Assignment 1.

1. Let f be the function
$$f(x) = x^2 - \ln x^8 \quad \text{where } x > 1$$

   (a) Use the sign pattern for $f'(x)$ to determine the intervals where f rises and where f falls. (5)

   (b) Determine the coordinates of the local extreme point(s). (2)

   (c) Find $f''(x)$ and determine where the graph of f is concave up and where it is concave down. (5)

[12]

2. You are designing a poster to contain 50 cm^2 of printing with margins of 4 cm each at the top and bottom and 2 cm at each side. What overall dimensions will minimize the amount of paper used? **[5]**

3. Use L'Hôpital's rule to evaluate:

   (a) $\lim_{x\to 0} \dfrac{1-\cos^3 x}{\sin^2 x}$ (5)

   (b) $\lim_{x\to\infty} (\cos \frac{1}{x})^x$ (5)

   (c) $\lim_{x\to\infty} \frac{x \ln x}{x^2-1}$ (5)

   (d) $\lim_{x\to 1} \frac{x \ln x}{x^2-1}$ (5)

[20]

4. Find the exact value of $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **[3]**

TOTAL: [40]

SOLUTIONS

1. Let f be the function defined by

$$f(x) = x^2 - \ln x^8 \quad \text{where } x > 1$$

Solutions:

(a)

Open Rubric

$$\begin{aligned} f'(x) &= \left(x^2 - 8\ln x\right)' \\ &= 2x - \frac{8}{x} \\ &= \frac{2(x^2-4)}{x} \\ &= \frac{2(x-2)(x+2)}{x} \text{ where } x > 1 \end{aligned}$$

We find the sign pattern for $x > 1$ so x in the denominator always has positive values and may be excluded from the sign pattern. We only look at the sign pattern for $x > 1$

$$\begin{array}{rc|c} y = x - 2 & - & + \\ y = x + 2 & + & + \\ f'(x) & - & + \\ \quad 1 & & 2 \end{array}$$

Thus f rises on $(2, \infty)$ and f falls on $(1, 2)$.

(b)

The local extreme point(s) occur at x when $f'(x) = 0$.Thus we must find where

$$f'(x) = \frac{2(x-2)(x+2)}{x} = 0 \text{ in the interval } x > 1$$

This can only happen when $x = 2$ and the local extreme point is $(2, f(2)) = (2, 2^2 - \ln 2^8) = (2, 4 - 8\ln 2)$ or $(2, 4 - \ln 256)$.

(c)

$$\begin{aligned} f''(x) &= 2 + \frac{8}{x^2} \quad \text{for } x > 1 \\ &= \frac{2x^2+8}{x^2} \quad \text{for } x > 1 \qquad (*) \end{aligned}$$

We see that $y = 2x^2 + 8$ in (*) is a parabola which have no roots and is always positive, Similarly $y = x^2$ is always positive for $x > 1$.

Thus the second derivative is always positive and so the function is concave up on the interval $(1, \infty)$.

2. Method 1

Let l and b be the length and width of the printing part of the poster: Thus

$$lb = 50$$

and we have

$$l = \frac{50}{b} \quad (*)$$

Let T be the area of the poster i.e.$T = (l+8)(b+4)$ so

$$\begin{aligned} & T(b) \\ = \; & (\frac{50}{b} + 8)(b+4) \text{ using } (*) \\ = \; & 82 + 8b + \frac{200}{b} \end{aligned}$$

Then

$$T^{'}(b) = 8 - \frac{200}{b^2}$$

For a minimum

$$T^{'}(b) = 8 - \frac{200}{b^2} = 0 \Leftrightarrow b^2 = \frac{200}{8} = 25$$

Thus $b = 5$ and $l = 10$ so that the poster has dimensions $18 \times 9cm^2$.
[To see that this values is indeed a minimum we take the second derivative of $T(b)$ i.e. $T^{''}(b) = 400b^{-3}$and we see that $T^{''}(5) = \frac{400}{5^3} > 0$
whihc implies immediately that these dimensions give a poster which meets the requirements.]
Method 2
Let l and b be the length and width of the poster: Thus the printing part will be

$$(l-8)(b-4) = 50$$

and we have

$$l = \frac{50}{b-4} + 8 \; (*)$$

Let T be the area of the poster i.e. $T = lb$ so

$$\begin{aligned} & T(b) \\ = \; & b(\frac{50}{b-4} + 8) \text{ using } (*) \\ = \; & \frac{50b}{b-4} + 8b \end{aligned}$$

