CONTENTS

PREFACE

Introduction

Welcome to MAT1503, the first year mathematics module on linear algebra. Topics studied in linear algebra, for example systems of linear equations and matrices, have a variety of applications in science, engineering and industry. Amongst these are games of strategy, computer graphics, economic models, forest management, cryptography, fractals, computed tomography, and a model for human hearing. So as you can see there are a number of exciting applications of linear algebra. However, before you can understand these applications you will have to familiarize yourself with the contents of this module.

MAT1503 prepares you for further studies in linear algebra at the second and third year levels. It also equips you with the basic tools of linear algebra which can be applied in various fields.

Now in order to study this module successfully you need a good working knowledge of algebra at matric level as well as the ability to think and work consistently.

Here are a few comments and advice from a colleague about **thinking**.

- Memorizing a theorem or a proof of a theorem, or memorizing an example or a method to do a certain type of exercise, is **not** thinking. Thinking starts with questions like: What is this module *really* about? What does this definition *mean*? What does Theorem x *mean*? Does Theorem x surprise me or would I have expected something like it to hold? What are the *ideas* behind the proof of Theorem x? If I tried to prove it myself without having yet seen a proof of it, how would I go about it? Would I get stuck? *Where* and *why*? Could I get around this obstacle? And so on.

- Thinking is *hard.* It is much easier to pass certain modules by memorizing a few facts than it is to think hard about and around the things you are studying. However, you do mathematics to make the concepts involved part of your mental vocabulary and your worldview, not just to obtain a credit.

- One does not understand mathematical concepts, definitions and theorems after thinking about them for 5 minutes. These concepts took hundreds of years to arrive at their current form. Mathematicians spend thousands of hours thinking about mathematics, sometimes even about a particular problem or concept!

- Do not be afraid to think. Thinking most often does *not* make you feel clever. On the contrary, it is usually a slow, halting process which most often makes you realize exactly how little you know, which usually makes you feel stupid and inadequate. You need to accept these feelings and think about things anyway. Eventually, it will become easier and you will realize its value. It does not matter if it takes a whole week/month/year to understand something. Once you have understood it, it is part of you.

- Sitting writing at your desk is often not the best way to understand new ideas/theorems/definitions. Read through the work and then mull over the ideas while you take a break/make tea/go outside. It is often in these times that one understands things.

- Really understanding something is usually not done on the first attempt. Your mind does not work that way. The best way to try to understand something difficult is firstly to ask yourself: What is this *about*? Then try to find the *main ideas* first before going through all the details. Do not try to force understanding, if after 15 minutes you have made no headway, take a break! But while you're on this break ask yourself where and why you're getting stuck, exactly *what* it is that you don't understand. Also, it is very easy to get stuck in a certain pattern of thought about something. Try to change your angle, as it were.

- Related to this is the fact that one does not study mathematics only by spending a fixed time each day or week in front of your desk. The ideas in maths textbooks are alive, and you should adopt them by thinking about and around them often, not just for an hour a day.

- If you think long enough and hard enough, you will eventually realize/understand/see something that you did not before. It is then that you will realize how exciting and enriching thinking can be, despite the fact that it can be so hard. (It can also become easy and fun, but only when you have thought really hard about the concepts already!)

- Mathematics is primarily about ideas and concepts. In 10 years time you will remember no details of any calculations done in this module but if you thought long and hard about the concepts, you will remember what we were doing, and why.

- Working through countless exercises is useless without knowing what you are doing and why you are doing it the way you are. We urge you to think about what you are doing at all times.

- You need to be brutally honest with yourself about what you do and do not understand. It is very tempting to ignore the things you don't understand and hope that they go away! Face these things, no matter how long it takes. When you finally do understand them, it will be a great thrill!

- A good exercise to do is the following: After each section, explain to a friend who does not do mathematics, what the section was about. If you cannot (that is, your friend does not understand) then you may not yet understand the work.

Prescribed Book

Title

The prescribed book for this module is

> Howard Anton & Chris Rorres: *Elementary Linear Algebra: Applications Version,* (11^{th} edition, 2014), John Wiley & Sons, Inc.

You cannot study MAT1503 without this book, so we advise you to purchase it as soon as possible.

Throughout this study guide we shall refer to the textbook as Anton.

Layout

Before you begin your studies, we suggest that you browse through Anton in order to familiarize yourself with its layout.

Each chapter is divided into various sections. At the end of each section is an exercise set, which begins with routine drill exercises and progresses to more difficult problems including theoretical problems. At the end of the chapter is a set of supplementary exercises that combines ideas and results from the whole chapter and not just from a specific section. This is followed by a set of technology exercises which are designed to be solved, using a technology utility, such as MatLab, Mathematica, Maple, Derive or Mathcard (or some other type of linear algebra software or a scientific calculator with linear algebra capabilities). The use of a technology utility to solve various linear algebra problems **DOES NOT** form part of the syllabus for this module. However, if you are interested and have a suitable technology utility you are welcome to do these exercises.

The final chapter, i.e. Chapter 11 contains 21 applications of linear algebra. Each application begins with a list of linear algebra prerequisites so that a reader can tell in advance if he or she has enough background to read the section. A table is given on p. x of Anton which classifies the difficulty of each of the applications. You may wish to browse through this chapter before you begin your study of the module, in order to become aware of the variety of the applications. Of special interest is the section (§ 11.4) on the earliest applications of linear algebra. This section shows how linear equations were used to solve practical problems in ancient Egypt, Babylonia, Greece, China and India.

Overview of the Module

This module deals with topics such as systems of linear equations, matrices, determinants and vectors; and consists of 16 sections from Chapters 1 to 4 of Anton. The diagram on the following page summarizes the contents of the module. We use the same numbering as that used in Anton.

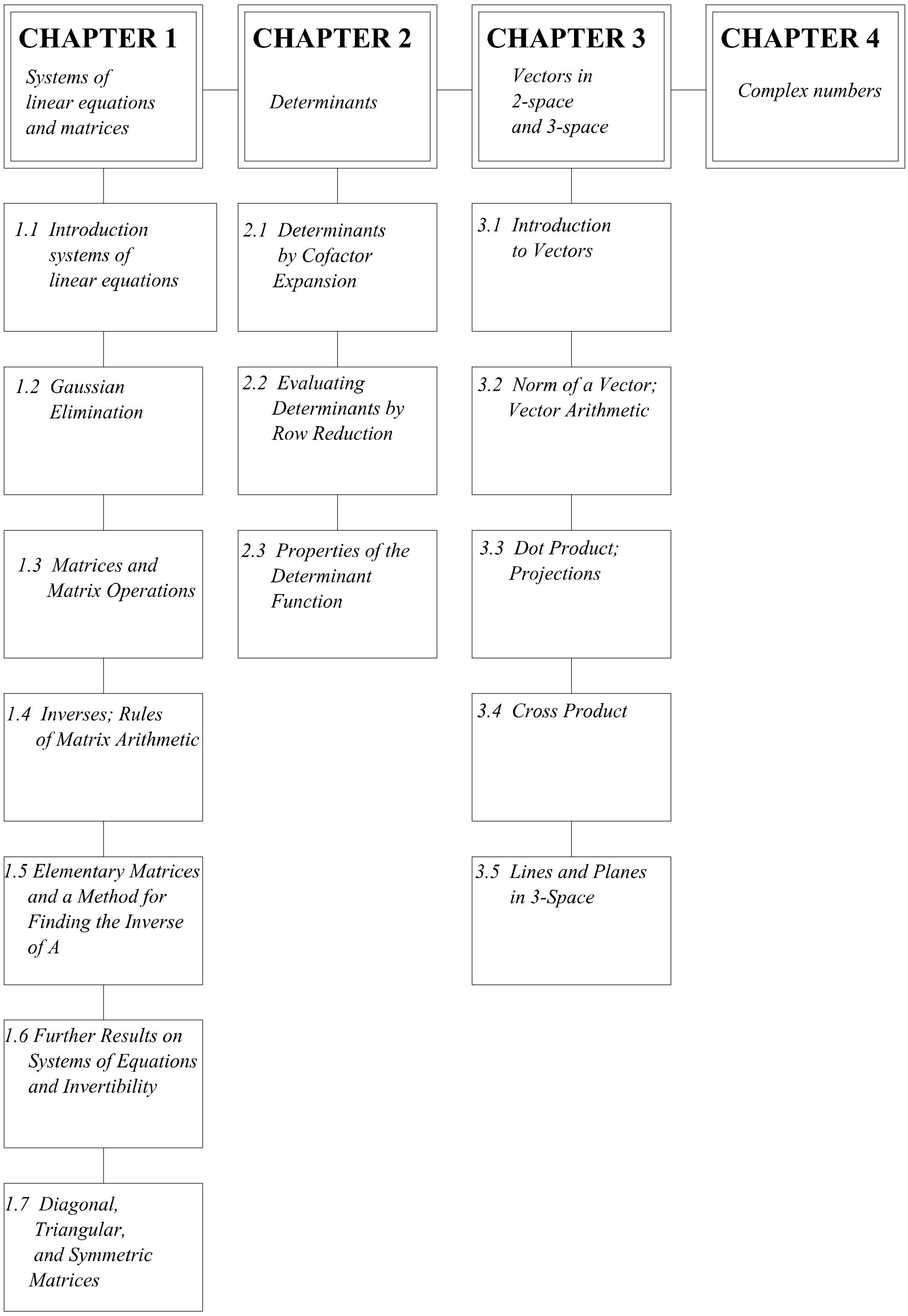


Diagram: Overview of the module

The outcomes for this module are

- To solve systems of linear equations.
- To perform basic matrix operations.
- To evaluate determinants and use them to solve certain systems of linear equations and to find inverses of invertible matrices.
- To perform various operations in 2-space, 3-space and n-space and to find equations for lines and planes in 3-space.

Study Guide

Each chapter starts with an introduction. The chapter is divided into a number of sections. Each section consists of the following:

Overview: This is a short description of the contents of the section.

Source: This gives the reference to the material to be studied directly from Anton.

Learning Outcomes: This lists the skills that you should acquire during your study of the section.

Additional notes: Where relevant these include extra explanations, worked out examples or information regarding parts of Anton that do not form part of the syllabus and need only be read for interest's sake.

List of important concepts: As you read through the list of concepts make sure that you are able to define or explain each of them to yourself or a friend. If not, we suggest that you study them again, as they form the building blocks of the module.

Activities: We provide a list of problems from the exercise sets in Anton for you to do. We suggest that you attempt only a selection of them when you first study a specific section. You can then do the remaining problems when you revise the work.

Summary: At the end of each section we have provided a blank space for you to write your own summary. It is extremely important to be able to summarize the main points from the textbook. For example, you may wish to include important definitions, results of main theorems, formulas, properties, methods for solving certain problems, etc. If the spaces provided are not large enough we suggest that you use a note book for your summaries. We provide an example of a summary for Section 1.1.

At the end of each chapter we provide a **"Review"** as well as a **"diagrammatic summary"** of some of the main ideas of the chapter.

At the end of the study guide we provide the **answers** to some of the questions given under Activities which are not given in Anton. However, note that no proofs or verification of statements are given.

Learning Strategies for Mathematics, and in particular for this Module

One of the prerequisites for successful distance learning is the ability to learn independently, and take responsibility for one's own learning. We hope that you will not study this module with the aim of only passing the exam; we hope that you will want to learn the material covered in this module for other reasons as well. Here are some suggestions that may assist you with your study of mathematics.

Have the right attitude!

Your past attitudes and feelings towards mathematics will affect your perception of the MAT1503 study material even before you begin to work through it. If studying mathematics was a pleasant experience for you before, you are probably looking forward to this work. If not, you probably have serious misgivings about being able to "make the grade".

A negative attitude undermines your ability to understand and enjoy what you are learning. Be conscious of negative attitudes, and try to put them aside. Expect that you will have to work hard (there are no short cuts to understanding mathematical concepts) but expect also that you will ultimately master the work.

Practise

You cannot learn mathematics by just reading the study material. There is a big difference between **reading** and **doing** mathematics. Try to answer questions, solve problems or verify statements to prove to yourself that you have actually understood the concepts involved. The questions in the textbook are structured in such a way that problems are graded from simple to more difficult, and it is a good idea to tackle the simpler ones first, before dealing with more complicated ones.

A good reason to practise regularly in this way is that as you gradually get correct answers more often, you will develop confidence in your own mathematical ability. Confidence in your own ability will help you to develop a positive attitude to your studies, and hence enjoy the work more and cope better.

Try to set aside a regular time and a suitable place for study

Because you need to concentrate in order to make sense of abstract concepts, it is usually unproductive to try to study mathematics when you are very tired, or when there are many distractions. Adult distance learners often have to cope with the demands of work and family and many other issues as well. Be realistic about your circumstances, but try to create the best possible conditions for your studies.

As we have said before, mathematics involves **doing**, so you need to work where you can write as well as read, and you need to have plenty of rough paper handy as you work. Having said that, though, it is also a good idea to be flexible, and accept that it is possible, at times, to study on the bus, or to try various ways of solving a problem while waiting in a queue!

We mentioned earlier the importance of having the right attitude. Part of this attitude involves a willingness to give the necessary time and effort to your studies.

Summarise

As we mentioned earlier it is important to make your own summary for each section. You may find it useful to refer to these summaries when you do your assignments or prepare for the exam.

Think

We started this preface by mentioning that your success at studying a mathematics module is dependent on your ability to think. To study mathematics you cannot be a **passive** reader or listener; you need to be an **active** thinker.

Understand – do not memorise

One of the best things about mathematics is that it has a logical structure. Often, understanding how a rule or formula is derived means that there is no need to memorise it, because you can work it out when you need it. However, there are certain facts or definitions which have to be memorised, so try to distinguish between the things you really do need to memorise and those you do not, and by regular practice develop confidence in your ability to reason things out if necessary.

Set realistic goals and give yourself enough time

You will make good progress with this module if you set goals for yourself. The closing dates for the assignments (see Tutorial Letter 101) already create a framework in which to set target dates. Try to keep to your own schedule. Students in general, and distance learners in particular, underestimate how long it will take them to master a new concept. Remember that several steps are involved. Firstly, you need to **read critically**. (Mathematical text is usually more time-consuming to read than other academic text, and even more so for learners who are not studying in their mother tongue.) You then need to **test your understanding of individual concepts** and **consolidate your understanding of how different concepts relate to one another** by doing the exercises. All this takes time – lots of it!

Be critical of your own answers

Question the validity of numerical answers. For example, if you calculate that someone runs at 100 kph, you have probably made a mistake! Try to find the mistake yourself.

Question the logic of your answers. When you solve a problem, read what you have written and ask yourself whether there is a logical relationship between the various steps of your solution. Or have you made deductions without having the necessary conditions? One way to avoid this is to keep the steps in your solution small, even if this means you need more of them. Also, avoid using rules or formulas that you have not understood, because the chances are that you will apply them incorrectly.

If possible test each answer, by substituting it into the original problem, to see whether it is valid.

Form a study group

Many students study by means of distance education because they are in remote areas, and in such cases **group work** is usually not possible. However, the majority of you will be near an established learning centre, or will be in an area where there is some venue available where you can arrange to meet regularly and share your ideas. The "Discussion Discovery" questions at the end of the exercise sets in Anton are ideal questions to be tackled in groups.

Different students will tackle the questions in different ways, and you can learn a lot from discussing different approaches. (Remember that you can contact the university to make arrangements to obtain addresses or telephone numbers of students in various geographic locations who are studying the same modules.) Team work is only effective if everyone in the team works, i.e. there is no room for passengers hoping for quick answers.

It is inevitable that students in a study group will discuss the questions in the assignments. In such cases, students must write up the assignments on their own and not copy other students' work.

Take part in the on-line discussion forum

If you are unable to form a study group and you have access to the internet, then you may find it beneficial to join in the discussion forum for MAT1503. You will first have to register on myUnisa. The web address is: https://my.unisa.ac.za. For more information regarding myUnisa please consult the booklet "Services and Procedures".

Join a tutorial class

Tutorial classes are presented at certain learning centres – you can obtain more information about these from the booklet "Services and Procedures".

Assessment

There are a number of assignments that you need to submit during the year. The marks for these will count towards a year mark which will be combined with your exam mark to give you your final mark.

Please consult your **first tutorial letter** (Tutorial Letter 101) in connection with the assignment questions, the closing dates for assignments, the exam admission procedure and other important information relating to this module.

We wish you success with MAT1503 and hope that you enjoy studying it.

CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS AND MATRICES

Introduction

According to Anton:

"Information in science and mathematics is often organized into rows and columns to form rectangular arrays, called "matrices" (plural of "matrix"). Matrices are often tables of numerical data that arise from physical observations, but they also occur in various mathematical contexts. For example, we shall see in this chapter that to solve a system of equations such as

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

all of the information required for the solution is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

and that the solution can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs to solve systems of linear equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a wide variety of applications. In this chapter we will begin the study of matrices."

The material in this chapter is divided into the following sections:

1.1 Introduction to Systems of Linear Equations

Overview

The study of systems of linear equations and their solutions is one of the major topics in linear algebra. In this section we introduce some basic terminology and discuss a method for solving such systems.

Source: Anton §1

Learning Outcomes

After studying this section you should be able to

- recognise a linear equation
- determine whether a given sequence of numbers or a given element is a solution of a linear equation (or system of linear equations)
- find the solution of a linear equation
- determine geometrically whether a system of linear equations in 2 unknowns has no solution, exactly one solution or infinitely many solutions
- state the three elementary row operations.

Additional notes

Use of the terms solution set and general solution

Anton uses the terms solution set and general solution interchangeably, and does not use set notation when writing down a solution set. We shall distinguish between the two terms in the following way: Consider the equation

$$x + 2y - z = 3.$$

The general solution (or the solution) of this equation is

$$\begin{aligned} x &= -2s + t + 3 \\ y &= s \qquad\qquad t, s \in \mathbb{R} \\ z &= t \end{aligned}$$

whereas the solution set is

$$\{(x, y, z) \mid x = -2s + t + 3,\ y = s,\ z = t \quad \text{where} \quad t, s \in \mathbb{R}\}.$$

We notice that each element in this set has three coordinates, namely x, y and z and we say that (x, y, z) is an element of $\mathbb{R}^3$. Likewise we call $(x_1, x_2, \ldots, x_n)$ which has n coordinates an element of $\mathbb{R}^n$.

