Tutorial letter 103/3/2022

Theoretical Computer Science 1

COS1501

School of Computing

This tutorial letter contains solutions to the self-assessment exercises in the study guide. They are also available in the Lessons on myUnisa.

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Dear Students,

This tutorial letter contains solutions to the self-assessment exercises in the study guide. Please note that all page references in this tutorial letter refer to the study guide.

We advise you to draw mind maps for all the study units. There is an example of a mind map for study unit 1 on page 2 of the study guide. You can extend this map by adding information provided in study unit 2.

The (CAI) tutorial *Relations* which is referred to in tutorial letter 101, can be downloaded – please see download instructions and instructions of how to run the tutorial in Tutorial letter 101. *Relations* explains sets and the main properties of relations, such as reflexivity, irreflexivity, symmetry, antisymmetry and transitivity, and also explains the properties of different types of relations. These explanations include animations and graphics. *Relations* offers examples for interactive practice.
Many students struggle with the concepts explained in *Relations* that are discussed in chapters 5 and 6 of the study guide and are covered in assignment 02. We recommend that you use this tutorial before attempting assignment 02. Please note that this tutorial is optional – no exam questions will be based directly on the content of the tutorial.

Everything of the best with your studies this year!
Regards,
The COS1501 team

STUDY UNIT 1

ACTIVITY 1-11:

1. *Factorising:* If we need to factorise an expression of the form $x^2 + ax + b$ or ($x^2 - ax - b$ or $x^2 + ax - b$ or $x^2 - ax + b$) we need to find some c and some d such that $a = c + d$ and $b = (c)(d)$ and $(x + c)(x + d) = x^2 + ax + b$.

(a) $x^2 + 6x + 9 = x{\cdot}x + (3x + 3x) + (3)(3)$
$= (x + 3)(x + 3)$

(b) $x^2 - x - 2 = x{\cdot}x + (x - 2x) + (1)(-2)$
$= (x + 1)(x - 2)$

(c) $x^2 - 5x + 6 = x{\cdot}x - 2x - 3x + (-2)(-3)$
$= (x - 2)(x - 3)$

(d) $x^2 + 4x - 12 = x{\cdot}x - 2x + 6x + (-2)(6)$
$= (x - 2)(x + 6)$

2. *Solve $x^2 - 4x + 4 = 0$ by factorising.*

Well, at school we learned that:
the '+' in front of the last term means that our factorised version
$x^2 - 4x + 4$ will be either of the form $(x + ?)(x + ?)$ or of the form $(x - ?)(x - ?)$,
and that the '–' in front of the middle term tells us that our factorised version must be of the form $(x - ?)(x - ?)$.

Experimenting with numbers that, when multiplied give us 4, we soon find that our factorised version has to be $(x - 2)(x - 2)$, or $(x - 2)^2$ if you prefer.
Our equation $x^2 - 4x + 4 = 0$ may therefore be rewritten as $(x - 2)^2 = 0$.
By Property 9, at least one of the factors on the left-hand side must be zero, and both factors are $(x - 2)$, so we get that $x - 2 = 0$
ie that $x = 2$.

3. *Complete the square to solve $x^2 - 4x = 12$.*

If	$x^2 - 4x$	=	12	
then	$x^2 - 4x + 4 - 4$	=	12	(by Property 8, since $4 - 4 = 0$)
ie	$x^2 - 4x + 4$	=	12 + 4	(by Property 6 with $k = 4$)
ie	$(x - 2)^2$	=	16	(factorise)
ie	$x - 2 = 4$ or $x - 2$	=	−4	(taking square roots)
ie	$x = 6$ or x	=	−2	(using Property 6 again, with $k = 2$).

4. *Is 21 a prime number?*

No. Refer to the definition of prime numbers on p 16.
The numbers 3 and 7 are factors of 21 ($3 \times 7 = 21$).

5. *What is the value of 5! (5 factorial)?*

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

STUDY UNIT 2

ACTIVITY 2-8

1. *Define the words "even" and "odd" for positive integers.*

Definitions:

- An integer n is *even* if n is a multiple of 2.

(We can say a positive integer n is even if $n = 2k$ for some positive integer k. You can think of even positive integers as numbers n of the form $n = 2k$, where k is some positive integer.)

- An integer n is *odd* if n is not even.

(Using the general form of an even positive integer, we can now say that n is odd if $n = 2k + 1$ for some positive integer k. You can think of odd positive integers as numbers n of the form $n = 2k + 1$, where k is some positive integer.)

2. *Is it the case that $m + (n \cdot k) = (m + n)(m + k)$ for all positive integers m, n and k?*

Substitute a few values and see whether the idea is plausible.
Take $m = 1$, $n = 2$, and $k = 3$, then the left-hand side is
$m + (n \cdot k) = 1 + (2 \cdot 3) = 7$
while the right-hand side becomes
$(m + n)(m + k) = (1 + 2) \cdot (1 + 3) = 3 \cdot 4 = 12$.
This is a counterexample to show that it is not the case that $m + (n \cdot k) = (m + n)(m + k)$ for all positive integers m, n and k.

3. *Are there any even prime numbers besides 2?*

No. Any even prime number other than 2 would have three factors: 1, because 1 is a factor of every number; 2, because the number we are talking about is supposedly even; and the number itself, because a number is always a factor of itself. But primes cannot have so many factors, which means that 2 is the only even prime number.

4. *If m and n are even positive integers, is m + n even?*

If m and n are even positive integers, then each is a multiple of 2, in other words

	m	$= 2k$ for some $k \in \mathbb{Z}^+$.
and	n	$= 2j$ for some $j \in \mathbb{Z}^+$.
So	$m + n$	$= 2k + 2j$
		$= 2(k + j)$

which means $m + n$ is also even.

5. *If m and n are odd positive integers, is m·n odd?*

If m and n are odd positive integers, then both m and n can be written in the following general form:

	m	=	$2k + 1$ for some k in $\mathbb{Z}^+$.
and	n	=	$2j + 1$ for some j in $\mathbb{Z}^+$.
So	m·n	=	$(2k + 1)(2j + 1)$
		=	$4kj + 2k + 2j + 1$
		=	$2(2kj + k + j) + 1$

which means that m·n is odd .

An additional exercise:

If m and n are prime, is m + n and m − n prime?

No, not usually.

It can occasionally happen that m + n is also prime: take m = 3 and n = 2 then m + n = 5.

But for other values m + n may not be prime: take m = 3 and n = 7, for instance. 3 + 7 = 10 which is not a prime number.

What about the difference between two prime numbers? The difference m − n will sometimes be prime and sometimes not. E.g. 5 – 3 = 2, which is a prime number, but 23 – 3 = 20, which is not prime.

STUDY UNIT 3

ACTIVITY 3-3

1. *In each of the following cases, describe the set more concisely, firstly using list notation and then using set-builder notation.*

(a) list notation: $\{0, 2, 4, 6, 8\}$.
set-builder notation: $\{x \in \mathbb{Z}^{\geq} \mid x$ is an even non-negative integer and $x < 10\}$ (property description)

(b) The roster method: $\{-11, -9, -7, -5, -3, -1\}$.
There is more than one way (we give only two) to describe this set using set-builder notation:
$\{x \in \mathbb{Z} \mid x$ is an odd negative integer and $x > -13\}$ (property description)
or, if you prefer:
$\{y \in \mathbb{Z} \mid y$ is odd and $-13 < y < 0\}$ (property description)

(c) The roster method: $\{\ \} = \emptyset$
(because there does not exist an integer that is, at the same time, positive and less than 1).
In set-builder notation:
One possibility is $\{x < 1 \mid x \in \mathbb{Z}^+\}$.

(d) Because of the nature of real numbers, ie between any two real numbers a and b one can always find another real number, namely $(a + b)/2$, it is not really possible to represent this set using the roster method. In set-builder notation: $\{x \in \mathbb{R} \mid x > 2\}$.

2. *In each of the following cases, give an unambiguous description in English.*

(a) $\{-1, 0, 1\}$:
One possibility is to speak of the set having −1, 0 and 1 as its only elements.
Another is to speak of the set of all integers greater than −2 and less than 2.

(b) $\{ x \in \mathbb{R} \mid 0 < x < 1 \}$: The set of all real numbers greater than 0 and less than 1.

(c) $\{0\}$:
Again we can give many descriptions of this set in English:
- The set having 0 as its only element.
- The set of all non-negative integers less than 1.
- The set of all integers greater than -1 and less than 1.
- The set of all integers that are simultaneously not positive and not negative.

(d) $\{\mathbb{Z}\}$: The set containing the set of integers, $\mathbb{Z}$, as its only member.

Does it bother you to have a set like $\mathbb{Z}$ as an element inside another set? Remember that the purpose of a set is just to group together the things we are interested in, and these things may well be sets themselves. If we are interested in the number sets, we may group together as elements not just $\mathbb{Z}$ but also $\mathbb{Z}^+$, $\mathbb{Z}^{\geq}$, $\mathbb{Q}$ and $\mathbb{R}$ to form the set $\{\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^{\geq}, \mathbb{Q}, \mathbb{R}\}$, and so on.

ACTIVITY 3-6

1. Let U = {1, 2, 3, 4, 5}, A = {1, 2, 3} and B = {3, 4, 5}. Determine the required sets.

(a) $A \cup B = \{1, 2, 3, 4, 5\} = B \cup A$ (1, 2, 3, 4 and 5 are elements of A or B or both.)

(b) $A \cap B = \{3\} = B \cap A$ (3 is an element of both A and B)

(c) $A - B = \{1, 2\}$ (1 and 2 are elements of A but not of B)
$B - A = \{4, 5\}$ (4 and 5 are elements of B but not of A)

(d) $A + B = \{1, 2, 4, 5\} = B + A$ (1, 2, 4 and 5 are elements that belong either to A or to B but not to both)

2. *Let U = {a, e, i, o, u}, A = {i, o, u} and B = {a, e, o, u}. Determine the following sets:*

(a) $A' = \{a, e\}$ (a and e are elements of U but not of A)
$(A')' = \{i, o, u\}$
$= A$

(b) $B' = \{i\}$
$(B')' = \{a, e, o, u\}$

(c) $A \cup B = \{a, e, i, o, u\}$
$(A \cup B)' = \emptyset$

(d) $A' \cap B' = \{a, e\} \cap \{i\}$
$= \emptyset$ (We can also refer to this set merely as $A' \cap B'$.)

(e) $A \cap B = \{o, u\}$
$(A \cap B)' = \{a, e, i\}$

(f) $A' \cup B' = \{a, e\} \cup \{i\}$
$= \{a, e, i\}$ (Can also be called $A' \cup B'$.)

(g) $A - B = \{i\}$
$B - A = \{a, e\}$

(h) $A \cap B' = \{i, o, u\} \cap \{i\}$
$= \{i\}$ (or $A \cap B'$)
$B \cap A' = \{a, e, o, u\} \cap \{a, e\}$
$= \{a,e\}$ (or $B \cap A'$)

(i) $A + B = \{i, o, u\} + \{a, e, o, u\}$
$= \{a, e, i\}$
$B + A = \{a, e, i\}$

3. *Let $U = \{1, 2, 3, 4, 5\}$, $A = \{3\}$ and $B = \{\{3\}, 4, 5\}$. Determine $\mathcal{P}(A)$ and $\mathcal{P}(B)$.*

***Note**: If the cardinality of some finite set C is n (ie $|C| = n \geq 0$), then a total of 2^n subsets of C can be formed, so $|\mathcal{P}(C)| = 2^n$. In the case of $B = \{\{3\}, 4, 5\}$, B has 3 elements namely $\{3\}$, 4 and 5, so the power set of B, namely $\mathcal{P}(B)$ has $2^3 = 8$ elements.*

$\mathcal{P}(A) = \{ \emptyset, \{3\} \}$ (All the subsets of $A = \{3\}$ are elements of $\mathcal{P}(A)$.)

$\mathcal{P}(B) = \{ \emptyset, \{\{3\}\}, \{4\}, \{5\}, \{\{3\}, 4\}, \{\{3\}, 5\}, \{4, 5\}, \{\{3\}, 4, 5\} \}$

Subsets of **B** are *members* of $\mathcal{P}$ **(B).** We determine two *members* of $\mathcal{P}$ **(B)**:

$\{3\}$, $\underline{4}$ and $\underline{5}$ are *members* of **B** $= \{\{3\}, \underline{4}, \underline{5}\}$.
We form subsets of B:
Keep the outside brackets of B then throw away the members $\underline{4}$ and $\underline{5}$ of B then we are left with the *subset* $\{\{3\}\}$ of B which is then a *member* of $\mathcal{P}$ **(B)**. (Note that $\{3\}$ is **not** a member of $\mathcal{P}$ (B).)
Keep the outside brackets of B then throw away the members $\{3\}$ and $\underline{4}$ of B then we are left with the *subset* $\{\underline{5}\}$ of B which is then a member of $\mathcal{P}$ (B). All the subsets of B are the members of $\mathcal{P}$ (B).

4. *Let U = {a, e, i, o, u}, A = {i, o, u} and B = {a, e, o, u}. Determine the following sets:*

(a) $\mathcal{P}$(A) = **{** ∅, {i}, {o}, {u}, {i, o}, {i, u}, {o, u}, {i, o, u} **}**

$\mathcal{P}$(B) = **{** ∅, {a}, {e}, {o}, {u}, {a, e}, {a, o}, {a, u},
{e, o}, {e, u}, {o, u}, {a, e, o}, {a, o, u},
{a, e, u}, {e, o, u}, {a, e, o, u} **}**

(b) $\mathcal{P}$(A ∩ B) = $\mathcal{P}$({o, u}) = **{** ∅, {o}, {u}, {o, u} **}**

$\mathcal{P}$(A) ∩ $\mathcal{P}$(B) = **{** ∅, {o}, {u}, {o, u} **}**

(c) $\mathcal{P}$(A′) = $\mathcal{P}$({a, e}) = **{** ∅, {a}, {e}, {a, e} **}**

In order to be able to determine ($\mathcal{P}$(A))′, we need to determine $\mathcal{P}$(U) first.

$\mathcal{P}$(U) = **{** ∅, {a}, {e}, {i}, {o}, {u},
{a, e}, {a, i}, {a, o}, {a, u}, {e, i},
{e, o}, {e, u}, {i, o}, {i, u}, {o, u},
{a, e, i}, {a, i, o}, {a, o, u}, {a, i, u},
{a, e, o}, {a, e, u}, {e, i, o}, {e, o, u},
{e, i, u}, {i, o, u}, {a, e, i, o}, {a, e, i, u},
{a, e, o, u}, {a, i, o, u}, {e, i, o, u},
{a, e, i, o, u} **}**

($\mathcal{P}$(A))′ = **{** {a}, {e}, {a, e}, {a, i}, {a, o}, {a, u},
{e, i}, {e, o}, {e, u}, {a, e, i}, {a, i, o},
{a, o, u}, {a, i, u}, {a, e, o}, {a, e, u},
{e, i, o}, {e, o, u}, {e, i, u}, {a, e, i, o},
{a, e, i, u}, {a, e, o, u}, {a, i, o, u},
{e, i, o, u}, {a, e, i, o, u} **}**

(d) $\mathcal{P}$(A) ∪ $\mathcal{P}$(B)
= **{** ∅, {i}, {o}, {u}, {a}, {e}, {i, o}, {i, u}, {o, u}, {a, e}, {a, o}, {a, u},
{e, o}, {e, u}, {i, o, u}, {a, e, o}, {a, o, u}, {a, e, u}, {e, o, u}, {a, e, o, u} **}**

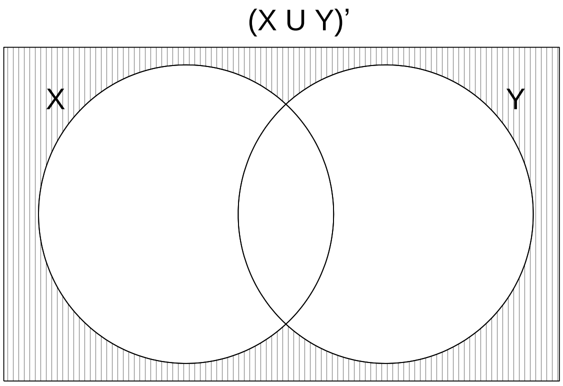
$\mathcal{P}$(A ∪ B) = $\mathcal{P}$({a, e, i, o, u})
= $\mathcal{P}$(U)

STUDY UNIT 4

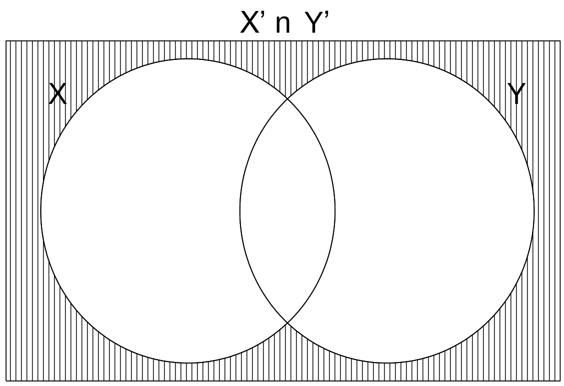
ACTIVITY 4-4

1. *Draw the following diagrams:*

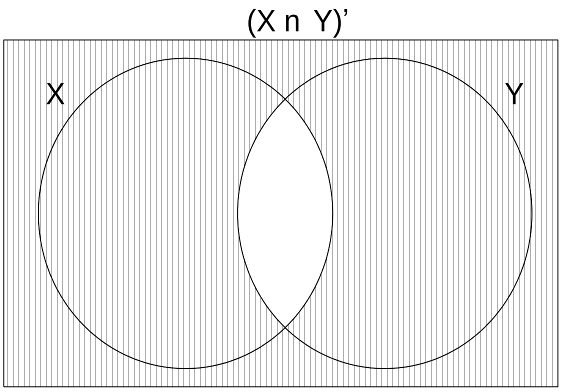
1(a) $(X \cup Y)'$ (First draw a Venn diagram for $X \cup Y$.)



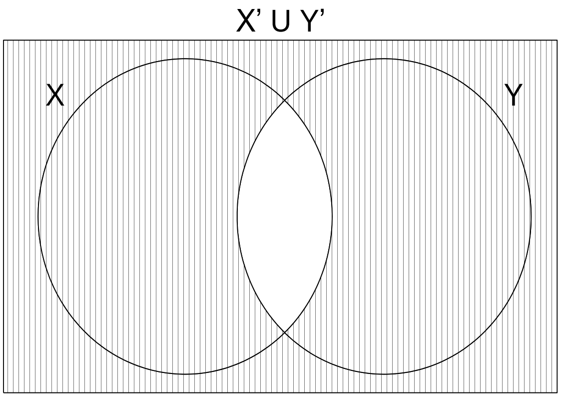
1(b) $X' \cap Y'$ (First draw Venn diagrams for X' and Y'.)



1(c) $(X \cap Y)'$ (First draw a Venn diagram for $X \cap Y$.)



1(d) $X' \cup Y'$

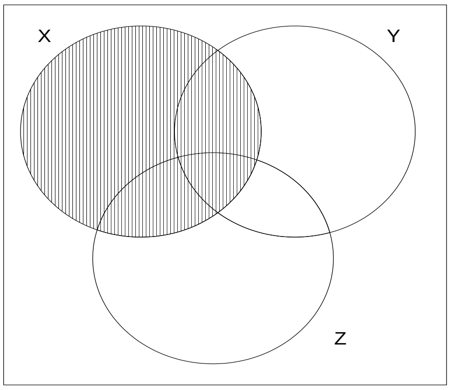


2. *Draw the following Venn diagrams:*

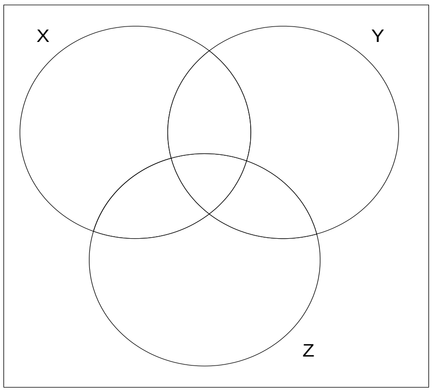
(a) **Note**: Draw the universal set for each diagram; draw all three sets in each diagram; name all three sets; provide a subscript for each diagram; colour in only the area relevant to the subscript.

X − (Y ∪ Z):

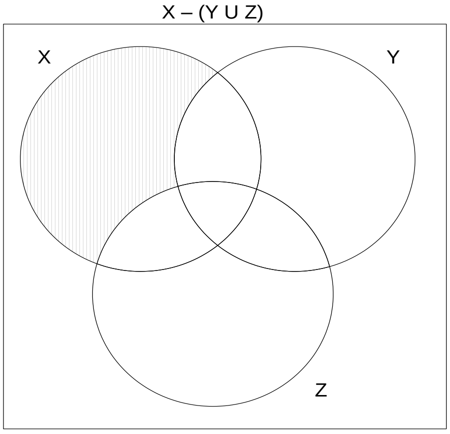
X



Y U Z

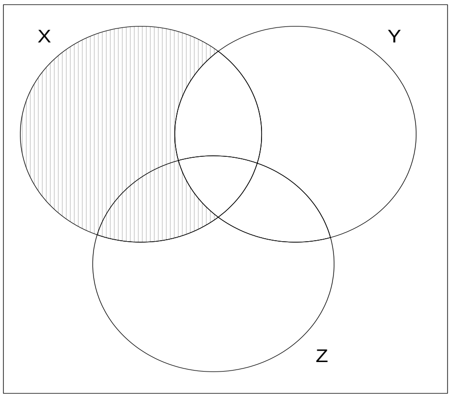


X – (Y U Z)

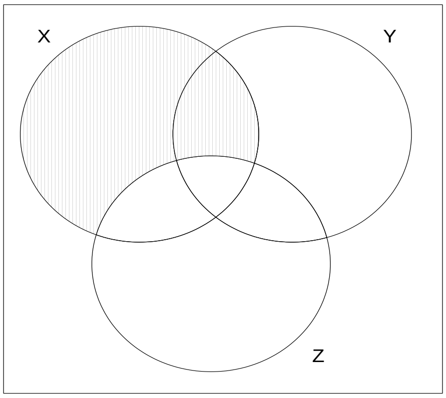


2(b) (X−Y) ∪ (X−Z)

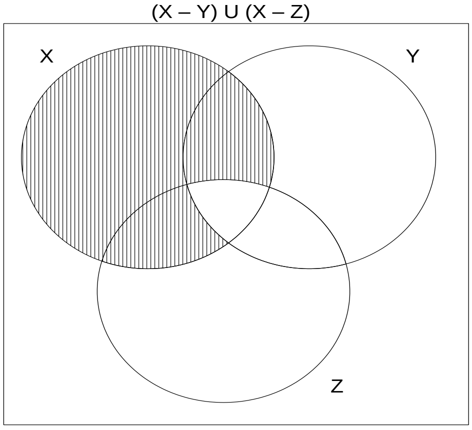
X – Y



X – Z

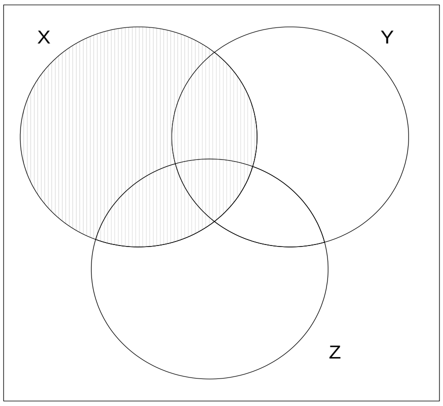


(X – Y) U (X – Z)

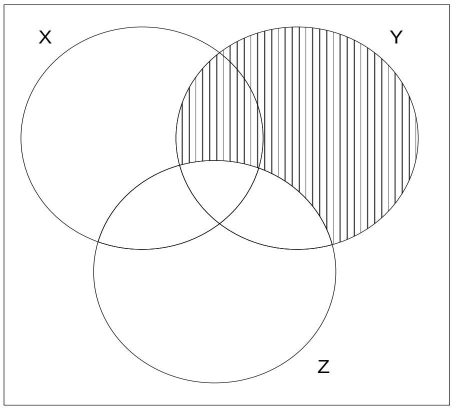


2(c) $X \cap (Y - Z)$

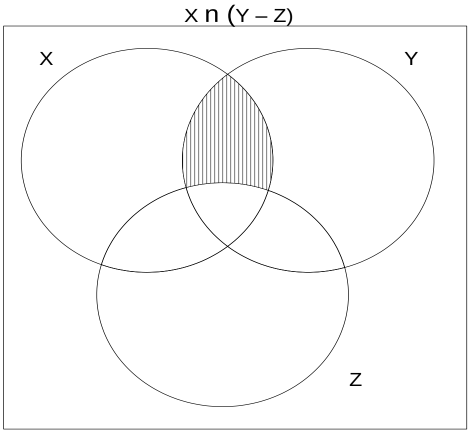
X



Y - Z

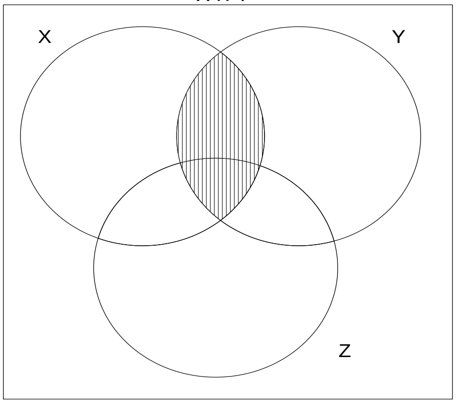


X n (Y – Z)

