then

$$x^2 = x$$

i.e.

$$x^2 - x = 0.$$

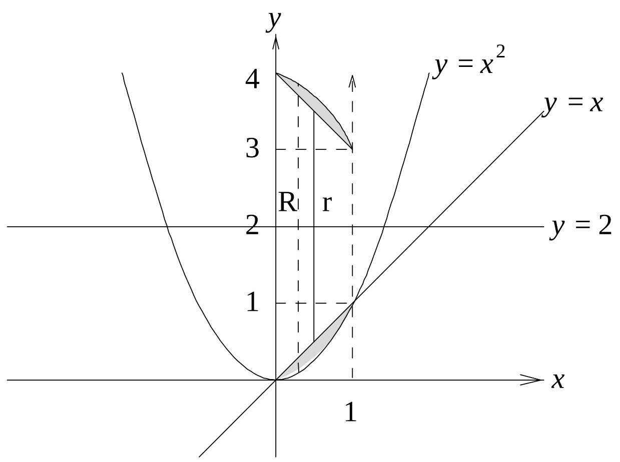
Thus

$$x(x-1) = 0.$$

Hence, the points of intersection are at $x = 0$ and $x = 1$. If $x = 0$, then $y = 0$ and if $x = 1$, then $y = 1$.

Solution

The outer radius is $R(x) = 2 - x^2$ and the inner radius $r(x) = 2 - x$ is shown in the next figure



The volume is thus

$$\begin{aligned} V &= \pi \int_0^1 [R(x)]^2 - (r(x))^2 \, dx \\ &= \pi \int_0^1 \left[\left(2 - x^2\right)^2 - (2 - x)^2 \right] dx \\ &= \pi \int_0^1 \left(x^4 - 5x^2 + 4x \right) dx \\ &= \pi \left[\frac{x^5}{5} - \frac{5x^3}{3} + \frac{4x^2}{2} \right]_0^1 \\ &= \frac{8\pi}{15}. \end{aligned}$$

Summary

After a learner has worked through all these examples, he/she will be able to evaluate areas between curves and evaluate volumes of solids by revolving plane regions about different types of axes, for instance (about the x-axis, y-axis, line $y = 2$ etc.).

Chapter 4

Advanced Techniques of Integration

Introduction

We know in theory how integrals are evaluated using antiderivatives. The more sophisticated our models become, however, the more involved our integrals become. We need to know how to change these more involved integrals into forms we can work with. The first goal of this chapter is to change unfamiliar integrals into integrals we can recognize.

Outcomes

After studying this chapter the learner should be able to:

- use substitution to obtain some basic integrals (integration by substitution is included in MAT1512). Make sure you understand and can apply it.
- use the following integration techniques:
  - integration by parts
  - integration by trigonometric substitution
  - integration by partial fractions
  - integration by z–substitution.
- identify the integration method that should be used for a specific problem.

4.1 BASIC INTEGRATION FORMULAS

When you want to solve an integral, first determine whether the integrand is in a standard form. Then you can directly apply the converse of differentiation, i.e. you ask of which function is the function under the integral sign the derivative for example $\int \cos x dx = \sin x + c$ because the function $\cos x$ is the derivative of the function $\sin x$.

In many cases it is also possible to change the integrand to a standard forms using algebraic manipulation or trigonometric identities.

Now try to do as many problems as possible in your textbook. The only way in which you can master the basic integration formulas and also memorise them is by regularly doing as many exercises as possible.

4.2 INTEGRATION BY PARTS

This is one of the main methods of integration and thus you must study this section thoroughly.

Do you still remember the formula for the differentiation of a product? If u and v are two differentiable functions then

$$\frac{d\,(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

(the first $\times$ derivative of the second $+$ the second $\times$ the derivative of the first). We write it in simple terms as

$$d\,(uv) = udv + vdu$$

Now integrate both sides of this formula to obtain the formula for integration by parts:

$$\int d\,(uv) = \int u\;dv + \int v\;du$$

so that

$$uv = \int u\;dv + \int v\;du.$$

When we rewrite this formula we obtain:

Formula for integration by parts

$$\boxed{\int u\;dv = uv - \int v\;du.}$$

Before discussing some examples, keep the following remarks in mind when you do integration by parts.

Remarks

1. *If the integrand consists of a product of two functions that are* ***not related,*** *for example* $\int x \sin x\;dx$ *(it is obvious that substitution cannot be used), then we use the technique of integration by parts.*

2. *Remember the* ***special cases*** *for which we use integration by parts:* ℓn, $\tan^{-1}$, $\sin^{-1}$, $\cos^{-1}$, *and* $\sec^{-1}$. *Note carefully that in every one of these cases, we must choose* u *and* dv *as follows:* $u = f(x)$ *where* $f(x)$ *is one of the above-mentioned functions and* $dv = dx$.

3. *Furthermore, we also write* $\sec^3 x = \sec x\;\sec^2 x$ *and* $cosec^3 x = cosec\;x\;cosec^2 x$ *and apply integration by parts.*

4. *The* **choice of** u **and** dv *is of major importance. We must make sure that we can in fact integrate the function that is chosen as* dv*. If possible, we must also choose* u *in such a way that it becomes simpler when we differentiate.*

   When we have a power of x *(or the variable which we work with) in the integrand and we use the method of integration by parts, we* ***usually*** *choose the power of* x *so that the power* ***decreases*** *(i.e. it becomes simpler) in other word if* $u = x^p$ *(p the power) then* $du = px^{p-1}dx$*.*

   The most important ***exclusion*** *is when the functions under the integrand are such that the one is a power of* x *and the other, one of the examples of remark 2 above i.e* ℓn, $\tan^{-1}$*, etc., i.e. a function of which we know the derivative but not the integral.*

   Now work through the following two examples

   Worked Examples

   (a) Determine

   $$\boxed{I = \int x^2 \sin x \, dx.}$$

   Solution:

   In this case, we let $u = x^2$ and $dv = \sin x \, dx$. Then $du = 2x \, dx$ and $v = -\cos x \, dx$ so that

   $$I = -x^2 \cos x + 2\int x \cos x \, dx.$$

   For the second integral that we have here i.e. $I_2 = \int x \cos x \, dx$ we *again* use $u = x$ and $dv = \cos x \, dx$, so that the derivative of x becomes a constant $= 1$.

   (b) Determine

   $$\boxed{I = \int x \sin^{-1} x \, dx.}$$

   Solution:

   In this case we set $u = \sin^{-1} x$ and $dv = x \, dx$ because the derivative of $\sin^{-1} x$ is known but the integral not.

   Then $du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = \frac{1}{2}x^2$. We use the formula for integration by parts and obtain

   $$I = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx.$$

   For the integral $I_2 = \int \frac{x^2}{\sqrt{1-x^2}}dx$ we use trigonometric substitution with $x = \sin\theta$ and also the identity $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

5. *In the case of repeated application of the formula, we may not change the* **order of** u **and** dv*. We will point this out again in the examples.*

6. *The **constant** is added only to the last integral. It can be shown that it makes no difference.*

7. *When you are solving problems, you would do well to remember the following:*

$$\int \sin\ ax\ dx = -\frac{\cos ax}{a} + c$$

and

$$D_x(\sin\ ax) = a\ \cos\ ax.$$

8. *When you use a repeated application of the method of integration by parts, you must **simplify** the integrand as far as possible after **each step**, i.e. you must take out all the constants and algebraic signs from the integrand. The following example illustrates this:*

Determine

$$\boxed{I = \int e^{2x} \cos\ 3x\ dx.}$$

Solution

Put $u = \cos 3x$ and $dv = e^{2x}dx$. Then $du = -3\ \sin 3x\ dx$ and $v = e^{2x}/2$. We use the formula for integration by parts:

$$I = \int u\ dv = uv - \int v\ du$$

and then

$$\begin{aligned}
I &= \cos 3x.\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}(-3\sin 3x)dx \\
&= \frac{e^{2x}\cos 3x}{2} + \frac{3}{2}\int e^{2x}\sin 3x\ dx \\
&\quad \text{Take out } -\frac{3}{2} \text{ and write it in front of the integral sign.} \\
&= \frac{e^{2x}\cos 3x}{2} + \frac{3}{2}\left[\sin\ 3x\frac{e^{2x}}{2} - \int 3\cos 3x\frac{e^{2x}}{2}dx\right] \\
&\quad \text{(Remember the bracket)} \\
&= \frac{e^{2x}\cos 3x}{2} + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x dx \\
&= \frac{e^{2x}\cos 3x}{2} + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I.
\end{aligned}$$

Now take similar terms to the left-hand side to obtain:

$$I + \frac{9}{4}I = \frac{13}{4}I = \frac{13}{4}\int e^{2x}\cos 3x\ dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin\ 3x.$$

Consequently

$$I = \int e^{2x}\cos 3x\ dx = \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + c.$$

9. *After (repeated) application(s) of the formula, one of the following can occur:*

   (a) *The integrand becomes simpler and the integral can be determined directly.*

   (b) *The integrand reappears in its original form.*

   (The first three of the following examples are of case (a) and the following two are of case (b).)

Further Worked Examples

1. Determine

$$\boxed{I = \int x^2 \cos\ x\ dx.}$$

   (See a similar problem in remark (4) above).