Solutions of systems of equations

An element is a solution of a given system of equations if it has the **same number of coordinates** as the **number of unknowns in the system** and if it satisfies **all** the questions in the system.
Therefore $(1,2)$ is not a solution of $2x_1 - x_2 + 0x_3 = 0$ because $(1,2) \in \mathbb{R}^2$, whereas the solutions of this equation are elements of $\mathbb{R}^3$.
Also $(-1,2,3)$ is not a solution of the system

$$\left.\begin{array}{ll} 2x_1 + x_2 + 0x_3 & = 0 \\ -x_1 + x_2 + 0x_3 & = -3 \end{array}\right\}$$

because $(-1,2,3)$ does not satisfy the second equation. In fact on substituting $x_1 = -1$, $x_2 = 2$ and $x_3 = 3$ in the equations we obtain

$$\begin{array}{rcl} \text{LHS } (1) & = & 2(-1) + 2 + 0(3) = -2 + 2 = 0 = \text{RHS } (1) \\ \text{and LHS } (2) & = & -(-1) + 2 + 0(3) = \ \ 1 + 2 = 3 \neq \ \text{RHS } (2). \end{array}$$

We also use the fact that an element is a solution of a system of equations if and only if it satisfies all the equations in the system in order to solve the following type of problem.

Example 1.1.1

For which value(s) of a is $(1,2,-1)$ a solution of the system

$$\left.\begin{array}{ll} x_1 - 2x_2 - 3x_3 & = 0 \\ x_1 + ax_2 & = 3 \\ x_1 + x_2 + (a+1)\,x_3 & = 2 \end{array}\right\}?$$

Solution:
In order for $(1,2,-1)$ to be a solution of the above system it must satisfy each of the equations. Now $(1,2,-1)$ satisfies the first equation for all values of a. In order for $(1,2,-1)$ to satisfy the second equation we must have

$$\begin{array}{lrcll} & 1 + 2a & = & 3, & \\ \text{i.e.} & a & = & 1 & \quad \text{... (i)} \end{array}$$

and in order to satisfy the third equation we must have

$$\begin{array}{lrcll} & 1 + 2 - (a+1) & = & 2, & \\ \text{i.e.} & a & = & 0 & \quad \text{... (ii)} \end{array}$$

As (i) and (ii) are not in agreement (i.e. a cannot be 1 and at the same time 0), there are no values for a for which $(1,2,-1)$ will be a solution of the above system. ■

Equivalent systems

On p. 3 of Anton we come across the phrase "resulting equivalent system". Do you know what this phrase means? We define the concept as follows.

Definition 1.1.2 Equivalent systems

Two systems of equations are **equivalent** if they have the same solution set.

Why is the above concept important? A very useful strategy in solving a linear system is to reduce it to an equivalent system which is easier to solve. The elementary row operations (stated on p. 5 of Anton) are used in producing such systems.

List of important concepts

a ***linear equation in*** n ***unknowns (variables)***
a ***solution*** of a linear equation
the ***solution set*** (***general solution***) of a linear equation
a ***system*** of ***linear equations*** (or ***linear system***)
a ***solution*** of a ***system of linear equations***
an ***inconsistent/consistent*** system of equations
elementary row operations
equivalent systems

Additional questions

A.1 Consider the system

$$\left.\begin{array}{rrrrl} x_1 + & x_2 - & x_3 + & 2x_4 & = 2 \\ 2x_1 & & + x_3 - & x_4 & = 1 \\ & -x_2 & & + 2x_4 & = 3 \end{array}\right\} . \tag{1}$$

Which of the following elements are solutions of the system?

Give reasons for your answers.

(a) $(1, -1, 0, 1)$

(b) $(1, -1, 0, 1, 0)$

(c) $(0, 1, 3, 1)$

(d) $(2, -3, 3, 0)$

A.2 Consider the system

$$\left.\begin{array}{rl} x_1 + 2x_2 + 0x_3 & = 1 \\ 3x_1 + ax_2 + 0x_3 & = 3 \\ x_1 + ax_2 + x_3 & = 2 \end{array}\right\} . \tag{2}$$

(a) For which value(s) of a is $(1, 0, 1)$ a solution of the system (2)?

(b) For which value(s) of a is $(1, 0, -1)$ a solution of the system (2)?

(c) For which value(s) of a is $(3, -1, 5)$ a solution of the system (2)?

We conclude by giving an example of a summary for this section. Note that there is no standard correct summary. You need to ensure that your summary contains all the important information and is useful to you.

Summary of Section

Linear equation in n unknowns:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b. \tag{1}$$

Examples:

$$\begin{array}{ll} 2x + 3y - z = 2 & \text{is } \textbf{linear} \\ 2\sqrt{x} + y = 2; & y = \cos x \text{ are } \textbf{not linear} \end{array}$$

The sequence of numbers $s_1, s_2, \ldots, s_n$ or the element $(s_1, s_2, \ldots, s_n)$ is a **solution** of (1) if and only if $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ satisfies (1).

Solution set (general solution) is the set of all solutions.

In the solution $x = t$, $y = 3 + t$, where $t \in \mathbb{R}$, t is called a **parameter.**

System of linear equations (linear system):

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \tag{2}$$

(m equations in n unknowns)

The sequence $s_1, s_2, \ldots, s_n$ or the element $(s_1, s_2, \ldots, s_n)$ is a solution of (2) if and only if it is a solution of each equation in (2).

Consistent system has a solution. **Inconsistent** system has no solution.

Every linear system has no solution, or has exactly one solution, or has infinitely many solutions.

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

is called the **augmented matrix** of (2).

Two systems are **equivalent** if they have the same solution set.

To solve a linear system perform the following types of operations on the equations (or rows of the augmented matrix) in order to produce an equivalent system.

1. Multiply an equation (row) by a nonzero constant.
2. Interchange two equations (rows).
3. Add a multiple of one equation (row) to another.

The above row operations are called **elementary row operations.**

1.2 Gaussian Elimination

Overview

In this section we consider a systematic procedure for solving systems of linear equations. It is based on the idea of reducing the augmented matrix of a system to a form that is simple enough that the solution of the system can be found by inspection.

Source: Anton §1

Learning Outcomes

After studying this section you should be able to

- recognise matrices that are in row-echelon form, reduced row-echelon form, or generalized row-echelon form
- solve a linear system by using Gauss-Jordan elimination (i.e. by reducing the augmented matrix to reduced row-echelon form)
- solve a linear system by using Gaussian elimination (i.e. by reducing the augmented matrix to row-echelon form)
- solve a linear system by reducing the augmented matrix to generalized row-echelon form
- determine if/when a linear system has no solution, exactly one solution or infinitely many solutions
- determine if/when a linear system of homogeneous equations has only the trivial solution (i.e. only one solution) or the trivial as well as nontrivial solutions (i.e. infinitely many solutions).

Additional notes

It can be rather time consuming to reduce the augmented matrix of a system to row-echelon or reduced row-echelon form. We now give a method in which we solve a system by reducing the augmented matrix to "generalized row-echelon form".

Definition 1.2.1 Generalized Row-Echelon Form

We say that a matrix is in **generalized row-echelon** form if all the rows consisting of zeros are grouped together at the bottom of the matrix and if each nonzero row begins with more zeros than the previous row.

Some authors use the above definition as the definition of row-echelon form.

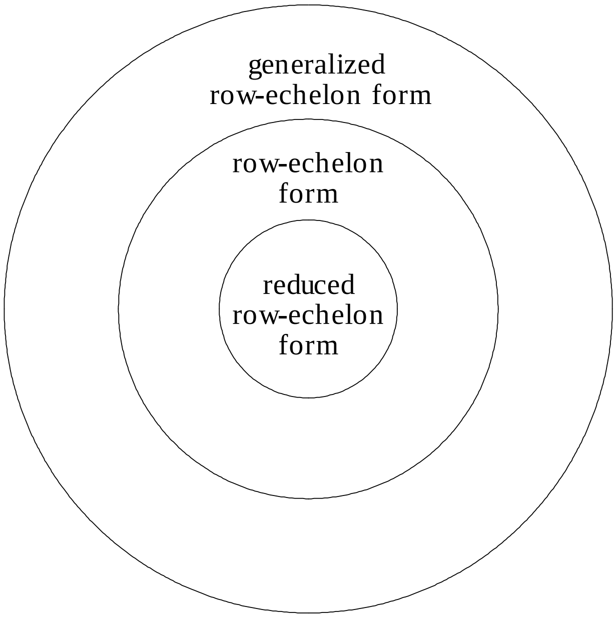
Example 1.2.2

The following matrices are in generalized row-echelon form.

$$\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \blacksquare$$

Are you able to distinguish between row-echelon form, reduced row-echelon form and generalized row-echelon form?

Note that matrices that are in row-echelon or reduced row-echelon form are in generalized row-echelon form. The following diagram gives the relationship between the various "row-echelon" forms.



Work through the following example in order to establish if you really understand the concept of generalized row-echelon form.

Example 1.2.3

For what values of a is each of the following matrices in generalized row-echelon form?

(i) $A = \begin{bmatrix} 2 & a & 1 \\ 0 & 3 & 2 \\ 0 & 0 & a+3 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & (a+1) & 3 \\ 0 & 0 & (a-3) \end{bmatrix}$

(iii) $C = \begin{bmatrix} a & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & a+3 \end{bmatrix}$ (iv) $D = \begin{bmatrix} 2 & 3 & 4 \\ 0 & (a-1) & 3 \\ 0 & 0 & (a-1)(a+2) \end{bmatrix}$

Solution:

(i) Each row begins with more zeros than the previous row, no matter what the value of a is. (If $a = -3$, then the last row consists of only zeros.) Thus A is in generalized row-echelon form for all real values of a.

(ii) If $a+1 \neq 0$ and $a-3 \neq 0$, then B is in generalized row-echelon form. However, we must consider what happens if either $a+1=0$ or $a-3=0$.

If $a+1=0$, i.e. if $a=-1$, then B becomes

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

which is **not** in generalized row-echelon form because rows 2 and 3 do not consist only of zeros and they begin with the same number of zeros.

If $a-3=0$, i.e. if $a=3$, then B becomes

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in generalized row-echelon form.

Thus B is in generalized row-echelon form for all real values of a, except $a=-1$, i.e. for $a \in \mathbb{R}-\{-1\}$.

(iii) From the form of C we see that if $a \neq 0$ and $a+3 \neq 0$, then C is in generalized row-echelon form. If $a=0$, then C becomes

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

which is not in generalized row-echelon form. (Why?)

If $a+3=0$, i.e. if $a=-3$, then C is in generalized row-echelon form. (Test this for yourself.)

Thus C is in generalized row-echelon form if $a \in \mathbb{R}-\{0\}$.

(iv) It is clear that if $a-1 \neq 0$ and $a+2 \neq 0$, then D is in generalized row-echelon form. If $a-1=0$, i.e. if $a=1$, then D becomes

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in generalized row-echelon form.

If $a+2=0$, i.e. if $a=-2$, then D becomes

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is again in generalized row-echelon form. Hence D is in generalized row-echelon form for all real values of a, i.e. for $a \in \mathbb{R}$. ■

Before solving a system by reducing the augmented matrix to generalized row-echelon form we introduce the following notation for elementary row operations so that we need not write out the relevant operation in words each time we use it.

Elementary row operation notation

1. Multiply row i by a **nonzero** constant k:

$$kR_i \to R_i$$

   (replace row i by k times row i)

2. Interchange rows i and j:

$$R_i \longleftrightarrow R_j$$

3. Add a multiple of row i to row j:

$$R_j + kR_i \to R_j$$

   (replace row j by row j plus k times row i)

Note that in 1 the value of k must be nonzero, but in 3 the value of k may be zero. If in 3 the value of k is zero, then we have

$$R_j \to R_j$$

and in fact no operation has been performed.

Warnings!

1. We often perform more than one elementary row operation at a time. We must take care that we do not use a row that no longer exists. For example, consider the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

   If you perform the two row operations $R_2 - R_1 \to R_2$ and $R_1 - R_2 \to R_1$ on the above matrix you may obtain the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (*)$$

   which is obviously not equivalent to the first matrix.

   We note that after we have performed the first operation we obtain the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   After the second operation we obtain the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   The error in obtaining $(*)$ occurred as follows:

   After the first operation, namely $R_2 - R_1 \to R_2$, the **second row changed**, but when the operation $R_1 - R_2 \to R_1$ was performed the *original* row 2 was used instead of the *new* row 2.

2. The row operation
$$kR_j + R_i \to R_j$$
is not an elementary row operation. Compare this with row operation 3 given under the heading "Elementary row operation notation".

In fact
$$kR_j + R_i \to R_j$$
consists of the row operation
$$kR_j \to R_j$$
followed by
$$R_j + R_i \to R_j \qquad \text{(remember } R_j \text{ is the "new" } R_j\text{)}$$
and the first operation is an elementary row operation only if $k \neq 0$.

Thus if you use the operation $kR_j + R_i \to R_j$ you must assume that $k \neq 0$, and then consider the case $k = 0$ separately.

We will discuss this further in Example 1.2.8 at the end of this section.

We are now ready to work through an example in which we solve a system of equations by reducing the augmented matrix to generalized row-echelon form.

Example 1.2.4

Solve the following system by reducing the augmented matrix to generalized row-echelon form.

$$\left.\begin{array}{rcr} x - 2y + z \qquad\quad &=& 2 \\ 2x \qquad\quad - z + 2w &=& 5 \\ x + 2y + z - 3w &=& -4 \end{array}\right\}$$

Solution:

The augmented matrix is
$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & 2 \\ 2 & 0 & -1 & 2 & 5 \\ 1 & 2 & 1 & -3 & -4 \end{array}\right].$$

The row operations $R_2 - 2R_1 \to R_2$ and $R_3 - R_1 \to R_3$ give
$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & 2 \\ 0 & 4 & -3 & 2 & 1 \\ 0 & 4 & 0 & -3 & -6 \end{array}\right].$$

The row operation $R_3 - R_2 \to R_3$ gives
$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 0 & 2 \\ 0 & 4 & -3 & 2 & 1 \\ 0 & 0 & 3 & -5 & -7 \end{array}\right].$$

The corresponding system is

$$\left.\begin{array}{rl} x - 2y + \quad z \qquad\quad &= \quad 2 \\ 4y - 3z + 2w &= \quad 1 \\ 3z - 5w &= -7 \end{array}\right\}.$$

We consider the last equation. Let $w = 3t$. (Why do we choose $3t$ and not t? We see that in order to find z we need to divide by 3 and by letting $w = 3t$ we will avoid at least one fraction. You may use $w = t$; this will just lead to extra fractions in the solution.)
Then

$$3z = 5w - 7 = 5 \times 3t - 7$$

i.e.

$$z = 5t - \frac{7}{3}.$$

From the second equation we get

$$\begin{aligned} 4y &= 3z - 2w + 1 \\ &= 3\left(5t - \frac{7}{3}\right) - 6t + 1 \\ &= 15t - 7 - 6t + 1 \\ &= 9t - 6 \end{aligned}$$

i.e.

$$y = \frac{9}{4}t - \frac{3}{2}.$$

From the first equation we obtain

$$\begin{aligned} x &= 2y - z + 2 \\ &= 2\left(\frac{9}{4}t - \frac{3}{2}\right) - \left(5t - \frac{7}{3}\right) + 2 \\ &= \frac{9}{2}t - 3 - 5t + \frac{7}{3} + 2 \\ &= -\frac{t}{2} + \frac{4}{3}. \end{aligned}$$

Thus the solution is

$$\left.\begin{array}{l} x = -\frac{1}{2}t + \frac{4}{3} \\ y = \frac{9}{4}t - \frac{3}{2} \\ z = 5t - \frac{7}{3} \\ w = 3t \end{array}\right\} t \in \mathbb{R}.$$

We check that this answer is correct by substituting it in each of the equations of the original system. (It is a good habit to check, whenever possible, answers to problems.)

$$\begin{aligned} \text{LHS equation (1)} &= -\frac{t}{2} + \frac{4}{3} - \frac{9}{2}t + 3 + 5t - \frac{7}{3} \\ &= -5t + 5t + 3 - 1 \\ &= 2 = \text{ RHS equation (1)} \end{aligned}$$

$$\begin{aligned}\text{LHS equation (2)} &= -t + \frac{8}{3} - 5t + \frac{7}{3} + 6t \\ &= -6t + 6t + \frac{15}{3} \\ &= 5 = \text{ RHS equation (2)}\end{aligned}$$

$$\begin{aligned}\text{LHS equation (3)} &= -\frac{t}{2} + \frac{4}{3} + \frac{9}{2}t - 3 + 5t - \frac{7}{3} - 9t \\ &= 4t + 5t - 9t - 3 - 1 \\ &= -4 = \text{ RHS equation (3)}\end{aligned}$$

■

Note that in the above example we used a vertical line before the last column in the augmented matrix. We use this to separate the coefficients from the constants of the equations.

Definition 1.2.5 System in Echelon form

We say that a **system of linear equations is in echelon form** if the augmented matrix of the system is in generalized row-echelon form.

Now according to the above definition if you are asked “to solve a certain linear system by reducing the system to echelon form” (without a specific method being stipulated), then you can use Gauss-Jordan elimination, Gaussian elimination or the method used in Example 1.2.4.

You may find the following result useful when solving a linear system:

> *Suppose you have a system in **echelon form** which has a solution. If it contains m nonzero equations in n unknowns then you need $n - m$ parameters in the solution.*

Now look at Examples 4, 5, 6 and 7 on pp. 14–17 of Anton. In each example, once the system has been reduced to echelon form, note the number of nonzero equations, the number of unknowns and the number of parameters in the solution.

We now consider examples in which we have to determine when a system has no solution, exactly one solution or infinitely many solutions. The strategy in such problems is first to produce an equivalent system in echelon form since such systems are easier to solve.