Then

$$\begin{aligned} T'(b) &= \frac{(b-4)50-50b}{(b-4)^2} + 8 \\ &= \frac{-200}{(b-4)^2} + 8 \end{aligned}$$

For a minimum

$$T'(b) = \frac{-200}{(b-4)^2} + 8 = 0 \Leftrightarrow (b-4)^2 = \frac{200}{8} = 25$$

Thus $b-4=5$ i.e. $b=9$ and $l=18$ so that the poster has dimensions $18 \times 9cm^2$.

3. (a) $\lim_{x\to 0} \frac{1-\cos^3 x}{\sin^2 x}$

Solution:

$$\begin{aligned} & \lim_{x\to 0} \frac{1-\cos^3 x}{\sin^2 x} \qquad \frac{0}{0} \text{ form} \\ = & \lim_{x\to 0} \frac{-3\cos^2 x(-\sin x)}{2\sin x \cos x} \\ = & \frac{3}{2} \lim_{x\to 0} \cos x \\ = & \frac{3}{2} \quad \text{since } \cos 0 = 1 \end{aligned}$$

(b) $\lim_{x\to\infty} (\cos \frac{1}{x})^x$

Solution:

We have a (1^∞) form so we need to use $e^{\ln x^*} = x^*$

$$\begin{aligned} & \ln \lim_{x\to\infty} (\cos \frac{1}{x})^x \\ = & \lim_{x\to\infty} \frac{\ln(\cos \frac{1}{x})}{\frac{1}{x}} \\ = & \lim_{x\to\infty} \frac{(\frac{1}{\cos \frac{1}{x}}. -\sin \frac{1}{x}. -x^{-2})}{-x^{-2}} \\ = & -\lim_{x\to\infty} (\tan \frac{1}{x}) \\ = & -\tan 0 \\ = & \ 0 \end{aligned}$$

Thus the answer is

$$\lim_{x\to\infty} (\cos \frac{1}{x})^x = e^0 = 1$$

(c) $\lim_{x\to\infty} \frac{x\ln x}{x^2-1}$
Solution:

$$\begin{aligned}
& \lim_{x\to\infty} \frac{x\ln x}{x^2-1} \qquad \frac{\infty}{\infty} \quad \text{form} \\
= & \lim_{x\to\infty} \frac{x.\frac{1}{x}+\ln x}{2x} \\
= & \lim_{x\to\infty} \frac{1+\ln x}{2x} \qquad \text{still an } \frac{\infty}{\infty} \text{ form} \\
= & \lim_{x\to\infty} \frac{\frac{1}{x}}{2} \\
= & \lim_{x\to\infty} \frac{1}{2x} \\
= & \ 0
\end{aligned}$$

(d) $\lim_{x\to 1} \frac{x\ln x}{x^2-1}$
Solution:

$$\begin{aligned}
& \lim_{x\to 1} \frac{x\ln x}{x^2-1} \qquad \frac{0}{0} \quad \text{form} \\
= & \lim_{x\to 1} \frac{x.\frac{1}{x}+\ln x}{2x} \\
= & \lim_{x\to 1} \frac{1+\ln x}{2x} \\
= & \frac{1}{2} \qquad \text{since } \ln 1 = 0
\end{aligned}$$

4. Find the exact value of $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$.

Solution:
We write $\sin^{-1}\left(-\frac{1}{2}\right) = \theta$ then $\sin\theta = -\frac{1}{2}$.
We know that $f(x) = \sin^{-1} x$ is only defined in the first and fourth quadrant and since it is negative in this case the angle θ is made with the $x-axis$ in the fouth quadrant.
Now the value of the side of the triangle on the $x-axis$ is $x = \sqrt{4-1} = \sqrt{3}$
The angle made with the x-axis in the fourt quadrant is thus $-\frac{\pi}{6}$
We then have

$$\tan\theta = \tan(-\frac{\pi}{6}) = -\frac{1}{\sqrt{3}}$$

THINGS TO REMEMBER DEALING WITH INVERSE TRIG FUNCTIONS:
1. The inverse functions are NEVER defined in the 3rd quadrant (see your study guide and look at the graphs and definitions there)
2. The angles θ are always as follows: $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.