   Solution

   Set $u = x^2$ and $dv = \cos x\ dx$. Then $du = 2xdx$ and $v = \sin x$. Hence

$$I = \int x^2 \cos x\ dx = x^2 \sin x - 2\int x \sin\ x\ dx \qquad (*)$$

   Now consider $\int x \sin x\ dx$. We may not change the order of u and dv if we want to apply the formula again. (Check, for yourself, what happens if you should in fact change the order, in other words by setting $u = \sin x$ and $dv = x\ dx$.) Therefore, we set $u = x$ and $dv = \sin x\ dx$. Then, again using the formula for integration by parts

$$\begin{aligned} \int x \sin x\ dx &= x(-\cos x) + \int \cos x\ dx \\ &= -x\cos x + \sin x. \end{aligned}$$

   Therefore it follows from $(*)$ that

$$\begin{aligned} I &= \int x^2 \cos x\ dx \\ &= x^2 \sin x - 2(-x\cos x + \sin x) + c \\ &= x^2 \sin x + 2x\cos x - 2\sin x + c. \end{aligned}$$

2. Determine

$$\boxed{I = \int x\sqrt{1+x}\ dx.}$$

   Solution

   Set $u = x$ and $dv = (1+x)^{1/2}dx$. Then

$$\begin{aligned} I = \int x\sqrt{1+x}dx &= x.\frac{(1+x)^{3/2}}{3/2} - \int \frac{(1+x)^{3/2}}{3/2}.1\ dx \\ &= \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\frac{(1+x)^{5/2}}{5/2} + c \\ &= \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + c. \end{aligned}$$

3. Determine

$$\boxed{I = \int_0^{\frac{\pi}{2}} x^3 \cos 2x \; dx.}$$

Solution

Use the formula $\int udv = uv - \int vdu$ for integration by parts with

$$u = x^3 \quad \text{and} \qquad dv = \cos 2x \; dx.$$

Thus

$$du = 3x^2 dx \quad \text{and} \qquad v = \frac{\sin 2x}{2}.$$

It gives

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x^3 \cos 2x \; dx &= \left[x^3 \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 3x^2 . \frac{\sin 2x}{2} .dx \\
&= 0 - \frac{3}{2} \int_0^{\frac{\pi}{2}} x^2 \sin 2x \; dx.
\end{aligned}$$

By applying integration by parts twice more we see that

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x^3 \cos 2x \; dx &= -\frac{3}{2} \left[\left(x^2 . \left(-\frac{\cos 2x}{2} \right) \right) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(2x . \left(-\frac{\cos 2x}{2} \right) \right) dx \right] \\
&= -\frac{3}{2} \left(\frac{\pi}{2} \right)^2 \left(\frac{1}{2} \right) - \frac{3}{2} \int_0^{\frac{\pi}{2}} x \cos 2x \; dx \\
&= -\frac{3}{16} \pi^2 - \frac{3}{2} \left[\left[x . \frac{\sin 2x}{2} \right] \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2} \right] dx \\
&= -\frac{3}{16} \pi^2 + 0 + \frac{3}{4} \int_0^{\frac{\pi}{2}} \sin 2x \; dx. \\
&= -\frac{3}{16} \pi^2 + \frac{3}{4} \left[-\frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}} \\
&= -\frac{3}{16} \pi^2 - \frac{3}{8} (-1 - 1) \\
&= -\frac{3}{16} \pi^2 + \frac{3}{4}.
\end{aligned}$$

4. Determine

$$\boxed{I = \int \sin^2 \theta \; d\theta.}$$

Solution

Let

$$I = \int \sin^2 \theta \; d\theta = \int \sin \theta . \sin \theta \; d\theta$$

Let $u = \sin\theta$ and $dv = \sin\theta\ d\theta$. Then $du = \cos\theta\ d\theta$ and $v = -\cos\theta$. Hence

$$\begin{aligned} I &= \int \sin^2\theta\ d\theta = \sin\theta(-\cos\theta) - \int(-\cos\theta\ d\theta).\cos\theta\ d\theta \\ &= -\sin\theta\ \cos\theta + \int \cos^2\theta\ d\theta \\ &= -\sin\theta\ \cos\theta + \int (1-\sin^2\theta)d\theta \\ &= -\sin\theta\ \cos\theta + \int 1\ d\theta - \int \sin^2\theta\ d\theta \\ &= -\sin\theta\ \cos\theta + \theta - \int \sin^2\theta\ d\theta \\ &= -\sin\theta\cos\theta + \theta - I. \end{aligned}$$

By collecting similar terms on the left-hand side, we find that

$$2I = 2\int \sin^2\theta\ d\theta = -\sin\theta\cos\theta + \theta,$$

and therefore

$$I = \int \sin^2\theta\ d\theta = -\frac{1}{2}\sin\theta\ \cos\theta + \frac{\theta}{2} + c.$$

- 5. Determine

$$\boxed{I = \int \frac{2\tan^{-1}x}{x^2}dx.}$$

(Since $\tan^{-1}$ is one of the special cases that can only be solved by integration by parts, this example is included here.)

Solution

Use the formula $\int udv = uv - \int vdu$ with

$$u = \tan^{-1}x \quad \text{and} \qquad dv = \frac{1}{x^2}dx.$$

Then

$$du = \frac{dx}{1+x^2} \quad \text{and} \qquad v = -\frac{1}{x}\ .$$

Thus

$$\begin{aligned} \int \frac{2\tan^{-1}x}{x^2} &= 2\left(-\frac{\tan^{-1}x}{x} - \int -\frac{1}{x}\cdot\frac{1}{1+x^2}dx\right) \\ &= 2\left(-\frac{\tan^{-1}x}{x} + \int \frac{dx}{x\,(1+x^2)}\right). \end{aligned}$$

Now consider

$$\begin{aligned} \frac{1}{x\,(1+x^2)} &= \frac{A}{x} + \frac{Bx+C}{1+x^2} \\ &= \frac{A\left(1+x^2\right) + x\,(Bx+C)}{x\,(1+x^2)}. \end{aligned}$$

Hence

$$1 = x^2(A+B) + xC + A.$$

By now comparing the coefficients of equal powers of x we see that:

$$\begin{aligned} x^o &: \ 1 = A \\ x^1 &: \ 0 = C \\ x^2 &: \ 0 = A + B \quad \text{which implies } B = -1. \end{aligned}$$

Thus

$$\begin{aligned} I &= \int \frac{2\tan^{-1}x}{x^2}dx = 2\left[-\frac{\tan^{-1}x}{x} + \int \frac{1}{x} - \left(\frac{x}{1+x^2}\right)dx\right] \\ &= \left[-\frac{2\tan^{-1}x}{x} + 2\ell n|x| - \ell n(x^2+1) + c\right] \\ &= -\frac{2\tan^{-1}x}{x} + \ell n\frac{x^2}{x^2+1} + c. \end{aligned}$$

4.3 TRIGONOMETRIC INTEGRALS (USING SUBSTITUTION)

When we have an integral of the form $\int \cos^4 x \sin x \ dx$, we immediately see that if we write this in the form $\int [\cos x]^4 \sin x \ dx$, that $\sin x$ is the derivative of $\cos x$. We use the **substitution formula:**

Substitution formula

$$\boxed{\int f\left(g\left(x\right)\right) \cdot g'\left(x\right) dx = \int f\left(u\right) du.}$$

where f and g are related.

In this case, we substitute $u = g\left(x\right) = \cos x$ so that $\frac{du}{dx} = g'\left(x\right)$ and $du = -\sin x \ dx$.

Then we obtain the integral

$$\begin{aligned} -\int u^4 du &= -\frac{1}{5}u^5 \\ &= -\frac{1}{5}\left(\cos x\right)^5 + c. \end{aligned}$$

In order to do more difficult examples we use reduction methods:

For example, we use $\sin^2 x = 1 - \cos^2 x$ to obtain again an integral where we can use the substitution formula above.

Worked example

Determine

$$\boxed{\int \cos^3 x \ \sin^4 x \ dx.}$$

Solution

Now

$$\begin{aligned}\int \cos^3 x \ \sin^4 x \ dx &= \int \cos \ x \ \cos^2 x \ \sin^4 x \ dx \\ &= \int \cos \ x(1-\sin^2 x)\sin^4 x \ dx \\ &= \int \cos \ x(\sin^4 x - \sin^6 x)dx.\end{aligned}$$

Now set

$$u = \sin \ x.$$

Then

$$du = \cos \ x \ dx,$$

and so it follows that

$$\begin{aligned}\int \cos^3 x \ \sin^4 x \ dx &= \int (u^4 - u^6)du \\ &= \frac{u^5}{5} - \frac{u^7}{7} + c \\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c.\end{aligned}$$

Remark

From the example, we see that integrals of the form $\int \sin^n x \ dx$, $\int \cos^n x \ dx$, $\int \cos^n x \ \sin^m x \ dx$ *and* $\int \sin^n x \ \cos^m x \ dx$, *with* n ***odd,*** *can be determined easily by first* ***splitting*** *the integrand and then writing* $\cos x$ *in terms of* $\sin x$, *or vice versa. Integrals of the form*

$$\int \sec^n x \ \tan^m x \ dx \ \textit{and} \ \int \ \mathrm{cosec}^n x \ \cot^m x \ dx$$

can be determined similarly using $\sec^2 x = 1 + \tan^2 x$ *or* $\mathrm{cosec}^2 x = 1 + \cot^2 x$.

See the following worked examples.