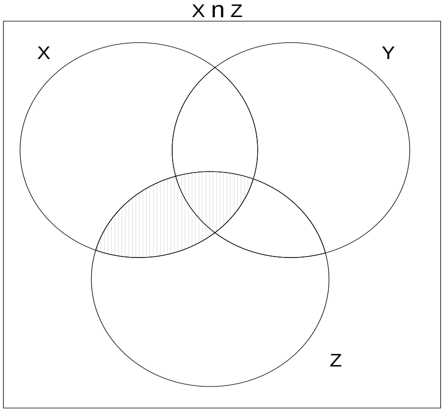


2(d) $(X \cap Y) - (X \cap Z)$

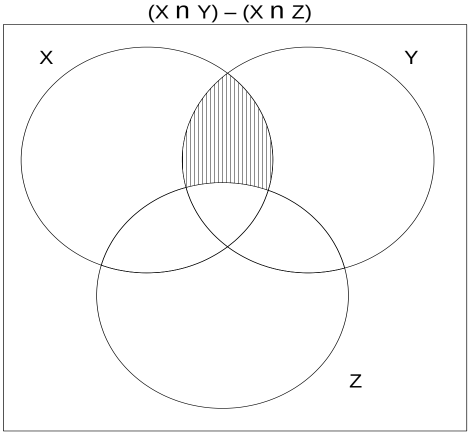
X n Y



X n Z

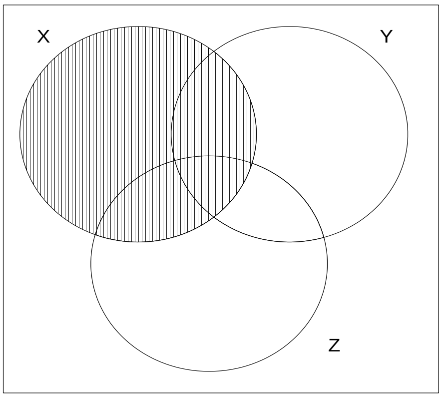


(X n Y) – (X n Z)

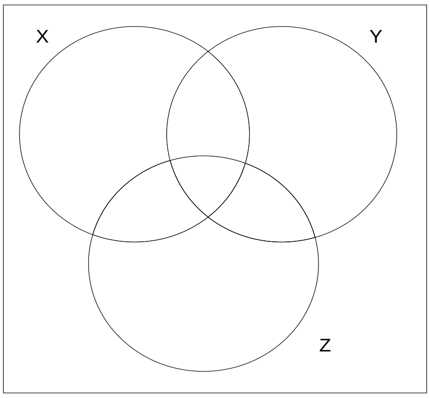


2(e) $X \cap (Y + Z)$

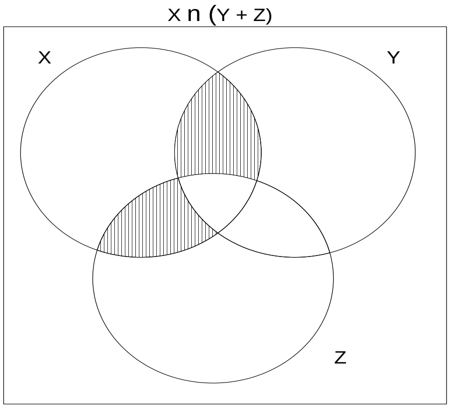
X



Y + Z

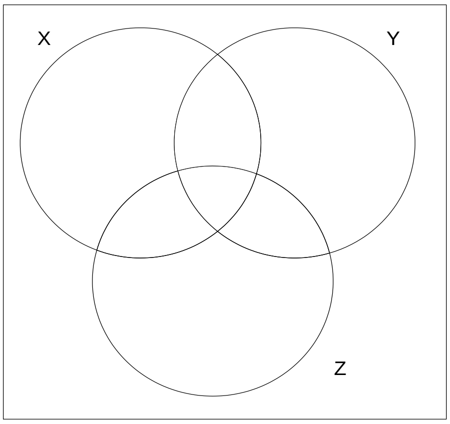


X n (Y + Z)

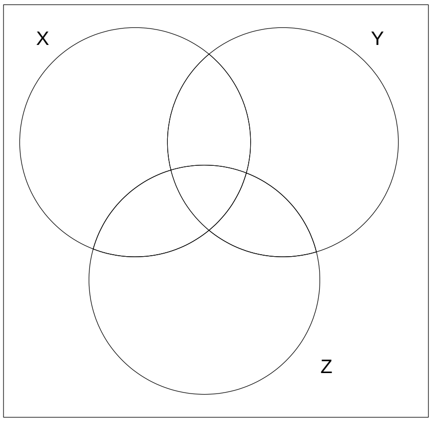


2(f) $(X \cap Y) + (X \cap Z)$

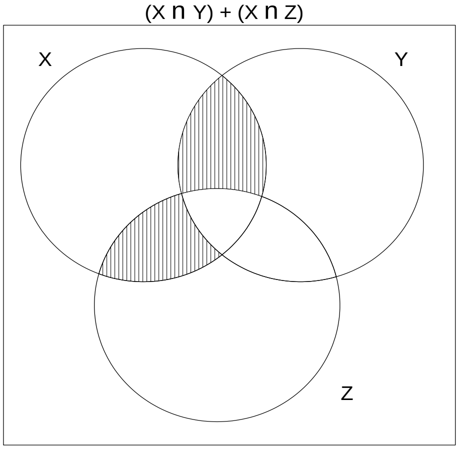
X n Y



X n Z



(X n Y) + (X n Z)

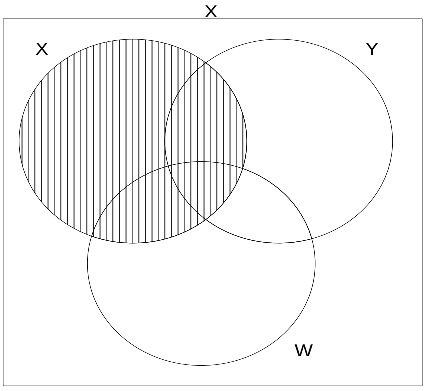


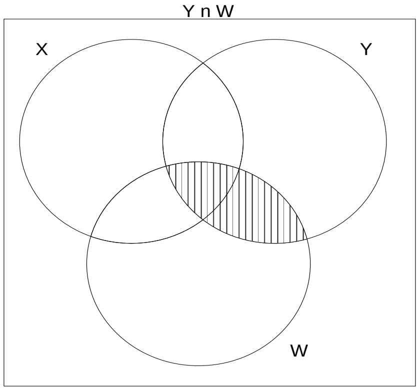
ACTIVITY 4-5

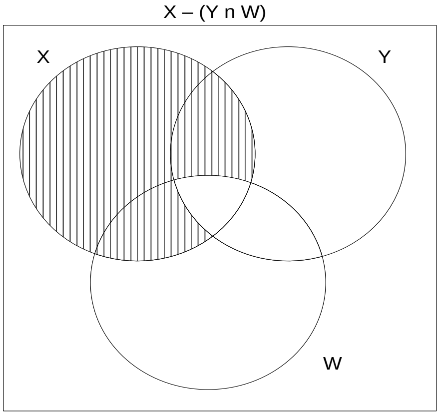
Use Venn diagrams to determine whether or not the given equations hold.

1. (a) Is $X - (Y \cap W) = (X - Y) \cup (X - W)$?

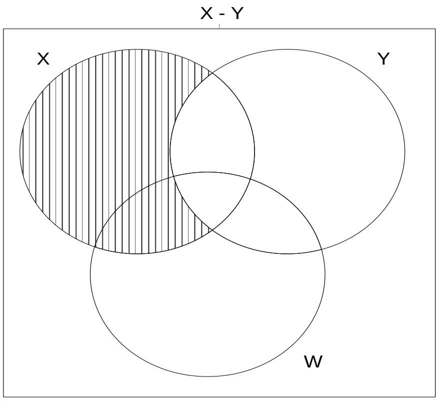
Left-hand side: $X - (Y \cap W)$

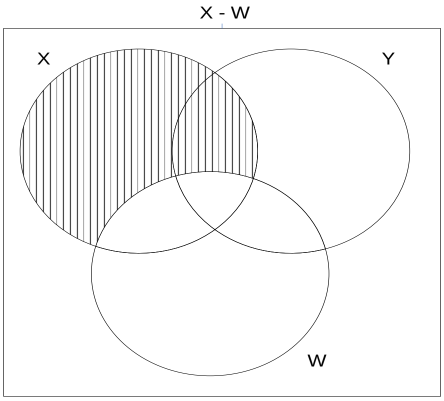


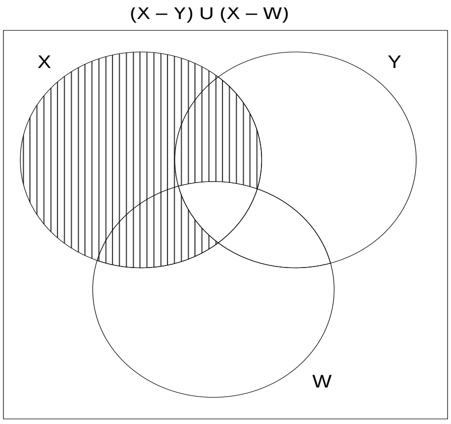




Right-hand side: $(X - Y) \cup (X - W)$



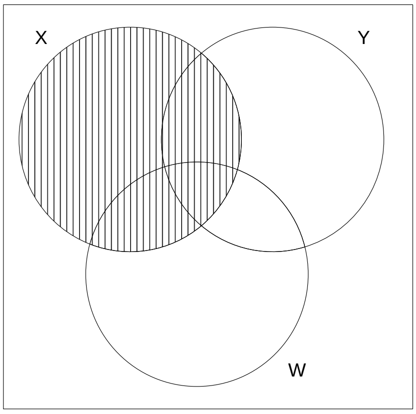




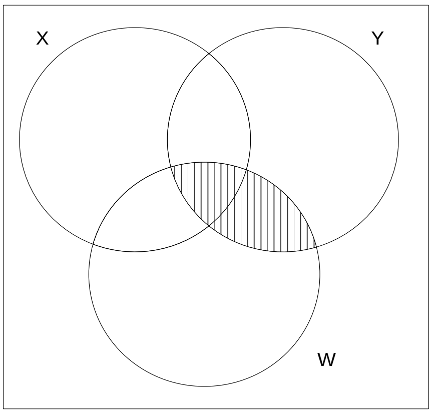
(b) Is $X \cap (Y \cap W) = (X \cap Y) \cap W$?

Left-hand side: $X \cap (Y \cap W)$

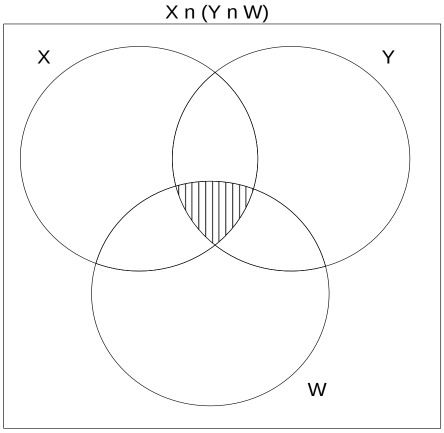
X



Y n W

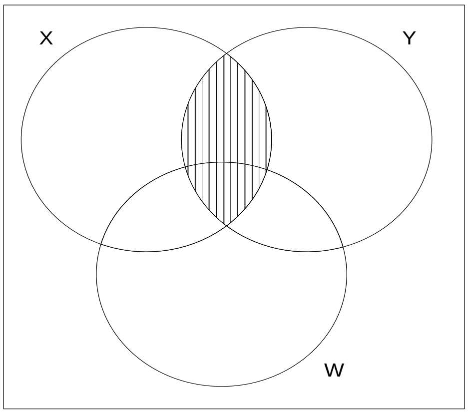


X n (Y n W)

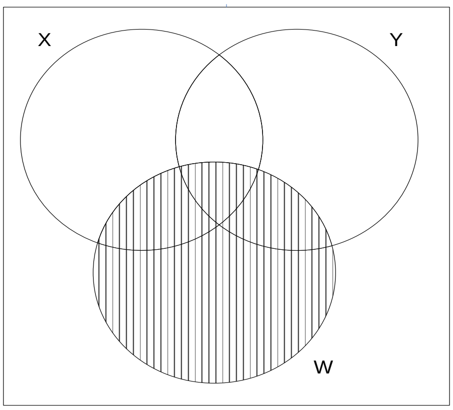


Right-hand side: $(X \cap Y) \cap W$

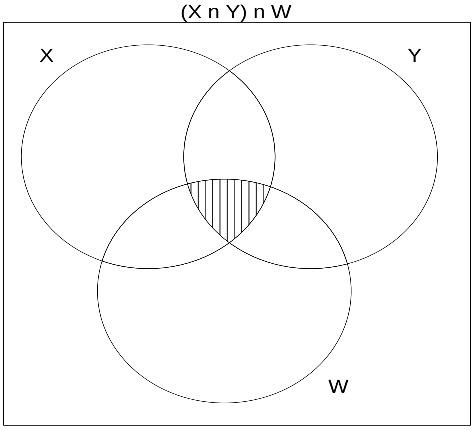
X n Y



W



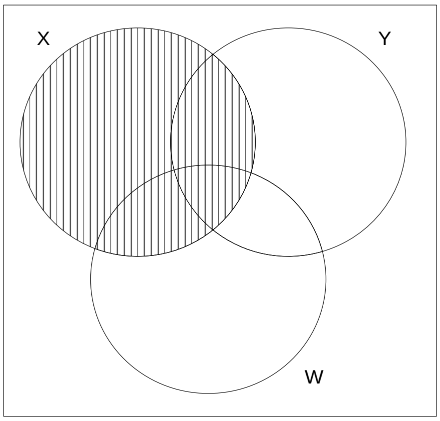
(X n Y) n W



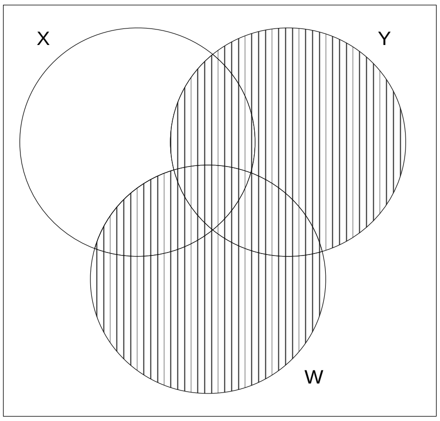
(c) Is $X \cap (Y \cup W) = (X \cap Y) \cup (X \cap W)$?

Left-hand side: $X \cap (Y \cup W)$

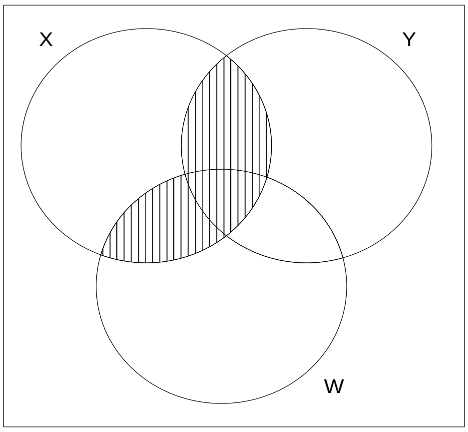
X



Y U W

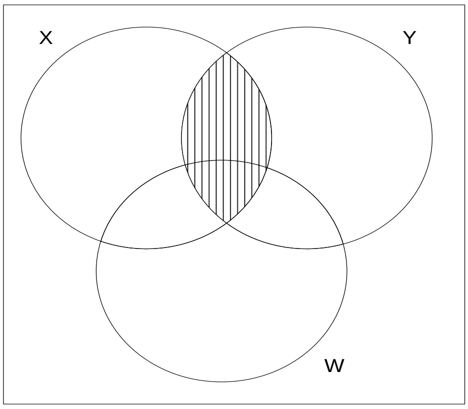


X n (Y U W)

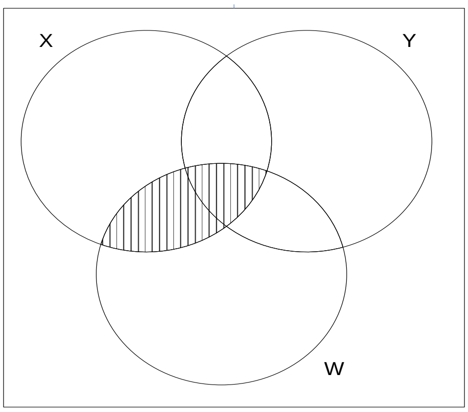


Right-hand side: $(X \cap Y) \cup (X \cap W)$

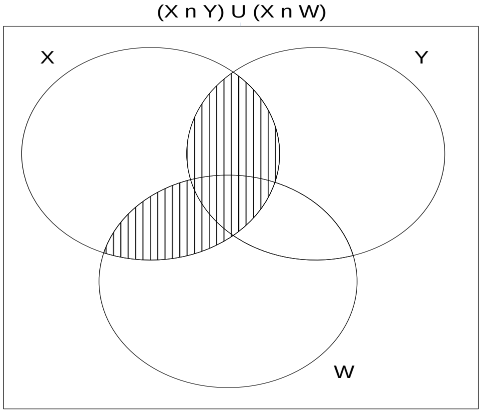
X n Y



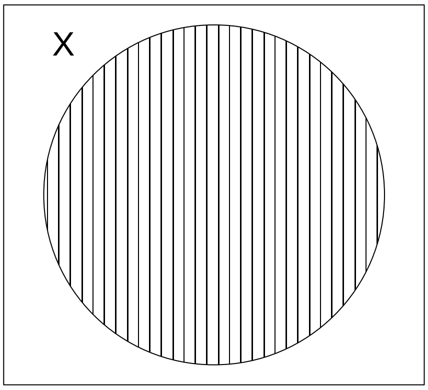
X n W



(X n Y) U (X n W)



(d) The Venn diagrams for both $(X')'$ and X look like this:



ACTIVITY 4-6

Using if and only if statements, write out a proof in words for each of the following identities, where X, Y and W are arbitrary subsets of a universal set U:

1(a) $(X')' = X$
It seems we should be able to produce a cast-iron proof.

$x \in (X')'$
iff $x \notin X'$
iff $x \in X$.
Therefore $(X')' = X$.

(b) $X - (Y \cap W) = (X - Y) \cup (X - W)$

$x \in X - (Y \cap W)$
iff $x \in X$ and $x \notin (Y \cap W)$
iff $x \in X$ and ($x \in Y'$ or $x \in W'$)
iff ($x \in X$ and $x \in Y'$) or ($x \in X$ and $x \in W'$)
iff $x \in (X - Y)$ or $x \in (X - W)$
iff $x \in (X - Y) \cup (X - W)$
Thus $X - (Y \cap W) = (X - Y) \cup (X - W)$ for all subsets X, Y and W of U.

(c) $X \cap (Y \cap W) = (X \cap Y) \cap W$

$x \in X \cap (Y \cap W)$
iff $x \in X$ and $x \in (Y \cap W)$
iff $x \in X$ and ($x \in Y$ and $x \in W$)
iff ($x \in X$ and $x \in Y$) and $x \in W$
iff $x \in (X \cap Y)$ and $x \in W$
iff $x \in (X \cap Y) \cap W$.

We can conclude that $X \cap (Y \cap W) = (X \cap Y) \cap W$ for all subsets X, Y and W of U.

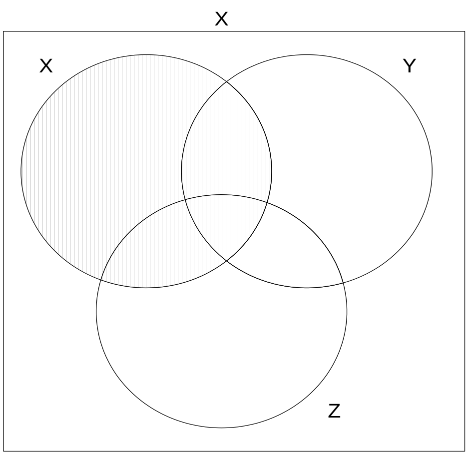
(d) $X \cap (Y \cup W) = (X \cap Y) \cup (X \cap W)$

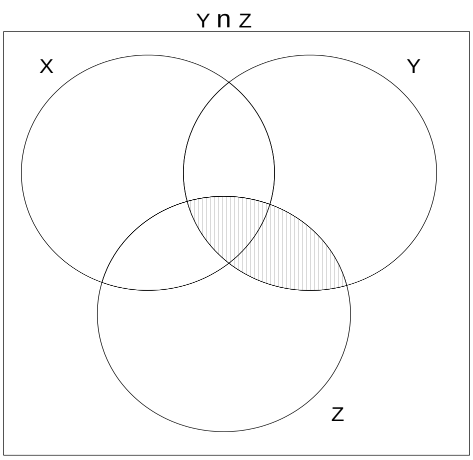
$x \in X \cap (Y \cup W)$
iff $x \in X$ and $x \in (Y \cup W)$
iff $x \in X$ and ($x \in Y$ or $x \in W$)
iff ($x \in X$ and $x \in Y$) or ($x \in X$ and $x \in W$)
iff ($x \in X \cap Y$) or ($x \in X \cap W$)
iff $x \in (X \cap Y) \cup (X \cap W)$.
Thus $X \cap (Y \cup W) = (X \cap Y) \cup (X \cap W)$.

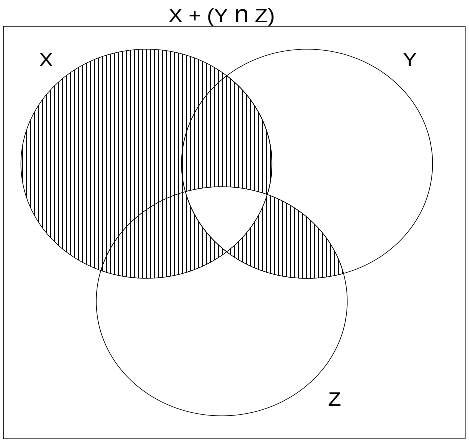
ACTIVITY 4-8

1. *Is it the case that for all $X, Y, Z \subseteq U$, $X + (Y \cap Z) = (X + Y) \cap (X + Z)$?*

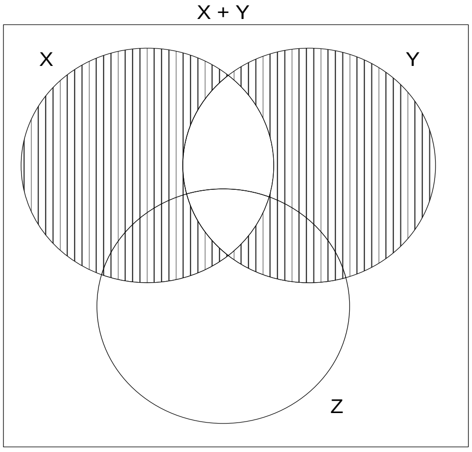
Determine the Venn diagram for $X + (Y \cap Z)$:

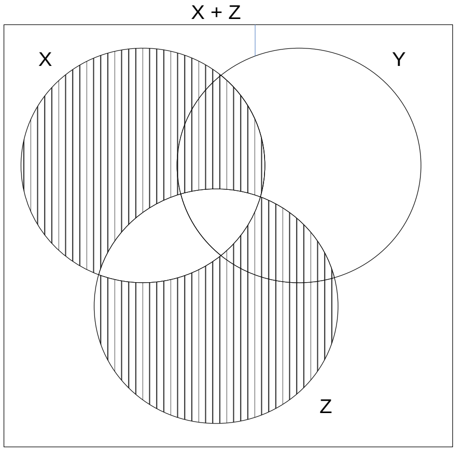


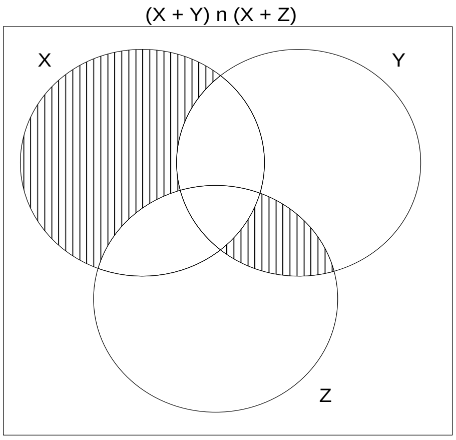




Determine the Venn diagram for (X + Y) $\cap$ (X + Z):







From the Venn diagrams it is clear that it is *not* the case that for all X, Y, Z $\subseteq$ U,
X + (Y $\cap$ Z) = (X + Y) $\cap$ (X + Z).