Note the following:

If one reduces a nonhomogeneous linear system to echelon form and deletes all the equations of the form

$$0x_1 + 0x_2 + \ldots + 0x_n = 0,$$

then the following cases can occur.

Case 1: The last equation is of the form

$$0x_1 + 0x_2 + \ldots + 0x_n = b, \qquad \text{where } b \neq 0.$$

In this case there is no solution.

Case 2: No equation of the form given in Case 1 occurs. Then if the number of equations is equal to the number of unknowns, then exactly one solution exists. If the number of equations is less than the number of unknowns, then an infinite number of solutions exist.

Example 1.2.6

Consider the system

$$\left.\begin{array}{r} x+2y \qquad\qquad +3z=a \\ 2x+5y+\quad (a+5)\,z=-2+2a \\ -y+\left(a^2-a\right)z=a^2-a \end{array}\right\}. \qquad \mathrm{I}$$

Find (if possible) the values of a for which the system has

(a) no solution,

(b) exactly one solution,

(c) infinitely many solutions.

Solution:

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 2 & 5 & a+5 & -2+2a \\ 0 & -1 & a^2-a & a^2-a \end{array}\right].$$

The row operation $R_2-2R_1 \to R_2$ gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & a-1 & -2 \\ 0 & -1 & a^2-a & a^2-a \end{array}\right].$$

$R_3+R_2 \to R_3$ gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & a-1 & -2 \\ 0 & 0 & a^2-1 & a^2-a-2 \end{array}\right].$$

The corresponding system is

$$\left.\begin{array}{r} x+2y \qquad\quad +3z= \qquad\qquad a \\ y+(a-1)z= \qquad\quad -2 \\ (a^2-1)z=a^2-a-2 \end{array}\right\}. \qquad (1)$$

(a) No solution exists if the last equation is of the form $0z=c$, where $c\neq 0$. Hence no solution exists if

$$a^2-1=0 \quad \text{and} \quad a^2-a-2\neq 0$$

i.e. if

$$(a-1)\,(a+1)=0 \quad \text{and} \quad (a+1)\,(a-2)\neq 0$$

i.e. if

$$a = 1 \quad \text{or} \quad a = -1 \quad \text{and} \quad a \neq -1 \quad \text{and} \quad a \neq 2,$$

i.e. if

$$a = 1.$$

(b) If $a \neq 1$, then the system has at least one solution. Exactly one solution exists if the number of nonzero equations in (1) is equal to the number of unknowns (i.e. the solution contains no parameters). Thus exactly one solution exists if

$$\left(a^2 - 1\right) \neq 0$$

i.e. if

$$a \neq -1 \quad \text{and} \quad a \neq 1$$

i.e. if

$$a \in \mathbb{R} - \{-1, 1\}.$$

(c) If $a \neq 1$, then infinitely many solutions exist if (1) has less nonzero equations than unknowns (i.e. the solution contains at least one parameter),

i.e. if

$$a^2 - 1 = 0 \quad \text{and} \quad a^2 - a - 2 = 0$$

i.e. if

$$(a-1)(a+1) = 0 \quad \text{and} \quad (a-2)(a+1) = 0$$

i.e. if

$$(a = 1 \quad \text{or} \quad a = -1) \quad \text{and} \quad (a = 2 \quad \text{or} \quad a = -1)$$

i.e. if

$$a = -1.$$

■

We now turn our attention to a homogeneous system of linear equations. Recall that in Anton that a homogeneous system of linear equations **always has a solution**, namely the trivial solution. Thus there is no homogeneous system that has no solution.

A homogeneous linear system having ***only*** *the trivial solution is equivalent to the system having exactly one solution.*

A homogeneous linear system having nontrivial solutions is equivalent to the system having infinitely many solutions.

Now work through the following example.

Example 1.2.7

Consider the homogeneous system

$$\left.\begin{aligned} x+2y \qquad\qquad &+3z=0\\ 2x+5y+ \quad (a+5)\,&z=0\\ -y+\left(a^2-a\right)&z=0 \end{aligned}\right\} \qquad \text{II}$$

Find the values of a for which the system has

(a) only the trivial solution,

(b) nontrivial solutions.

Solution:

Firstly note that the coefficients of system II above are the same as those of the nonhomogeneous system I given in Example 1.2.6. System II is called the **associated homogeneous system** of System I.

The augmented matrix of II is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0\\ 2 & 5 & (a+5) & 0\\ 0 & -1 & a^2-a & 0 \end{array}\right].$$

By performing the same row operations that we used in Example 1.2.6 we obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0\\ 0 & 1 & a-1 & 0\\ 0 & 0 & a^2-1 & 0 \end{array}\right].$$

(Check this yourself.)

The corresponding system is

$$\left.\begin{aligned} x+2y \qquad\qquad &+3z=0\\ y+\quad (a-1)\,&z=0\\ \left(a^2-1\right)&z=0 \end{aligned}\right\}. \qquad (2)$$

(a) Only the trivial solution (i.e. only one solution) exists if the number of nonzero equations in (2) is equal to the number of unknowns. Thus **only** the trivial solution exists if

$$a^2-1\neq 0$$

i.e. if

$$a\neq -1 \quad \text{and} \quad a\neq 1$$

i.e. if

$$a\in\mathbb{R}-\{-1,1\}.$$

(b) Nontrivial solutions (i.e. infinitely many solutions) exist if the number of nonzero equations in (2) is less than the number of unknowns. Thus nontrivial solutions exist if

$$a^2 - 1 = 0$$

i.e. if

$$a = -1 \quad \text{or} \quad a = 1$$

i.e. if

$$a \in \{-1, 1\}.$$

■

In the following example we illustrate Warning 2 which is given just before Example 1.2.4.

Example 1.2.8

Consider the following homogeneous system

$$\left.\begin{aligned} (\lambda + 1)\, x + y &= 0 \\ 3x + (\lambda - 1)\, y &= 0 \end{aligned}\right\}. \qquad (3)$$

(λ is a letter of the Greek alphabet, called lambda. A list of the Greek alphabet is given in the appendix at the end of this study guide.)
If we substitute $\lambda = -1$ in (3) we obtain

$$\left.\begin{aligned} 0x + y &= 0 \\ 3x - 2y &= 0 \end{aligned}\right\}.$$

From the first equation we obtain $y = 0$ and hence from the second $x = 0$. Thus (3) has only the trivial solution for $\lambda = -1$.

The augmented matrix of (3) is

$$\left[\begin{array}{cc|c} \lambda + 1 & 1 & 0 \\ 3 & \lambda - 1 & 0 \end{array}\right].$$

If we perform $(\lambda + 1)\, R_2 - 3R_1 \to R_2$ we obtain

$$\left[\begin{array}{cc|c} \lambda + 1 & 1 & 0 \\ 0 & \lambda^2 - 4 & 0 \end{array}\right].$$

The corresponding system is

$$\left.\begin{aligned} (\lambda + 1)\, x + y &= 0 \\ \left(\lambda^2 - 4\right) y &= 0 \end{aligned}\right\}. \qquad (4)$$

If $\lambda = -1$, then (4) becomes

$$\left.\begin{aligned} 0x + y &= 0 \\ -3y &= 0 \end{aligned}\right\}$$

which has solutions $y = 0$ and $x = t$, where $t \in \mathbb{R}$. Thus (4) has the trivial as well as nontrivial solutions for $\lambda = -1$.

What has gone wrong? Why are systems (3) and (4) **not** equivalent?
The answer lies in the row operation that we used, namely

$$(\lambda + 1)\, R_2 - 3R_1 \to R_2.$$

This operation consists of

$$(\lambda + 1)\, R_2 \to R_2$$

followed by

$$R_2 - 3R_1 \to R_2.$$

The first of these operations is an elementary row operation only if $\lambda + 1 \neq 0$, i.e. only if $\lambda \neq -1$.

Thus systems (3) and (4) are equivalent for all values of λ not equal to -1. Thus (3) and (4) may have different solution sets for $\lambda = -1$.

In order to produce a system in echelon form which is equivalent to (3) for all values of λ we can proceed as follows:
The augmented matrix of (3) is

$$\left[\begin{array}{cc|c} \lambda+1 & 1 & 0 \\ 3 & \lambda-1 & 0 \end{array}\right].$$

The row operation $R_1 \leftrightarrow R_2$ gives

$$\left[\begin{array}{cc|c} 3 & \lambda-1 & 0 \\ \lambda+1 & 1 & 0 \end{array}\right].$$

The row operation $3R_2 \to R_2$ gives

$$\left[\begin{array}{cc|c} 3 & \lambda-1 & 0 \\ 3\,(\lambda+1) & 3 & 0 \end{array}\right]$$

and finally $R_2 - (\lambda + 1)\, R_1 \to R_2$ (WHICH IS AN ELEMENTARY ROW OPERATION FOR ALL VALUES OF λ – see 3 under "Elementary row operation notation") produces

$$\left[\begin{array}{cc|c} 3 & \lambda-1 & 0 \\ 0 & 4-\lambda^2 & 0 \end{array}\right].$$

The corresponding system is

$$\left.\begin{array}{r} 3x + (\lambda - 1)\, y = 0 \\ \left(4 - \lambda^2\right) y = 0 \end{array}\right\} \qquad (5)$$

which is a system in echelon form which is equivalent to system (3) for **all** real values of λ. From (5) we can deduce, for example, that (3) has only the trivial solution for $\lambda \neq -2$ and $\lambda \neq 2$, and nontrivial solutions if $\lambda = -2$ or $\lambda = 2$. ■

List of important concepts

reduced row-echelon form of a matrix
row-echelon form of a matrix
generalized row-echelon form of a matrix
a ***linear system in echelon form***
Gauss-Jordan elimination

Gaussian elimination

a ***homogeneous system*** of linear equations

associated homogeneous system

the trivial solution

a nontrivial solution

Summary of Section

Now write your own summary.

1.3 Matrices and Matrix Operations

Overview

Rectangular arrays of real numbers occur in many contexts other than as augmented matrices for systems of linear equations. We now begin the study of matrices as objects in their own right. In this section we give some fundamental definitions and see how matrices can be combined using the arithmetic operations of addition, subtraction and multiplication.

Source: Anton §1

Learning Outcomes

After studying this section you should be able to

- explain what is meant by a matrix and the entries of a matrix
- determine the size of a matrix
- determine if/when two matrices are equal
- find the sum and difference of two matrices
- recognise if matrix addition and subtraction are defined or undefined
- multiply a matrix by a scalar
- multiply two matrices
- recognise if the product of two matrices is defined or undefined
- write the product of certain types of matrices as a linear combination of row or column matrices
- determine the transpose of a matrix
- find the trace of a matrix.

Additional notes

Study Anton and make quite sure that you are able to multiply two matrices. Initially, many students find matrix multiplication difficut, but they quickly master the method after doing a few examples. Remember, practice makes perfect.

Matrix multiplication has an important application to systems of linear equations in that a system of m equations in n unknowns can be represented by a single matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

The following example illustrates a simple everyday application of matrices.

Example 1.3.1

Suppose the array

$$\begin{bmatrix} 4 & 3 & 3 \\ 2 & 1 & 0 \\ 4 & 4 & 2 \end{bmatrix}$$

represents the orders placed by three students at a take-away restaurant. The first student orders 4 hamburgers, 3 cooldrinks and 3 packets of chips; the second orders 2 hamburgers and 1 cooldrink and the third orders 4 hamburgers 4 cooldrinks and 2 packets of chips. Hamburgers cost R15 each, cooldrinks R6 each and a packet of chips costs R10.

(a) Set up a column matrix that represents the cost of each item.

(b) Perform a suitable matrix multiplication to determine the amount owed by each student.

Solution:

(a) Cost per item matrix is $\begin{bmatrix} 15 \\ 6 \\ 10 \end{bmatrix}$

(b) $$\begin{bmatrix} 4 & 3 & 3 \\ 2 & 1 & 0 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} 15 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \times 15 + 3 \times 6 + 3 \times 10 \\ 2 \times 15 + 1 \times 6 \\ 4 \times 15 + 4 \times 6 + 2 \times 10 \end{bmatrix} = \begin{bmatrix} 108 \\ 36 \\ 104 \end{bmatrix}.$$

Thus the first student owes R108, the second R36 and the third R104. ■

List of important concepts

a ***matrix***
the ***entries*** of a matrix
the ***size*** of a matrix
a ***column matrix*** (column vector)
a ***row matrix*** (row vector)
a ***scalar***
a ***square matrix*** of order n
the entries on the ***main diagonal*** of a square matrix
equality of matrices
the ***sum*** of two matrices
the ***difference*** of two matrices
scalar product of a matrix
the ***product*** of two matrices
a ***linear combination of matrices***
the ***coefficient matrix*** of a system of linear equations
the ***transpose*** of a matrix
the ***trace*** of a matrix

Summary of Section

1.4 Inverses; Rules of Matrix Arithmetic

Overview

In this section we discuss some properties of the arithmetic operations on matrices. We see that most of the basic rules of arithmetic for real numbers also hold for matrices, but that there are some exceptions when it comes to matrix multiplication. We define the inverse of a matrix and give some properties of inverses and transposes of matrices. We give the formula for the inverse of a 2×2 invertible matrix.

Source: Anton §1

Learning Outcomes

After studying this section you should be able to

- apply the various properties of matrix arithmetic
- state which basic rules of arithmetic for real numbers do not hold for matrices
- explain what is meant by an invertible (nonsingular) matrix
- find the inverse of a 2×2 invertible matrix
- apply the properties of inverse matrices and laws of matrix exponents
- apply the properties of the transpose of a matrix.

Additional notes

Take careful note of the basic rules of arithmetic for real numbers that **do not apply** to matrices, namely the following:

(1) The commutative law for multiplication does not hold, i.e. AB need not be equal to BA.

(2) The cancellation law need not hold, i.e.

$$AB = AC \text{ need not imply that } B = C.$$

When do you think that the cancellation law will hold?

(3) The product of two matrices can be zero without either of the matrices being zero, i.e.

$$AB = 0 \text{ need not imply that } A = 0 \quad \text{or} \quad B = 0.$$

Under what conditions will $AB = 0$ imply $A = 0$ or $B = 0$?

Note that we also use the term **nonsingular matrix** for **invertible matrix**.

List of important concepts

zero matrix

identity matrix

an ***invertible (nonsingular)*** matrix

a ***singular*** matrix

the inverse of a matrix

nonnegative integer powers of a matrix

negative integer powers of a matrix

Summary of Section

1.5 Elementary Matrices and a Method for Finding the Inverse of A

Overview

In this section we develop an algorithm for finding the inverse of an invertible matrix. We also discuss some of the basic properties of invertible matrices.

Source: Anton §1

Learning Outcomes

After studying this section you should be able to

- explain what is meant by an elementary matrix
- find for any invertible matrix A, a sequence $E_1, E_2, \dots E_k$ of elementary matrices such that

$$E_k \dots E_2 E_1 A = I_n$$

- determine the inverse A^{-1} of any invertible matrix A by using the method (which we call the **matrix inverse algorithm**) used in Example 4 on of Anton.

Additional notes

Suppose A is an invertible matrix and $E_1, E_2, \dots E_k$ is a sequence of elementary matrices such that

$$E_k \dots E_2 E_1 A = I_n.$$

Then note that

$$A = E_1^{-1} E_2^{-1} \dots E_k^{-1}$$

and

$$A^{-1} = E_k \dots E_2 E_1.$$

We use these results in the following example.

Example 1.5.1

Suppose A is a 2×2 matrix and E_1, E_2 and E_3 are elementary matrices such that

$$E_3 E_2 E_1 A = I_2,$$

where

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

(a) Write down E_1^{-1}, E_2^{-1} and E_3^{-1}.

(b) Write A as a product of elementary matrices and then determine A.

(c) Write A^{-1} as a product of elementary matrices and then determine A^{-1}.

(d) Show that $AA^{-1} = A^{-1}A = I_2$.

Solution:

(a) By applying Theorem 1.5.2 and Table 1 on p. 53 of Anton we have:

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (E_1 \text{ is obtained by performing } R_1 \leftrightarrow R_2 \text{ on } I_2.)$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad (E_2 \text{ is obtained by performing } \tfrac{1}{3}R_2 \to R_2 \text{ on } I_2.)$$

$$E_3^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad (E_3 \text{ is obtained by performing } R_1 - 4R_2 \to R_1 \text{ on } I_2.)$$

(b)

$$\begin{aligned}
& E_3E_2E_1A = I_2 \\
\Rightarrow\ & E_3^{-1}E_3E_2E_1A = E_3^{-1}I_2 \\
\Rightarrow\ & E_2E_1A = E_3^{-1} \\
\Rightarrow\ & E_2^{-1}E_2E_1A = E_2^{-1}E_3^{-1} \\
\Rightarrow\ & E_1A = E_2^{-1}E_3^{-1} \\
\Rightarrow\ & E_1^{-1}E_1A = E_1^{-1}E_2^{-1}E_3^{-1} \\
\Rightarrow\ & A = E_1^{-1}E_2^{-1}E_3^{-1} \\
\Rightarrow\ & A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \\
\Rightarrow\ & A = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \\
\Rightarrow\ & A = \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix}.
\end{aligned}$$

(c)

$$\begin{aligned}
& E_3E_2E_1A = I_2 \\
\Rightarrow\ & A^{-1} = E_3E_2E_1 \qquad (\text{Note: } BA = I_2 \Rightarrow A^{-1} = B) \\
\Rightarrow\ & A^{-1} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\Rightarrow\ & A^{-1} = \begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\Rightarrow\ & A^{-1} = \begin{bmatrix} -\frac{4}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix}.
\end{aligned}$$

(d)

$$\begin{aligned}
AA^{-1} &= \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -\frac{4}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times \left(-\frac{4}{3}\right) + 3\left(\frac{1}{3}\right) & 0 \times 1 + 3 \times 0 \\ 1 \times \left(-\frac{4}{3}\right) + 4\left(\frac{1}{3}\right) & 1 \times 1 + 4 \times 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \\
A^{-1}A &= \begin{bmatrix} -\frac{4}{3} & 1 \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 4 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{4}{3} \times 0 + 1 \times 1 & -\frac{4}{3}(3) + 1 \times 4 \\ \frac{1}{3} \times 0 + 0 \times 1 & \frac{1}{3}(3) + 0 \times 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

■

Carefully study the equivalent statements concerning invertible matrices given in Theorem 1.5.3.