Worked Examples

1. Determine

$$\boxed{I = \int \tan^2 x \ \sec^4 x \ dx.}$$

Solution

$$\begin{aligned} I &= \int \tan^2 x \ \sec^2 x \ \sec^2 x \ dx \\ &= \int \tan^2 x(\tan^2 x + 1)\sec^2 x \ dx \\ &= \int (\tan^4 x + \tan^2 x)\sec^2 x \ dx. \end{aligned}$$

Now set

$$u = \tan \ x,$$

then

$$du = \sec^2 x \ dx,$$

and

$$\begin{aligned} I &= \int \tan^2 x \ \sec^4 x \ dx = \int (u^4 + u^2) du \\ &= \frac{u^5}{5} + \frac{u^3}{3} + c \\ &= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + c. \end{aligned}$$

2. Determine

$$\boxed{I = \int \cot^3 x \, \text{cosec}^5 x \ dx.}$$

Solution

$$\begin{aligned} I &= \int \cot^3 x \text{cosec}^5 x \ dx = \int (\cot^2 x \ \text{cosec}^4 x)(\cot x \ \text{cosec} \ x) dx \\ &= \int (\text{cosec}^2 x - 1)\text{cosec}^4 x \ \cot x \ \text{cosec} x dx \\ &= \int (\text{cosec}^6 x - \text{cosec}^4 x) \cot x \ \text{cosec} x \ dx. \end{aligned}$$

Therefore set

$$u = \text{cosec} \ x,$$

then

$$du = - \ \text{cosec} \, x \ \cot \ x \, dx,$$

and

$$\begin{aligned} I &= \int \cot^3 x \ \text{cosec}^5 x \ dx = -\int (u^6 - u^4) du \\ &= -\frac{u^7}{7} + \frac{u^5}{5} + c \\ &= -\frac{\text{cosec}^7 x}{7} + \frac{\text{cosec}^5 x}{5} + c. \end{aligned}$$

Now try to do the following exercises yourself:

Exercises
Determine the following integrals:

1. $\displaystyle\int \sin^5 x \ dx$

2. $\displaystyle\int \tan^3 x \ \sec^3 x \ dx$

3. $\displaystyle\int \sqrt{\tan x} \sec^4 x \ dx$

4. $\displaystyle\int \sec^6 x \ dx.$

Answers:

1. $-\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + c$

$$\text{Hint: Write } \int \sin^5 x \ dx = \int \left(1 - \cos^2 x\right)^2 \sin x \ dx$$
$$= \int \left(\left(1 - 2\cos^2 x + \cos^4 x\right) \sin x \ dx\right)$$

2. $\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + c$

$$\text{Hint: Write } \int \tan^3 x \ \sec^3 x \ dx = \int \left(\sec^4 x - \sec^2 x\right) \tan x \ \sec x \ dx$$

3. $\frac{2}{7} \tan^{7/2} x + \frac{2}{3} \tan^{3/2} x + c$

$$\text{Hint: Write } \int (\tan x)^{\frac{1}{2}} \sec^4 x \ dx = \int (\tan x)^{\frac{1}{2}} \left(\tan^2 x + 1\right) \sec^2 x \ dx$$

4. $\tan x + \frac{2}{3} \tan^3 x + \frac{1}{5} \tan^5 x + c.$

$$\text{Hint: Write } \int \sec^6 x \ dx = \int \left(\tan^2 x + 1\right)^2 \sec^2 x \ dx$$

4.4 TRIGONOMETRIC SUBSTITUTION

This substitution is easy to recognise and you will encounter it frequently. Thus, make sure that you understand this section well.

Trigonometric substitutions enable us to replace the binomials $a^2 + x^2$, $a^2 - x^2$ and $x^2 - a^2$ by single variables.

We once again discuss this method so that you may ensure that you understand it well.

By respectively making use of the three identities

- $1 - \sin^2\theta = \cos^2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

we obtain the following three substitutions for $a > 0$:

For	use	and obtain
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$a\sqrt{1-\sin^2\theta} = \lvert a\cos\theta\rvert$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$a\sqrt{1+\tan^2\theta} = \lvert a\sec\theta\rvert$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$a\sqrt{\sec^2\theta - 1} = \lvert a\tan\theta\rvert$

When we make substitutions we always want them to be reversible so that we can change back to the original variables after we are done. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}\frac{x}{a}$ after integration have taken place. As we know from section 2.3.1(S), the functions in these substitutions have inverses only for selected values of θ.

For reversibility

$$\begin{array}{llll}
x = a\tan\theta & \text{requires} & \theta = \tan^{-1}\frac{x}{a} & \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
x = a\sin\theta & \text{requires} & \theta = \sin^{-1}\frac{x}{a} & \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\
x = a\sec\theta & \text{requires} & \theta = \sec^{-1}\frac{x}{a} & \text{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if} \quad \frac{x}{a} \ge 1 \quad \text{i.e. } x \ge a \\ \frac{\pi}{2} < \theta \le \pi & \text{if} \quad \frac{x}{a} \le -1 \quad \text{i.e. } x \le -a \end{cases}
\end{array}$$

It is important at each substitution to state clearly what the limits of θ are. The integral only exists where $\sin^{-1}$, $\tan^{-1}$, *or* $\sec^{-1}$ *is defined and where it is one-to-one. (See examples 1 to 3 that follows).*

Worked Examples

1. Determine

$$\boxed{I = \int \frac{\left(16 - 9x^2\right)^{3/2}}{x^6}\,dx.}$$

Solution

Set

$$x = \frac{4}{3}\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(Remember we must restrict θ so that we have a one-to-one function whose inverse exists.)

Then

$$dx = \frac{4}{3}\cos\theta\ d\theta,$$

and

$$\sqrt{16-9x^2} = |4\cos\theta| = 4\ \cos\theta,$$

since

$$\cos\theta \geq 0 \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

Therefore

$$\begin{aligned} I &= \int \frac{64\cos^3\theta \frac{4}{3}\cos\theta}{\frac{4096}{729}\sin^6\theta} d\theta \\ &= \frac{243}{16}\int \frac{\cos^4\theta}{\sin^6\theta} d\theta \\ &= \frac{243}{16}\int \cot^4\theta \ \text{cosec}^2\theta \ d\theta \end{aligned}$$

By making use of the first substitution rule, we therefore set

$$u = \cot\theta.$$

Then

$$du = -\ \text{cosec}^2\theta \ d\theta$$

and therefore

$$\begin{aligned} I &= \int \frac{\left(16-9x^2\right)^{\frac{3}{2}}}{x^6} dx = \frac{243}{16}\int u^4 \left(-du\right) \\ &= -\frac{243}{16}\frac{\cot^5\theta}{5} + c. \end{aligned}$$

In order to write the integral back in terms of x we look at the following representation of the angle θ in terms of x.

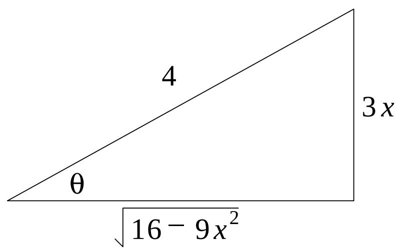
From

$$x = \frac{4}{3}\sin\theta, \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

we have

$$\sin\theta = \frac{3x}{4}, \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

so



Therefore

$$\begin{aligned} \int \frac{\left(16-9x^2\right)^{\frac{3}{2}}}{x^6} dx &= \frac{243}{16} \cdot \frac{1}{5} \frac{\left(16-9x^2\right)^{\frac{5}{2}}}{243x^5} + c \\ &= -\frac{1}{80}\frac{\left(16-9x^2\right)^{\frac{5}{2}}}{x^5} + c. \end{aligned}$$

2. Determine

$$\boxed{\int \frac{dx}{\sqrt{25x^2-9}} \qquad \text{when } x > \tfrac{3}{5}.}$$

Solution

We rewrite $\sqrt{25x^2-9}$:

$$\begin{aligned}\sqrt{25x^2-9} &= \sqrt{25\left(x^2-\frac{9}{25}\right)} \\ &= 5\sqrt{x^2-\left(\frac{3}{5}\right)^2}\end{aligned}$$

We now have a form $x^2 - a^2$.

We now use substitution with

$$x = \frac{3}{5}\sec\theta, \quad dx = \frac{3}{5}\sec\theta\tan\theta d\theta,\ \theta \in \left(0, \frac{\pi}{2}\right)$$

[Note that $\theta \in \left(0, \frac{\pi}{2}\right)$ since $x > \frac{3}{5}$].