2. *Find examples of sets A and B such that $\mathcal{P}$(A $\cup$ B) is not a subset of $\mathcal{P}$(A) $\cup$ $\mathcal{P}$(B).*

This means we need to find a counterexample to show that it is not the case that $\mathcal{P}$(A $\cup$ B) is a subset of $\mathcal{P}$(A) $\cup$ $\mathcal{P}$(B) for all sets A and B of U.

Let a universal set be U = {1, 2 } and let A = {1} and B = {2}.

$\mathcal{P}$(A) = **{** $\emptyset$, {1} **}**

$\mathcal{P}$(B) = **{** $\emptyset$, {2} **}**

$\mathcal{P}$(A) $\cup$ $\mathcal{P}$(B)= **{** $\emptyset$, {1}, {2}**}**

$\mathcal{P}$(A $\cup$ B) = $\mathcal{P}$({1, 2})

= { $\emptyset$, {1}, {2}, {1, 2}**}**

3. *Is it the case that, for all X, Y $\subseteq$ U, $\mathcal{P}(X) \cap \mathcal{P}(Y) = \mathcal{P}(X \cap Y)$? Justify your answer.*

S $\in \mathcal{P}$(X) $\cap\ \mathcal{P}$(Y)

iff S $\in \mathcal{P}$(X) and S $\in \mathcal{P}$(Y)

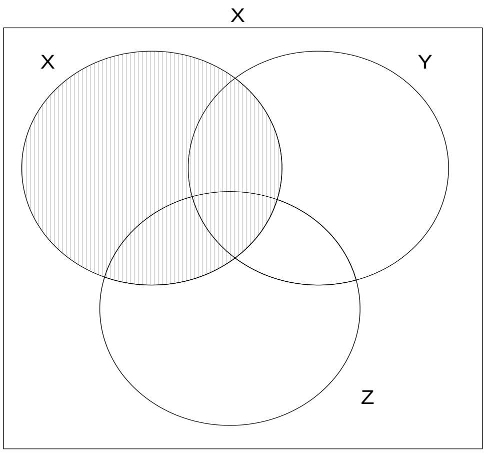
iff S $\subseteq$ X and S $\subseteq$ Y
iff the elements of S all belong to X and all belong to Y
iff the elements of S all belong to X $\cap$ Y
iff S $\subseteq$ X $\cap$ Y
iff S $\in \mathcal{P}$(X $\cap$ Y).

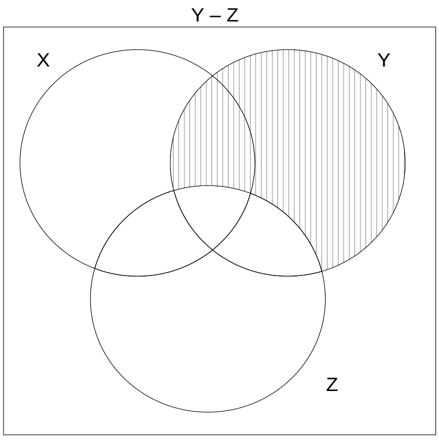
This proves that $\mathcal{P}(X) \cap \mathcal{P}(Y) = \mathcal{P}(X \cap Y)$.

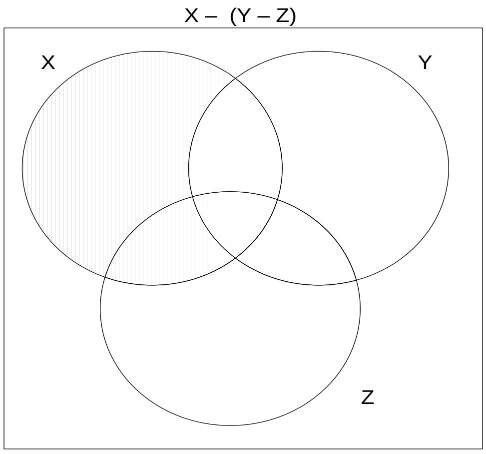
4. *Use Venn diagrams to investigate whether or not, for all sets X, Y, Z $\subseteq$ U,*
X – (Y – Z) = (X – Y) – Z. If the statement appears to hold, give a proof; if not, give a counterexample.

We draw the Venn diagrams as follows:

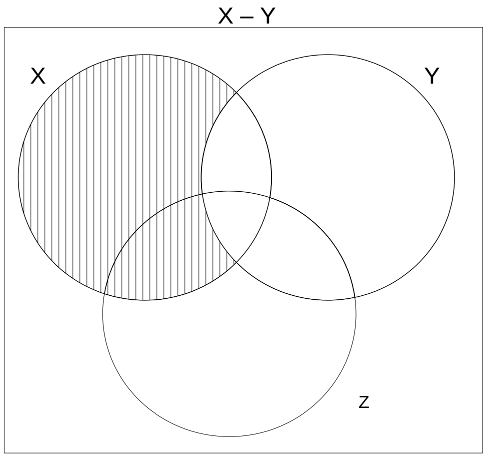
Left-hand side:

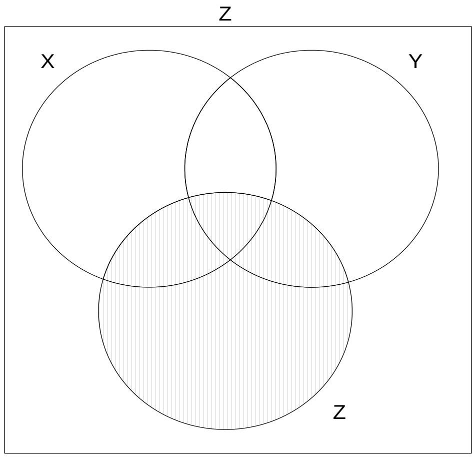


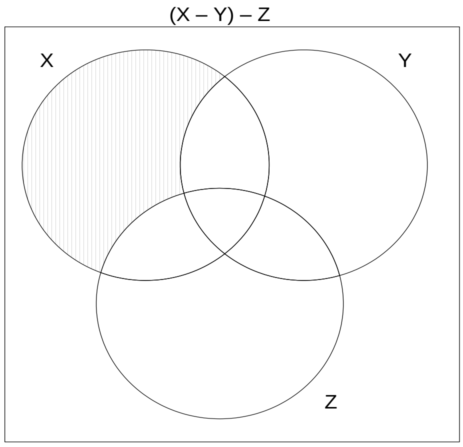




Right-hand side:







The shaded areas representing X − (Y − Z) and (X − Y) − Z differ, and therefore it seems as if the claim that X − (Y − Z) = (X − Y) − Z is not always true. We have to give a counterexample with specific values for X, Y, and Z. The two final Venn diagrams differ in the region $X \cap Z$, so we choose, for example, $1 \in X$ and $1 \in Z$.

Let X = {1, 2}, Y = {2, 3}, and Z = {1, 3},
then X − (Y − Z) = {1, 2} - {2} = {1}.
On the other hand, (X - Y) - Z = {1} - {1,3} = { }.

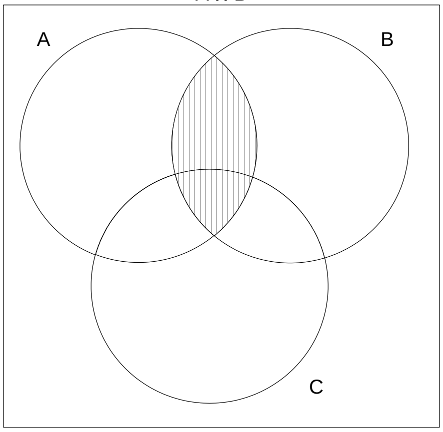
So, in this case, X − (Y−Z) ≠ (X−Y) − Z.

5. *Use Venn diagrams to investigate whether or not, for all subsets A, B and C of U,*
$(A \cap B) + (C \cap A) = (A \cap B') \cup (B - C)$. If the statement appears to hold, give a proof; if not, give a counterexample.

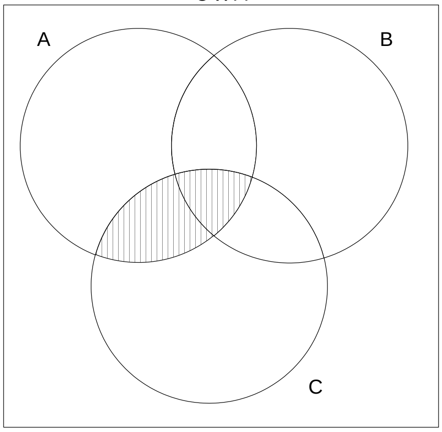
We draw the Venn diagrams as follows:

Left-hand side:

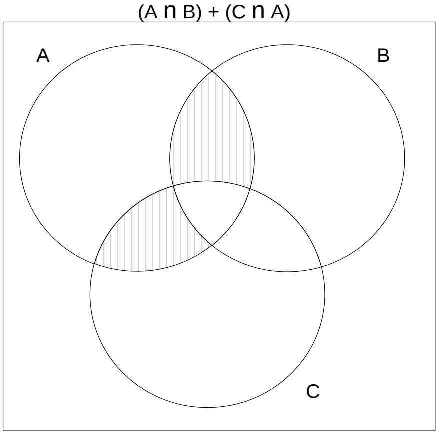
$A \cap B$



$C \cap A$

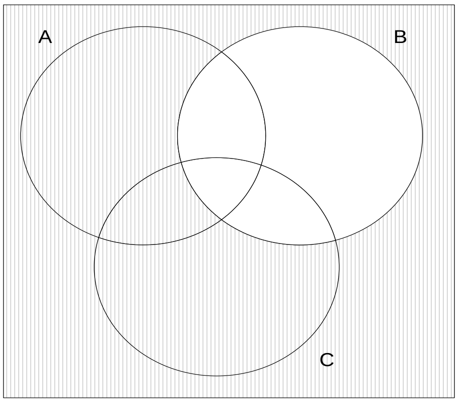


$(A \cap B) + (C \cap A)$

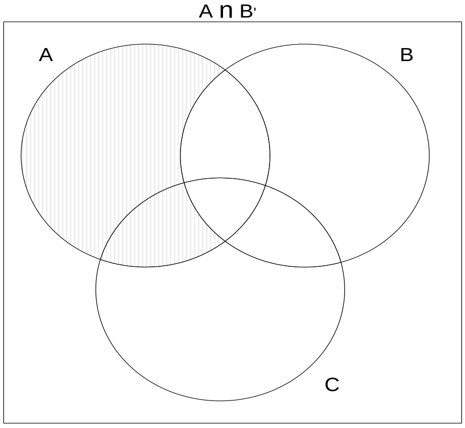


Right-hand side:

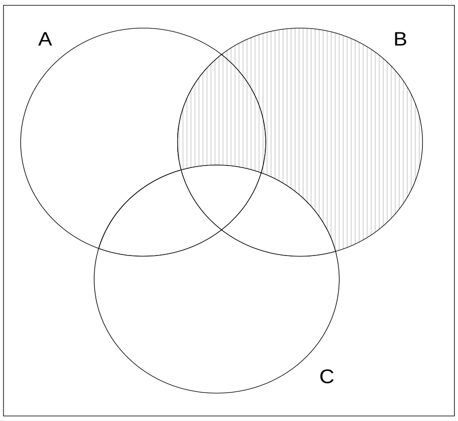
B'



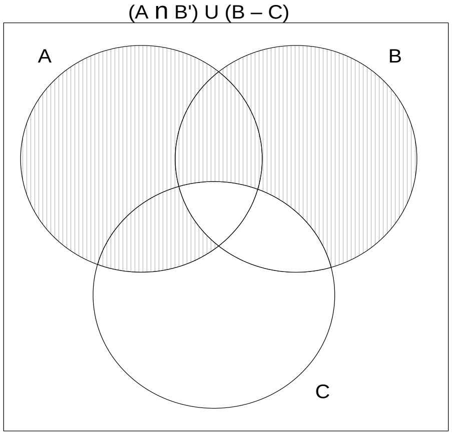
$A \cap B'$



$B - C$



$(A \cap B') \cup (B - C)$



It appears that the expression is not an identity, so we need a counterexample. That is, we want a concrete example of sets which shows that the left-hand side is different from the right-hand side.

Counterexample:

The final diagrams differ in some areas of B. We choose the element 2 that resides in set B only.

Let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 3\}$ and $U = \{1, 2, 3\}$ with U as the universal set.

Determine B′:

$B' = U - B = \{3\}$

Now $(A \cap B) + (C \cap A)$ (Determine which members reside in either $A \cap B$ or $C \cap A$,

$= \{1\} + \{1\}$ but not in both $A \cap B$ and $C \cap A$.)

$= \{\ \}$

while $(A \cap B') \cup (B - C)$ (Determine which members reside in $A \cap B'$ or $B - C$.)

$= \{\ \} \cup \{2\}$

$= \{2\}$.

In this example it is not the case that $(A \cap B) + (C \cap A) = (A \cap B') \cup (B - C)$.

ACTIVITY 4-10

1. *Suppose that of 1000 first-year students, 700 take Mathematics, 400 take Computer Science and 800 take Mathematics or Computer Science.*

(a) *How many take Mathematics and Computer Science?*

Let U be the set of first-year students, M the set of those taking Mathematics, and C the set of those taking Computer Science. Then

$|U| = 1000$,

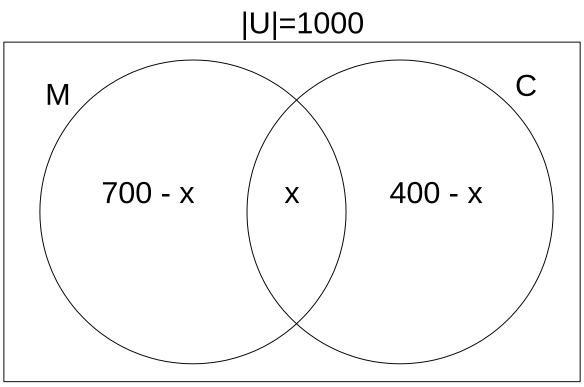
$|M| = 700$,

$|C| = 400$,

$|M \cup C| = 800$ and

$|M \cap C| = x$ (We do not know how many take Mathematics and Computer Science.)

By using this information, we can fill in the number of elements that reside in each region of the two sets, starting with the region in the middle that has x elements, and then $|M - C| = 700 - x$ and $|C - M| = 400 - x$.



We add the number of elements living in the three regions of the sets M and C, and since the total number of elements that reside in these regions is $|M \cup C| = 800$. We can determine x.

$|M - C| + x + |C - M| = 800$

ie$700 - x + x + 400 - x = 800$

ie $x = 300$, ie 300 students take Mathematics and Computer Science.

(b) *How many students take Mathematics but not Computer Science?*

$|M - C| = 700 - x = 700 - 300 = 400$.

(c) *How many students do not take any of the two subjects?*

There are 1000 students and 800 take Mathematics or Computer Science, so

$|(M \cup C)'| = |U| - |M \cup C| = 1000 - 800 = 200$ do not take any of the two subjects.

2. *A builder has a team of 64 construction workers. Of these, 45 are trained in the use of heavy machinery, ie cranes, bulldozers and backhoes. A total of 22 can operate cranes, 26 can operate backhoes, 4 can operate cranes and bulldozers, 6 can operate backhoes and bulldozers, 8 can operate cranes and backhoes, and 1 can operate all three kinds of machine. How many can operate bulldozers?*

First we set out the available information neatly. Let U be the set of all workers in the team. Let C be the set of those who can operate cranes, B those who can operate backhoes and D those who can operate dozers. Then

$|U| = 64$, $|C| = 22$, $|B| = 26$,

$|D| = x$, the unknown we want to solve, ie those who operate bulldozers

$|C \cap D| = 4$,

$|B \cap D| = 6$,

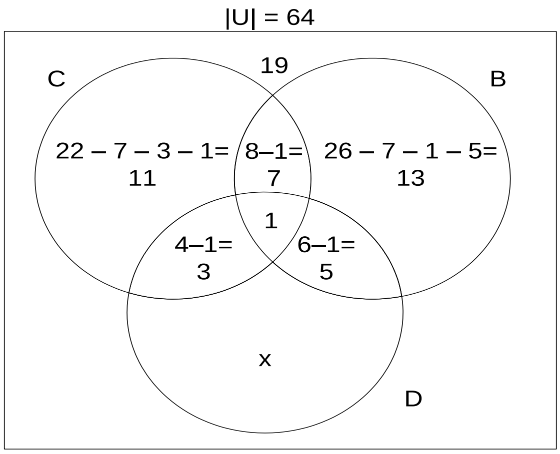
$|C \cap B| = 8$,

$|C \cap D \cap B| = 1$,

$|C \cup D \cup B| = 45$ and

$|(C \cup D \cup B)'| = 64 - 45 = 19$.

Now we can fill in the various regions. We initially fill in x for the value of $|D - (B \cup C)|$.



$|C \cup B \cup D| = 45 = 11 + 7 + 1 + 3 + 5 + 13 + x$

ie $45 = 40 + x$

ie $x = 5$, ie 5 workers can operate dozers only.
Thus $3 + 1 + 5 + 5 = 14$ workers can operate bulldozers.

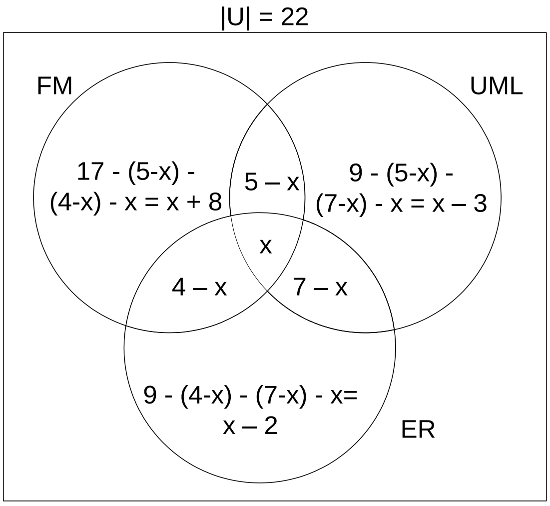
3. *A large software company employs 22 software engineers for the design of systems. Of these engineers, 17 are well versed in the secrets of a formal method (FM), 9 can use the Unified Modelling Language (UML), and 9 are familiar with the use of entity-relationship (ER) diagrams. If 5 engineers can use both an FM and UML, 4 both an FM and ER diagrams and 7 both UML and ER diagrams, answer the following:*

(a) How many engineers can use all 3 techniques, namely an FM, UML and ER diagrams?
Let U denote the number of software engineers. Let FM be the engineers who are well versed in the secrets of a formal method, UML those who can use the Unified Modelling Language and ER those who are familiar with the use of entity-relationship diagrams. Then

$|U| = 22$,
$|FM| = 17$,
$|UML| = 9$,
$|ER| = 9$,
$|FM \cap UML| = 5$,
$|FM \cap ER| = 4$,
$|UML \cap ER| = 7$ and
$|FM \cap UML \cap ER| = x$, ie the number of engineers who can use all 3 techniques.

Start by filling x into the intersection of the three circles. Then fill in the intersections of each pair of circles, e.g. 5 engineers can use both FM and UML, so insert $5 – x$ in the remaining overlap between FM and UML users, and so on ...

Now fill in the outstanding figures for each individual technique by subtracting the numbers already inside a particular circle from the total who uses that technique, e.g. in total, 9 use UML, so fill in: $9 – (5–x) – x – (7–x) = x–3$ in the open region of the UML circle, and so on ...



Now let us solve for x:
$(x + 8) + (5 – x) + (x – 3) + x + (4 – x) + (7 – x) + (x – 2) = 22$
ie $x + 19 = 22$
ie $x = 3$

So 3 engineers can use all three techniques.

(b) *How many engineers can use UML only?*
$x - 3 = 3 - 3 = 0$, ie no engineer uses UML only.

ACTIVITY 4-11

Prove the given sets equal.

1. $\{y \in \mathbb{Z}^+ \mid y \text{ is an even prime number}\} = \{u \in \mathbb{Z}^+ \mid u^2 = 4\}$

Proof:
If $x \in \{y \in \mathbb{Z}^+ \mid y \text{ is an even prime number}\}$
then $x \in \mathbb{Z}^+$ and x is an even prime number
ie $x \in \mathbb{Z}^+$ and $x = 2$ (since 2 is the only even prime number)
ie $x \in \{u \in \mathbb{Z}^+ \mid u^2 = 4\}$.

Conversely, if $x \in \{u \in \mathbb{Z}^+ \mid u^2 = 4\}$
then $x \in \mathbb{Z}^+$ and $x = 2$ (since $2 \in \mathbb{Z}^+$) (this excludes -2)
ie $x \in \mathbb{Z}^+$ and x is an even prime
ie $x \in \{y \in \mathbb{Z}^+ \mid y \text{ is an even prime}\}$.

2. $\mathcal{P}(\{0,1\}) = \{\emptyset\} \cup \{\{0\}\} \cup \{\{1\}\} \cup \{\{0,1\}\}$

Proof:
$S \in \mathcal{P}(\{0,1\})$
iff $S \in \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
iff $S = \emptyset$ or $S = \{0\}$ or $S = \{1\}$ or $S = \{0,1\}$
iff $S \in \{\emptyset\}$ or $S \in \{\{0\}\}$ or $S \in \{\{1\}\}$ or $S \in \{\{0,1\}\}$
iff $S \in \{\emptyset\} \cup \{\{0\}\} \cup \{\{1\}\} \cup \{\{0,1\}\}$.

Note: The following rules have to be applied when doing exercises 3 – 5.

Let *a* and *b* be two factors, then consider the following options:

(i) If $ab < 0$, i.e ab is a negative number,
then *a* is a negative number and *b* is a positive number (since a minus times a plus gives a minus)
OR
a is a positive number and *b* is a negative number (since a plus times a minus gives a minus)

(ii) If $ab > 0$, i.e ab is a positive number,
then *a* is a negative number and *b* is a negative number (since a minus times a minus gives a plus)

OR

a is a positive number and *b* is a positive number (since a plus times a plus gives a plus)

3. $\{x \in \mathbb{R} \mid x^2 + 6x + 5 < 0\} = \{x \in \mathbb{R} \mid -5 < x < -1\}$

Proof:

$y \in \{x \in \mathbb{R}: x^2 + 6x + 5 < 0\}$

iff $y \in \mathbb{R}$ and $y^2 + 6y + 5 < 0$

iff $y \in \mathbb{R}$ and $(y + 1)(y + 5) < 0$

iff $y \in \mathbb{R}$ and either $(y + 1 > 0$ and $y + 5 < 0)$ or $(y + 1 < 0$ and $y + 5 > 0)$

iff $y \in \mathbb{R}$ and either $(y > -1$ and $y < -5)$ or $(y < -1$ and $y > -5)$

iff $y \in \mathbb{R}$ and $-5 < y < -1$ (since there is no real number simultaneously less than -5 and greater than -1)

iff $y \in \{x \in \mathbb{R} \mid -5 < x < -1\}$

4. $\{x \in \mathbb{Z} \mid x^2 - 5x + 4 < 0\} = \{x \in \mathbb{Z}^+ \mid x \text{ is a prime factor of } 6\}$

Proof:

$y \in \{x \in \mathbb{Z} \mid x^2 - 5x + 4 < 0\}$

iff $y \in \mathbb{Z}$ and $y^2 - 5y + 4 < 0$

iff $y \in \mathbb{Z}$ and $(y - 1)(y - 4) < 0$

iff $y \in \mathbb{Z}$ and either $(y - 1 < 0$ and $y - 4 > 0)$ or $(y - 1 > 0$ and $y - 4 < 0)$

iff $y \in \mathbb{Z}$ and either $(y < 1$ and $y > 4)$ or $(y > 1$ and $y < 4)$

iff $y \in \mathbb{Z}$ and $1 < y < 4$

iff $y \in \mathbb{Z}$ and $y \in \{2, 3\}$

iff $y \in \mathbb{Z}^+$ and $y \in \{2, 3\}$ (since 2 and 3 are positive integers.)

iff $y \in \{x \in \mathbb{Z}^+ \mid x \text{ is a prime factor of } 6\}$

5. $\{x \in \mathbb{R} \mid x^2 + x - 2 > 0\} = \{x \in \mathbb{R} \mid x < -2 \text{ or } x > 1\}$

Note: There is a mistake in the exercise given in the study guide.

Proof:

$y \in \{x \in \mathbb{R} \mid x^2 + x - 2 > 0\}$

iff $y \in \mathbb{R}$ and $y^2 + y - 2 > 0$

iff $y \in \mathbb{R}$ and $(y - 1)(y + 2) > 0$

iff $y \in \mathbb{R}$ and either $(y - 1 < 0$ and $y + 2 < 0)$ or $(y - 1 > 0$ and $y + 2 > 0)$

iff $y \in \mathbb{R}$ and either $(y < 1$ and $y < -2)$ or $(y > 1$ and $y > -2)$

iff $y \in \mathbb{R}$ either $y < -2$ or $y > 1$

iff $y \in \{x \in \mathbb{R} \mid x < -2 \text{ or } x > 1\}$

Activity 4-12

1. *Determine whether or not for V, W, Z $\subseteq$ U, if V $\subseteq$ W, then V $\cup$ Z $\subseteq$ W $\cup$ Z and V $\cap$ Z $\subseteq$ W $\cap$ Z. Provide either a proof or a counterexample, whichever is appropriate.*

Let us first try to prove that if V $\subseteq$ W then V $\cup$ Z $\subseteq$ W $\cup$ Z.