Note that the concept of an elementary matrix together with Theorem 1.5.3 gives us a method for determining the inverse of an invertible matrix. We call this method, the **matrix inverse algorithm**.

List of important concepts

elementary matrix
inverse elementary row operations
row equivalent matrices

Summary of Section

1.6 Further Results on Systems of Equations and Invertibility

Overview

In this section we establish further results about systems of linear equations and invertibility of matrices. We also give another method for solving n linear equations in n unknowns.

Source: Anton §1

Learning Outcomes

After studying this section you should be able to

- solve certain linear systems by using the inverse of a matrix
- solve more than one linear system at once, if the coefficients of the systems are the same
- determine the consistency of a linear system by elimination
- apply different equivalent statements for the fact that a matrix is invertible, in various problems

Additional notes

In this section a method is given for solving a system of linear equations $A\mathbf{x} = \mathbf{b}$, where the coefficient matrix A is invertible. The solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$.

Once again carefully study the equivalent statements concerning invertible matrices given in Theorem 1.6.4.

Summary of Section

1.7 Diagonal, Triangular, and Symmetric Matrices

Overview

In this section we define certain classes of square matrices that have special forms. These are amongst some of the most important matrices encountered in linear algebra.

Source: Anton §1

Learning Outcomes

After studying this section you should be able to

- recognise a diagonal matrix, a triangular matrix and a symmetric matrix
- write down the inverse of an invertible diagonal matrix.

Additional notes

You need not know Theorems 1.7.1–1.7.4, except the result of Theorem 1.7.1(c) which will be proved in the next chapter.

List of important concepts

a ***diagonal*** matrix
a ***lower triangular*** matrix
an ***upper triangular*** matrix
a ***triangular*** matrix
a ***symmetric*** matrix

Activities

Anton, Chapter 1, Exercise Set 75, problems $1 - 220$.

Summary of Section

Review of Chapter 1

We now review the main points of this chapter.

We started with systems of linear equations and saw how we could represent them in terms of matrices. We obtained equivalent systems in echelon form by performing elementary row operations on the matrices. These equivalent systems were easier to solve than the original ones.

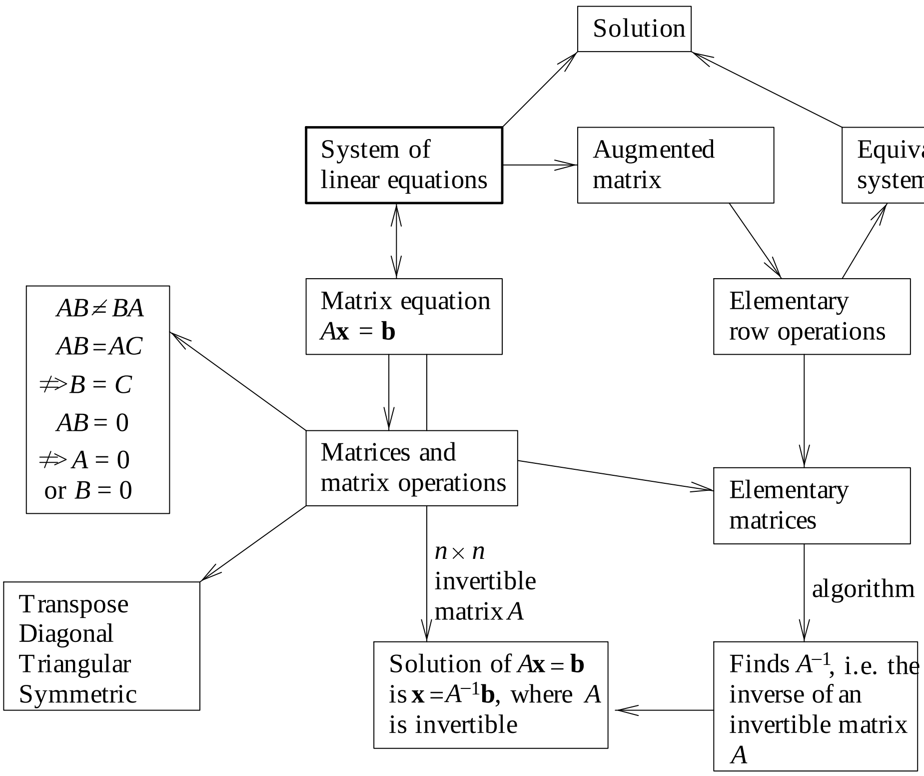
We then considered matrices as objects in their own right and saw how they could be combined by using various arithmetic operations. We took careful note of the basic rules of arithmetic of real numbers that do not hold for matrices.

Elementary matrices were defined in terms of elementary row operations. These special matrices were used to develop a method (the matrix inverse algorithm) for obtaining the inverse of an invertible matrix.

We then considered special types of linear systems, namely systems of the form $A\mathbf{x} = \mathbf{b}$, where A is invertible. In such cases, the inverse of A, i.e. A^{-1} can be used to obtain the solution since $\mathbf{x} = A^{-1}\mathbf{b}$.

Also, in this chapter, important equivalent statements for a matrix A to be invertible are given. Finally the chapter ends by introducing certain classes of matrices which have special forms.

The following is a diagrammatic summary of the main points.



Chapter 1

Equivalent Statements

If A is an $n \times n$ matrix, then the following are equivalent:

(a) *A is invertible.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced row–echelon form of A is I_n.*

(d) *A is expressible as a product of elementary matrices.*

(e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*

(f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*

CHAPTER 2

DETERMINANTS

Introduction

You have already encountered real valued functions $f(x)$, for example $f(x) = 2x - 3$ and $f(x) = \sin x$, which associate a real number $f(x)$ with a real value of the variable x. In this chapter we introduce and study the "determinant function" $\det(X)$ which associates a *real number* $\det(X)$ (which we call the determinant of X) with a *square matrix* X. This function plays an important role in linear algebra. For example, the determinant of a square matrix provides an efficient method to determine if the matrix concerned is invertible or singular. Determinants can also be used to find the inverse of an invertible matrix, and be used to solve systems of the form $A\mathbf{x} = \mathbf{b}$, where A is invertible.
In the next chapter you will also see that a determinant can be used to find the area of a parallelogram in 2-space, the volume of a parallelepiped in 3-space and a vector perpendicular to a plane in 3-space.

The material in this chapter is divided into the following sections:

2.1 Determinants by Cofactor Expansion

Overview

A determinant is a function that associates a real number with a square matrix. In this section we define this function in terms of cofactor expansion along the first row of a square matrix. We also obtain a formula for the inverse of an invertible matrix, as well as a formula (known as Cramer's rule) for the solution to certain systems of linear equations in terms of determinants.

Source: Anton §2

Learning Outcomes

After studying this section you should be able to

- find the minors and cofactors of a square matrix
- evaluate the determinant of a matrix by cofactor expansion along any row or column of the matrix
- determine the inverse of an invertible matrix by using its adjoint
- solve certain linear systems by using Cramer's rule.

Additional notes

The section in Anton is self-contained. Work through the examples carefully and make sure that you follow each of the steps involved. Remember to summarize the main results in the Summary at the end of this section and to practise the various techiques. Note the following points:

- The determinant of a square matrix A is defined in terms of cofactor expansion along the *first row* of A. However, according to Theorem 2.1.1 you can compute the determinant by using cofactor expansion along *any row or column* of the matrix.
- Although the result

  "A square matrix A is invertible if and only if $\det(A) \neq 0$"

  is proved only in Section 2.3, you may use it in this section. In other words, you may deduce that A^{-1} exists by showing that $\det(A) \neq 0$, rather than using the more tedious methods which were used in Chapter 1.

List of important concepts

the ***minor*** of entry a_{ij}
the ***cofactor*** of entry a_{ij}
cofactor expansion along a row or column

Summary of Section

2.2 Evaluating Determinants by Row Reduction

Overview

In this section we consider the effect that each type of elementary row operation has on the value of the determinant of the resulting matrix in terms of the determinant of the original matrix. We also show that the determinant of a square matrix can be evaluated by reducing the matrix to row-echelon form (or generalized row-echelon form). This method is important since it is the most computationally efficient way to evaluate the determinant of a general matrix.

Source: Anton §2

Learning Outcomes

After studying this section you should be able to

- state the effect that each elementary row operation has on the value of the determinant of the resulting matrix
- evaluate, by inspection, the determinants of elementary matrices
- evaluate the determinant of a matrix by using elementary row operations to reduce the given matrix to an upper triangular matrix
- evaluate the determinant of a matrix by using elementary column operations to reduce the given matrix to a lower triangular matrix
- evaluate a determinant by using a combination of row or column operations and cofactor expansion
- evaluate a determinant in terms of a related determinant.

Additional notes

The section in Anton is self-contained and contains a number of examples for you to work through. We include an extra example here to illustrate the last outcome stated above.

Remember to summarize the main points.

Example 2.2.1

Suppose

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3.$$

Find the value of

$$\begin{vmatrix} d+4g & e+4h & f+4i \\ a & b & c \\ 2g & 2h & 2i \end{vmatrix}.$$

Solution:

$$
\begin{aligned}
&\begin{vmatrix} d+4g & e+4h & f+4i \\ a & b & c \\ 2g & 2h & 2i \end{vmatrix} \\
= \;& 2\begin{vmatrix} d+4g & e+4h & f+4i \\ a & b & c \\ g & h & i \end{vmatrix} \quad \text{(Theorem 2.2.3(a))} \\
= \;& -2\begin{vmatrix} a & b & c \\ d+4g & e+4h & f+4i \\ g & h & i \end{vmatrix} \quad \text{(Theorem 2.2.3(b))} \\
= \;& -2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \text{(Theorem 2.2.3(c))} \\
= \;& -2 \times -3 \\
= \;& 6.
\end{aligned}
$$

■

Summary of Section

Summary of Section (cont.)

2.3 Properties of the Determinant Function

Overview

In this section we develop some of the fundamental properties of the determinant function. We investigate the relationship between a square matrix and its determinant and give the determinant test for the invertibility of a matrix. We introduce the concept of eigenvalue and eigenvector.

Source: Anton §2

Learning Outcomes

After studying this section you should be able to

- evaluate a determinant in terms of a related determinant (or related determinants)
- determine if a square matrix is invertible by evaluating its determinant
- find the eigenvalues and corresponding eigenvectors of a matrix
- know and apply different statements equivalent for the fact that a matrix is invertible, in various problems.

Additional notes

The material should be studied directly from Anton. Pay particular attention to formula (1) given, namely

$$\det(kA) = k^n \det(A),$$

where A is an $n \times n$ matrix and k a scalar.

Example 2.3.1

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \det(A) = 5.$$

Then

$$\det(3A) = 3^2 \det(A) = 9 \times 5 = 45$$

because

$$\begin{aligned} \det(3A) &= \begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix} \\ &= 3 \begin{vmatrix} a & b \\ 3c & 3d \end{vmatrix} \\ &= 3 \times 3 \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{(by repeated application of Theorem 2.2.3(a))} \\ &= 3^2 \det(A). \end{aligned}$$

Note that $\det(3A) \neq 3\det(A)$. ■

Now study the following example in which a number of properties of determinants that are given in Sections 2.2 and 2.3 are applied.

Example 2.3.2

Suppose A, B and C are 2×2 matrices such that

$$\det(A) = 3, \quad \det(B^{-1}) = -2 \quad \text{and} \quad \det(C^T) = 4.$$

Evaluate

(a) $\det(ABC)$

(b) $\det\left(\left[C^2\right]^T\right)$

(c) $\det(-4A)$

(d) $\det\left(-4A^{-1}\right)$

(e) $\det\left([-4A]^{-1}\right)$.

Solution:
Since $\det(B^{-1}) = -2$, it follows from Theorem 2.3.5 of Anton that

$$\det(B) = \frac{1}{\det(B^{-1})} = -\frac{1}{2}.$$

Also, since $\det(C^T) = 4$, it follows from Theorem 2.2.2 of Anton that

$$\det(C) = \det(C^T) = 4.$$

(a) According to Theorem 2.3.4 of Anton

$$\begin{aligned} \det(ABC) &= \det(A)\det(B)\det(C) \\ &= 3 \times -\frac{1}{2} \times 4 \\ &= -6. \end{aligned}$$

(b)

$$\begin{aligned} \det\left(\left[C^2\right]^T\right) &= \det\left(C^2\right) && \text{(Theorem 2.2.2)} \\ &= \det(C) \times \det(C) && \text{(Theorem 2.3.4)} \\ &= 16. \end{aligned}$$

(c) According to (1) of Anton

$$\begin{aligned}\det(-4A) &= (-4)^2 \det(A)\\ &= 16 \times 3\\ &= 48.\end{aligned}$$

(d)

$$\begin{aligned}\det\left(-4A^{-1}\right) &= (-4)^2 \det(A^{-1}) && (1),\\ &= 16\frac{1}{\det(A)} && \text{(Theorem 2.3.5)}\\ &= \frac{16}{3}.\end{aligned}$$

(e)

$$\begin{aligned}\det\left([-4A]^{-1}\right) &= \frac{1}{\det(-4A)} && \text{(Theorem 2.3.5)}\\ &= \frac{1}{(-4)^2\det(A)} && \text{((1), p. 103 of Anton)}\\ &= \frac{1}{16\times 3}\\ &= \frac{1}{48}.\end{aligned}$$

■

The last important point to note about this section is the inclusion of an equivalent statement, in terms of $\det(A)$ in the list of equivalent statements that we have so far, concerning the invertibility of a matrix A. This statement says "A is invertible if and only if $\det(A) \neq 0$". (See Theorem 2.3.6 of Anton.)

List of important concepts

eigenvalue (characteristic value) of a matrix
eigenvectors of a matrix corresponding to an eigenvalue
characteristic equation of a matrix

Summary of Section

Review of Chapter 2

We started off by defining the determinant of a square matrix in terms of cofactor expansion along the first row. We saw that we can evaluate a determinant by cofactor expansion along *any row* or *column*. It thus makes sense to expand along a row or column that has the largest number of zeros.

Determinants were then used in formulas to find the inverse of an invertible matrix A, and to solve systems of equations of the form $A\mathbf{x} = \mathbf{b}$, where A is invertible.

We then looked at the effect that each type of elementary row operation has on the value of the determinant of the resulting matrix in terms of the determinant of the original matrix. Since $\det(A) = \det\left(A^T\right)$, it follows that an elementary column operation performed on a square matrix A has the same effect on the determinant value as when the corresponding elementary row operation is performed on A. For example, suppose the second column of a square matrix A is multiplied by a scalar k to produce the matrix B, and that the second row of A is mulitplied by the same scalar k to produce the matrix C. Then

$$\det(B) = k\det(A) = \det(C)$$

but B need not be the same as C. Thus an effective method for evaluating determinants is provided by combining row and column operations together with cofactor expansion.

In Section 2.3 a number of properties of determinants were proved. Elementary matrices, which were introduced in the first chapter, were used to prove the following two important theorems:

- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B).$$

A combination of these two theorems leads to the following important result:

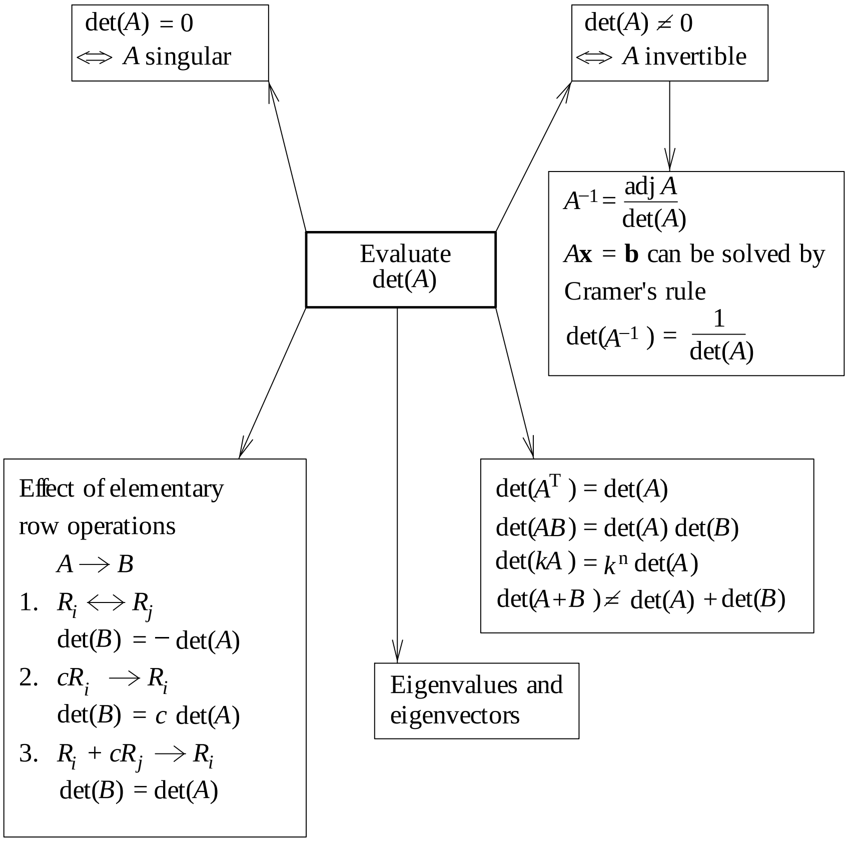
- If A is invertible, then

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.$$

Near the end of the first chapter five results were listed that are equivalent to the invertibility of a matrix A. We can now add the result that "A is invertible if and only if $\det(A) \neq 0$" to this list.

Finally eigenvalues and the corresponding eigenvectors of a square matrix A were introduced.

A diagrammatic summary of the main points is given on the following page.