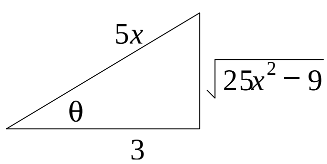
Also

$$\begin{aligned}x^2-\left(\frac{3}{5}\right)^2 &= \frac{9}{25}\sec^2\theta - \frac{9}{25} \\ &= \frac{9}{25}\left(\sec^2\theta - 1\right) \\ &= \frac{9}{25}\tan^2\theta\end{aligned}$$

so that

$$\sqrt{x^2-\left(\frac{3}{5}\right)^2} = \left|\frac{3}{5}\ \tan\theta\right|.$$

If $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2}$, then $\theta = \sec^{-1}\frac{5x}{3}$ and we can read the values of the other trigonometric functions of θ from the right triangle:

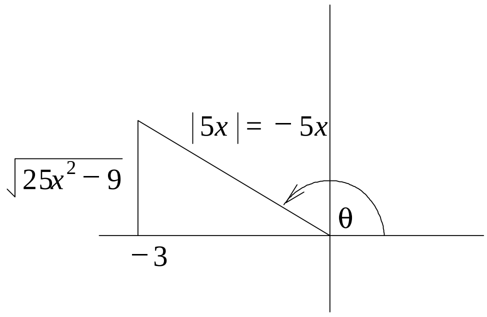


We now use the substitutions above to obtain

$$\begin{aligned}\int \frac{dx}{\sqrt{25x^2-9}} &= \int \frac{dx}{5\sqrt{x^2-\left(\frac{3}{5}\right)^2}}\\ &= \int \frac{\frac{3}{5}\sec\theta\tan\theta d\theta}{5\left(\frac{3}{5}\right)\tan\theta} \qquad |\tan\theta| = \tan\theta\\ &= \frac{1}{5}\int \sec\theta d\theta\\ &= \frac{1}{5}\ln|\sec\theta+\tan\theta| + c\\ &= \frac{1}{5}\ln\left|\frac{5x}{3}+\frac{\sqrt{25x^2-9}}{3}\right| + c\end{aligned}$$

Remark

If $x < -\frac{3}{5}$, we use the same substitutions i.e. $x = \frac{3}{5}\sec\theta,\ dx = \frac{3}{5}\sec\theta\tan\theta d\theta$ with $\theta \in \left(\frac{\pi}{2}, \pi\right)$ (see p. 72(S)). Also $|\tan\theta| = -\tan\theta$. The reference triangle for $\theta \in \left(\frac{\pi}{2}, \pi\right)$ is then



Then the integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{25x^2-9}} &= -\frac{1}{5}\int \sec\theta d\theta\\ &= -\frac{1}{5}\ln|\sec\theta+\tan\theta| + c\\ &= -\frac{1}{5}\ln\left|\frac{5x}{3}-\frac{\sqrt{25x^2-9}}{3}\right| + c.\end{aligned}$$

3. Determine

$$\boxed{I = \int \frac{dx}{(4x^2-24x+27)^{3/2}}.}$$

We do not recognise any of the standard forms for trigonometric substitution in this integral, but by completing the square, we obtain

$$4x^2 - 24x + 27 = 4(x-3)^2 - 9.$$

Therefore, we set

$$x - 3 = \frac{3}{2}\sec\theta,\ \theta \in (0,\pi) - \left\{\frac{\pi}{2}\right\},$$

then

$$dx = \frac{3}{2}\sec\theta\ \tan\theta\ d\theta$$

and

$$\sqrt{4x^2 - 24x + 27} = 3\left|\tan\theta\right|.$$

Therefore

$$\begin{aligned} I &= \int \frac{dx}{(4x^2-24x+27)^{\frac{3}{2}}} = \int \frac{\frac{3}{2}\sec\theta\tan\theta}{27\tan^2\theta\left|\tan\theta\right|}d\theta \\ &= \frac{1}{18}\int \frac{\left|\cos\theta\right|}{\sin^2\theta}d\theta. \end{aligned}$$

Note that $|\tan\theta| = \left|\frac{\sin\theta}{\cos\theta}\right| = \frac{\sin\theta}{|\cos\theta|}$ for $\theta \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$. Therefore

$$I = \int \frac{dx}{(4x^2-24x+27)^{\frac{3}{2}}} = \begin{cases} \frac{1}{18}\int \frac{\cos\theta}{\sin^2\theta}d\theta, \text{ if } \theta \in (0,\frac{\pi}{2}), \\ -\frac{1}{18}\int \frac{\cos\theta}{\sin^2\theta}d\theta, \text{ if } \theta \in (\frac{\pi}{2},\pi). \end{cases}$$

Now set

$$u = \sin\theta,$$

then

$$du = \cos\theta\ d\theta$$

and

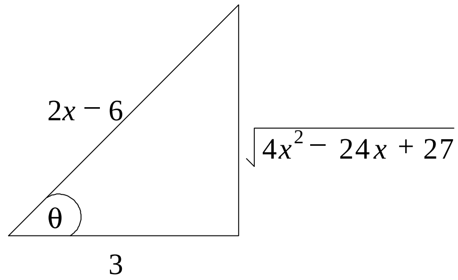
$$\begin{aligned} I &= \int \frac{dx}{(4x^2-24x+27)^{\frac{3}{2}}} = \pm\frac{1}{18}\int \frac{du}{u^2} \\ &= \mp\frac{1}{18}\frac{1}{u} \\ &= \mp\frac{1}{18}\ \mathrm{cosec}\theta + c. \end{aligned}$$

Now

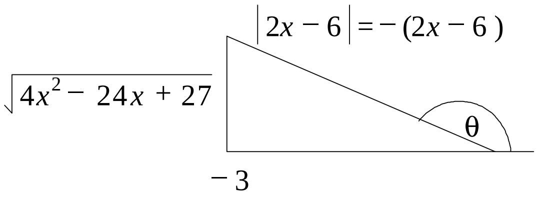
$$\sec\theta = \frac{2(x-3)}{3}$$

which can be represented as follows:

For $\theta \in \left[0,\frac{\pi}{2}\right)$:



For $\theta \in \left(\frac{\pi}{2}, \pi\right]$:



From this, it follows that

$$\operatorname{cosec}\theta = \begin{cases} \frac{2x-6}{\sqrt{4x^2-24x+27}} & \text{if } \theta \in \left[0, \frac{\pi}{2}\right), \text{ i.e. } x > \frac{3}{2} + 3, \\ \frac{-(2x-6)}{\sqrt{4x^2-24x+27}} & \text{if } \theta \in \left(\frac{\pi}{2}, \pi\right], \text{ i.e. } x < -\frac{3}{2} + 3, \end{cases}$$

Therefore

$$\begin{aligned} I &= \int \frac{dx}{(4x^2 - 24x + 27)^{\frac{3}{2}}} = \mp \frac{1}{18} \frac{\pm 2\,(x-3)}{\sqrt{4x^2 - 24x + 27}} + c \\ &= -\frac{1}{9} \frac{x-3}{\sqrt{4x^2 - 24x + 27}} + c. \end{aligned}$$

Summary

We have shown how to do trigonometric substitution.

4.5 PARTIAL FRACTIONS

A *rational function* has the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x) \neq 0$. Such a fraction is proper if the degree of $P(x)$ is smaller than the degree of $Q(x)$ and improper if the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$:

$$\frac{x+1}{3x^2 + 5x - 1} \text{ is a proper rational function}$$

(the highest power of x in the numerator is smaller than the highest power of x in the denominator) and

$$\frac{x^3 + 5}{(x+1)^3} \text{ is an improper rational function.}$$

(the highest power of x in the numerator is equal to the highest power of x in the denominator).

An *improper fraction* is reduced to a polynomial plus a proper fraction by long division, for example

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

It is very important that the rational function that needs to be integrated must be in the form of a polynomial plus a **proper** rational function.
The process by which a fraction such as $\frac{x+7}{x^2+8x+15}$ can be decomposed into the fractions $\frac{2}{x+3} - \frac{1}{x+5}$ is called splitting into **partial fractions**.

Remarks

- *Note that the examples on partial fractions always have positive integers as powers of x (or any variable that we work with) in both the numerator and the denominator.*
- *We have two kinds of factors in $Q(x)$, namely linear factors (i.e. of the form $ax+b$), for example $2x+3,\ x+1$ etc., and irreducible factors (i.e. of the form ax^2+bx+c) that cannot be decomposed into further linear factors (i.e. those where $b^2-4ac<0$), for example $2x^2+3,\ 3x^2+2x+1$, etc.*

Let's consider a number of specific cases of linear factors, repeated linear factors, irreducible quadratic factors and repeated irreducible factors in the denominator.

4.5.1 LINEAR FACTORS IN THE DENOMINATOR

Consider a proper rational function $\frac{P(x)}{Q(x)}$ where $Q(x)$ can be resolved into linear factors

$$(a_1x+b_1)(a_2x+b_2)(a_nx+b_n)$$

Then

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)}\\ &= \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}\cdots+\frac{A_n}{a_nx+b_n}.\end{aligned}$$

Worked Example

Split

$$\boxed{\frac{x+35}{x^2-25}}$$

into partial fractions.

Solution

$$\begin{aligned}\frac{x+35}{x^2-25} &= \frac{x+35}{(x-5)(x+5)}\\ &= \frac{A}{x+5}+\frac{B}{x-5}.\end{aligned}$$

To determine A and B, we add the fractions once more and find that

$$\begin{aligned}\frac{x+35}{(x-5)(x+5)} &= \frac{A}{x+5}+\frac{B}{x-5}\\ &= \frac{A(x-5)+B(x+5)}{(x+5)(x-5)}\end{aligned}$$

Therefore

$$x+35=A(x-5)+B(x+5). \qquad ...(1)$$

The unknown constants can be determined using many different methods. Here we show two, namely, fixed values or equal powers of x.

(i) Choose fixed values for x (usually the most convenient ones for which some terms become zero).

In (1) let $x = 5$, then

$$40 = A.0 + B.10,$$

and therefore

$$B = 4.$$

Also let $x = -5$, then

$$30 = A.(-10) + B.0,$$

and therefore

$$A = -3.$$

(ii) Comparing the coefficients of equal powers of x.

From (1), it follows that

$$\begin{aligned} x + 35 &= Ax - 5A + Bx + 5B \\ &= (A + B)\,x + (-5A + 5B)\,. \end{aligned}$$

By now comparing the coefficients, we see that

$$\begin{aligned} x &: \ 1 = A + B \qquad &\text{...(a)} \\ \text{Constant } (x^0) &: \ 35 = -5A + 5B \qquad &\text{...(b)} \end{aligned}$$

$(a) \times 5 + (b)$ gives that $40 = 10B$ and therefore, $B = 4$. By setting $B = 4$ in (a), we obtain that $A = -3$.