Suppose V $\subseteq$ W.
Let x $\in$ V $\cup$ Z,
then x $\in$ V or x $\in$ Z
ie x $\in$ W or x $\in$ Z (V $\subseteq$ W, so if x $\in$ V then x $\in$ W)
ie x $\in$ W $\cup$ Z.
Therefore V $\cup$ Z $\subseteq$ W $\cup$ Z.

Let us now consider whether it is the case that if V $\subseteq$ W, then V $\cap$ Z $\subseteq$ W $\cap$ Z.
Suppose V $\subseteq$ W.
Let x $\in$ V $\cap$ Z,
then x $\in$ V and x $\in$ Z
ie x $\in$ W and x $\in$ Z (because V $\subseteq$ W)
ie x $\in$ W $\cap$ Z.
We can conclude that if V $\subseteq$ W then V $\cap$ Z $\subseteq$ W $\cap$ Z.

2. *Is it the case that, for all subsets X, Y, W $\subseteq$ U, if X = Y and Y = W, then X = W, and if X $\subset$ Y and Y $\subset$ W, then X $\subset$ W? Justify your answer.*

In general, what does 'A $\subset$ B' mean? It means that A is a subset of (but not equal to) B,
ie A is a proper subset of B.
First we have to attempt to prove that if X = Y and Y = W, then X = W.
If X = Y and Y = W then we know that X has precisely the same elements as Y and that Y has exactly the same elements as W.
Therefore X and W contain exactly the same elements and hence X = W.

Also, if X $\subset$ Y and Y $\subset$ W we can try to prove that X $\subset$ W as follows:
Suppose X $\subset$ Y and Y $\subset$ W.
Let x $\in$ X,
then x $\in$ Y (since X $\subset$ Y)
ie x $\in$ W (because Y $\subset$ W).
So X $\subseteq$ W.
But Y has at least one element not in X (since X $\subset$ Y)
and W has at least one element not in Y (since Y $\subset$ W),
so W has at least two elements not in X
ie X $\neq$ W,
so X $\subset$ W.

What does this tell us about $\subseteq$? We now know that if $X \subseteq Y$ and $Y \subseteq W$, ie ($X \subset Y$ or $X = Y$) and ($Y \subset W$ or $Y = W$), then ($X \subset W$ or $X = W$) (from the above proofs)
ie $X \subseteq W$.
In other words, the fact that $=$ and $\subset$ satisfy transitive laws for sets tells us that $\subseteq$ is also transitive.

3. *Is it the case that, for all subsets X of U, $X \cup \emptyset = X$? Justify your answer.*
We know that the set we obtain when we determine the union of two sets Y and Z contains all the elements of Y and all the elements of Z. So when we form the union of a subset X and $\emptyset$, the new set, $X \cup \emptyset$, contains all the elements of X and all the elements of $\emptyset$.

But since $\emptyset$ has no elements, $X \cup \emptyset$ will only contain the elements of X.
Therefore $X \cup \emptyset = X$ for all subsets X of U.
In the style of our other proofs, we may also argue as follows:
Let $x \in X \cup \emptyset$,
then $x \in X$ or $x \in \emptyset$
ie $x \in X$ (because it is impossible for x to reside in $\emptyset$).
Thus $X \cup \emptyset \subseteq X$.

Conversely, let $x \in X$,
then $x \in X$ or $x \in \emptyset$
ie $x \in X \cup \emptyset$.
Thus $X \subseteq X \cup \emptyset$.
It follows that $X \cup \emptyset = X$.

4. *Is it the case that, for all subsets V and W of U, $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$? Justify your answer.* This claim has two parts. These are
if $V = \emptyset$ or $W = \emptyset$ then $V \cap W = \emptyset$, and
if $V \cap W = \emptyset$ then $V = \emptyset$ or $W = \emptyset$.
Both these parts must hold for the claim to be true.
Let us consider the first part. Suppose $V = \emptyset$ or $W = \emptyset$.
We know that, when the intersection of the two sets V and W is formed, the set $V \cap W$ contains the elements common to both V and W.
In this case at least one of V or W is empty so there is no element common to V and W.
Therefore $V \cap W$ is also empty.

Looking at the second part of the claim, we have to decide whether it is necessarily the case whenever $V \cap W = \emptyset$, it follows that $V = \emptyset$ or $W = \emptyset$
Well, if $V \cap W = \emptyset$ all we know is that V and W have no elements in common.
It is not necessarily the case that one of them is empty.

Consider the example $V = \{1, 2\}$ and $W = \{3, 4\}$.
It is clear that $V \cap W = \emptyset$ although neither V nor W is empty.
We can conclude that if $V = \emptyset$ or $W = \emptyset$ then $V \cap W = \emptyset$,
but if $V \cap W = \emptyset$, then it is not necessarily the case that $V=\emptyset$ or $W=\emptyset$.

Therefore it is not the case that, for all subsets V and W, $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$.

5. *Is it the case that for every subset X of U there exists a subset Y of U such that $X \cup Y = \emptyset$? Justify your answer.*

No. We give a counterexample.

We know that the set $X \cup Y$ contains all the elements of X as well as those of Y.
So if U is the set {1, 2} and X is the subset {1}, then there is no subset Y of U such that $X \cup Y = \emptyset$.
To see this, note that there are just four possible values for Y,
namely $Y = \emptyset$, Y = {1}, Y = {2}, and Y = {1, 2}.
In each case, $X \cup Y$ contains at least the element 1, so $X \cup Y \neq \emptyset$.

6. *Is it the case that for every subset X of U there is some subset Y such that $X \cap Y = U$? Justify your answer.*

No. We give a counterexample.

We know that the intersection $X \cap Y$ contains the elements that are common to X and Y.
So if $X \cap Y = U$ it must be the case that X and Y have all the elements of U in common. It is not usually the case.
Take U = {1, 2} and X = {1}.
Then there is no subset Y of U such that $X \cap Y = U$.
To see this, note that Y can have four possible values,
namely $Y = \emptyset$, Y = {1}, Y = {2}, and Y = {1, 2}.
In none of these four cases does $X \cap Y$ contain the element 2.

7. *Using "if and only if" statements, determine the following*:

(a) *Is it the case that $X + Y = Y + X$ for all $X, Y \subseteq U$?*

$x \in X + Y$
iff $x \in X$ or $x \in Y$ but not both
iff $x \in Y$ or $x \in X$ but not both
iff $x \in Y + X$. We conclude that $X + Y = Y + X$.

(b) *Is it case that $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ for all $X, Y, Z \subseteq U$?*

$x \in X \cap (Y + Z)$
iff $x \in X$ and $x \in (Y + Z)$
iff $x \in X$ and ($x \in Y$ or $x \in Z$ but not both)
iff x is in X and in exactly one of Y or Z
iff either ($x \in X$ and $x \in Y$) or ($x \in X$ and $x \in Z$) but not both
iff $x \in X \cap Y$ or $x \in X \cap Z$ but not both
iff $x \in (X \cap Y) + (X \cap Z)$.
Therefore $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$.

STUDY UNIT 5

ACTIVITY 5-4

Suppose $A = \{1, 2, 3, 4\}$, $B = \{2, 5\}$, $C = \{3, 4, 7\}$ and the universal set $U = \{1, 2, 3, 4, 5, 7\}$.
Write out the following Cartesian products by using list notation:

(a) $A \times B = \{(1, 2), (1, 5), (2, 2), (2, 5), (3, 2), (3, 5), (4, 2), (4, 5)\}$
(b) $B \times A = \{(2, 1), (2, 2), (2, 3), (2, 4), (5, 1), (5, 2), (5, 3), (5, 4)\}$
(c) $A \cup B = \{1, 2, 3, 4, 5\}$
$(A \cup B) \times C$
$= \{(1, 3), (1, 4), (1, 7), (2, 3), (2, 4), (2, 7), (3, 3), (3, 4), (3, 7), (4, 3), (4, 4), (4, 7), (5, 3), (5, 4), (5, 7)\}$
(d) $A + B = \{1, 3, 4, 5\}$
$(A + B) \times B = \{(1, 2), (1, 5), (3, 2), (3, 5), (4, 2), (4, 5), (5, 2), (5, 5)\}$

ACTIVITY 5-8

1. *Let P and R be relations on $A = \{ 1, 2, 3, \{1\}, \{2\} \}$ given by*
$P = \{ (1, \{1\}), (1, 2) \}$ and
$R = \{ (1, \{1\}), (1, 3), (2, \{1\}), (2, \{2\}), (\{1\}, 3), (\{2\}, \{1\}) \}$. Investigate the following:

(a) **Irreflexivity:**
Is it the case that for all $x \in A$, $(x, x) \notin R$?
Yes, R is irreflexive, since the first and second co-ordinates differ from each other in each ordered pair of R. We do not have all the following as elements of R: $(1, 1), (2, 2), (3, 3), (\{1\}, \{1\}), (\{2\}, \{2\})$.

(b) **Reflexivity:**
Is it the case that for all $x \in A$, $(x, x) \in R$?
No, R is not reflexive, we give a counterexample:
$(1, 1) \notin R$.

(c) **Symmetry:**
If $(x, y) \in R$, is it the case that $(y, x) \in R$?
No, R is not symmetric. We give a counterexample:
$(1, \{1\}) \in R$, but its mirror image, $(\{1\}, 1) \notin R$.

(d) **Antisymmetry:**
If $x \neq y$ and $(x, y) \in R$, is it the case that $(y, x) \notin R$?
Yes, R is antisymmetric, since no member of R has its mirror image also living in R.
For each ordered pair (x, y) living in R, we have that $(y, x) \notin R$.

(e) **Transitivity:**
If $(x, y) \in R$ and $(y, z) \in R$, is it the case that $(x, z) \in R$?
No, R is not transitive. We give a counterexample:

$(2, \{1\}) \in R$, and $(\{1\}, 3) \in R$, but $(2, 3) \notin R$.

(f) **Trichotomy**:

Is it the case for all $x, y \in A$, if $x \neq y$ then $(x, y) \in R$ or $(y, x) \in R$?

No, R does not satisfy trichotomy. We give a counterexample:

$(2, 3) \in R$ and also $(3, 2) \in R$.

This means that $2 \neq 3$ but we cannot compare the elements 2 and 3 of A in terms of R because these 2 elements do not appear together in any ordered pair of R.

(g) **R ○ R:** $(x, w) \in R \circ R$ iff for some y there exist pairs $(x, y) \in R$ and $(y, w) \in R$.

$R = \{ (1, \{1\}), (1, 3), (2, \{1\}), (2, \{2\}), (\{1\}, 3), (\{2\}, \{1\}) \}$

We have that $(1, \{1\}) \in R$ and $(\{1\}, 3) \in R$, hence $(1, 3) \in R \circ R$.

Also: $(2, \{1\}) \in R$ and $(\{1\}, 3) \in R$, hence $(2, 3) \in R \circ R$.

$(2, \{2\}) \in R$ and $(\{2\}, \{1\}) \in R$, hence $(2, \{1\}) \in R \circ R$.

$(\{2\}, \{1\}) \in R$ and $(\{1\}, 3) \in R$, hence $(\{2\}, 3) \in R \circ R$.

Thus $R \circ R = \{ (1, 3), (2, 3), (2, \{1\}), (\{2\}, 3) \}$.

(h) **R ○ P:** $(x, w) \in R \circ P$ iff for some y there exist pairs $(x, y) \in P$ and $(y, w) \in R$.

We have that $(1, \{1\}) \in P$ and $(\{1\}, 3) \in R$, hence $(1, 3) \in R \circ P$.

Also: $(1, 2) \in P$ and $(2, \{1\}) \in R$, hence $(1, \{1\}) \in R \circ P$.

$(1, 2) \in P$ and $(2, \{2\}) \in R$, hence $(1, \{2\}) \in R \circ P$.

Thus $R \circ P = \{ (1, 3), (1, \{1\}), (1, \{2\}) \}$.

(i) **T:** T is a subset of R, so T is also a relation on A.

We have $(a, B) \in T$ iff $a \in B$.

In each ordered pair in T, the first co-ordinate must be a member of the second co-ordinate.

We have $T = \{ (1, \{1\}), (2, \{2\}) \}$.

2. *Let $A = \{a, b\}$. For each of the specifications given below, find suitable examples of relations on $\mathcal{P}(A)$.*

First of all let us write down $\mathcal{P}(A)$:

$\mathcal{P}(A) = \{ \emptyset, \{a\}, \{b\}, \{a,b\} \}$,

and $\mathcal{P}(A) \times \mathcal{P}(A) = \{ (\emptyset, \emptyset), (\emptyset, \{a\}), (\emptyset, \{b\}), (\emptyset, \{a,b\}), (\{a\}, \emptyset), (\{a\}, \{a\}), (\{a\}, \{b\}), (\{a\}, \{a,b\}), (\{b\}, \emptyset), (\{b\}, \{a\}), (\{b\}, \{b\}), (\{b\}, \{a,b\}), (\{a,b\}, \emptyset), (\{a,b\}, \{a\}), (\{a,b\}, \{b\}), (\{a,b\},\{a,b\}) \}$.

(a) R is reflexive on $\mathcal{P}(A)$, symmetric, and transitive:

Two examples of relations that satisfy this specification are **{** $(\emptyset, \emptyset)$, ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}) **}** and **{** $(\emptyset, \emptyset)$, ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}), ($\emptyset$, {a}), ({a}, $\emptyset$) }.

(b) R is reflexive on $\mathcal{P}$(A) and symmetric, but not transitive:

{ $(\emptyset, \emptyset)$, ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}), ($\emptyset$, {a}), ({a}, $\emptyset$), ({a}, {a,b}), ({a,b}, {a}) **}**

This relation is not transitive because it contains both ($\emptyset$, {a}) and ({a}, {a,b}) but not ($\emptyset$, {a,b}). Another counterexample to transitivity is that both ({a,b}, {a}) and ({a}, $\emptyset$) belong to the relation, but not ({a,b}, $\emptyset$).

(c) R is reflexive on $\mathcal{P}$(A), transitive, but is not symmetric and not antisymmetric:

{ $(\emptyset, \emptyset)$, ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}), ($\emptyset$, {a}), ($\emptyset$, {b}), ({a}, {b}), ({a,b}, $\emptyset$), ({a,b}, {a}), ({a,b}, {b}), ({b}, {a}) **}**

(d) R is simultaneously symmetric and antisymmetric:
{ $(\emptyset, \emptyset)$, ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}) **}**

(e) R is irreflexive, antisymmetric, transitive:
{ ($\emptyset$, {a}), ($\emptyset$, {b}), ($\emptyset$, {a,b}), ({a}, {b}), ({a}, {a,b}), ({b}, {a,b}).}

3. *Prove that if R is a relation on X, then R is transitive iff $R \circ R \subseteq R$.*

First we attempt to prove that if R is transitive, then we prove that $R \circ R \subseteq R$.

Assume R is transitive.

Suppose $(x, z) \in R \circ R$, then, according to the definition of composition, there exists some

$y \in X$ such that $(x, y) \in R$ and $(y, z) \in R$.
Because R is transitive, it follows that $(x, z) \in R$.

This completes the proof that if R is transitive, then $R \circ R \subseteq R$.

Now we have to prove the converse.
Assume $R \circ R \subseteq R$.

Suppose $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R \circ R$ according to the definition of composition.

Because $R \circ R \subseteq R$, it follows that $(x, z) \in R$.

Therefore if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.
Hence R is transitive.

We can now conclude that if R is a relation on X, then R is transitive iff $R \circ R \subseteq R$.

STUDY UNIT 6

ACTIVITY 6.4

For each of the following relations, determine whether or not the relation is a weak partial order (reflexive, antisymmetric and transitive) on the given set:

(a) Let A = {a, b, {a, b} }. The relation S on A is defined by (c, B) ∈ S iff c ∈ B.
We see that a ∈ {a, b} and b ∈ {a, b} and thus S = { (a, {a, b}), (b, {a, b}) }.

Reflexivity:
Is it the case that for all x ∈ A, (x, x) ∈ S?
No, we give a counterexample: (a, a) ∉ S. (It is also the case that (b, b) ∉ S and ({a, b}, {a, b}) ∉ S.)

S is not reflexive therefore we can say that it is not a weak partial order.
(You can test whether or not S is antisymmetric and transitive.)

(b) Define R ⊆ $\mathbb{Z} \times \mathbb{Z}$ by x R y iff x + y is even.

If x + y is even then we can say that x + y = 2k for some integer k.

Reflexivity:
Is it the case that for all x ∈ $\mathbb{Z}$, (x, x) ∈ R?
x + x = 2x, ie x + x is an even number for any x ∈ $\mathbb{Z}$.
Thus R is reflexive on $\mathbb{Z}$.

Antisymmetry:
If (x, y) ∈ R, is it the case that (y, x) ∉ R?
Suppose (x, y) ∈ R
then x + y = 2k
ie y + x = 2k, but this means that (y, x) ∈ R.
Thus R is not antisymmetric. R is actually symmetric.
Because R is not antisymmetric it is not a weak partial order.

For interest's sake, let's test whether R is transitive:
Transitivity:
If (x, y) ∈ R and (y, z) ∈ R, is it the case that (x, z) ∈ R?
Suppose (x, y) ∈ R and (y, z) ∈ R
then x + y = 2k and y + z = 2m for some k, m ∈ $\mathbb{Z}$.
ie x = 2k – y and z = 2m – y
ie x + z = 2k – y + 2m – y
ie x + z = 2 (k + m – y)
ie x + z = 2t for some integer t
ie (x, z) ∈ R. Thus R is transitive.

(c) Define R on $\mathbb{Z} \times \mathbb{Z}$ by (a, b) R (c, d) if either a < c or else (a = c and b ≤ d).

Reflexivity:

Is it the case that for all (a, b) $\in \mathbb{Z} \times \mathbb{Z}$, (a, b) R (a, b)?

For any (a, b) $\in \mathbb{Z} \times \mathbb{Z}$ it is not the case that a < a but
(a = a and b ≤ b)
ie (a, b) R (a, b).
Thus R is reflexive.

Antisymmetry:

If (a, b) ≠ (c, d) and ((a, b), (c, d)) ∈ R, is it the case that ((c, d), (a, b)) ∉ R?

Suppose (a, b) ≠ (c, d) and (a, b) R (c, d), then a < c or else (a = c and b ≤ d).
Firstly, we do not have c < a and secondly we do not have that c = a and d ≤ b, thus ((c, d), (a, b)) ∉ R.
(In the second case we cannot have c = a and d = b because we assumed that (a, b) ≠ (c, d).
Furthermore, we cannot have c = a and d < b because by our assumption, b ≤ d.)
We can safely say that ((c, d), (a, b)) ∉ R, thus R is antisymmetric.

Transitivity:

If ((a, b), (c, d)) ∈ R and ((c, d), (e, f)) ∈ R, is it the case that ((a, b), (e, f)) ∈ R?

Suppose ((a, b), (c, d)) ∈ R and ((c, d), (e, f)) ∈ R
ie a < c or else (a = c and b ≤ d), and c < e or else (c = e and d ≤ f).
We can look at the following cases:
a < c and c < e,
a < c and c = e and d ≤ f,
a = c and b ≤ d and c < e, or
a = c and b ≤ d and c = e and d ≤ f.
In the first three cases we have a < e and in the last case we have a = e and b ≤ f.
We can deduce that ((a, b), (e, f)) ∈ R, thus R is transitive.

Because R is reflexive, antisymmetric and transitive we can say that R is a weak partial order.

ACTIVITY 6.5

For each of the following relations, determine whether or not the relation is a strict partial order (irreflexive, antisymmetric and transitive) on the given set:

Let A = {a, {a}, {b}} and let S on A be the relation S = { (a, {a}), (a, {b})}.

Irreflexivity:

Is it the case that for all x ∈ A, (x, x) ∉ S?
Yes. (a, a) ∉ S, ({a}, {a}) ∉ S and ({b}, {b}) ∉ S.

Antisymmetry:

If (x, y) ∈ S, is it the case that (y, x) ∉ S?
Yes. (a, {a}) and (a, {b}) are the elements of S but ({a}, a), ({b}, a) ∉ S.

Transitivity:

If $(x, y) \in S$ and $(y, z) \in S$, is it the case that $(x, z) \in S$?
No two ordered pairs in S are such that $(x, y) \in S$ and $(y, z) \in S$, so we need not find the pair (x, z) in S. S is thus transitive.
(We cannot prove that S is not transitive. Such a proof actually has a special name: it is vacuously true that S is transitive.)

Because S is irreflexive, antisymmetric and transitive we can say that S is a strict partial order.

(b) Define $R \subseteq (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$ by $(a, b)\ R\ (c, d)$ iff $a < c$.

Irreflexivity:

Is it the case that for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $((a, b), (a, b)) \notin R$?
Yes. It is not true that $a < a$ therefore $((a, b), (a, b)) \notin R$.

Antisymmetry:

If $(a, b) \neq (c, d) \in R$ and $((a, b), (c, d)) \in R$, is it the case that $((c, d), (a, b)) \notin R$?
Yes. Suppose $(a, b) \neq (c, d)$ and $(a, b)\ R\ (c, d)$, then $a < c$.
This means that it is not possible that $c < a$, therefore $((c, d), (a, b)) \notin R$.

Transitivity:

If $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, is it the case that $((a, b), (e, f)) \in R$?
Suppose $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$
then $a < c$ and $c < e$
ie $a < e$
We can deduce that $((a, b), (e, f)) \in R$, thus R is transitive.

Because R is irreflexive, antisymmetric and transitive we can say that R is a strict partial order.

ACTIVITY 6-7

1. *Let X = {a,b,c}. Write down all strict partial orders on X. Which of them are linear?*

Strict partial orders on X are irreflexive, antisymmetric and transitive. The strict partial orders on X are:
∅, { (a,b) }, { (a,c) }, { (b,a) }, { (b,c) }, { (c,a) }, { (c,b) },
{ (a,b), (a,c) }, { (a,b), (c,b) }, { (a,c), (b,c) }, { (b,a), (b,c) }, { (b,a), (c,a) }, { (c,a), (c,b) },
{ (a,b), (b,c), (a,c) }, { (b,a), (a,c), (b,c) }, { (b,c), (c,a), (b,a) }, { (a,c), (c,b), (a,b) }, { (c,a), (a,b), (c,b) }, and
{ (c,b), (b,a), (c,a) }.
All the relations containing three elements satisfy trichotomy and are therefore linear.
Note: How do we know we have found all the strict partial orders on X? Well, we were systematic. We wrote down those with one pair then we listed all the ways to add another pair without losing properties like transitivity. Lastly we listed the relations containing 3 pairs, and each of these can be viewed as a 2-step journey together with the contraction of that journey required by transitivity. It is not possible to have more than 3 pairs without losing irreflexivity or antisymmetry.

2. *In each of the following cases, determine whether or not R is some sort of order relation on the given set X (weak partial, weak total, strict partial, or strict total). Justify your answer.*

To determine whether R is some sort of order relation on X, we have to examine the relevant properties of R in each case.

(a) X = { ∅, {0}, {2} } and R = { (∅, {0}), (∅, {2}) }:

Reflexivity:
R is not reflexive so we provide a counterexample:
(∅, ∅) ∉ R (We also have that ({0}, {0}) ∉ R and ({2}, {2}) ∉ R.)

Irreflexivity:
For all x ∈ X we have that (x, x) ∉ R:
(∅, ∅) ∉ R, ({0}, {0}) ∉ R and ({2}, {2}) ∉ R.
Hence R is irreflexive.

Antisymmetry:
R is antisymmetric because the mirror images of (∅, {0}) and (∅, {2}) are not in R.

Transitivity:
R is transitive because it does not contain any members with first co-ordinates {0} and {2}, so there are no 2-step journeys to worry about.
(Does this proof bother you? If so, remember the definition of transitivity, '**if** (x, y) ∈R and (y, z) ∈R, **then** ...'. When the **if** part does not apply, we have no more work to do!)

Trichotomy:
There is no ordered pair comparing {0} and {2}.
So R does not satisfy trichotomy.
R is a strict partial order on X because R is irreflexive, antisymmetric and transitive.

(b) *X = {∅, {∅}, {{∅}}} and R = ⊆ (R is the relation of all ordered pairs where each first co-ordinate is a subset of the second co-ordinate, and R ⊆ X × X.)*
For example, ∅ ⊆ {∅}, so (∅, {∅}) ∈ ⊆ or (∅, {∅}) ∈ R.
Because X has only three elements we can describe R in list notation.
⊆ = { (∅, ∅), (∅, {∅}), (∅, {{∅}}), ({∅}, {∅}), ({{∅}}, {{∅}}) }.