Chapter 2

Equivalent Statements

If A is an $n \times n$ matrix, then the following are equivalent:

(a) A *is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row-echelon form of* A *is* I_n.

(d) A *can be expressed as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

CHAPTER 3

VECTORS IN 2-SPACE AND 3-SPACE

Introduction

Many physical notions are completely described by magnitude, for example, the notions of mass (the mass of the man is 70 kg), distance (the towns are 120 km apart), and temperature (the temperature is 30°C). However, there are some physical notions that require both magnitude and direction to be fully understood. To describe these notions we introduce the concept of **vectors**. For example, velocity is a vector since it is described by both speed (magnitude) and direction. Wind movement too has the attributes of speed and direction, for example, a wind of 30 kph south-east. Vectors can also be used to model many other physical notions, such as force, displacement, acceleration, electric and magnetic fields, momentum and angular momentum.

In this chapter we introduce vectors in 2-space and 3-space. We define certain operations on vectors and establish various properties of these operations. Amongst the concepts defined are the norm (length) of a vector, the distance between two points, and the dot product and cross product of vectors (special types of vector multiplication). Vectors are then used to derive the equations for lines and planes in 3-space.

The material in this chapter is divided into the following sections:

3.1 Introduction to Vectors

Overview

In this section we introduce vectors in 2-space and 3-space geometrically. We define arithmetic operations on vectors and give some basic properties of these operations.

Source: Anton §3

Learning Outcomes

After studying this section you should be able to

- plot points and sketch vectors in 2-space and 3-space
- find the components of a vector given the initial point and terminal point of the vector
- find the initial or terminal point of a vector given certain information about the vector
- perform various arithmetic operations on vectors.

Additional notes

Notation

2-space and 3-space are often denoted by $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively,

Points and Vectors

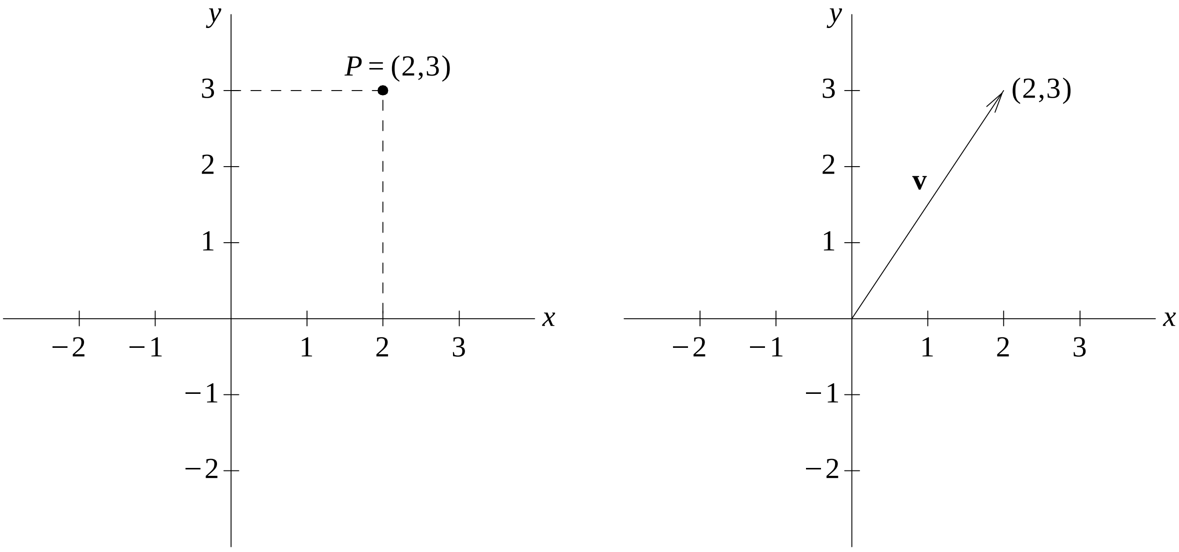
We discuss the plotting of points and sketching of vectors in 2-space and 3-space by means of examples.

Example 3.1.1

(a) Plot the point $P = (2, 3)$.

(b) Sketch the vector $\mathbf{v} = (2, 3)$ with initial point at the origin.

Solution:



(a) The point $P = (2, 3)$ (b) The vector $\mathbf{v} = (2, 3)$

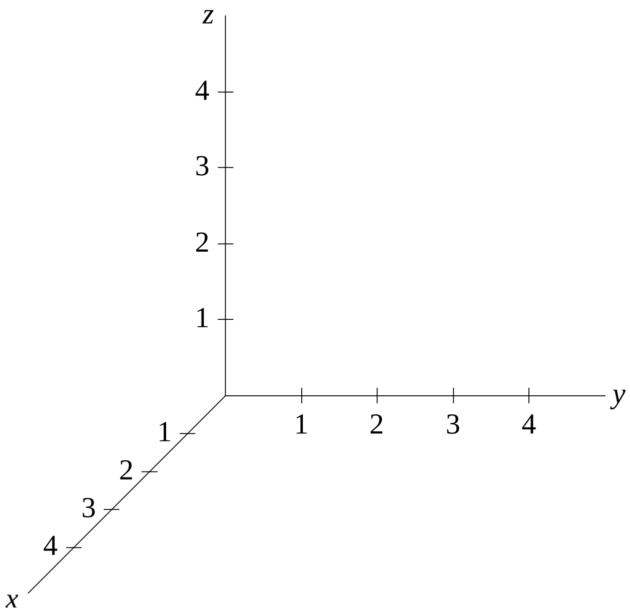
Note that when we plot $(2, 3)$ we first move two units from the origin along the positive x-axis to the point $(2, 0)$. From there we move three units upwards, parallel to the y-axis which brings us to the point $(2, 3)$.
■

Example 3.1.2

(a) Plot the point $P = (2, 3, 4)$.

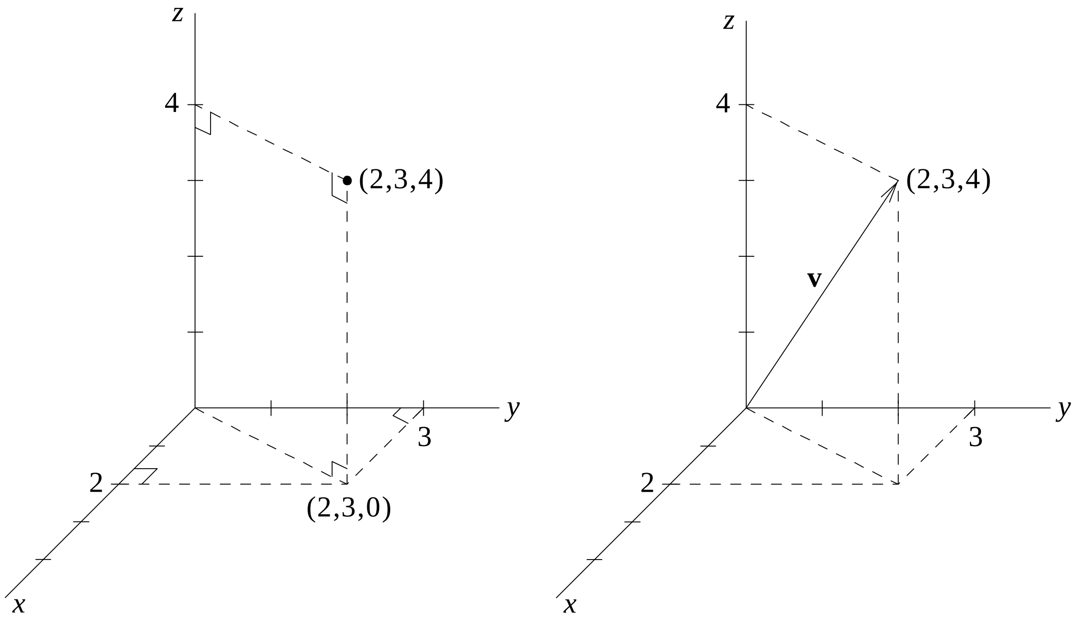
(b) Sketch the vector $\mathbf{v} = (2, 3, 4)$ with initial point at the origin.

Solution:
Consider the (right-handed) rectangular coordinate system sketched below.



Rectangular coordinate system in 3-space

Note that since the x-axis points towards the viewer, units on the x-axis **appear shorter** than on the y- and z-axes. This phenomenon is called **foreshortening**.
To plot the point $(2,3,4)$ we first move two units from the origin along the positive x-axis to the point $(2,0,0)$. From there we move three units to the right, parallel to the y-axis to the point $(2,3,0)$. From there we move four units upwards, parallel to the z-axis, which finally brings us to the point $(2,3,4)$. Note that the point $(2,3,4)$ lies vertically above the point $(2,3,0)$, at a height of four units above the xy-plane.



(a) The point $P = (2,3,4)$ (b) The vector $\mathbf{v} = (2,3,4)$

■

The coordinate planes in 3-space

The following table gives the equation for each of the coordinate planes.

Coordinate plane	Equation of plane
xy-plane	$z = 0$
xz-plane	$y = 0$
yz-plane	$x = 0$

Translation of Axes

You need only read this section of Anton for interest's sake.

List of important concepts

a ***vector*** in 2-space or 3-space
initial point of a vector
terminal point of a vector
a ***scalar***
components of a vector
coordinates of a point
equivalent vectors (***equal*** vectors)

sum and difference of two vectors
zero vector
negative of a vector
scalar multiple of a vector

Summary of Section

3.2 Norm of a Vector; Vector Arithmetic

Overview

In this section we establish the basic rules of vector arithmetic and introduce the concept of the norm of a vector.

Source: Anton §3

Learning Outcomes

After studying this section you should be able to

- apply the properties of vector arithmetic in 2-space and 3-space
- find the norm of a vector in 2-space and 3-space
- find the distance between two points in 2-space and 3-space.

Additional notes

The material in Anton is self-contained. Study this section directly from the textbook.

List of important concepts

norm of a vector
a ***unit vector***
distance between two points

Summary of Section

3.3 Dot Product; Projections

Overview

In this section we introduce the concept of the dot product (an important type of product) of two vectors in 2-space and 3-space. We also discuss some geometric implications of this concept.

Source: Anton §3

Learning Outcomes

After studying this section you should be able to

- determine the dot product of two vectors in 2-space or 3-space
- use the dot product to find the angle (or cosine of the angle) between two vectors in 2-space or 3-space
- determine if two vectors in 2-space or 3-space are orthogonal (perpendicular)
- find the orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$ (the vector component of $\mathbf{u}$ along $\mathbf{a}$) and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$
- find the distance between a point and a line in 2-space.

Additional notes

Note that the **dot product** of two vectors is a **real number**.

Formula for dot product

In this section the dot product of two vectors is defined in two different ways.
If you are given the **components** of two vectors, then use formula in Anton to determine the dot product.
If you are given the **lengths** of two vectors and the **angle** between the vectors, then use formula in Anton to determine the dot product.

Formula for the distance between a point and a line in 2-space

You must know and be able to apply the formula for the distance between a point and a line. You need not know how to derive the formula. However, work through the derivation of the formula and note how the concept of the orthogonal projection of a vector on another vector is used in the derivation.

List of important concepts

the ***dot product*** (***Euclidean inner product***) of two vectors
orthogonal vectors
the ***orthogonal projection of*** $\mathbf{u}$ ***on*** $\mathbf{a}$ (***vector component of*** $\mathbf{u}$ ***along*** $\mathbf{a}$)
the ***vector component of*** $\mathbf{u}$ ***orthogonal to*** $\mathbf{a}$.

Activities

Anton, Exercise 3

Summary of Section

3.4 Cross Product

Overview

In many applications of vectors to problems in geometry, physics and engineering, we need to construct a vector in 3-space that is perpendicular to two given vectors. In this section we show how to do this by introducing the concept of the cross product (another special type of product) of two vectors.

Source: Anton §3

Learning Outcomes

After studying this section you should be able to

- determine the cross product of two vectors in 3-space
- use the cross product to find the angle (or sine of the angle) between two vectors in 3-space
- use the cross product to find the area of a parallelogram in 3-space
- use determinants to find the area of a parallelogram in 2-space and the volume of a parallelepiped in 3-space.

Additional notes

First note that the cross product is only defined **for vectors in 3-space** and that the **cross product** of two vectors is again a **vector**. Compare this with the definition of the dot product which was given in the previous section. Possibly, the easiest way of determining the cross product is to use the equation

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

and we consider “expanding the ”determinant” by the first row”. Note that the above is not really a determinant as the entries in the first row are vectors.

We now add a few steps to the example that is done just below equation (4).
If $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$, then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \\ & \quad \text{“expand by the first row”} \\ &= 2\mathbf{i} - (1+6)\,\mathbf{j} - 6\mathbf{k} \\ &= 2\,(1,0,0) - 7\,(0,1,0) - 6\,(0,0,1) \\ &= (2, -7, -6)\,. \end{aligned}$$

One of the most important facts to note is that the cross product $\mathbf{u} \times \mathbf{v}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector which is **perpendicular** to both $\mathbf{u}$ and $\mathbf{v}$. We shall use this result in the next section.

List of important concepts

the ***cross product*** of two vectors in 3-space

the ***standard unit vectors*** in 3-space

Summary of Section

Summary of Section

3.5 Lines and Planes in 3-Space

Overview

In this section we use vectors to derive equations of lines and planes in 3-space. We then use these equations to solve some basic geometric problems.

Source: Anton §3

Learning Outcomes

After studying this section you should be able to

- find the equations of lines and planes in 3-space
- find the distance between a point and a plane or between two parallel planes in 3-space
- solve various geometric problems involving lines and planes in 3-space.

Additional notes

Equations of planes in 3-space

The various types of equations that define a plane are summarized below.

Equations of planes in 3-space	
Point-normal:	$a\,(x-x_0)+b\,(y-y_0)+c\,(z-z_0)=0$
General form:	$ax+by+cz+d=0$
Vector form:	$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0$, i.e. $(a,b,c)\cdot(x-x_0,\ y-y_0,\ z-z_0)=0$

We note that in the equations above, (a,b,c) is a vector **perpendicular** to the plane and (x_0,y_0,z_0) is a **specific point** on the plane.

> *Thus when we determine the equation of a* ***plane*** *we need to find, from the information provided, a* ***point*** *on the plane and a* ***vector perpendicular*** *to the plane.*

Equations of lines in 3-space

Two types of equations that define a line are given below.

Equations of lines in 3-space		
Parametric equations:	$x = x_0 + at$	
	$y = y_0 + bt$	$-\infty < t < \infty$
	$z = z_0 + ct$	
Vector form:	$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t,$	$-\infty < t < \infty$
i.e.	$(x, y, z) = (x_0, y_0, z_0) + (a, b, c)\, t$	

We note that in the equations above, (a, b, c) is a vector **parallel** to the line and (x_0, y_0, z_0) is a **specific point** on the line.

*Thus when we determine the equation(s) of a **line** we need to find, from the information provided, a **point** on the line and a **vector parallel** to the **line**.*

*The important point to notice here is that when we are finding the equation of a plane we need a **vector perpendicular** to the **plane**, and when we are finding the equation(s) of a line we need a **vector parallel** to the **line**.*

Formula for the distance between a point and a plane in 3-space

Note that the formula for the distance between a point and a plane is derived in a similar manner to the formula for the distance between a point and a line in 2-space. Also note the similarity between the two formulas.

Further examples

When solving a problem concerning lines and planes in 3-space, try to visualize the situation in 3-space. You may find a rough sketch of the situation useful.

We now give a few more examples.

Example 3.5.1

Suppose L is the line in 3-space defined by

$$\left.\begin{aligned} x &= 3t \\ y &= 1 + 2t \\ z &= 2 - t \end{aligned}\right\} \quad -\infty < t < \infty$$

and V is the plane in 3-space defined by

$$4x - y + 2z = 1.$$

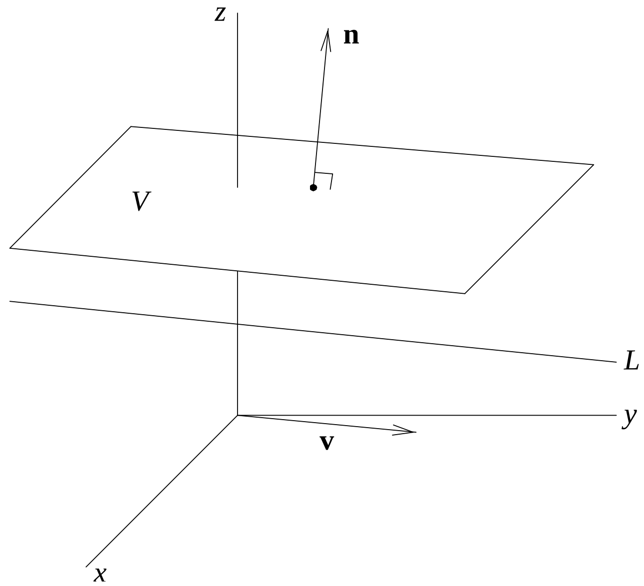
Determine whether L and V are parallel.

Solution:

The vector form of the equation of L is

$$(x, y, z) = (0, 1, 2) + t\,(3, 2, -1)\,.$$

Hence $\mathbf{v} = (3, 2, -1)$ is parallel to L.
From the equation of V it follows that $\mathbf{n} = (4, -1, 2)$ is a normal of V.