From the above, we therefore see that

$$\frac{x + 35}{x^2 - 25} = \frac{-3}{x + 5} + \frac{4}{x - 5}.$$

At this stage method (i) is simpler than method (ii), but later both methods will have to be used to determine certain constants. Next we use the method of splitting into the partial fractions to solve integrals where the integrand is a rational function:

Worked Examples

1. Determine

$$\boxed{I = \int \frac{23 - 2x}{2x^2 + 9x - 5}dx.}$$

Solution

$$\begin{aligned} \frac{23 - 2x}{2x^2 + 9x - 5} &= \frac{23 - 2x}{(2x - 1)\,(x + 5)} \\ &= \frac{A}{2x - 1} + \frac{B}{x + 5} \\ &= \frac{A\,(x + 5) + B\,(2x - 1)}{(2x - 1)\,(x + 5)} \end{aligned}$$

and therefore

$$23 - 2x = A(x+5) + B(2x-1).$$

By setting $x = -5$, we obtain that $B = -3$ and by setting $x = \frac{1}{2}$, we obtain that $A = 4$. Therefore

$$\begin{aligned} I &= \int \left(\frac{4}{2x-1} - \frac{3}{x+5} \right) dx \\ &= 4.\frac{1}{2}\ell n\, |2x-1| - 3\ell n\, |x+5| + c \\ &= \ell n \frac{(2x-1)^2}{|x+5|^3} + c. \end{aligned}$$

2. Determine

$$\boxed{I = \int \frac{x^2+10x+6}{x^2+2x-8} dx.}$$

Solution

We must first apply long division. After division

$$\frac{x^2+10x+6}{x^2+2x-8} = 1 + \frac{8x+14}{x^2+2x-8}.$$

Furthermore

$$\begin{aligned} \frac{8x+14}{x^2+2x-8} &= \frac{8x+14}{(x-2)\,(x+4)} \\ &= \frac{A}{x-2} + \frac{B}{x+4} \\ &= \frac{A\,(x+4) + B\,(x-2)}{(x-2)\,(x+4)}, \end{aligned}$$

and therefore

$$8x + 14 = A(x+4) + B(x-2).$$

By setting $x = -4$, we obtain that $B = 3$ and by setting $x = 2$ we obtain that $A = 5$.

Therefore

$$\begin{aligned} I &= \int \left(1 + \frac{5}{x-2} + \frac{3}{x+4} \right) dx \\ &= x + 5\ell n\, |x-2| + 3\ \ell n\ \ |x+4| + c. \end{aligned}$$

The integrals that occur most generally after splitting into partial fractions are

$$\boxed{\begin{aligned} &\int \frac{dx}{ax+b} = \frac{1}{a}\ \ell n\ \ |ax+b| + c \\ &\int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + c,\ n \neq 1. \end{aligned}}$$

4.5.2 REPEATED LINEAR FACTORS IN THE DENOMINATOR

Consider

$$\frac{3}{(x-1)^2} - \frac{2}{(x-1)}.$$

Following addition

$$\begin{aligned} \frac{3}{(x-1)^2} - \frac{2}{x-1} &= \frac{3-2(x-1)}{(x-1)^2} \\ &= \frac{-2x+5}{(x-1)^2}. \end{aligned}$$

If the reverse process must be performed, namely to split $\frac{-2x+5}{(x-1)^2}$ into partial fractions, provision must be made for two fractions, in other words

$$\frac{-2x+5}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1}.$$

Similarly, for each factor $(ax+b)^r$ that occurs in the denominator of a proper rational function $\frac{P(x)}{Q(x)}$, provision must be made by means of r terms in the decomposition, namely:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}.$$

Worked Examples

1. Determine

$$\boxed{I = \int \frac{3x+1}{(x+1)^2} dx.}$$

Solution

$$\frac{3x+1}{(x+1)^2} = \frac{A}{(x+1)^2} + \frac{B}{x+1}.$$

Therefore

$$3x+1 = A + B(x+1).$$

By setting $x=-1$ we obtain that $A=-2$ and by setting $x=0$ we obtain that $A+B=1$, from which it follows that $B=3$. (We can set $x=0$ or use any other convenient value.) Therefore

$$\begin{aligned} I &= \int \left[\frac{-2}{(x+1)^2} + \frac{3}{x+1} \right] dx \\ &= \int \left[-2(x+1)^{-2} + \frac{3}{x+1} \right] dx \\ &= \frac{-2(x+1)^{-1}}{-1} + 3\,\ell n\ |x+1| + c \\ &= \frac{2}{x+1} + 3\,\ell n\ |x+1| + c. \end{aligned}$$

2. Determine

$$\boxed{I = \int \frac{x^4 - x^3 - x - 1}{x^3 - x^2} dx.}$$

Solution

The integrand is improper. After division

$$\frac{x^4 - x^3 - x - 1}{x^3 - x^2} = x - \frac{x+1}{x^3 - x^2}.$$

Furthermore

$$\begin{aligned} \frac{x+1}{x^3 - x^2} &= \frac{x+1}{x^2(x-1)} \\ &= \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1} \\ &= \frac{A(x-1) + Bx(x-1) + Cx^2}{x^2(x-1)}. \end{aligned}$$

Therefore

$$x + 1 = A(x-1) + Bx(x-1) + Cx^2.$$

By setting $x = 1$ we obtain that $C = 2$, by setting $x = 0$ we obtain that $A = -1$ and then by setting x equal to any convenient value, say $x = -1$, we obtain that $B = -2$. Therefore

$$\begin{aligned} I &= \int \left[x - \left(\frac{-1}{x^2} - \frac{2}{x} + \frac{2}{x-1} \right) \right] dx \\ &= \frac{x^2}{2} - \frac{1}{x} + 2\,\ell n\ |x| - 2\,\ell n\ |x-1| + c \\ &= \frac{x^2}{2} - \frac{1}{x} + \ell n\ \left(\frac{x}{x-1} \right)^2 + c. \end{aligned}$$

4.5.3 IRREDUCIBLE QUADRATIC FACTORS IN THE DENOMINATOR

For each irreducible factor

$$\boxed{ax^2 + bx + c \text{ where } \left(b^2 - 4ac < 0\right)}$$

which occurs in the denominator of $\frac{P(x)}{Q(x)}$, a proper fraction $\frac{Ax+B}{ax^2+bx+c}$ may occur in the decomposition.

Worked Example

Determine

$$\boxed{I = \int \frac{x-1}{(x+1)(x^2+1)} dx.}$$

Solution

$$\begin{aligned}\frac{x-1}{(x+1)(x^2+1)} &= \frac{A}{x+1}+\frac{Bx+C}{x^2+1}.\\ &= \frac{A\left(x^2+1\right)+(Bx+C)(x+1)}{(x+1)(x^2+1)}.\end{aligned}$$

Therefore

$$\begin{aligned}x-1 &= A(x^2+1)+(Bx+C)(x+1) &\quad ...(a)\\ &= (A+B)x^2+(B+C)x+A+C. &\quad ...(b)\end{aligned}$$

First set $x=-1$ in (a). Then it follows that $A=-1$. Now compare the coefficients of equal powers in (b):

$$x^2 : 0 = A+B, \text{ which implies that } \; B=1,$$

and

$$x : 1 = B+C, \text{ which implies that } \; C=0.$$

Therefore

$$\begin{aligned}I &= \int\left(\frac{-1}{x+1}+\frac{x}{x^2+1}\right)dx\\ &= -\ell n\,|x+1|+\frac{1}{2}\int\frac{du}{u}, \quad \text{where } u=x^2+1,\\ &= -\ell n\,|x+1|+\frac{1}{2}\,\ell n\,\left|x^2+1\right|+c\\ &= \ell n\left|\frac{\sqrt{x^2+1}}{x+1}\right|+c.\end{aligned}$$

4.5.4 REPEATED IRREDUCIBLE QUADRATIC FACTORS IN THE DENOMINATOR

As in the case of repeated linear factors in the denominator, for each factor of the form

$$\boxed{(ax^2+bx+c)^r \text{ with } b^2-4ac<0,}$$

which occurs in the denominator of a proper rational function, provision must be made by means of r terms in the decomposition, namely:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Worked Example

Determine

$$\boxed{I=\int\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx.}$$

Solution

$$\begin{aligned}\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} &= \frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}\\ &= \frac{(Ax+B)\left(x^2+1\right)+Cx+D}{(x^2+1)^2}.\end{aligned}$$

Therefore

$$5x^3-3x^2+7x-3 = Ax^3+Bx^2+(A+C)x+(B+D).$$

By now comparing the coefficients of equal powers of x, we obtain that

$$\begin{aligned} x^3 &: \quad 5 = A,\\ x^2 &: \quad -3 = B,\\ x &: \quad 7 = A+C, \text{ from which it follows that } C=2.\\ x^0 &: \quad -3 = B+D, \text{ from which it follows that } D=0.\end{aligned}$$

Therefore

$$\begin{aligned} I &= \int\left[\frac{5x-3}{x^2+1}+\frac{2x}{(x^2+1)^2}\right]dx\\ &= -3\int\frac{dx}{x^2+1}+\frac{5}{2}\int\frac{2x}{x^2+1}dx+\int\frac{2x}{(x^2+1)^2}dx.\end{aligned}$$

By applying the substitution $u=x^2+1$ in the last two integrals on the right, we therefore see that

$$\int\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx = -3 \text{ arctan } \ x+\frac{5}{2}\ell n(x^2+1)-\frac{1}{x^2+1}+c.$$

Summary of the Method of Splitting into Partial Fractions

The following is a very short summary of the method of splitting into partial fractions.