Note: ({∅}, {{∅}}) ∉ ⊆ because {∅} is not a subset of {{∅}}. *{∅} is not a subset of {{∅}} because each element of {∅} is not an element of {{∅}} – the only element of {∅} is ∅ and the only element of {{∅}} is {∅}. We see that ∅ ∈ {∅} but ∅ ∉ {{∅}} thus {∅} is not a subset of {{∅}}.*

Reflexivity:
It is clear that for every $x \in X$, $(x, x) \in R$: $(\emptyset, \emptyset)$, $(\{\emptyset\}, \{\emptyset\})$ and $(\{\{\emptyset\}\}, \{\{\emptyset\}\})$ are all members of R. So R (or $\subseteq$, if you prefer) is reflexive on X.

Antisymmetry:
By inspection of R it is clear that for all $x, y \in X$, if $x \neq y$ and $(x, y) \in R$ then $(y, x) \notin R$.
Or to say the same thing in different words, whenever both $x \subseteq y$ and $y \subseteq x$, then $x = y$.
So R (or $\subseteq$, if you prefer) is antisymmetric.

Transitivity:
A typical 2-step journey is $(\emptyset, \{\emptyset\})$ followed by $(\{\emptyset\}, \{\emptyset\})$, and its corresponding 1-step journey $(\emptyset, \{\emptyset\})$ is also in R, playing a double role.
Clearly every 2-step journey in R can be performed in a single step.
Can you find them all?
Therefore R (or $\subseteq$, if you prefer) is transitive.

Irreflexivity:
There are several x in X such that $(x, x) \in \subseteq$, for example $x = \emptyset$, where $(\emptyset, \emptyset) \in \subseteq$.
So $\subseteq$ is not irreflexive.

Trichotomy:
$\subseteq$ does not satisfy trichotomy, because it is not the case that for all $x, y \in X$, if $x \neq y$ then $(x, y) \in \subseteq$ or $(y, x) \in \subseteq$, for example, there is no ordered pair containing both $\{\emptyset\}$ and $\{\{\emptyset\}\}$.

Because $\subseteq$ is reflexive, antisymmetric and transitive, it is a weak partial order.

(c) $X = \mathbb{Z}$ and $R = \leq$:

Reflexivity:
For all $x \in \mathbb{Z}$, $(x, x) \in \leq$, because $x \leq x$ for every integer x.
So R is reflexive.

Antisymmetry:
For any $x, y \in \mathbb{Z}$, if $x \neq y$ and $(x, y) \in \leq$ then $x \leq y$.
Therefore it is not the case that $y \leq x$. So $(y, x) \notin \leq$.
So $\leq$ is antisymmetric.
Another way to say the same thing is that if $x \leq y$ and, at the same time, $y \leq x$, then it must be the case that $x = y$, because if the value of x appears to the left of the value of y on the number line and the value of y appears to the left of that of x, then they must lie on the same position.

Transitivity:
If $(x, y) \in \leq$ and $(y, z) \in \leq$ for any $x, y, z \in \mathbb{Z}$, then $x \leq y$ and $y \leq z$,
ie x appears to the left of y and y appears to the left of z.
Thus x appears to the left of z, ie $x \leq z$, ie $(x, z) \in$ v. Therefore $\leq$ is transitive.

Irreflexivity:
$\leq$ is not irreflexive, because we can find values of x such that $(x, x) \in \leq$,
for example x = 113, or if you prefer simple values, x = 2.

Trichotomy: For all $x, y \in \mathbb{Z}$, if $x \neq y$ then either $x > y$ or $y > x$,
because the one must appear to the left of the other on the number line.
Therefore either $(x, y) \in \leq$ or $(y, x) \in \leq$. Hence $\leq$ satisfies trichotomy.
Because $\leq$ is reflexive, antisymmetric and transitive, it is a weak partial ordering,
and because $\leq$ is a weak partial ordering satisfying trichotomy, it is also a weak total (linear) ordering.

(d) $X = \mathbb{Z}$ and $R = >$:

Reflexivity:
There are values of x such that $(x, x) \notin >$, since if x = 113, say, then it is not the case that $x > x$. So $>$ is not reflexive on $\mathbb{Z}$.

Antisymmetry:
If $(x, y) \in >$, then $x > y$, ie x lies to the right of y on the number line.
Thus it cannot be the case that $y > x$.
So $(y, x) \notin >$, and $>$ is therefore antisymmetric.

Transitivity:
Suppose $(x, y) \in >$ and $(y, z) \in >$, ie $x > y$ and $y > z$.
This means that x lies to the right of y and y to the right of z on the number line.
Therefore x lies to the right of z, ie $x > z$, ie $(x, z) \in >$.
Thus $>$ is transitive.

Irreflexivity:
For all $x \in \mathbb{Z}$ it is not the case that x lies to the right of itself, ie it will never be the case that $x > x$.
So $(x, x) \notin >$ for all $x \in \mathbb{Z}$, and hence $>$ is irreflexive.

Trichotomy:
For all $x, y \in \mathbb{Z}$, if $x \neq y$ then either x lies to the right of y (ie $x > y$) or x lies to the left of y (ie $y > x$). So either $(x, y) \in >$ or $(y, x) \in >$. Therefore $>$satisfies trichotomy.
We can conclude that $>$ is a strict partial order relation because it is irreflexive, antisymmetric and transitive. What is more, $>$ is a strict linear ordering because it satisfies trichotomy as well.
Note: Any linear order is also a partial order, but not vice versa.

(e) *$X = \mathbb{Z}^+$ and R is defined by: x R y iff x divides y with zero remainder, ie $y = kx$ for some $k \in \mathbb{Z}^+$.*
(x R y is another way of saying $(x, y) \in R$.)

This means that x is a factor of y and y is a multiple of x.
Let us synthesize some ordered pairs that belong to R:
How about (2, 6), (3, 6), (5, 35) and (4, 24)?
All of these meet the requirement that $y = kx$ for some $k \in \mathbb{Z}^+$.

Reflexivity:
For each $x \in \mathbb{Z}^+$ we have that $x = 1x$ and $1 \in \mathbb{Z}^+$, so $(x, x) \in R$.
R is therefore reflexive on $\mathbb{Z}^+$.

Antisymmetry:
Suppose $x \neq y$ and $(x, y) \in R$.
Can (y, x) qualify to belong to R?
If $(x, y) \in R$, $y = kx$ ① for some $k \in \mathbb{Z}^+$.

Does it ever happen that $(y, x) \in R$, ie $x = my$ ② for some $m \in \mathbb{Z}^+$?

Substitute ② into ①:

$y = kx = k(my) = (km)y$, ie $y = (km)y$, which means $km=1$.
Hence $k = m = 1$, so $x = y$.
But we specifically assumed that $x \neq y$,
so it can never happen that $(y, x) \in R$, which means that $(y, x) \notin R$.
Therefore R is antisymmetric.

Transitivity:
Suppose $(x, y) \in R$ and $(y, z) \in R$.
Then $y = kx$ for some $k \in \mathbb{Z}^+$.
and $z = my$ for some $m \in \mathbb{Z}^+$.
Hence $z = my = m(kx) = (mk)x$,
ie $(x, z) \in R$.
Thus R is transitive.

Irreflexivity:
Since we can find values of x such that $(x, x) \in R$,
for example $x = 113$, where $(113, 113) \in R$,
R cannot be irreflexive.

Trichotomy:
R does not satisfy trichotomy.
Take $x = 2$ and $y = 3$, then there do not exist some $k, m \in Z^+$ such that $3 = k(2)$ or $2 = m(3)$, so neither $(2, 3) \in R$ nor $(3, 2) \in R$.
Thus R is a weak partial order on $\mathbb{Z}^+$.

ACTIVITY 6-10

1. *Let X = {a, b, c}. Write down all equivalence relations on X.*

Equivalence relations on X must be reflexive, symmetric and transitive:
R_1 = { (a, a), (b, b), (c, c) },
R_2 = { (a, a), (b, b), (c, c), (a, b), (b, a) },
R_3 = { (a, a), (b, b), (c, c), (a, c), (c, a) },

$R_4 \quad = \quad \{ (a, a), (b, b), (c, c), (b, c), (c, b) \}$,

$R_5 \quad = \quad \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b) \}$,

2. *In each of the following cases, determine whether or not the given relation is an equivalence relation. If it is, describe the equivalence class(es) of R. Justify your reasoning.*

(a) $X = \{a, b, c\}$ and $R = \{ (c, c), (b, b), (a, a) \}$:

Reflexivity:
R is reflexive because for every $x \in X$, we have $(x, x) \in R$, as we can see by inspecting R. The ordered pairs (a, a), (b, b) and (c, c) are all present in R.

Symmetry:
R is also symmetric, because there is no pair $(x, y) \in R$, $x \neq y$, such that $(y, x) \notin R$,
in other words, for every $(x, y) \in R$ it is also the case that $(y, x) \in R$, since the first co-ordinate is equal to the second co-ordinate in all the ordered pairs belonging to R. Each pair in R, namely, (a, a), (b, b) and (c, c), plays a double role; each plays the part of (x, y) as well as (y, x).

Transitivity:
Is it the case that for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$?
This is surely the case, because the only 2-step journeys are trivial ones like 'From b go to b, and then go from b to b'.
If fact, each pair in R plays a triple role; it plays the parts of (x, y), (y, z) and (x, z).

To illustrate, let's consider one specific example, say (b, b) in this triple role:

x y   y z   x z
ie (b,b) and (b,b) and (b,b).
This means that R is transitive.
R is the equality (or identity) relation on X.

What are the equivalence classes of R?
Because X has only three elements we can consider each element individually:

$[a] \quad = \quad \{y \mid (a, y) \in R\}$
$\quad\quad = \quad \{a\}$.

Similarly, $[b] = \{b\}$ and $[c] = \{c\}$.

(b) *$X = \{a, b, c\}$ and $R = X \times X$:*
It is easy to describe R in list notation because X has only three elements.
$R = \{ (a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c) \}$.

Reflexivity:
R is reflexive, because (a, a), (b, b) and (c, c) are all contained in R.

Symmetry:
R is symmetric, because for each pair (x, y), its mirror image (y, x) is also in R. This can be checked by inspecting R. We find $(a, b) \in R$ and $(b, a) \in R$; $(a, c) \in R$ and $(c, a) \in R$; and $(b, c) \in R$ and $(c, b) \in R$. Furthermore the ordered pairs (a, a), (b, b) and (c, c) each play a double role, being itself and its own mirror image.

Transitivity:
Scrutinising R very carefully we see that for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.
For example, among the various 2-step journeys found in R is (b, c) followed by (c, a).
Since (b, a) is in R, the 2-step journey can be contracted to a single step.
We also have (a, a) and (a, b) in R, and (a, b) the single-step journey is there, playing a double role.
Similarly we find (b, a), (a, b) and (b, b).
Can you spot **all** the other 2-step journeys?
All must be tested, and the associated single-step journeys must be found to be present, before we can confirm transitivity.
This can be done; thus R is transitive.

We can now conclude that R is an equivalence relation.

What are the equivalence classes of R?

$[a] = \{y \mid (a, y) \in R\}$
$= \{a, b, c\}$.

We do not even bother to work out [b] and [c], because b and c are both in [a],
so we know that [a] = [b] = [c] = X.
In other words, R says 'All the elements of X are equivalent to one another',
and there is only one equivalence class.

(c) *$X = \mathcal{P}(Y)$ where $Y = \{1, 2, 3\}$ and*
R consists of all pairs (C, D) such that $C \cap \{2\} = D \cap \{2\}$:

We can use a brute force approach to this problem, because X and R are small sets:
$\mathcal{P}(Y) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$ and
we can work out what R is by noting that the only possible outcome of
$C \cap \{2\}$ is $\emptyset$, if $2 \notin C$,
and $C \cap \{2\}$ is $\{2\}$, if $2 \in C$.

So all subsets of Y that do not contain the member 2 are related to one another by R, and all subsets of Y that do contain the element 2 are related to one another by R. Thus
$R = \{ (\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{3\}), (\emptyset, \{1, 3\}), (\{1\}, \emptyset), (\{1\}, \{1\}), (\{1\}, \{3\}), (\{1\}, \{1, 3\}),$
$(\{3\}, \emptyset), (\{3\}, \{1\}), (\{3\}, \{3\}), (\{3\}, \{1, 3\}), (\{1, 3\}, \emptyset), (\{1, 3\}, \{1\}), (\{1, 3\}, \{3\}), (\{1, 3\}, \{1, 3\}),$
$(\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{2\}, \{2, 3\}), (\{2\}, Y), (\{1, 2\}, \{2\}), (\{1, 2\}, \{1, 2\}), (\{1, 2\}, \{2, 3\}), (\{1, 2\}, Y),$
$(\{2, 3\}, \{2\}), (\{2, 3\}, \{1, 2\}), (\{2, 3\}, \{2, 3\}), (\{2, 3\}, Y), (Y, \{2\}), (Y, \{1,2\}), (Y, \{2,3\}), (Y, Y) \}$.

By inspection R is reflexive on X, symmetric, and transitive. So R is an equivalence relation.

The equivalence classes of R are
$[\emptyset] = \{ \emptyset, \{1\}, \{3\}, \{1, 3\} \}$ and
$[\{2\}] = \{ \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\} \}$.

Note: Of course we could call $[\emptyset]$ by the name $[\{1\}]$ instead, or $[\{2\}]$ by the name $[\{2, 3\}]$ and so on.

3. *Let R be the relation on $\mathbb{Z}$ such that $(x, y) \in R$ iff $x - y$ is a multiple of 4.*
We can define R as follows: $(x, y) \in R$ iff $x - y = 4k$ for some integer k.

(a) **Reflexivity:**
Is it true that $(x, x) \in R$ for all $x \in \mathbb{Z}$?
Yes, for all $x \in \mathbb{Z}$ we have $x - x = 0 = 4 \cdot 0$, which is a multiple of 4,
thus R is reflexive on $\mathbb{Z}$.

Irreflexivity:
Is it the case that for all $x \in \mathbb{Z}$, $(x, x) \notin R$?
No, there is no integer x such that $(x, x) \notin R$.
We give a counterexample:
$(1, 1) \in R$ since $1 - 1 = 0 = 4 - 0$,
thus R is not irreflexive.

Symmetry:
If $(x, y) \in R$, is it true that $(y, x) \in R$?
Suppose $(x, y) \in R$, ie $x - y$ is a multiple of 4,
ie $x - y = 4k$ for some $k \in Z$.
ie $y - x = -(x - y) = -4k = 4(-k)$.
So $y - x$ is a multiple of 4, hence $(y, x) \in R$.
Thus R is symmetric.

Antiymmetry:
No, if $(x, y) \in R$, it is not necessarily true that $(y, x) \notin R$.
We give a counterexample:
$(5, 1) \in R$ since $5 - 1 = 4$ which is a multiple of 4, but $1 - 5 = -4$ which is also a multiple of 4,
so $(1, 5) \in R$.
Thus R is not antisymmetric.

Transitivity:
If $(x, y) \in R$ and $(y, z) \in R$, is it true that $(x, z) \in R$?
Suppose $(x, y) \in R$,
then $x - y = 4k$ for some $k \in \mathbb{Z}$ ①
and suppose $(y, z) \in R$,
then $y - z = 4m$ for some $m \in \mathbb{Z}$. ②
① + ②, then $(x - y) + (y - z) = 4k + 4m$
ie $x - z = 4(k + m)$, which is a multiple of 4, so $(x, z) \in R$.

Thus R is transitive.

Trichotomy:

Is it true that for all positive integers x, y if $x \neq y$, then either $(x, y) \in R$ or $(y, x) \in R$?

No, it is not true! We show this by using a counterexample:

Choose $1, 2 \in \mathbb{Z}$,

then $2 - 1 = 1$ and $1 - 2 = -1$ and these are not multiples of 4,

so $(2, 1) \notin R$ and $(1, 2) \notin R$.

Thus R does not satisfy trichotomy.

(b) *What kind of relation is R?*

Since R is reflexive on $\mathbb{Z}$, symmetric and transitive, it follows that R is an equivalence relation.

(c) *R is an equivalence relation, so we can give the equivalence classes of R:*

$[x] = \{y \mid (x, y) \in R\}$

We know $(v, w) \in R$ iff $v - w = 4k$, therefore $[x] = \{y \mid x - y = 4k \text{ for some } k \in \mathbb{Z}\}$

$$
\begin{aligned}
[0] &= \{y \mid 0 - y = 4k \text{ for some } k \in \mathbb{Z}\} \\
&= \{y \mid y = -4k\} \\
&= \{..., -8, -4, 0, 4, 8, ...\} \\
[1] &= \{y \mid 1 - y = 4k \text{ for some } k \in \mathbb{Z}\} \\
&= \{y \mid y = -4k + 1\} \\
&= \{... -3, 1, 5, 9, ...\} \\
[2] &= \{y \mid 2 - y = 4k \text{ for some } k \in \mathbb{Z}\} \\
&= \{y \mid y = -4k + 2\} \\
&= \{... -2, 2, 6, 10, ...\} \\
[3] &= \{y \mid 3 - y = 4k \text{ for some } k \in \mathbb{Z}\} \\
&= \{y \mid y = -4k + 3\} \\
&= \{... -1, 3, 7, 11, ...\}
\end{aligned}
$$

[−4], [4], etc. are identical to [0],

similarly [−3], [5], etc. are the same as [1],

similarly [−2], [6], etc. are the same as [2],

and similarly [−1], [7], etc. are the same as [3].

So R has four different equivalence classes namely, [0], [1], [2] and [3].

4. *Suppose $\mathbb{Q}^+$ is the set of all positive quotients n/m, where $n, m \in \mathbb{Z}^+$*
ie $\mathbb{Q}^+$ is the set of positive rational numbers.

Let R be the relation on $\mathbb{Q}^+$, defined by $(x, y) \in R$ iff $y = (a \cdot x) / b$ for some $a, b \in \mathbb{Z}^+$.

Prove that R is an equivalence relation and describe the equivalence classes of R.

We can get the 'feel' of a relation by writing down some of its members. Let us do this with R:

The members of R are ordered pairs of positive rational numbers, such as (1/2, 3/5),

ie $x = 1/2$ and $y = 3/5$.

Does this pair meet the entrance requirement for R, namely that $y = (a \cdot x)/b$?

Yes, where $a = 6$ and $b = 5$.

y a x b

(Test it to check: $3/5 = (6 \cdot (1/2)) / 5$.)

Another member of R is (4, 5).

(It meets the requirement $y = (a \cdot x) / b$, because $5 = (5 \cdot 4) / 4$, where $a = 5$ and $b = 4$.)

Now we want to prove that R is an equivalence relation:

In this kind of proof, we often need to determine whether a certain ordered pair belongs to R. Say, for example, we need to show that $(x, y) \in R$. We must then demonstrate that **x and y meet the requirements of the definition of R**, ie that $y = a \cdot x/b$. (Make sure you use the appropriate sequence for x and y!)

In order to be an equivalence relation, R must be reflexive on $\mathbb{Q}^+$, symmetric and transitive. This gives our 'agenda'.

Reflexivity:

Goal: to show that for every $x \in \mathbb{Q}^+$, $(x, x) \in R$.

Let us relate the definition of reflexivity to the definition of the specific relation R on $\mathbb{Q}^+$,

ie to show that $(\mathbf{x}, \mathbf{x}) \in R$, we must show that $\mathbf{x} = (a \cdot \mathbf{x}) / b$ for some $a, b \in \mathbb{Z}^+$.)

For all $x \in \mathbb{Q}^+$, we know that $x = x$,

ie $\mathbf{x} = 1 \cdot \mathbf{x}/1$ and $1 \in \mathbb{Z}^+$.

Thus $(x, x) \in R$, and so R is reflexive on $\mathbb{Q}^+$.

Symmetry:

Goal: We assume that $(x, y) \in R$, ie $y = (a \cdot x) / b$, and we want to **use** this to demonstrate that $(y, x) \in R$.

Suppose $(x, y) \in R$,

then $y = (a \cdot x) / b$ for some $a, b \in \mathbb{Z}^+$

ie $b \cdot y = a \cdot x$

ie $(b \cdot y) / a = x$

ie $x = (b \cdot y) / a$.

Thus $(y, x) \in R$ and so R is symmetric.

Transitivity:

Goal: We assume that $(x, y) \in R$, ie $y = (a \cdot x) / b$ and that $(y, z) \in R$,

ie $z = (c \cdot y) / d$, and then set out to **use** these facts to prove that $(x, z) \in R$.

Suppose $(x, y) \in R$, then $y = (a \cdot x) / b$ ① for some $a, b \in \mathbb{Z}^+$, and

suppose $(y, z) \in R$, then $z = (c \cdot y) / d$ ② for some $c, d \in \mathbb{Z}^+$.

Substitute ① into ②, then

$z = c \cdot (a \cdot x/b) / d$

ie $z = (ca \cdot x) / bd$

ie $z = (e \cdot x) / f$ where $e = ca$ and $f = bd$ for some $e, f \in \mathbb{Z}^+$.

Thus $(x, z) \in R$ and so R is transitive.

Since R is reflexive on $\mathbb{Q}^+$, symmetric and transitive, R is an equivalence relation.
Next we look at the equivalence classes of R:

Note: Remember that equivalence classes are determined by considering sets of the following format:
$[x] = \{y \mid (x, y) \in R\}$ for all $x \in \mathbb{Q}^+$.
In this case it means that:
$[x] = \{y \mid y = (a \cdot x) / b\}$
Consider x = 1 in the above equation.
$[1] = \{y \mid y = (a \cdot 1) / b\}$
$= \{y \mid y = a/b\}$
This is the set of all positive rational numbers, for example, [1] = [2] = [1/2] = [3/4] = ... etc.
In this example, each equivalence class is equal to every other equivalence class, so there is only one equivalence class in R.

5. *Prove that if R is a relation on $\mathbb{Z}^+$, then R is symmetric iff $R = R^{-1}$.*
Let us first try to prove that if R is symmetric, then $R = R^{-1}$.

Assume R is symmetric. We want to show that $R = R^{-1}$.
Suppose $(x, y) \in R$,
then $(y, x) \in R$ because R is symmetric.
So $(x, y) \in R^{-1}$ according to the definition of R^{-1}.
Hence $R \subseteq R^{-1}$.

Conversely, suppose $(x, y) \in R^{-1}$,
then $(y, x) \in R$, and because R is symmetric, $(x, y) \in R$.
Hence $R^{-1} \subseteq R$.

Since $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$, we can conclude that $R = R^{-1}$.

Next we assume that $R = R^{-1}$. Now we need to show that R is symmetric, ie that if $(x, y) \in R$, then $(y, x) \in R$.
Suppose $(x, y) \in R$,
then $(y, x) \in R^{-1}$.
But because $R = R^{-1}$, $(y, x) \in R$.
Therefore R is symmetric.
This completes the proof that if R is a relation on $\mathbb{Z}^+$, then R is symmetric iff $R = R^{-1}$.

ACTIVITY 6-12

Determine whether P is a partition of X in each of the following cases. If so, describe the corresponding equivalence relation.

(a) $X = \{1, 2, 3\}$ and $P = \{ \emptyset, \{1\}, \{2, 3\} \}$:
A partition of X must consist of nonempty subsets of X.
So P is not a partition of X because $\emptyset \in P$.

(b) $X = \{1, 2, 3\}$ and $P = \{ \{1\}, \{2\}, \{1, 3\} \}$:
P is not a partition of X since $\{1\} \cap \{1, 3\} = \{1\} \neq \emptyset$.

(c) $X = \{1, 2, 3\}$ and $P = \{ \{1,3\}, \{2\} \}$:
P satisfies all the requirements to be a partition of X:
P is a collection of nonempty subsets of X,
and for each $x \in X$ there is some $Y \in P$ such that $x \in Y$,
and for all $Y, W \in P$, if $Y \neq W$ then $Y \cap W = \emptyset$.
The equivalence classes of the corresponding equivalence relation (that we call R) are:
$[2] = \{2\}$, so $(2, 2) \in R$, and
$[1] = [3] = \{1, 3\}$, so $(1, 1)$, $(3, 3)$, $(1, 3)$ and $(3, 1)$ must all be in R.
Therefore $R = \{ (1, 1), (2, 2), (3, 3), (1, 3), (3, 1) \}$.

(d) $X = \{1, 2, 3\}$ and $P = \{ \{1\}, \{2\} \}$:
P is not a partition of X because there is no $Y \in P$ such that $3 \in Y$.

(e) $X = \mathbb{Z}$ and $P = \{\{0\}, \mathbb{Z}^+, Neg\}$ where $Neg = \{x \mid x \in \mathbb{Z} \text{ and } x < 0\}$:
P is a partition of $\mathbb{Z}$ with the equivalence classes $\{0\}$, $\mathbb{Z}^+$ and Neg.
The corresponding equivalence relation is:
$\{(x, y) \mid (x = 0 \text{ and } y = 0) \text{ or } (x \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}^+) \text{ or } (x \in Neg \text{ and } y \in Neg)\}$.

(f) $X = \mathbb{Z}$ and $P = \{ [0], [1], [2], [3], [4] \}$ where
$[0] = \{x \mid x - 0$ is divisible by 5 with zero remainder$\}$
$[1] = \{x \mid x - 1$ is divisible by 5 with zero remainder$\}$
$[2] = \{x \mid x - 2$ is divisible by 5 with zero remainder$\}$
$[3] = \{x \mid x - 3$ is divisible by 5 with zero remainder$\}$
$[4] = \{x \mid x - 4$ is divisible by 5 with zero remainder$\}$.
P is a partition of $\mathbb{Z}$. The reasons are:

- Every element of P is a nonempty subset of $\mathbb{Z}$. Each of them contains at least the representative given between square brackets.
- For all $Y, W \in P$, if $Y \neq W$, then $Y \cap W = \emptyset$, ie different classes do not have any elements in common. No integer can be in two different sets $Y, W \in P$, because no integer gives two different remainders on integer division by 5. (*Note*: If, say, x - 3 is divisible by 5 with zero remainder, then it means x itself leaves 3 as remainder when divided by 5.)
- For each $x \in \mathbb{Z}$, there is some $Y \in P$ such that $x \in Y$, because, after all, any integer x will, when divided by 5, give a remainder of 0, 1, 2, 3 or 4. Subtracting this remainder from x results in a value which is divisible by 5 with zero remainder.

The corresponding equivalence relation is: $\{(x, y) \mid x - y = 5k$, or some integer $k\}$.

ACTIVITY 6-14

1. *Give 5 functions from A = {1, 2, 3, 4} to B = {a, b, c}.*

Think of building each function by filling in the template
$\{ (1,\quad), (2,\quad), (3,\quad), (4,\quad) \}$
with elements of B.

Some possibilities are:

$f_1 = \{ (1, \mathbf{a}), (2, \mathbf{a}), (3, \mathbf{a}), (4, \mathbf{a}) \}$
$f_2 = \{ (1, \mathbf{b}), (2, \mathbf{b}), (3, \mathbf{b}), (4, \mathbf{b}) \}$
$f_3 = \{ (1, \mathbf{c}), (2, \mathbf{c}), (3, \mathbf{c}), (4, \mathbf{c}) \}$
$f_4 = \{ (1, \mathbf{a}), (2, \mathbf{b}), (3, \mathbf{a}), (4, \mathbf{b}) \}$
$f_5 = \{ (1, \mathbf{c}), (2, \mathbf{a}), (3, \mathbf{b}), (4, \mathbf{c}) \}$.

2. *Give all the functions from $A = \{a, b\}$ to $B = \{5, 6, 7\}$.*

We can get the functions by filling in the template $\{ (a, \quad), (b, \quad) \}$ with second co-ordinates chosen from B.

$f_1 = \{ (a, 5), (b, 5) \}$
$f_2 = \{ (a, 6), (b, 6) \}$
$f_3 = \{ (a, 7), (b, 7) \}$
$f_4 = \{ (a, 5), (b, 6) \}$
$f_5 = \{ (a, 5), (b, 7) \}$
$f_6 = \{ (a, 6), (b, 5) \}$
$f_7 = \{ (a, 6), (b, 7) \}$
$f_8 = \{ (a, 7), (b, 5) \}$
$f_9 = \{ (a, 7), (b, 6) \}$.

3. *Give 3 functions from $A \times A$ to B if $A = \{a, b\}$ and $B = \{5, 6, 7\}$.*

Each function has as domain the set

$A \times A = \{ (a, a), (a, b), (b, a), (b, b) \}$.

To build such a function, just fill in the template

$\{ \mathbf{(}(a, a)\mathbf{,} \quad \mathbf{)}, \mathbf{(}(a, b)\mathbf{,} \quad \mathbf{)}, \mathbf{(}(b, a)\mathbf{,} \quad \mathbf{)}, \mathbf{(}(b, b)\mathbf{,} \quad \mathbf{)} \}$

with second co-ordinates chosen from B. Three examples are:

$f_1 = \{ \mathbf{(}(a, a)\mathbf{, 5)}, \mathbf{(}(a, b)\mathbf{, 5)}, \mathbf{(}(b, a)\mathbf{, 5)}, \mathbf{(}(b, b)\mathbf{, 5)} \}$
$f_2 = \{ \mathbf{(}(a, a)\mathbf{, 5)}, \mathbf{(}(a, b)\mathbf{, 6)}, \mathbf{(}(b, a)\mathbf{, 7)}, \mathbf{(}(b, b)\mathbf{, 5)} \}$
$f_3 = \{ \mathbf{(}(a, a)\mathbf{, 7)}, \mathbf{(}(a, b)\mathbf{, 6)}, \mathbf{(}(b, a)\mathbf{, 5)}, \mathbf{(}(b, b)\mathbf{, 6)} \}$.

4. *Let R be a relation on $A = \{1, 2, 3, \{1\}, \{2\}\}$ defined by $R = \{ (1, \{1\}), (1, 3), (2, \{1\}), (2, \{2\}), (\{1\}, 3), (\{2\}, \{1\}) \}$.*

(a) *Is R a function from A to A?*

First we have to ask: is R functional? (ie if $(x, y) \in R$ and $(x, z) \in R$ is $y = z$?) and is $dom(R) = A$?

No, R is not a function. We give a *counterexample*:

$(1, \{1\}) \in R$ and $(1, 3) \in R$,

so 1 appears twice as first co-ordinate, but with different second co-ordinates, namely $\{1\}$ and 3 as partners, so R is not functional and thus not a function.

(We also have that $dom(R) \neq A$ since $3 \notin dom(R)$.)

(b) *Is ran(R) equal to the codomain of R?*

The codomain $A = \{1, 2, 3, \{1\}, \{2\}\}$ is not equal to the range of R

ie $ran(R) = \{ 3, \{1\}, \{2\} \} \neq A$.

5. *Consider the set $\mathcal{P}(A) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}$.*

Show that the relations f, g, and h described below are functional and have as domains $\mathcal{P}(A)$, $\mathcal{P}(A) \times \mathcal{P}(A)$, and $\mathcal{P}(A) \times \mathcal{P}(A)$ respectively.

(a) Let $f = \{(x, y) \mid x, y \in \mathcal{P}(A) \text{ and } y = x'\}$.

We can follow either of two approaches. The brute force approach involves writing out in list notation the set f, so that we can verify by inspection that f is functional and that each element of $\mathcal{P}(A)$ occurs as a first co-ordinate. This approach is suitable only for smallish sets and the set $\mathcal{P}(A)$ is just barely small enough. A more sophisticated approach would involve abstract reasoning with the help of variables.

Let us use the brute force approach here and the abstract approach for subsequent questions where the domain is bigger.

$F = \{(\emptyset, \{a, b, c\}), (\{a\}, \{b, c\}), (\{b\}, \{a, c\}), (\{c\}, \{a, b\}), (\{a, b\}, \{c\}), (\{a, c\}, \{b\}), (\{b, c\}, \{a\}), (\{a, b, c\}, \emptyset) \}$.

By inspection it is clear that $\text{dom}(f) = \mathcal{P}(A)$ and that every element of $\mathcal{P}(A)$ occurs exactly once as a first co-ordinate, so $f: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$.

(b) Let $g = \{ ((u, v), y) \mid (u, v) \in \mathcal{P}(A) \times \mathcal{P}(A) \text{ and } y = u \cup v \}$.

This time we use abstract reasoning.

$\text{Dom}(g) \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$, because g is a relation from $\mathcal{P}(A) \times \mathcal{P}(A)$ to $\mathcal{P}(A)$.

Is it the case that $\mathcal{P}(A) \times \mathcal{P}(A) \subseteq \text{dom}(g)$?

Yes, because if $(u, v) \in \mathcal{P}(A) \times \mathcal{P}(A)$, then $u \cup v$ is a subset of A, ie for each element (u, v) of the set $\mathcal{P}(A) \times \mathcal{P}(A)$ we can find an element y of $\mathcal{P}(A)$ such that $y = u \cup v$.

Since $\text{dom}(g) \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ and $\mathcal{P}(A) \times \mathcal{P}(A) \subseteq \text{dom}(g)$, it follows that $\text{dom}(g) = \mathcal{P}(A) \times \mathcal{P}(A)$.

Is g functional?

Suppose $(x, y) \in g$ and $(x, z) \in g$.

Then $x = (u, v)$ for some $u \subseteq A$ and some $v \subseteq A$,

and $y = u \cup v = z$, so g is indeed functional.

Since $\text{dom}(g) = \mathcal{P}(A) \times \mathcal{P}(A)$ and g is functional, it follows that

$g : \mathcal{P}(A) \times \mathcal{P}(A) \rightarrow \mathcal{P}(A)$, ie g is a function from $\mathcal{P}(A) \times \mathcal{P}(A)$ to $\mathcal{P}(A)$.

(c) Let $h = \{ ((u, v), y) \mid (u, v) \in \mathcal{P}(A) \times \mathcal{P}(A) \text{ and } y = u \cap v \}$.

$\text{Dom}(h) \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ because h is a relation from $\mathcal{P}(A) \times \mathcal{P}(A)$ to $\mathcal{P}(A)$.

Is it the case that $\mathcal{P}(A) \times \mathcal{P}(A) \subseteq \text{dom}(h)$?

Yes, for if $(u, v) \in \mathcal{P}(A) \times \mathcal{P}(A)$ then $u \cap v$ is a subset of A,

so we can find an element y of $\mathcal{P}(A)$ such that $((u, v), y) \in h$, namely $y = u \cap v$.

Since $dom(h) \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ and $\mathcal{P}(A) \times \mathcal{P}(A) \subseteq dom(h)$, it follows that $dom(h) = \mathcal{P}(A) \times \mathcal{P}(A)$.

Is h functional?
If $(x, y) \in h$ and $(x, z) \in h$,
then x has the form (u, v) for some subsets u and v of A, and $y = u \cap v = z$.

Since $dom(h) = \mathcal{P}(A) \times \mathcal{P}(A)$ and h is functional, it follows that

$h: \mathcal{P}(A) \times \mathcal{P}(A) \to \mathcal{P}(A)$, ie h is a function from $\mathcal{P}(A) \times \mathcal{P}(A)$ to $\mathcal{P}(A)$.

6. *For each of the following relations from X to Y, determine whether or not the relation may be regarded as a function from X to Y, then determine the range of the relation.*

A relation R from X to Y is a function iff R is **functional** and **dom(R) = X**.
In each of the following examples we investigate whether or not R (or S) has these two properties, then we determine ran(R) or ran(S).

(a) $X = Y = \mathbb{Z}$ and $R = \{(x, y) \mid y = x\}$.
Determine dom(R):
$Dom(R)= \{x \mid \text{for some } y \in Y, (x, y) \in R\}$
$= \{x \mid \text{for some } y \in \mathbb{Z}, y = x\}$
$= \{x \mid x \text{ is an integer}\}$
$= \mathbb{Z}$.

Next we investigate functionality.
Suppose $(x, y) \in R$ and $(x, z) \in R$
ie $y = x$ and $z = x$
ie $y = z$.
So every x in $\mathbb{Z}$ that appears as a first co-ordinate does so in only one pair.
Hence R is functional.

Because R is functional and $dom(R) = \mathbb{Z}$, it follows that R is a function, so we may write $R: \mathbb{Z} \to \mathbb{Z}$.

(R is in fact a very important function, namely the **identity** function on Z. Informally, R is the function that instructs us, no matter in which city we find ourselves, not to go anywhere else but to stay just where we are.)

Determine ran(R):
$ran(R) = \{y \mid \text{for some } x \in X, (x, y) \in R\}$
$= \{y \mid \text{for some } x \in \mathbb{Z}, y = x\}$
$= \{y \mid y \text{ is an integer}\}$
$= \mathbb{Z}$.

(b) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = x + 1}.

Determine dom(R):
Note: We show that dom(R) $\subseteq$ Z and $\mathbb{Z} \subseteq$ dom(R), ie dom(R) = $\mathbb{Z}$.

By definition we know that R $\subseteq \mathbb{Z} \times \mathbb{Z}$, so dom(R) $\subseteq \mathbb{Z}$.

But we also have that $\mathbb{Z} \subseteq$ dom(R) since for **every** x in $\mathbb{Z}$ there is an **integer** y of the form x + 1, so we have a pair of the form (x, x + 1) in R and therefore x $\in$ dom(R).

Since dom(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ dom(R), it follows that dom(R) = $\mathbb{Z}$.

Next we look at functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = x + 1 and z = x + 1
ie y = x + 1 = z
ie y = z
Hence R is functional.

Since R is functional and dom(R) = $\mathbb{Z}$, R is a function from Z to Z.

Note: R is called the successor function because it tells us to go from x to x + 1.

Determine ran(R):
Note: We show that ran(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ ran(R), ie ran(R) = $\mathbb{Z}$.

Now, we know that ran(R) $\subseteq \mathbb{Z}$, because R $\subseteq \mathbb{Z} \times \mathbb{Z}$.
It is also the case that $\mathbb{Z} \subseteq$ ran(R) because
for **every** integer y we can find an **integer** x such that y = x + 1,
(just take x to be y – 1, then x + 1 = (y – 1) + 1 = y then (y – 1, y) $\in$ R, so y $\in$ ran(R).
Since ran(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ ran(R), it follows that ran(R) = $\mathbb{Z}$.

(c) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = 3 – x}.
Determine dom(R):
dom(R) = {x | for some y $\in$ Y, (x, y) $\in$ R}
= {x | for some y $\in$ Z, y = 3 – x}
= {x | 3 – x is an integer}
= $\mathbb{Z}$

Now for functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = 3 – x and z = 3 – x
ie y = 3 – x = z
ie y = z.
So R is functional.

Since R is functional and dom(R) = $\mathbb{Z}$, R: $\mathbb{Z} \to \mathbb{Z}$.
Determine ran(R):

$$\begin{aligned}
\text{ran(R)} &= \{y \mid \text{for some } x \in X, (x, y) \in R\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = 3 - x\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, x = 3 - y\} \\
&= \{y \mid 3 - y \text{ is an integer}\} \\
&= \mathbb{Z}
\end{aligned}$$

(d) X = Y = $\mathbb{Z}$ and R = $\{(x, y) \mid y = \sqrt{x}\}$, where the notation $\sqrt{x}$ refers to the positive square root of x.
Determine dom(R):
dom(R) = $\{x \mid \text{for some } y \in Y, (x, y) \in R\}$
= $\{x \mid \text{for some } y \in \mathbb{Z}, y = \sqrt{x}\}$
Now, we know that dom(R) $\subseteq \mathbb{Z}$ since R $\subseteq \mathbb{Z} \times \mathbb{Z}$.

Is it also the case that $\mathbb{Z} \subseteq$ dom(R)? Alas, no. Let us find a counterexample:
Take the integer 2. As we saw in study unit 2, $\sqrt{2}$ is irrational, so there can be no integer y equal to $\sqrt{2}$, and hence 2 $\notin$ dom(R).
(Other counterexamples are -1, -2, -3, etc.)
As far as dom(R) is concerned, therefore, we can do no better than to describe dom(R) as $\{x \mid \text{for some } y \in \mathbb{Z}, y = \sqrt{x}\}$.

Equivalently, saying the same thing in different words,
dom(R) = $\{x \mid x = y^2 \text{ for some } y \in \mathbb{Z}\}$.
Note: The advantage of the latter description is that it suggests a way to generate the members of dom(R) in the following way: Start with y = 1, form y^2, then take y = 2 and form y^2, and so on.

Now for functionality.
Suppose $(x, y) \in R$ and $(x, z) \in R$
ie $y = \sqrt{x}$ and $z = \sqrt{x}$
ie $y = \sqrt{x} = z$.
Thus y = z and R is functional.

However, we saw earlier that dom(R) $\neq \mathbb{Z}$, so R is **not** a function from $\mathbb{Z}$ to $\mathbb{Z}$.

Determine ran(R):

$$\begin{aligned}
\text{ran(R)} &= \{y \mid \text{for some } x \in X, (x, y) \in R\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = \sqrt{x}\}.
\end{aligned}$$

We know that ran(R) $\subseteq \mathbb{Z}$ since R $\subseteq \mathbb{Z} \times \mathbb{Z}$.

Is it the case that $\mathbb{Z} \subseteq$ ran(R)?
Well, is it the case that every integer is the square root of some other integer?
Only for integers living in $\mathbb{Z}^{\geq}$, since for every $y \in \mathbb{Z}^{\geq}$, y is the square root of the integer y^2,

ie y = $\sqrt{x}$ for some integer x (just take x = y^2), ie y $\in$ ran(R).
But no negative integer can be written in the form $\sqrt{x}$, because $\sqrt{x}$ refers to the positive square root of x.
As a counterexample, $\sqrt{4} = 2$, whereas −2 can only be indicated by $-\sqrt{4}$, so
$-2 \notin$ ran(R).
Thus ran(R) $\neq \mathbb{Z}$.

(e) X = Y = $\mathbb{Z}$ and R = {(x, y) | $y^2 = x$}.
You may have been strongly tempted to say that this relation is the same as the one we dealt with in (d), but in fact it is not. The present relation contains pairs like (4, 2) **as well as (4, −2)**, since $(-2)^2 = 4$. With every integer x in its domain, the present relation associates both the positive square root $\sqrt{x}$ and the negative square root $-\sqrt{x}$, so we could in fact have described the relation by
R = {(x, y) | x $\in$ Z and y = $\sqrt{x}$ or y = $-\sqrt{x}$}.

Determine dom(R):
dom(R) = {x | for some y $\in$ Y, (x, y) $\in$ R}
= {x | for some y $\in$ Z, $y^2 = x$}.
= {x | $+\sqrt{x}$ and $-\sqrt{x}$ are integers}

Is dom(R) = $\mathbb{Z}$?
Well, we know that dom(R) $\subseteq \mathbb{Z}$, but just as in the previous question, integers like 2 or −5 do not belong to dom(R), so dom(R) $\neq \mathbb{Z}$.

Now for functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R.
Is it necessarily the case that y = z?
No! Take x = 4, as a *counterexample*, then we may think of y as 2 and z as −2. It is clear that y $\neq$ z.

(Informally, R is not functional because it often gives us conflicting instructions: if we are in city number 4, say, R tells us to go directly to city 2 and also to go directly to city −2, and we cannot do both at the same time.)

Determine ran(R):
ran(R) = {y | for some x $\in$ X, (x, y) $\in$ R}
= {y | for some x $\in \mathbb{Z}$, $y^2 = x$}
= {y | y^2 is an integer}
= $\mathbb{Z}$.

(f) X = Y = $\mathbb{R}$ and S = {(x, y) | $x^2 + y^2 = 1$}.
If you wish, you may visualise S as the circle with its centre at the origin and with a radius of one unit.

Determine dom(S):
dom(S) = {x | for some y $\in$ Y, (x, y) $\in$ S}
= {x | for some y $\in$ R, $x^2 + y^2 = 1$}

$= \{x \mid -1 \leq x \leq 1$

Is S functional?
No. We can find a counterexample. Take x = 0, for example.
Then $(0, 1) \in S$ because $0^2 + 1^2 = 1$, and $(0, -1) \in S$ because $0^2 + (-1)^2 = 1$.
So the element 0 of the domain is used more than once as a first co-ordinate.
We can conclude that S is not a function since it is not functional and $dom(S) \neq \mathbb{R}$.

Determine ran(S):

$$\begin{aligned} ran(S) &= \{y \mid \text{for some } x \in X, (x, y) \in S\} \\ &= \{y \mid \text{for some } x \in R, x^2 + y^2 = 1\} \\ &= \{y \mid -1 \leq y \leq 1\}. \end{aligned}$$

7. *Is the relation R on $\mathbb{Z}^+$, which consists of all pairs (x, y) such that $y = x - 1$, a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$?*

Well, R certainly seems functional but the problem lies with dom(R).
Recall that $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$. Is $1 \in dom(R)$, ie can we find an appropriate second co-ordinate inside $\mathbb{Z}^+$ to match with 1?
No, because the only second co-ordinate that is suitable is 0, and $0 \notin \mathbb{Z}^+$. So $dom(R) \neq \mathbb{Z}^+$.
So R is not a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$.

8. *Let A = {a, b, c}. Consider all the equivalence relations on A. (See Activity 6-**10**(1)). How many relations are also functions from A to A?*

We use brute force and then abstract reasoning in our answer. We recommend the latter.

Brute force:

From the previous example we have all the equivalence relations on A. Now we can inspect them to see which are functional:

R_1 = { (a, a), (b, b), (c, c) }
R_2 = { (a, a), (b, b), (c, c), (a, b), (b, a) }
R_3 = { (a, a), (b, b), (c, c), (b, c), (c, b) }
R_4 = { (a, a), (b, b), (c, c), (a, c), (c, a) }
R_5 = { (a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b), (a, c), (c, a) }

Clearly R_1 is the only function from A to A.

Abstract reasoning:

If R is any equivalence relation on A, then R is reflexive on A and therefore dom(R) = A.
But if R is any equivalence relation other than the identity relation {(a, a), (b, b), (c, c)}, then R will fail to be functional.
To see this, note that every equivalence relation on A must contain the pairs (a, a), (b, b) and (c, c). Any additional pair such as, for example, (a, b) will result in a member of A being used more than once as a first co-ordinate.

9. *Let A = {a, b, c}. (See Activity 6-**7**(1))*

(In the answers to the questions that follow, we do everything twice, first using brute force and then using abstract reasoning. We recommend the latter.)

(a) *How many weak partial orders on A (reflexive, antisymmetric and transitive relations) are also functions from A to A?*

Brute force:

Here are all the weak partial orders on A. We can inspect them to see which are functional:

S_1 = { (a,a), (b,b), (c,c) }
S_2 = { (a,a), (b,b), (c,c), (a,b) }
S_3 = { (a,a), (b,b), (c,c), (a,c) }
S_4 = { (a,a), (b,b), (c,c), (b,a) }
S_5 = { (a,a), (b,b), (c,c), (b,c) }
S_6 = { (a,a), (b,b), (c,c), (c,a) }
S_7 = { (a,a), (b,b), (c,c), (c,b) }
S_8 = { (a,a), (b,b), (c,c), (a,b), (a,c) }
S_9 = { (a,a), (b,b), (c,c), (a,b), (c,b) }
S_{10} = { (a,a), (b,b), (c,c), (b,a), (b,c) }
S_{11} = { (a,a), (b,b), (c,c), (b,a), (c,a) }
S_{12} = { (a,a), (b,b), (c,c), (c,a), (c,b) }
S_{13} = { (a,a), (b,b), (c,c), (a,c), (b,c) }
S_{14} = { (a,a), (b,b), (c,c), (a,b), (b,c), (a,c) }
S_{15} = { (a,a), (b,b), (c,c), (a,c), (c,b), (a,b) }
S_{16} = { (a,a), (b,b), (c,c), (b,a), (a,c), (b,c) }
S_{17} = { (a,a), (b,b), (c,c), (b,c), (c,a), (b,a) }
S_{18} = { (a,a), (b,b), (c,c), (c,a), (a,b), (c,b) }
S_{19} = { (a,a), (b,b), (c,c), (c,b), (b,a), (c,a) }

Clearly S_1 is the only function from A to A.

(Incidentally, how do we know that we have found all the weak partial orders on A? Well, because we were systematic: S_1 is the smallest possible chap, with just 3 elements; then we wrote down those with 4 elements; then 5; and finally those with 6.)

Abstract reasoning:

Every weak partial order is reflexive, so every element of A already appears in an ordered pair as first co-ordinate.

The moment a weak partial order has more than just the pairs needed for reflexivity, ie has more pairs than the identity relation on A, some element of A will occur more than once as a first co-ordinate, so the relation will not be functional.

Thus the identity relation on A is the only weak partial order that is also a function from A to A.

(b) *How many strict partial orders on A (irreflexive, antisymmetric and transitive relations) are also functions from A to A?*

Brute force:

Here are all the strict partial orders on A, so that we can inspect them to see which are functional:

T_1 = { }
T_2 = { (a,b) }
T_3 = { (a,c) }
T_4 = { (b,a) }
T_5 = { (b,c) }
T_6 = { (c,a) }
T_7 = { (c,b) }
T_8 = { (a,b), (a,c) }
T_9 = { (a,b), (c,b) }
T_{10} = { (b,a), (b,c) }
T_{11} = { (b,a), (c,a) }
T_{12} = { (c,a), (c,b) }
T_{13} = { (a,c), (b,c) }
T_{14} = { (a,b), (b,c), (a,c) }
T_{15} = { (a,c), (c,b), (a,b) }
T_{16} = { (b,a), (a,c), (b,c) }
T_{17} = { (b,c), (c,a), (b,a) }
T_{18} = { (c,a) ,(a,b), (c,b) }
T_{19} = { (c,b), (b,a), (c,a) }

Clearly none of the relations are functions from A to A. Many of the relations are functional, but none have A as domain.
How do we know we have found all the strict partial orders on A? Easy. Each of them is just the corresponding weak partial order with the reflexive pairs thrown away.

Abstract reasoning:

If a strict partial order T is to be a function from A to A, then it must have A as its domain and be functional.
So every member of A must occur as first co-ordinate of exactly one pair in T.
This means that T must be obtainable by filling in the gaps in the template
{ (a,), (b,), (c,) }
in such a way that the result is irreflexive, antisymmetric, and transitive.

Let us try to fill the gaps:
To assign to **a** the value **a** would violate irreflexivity.
So suppose we start by assigning to **a** the value **b**:
{ (a, **b**), (b,), (c,) }.
Then we cannot in the next pair assign to **b** the value **a** (because that would violate antisymmetry), nor can we assign to **b** the value **b** (because that would violate irreflexivity), so we must assign to **b** the value **c**: { (a, **b**), (b, **c**), (c,) }.

To maintain transitivity, we are now forced to add the pair (a, c) even before we can think about filling the gap in the pair (c,). But then we have two different pairs starting with **a** and this violates functionality.
Our last hope is to start by assigning to **a** the value **c**: { (a, **c**), (b,), (c,) }.
But then, reasoning as before, we find that to **c** we must assign the value **b**, so that transitivity demands the inclusion of the pair (a, b), and the relation loses functionality.
From this we conclude that there is no way to fill in the template so as to produce a relation that is both a strict partial order and a function.

STUDY UNIT 7

ACTIVITY 7-4

1. *Write down the possible surjective functions from X to Y:*

(b) X = {a, b} and Y = {c, d}:
To obtain a surjective function from X to Y, we must try to fill in, using all the members of Y, the template { (a,), (b,) }.
This can be done in two ways:
g_1 = { (a, c), (b, d) }
and g_2 = { (a, d), (b, c) }.

(c) X = {a, b} and Y = {c, d, e}:
The template
{ (a,), (b,) }
cannot be completed in a way that uses all the elements of Y, because Y has 3 members and there are only two gaps to be filled, so there are no surjective functions from X to Y in this case.

2. *Let f: $\mathbb{Z} \to \mathbb{Z}$ be defined by f(x) = x + 1.*

(a) *Determine f[$\mathbb{Z}$] (= ran(f)) . (Do not give specific examples.)*

$$
\begin{aligned}
f[\mathbb{Z}] &= \{y \mid \text{for some } x \in \mathbb{Z}, (x, y) \in f\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = x + 1\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, x = y - 1\} \\
&= \{y \mid y - 1 \text{ is an integer}\} \\
&= \mathbb{Z}
\end{aligned}
$$

(b) *Is f surjective? If f is not surjective, provide a counterexample to show why it is not surjective.*
f is surjective because the range of f is equal to the codomain of f: f[$\mathbb{Z}$] = $\mathbb{Z}$.

3. *Let g: $\mathbb{Z} \to \mathbb{Z}$ be defined by g(x) = 4x + 8.*

(a) *Determine g[$\mathbb{Z}$] (= ran(g)) . (Do not give specific examples.)*

$$
\begin{aligned}
g[\mathbb{Z}] &= \{y \mid \text{for some } x \in \mathbb{Z}, (x, y) \in g\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = 4x + 8\}
\end{aligned}
$$

$= \{y \mid \text{for some } x \in \mathbb{Z}, x = (y-8)/4\}$
$= \{y \mid (y-8)/4 \text{ is an integer}\}$
$\neq \mathbb{Z}$

(b) *Is g surjective? If g is not surjective, provide a counterexample to show why it is not surjective.*
We give a counterexample: Let y = 9 ($9 \in \mathbb{Z}$), then $x = (y-8)/4 = (9-8)/4 = ¼ \notin \mathbb{Z}$, so if y = 9, no integer x can be found such that $(x, 9) \in g$. Thus $9 \notin g[\mathbb{Z}]$.
This means that $\mathbb{Z}$ is not a subset of $g[\mathbb{Z}]$ because each element of $\mathbb{Z}$ is not an element of $g[\mathbb{Z}]$, so $g[\mathbb{Z}] \neq \mathbb{Z}$. Thus g is not surjective because the range of g is not equal to the codomain.

ACTIVITY 7-5

1. *Write down the injective (one-to-one) functions from X to Y.*

(b) X = {2, 4} and Y = {1, 3}:
We can fill in the gaps in the template
{ (2,), (4,) }
so that different pairs contain different elements of Y in two ways, giving the injective functions:
$f_1 = \{ (2, 1), (4, 3) \}$
and $f_2 = \{ (2, 3), (4, 1) \}$.

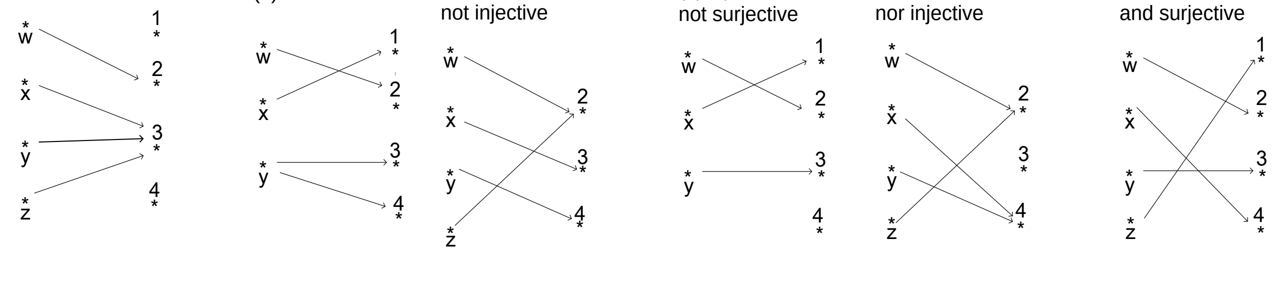
(c) X = {2, 4} and Y = {1, 3, 5}:
We can fill in the gaps in the template
{ (2,), (4,) }
so that different pairs contain different elements of Y in several ways.
$f_1 = \{ (2, 1) , (4, 3) \}$
$f_2 = \{ (2, 3) , (4, 1) \}$
$f_3 = \{ (2, 1) , (4, 5) \}$
$f_4 = \{ (2, 5) , (4, 1) \}$
$f_5 = \{ (2, 3) , (4, 5) \}$
$f_6 = \{ (2, 5) , (4, 3) \}$.

2. *Consider h: $\mathbb{Z} \to \mathbb{Z}$ be defined by g(x) = 2x – 5. Is h injective?*
Assume g(u) = g(v)
then 2u – 5 = 2v – 5
ie u = v.
Therefore h is injective.

ACTIVITY 7-6

For each of the following diagrams, write down the corresponding relation it represents, then provide the reason(s) why the relation has the given property or properties.

(a) a function (b) not a function (c) surjective, not injective (d) injective, not surjective (e) neither surjective, nor injective (f) injective, and surjective



Note: The arrows that should be present in the graphs are not present in the graphs in the study guide.

{(w, 2), (x, 3), (y, 3), (z, 3)} is a function: for each first co-ordinate there is only one corresponding second co-ordinate and the domain is the set {w, x, y, z}.

{(w, 2), (x, 1), (y, 3), (y, 4)} is not a function since it is not functional: y appears twice as first co-ordinate, but y does not have only one corresponding second co-ordinate.
{(w, 2), (x, 3), (y, 4), (z, 2)} is surjective since the range is the set {2, 3, 4} = codomain but it is not injective since w and z share the same corresponding second co-ordinate namely 2.

{(w, 2), (x, 1), (y, 3)} is injective since each first co-ordinate has a unique corresponding second co-ordinate, but not surjective since 4 is an element of the codomain {1, 2, 3, 4} but not an element of the range {1, 2, 3}.

{(w, 2), (x, 4), (y, 4), (z, 2)} is not injective since x and y share the same corresponding second co-ordinate namely 4 and w and z share the same corresponding second co-ordinate namely 2, and the relation is not surjective since 3 is not an element of the range {2, 4} but 3 is an element of the codomain {2, 3, 4}.

{(w, 2), (x, 4), (y, 3), (z, 1)} is injective since each first co-ordinate has a unique corresponding second co-ordinate, and it is surjective since the range is the set {1, 2, 3, 4} = codomain.

ACTIVITY 7-10

Determine f ○f, g ○g, g ○f, and f ○g in the following cases:

(a) f: Z → Z is defined by the rule f(x) = x + 1 and g: Z → Z is defined by the rule g(x) = x – 1:

f ○ f: Z → Z is defined by (f ○ f)(x):

(f ○ f)(x)
= f(f(x))
= f(**x** + **1**) (replace f(x) by **x** + **1**)
= (**x** + **1**) + 1 (f(**x**) = **x** + 1, so f(**x** + **1**) = (**x** + **1**) + 1)
= x + 2.

Note: If you want to express what this means in words: if you feed f ○ f an element x, it spits out the same thing you get if you feed x to f and then feed the result to f again.
First you feed **x** to f to get **x** + 1. Now feeding **x + 1** to f gives you (**x + 1**) + 1 = x + 2, because f takes anything you feed it and adds 1 to it.

g ○ g: Z → Z is defined by (g ○ g)(x) = g(g(x))
= g(x – 1)
= (x – 1) – 1
= x – 2.

g ○ f: Z → Z is defined by (g ○ f)(x) = g(f(x))
= g(**x** + **1**) (replace f(x) by **x** + **1**)
= (**x** + **1**) – 1 (g(**x**) = **x** – 1, so g(**x** + **1**) = (**x** + **1**) – 1)

$= x.$

$f \circ g: Z \to Z$ is defined by $(f \circ g)(x) = f(g(x))$

$= f(x - 1)$
$= (x - 1) + 1$
$= x.$

(b) $f: R \to R$ is defined by $f(x) = 3x - 2$, and
$g: R \to R$ is defined by $g(x) = x^2 + x$:

$f \circ f: R \to R$ is defined by $(f \circ f)(x) = f(f(x))$

$= f(\mathbf{3x} - \mathbf{2})$ (replace $f(x)$ by $\mathbf{3x} - \mathbf{2}$)
$= 3(\mathbf{3x} - \mathbf{2}) - 2$ ($f(\mathbf{x}) = 3\mathbf{x} - 2$, so $f(\mathbf{3x} - \mathbf{2}) = 3(\mathbf{3x} - \mathbf{2}) - 2$)
$= 9x - 8.$

$g \circ g: R \to R$ is defined by $(g \circ g)(x) = g(g(x))$

$= g(x^2 + x)$
$= (x^2 + x)^2 + (x^2 + x)$
$= (x^2 + x)(x^2 + x) + (x^2 + x)$
$= x^4 + 2x^3 + x^2 + x^2 + x$
$= x^4 + 2x^3 + 2x^2 + x.$

$g \circ f: R \to R$ is defined by $(g \circ f)(x) = g(f(x))$

$= g(3x - 2)$
$= (3x - 2)^2 + (3x - 2)$
$= 9x^2 - 12x + 4 + 3x - 2$
$= 9x^2 - 9x + 2.$

$f \circ g: R \to R$ is defined by $(f \circ g)(x) = f(g(x))$

$= f(x^2 + x)$
$= 3(x^2 + x) - 2$
$= 3x^2 + 3x - 2.$

(c) $f: \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $f(x) = 113$, and
$g: \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $g(x) = x + 1$:

$f \circ f: \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $(f \circ f)(x) = f(f(x))$

$= f(113) = 113.$

$g \circ g: \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $(g \circ g)(x) = g(g(x))$

$= g(x + 1)$
$= (x + 1) + 1$

$= x + 2.$

$g \circ f: \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $(g \circ f)(x) = g(f(x))$

$= g(113)$
$= 113 + 1$
$= 114$

$f \circ g: \mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by $(f \circ g)(x) = f(g(x))$

$= f(x + 1)$
$=113.$ (because f does not care what you feed it is constantly going to spit out 113 nothing else).

ACTIVITY 7-12

1. *Write down the bijective (one-to-one correspondence) functions from X to Y in each case:*

(a) $X = \{ \emptyset, \{113\} \}$ and $Y = \{ \{1\} \}$:
To obtain a bijective function from X to Y we must fill in the template
$\{ (\emptyset, \quad), (\{113\}, \quad) \}$
in such a way that different pairs get different elements of Y (for injectivity) and all elements of Y are used up (for surjectivity).
This is not possible, since Y has only one element and there are two pairs needing *different* second co-ordinates.
So there is no bijective function from X to Y in this case.

(b) $X = \{ \emptyset, \{113\} \}$ and $Y = \{ \{1\}, \{2\} \}$:
We need to fill in the template
$\{ (\emptyset, \quad), (\{113\}, \quad) \}$
so that we use, once only, each element of Y. This is possible in 2 ways:
$h_1 = \{ (\emptyset, \{1\}) , (\{113\}, \{2\}) \}$
and $h_2 = \{ (\emptyset, \{2\}) , (\{113\}, \{1\}) \}$.

(c) $X = \{ \emptyset, \{113\} \}$ and $Y = \{ \{1\}, \{2\}, \{7\} \}$:
There is no way to fill in the template $\{ (\emptyset, \quad), (\{113\}, \quad) \}$ so that we use each member of Y exactly once, because there are 3 elements in Y and only 2 gaps to be filled in.
So there are no bijective functions from X to Y in this case.

2. *Check the following functions for injectivity (one-to-one), surjectivity (onto) and bijectivity (a one-to-one correspondence) (ie functions that are both injective and surjective), and give the inverse function of each:*

(a) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = x + 1$:
f is **injective**, because
if $f(u) = f(v)$
then $u + 1 = v + 1$
ie $u = v$.

f is **surjective**, because

$$\begin{aligned}
\mathrm{ran}(f) &= \{y \mid \text{for some } x \in \mathbb{Z}, (x, y) \in f\}\\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = x + 1\}\\
&= \{y \mid \text{for some } x \in \mathbb{Z}, x = y - 1\}\\
&= \{y \mid y - 1 \text{ is an integer}\} = \mathbb{Z}
\end{aligned}$$

Since f: $\mathbb{Z} \to \mathbb{Z}$ is **bijective**, f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{Z}$.
We can determine the inverse function f^{-1}:

$$\begin{aligned}
(y, x) \in f^{-1} &\text{ iff } (x, y) \in f\\
&\text{ iff } y = x + 1\\
&\text{ iff } x = y - 1
\end{aligned}$$

so $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(y) = y - 1$.

Note: We can also write this as
$f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(x) = x - 1$ or
$f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(z) = z - 1$, etc.
It is important to note that it does not matter which variable we use.

(b) f: $\mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = x^2$:

f is **not injective**, because we can find a counterexample:
If we choose $u = 2$ and $v = -2$, then $u \neq v$ but
$f(u) = f(2) = 4$ and $f(v) = f(-2) = 4$
ie $f(u) = f(v)$.

f is **not surjective**, because we can find a counterexample:
If we choose $y = -9$,
then there is no $x \in \mathbb{Z}$ such that $x^2 = y$ (since $x^2 \geq 0$ for every integer x),
so $-9 \notin \mathrm{ran}(f)$.
Hence $\mathrm{ran}(f) \neq \mathbb{Z}$. Since f: $\mathbb{Z} \to \mathbb{Z}$ is **not bijective**, f^{-1} is not a function from $\mathbb{Z}$ to $\mathbb{Z}$.

(c) f: $\mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = 3 - x$:

f is **injective** because

$$\begin{aligned}
\text{if } && f(u) &= f(v)\\
\text{then } && 3 - u &= 3 - v\\
\text{ie } && u &= v.
\end{aligned}$$

f is **surjective** because

$$\begin{aligned}
\mathrm{ran}(f) &= \{y \mid \text{for some } x \in \mathbb{Z}, (x, y) \in f\}\\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = 3 - x\}\\
&= \{y \mid \text{for some } x \in \mathbb{Z}, x = 3 - y\}\\
&= \{y \mid 3 - y \text{ is an integer}\}
\end{aligned}$$

$= \mathbb{Z}$

Since f: $\mathbb{Z} \to \mathbb{Z}$ is **bijective**, f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

$(y, x) \in f^{-1}$ iff $(x, y) \in f$
iff $y = 3 - x$
iff $x = 3 - y$.
So $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(y) = 3 - y$.

(d) f: $\mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = 4x + 5$:

f is **injective** because
if $f(u) = f(v)$
then $4u + 5 = 4v + 5$
ie $4u = 4v$
ie $u = v$.

f is **not surjective** because, if we choose y to be even, say $y = 8$,
then y cannot be written in the form $4x + 5$ (Can you remember why not?)
ie $y \notin \text{ran}(f)$.
Hence $\text{ran}(f) \neq \mathbb{Z}$ (although $\text{ran}(f) \subseteq \mathbb{Z}$).
So f: $\mathbb{Z} \to \mathbb{Z}$ is **not bijective**.

Note: If f were defined to be a function from $\mathbb{R}$ to $\mathbb{R}$, then f would be bijective (can you show this?) with an inverse calculated as follows:

$(y, x) \in f^{-1}$ iff $(x, y) \in f$
iff $y = 4x + 5$
iff $y - 5 = 4x$
iff $(y - 5)/4 = x$.
Hence $f^{-1}: \mathbb{R} \to \mathbb{R}$ is defined by the rule $f^{-1}(y) = (y - 5)/4$.

3. *Consider an identity function $i_C: C \to C$.*

(a) *Prove that $i_C: C \to C$ is bijective.*
A function is **bijective** if it is **injective** and **surjective**.

$i_C: C \to C$ is defined by the rule $i_C(x) = x$ (i_C is the identity function on C):

i_C is **injective**, because
if $i_C(u) = i_C(v)$
then $u = v$.

i_C is **surjective**, because

$$\begin{aligned} ran(i_C) &= \{y \mid \text{for some } x \in C, (x, y) \in i_C\} \\ &= \{y \mid \text{for some } x \in C, y = x\} \\ &= \{y \mid y \in C\} \quad = \quad C \end{aligned}$$

i_C is injective and surjective, thus it is a bijective function.

(c) *Prove that i_C is an equivalence relation on C.*

In order to prove that i_C is an equivalence relation, we have to prove that i_C **is reflexive, symmetric and transitive.**

Reflexivity:
Is it the case that for all $x \in C$, $(x, x) \in i_C$?
Yes, for any $x \in C$, $x = x$,
ie $(x, x) \in i_C$.
Thus i_C is reflexive.

Symmetry:
If $(x, y) \in i_C$, is it the case that $(y, x) \in i_C$?
Suppose $(x, y) \in i_C$,
then $y = x$
ie $x = y$,
therefore $(y, x) \in i_C$.
Thus i_C is symmetric.

Transitivity:
If $(x, y) \in i_C$ and $(y, z) \in i_C$, is it the case that $(x, z) \in i_C$?
*(Assume $(x, y) \in i_C$ and $(y, z) \in i_C$ then **use** this information to prove that $(x, z) \in i_C$.)*

Suppose $(x, y) \in i_C$,
then $y = x$ ①
and
suppose $(y, z) \in i_C$,
then $z = y$. ②

From ① and ② it follows that:
$z = y = x$
ie $z = x$,
therefore $(x, z) \in i_C$.
Thus i_C is transitive.

i_C is reflexive, symmetric and transitive, thus it is an equivalence relation.

STUDY UNIT 8

ACTIVITY 8-3

1. Let X be the set {2, 7}.

(a) *Give 3 binary operations on X, both in list notation and in tabular form.*
Let us compile the table first and then the set of ordered pairs in each case.
There are many possible examples, of which we give just three.

+	**2**	**7**
2	2	2
7	2	2

This table represents the very simple operation that does not care what you feed it, because it has made up its mind to constantly spit out the value 2.

Note: Although we have chosen to call the operation '+', it has *no* connection with ordinary addition whatsoever.
In list notation, + = **{ (**(2, 2), 2**)**, **(**(2, 7), 2**)**, **(**(7, 2), 2**)**, **(**(7, 7), 2**)** }.

Our next example is:

∗	**2**	**7**
2	7	2
7	2	7

In list notation, ∗ = { **(**(2, 2), 7**)**, **(**(2, 7), 2**)**, **(**(7, 2), 2**)**, **(**(7, 7), 7**)** }.

Our final example is:

☐	**2**	**7**
2	7	7
7	2	7

In list notation: ☐ = {**(**(2, 2), 7**)**, **(**(2, 7), 7**)**, **(**(7, 2), 2**)**, **(**(7, 7), 7**)** }.

(b) *Check these operations for commutativity and associativity.*

Commutativity:
+ and * are commutative, whereas ☐ is not.
A quick way to see this is to look at the tables and see which are symmetric about the diagonal from the top left to the bottom right. In the case of ☐, the triangle above the diagonal is not a mirror image of the triangle below, because, for example, 2 ☐ 7 = 7 whereas 7 ☐ 2 = 2.

Associativity:
+ must be associative, because it always spits out 2,
so for all x, y, and z in X, x + (y + z) = 2 = (x + y) + z.
To see that * is associative we have to check all the various cases (8 of them), and if you do, you will find that everything works out.

☐ fails to be associative. A counterexample is provided by the values x=2, y=2, and z=2:

x □ (y □ z) = 2 □ (2 □ 2) = 2 □ 7 = 7, but (x □ y) □ z = (2 □ 2) □ 2 = 7 □ 2 = 2.

2. *Give 2 binary operations on X = {a, b, c} and check them for commutativity and associativity.*

Bear in mind there are many examples of such operations, and we will choose two more or less at random.
First, an example using list notation. Just fill in the template

{ ((a, a),), ((a, b),), ((a, c),), ((b, a),), ((b, b),), ((b, c),), ((c, a),), ((c, b),), ((c, c),) }

to get, for instance,

{ ((a, a), **b**) , ((a, b), **b**) , ((a, c), **b**) , ((b, a), **b**) , ((b, b), **b**) , ((b, c), **b**), ((c, a), **b**) , ((c, b), **b**) , ((c, c), **b**) }.

This operation is commutative and associative - do you agree? Think about it.

Finally, the following is an example of an operation that fails to be either commutative or associative:

*	**a**	**b**	**c**
a	b	c	b
b	a	b	b
c	b	b	b

Commutativity fails, because a * b = c whereas b * a = a.
Associativity fails, because a * (a * a) = a * b = c whereas (a * a) * a = b * a = a.

3. *Consider the dot operation, "•", defined in section 8.1. Let us compare the dot operation on {a, b, c, d} with ordinary multiplication.*

•	**a**	**b**	**c**	**d**
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(a) *We know that ordinary multiplication on R is commutative. Examine x • y and y • x for each x, y ∈ A. What do you conclude?*

a • b = b = b • a
a • c = c = c • a
a • d = d = d • a
b • a = b = a • b
b • c = d = c • b
b • d = c = d • b
c • d = b = d • c

The *dot* operation is commutative.

(b) *We know that R has an identity for multiplication, namely 1.*
This means that 1 • x = x = x • 1 for all x ∈ R. Does A have an element that behaves similarly?

a • a = a
a • b = b
a • c = c
a • d = d
b • a = b
c • a = c

d • a = d It appears that the element a is an identity element for the *dot* operation.

ACTIVITY 8-6

Consider the following vectors: u = (3, 1), v = (−4, −4), and w = (0, −1). Determine the following:

(a) 2u + v = 2(3, 1) + (−4, −4)
= (6, 2) + (−4, −4)
= (2, −2)

(b) u − 3v = (3, 1) − 3(−4, −4)
= (3, 1) + (12, 12)
= (15, 13)

(c) −3(v + w) = −3[(−4, −4) + (0, −1)]
= −3(−4, −5)
= (12, 15)

ACTIVITY 8-7

Consider the following vectors: u = (1, 2, 5) and v = (2, 3, 5). Determine the following:

(a) u · v = (1, 2, 5) · (2, 3, 5) = 1·2 + 2·3 + 5·5 = 2 + 6 + 25 = 33.

(b) v(2u) = (1, 2, 5) · (2×(2, 3, 5)) = (1, 2, 5) · (4, 6, 10) = 1·4 + 2·6 + 5·10 = 4 + 12 + 50 = 66.

ACTIVITY 8-8

(a) A + B = $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 5 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 4 & 0 \end{bmatrix}$

(b) This addition is impossible because the sizes of the matrices (2 × 2 and 2 × 3) do not correspond.

(c) A + B = $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 7 & 7 \end{bmatrix}$

(d) This addition is impossible because the sizes of the matrices (2 × 2 and 2 × 3) do not correspond.

ACTIVITY 8-9

$$2\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 26 \end{bmatrix}$$

Activity 8-10

Two matrices A and B can only be multiplied if the sizes of A and of B match up in the following way:

Schematically:

$$\underset{m\times n}{A} \cdot \underset{n\times k}{B} = \underset{m\times k}{C}$$

(equal)

Determine the following:

1. $\begin{bmatrix} 31 & -3 & 2 \\ 2 & 5 & 1 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix}$

2. $\begin{bmatrix} 9 & 3 \\ 1 & 5 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 5 & 1 \end{bmatrix} = ?$

This multiplication cannot be performed because the sizes of the matrices do not match up as explained above. Both matrices are 3×2.