L is parallel to V if L is perpendicular to $\mathbf{n}$, i.e. if $\mathbf{v}$ is perpendicular to $\mathbf{n}$.
Now

$$\begin{aligned}\mathbf{v} \cdot \mathbf{n} &= (3, 2, -1) \cdot (4, -1, 2)\\ &= 12 - 2 - 2\\ &= 8 \neq 0.\end{aligned}$$

Hence $\mathbf{v}$ is not perpendicular to $\mathbf{n}$ and hence L is not parallel to V. ■

Example 3.5.2

Suppose L is the line in 3-space defined by

$$\left.\begin{array}{l} x = -1 + 3t \\ y = 5 + 2t \\ z = 2 - t \end{array}\right\} \quad -\infty < t < \infty$$

and V_1 is the plane in 3-space defined by

$$2x - 4y + 2z = 9.$$

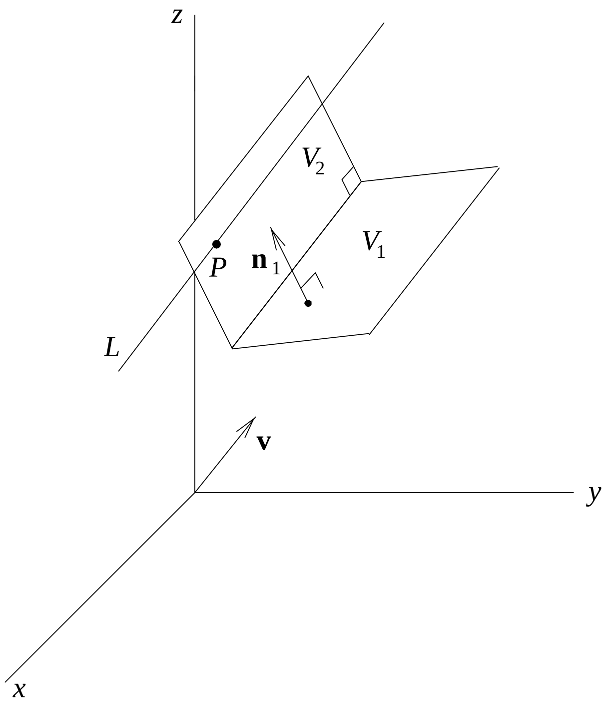
Find an equation for the plane V_2 which contains L and is perpendicular to V_1.

Solution:

The vector form of the equation for L is

$$(x, y, z) = (-1, 5, 2) + t\,(3, 2, -1)\,.$$

Hence $P = (-1, 5, 2)$ is a point on L and $\mathbf{v} = (3, 2, -1)$ is a vector parallel to L. Since V_2 contains L it follows that $P = (-1, 5, 2)$ is a point in V_2 and $\mathbf{v} = (3, 2, -1)$ is parallel to V_2.
From the equation of V_1 it follows that a vector perpendicular to V_1 is $\mathbf{n}_1 = (2, -4, 2)\,.$
Since V_1 is perpendicular to V_2 it follows that $\mathbf{n}_1 = (2, -4, 2)$ is parallel to V_2.



Thus $\mathbf{v}$ and $\mathbf{n}_1$ are two vectors parallel to V_2.
Thus $\mathbf{n}_2 = \mathbf{v} \times \mathbf{n}_1$ is a vector perpendicular to V_2.
Now

$$\begin{aligned}
\mathbf{n}_2 = \mathbf{v} \times \mathbf{n}_1 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 2 & -4 & 2 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} \\
&= (4-4)\,\mathbf{i} - (6+2)\,\mathbf{j} + (-12-4)\,\mathbf{k} \\
&= 0\mathbf{i} - 8\mathbf{j} - 16\mathbf{k} \\
&= (0, -8, -16) \\
&= -8\,(0, 1, 2)\,.
\end{aligned}$$

Hence $\mathbf{n} = (0, 1, 2)$ is a vector perpendicular to V_2.
Thus V_2 is a plane containing the point $P = (-1, 5, 2)$ and a normal to V_2 is $\mathbf{n} = (0, 1, 2)\,.$

Thus an equation of V_2 is

$$\begin{aligned} 0\,(x+1)+1\,(y-5)+2\,(z-2) &= 0 \qquad \text{(point-normal)} \\ \text{ie.} \qquad y+2z-9 &= 0. \qquad \text{(general form)} \end{aligned}$$

■

Example 3.5.3 (Anton, Exercise Set 3.5, Question 35)
Show that the lines L_1 and L_2, where

$$L_1 \quad : \quad \begin{cases} x-3=4t \\ y-4=t \\ z-1=0 \end{cases} \qquad -\infty < t < \infty$$

and

$$L_2 \quad : \quad \begin{cases} x+1=12t \\ y-7=6t \\ z-5=3t \end{cases} \qquad -\infty < t < \infty$$

intersect, and find the point of intersection.

Solution:
We rewrite the equations that define L_1 and L_2 in the following form:

$$L_1 \quad : \quad \begin{cases} x=3+4t \\ y=4+t \\ z=1 \end{cases} \qquad -\infty < t < \infty$$

$$L_2 \quad : \quad \begin{cases} x=-1+12t \\ y=7+6t \\ z=5+3t \end{cases} \qquad -\infty < t < \infty$$

The point (a, b, c) is a point of intersection of L_1 and L_2 if there exist numbers r and s such that

$$\left.\begin{aligned} a &= 3+4r \\ b &= 4+r \\ c &= 1 \end{aligned}\right\} \qquad (1)$$

and

$$\left.\begin{aligned} a &= -1+12s \\ b &= 7+6s \\ c &= 5+3s \end{aligned}\right\} \qquad (2)$$

We must now determine if r and s exist such that

$$\left.\begin{aligned} 3+4r &= -1+12s \\ 4+r &= 7+6s \\ 1 &= 5+3s \end{aligned}\right\} \qquad (3)$$

i.e. such that

$$\begin{aligned} 4r - 12s &= -4 \qquad (4) \\ r - 6s &= 3 \qquad (5) \\ -3s &= 4 \qquad (6) \end{aligned}$$

From (6) we have $s = -\frac{4}{3}$.
On substituting $s = -\frac{4}{3}$ in (5) we obtain

$$r - 6\left(-\frac{4}{3}\right) = 3$$

i.e.

$$r = 3 - 8 = -5$$

and on substituting $s = -\frac{4}{3}$ in (4) we obtain

$$4r - 12\left(-\frac{4}{3}\right) = -4$$

i.e.

$$4r = -4 - 16$$

i.e.

$$r = -5.$$

Hence system (3) has a solution and thus L_1 and L_2 intersect. We find the point of intersection (a, b, c) by substituting $r = -5$ in (1) or $s = -\frac{4}{3}$ in (2). In both cases we obtain $(a, b, c) = (-17, -1, 1)$. Hence the point of intersection of L_1 and L_2 is $(-17, -1, 1)$. ■

Example 3.5.4

Show that if a, b and c are nonzero, then the line

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \qquad (-\infty < t < \infty)$$

consists of all points (x, y, z) that satisfy

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

These are called **symmetric equations** for the line.

Solution:

$$\begin{aligned} x &= x_0 + at \Rightarrow t = \frac{x - x_0}{a}. \\ y &= y_0 + bt \Rightarrow t = \frac{y - y_0}{b}. \\ z &= z_0 + ct \Rightarrow t = \frac{z - z_0}{c}. \end{aligned}$$

Hence

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$ ■

List of important concepts

a ***normal*** to a plane

point-normal form of the equation of a plane

general form of the equation of a plane

vector form of the equation of a plane

parametric equations of a line

vector form of the equation of a line

symmetric equations for a line.

Activities

Anton, Chapter 3, Exercise set from pp 161, problem $1 - 174$

Summary of Section

Review of Chapter 3

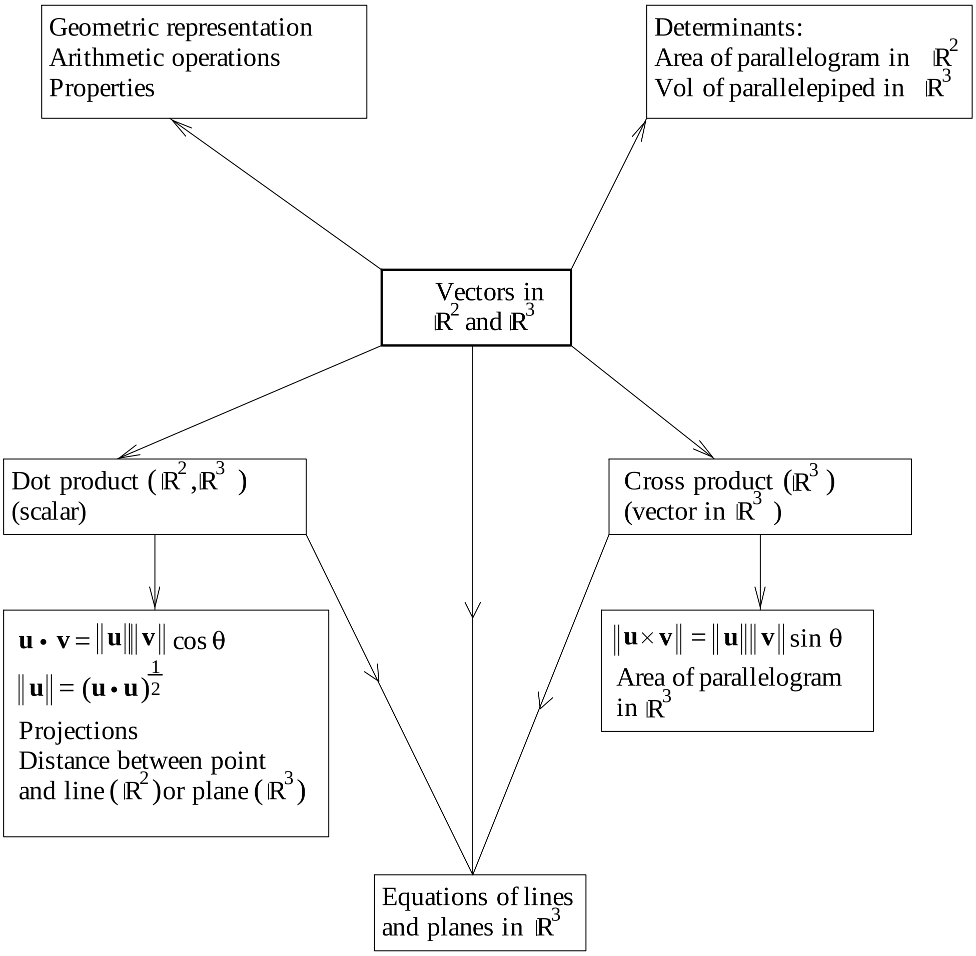
We first defined vectors in 2-space and 3-space geometrically and saw how to add and subtract two vectors and to multiply a vector by a scalar. We then introduced a rectangular coordinate system and defined the above operations in terms of components of vectors. We investigated properties of vector operations. We defined the norm (length) of a vector and saw that the distance between two points is equal to the norm of the vector between the two points.

We defined the dot product of two vectors (as a scalar) in terms of the norms of the two vectors and the angle between the vectors, and also derived a formula that expresses the dot product in terms of the components of the two vectors. We were then able to find the relationship between the norm of a vector and the dot product of the vector with itself. We obtained formulas for the orthogonal projection of a vector $\mathbf{u}$ on a vector $\mathbf{a}$, (i.e. the vector component of $\mathbf{u}$ along $\mathbf{a}$) as well as for the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$. The orthogonal projection was used to derive a formula for the distance between a point and a line in 2-space, and later also for the distance between a point and a plane in 3-space.

We defined the cross product of two vectors in 3-space (as a vector in 3-space) in terms of determinants. We deduced that the area of a parallelogram determined by two vectors in 3-space is equal to the norm of the cross product of the two vectors. This result was used to provide a geometric interpretation of the determinants of 2×2 and 3×3 matrices. If the rows of a 2×2 matrix are considered as vectors, then the absolute value of the determinant of the 2×2 matrix is equal to the area of the parallelogram (in 2-space) which is determined by the above mentioned two vectors. If the rows of a 3×3 matrix are considered as vectors, then the absolute value of the determinant of the 3×3 matrix is equal to the volume of the parallelepiped (in 3-space) which is determined by the above mentioned vectors.

Lastly, we used vectors, the dot product and the cross product to find equations of lines and planes in 3-space and to solve some basic geometric problems.

A diagrammatic summary of the main points is given on the following page.


Geometric representation
Arithmetic operations
Properties
Determinants:
Area of parallelogram in $\mathbb{R}^2$
Vol of parallelepiped in $\mathbb{R}^3$
Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$
Dot product ($\mathbb{R}^2$, $\mathbb{R}^3$)
(scalar)
Cross product ($\mathbb{R}^3$)
(vector in $\mathbb{R}^3$)
$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$
$\|\mathbf{u}\| = (\mathbf{u}\cdot\mathbf{u})^{\frac{1}{2}}$
Projections
Distance between point and line ($\mathbb{R}^2$) or plane ($\mathbb{R}^3$)
$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$
Area of parallelogram in $\mathbb{R}^3$
Equations of lines and planes in $\mathbb{R}^3$

Chapter 3

CHAPTER 4

COMPLEX NUMBERS (in addition to Sections 10.1-10.3 of Anton)

Introduction

According to the quadratic formula, the solutions of the equation $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad , \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

1. If $b^2 - 4ac < 0$ we cannot get the square root and the equation $ax^2 + bx + c = 0$ has no real solutions.

   It would be convenient if we could solve the quadratic equation $ax^2 + bx + c = 0$ in all cases. In order that this be possible we introduce new algebraic quantities called **complex numbers**. We shall see that, although the equation $ax^2 + bx + c = 0$ may have no **real** solutions, we can always solve it if we allow solutions which are complex numbers.

4.1 Complex numbers

We begin by introducing a new symbol i which is not a real number and we assign to i the property that

$$i^2 = -1.$$

Definition 4.1.1

1. A **complex number** is an expression of the form $x + yi$, where x and y are real numbers and i is as above.

Examples of complex numbers are $3 + 5i$, $-2 + 4i$, $7 - 2i$, $7i$ and i, where $7 - 2i = 7 + (-2)\, i$, $7i = 0 + 7i$ and $i = 0 + 1i$. Every real number x is a complex number since we may write $x = x + 0i$. All of the usual rules for addition, subtraction, multiplication and division extend to the complex numbers. When simplifying expressions which involve complex numbers, account must be taken of the fact that $i^2 = -1$.

Example 4.1.2

Simplify each of the following and give your answer in the form $x + yi$, where x and y are real.
(a) $(2 + 3i)\,(1 + 2i)$
(b) $2i\,(4 - 2i)\,(3 + i)$

Solution:

$$\begin{aligned}
\text{(a)}\quad & (2 + 3i)\,(1 + 2i) = 2 + 4i + 3i + 6i^2 \\
& = 2 + 4i + 3i - 6 \quad (\text{using } i^2 = -1) \\
& = -4 + 7i
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad & 2i\,(4 - 2i)\,(3 + i) = 2i\left(12 + 4i - 6i - 2i^2\right) \\
& = 2i\,(12 + 4i - 6i + 2) \quad (\text{again using } \; i^2 = -1) \\
& = 2i\,(14 - 2i) = 28i - 4i^2 \\
& = 28i + 4 = 4 + 28i
\end{aligned}$$

We generally use the symbols $z,\ w$ for complex numbers and use $x,\ y,\ u,\ v$ for real numbers. If $z = x + yi$ is a complex number we refer to x as the **real part** of z and write Re $z = x$. We refer to y (**not** yi) as the **imaginary part** of z and write Im $z = y$.

For example

Re $(3 + 4i) = 3$, Im $(3 + 4i) = 4$
Re $(2 - 7i) = 2$, Im $(2 - 7i) = -7$
Re $(6i)$ =Re $(0 + 6i) = 0$, Im $(6i)$ =Im $(0 + 6i) = 6$
Re(-2) =Re$(-2 + 0i) = -2$, Im(-2) =Im$(-2 + 0i) = 0$.

If the complex number z is real then z is of form $x + 0i$ so Im $z = 0$.
A complex number of form yi, where y is real is called **pure imaginary**. Examples of pure imaginary complex numbers are $6i,\ i,\ -3i$.

If z is a pure imaginary number, then z is of form $yi = 0 + yi$ and so Re $z = 0$.

Definition 4.1.3

Let $z = x+yi$ be a complex number. The **complex conjugate** of z is defined to be the complex number $\overline{z} = x - yi$.

Illustrations:

$$\begin{aligned} \overline{2+3i} &= 2-3i \quad , \quad \overline{6-5i} = 6+5i, \\ \overline{3i} &= \overline{0+3i} \quad = 0-3i \quad = \quad -3i, \\ \overline{4} &= \overline{4+0i} \quad = 4-0i \quad = \quad 4. \end{aligned}$$

Proposition 4.1.4

(a) $\overline{\overline{z}} = z$.

(b) z is real if and only if $z = \overline{z}$.

(c) $z + \overline{z} = 2$ Re z and $z - \overline{z} = 2i$ Im z.

(d) $z\overline{z}$ is always real.

Proof.

(a) Let $z = x + yi$. Then $\overline{z} = x - yi$.
But then $\overline{\overline{z}} = \overline{x - yi} = x + yi = z$.

(b) If z is real then $z = x + 0i$ so $\overline{z} = x - 0i = x = x + 0i = z$.
For the converse, let $z = \overline{z}$. Put $z = x + yi$.
Then $z = \overline{z} \Rightarrow x + yi = x - yi \Rightarrow 2yi = 0 \Rightarrow y = 0 \Rightarrow z = x + 0i = x$ so z is real.

(c) Let $z = x + yi$. Then $z + \overline{z} = x + yi + x - yi = 2x = 2$ Re z. Similarly $z - \overline{z} = 2iy = 2i$ Im z.

(d) If $z = x + yi$ then $z\,\overline{z} = (x + yi)(x - yi) = x^2 - xyi + xyi - y^2i^2 = x^2 + y^2$
which is a real number (x and y are real).

Division by a complex number may be accomplished by multiplying above and below by the complex conjugate of the denominator.

Example 4.1.5

Express $\dfrac{2-i}{3+2i}$ and $\dfrac{1}{i}$ in the form

$x + yi$, where x and y are real.

Solution.

$$\begin{aligned} \frac{2-i}{3+2i} &= \frac{(2-i)(3-2i)}{(3+2i)(3-2i)} &= \frac{6-4i-3i+2i^2}{9-6i+6i-4i^2} \\ &= \frac{6-7i-2}{9+4} &= \frac{4}{13} - \frac{7i}{13} = \frac{4}{13} + \left(-\frac{7}{13}\right)i \end{aligned}$$

Since $\overline{i} = -i$, we get $\dfrac{1}{i} = \dfrac{-i}{i(-i)} = \dfrac{-i}{1} = -i = 0 + (-1)\,i$.