1. If the degree of $P(x)$ is not smaller than that of $Q(x)$, use long division to obtain the desired form.

2. Express $Q(x)$ as a product of linear and/or irreducible quadratic factors.

3. (a) For each factor of the form $(ax+b)$ which occurs only once in $Q(x)$, there is a term of the form

$$\frac{A}{ax+b}.$$

   (b) For each factor of the form $(ax+b)$, which occurs r times in $Q(x)$, there are r terms of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_r}{(ax+b)^r}.$$

   (c) For each factor of the form (ax^2+bx+c) with $b^2-4ac<0$, which occurs only once in $Q(x)$, there is a term of the form

$$\frac{Ax+B}{ax^2+bx+c}.$$

(d) For each factor of the form $(ax^2 + bx + c)$ with $b^2 - 4ac < 0$, which occurs r times in $Q(x)$, there are r terms of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}.$$

Now try to do the following exercises yourself:

Exercises

1. Determine

$$\boxed{\int \frac{11x + 17}{2x^2 + 7x - 4}dx.}$$

Solution

$$\frac{5}{2}\ \ell n|2x - 1| + 3\ \ell n|x + 4| + c.$$

2. Determine

$$\boxed{\int \frac{3x^2 - 10}{x^2 - 4x + 4}dx.}$$

Solution

$$3x - \frac{2}{x - 2} + 12\ \ell n|x - 2| + c.$$

3. Determine

$$\boxed{\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4}dx.}$$

Solution

$$\frac{3}{2}\ell n(x^2 + 4) + \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{5}{2}\ell n|2x - 1| + c.$$

4. Determine

$$\boxed{\int \frac{x^3 - 4x - 1}{x\,(x - 1)^3}dx.}$$

Solution

$$\ell n|x| - \frac{3}{x - 1} + \frac{2}{(x - 1)^2} + c.$$

5. Determine

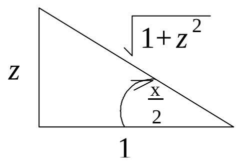
$$\boxed{\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} dx.}$$

Solution

$$2\ell n|x| + \frac{1}{x} + \frac{3}{2}\ell n(x^2+1) - 2\ \tan^{-1} x + c.$$

4.6 z-SUBSTITUTION

This method (using $z = \tan\frac{x}{2}$) reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This is only used when simpler methods have failed. Let $z = \tan\frac{x}{2}$, then $x = 2\tan^{-1} z$ and $dx = \frac{2}{1+z^2}dz$. From the following figure:



we have

$$\sin\frac{x}{2} = \frac{z}{\sqrt{1+z^2}} \text{ and } \cos\frac{x}{2} = \frac{1}{\sqrt{1+z^2}}$$

Using the identities

$$\begin{aligned} \sin 2x &= 2\sin x\cos x \quad \text{and} \\ \cos 2x &= \cos^2 x - \sin^2 x \end{aligned}$$

we have

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{z}{\sqrt{1+z^2}}\cdot\frac{1}{\sqrt{1+z^2}} = \frac{2z}{1+z^2}$$

and

$$\begin{aligned} \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} &= \frac{1}{1+z^2} - \frac{z^2}{1+z^2} \\ &= \frac{1-z^2}{1+z^2} \end{aligned}$$

We thus have the following three important formulas:

$$\boxed{\begin{gathered} dx = \tfrac{2}{1+z^2}dz \\ \sin x = \tfrac{2z}{1+z^2} \\ \cos x = \tfrac{1-z^2}{1+z^2}. \end{gathered}}$$

You must know these formulas well. We will now do one example. Note the method for changing the limits of integration in the example:

Worked Example
Calculate

$$\boxed{I = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.}$$

Solution
Set $z = \tan\frac{x}{2}$, $-\pi < x < \pi$. If $x = 0$, then $z = \tan 0 = 0$, and if $x = \pi/2$, then $z = \tan\frac{\pi}{4} = 1$.

Therefore

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x} = \int_0^1 \frac{\frac{2}{1+z^2}dz}{\frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}} \\ &= \int_0^1 \frac{2}{-z^2+2z+1}dz \\ &= \int_0^1 \frac{2dz}{2-(z-1)^2}, \end{aligned}$$

by completion of the square.

Now set

$$z - 1 = \sqrt{2}\sin\theta, \text{ with } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

then

$$dz = \sqrt{2}\cos\theta \; d\theta.$$

Furthermore, if $z = 0$, then $\sqrt{2}\sin\theta = -1$, in other words $\theta = \arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$, and if $z = 1$, then $\sin\theta = 0$, in other words $\theta = 0$.
Therefore

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = 2\int_{-\pi/4}^0 \frac{\sqrt{2}\cos\theta}{2\cos^2\theta}d\theta \\ &= \sqrt{2}\int_{-\pi/4}^0 \sec\theta d\theta \\ &= \sqrt{2}\ell n\,|\sec\theta + \tan\theta|]_{-\pi/4}^0 \\ &= \sqrt{2}\left(\ell n\,(1+0) - \ell n\left(\sqrt{2}-1\right)\right) \\ &= -\sqrt{2}\ell n\left(\sqrt{2}-1\right). \end{aligned}$$

Exercises:
Determine the following integrals by using z–substitution:

1. $\displaystyle\int \frac{dx}{1+\sin x}$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x+\cos x}$

3. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{3+2\cos x}$

4. $\displaystyle\int \frac{dx}{\sin x - \cos x}$

5. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1-\cos x}$

6. $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{\cos x dx}{\sin x \cos x + \sin x}$

7. $\displaystyle\int \frac{\cos x dx}{1-\cos x}$

Answers:

1. $\displaystyle\frac{-2}{\tan\frac{x}{2}+1} + c$

2. $\ln(2)$

3. $\frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{1}{\sqrt{15}}\right)$

4. $\frac{1}{\sqrt{2}} \ln\left(\frac{(\tan\frac{x}{2})+1-\sqrt{2}}{(\tan\frac{x}{2})+1+\sqrt{2}}\right) + c$

5. $-1+\sqrt{3}$

6. $\frac{1}{2}\ln\sqrt{3} - \frac{1}{2}$

7. $-\cot\left(\frac{x}{2}\right) - x + c$

Summary (of the Techniques of Integration)

At this point we have concluded the integration techniques. Whenever you have to determine a given integral, you must first decide which method to use. Always keep in mind that there may be more than one correct method to determine an integral. Sometimes the one method could involve simpler calculations than the other. Try, for example, to evaluate $\int_1^{3/2} \frac{(x-1)dx}{\sqrt{2x-x^2}}$ by (i) the substitution $u = 2x - x^2$ and (ii) completing the square and trigonometric substitution. (The answer is $\frac{2-\sqrt{3}}{2}$.)

Remember, there are only *two general* methods of integration:

I. Integration by substitution and

II. Integration by parts.

All other methods constitute algebraic manipulation followed by the application of one of the above-mentioned two methods.

1. When you now have a given integral, first try to spot a *suitable substitution* that will transform the integrand into a simpler form or standard form. Here you need to take particular note of the following:

   1.1 Always determine first whether the integrand does not perhaps consist of the product of two functions one of which is the *derivative* of the other. (Then you can apply the method of substitution.)

   1.2 Integrals in which *trigonometric substitutions* are employed are usually easily recognisable.

2. Integration by parts: This method, as well as the circumstances in which it should be used was discussed in paragraph 4.2(S).

3. You also need to keep in mind a number of *special methods*:

   3.1 For *trigonometric integrals* you may try:

   (i) Using the *splitting method* or by using *identities,* you can rewrite the integrand. (See the worked examples in 4.3(S)).

   (ii) By using the *substitution* $z = \tan \frac{x}{2}$, where a rational function of sin x and cos x occurs, you obtain a rational function of z. (See p. 4.6(S)). $\int \frac{\cos t}{1-\cos t} dt$ is such an example.

   (iii) By using *identities*, you can rewrite a function to get rid of a square root (e.g. $\sqrt{1+\cos 2\theta} = \sqrt{2\cos^2\theta} = \sqrt{2}\cos\theta$ since $\cos^2\theta = \frac{1+\cos 2\Theta}{2}$).

   (iv) As a last resort, you can rewrite all trigonometric functions in terms of cos and sin.

   3.2 For integrals where *rational functions* play a part, you should keep the following in mind:

   (i) Always use *division* if the integrand is an improper fraction as in the worked example 2 in 4.5.2(S).

   (ii) If possible, factorise the denominator and then decompose the integrand into *rational fractions.*

   (iii) When you are confronted with an irreducible quadratic function (or the square root of a quadratic function), try *completion of the square.*

Summary

The learner should now be able to identify which method to use for a specific problem and he/she must also be able to apply all the integration methods.

Chapter 5

The Improper Integral

Introduction

Up to now, we have required our definite integrals to have two properties. First, that the domain of integration, from a to b must be finite and second that the range of the integrand must be finite on this domain. In practice, however, we frequently encounter problems that fail to meet one or both of these conditions. These integrals are called *improper integrals*. Thus in this chapter we are concerned with three types of integrals, namely where the *integrand is unbounded*, *the interval of integration is unbounded, or both*. We determine if such an integral converges or diverges and if possible, we evaluate the integral.
It is very important that you must use limit(s) whenever you have an improper integral

Outcomes

After studying this chapter the learner should be able to:

- evaluate improper integrals (i.e integrals with infinite limits of integration and integrals that become infinite at a point within the interval of integration).
- use the limit comparison test to determine whether an integral converges or diverges.