3. $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 & 3 \\ 1 & 1/3 & 1 \\ 1/2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 0 \\ 8 & 22 & 6 \\ 3/2 & 12 & 9 \end{bmatrix}$

4. *Provide examples of matrices X and Y such that XY is a 3×3 matrix but YX is a 2×2 matrix.*

Let X = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

and Y = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ for example, then

XY = $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ and YX = $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

5. *Provide examples of matrices X and Y such that both X and Y contain at least some nonzero entries, but XY is the 2×2 zero matrix,*

ie $XY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Take X = $\begin{bmatrix} 3 & 0 \\ 5 & 0 \end{bmatrix}$ and Y = $\begin{bmatrix} 0 & 0 \\ 6 & 2 \end{bmatrix}$ for example.

6. *Prove that addition is a commutative operation on the set of 2×2 matrices and that there is a 2×2 matrix that acts as an identity element in respect of addition.*

Let X = $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and Y = $\begin{bmatrix} s & t \\ u & v \end{bmatrix}$.

Then X + Y = $\begin{bmatrix} x+s & y+t \\ z+u & w+v \end{bmatrix}$

= $\begin{bmatrix} s+x & t+y \\ u+z & v+w \end{bmatrix}$

= Y + X

Notice we use the commutativity of ordinary addition inside the matrix.

Let O be the identity element: O = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

7. *Prove that multiplication is **not** a commutative operation on the set of 2×2 matrices, and that there is a 2×2 matrix that acts as an identity element in respect of multiplication.*

Let X = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and Y = $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then XY = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ but YX = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Let I be the identity element: I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

STUDY UNIT 9

ACTIVITY 9-5

1. *Suppose that p represents the statement "It is sunny" and q the statement "It is humid". Write each of the following in abbreviated form:*

(a) *It is sunny and it is not humid*: $p \wedge \neg q$

(b) *It is humid or it is sunny*: $q \vee p$

(c) *It is false that it is humid*: $\neg q$

(d) *It is false that it is sunny and humid*: $\neg (p \wedge q)$

(e) *It is neither sunny nor humid*: $\neg p \wedge \neg q$

(f) *It is not the case that if it is sunny then it is humid*: $\neg (p \rightarrow q)$

(g) *It is humid if it is sunny*: $p \rightarrow q$

(h) *It is humid only if it is sunny*: $q \rightarrow p$

(i) *It is sunny if and only if it is humid*: $p \leftrightarrow q$

(j) *If it is false that it is either sunny or humid, then it is not sunny*: $\neg [(p \vee q) \wedge \neg (p \wedge q)] \rightarrow \neg p$

2. *Construct the truth tables for the following compound statements*:

(a) $[(\neg q) \rightarrow (\neg p)] \rightarrow (p \rightarrow q)]$

p	**q**	**¬p**	**¬q**	**$(\neg q) \rightarrow (\neg p)$**	**$p \rightarrow q$**	**$[(\neg q) \rightarrow (\neg p)] \rightarrow (p \rightarrow q)$**
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(b) $[\neg p \rightarrow (q \wedge (\neg q))] \rightarrow p$

p	**q**	**¬ p**	**¬q**	**$q \wedge (\neg q)$**	**$\neg p \rightarrow (q \wedge (\neg q))$**	**$[\neg p \rightarrow (q \wedge (\neg q))] \rightarrow p$**
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	F	T
F	F	T	T	F	F	T

(c) $p \vee (\neg p)$

p	**¬p**	**$p \vee (\neg p)$**
T	F	T
F	T	T

(d) $[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(e) $(p \vee q) \wedge (\neg p \vee \neg q)$

p	q	$p \vee q$	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$(p \vee q) \wedge (\neg p \vee \neg q)$
T	T	T	F	F	F	F
T	F	T	F	T	T	T
F	T	T	T	F	T	T
F	F	F	T	T	T	F

(f) $(\neg p \rightarrow [\, q \wedge r \,]) \vee r$

p	q	r	$\neg p$	$q \wedge r$	$\neg p \rightarrow [\, q \wedge r \,]$	$(\neg p \rightarrow [\, q \wedge r \,]) \vee r$
T	T	T	F	T	T	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	T
F	F	F	T	F	F	F

(g) $(p \to [\, q \wedge r\,]\,) \leftrightarrow (\,[\, p \to q\,] \vee [\, p \to r\,]\,)$

p	**q**	**r**	**q ∧ r**	**p→ [q ∧ r]**	**p → q**	**p → r**	**[p→q] ∨ [p→r]**	**(p→[q ∧ r])↔ ([p→q] ∨ [p→r])**
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T	F
T	F	T	F	F	F	T	T	F
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Activity 9-6

1. *Express the following sentence symbolically and then determine whether or not it is a tautology*:
If demand has remained constant and prices have been increased, then turnover must have decreased.

*Use **p** to represent the sentence 'demand has remained constant', let **q** represent 'prices have been increased' and let **r** represent 'turnover must have decreased',* then this sentence as a whole is represented by $(p \wedge q) \to r$.

To determine whether this is a tautology, one can compile a truth table. Sometimes, as in this case, there is also a faster way: one works backward from a truth value of F for the whole sentence and determines whether truth values for the sentences can be found to support it.

From our knowledge of the truth table of '→', we know that
if $(p \wedge q) \to r$ is F then $p \wedge q$ is T and r is F, ie p is T, q is T and r is F.

So the sentence $(p \wedge q) \to r$ is not a tautology, because allocating the values mentioned above to p, q, and r would make the sentence as a whole false.

2. *Refer to the truth tables in Activity 9-5. Determine whether each of the statements is a tautology, a contradiction or neither of the two.*

(a) $[(\neg q) \to (\neg p)] \to (p \to q)]$ is a tautology. For all possible combinations of the truth values p and q, $[(\neg q) \to (\neg p)] \to (p \to q)$ is true as can be seen from the final column of the table.

(b) $[\neg p \to (q \wedge (\neg q))] \to p$ is a tautology. An interesting observation can be made from the fourth column where all the truth values are F: The statement $q \wedge (\neg q)$ is a contradiction.

(c) $p \vee (\neg p)$ is a tautology. Feel free to practice your truth table technique on this one. However, here is a second method: For $p \vee (\neg p)$ to be F, both p and $\neg p$ have to be F, which is impossible, so $p \vee (\neg p)$ is always T.

(d) $[p \wedge (p \to q)] \to q$ is a tautology.
In order for the statement as a whole to assume the value F, q must be F while
$p \wedge (p \to q)$ is T.

For $p \wedge (p \to q)$ to be T, p must be T and $p \to q$ must be T.
But since q is F, $p \to q$ can only be T if p is F, whereas we know p is T.
So it is impossible for $[p \wedge (p \to q)] \to q$ to be F.

(e) From the truth table it is clear that $(p \vee q) \wedge (\neg p \vee \neg q)$ is neither a tautology nor a contradiction. Let p be T and q be F, then $(p \vee q) \wedge (\neg p \vee \neg q)$ is T, whereas if p is T and q is also T, then
$(p \vee q) \wedge (\neg p \vee \neg q)$ is F.

(f) This is neither a tautology nor a contradiction. There are F and T truth values in the final column.

(g) The final column tells us that this is not a tautology. We also know that two statements p and q are **logically equivalent** iff the statement $p \leftrightarrow q$ is a tautology.
The two given statements are not logically equivalent because the eighth and ninth columns are not identical, ie $(p \to [q \wedge r])$ is not logically equivalent to $([p \to q] \vee [p \to r])$.
The left-hand side (of the biconditional, $\leftrightarrow$) does not have exactly the same truth value as the right-hand side.

ACTIVITY 9-9

1. *Rewrite $p \leftrightarrow q$ as a statement built up using only $\neg$, $\wedge$ and $\vee$:*

$$
\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\
&\equiv (\neg p \vee q) \wedge (\neg q \vee p)
\end{aligned}
$$

2. *Refer to study unit 6 for the definition of equivalence relations. This kind of relation is reflexive, symmetric and transitive. Show that $\equiv$ is an equivalence relation.*

Let's think about the meaning of the symbol '$\equiv$'. In the context of this question, it states that $p \equiv q$ means that p is equivalent to q, where p and q are two statements.
It must be the case that $p \equiv p$ (p is equivalent to itself), so the relation is reflexive.
If $p \equiv q$, it is also the case that $q \equiv p$ (p and q are equivalent statements), so the relation is symmetric.
If $p \equiv q$ and $q \equiv r$, it must be the case that $p \equiv r$, so the relation is transitive.
This means that $\equiv$ is an equivalence relation on statements.

3. *Truth table for the exclusive OR (XOR)*:

p	**q**	**$p \oplus q$**
T	T	F
T	F	T
F	T	T
F	F	F

4.

p	**q**	**$\neg p$**	**$\neg q$**	**$p \vee \neg q$**	**$\neg (p \vee \neg q)$**	**$\neg p \wedge q$**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

5. *Use the property of double negation and De Morgan's laws to rewrite the following statements so that the not symbol (¬) does not appear outside parentheses:*

(a) $\neg [(p \vee q \vee \neg q) \wedge (q \wedge \neg p)]$

$\equiv \neg (p \vee q \vee \neg q) \vee \neg (q \wedge \neg p)$ *De Morgan's law*

$\equiv (\neg p \wedge \neg q \wedge \neg\neg q) \vee (\neg q \vee \neg\neg p)$ *De Morgan's law*

$\equiv (\neg p \wedge \neg q \wedge q) \vee (\neg q \vee p)$ *Double negation*

(b) $\neg [(p \vee (p \rightarrow q)) \vee (p \wedge q)]$

$\equiv \neg [(p \vee (\neg p \vee q)) \vee (p \wedge q)]$ *Refer to Study Guide, comment p 147, and second table p 148.*

$\equiv \neg (p \vee (\neg p \vee q)) \wedge \neg (p \wedge q)$

$\equiv (\neg p \wedge \neg (\neg p \vee q)) \wedge (\neg p \vee \neg q)$ De Morgan's law

$\equiv (\neg p \wedge (\neg\neg p \wedge \neg q)) \wedge (\neg p \vee \neg q$ De Morgan's law

$\equiv (\neg p \wedge (p \wedge \neg q)) \wedge (\neg p \vee \neg q)$ Double negation

6. *Determine whether or not the following statements are equivalent: $\neg p \wedge (\neg p \wedge \neg q)$ and $\neg (p \vee (p \rightarrow q))$.*

$\neg (p \vee (p \rightarrow q))$

$\equiv \neg (p \vee (\neg p \vee q))$ *Refer to Study Guide, comment p 147, and second table p 148.*

$\equiv \neg p \wedge \neg (\neg p \vee q)$ *De Morgan's law*

$\equiv \neg p \wedge (\neg\neg p \wedge \neg q)$ *De Morgan's law*

$\equiv \neg p \wedge (p \wedge \neg q)$ *Double negation*

Clearly the two given expressions are equivalent.

STUDY UNIT 10

ACTIVITY 10-3

1. *Write down the English equivalent of each of the following statements. Give an opinion on whether or not the statement is true.*

(a) $\exists\ y \in \mathbb{Q}, y = \sqrt{2}$

There exists some rational number y which is equal to $\sqrt{2}$.
This is not true, since $\sqrt{2}$ is not a rational number.

(b) $\forall\ x \in \mathbb{R}, 2x < x^2$

For all real numbers it holds that $2x < x^2$.
We give a counterexample to show that this statement does not hold.
Choose $x = 0$. $2{\cdot}0 = 0$ and $0^2 = 0$. In this case it does not hold that $2x < x^2$.

(c) $\forall\ x \in \mathbb{Z}, x > 0$

For all integers x, it holds that $x > 0$. The set $\mathbb{Z}$ includes all integers, so if we choose $x = 0$ or x equal any negative integer, the statement does not hold.

(d) $\exists\ x \in \mathbb{Z}^+, x = 0$

There exists a positive integer which is equal to 0.
The set $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. The value 0 does not belong to $\mathbb{Z}^+$, so the statement is not true.

ACTIVITY 10-4

Prove by means of truth tables that $\neg (p \wedge q) \equiv (\neg p) \vee (\neg q)$.

p	**q**	**¬ p**	**¬ q**	**p ∧ q**	**¬ (p ∧ q)**	**(¬ p) ∨ (¬ q)**
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

ACTIVITY 10-5

Prove by means of truth tables that $\neg (p \vee q) \equiv (\neg p) \wedge (\neg q)$.

p	**q**	**¬ p**	**¬ q**	**p ∨ q**	**¬ (p ∨ q)**	**(¬ p) ∧ (¬ q)**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

ACTIVITY 10-6

Write down the negations of the following in a useful form.

(a) The negation of $\forall x \in \mathbb{Z}^+, x > 3$:

$\neg (\forall x \in \mathbb{Z}^+, x > 3)$

$\equiv \exists x \in \mathbb{Z}^+, \neg (x > 3)$

$\equiv \exists x \in \mathbb{Z}^+, x \leq 3$

(b) The negation of $\exists x \in \mathbb{R}, 2x = x^2$:

$\neg (\exists x \in \mathbb{R}, 2x = x^2)$

$\equiv \forall x \in \mathbb{R}, \neg (2x = x^2)$

$\equiv \forall x \in \mathbb{R}, 2x \neq x^2$

(c) The negation of $\forall x \in \mathbb{Z}, (x > 0) \vee (x^2 > 0)$:

$\neg [\forall x \in \mathbb{Z}, (x > 0) \vee (x^2 > 0)$

$\equiv \exists x \in \mathbb{Z}, \neg [(x > 0) \vee (x^2 > 0)]$

$\equiv \exists x \in \mathbb{Z}, \neg (x > 0) \wedge \neg (x^2 > 0)$

$\equiv \exists x \in \mathbb{Z}, (x \leq 0) \wedge (x^2 \leq 0)$

(d) The negation of $\exists y \in \mathbb{Z}^+, (y \leq 10) \wedge (y \neq 0)$:

$\neg [\exists y \in \mathbb{Z}^+, (y \leq 10) \wedge (y \neq 0)]$

$\equiv \forall y \in \mathbb{Z}^+, \neg [(y \leq 10) \wedge (y \neq 0)]$

$\equiv \forall y \in \mathbb{Z}^+, \neg (y \leq 10) \vee \neg (y \neq 0)$

$\equiv \forall y \in \mathbb{Z}^+, (y > 10) \vee (y = 0)$

(e) The negation of $\exists x \in A, P(x) \wedge Q(x)$:

$\neg (\exists x \in A, P(x) \wedge Q(x))$

$\equiv \forall x \in A, \neg (P(x) \wedge Q(x))$

$\equiv \forall x \in A, \neg P(x) \vee \neg Q(x)$

(f) $\forall x \in \mathbb{Z}^+, (x \leq 3) \rightarrow (x^3 \geq 1)$

$\neg (\forall x \in \mathbb{Z}^+, (x \leq 3) \rightarrow (x^3 \geq 1))$

$\equiv \neg (\forall x \in \mathbb{Z}^+, \neg (x \leq 3) \vee (x^3 \geq 1))$

$\equiv \exists x \in \mathbb{Z}^+, \neg (\neg (x \leq 3) \vee (x^3 \geq 1))$

$\equiv \exists x \in \mathbb{Z}^+, \neg \neg (x \leq 3) \wedge \neg (x^3 \geq 1)$

$\equiv \exists x \in \mathbb{Z}^+, (x \leq 3) \wedge (x^3 < 1)$

ACTIVITY 10-7

For each of (a) to (d) of Activity 10-6, try to decide whether the original statement is true or whether its negation is true or whether neither of the two is true.

(a) The original statement is false. 1, 2 and 3 are elements of $\mathbb{Z}^+$ and they are not greater than 3. The negation of the statement is true for x = 1, 2 and 3.

(b) The original statement is true. Choose x = 2. Then 2x = 4 and $x^2 = 4$.

(c) The negation of the statement is true. Choose x = 0. Then $x \leq 0$ and $x^2 \leq 0$.

(d) The original statement is true. It holds for the elements 1, 2, …, 10. All elements of $\mathbb{Z}^+$ are > 0.

ACTIVITY 10-10

1. *Prove each of the following by direct proof, contrapositive, and contradiction (reductio ad absurdum) respectively. Which strategy worked best?*

(a) If $x^2 - 3x + 2 < 0$ then $x > 0$.

Direct proof:

Suppose $x^2 - 3x + 2 < 0$,
then $(x - 1)(x - 2) < 0$
The result is < 0, so one factor must be < 0 and the other > 0.
ie either $x - 1 < 0$ and simultaneously $x - 2 > 0$
OR else $x - 1 > 0$ and simultaneously $x - 2 < 0$
ie either $x < 1$ and simultaneously $x > 2$ (which is impossible)
OR else $x > 1$ and simultaneously $x < 2$ ie $1 < x < 2$
We can conclude that $1 < x < 2$
Thus $x > 0$.

Contrapositive:

Suppose $x \leq 0$,
then $x^2 \geq 0$ and $-3x \geq 0$ (A minus times a minus, remember?)
thus $x^2 - 3x + 2 \geq 0$ (The sum of non-negative numbers is also ≥ 0).

Contradiction (*Reductio ad absurdum*):

Suppose $x^2 - 3x + 2 < 0$,
then $(x - 1)(x - 2) < 0$
and this means that one of the factors is < 0 and the other is > 0.

Now there are just 2 possibilities for x: either $x > 0$ or $x \leq 0$.
The former is the good possibility, so let us try to eliminate $x \leq 0$.

Suppose, just for the moment, that $x \leq 0$ (bad possibility).
Then $x - 1 < 0$ and $x - 2 < 0$, but this contradicts the original fact that one of these factors must be > 0, so we can discard the bad possibility.
Thus we conclude that $x > 0$.

Which worked best? Well, contrapositive was the shortest, but the one that worked best is the one you feel most comfortable with. You decide. (Most beginners feel safest with direct proof; after a year or two of practice, many people grow to love *reductio ad absurdum* (contradiction), perhaps because so many of the opinions you meet in daily life lead to some absurd conclusion!)

(b) If $x^2 - x - 6 > 0$ then $x \neq 1$:

Direct proof:

Suppose $x^2 - x - 6 > 0$,
then $(x - 3)(x + 2) > 0$
ie either $x - 3 < 0$ and $x + 2 < 0$ (both factors are negative)
OR
$x - 3 > 0$ and $x + 2 > 0$ (both factors are positive)
ie either $x < 3$ and $x < -2$, ie $x < -2$
OR
$x > 3$ and $x > -2$, ie $x > 3$
ie $x < -2$ or $x > 3$
Thus $x \neq 1$.

Contrapositive:

Suppose $x = 1$,
then $x^2 - x - 6 = 1 - 1 - 6 = -6$
ie $x^2 - x - 6 \leq 0$

Contradiction:
Suppose $x^2 - x - 6 > 0$.

Now there are just 2 possibilities for x: either $x = 1$ or $x \neq 1$.
The latter is the good possibility, so let us eliminate $x = 1$.

Suppose, just for the moment, that $x = 1$.
Then $x^2 - x - 6 = 1-1-6 = -6$
ie $x^2 - x - 6 < 0$. But this contradicts our starting assumption.

Hence we conclude that $x \neq 1$.

The easiest? Direct proof, for most people.

(c) *If a + b is odd, exactly one of a and b is odd.*

Direct proof:
Suppose $a + b$ is odd (assume $a, b \in \mathbb{Z}$),
then $a + b = 2n + 1$ for some integer n.
Now there are exactly 2 cases: a is either even or odd.
Case 1: Suppose a is even,
then $a = 2k$ for some integer k
so $b = (a + b) - a$
$= 2n + 1 - 2k$
$= 2(n - k)+1$
Thus b is odd.
Case 2: Suppose a is odd,
then $a = 2k + 1$ for some integer k.
Now $b = (a + b) - a$
$= 2n + 1 - (2k + 1)$
$= 2(n - k)$
Thus b is even.

(Note that we have considered the two cases; not in order to eliminate one of them, as we would in a proof by contradiction, but in order to show that **in both cases** exactly one of a and b is odd.)

Contrapositive:
Suppose a and b are both even or both odd,
then either $a = 2m$ and $b = 2k$ for some integers m and k,
or else $a = 2m + 1$ and $b = 2k + 1$ for some $m, k \in Z$.
So either $a + b = 2m + 2k = 2(m + k)$
which means that $a + b$ is even,
or else $a + b = (2m + 1) + (2k + 1) = 2m + 2k + 2 = 2(m + k + 1)$
which also means that $a + b$ is even.

Contradiction:

Suppose a + b is odd.

There are exactly two possibilities:
either it is the case that exactly one of a and b is odd, or this is not the case.
The former is the good possibility, so we eliminate the latter.

Suppose, just for a moment, that it is **not** the case that exactly one of a and b is odd,
then it may be that a and b are both odd, or else it may be that a and b are both even.

Suppose that a and b are both odd,
then we may write $a = 2m+1$ and $b = 2k + 1$ for some integers m and k.
Now $a + b = 2m + 2k + 2 = 2(m + k + 1)$, ie an even number, contradicting the fact that
a + b is odd.

On the other hand, suppose that a and b are both even, ie $a = 2m$ and $b = 2k$, for some integers m and k.
Then $a + b = 2(m + k)$, ie an even number, also contradicting the fact that a + b is odd.
So exactly one of a and b must be odd.
(Part of this proof is rather similar to the proof by contrapositive.)

The easiest method? Maybe proof by contrapositive.

(d) *If x is even then $x^2 + 4x + 2$ is even (assume $x \in \mathbb{Z}$).*

Direct Proof:

Suppose x is even. Then $x = 2k$ for some integer k.

$$\begin{aligned} \text{So } x^2 + 4x + 2 &= (2k)^2 + 4(2k) + 2 \\ &= 4k^2 + 8k + 2 \\ &= 2(2k^2 + 4k + 1) \end{aligned}$$

Thus $x^2 + 4x + 2$ is even.

Contrapositive:

Suppose $x^2 + 4x + 2$ is odd.

So $x^2 + 4x + 2 = 2m + 1$ for some integer m.
ie $x^2 + 4x + 4 = 2m + 2 + 1$ (completing the square)
ie $(x + 2)(x + 2) = 2(m + 1) + 1$
This means that x + 2 must be odd. (Product of two integers is odd, means both integers must be odd.)

Write this as $x + 2 = 2k + 1$ for some integer k.
ie $x = 2(k - 1) + 1$, which means that x is odd.
We can conclude that, if x is even then $x^2 + 4x + 2$ is even.

Contradiction:
Suppose x is even.
Then $x = 2k$ for some integer k.

There are two possibilities: either $x^2 + 4x + 2$ is even (the good possibility) or it is odd.
Let's eliminate the bad possibility. Assume $x^2 + 4x + 2$ is odd.
Then $x^2 + 4x + 2 = 2m + 1$ for some integer m.
ie $x^2 + 4x + 4 = 2m + 2 + 1$ (completing the square)
ie $(x + 2)(x + 2) = 2(m + 1) + 1$
This means that $x + 2$ must be odd. (Product of two integers is odd, means both integers must be odd.)
Write this as $x + 2 = 2k + 1$ for some integer k.
ie $x = 2(k - 1) + 1$, which means that x is odd.
This contradicts our initial assumption.
We can conclude that $x^2 + 4x + 2$ is even.

(e) *If n is a multiple of 3 then $n^3 + n^2$ is a multiple of 3.*

Direct Proof:
Suppose n is a multiple of 3.
Then $n = 3k$ for some integer k.

$$\begin{aligned} \text{So } n^3 + n^2 &= (3k)^3 + (3k)^2 \\ &= 3(9k^3) + 3(3k^2) \\ &= 3(9k^3 + 3k^2) \end{aligned}$$

It follows that $n^3 + n^2$ is a multiple of 3.

Contrapositive:
Suppose $n^3 + n^2$ is a not a multiple of 3
Then $n(n^2 + n)$ can be written as $3k + 1$, or $3k + 2$ for some integer k.
Let's look at the first alternative:
$n(n^2 + n) = 3k + 1$
This means that both n and $(n^2 + n)$ are not multiples of 3.
We can conclude that n is not a multiple of 3.

Contradiction:
Suppose n is a multiple of 3.
Then $n = 3k$ for some integer k.

Now there are two possibilities: Either $n^3 + n^2$ is a multiple of 3 (the good possibility) or $n^3 + n^2$ is not a multiple of 3.

Let's eliminate the bad possibility, so we assume that $n^3 + n^2$ is not a multiple of 3
then $n(n^2 + n)$ can be written as $3k + 1$, or $3k + 2$ for some integer k.
Let's look at the first alternative:
$n(n^2 + n) = 3k + 1$
This means that both n and $(n^2 + n)$ are not multiples of 3.

We can conclude that n is not a multiple of 3.
But this contradicts our initial assumption.
We can conclude that $n^3 + n^2$ is a multiple of 3.

2. *Provide a counterexample to show that the following is not true for all integers $x > 0$:*
If $x > 0$, then $x^2 - 3x + 1 < 0$.
Choose $x = 3$.
Then $(3)^2 - 3(3) + 1 = 9 - 9 + 1$, which is greater than 0.