The procedure used in this example will always work because of the fact that $z\,\overline{z}$ is real. We will be left with a real denominator in all cases.

Proposition 4.1.6

Let z and w be complex numbers.

(a) $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{z-w} = \overline{z} - \overline{w}$

(b) $\overline{zw} = \overline{z}\;\overline{w}$

(c) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$ provided $w \neq 0$

Proof.

Let $z = x + yi$, $w = u + vi$ where $x,\ y,\ u$ and v are real.

(a)
$$\begin{aligned} \overline{z+w} &= \overline{x+yi+u+vi} = \overline{(x+u)+(y+v)\,i} \\ &= (x+u)-(y+v)\,i = (x-yi)+(u-vi) = \overline{z}+\overline{w} \end{aligned}$$

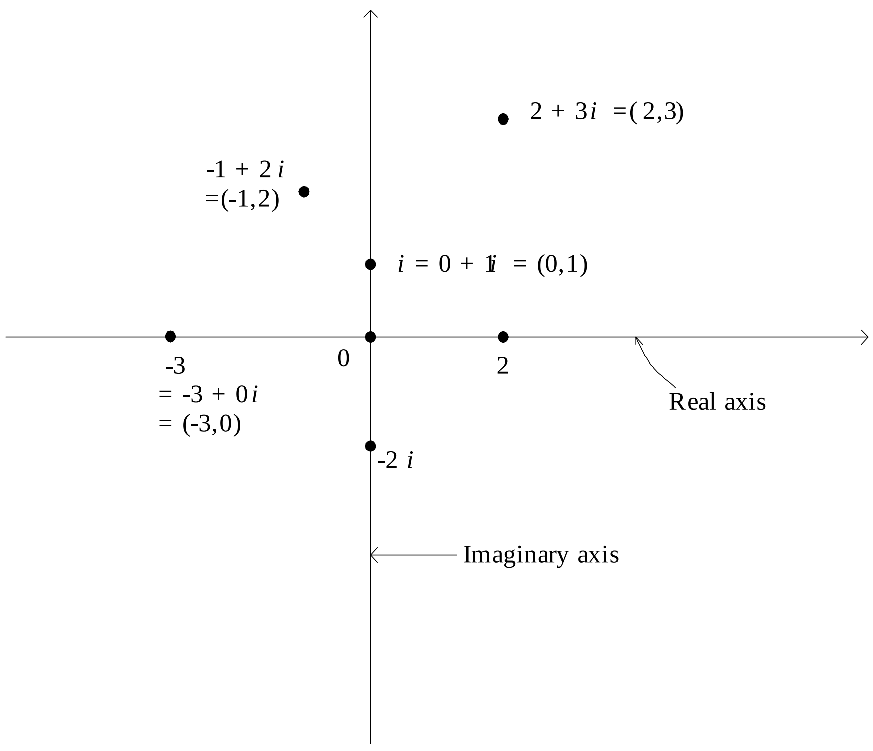
Similarly (exercise) $\overline{z-w} = \overline{z} - \overline{w}$.

(b)
$$\begin{aligned} z\,w &= (x+yi)\,(u+vi) = xu + yui + xvi - yv \\ &= (xu-yv)+(yu+xv)\,i \\ \Rightarrow \overline{zw} &= (xu-yv)-(yu+xv)\,i \\ \overline{z}\,\overline{w} &= (x-yi)\,(u-vi) = xu - yui - xvi - yv \\ &= (xu-yv)-(yu+xv)\,i = \overline{zw} \end{aligned}$$

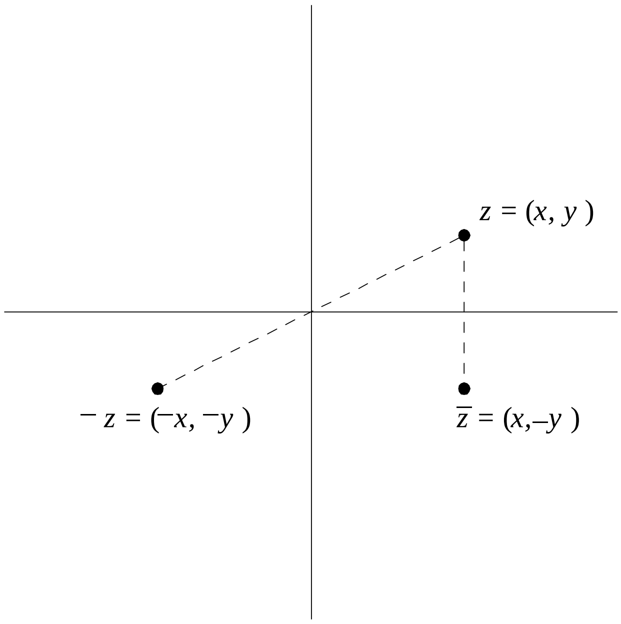
(c) Exercise

4.2 Graphical representation

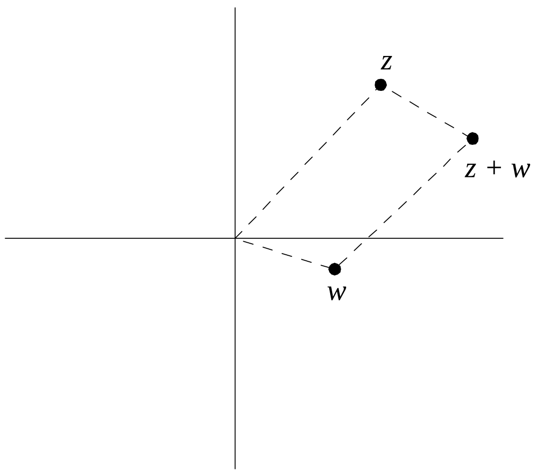
The real numbers may be represented graphically as points on a straight line, the real line. Two real numbers x, y are needed to specify an arbitrary complex number $z = x + yi$. We therefore represent the complex numbers as points on a plane. The complex number $z = x + yi$ is represented as the point with coordinates (x, y). A **real number** x is of form $x + 0i$ and is represented by the point $(x, 0)$ on the horizontal axis. A **pure imaginary** complex **number** yi is of form $0 + yi$ and is represented by the point $(0, y)$ on the vertical axis. For these reasons, the horizontal axis is referred to as the **real axis** and the vertical axis is referred to as the **imaginary axis**. When a plane is used in this way to represent complex numbers, it is referred to as the **complex plane** or the **Argand diagram**. From a geometric or graphical point of view the complex number $z = x + yi$ is identical to the point (x, y) in the plane.



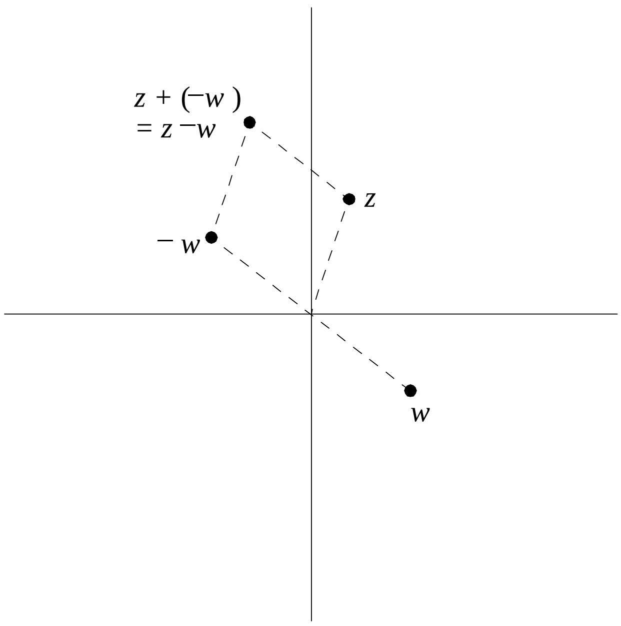
If $z = x + yi = (x, y)$ then $-z = -x - yi = (-x, -y)$ which is the point obtained by reflecting (x, y) through the origin. Also $\overline{z} = x - yi = (x, -y)$ which is obtained by reflecting z in the real axis, as shown in the diagram below.



If $z = x + yi = (x, y)$ and $w = u + vi = (u, v)$, then $z + w = x + u + (y + v)\, i = (x + u, y + v)$. The point $z + w$ can be found using the **parallelogram law of addition.** Use z, w and the origin as three vertices of a parallelogram. If this parallelogram is completed, the fourth vertex is $z + w$.



To find $z - w$, treat it as $z + (-w)$. First find $-w$ by reflecting w through the origin as shown in the sketch preceding the sketch above. Then use the parallelogram law of addition to get $z + (-w) = z - w$.



Definition 4.2.1

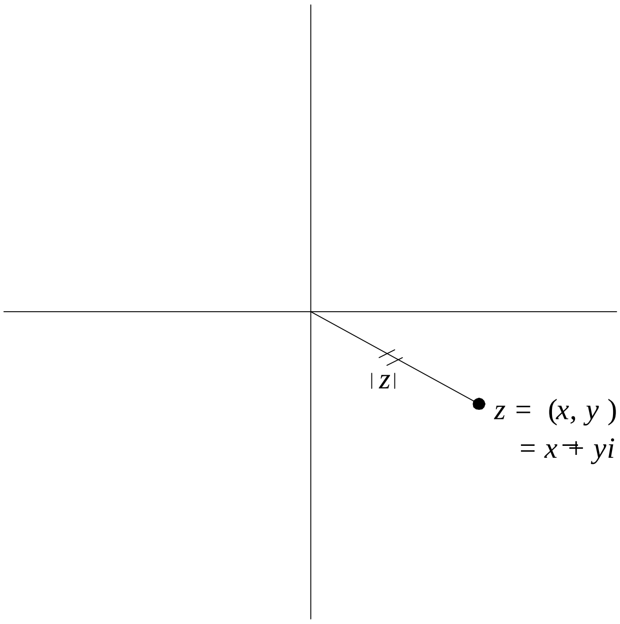
The **modulus** or **absolute value** of the complex number $z = x + yi$ is defined to be $|z| = \sqrt{x^2 + y^2}$.

We note that, since x, y are real, $x^2 + y^2 \geq 0$ so the square root exists. $|z|$ is always a **real number** and $|z| \geq 0$.

For example, $|3 + 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$, $|i| = |0 + 1i| = \sqrt{0^2 + 1^2} = 1$, $|4 - 3i| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$.

If $z = x$ is real, then $|z| = |x + 0i| = \sqrt{x^2 + 0^2} = \sqrt{x^2}$ which is equal to the usual absolute value of the real number x.

On a sketch, $|z| = \sqrt{x^2 + y^2} = \sqrt{(x - 0)^2 + (y - 0)^2}$ represents the distance from the point $z = (x, y)$ to the origin.



THEOREM 4.2.2

Let $z = x + yi$ be any complex number.

(a) $|z| = 0$ if and only if $z = 0$

(b) $|\overline{z}| = |z|$

(c) $z\overline{z} = |z|^2$

Proof

(a) Let $z = x + yi$.
If $|z| = 0$ then $|z|^2 = 0$ so $x^2 + y^2 = 0$.
But $0 \leq x^2 \leq x^2 + y^2$, i.e. $0 \leq x^2 \leq 0$, i.e. $x^2 = 0$, i.e. $x = 0$.
Similarly $y = 0$. Therefore $z = x + yi = 0 + 0i = 0$.
For the converse, if $z = 0 = 0 + 0i$
then $|z| = \sqrt{0^2 + 0^2} = 0$. This completes (a).

(b) Let $z = x + yi$. Then $|\overline{z}| = |x - yi| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$.

(c) $z\overline{z} = (x + yi)(x - yi) = x^2 + xyi - xyi + y^2 = x^2 + y^2 = |z|^2$

THEOREM 4.2.3

Let $z = x + yi$ be any complex number. Then $|\text{Re } z| \leq |z|$ and $|\text{Im } z| \leq |z|$.

Proof. $|\text{Re } z|^2 = |x|^2 = x^2 \leq x^2 + y^2 = |z|^2$

Since $|\text{Re } z| \geq 0$ and $|z| \geq 0$, we may take square roots and get $|\text{Re } z| \leq |z|$. Similarly $|\text{Im } z| \leq |z|$.

Illustration. If $z = -4 + 2i$, then $|\text{Re } z| = |-4| = 4$ and $|\text{Im } z| = |2| = 2$ while $|z| = \sqrt{(-4)^2 + 2^2} = \sqrt{20}$. The theorem works since $4 \leq \sqrt{20}$ and $2 \leq \sqrt{20}$ are true.

THEOREM 4.2.4

Let z and w be complex numbers.

(a) $|zw| = |z| \ |w|$

(b) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ provided $w \neq 0$

(c) $|z + w| \leq |z| + |w|$ (Triangle Inequality)

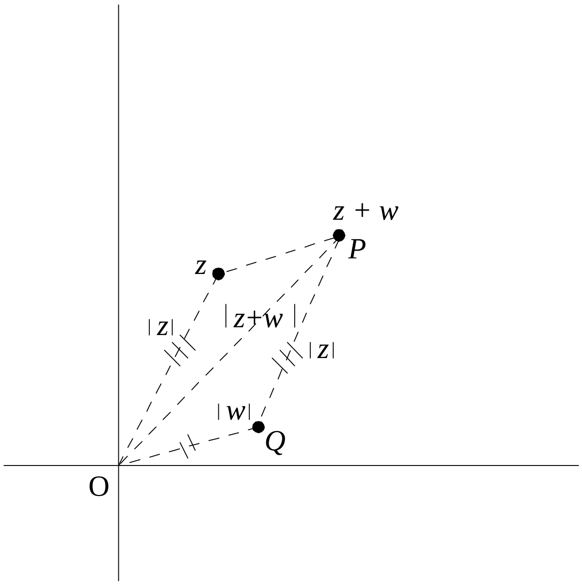
Proof

(a)
$$\begin{aligned} |zw|^2 &= (zw)\,(\overline{zw}) && \text{(Theorem 5.2.2(c))} \\ &= zw\ \overline{z}\ \overline{w} && \text{(Proposition 5.1.6(b))} \\ &= z\ \overline{z}\ w\ \overline{w} = |z|^2\ \ |w|^2 = (|z|\ \ |w|)^2 \end{aligned}$$
Take square roots to get $|zw| = |z|\,|w|$.

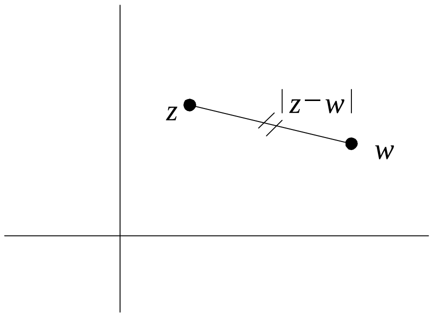
(b) Similar to (a) $-$ exercise

(c)
$$\begin{aligned} |z+w|^2 &= (z+w)\,(\overline{z+w}) \\ &= (z+w)\,(\overline{z}+\overline{w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + z\overline{w} + \overline{z}w + |w|^2 && \text{(Theorem 5.2.2}\,(c)) \\ &= |z|^2 + z\overline{w} + \overline{z\overline{w}} + |w|^2 && (\overline{z\overline{w}} = \overline{z}\ \ \overline{\overline{w}} = \overline{z}w) \\ &= |z|^2 + 2\,\text{Re }\ z\overline{w} + |w|^2 \\ &\leq |z|^2 + 2\,|\text{Re }\ z\overline{w}| + |w|^2 && (x \text{ real} \Rightarrow x \leq |x|) \\ &\leq |z|^2 + 2\,|z\overline{w}| + |w|^2 && \text{(Theorem 5.1.4(a))} \\ &= |z|^2 + 2\,|z|\,|\overline{w}| + |w|^2 && \text{(Theorem 5.2.4 (a))} \\ &= |z|^2 + 2\,|z|\,|w| + |w|^2 && \text{(Theorem 5.2.2 (b))} \\ &= (|z| + |w|)^2 \end{aligned}$$
Thus $|z+w|^2 \leq (|z| + |w|)^2$. Take square roots to get $|z+w| \leq |z| + |w|$.

On the following sketch, z, w and $z + w$ are shown. $|z|, |w|$ and $|z+w|$ are also shown. The triangle inequality
$|z + w| \leq |z| + |w|$ represents the fact that the length of OP is less than or equal to the sum of the lengths of OQ and QP $-$ a well known property of triangles.



Before we proceed, note that if $z = x+yi,\ w = u+vi$ then $|z-w| = |x+yi-(u+vi)| = |(x-u)+(y-v)\,i| = \sqrt{(x-u)^2+(y-v)^2}$ which is the distance from the point $z = (x, y)$ to the point $w = (u, v)$.



4.3 Equality of Complex numbers

We regard two complex numbers $z = x + yi$ and $w = u + vi$ as equal if they are identical i.e. if $x = u$ and $y = v$. Thus the single equation $z = w$ involving complex numbers gives two equations involving real numbers. If we know that $z = w$, we can state that $\mathrm{Re}\ z = \mathrm{Re}\ w$ and also $\mathrm{Im}\ z = \mathrm{Im}\ w$.

Example 4.3.1

If a and b are real and $2a + (3b - a)\,i = b - 4i$, find a and b.

Solution.
Equate real and imaginary parts on both sides and get $2a = b,\ 3b - a = -4$.
Solving these simultaneous equations for a and b gives $a = -\dfrac{4}{5}$, $b = -\dfrac{8}{5}$.

Example 4.3.2

Find all complex numbers z such that $z^2 = -1$.

Solution. We put $z = x + yi$ and find x and y. $z^2 = (x+yi)^2 = x^2 + 2xyi + (yi)^2 = x^2 - y^2 + 2xyi$.
The equation $z^2 = -1$ gives $x^2 - y^2 + 2xyi = -1 = -1 + 0i$.