We are concerned with three types of integrals:

5.1 integrals of which the **integrand is unbounded** over the integration interval

5.2 integrals of which the **integration interval is unbounded**

5.3 integrals **for which the integrand is unbounded and the integration interval is unbounded.**

We will discuss here the three types of improper integrals. Please study the relevant sections in your prescribed textbook.

5.1 INTEGRALS OF WHICH THE INTEGRAND IS UNBOUNDED OVER THE INTEGRATION INTERVAL

For example, $\int_0^3 (x-2)^{-1}\,dx$ is unbounded in $x = 2$.
Consider the integral $\int_a^b f(x)dx$. Suppose $f(x)$ becomes infinitely large at the point $c \in [a,b]$. Then we will investigate the following:

$$\boxed{\lim_{t\to c^-}\int_a^t f(x)dx \quad \text{and} \quad \lim_{u\to c^+}\int_u^b f(x)dx.}$$

If both these limits exist and they are finite, we say that $\int_a^b f(x)dx$ converges and

$$\boxed{\int_a^b f(x)dx = \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{u\to c^+}\int_u^b f(x)dx.}$$

If we write the given example into this notation, we obtain

$$\boxed{\int_0^3 (x-2)^{-1}\,dx = \lim_{t\to 2^-}\int_0^t (x-2)^{-1}\,dx + \lim_{u\to 2^+}\int_u^3 (x-2)^{-1}\,dx.}$$

Worked Example
Determine whether

$$\boxed{\int_0^3 \frac{dx}{(x-1)^{2/3}}}$$

converges and, if so, calculate the integral.

Solution
$f(x) = \frac{1}{(x-1)^{2/3}}$ becomes infinitely large at the point $x = 1$. Therefore we consider the following two integrals:

$$\int_0^a \frac{dx}{(x-1)^{2/3}} \quad \text{and} \quad \int_b^3 \frac{dx}{(x-1)^{2/3}}$$

and we investigate the following:

$$\begin{aligned}
\lim_{a\to 1^-}\int_0^a \frac{dx}{(x-1)^{2/3}} &= \lim_{a\to 1^-}\left[3\,(x-1)^{1/3}\right]_0^a \\
&= \lim_{a\to 1^-}\left[3\,(a-1)^{1/3} - 3\,(0-1)^{1/3}\right] \\
&= 3,
\end{aligned}$$

as well as

$$\begin{aligned}
\lim_{b\to 1^+}\int_b^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^+}\left[3\,(x-1)^{1/3}\right]_b^3 \\
&= \lim_{b\to 1^+}\left[3\,(3-1)^{1/3} - 3\,(b-1)^{1/3}\right] \\
&= 3\sqrt[3]{2}.
\end{aligned}$$

Both limits exist, therefore the integral converges and has the value

$$\left(3+3\sqrt[3]{2}\right).$$

5.2 INTEGRALS OF WHICH THE INTEGRATION INTERVAL IS UNBOUNDED

In this case we again use limits to determine whether the integrals are convergent or divergent and, provided the integral is convergent, we can also determine the value of the integral. The three cases which we may find, are the following:

- $\displaystyle\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$
- $\displaystyle\int_{-\infty}^a f(x)dx = \lim_{b\to-\infty}\int_b^a f(x)dx.$
- $\displaystyle\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\textstyle\int_0^b f(x)dx.$

Worked Example

Ascertain whether

$$\boxed{\int_2^\infty \frac{dx}{(x-1)^2}}$$

converges and, if so, calculate the integral.

Solution

$$\begin{aligned}
\int_2^\infty \frac{dx}{(x-1)^2} &= \lim_{a\to\infty}\int_2^a \frac{dx}{(x-1)^2} \\
&= \lim_{a\to\infty}\left[\frac{-1}{x-1}\right]_2^a \\
&= \lim_{a\to\infty}\left(\frac{-1}{a-1}+\frac{1}{2-1}\right) \\
&= 0+1 \\
&= 1.
\end{aligned}$$

Therefore the integral converges and has the value 1.

5.3 INTEGRALS FOR WHICH BOTH (5.1(S)) AND (5.2(S)) HOLD

We start with a worked example.

Worked Example

Determine whether

$$\boxed{\int_0^\infty \frac{1}{\sqrt{x}}dx}$$

converges and, if so, calculate the integral.

Solution

This integral has both an unbounded integrand over the integration interval, and an unbounded integration interval. In this case, we express the integral as the sum of improper integrals, each of which is of one of the types mentioned above.

Consider

$$\int_0^\infty \frac{1}{\sqrt{x}}dx.$$

The integrand is unbounded at 0, therefore we choose *any* number greater than 0, for example 1, and we write

$$\int_0^\infty \frac{1}{\sqrt{x}}dx = \int_0^1 \frac{1}{\sqrt{x}}dx + \int_1^\infty \frac{1}{\sqrt{x}}dx. \qquad (*)$$

The first integral on the right-hand side of (*) is of type (5.1(S)) and therefore we do it as follows:

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}}dx &= \lim_{b\to 0^+} \int_b^1 \frac{1}{\sqrt{x}}dx \\
&= \lim_{b\to 0^+} \left[2\sqrt{x}\right]_b^1 \\
&= \lim_{b\to 0^+} \left[2 - 2\sqrt{b}\right] \\
&= 2.
\end{aligned}$$

The second integral on the right-hand side of (*) is of type (5.2(S)), and therefore we do it as follows:

$$\begin{aligned}
\int_1^\infty \frac{1}{\sqrt{x}}dx &= \lim_{b\to \infty} \int_1^b \frac{1}{\sqrt{x}}dx \\
&= \lim_{b\to \infty} \left[2\sqrt{x}\right]_1^b \\
&= \lim_{b\to \infty} \left[2\sqrt{b} - 2\right].
\end{aligned}$$

It is clear that this integral diverges and thus $\int_0^\infty \frac{1}{\sqrt{x}}dx$ also diverges.

You must be very **careful** when determining an *improper integral.* Before you determine any integral, first determine whether the integrand and the integration interval are bounded. If that is not the case, you must use the method for improper integrals.

Extra Worked Example
Determine

$$\boxed{\int_0^3 \frac{dx}{(1-x)^2}.}$$

If you do not consider the fact that the integrand is unbounded at $x = 1$, you will get the wrong answer of $-\frac{3}{2}$. If you take this as an improper integral you will get the correct answer, i.e., that the integral diverges.

Solution

$$\int_0^3 \frac{dx}{(1-x)^2} = \lim_{t\to 1^-} \int_0^t \frac{1}{(1-x)^2}dx + \lim_{u\to 1^+} \int_u^3 \frac{1}{(1-x)^2}dx \qquad (*)$$

Let $v = 1 - x$ then

$$\frac{dv}{dx} = -1$$

so that $dx = (-1)\,dv$.

$$\begin{aligned} &\text{If } x = 0 \text{ then } v = 1 \\ &\text{If } x = t \text{ then } v = 1 - t \\ &\text{If } x = u \text{ then } v = 1 - u \\ &\text{If } x = 3 \text{ then } v = -2. \end{aligned}$$

Then (*) becomes

$$\begin{aligned} \int_0^3 \frac{dx}{(1-x)^2} &= (-1) \lim_{t\to 1^-} \int_1^{1-t} v^{-2}dv \\ &\quad + (-1) \lim_{u\to 1^+} \int_{1-u}^{-2} v^{-2}dv \\ &= \lim_{t\to 1^-} \left(\left[v^{-1}\right]_1^{1-t} \right) + \lim_{u\to 1^+} \left(\left[v^{-1}\right]_{1-u}^{-2} \right) \\ &= \lim_{t\to 1^-} \left[(1-t)^{-1} - 1\right] \\ &\quad + \lim_{u\to 1^+} \left[-\frac{1}{2} - (1-u)^{-1} \right]. \end{aligned}$$

Now

$$\lim_{t\to 1^-} \left(\frac{1}{1-t} - 1 \right) = \infty.$$

Also

$$\lim_{u\to 1^+} \left(-\frac{1}{2} - \frac{1}{(1-u)} \right) = \infty.$$

Thus this integral diverges.

5.4 THE LIMIT COMPARISON TEST

You have to study the relevant section in your prescribed textbook.

Summary

The improper integral can sometimes be evaluated directly using the methods specified in paragraph 5.1(S), 5.2(S) and 5.3(S). However, when an improper integral cannot be evaluated directly we use the comparison test. The idea of the comparison test is to compare a given improper integral to another improper integral whose convergence or divergence is already known.

Chapter 6

Infinite Sequences and the Taylor Series

Introduction

Informally, a sequence is any list (or ordered collection) of things, but in this chapter the things will usually be numbers.

It is also explained here what is meant by convergence and divergence of a sequence. We also develop one of the most **remarkable formulas in all of mathematics, a formula that enables us to express many functions as "infinite polynomials"**. In addition to providing effective polynomial approximations of differentiable functions, these infinite polynomials (called power series) have many uses. For instance, they provide an efficient way to evaluate non–elementary integrals and they solve differential equations that give insight into vibration, chemical diffusion and signal transmission.

Outcomes

After studying this chapter the learner should be able to:

- use limit techniques to determine the limit of a sequence.
- write out the Taylor series generated by f at a.
- obtain the Taylor polynomial $P_n(x)$ of any given order n generated by f at a point, where f is, for example of the form $f(x) = e^x,\ f(x) = \ln x$.

6.1 LIMITS OF INFINITE SEQUENCES

Study this section in your textbook.

6.2 TAYLOR SERIES

Please note that you don't need to be able to find the remainder for a Taylor series. This will be done in the second year.

In the previous section, the concept of an infinite sequence was discussed. When the terms of such a sequence are added, we obtain an infinite series, one of which we discuss in more detail in this section.

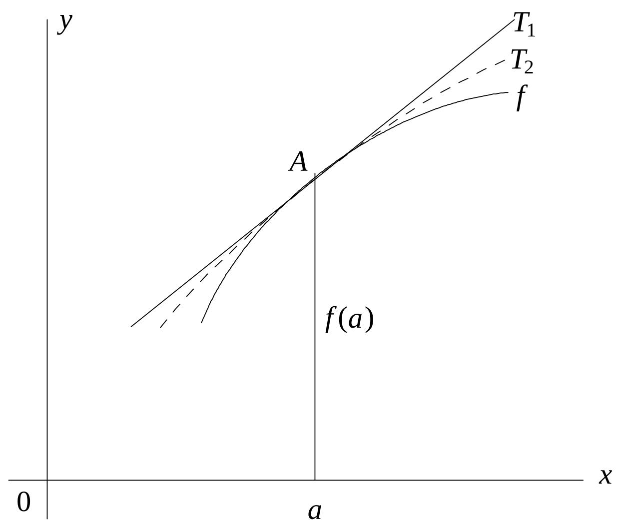
The values of polynomial functions can easily be calculated in a given point. If, for example,

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

we can easily calculate $f(r)$ for any number r. However, at this stage we do not yet have a method of calculating the values of functions such as $\ell n\ x$, $\sin x$, e^x, etc. In this section, we become acquainted with infinite series. We convert functions such as $\ell n\ x$, $\sin x$, e^x, etc., to infinite series, which then reduces the approximate determination of the value of such functions in a given point to the calculation of a polynomial. Computers, for example, also employ this method for calculating the above-mentioned functions.

We will therefore try to reduce the evaluation of $f(x)$ for certain functions f to the computation of the function values of polynomial functions. The method entails the determination of polynomial functions that are good approximations of f on an interval that is contained in the domain of f.

To begin with, assume that f is differentiable at the point $A = (a, f(a))$. We expect that the tangent to the graph of f at A provides us with a good approximation of f in the *immediate* neighbourhood of A. This tangent is the graph of a polynomial function of the first-degree that passes through A and possesses the same derivative as f at A.



The equation of the tangent is

$$P_1(x) = f(a) + f'(a)(x - a).$$

We cannot expect that the tangent at A will continue to yield good approximations of the function values of f for values of x further away from a. On the other hand, we can however try to get a better approximation of f by means of a second-degree polynomial. Suppose f possesses first and second derivatives at A. Then we choose this second-degree polynomial in such a way that its graph passes through $A = (a, f(a))$ and, furthermore, is such that its first and second derivatives at A coincide with those of f. Suppose this polynomial is

$$P_2(x) = a_0 + a_1(x - a) + a_2(x - a)^2.$$

Then

$$\begin{aligned} P_2(x) &= a_0 + a_1(x - a) + a_2(x - a)^2, && \text{and hence } P_2(a) = a_0; \\ P_2'(x) &= a_1 + 2a_2(x - a), && \text{and hence } P_2'(a) = a_1; \\ P_2''(x) &= 2a_2, && \text{and hence } P_2''(a) = 2a_2. \end{aligned}$$

Now we choose a_0, a_1 and a_2, such that

$$\begin{aligned}
P_2(a) &= f(a), && \text{in other words, we choose } a_0 = f(a);\\
P_2'(a) &= f'(a), && \text{in other words, we choose } a_1 = f'(a);\\
P_2''(a) &= f''(a), && \text{in other words, we choose } a_2 = \tfrac{1}{2}f''(a).
\end{aligned}$$

Therefore the polynomial is

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

We can now proceed in this manner and find higher degree polynomials of which we hope that they will be still better approximations of f. Then we find that the polynomial $P_n(x)$ is as follows:

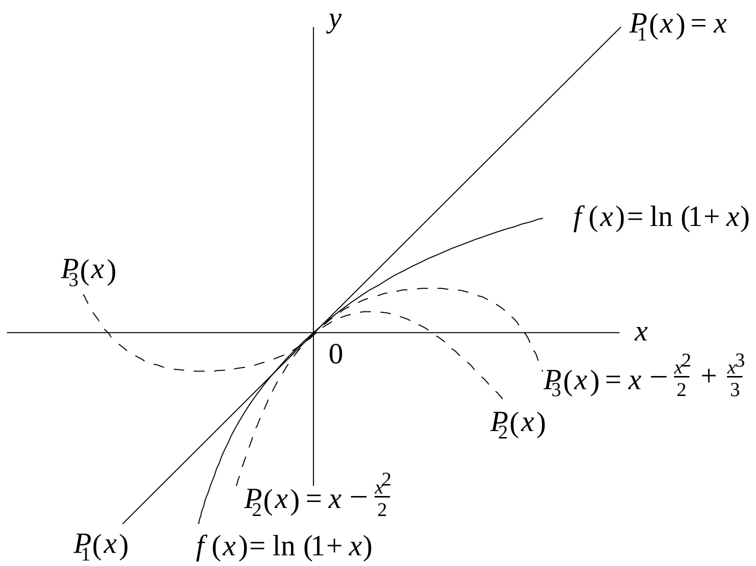
Polynomial $\mathbf{P_n\,(x)}$

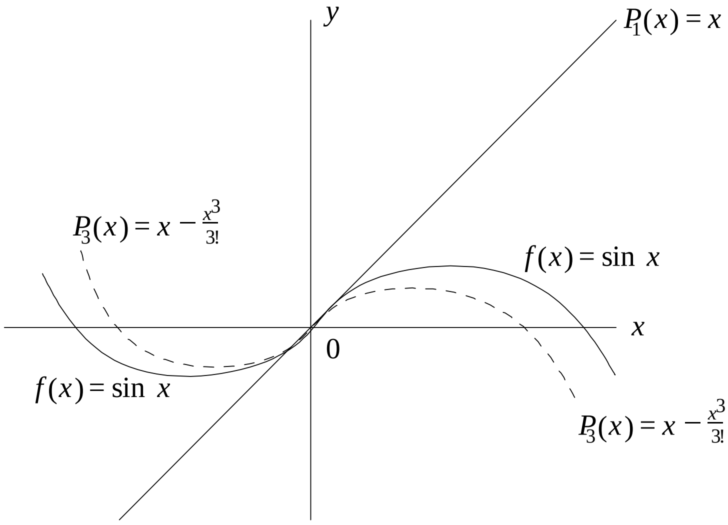
$$P_n(x) = f(a) + f'(a)(x-a) + \tfrac{f''(a)}{2!}(x-a)^2 + \ldots + \tfrac{f^{(n)}(a)}{n!}(x-a)^n.$$

The polynomial $P_n(x)$ is called the *nth degree Taylor polynomial of f at the point a.* When $a = 0$, the polynomial is said to be the *nth degree Maclaurin polynomial of f.*

When working with infinite series, the convergence of such series must be considered. At this stage we will discuss the convergence of series – this will only be done in your second year Mathematics course.

We now show the approximations for $\ell n(1+x)$ and sin x, respectively, at $a = 0$ where $P_1(x), P_2(x)$ and $P_3(x)$ were obtained from the method above.





Worked Example

Determine the Taylor polynomial of order 3 generated by $f(x) = \sqrt{x}$ about the point $a = 4$.

Solution

First obtain the derivatives $f'(x)$, $f''(x)$ and $f'''(x)$ and then the values in the point 4.

$$f(x) = x^{\frac{1}{2}} \qquad f\left(2^2\right) = 2$$

$$f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}} \qquad f'\left(2^2\right) = \tfrac{1}{4}$$

$$f''(x) = -\tfrac{1}{4}x^{-\frac{3}{2}} \qquad f''\left(2^2\right) = -\tfrac{1}{32}$$

$$f'''(x) = \tfrac{3}{8}x^{-\frac{5}{2}} \qquad f'''\left(2^2\right) = \tfrac{3}{256}$$

Write out the Taylor polynomial of order 3 and substitute the values above:

$$\begin{aligned}
P_3(x) &= f(4) + \frac{(x-4)\,f'(4)}{1!} + \frac{(x-4)^2\,f''(4)}{2!} + \frac{(x-4)^3\,f'''(4)}{3!} \\
&= 2 + \frac{1}{4}(x-4) - \frac{1}{32\times 2}(x-4)^2 + \frac{3}{256\times 3\times 2}(x-4)^3 \\
&= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 .
\end{aligned}$$

Exercises:
Find the Taylor polynomial of any order (say 3) or of order n for as many functions as possible. See the exercises in your prescribed textbook. Remember, you do not need to be able to find the remainder for the Taylor series. This will be done in the second year.
Also, try the following:

1. Expand the given polynomial $f(x) = 10 - 20x + 15x^2 - 4x^3$ as a polynomial about the point 1 (or as a polynomial in $(x - 1)$).
2. If $f(x) = \frac{1}{1-x}$, determine $P_5(x)$ about the point 0.

Answers

1. $1 - 2(x - 1) + 3(x - 1)^2 - 4(x - 1)^3$.
2. $1 + x + x^2 + x^3 + x^4 + x^5$.

Summary

The learner should now have an intuitive understanding of sequences and be able to determine the limit of a sequence when it exists. The learner should also be able to demonstrate a basic understanding of the Taylor series and how to construct Taylor polynomials for certain functions.