Equating real and imaginary parts, we get $x^2 - y^2 = -1 \cdots$ (1) and $2xy = 0 \cdots (2)$.
From (2), either $x = 0$ or $y = 0$. Now $y = 0$ is impossible since substituting $y = 0$ in (1) gives $x^2 = -1$ which is impossible, since x in real. Therefore $x = 0$. In this case (1) becomes $-y^2 = -1$ i.e. $y^2 = 1$ so $y = \pm 1$.
The possible values of z are $0 + 1i = i$ and $0 - 1i = -i$. Thus the solutions are $z = \pm i$.

Exercise 4.3.3

Find all complex numbers z such that $z^2 = 3 - 4i$.

4.4 Remark

Because of the fact that $i^2 = -1$, we may think of i as being $\sqrt{-1}$. The solutions of the equation $z^2 = -1$ could then be found by simply writing $z = \pm\sqrt{-1} = \pm i$ which, according to the above example, are the correct answers. The procedure will work with any negative real number on the right e.g. given $z^2 = -3$ we get $z = \pm\sqrt{-3} = \pm\sqrt{3}\sqrt{-1} = \pm\sqrt{3}i$. The method of the above examples can be used to show that these answers are correct.

Example 4.4.1

Find all solutions of the equation $z^2 + z + 1 = 0$.

Solution. The discriminant of the given equation is $1^2 - 4(1)(1) = 1 - 4 = -3$ so there will be no real solutions. However, there are complex solutions. We have

$$\begin{aligned}
& z^2 + z + 1 = 0 \\
\Rightarrow\quad & z^2 + z + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = 0 \text{ (completing the square)} \\
\Rightarrow\quad & \left(z + \frac{1}{2}\right)^2 + \frac{3}{4} = 0 \\
\Rightarrow\quad & \left(z + \frac{1}{2}\right)^2 = -\frac{3}{4} \\
\Rightarrow\quad & z + \frac{1}{2} = \pm\sqrt{-\frac{3}{4}} = \pm\sqrt{\frac{3}{4}}\sqrt{-1} = \pm\frac{\sqrt{3}}{2}i \\
\Rightarrow\quad & z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \text{ The solutions are } -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ or } -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned}$$

Any quadratic equation with real coefficients can be solved in this way.

4.5 Polynomial equations

A polynomial is an expression of the form $P(z) = a_0 + a_1\, z + a_2\, z^2 + \cdots + a_n\, z^n$ where $a_0, a_1, \cdots a_n$ are constants which may be complex numbers. A **zero** of the polynomial P is a solution of the equation $P(z) = 0$ i.e. a complex number z_0 such that $a_0 + a_1\, z_0 + a_2\, z_0^2 + \cdots + a_n\, z_0^n = 0$.

For example, i is a zero of the polynomial $P(z) = z^3 + z^2 + z + 1$ since, by direct substitution, $P(i) = i^3 + i^2 + i + 1 = i^2\, i - 1 + i + 1 = -i - 1 + i + 1 = 0$.

THEOREM 4.5.1

Let $P(z) = a_0 + a_1\, z + a_2\, z^2 + \cdots + a_n\, z^n$ be a polynomial with **real** coefficients (i.e. $a_0,\ a_1, \cdots a_n$ are real). If z_0 is a zero of P then so is $\overline{z}_0$.
Proof. Since z_0 is a zero of P we have $a_0 + a_1\, z_0 + a_2\, z_0^2\ + \cdots + a_n\, z_0^n = 0$.
Take the complex conjugate of both sides. Then

$$\begin{array}{lll} & \overline{a_0 + a_1\, z_0 + a_2\, z_0^2 + \cdots + a_n\, z_0^n} & = \overline{0} \\ \text{i.e.} & \overline{a}_0 + \overline{a_1\, z_0} + \overline{a_2\, z_0^2} + \cdots + \overline{a_n\, z_0^n} & = 0 \\ \text{i.e.} & \overline{a}_0 + \overline{a}_1\, \overline{z}_0 + \overline{a_2}\, \overline{z_0^2} + \cdots + \overline{a_n}\, \overline{z_0^n} & = 0 \\ \text{i.e.} & a_0 + a_1\, \overline{z}_0 + a_2\, \overline{z}_0^2 + \cdots + a_n\, \overline{z}_0^n & = 0 \end{array}$$

where we have used the fact that $a_0,\ a_1, \cdots a_n$ are real so $\overline{a}_0 = a_0$, $\overline{a}_1 = a_1$, etc. Thus $P(\overline{z}_0) = 0$ so $\overline{z}_0$ is a zero of P.

If we are given a polynomial equation $P(z) = 0$, where P has real coefficients, and we know that z_0 is a solution, then, by the above, $\overline{z}_0$ is also a solution. As in the real case, this means that $z - z_0$ and $z - \overline{z}_0$ are factors of $P(z)$, and hence $(z - z_0)(z - \overline{z}_0)$ is a factor of $P(z)$. By dividing this factor into $P(z)$ we may find another factor. Provided the degree of P is not too high, this procedure may allow us to find all solutions of the equation $P(z) = 0$.

Example 4.5.2

Given that $-2 + 3i$ is a solution of the equation $z^4 + 3z^3 + 10z^2 - 9z + 13 = 0$, find all solutions.

Solution. The polynomial $z^4 + 3z^3 + 10z^2 - 9z + 13$ has real coefficients. Given $-2 + 3i$ is a solution, $-2 - 3i$ is also a solution. Therefore $z - (-2 + 3i)$ and $z - (-2 - 3i)$ are factors of the given polynomial i.e. $z + 2 - 3i$ and $z + 2 + 3i$ are factors. It follows that their product $(z + 2 - 3i)(z + 2 + 3i) = (z + 2)^2 - (3i)^2 = z^2 + 4z + 4 + 9 = z^2 + 4z + 13$ is also a factor. To find another factor, use long division. We find that $z^4 + 3z^3 + 10z^2 - 9z + 13 = \left(z^2 + 4z + 13\right)\left(z^2 - z + 1\right)$. The remaining solutions of the equation $z^4 + 3z^3 + 10z^2 - 9z + 13 = 0$ are obtained by putting $z^2 - z + 1$ equal to zero.

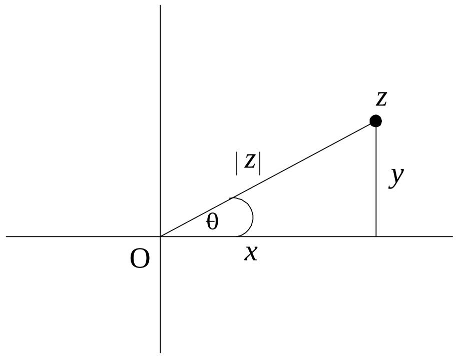
$$z^2 - z + 1 = 0 \Rightarrow z^2 - z + \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 + 1 = 0$$

$$\Rightarrow \quad \left(z - \frac{1}{2}\right)^2 + \frac{3}{4} = 0 \Rightarrow \left(z - \frac{1}{2}\right)^2 = -\frac{3}{4} \Rightarrow z - \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i$$

$$\Rightarrow \quad z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

The four solutions are $-2 + 3i,\ -2 - 3i,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

4.6 Polar form of a complex number



Let $z = x + yi$ be complex. Let θ be the angle (in radians) between Oz and the positive real axis. The distance from the origin to z is $|z|$ so, from the sketch, $x = |z| \cos\theta$, $y = |z| \sin\theta$. It is easy to check that these relations hold for the other three quadrants as well as the first as sketched. The construction can be done for any complex z except $z = 0$. (If $z = 0$, no angle θ can be constructed as described above.)

If θ is as above then $z = x + yi = |z| \cos\theta + i\,|z| \sin\ \theta = |z| (\cos\theta + i \sin\theta)$.

It is usual to write r for $|z|$ giving $z = r (\cos\theta + i \sin\theta)$.

This is called the **polar form** of the complex number z. Every complex number except 0 can be written in polar form.

The relations $x = |z| \cos\theta = r \cos\theta$ and $y = |z| \sin\theta = r \sin\theta$ give $\cos\theta = \dfrac{x}{|z|} = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{|z|} = \dfrac{y}{r}$.

Example 4.6.1

Express $1 + i\sqrt{3}$ and $2 - 2i$ in polar form.

Solution.
For $z = 1 + i\sqrt{3}$, we have $x = 1,\ y = \sqrt{3}$.
Hence $r = |z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = 2$
and $\cos\theta = \dfrac{x}{r} = \dfrac{1}{2}$, $\sin\theta = \dfrac{y}{r} = \dfrac{\sqrt{3}}{2}$.

We can take $\theta = \dfrac{\pi}{3}$ and get $1 + i\sqrt{3} = r (\cos\theta + i \sin\theta) = 2 \left(\cos\dfrac{\pi}{3} + \sin\dfrac{\pi}{3}\right)$.

For $z = 2 - 2i$, we have $x = 2$, $y = -2$.
Thus $r = |z| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$
and $\cos\theta = \dfrac{x}{r} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}$, $\sin\theta = \dfrac{y}{r} = \dfrac{-2}{2\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$.

We can take $\theta = -\dfrac{\pi}{4}$ and get $2 - 2i = r (\cos\theta + i\ \sin\theta) = 2\sqrt{2} \left(\cos\left(-\dfrac{\pi}{4}\right) + i\ \sin\left(-\dfrac{\pi}{4}\right)\right)$.

The number θ in the polar form $z = r (\cos\theta + i\ \sin\theta)$ is called an **argument** of z. It is not unique. If θ satisfies $\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$, then $\theta + 2k\pi$, k an integer, also satisfies $\cos(\theta + 2k\pi) = \dfrac{x}{r}$, $\sin(\theta + 2k\pi) = \dfrac{y}{r}$.

Thus a nonzero complex number has infinitely many arguments. We chose $\theta = \dfrac{\pi}{3}$ when we were dealing with $1 + i\sqrt{3}$ above but we could equally well take $\dfrac{\pi}{3} \pm 2\pi$, $\dfrac{\pi}{3} \pm 4\pi, \cdots$ and get the same values for $\sin\theta$ and $\cos\theta$.

If z is a nonzero complex number, exactly one argument of z lies in the range $-\pi < \theta \leq \pi$. This is called the **principal argument** of z. The principal argument of $1 + i\sqrt{3}$ is $\dfrac{\pi}{3}$ and the principal argument of $2 - 2i$ is $-\dfrac{\pi}{4}$.

Example 4.6.2

Express 3, i, -1 in polar form with **principal argument**.

Solution.

For $3 = 3 + 0i$ we have $x = 3,\ y = 0$.
Thus $r = |z| = \sqrt{9+0} = 3$, $\cos\theta = \dfrac{x}{r} = \dfrac{3}{3} = 1$ and $\sin\theta = \dfrac{y}{r} = \dfrac{0}{3} = 0$.
The principal argument is $\theta = 0$.
Thus $3 = 3\left(\cos 0 + i\ \sin 0\right)$.

For $i = 0 + 1i$ we have $x = 0,\ \ y = 1$.
Hence $r = |z| = \sqrt{0^2 + 1^2} = 1$, $\cos\theta = \dfrac{x}{r} = \dfrac{0}{1} = 0$ and $\sin\theta = \dfrac{y}{r} = \dfrac{1}{1} = 1$.
The principal argument is $\theta = \dfrac{\pi}{2}$.
Hence $i = 1\left(\cos\dfrac{\pi}{2} + i\ \sin\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} + i\ \sin\dfrac{\pi}{2}$.

Show as an exercise that $-1 = 1\left(\cos\pi + i\ \sin\ \pi\right) = \cos\pi + i\ \sin\pi$.

THEOREM 4.6.3

(a) If $z = r\left(\cos\theta + i\sin\theta\right)$, then $\overline{z} = r\left(\cos(-\theta) + i\ \sin(-\theta)\right)$.

(b) If $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$,
then $z_1\ z_2 = r_1\ r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$ and $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$.

Proof.
(a)

$$
\begin{aligned}
z &= r\left(\cos\theta + i\ \sin\theta\right) = r\cos\theta + i\ r\sin\theta \\
\Rightarrow \overline{z} &= r\cos\theta - i\ r\ \sin\theta = r\left(\cos\theta - i\ \sin\theta\right) \\
&= r\ \left(\cos(-\theta) + i\sin(-\theta)\right).
\end{aligned}
$$

(b)

$$
\begin{aligned}
z_1 &= r_1\left(\cos\theta_1 + i\ \sin\ \theta_1\right) \text{ and } z_2 = r_2\left(\cos\theta_2 + i\ \sin\theta_2\right) \\
\Rightarrow \quad z_1\ z_2 &= r_1\ r_2\left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 + i\ \sin\theta_2\right) \\
&= r_1\ r_2\left(\cos\theta_1\cos\theta_2 + i\ \sin\theta_1\cos\theta_2 + i\ \cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2\right) \\
&= r_1\ r_2\ \left(\left(\cos\theta_1\ \cos\theta_2 - \sin\theta_1\sin\theta_2\right) + i\left(\sin\theta_1\ \cos\theta_2 + \cos\theta_1\ \sin\theta_2\right)\right) \\
&= r_1\ r_2\left(\cos(\theta_1 + \theta_2) + i\ \sin(\theta_1 + \theta_2)\right)
\end{aligned}
$$

It is left as an exercise to deal with $\dfrac{z_1}{z_2}$.

THEOREM 4.6.4 (de Moivre's Theorem)

If $z = r(\cos\theta + i\ \sin\theta)$ then $z^n = r^n(\cos\ n\theta + i\ \sin\ n\theta)$ for every natural number n.

Proof. The theorem is proved by induction. The given statement is true for $n = 1$ since the equation $z^1 = r^1(\cos 1\theta + i \sin 1\theta)$ reduces to $z = r(\cos\theta + i\ \sin\theta)$ which is given.

Assume the statement true for $n = k$, i.e. assume that if $z = r(\cos\theta + i\sin\theta)$ then $z^k = r^k(\cos\ k\theta + i\sin k\theta)$.
Using Theorem 5.6.3 (b) above, we obtain
$z^{k+1} = z^k z = r^k r(\cos(k\theta+\theta) + i\ \sin(k\theta+\theta)) = r^{k+1}(\cos(k+1)\theta + i\ \sin(k+1)\theta)$.
Hence the statement is true for $n = k+1$.

By induction the given statement is true for every natural number n.

Example 4.6.5

Find $(-1+i)^{134}$ in the form $x + yi$.

Solution. First write $-1+i$ in polar form. We have $r = \sqrt{1+1} = \sqrt{2}$; $\cos\theta = \dfrac{-1}{\sqrt{2}}$, $\sin\theta = \dfrac{1}{\sqrt{2}}$.
Thus the principal argument is $\theta = \dfrac{3\pi}{4}$ and $-1+i = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$.

By de Moivre's Theorem $(-1+i)^{134} = \left(\sqrt{2}\right)^{134}\left(\cos 134\left(\dfrac{3\pi}{4}\right) + i\sin 134\left(\dfrac{3\pi}{4}\right)\right)$
$= 2^{67}\left(\cos\dfrac{402\pi}{4} + i\ \sin\dfrac{402\pi}{4}\right) = 2^{67}\left(\cos\left(100\pi + \dfrac{\pi}{2}\right) + i\ \sin\left(100\pi + \dfrac{\pi}{2}\right)\right)$
$= 2^{67}\left(\cos\dfrac{\pi}{2} + i\ \sin\dfrac{\pi}{2}\right) = 2^{67}(0+i) = 2^{67}i.$

Example 4.6.6

Express $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.

Solution. The method is to find $(\cos\theta + i\ \sin\theta)^4$ in the form $x + yi$ using (i) de Moivre's Theorem and (ii) the Binomial Theorem and then equate real and imaginary parts.

(i) $(\cos\theta + i\ \sin\theta)^4 = \cos 4\theta + i\ \sin 4\theta$ by de Moivre's Theorem.

(ii)
$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= \binom{4}{0}\cos^4\theta\,(i\ \sin\theta)^0 + \binom{4}{1}\cos^3\theta\,(i\sin\theta)^1 \\
&\quad + \binom{4}{2}\cos^2\theta\,(i\ \sin\theta)^2 + \binom{4}{3}\cos\theta\,(i\ \sin\theta)^3 + \binom{4}{4}(\cos\theta)^0\ (i\ \sin\theta)^4 \\
&= \cos^4\theta + 4i\ \cos^3\theta\ \sin\theta + 6\cos^2\theta\,(-1)\sin^2\theta + 4\ \cos\theta\,(-i)\ \sin^3\theta + i^4\sin^4\theta\ \ \left(\text{using } i^2 = -1\right) \\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\ \cos\theta\ \sin^3\theta + \sin^4\theta \\
&= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\ \sin^3\theta\right)
\end{aligned}$$

This expression is equal to $\cos 4\theta + i\sin 4\theta$ by (i).
Equating real parts gives:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \text{ as required.}$$

Equating imaginary parts gives:

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta, \text{ as required.}$$

In the special case where $r = 1$ and $z = \cos\theta + i\sin\theta = 1(\cos\theta + i\sin\theta)$, de Moivre's Theorem gives $z^n = 1^n(\cos n\theta + i\sin n\theta) = \cos n\theta + i\ \sin n\theta$.

In this case

$$\begin{aligned}\frac{1}{z^n} &= \frac{1}{\cos n\theta + i\ \sin n\theta} = \frac{\cos n\theta\ -\ i\ \sin n\theta}{(\cos n\theta\ +\ i\ \sin n\theta)(\cos n\theta\ -\ i\ \sin n\theta)} \\ &= \frac{\cos n\theta\ -\ i\ \sin n\theta}{\cos^2 n\theta\ +\ \sin^2 n\theta} = \frac{\cos n\theta\ -\ i\ \sin n\theta}{1} = \cos n\theta - i\ \sin n\theta.\end{aligned}$$

Thus, in the special case we have

$$z^n = \cos n\theta +\ i\ \sin n\theta \text{ and } \frac{1}{z^n} = \cos n\theta - i\ \sin n\theta.$$

Adding these equations gives $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ and subtracting the two equations gives $z^n - \dfrac{1}{z^n} = 2i\ \sin n\theta$.

APPENDIX

GREEK ALPHABET

Capital letter	Small letter	Name
A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ε	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
I	ι	iota
K	κ	kappa
Λ	λ	lambda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Υ	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega