# Project 3 Report

ROB-GY 6213 Robot Localization and Navigation

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## Contents

# 1 Introduction

In this project, we implemented a Unscented Kalman Filter for state estimation of a quadrotor. In part 1, we used the position and orientation of the body as measurements. In part 2, we used the linear velocity of the camera as a measurement.

# 2 Unscented Kalman Filter

The Taylor series expansion applied by the EKF is only one way to linearize the transformation of a Gaussian. Another linearization method is applied by the unscented Kalman filter, or UKF, which performs a stochastic linearization through the use of a weighted statistical linear regression process.

Instead of approximating the function g by a Taylor series expansion, the UKF deterministically extracts so-called sigma points from the Gaussian and passes these through the non linear model. In the general case, these sigma points are located at the mean and symmetrically along the main axes of the covariance (two per dimension). For an n-dimensional Gaussian with mean $\mu$ and covariance $\Sigma$, the resulting 2n + 1 sigma points $\mathcal{X}^{[i]}$ are chosen according to the following rule:

$$\begin{aligned}
\mathcal{X}^{[0]} &= \mu \\
\mathcal{X}^{[i]} &= \mu + (\sqrt{(n+\lambda)\Sigma})_i \quad \text{for } i = 1, \dots, n \\
\mathcal{X}^{[i]} &= \mu - (\sqrt{(n+\lambda)\Sigma})_{i-n} \quad \text{for } i = n+1, \dots, 2n
\end{aligned} \tag{1}$$

Here $\lambda = \alpha^2(n+\kappa) - n$, with $\alpha$ and $\kappa$ being scaling parameters that determine how far the sigma points are spread from the mean. Each sigma point $\mathcal{X}[i]$ has two weights associated with it. One weight, $w_m^{[i]}$, is used when computing the mean, the other weight, $w_c^{[i]}$, is used when recovering the covariance of the Gaussian.

$$\begin{aligned}
w_m^{[0]} &= \frac{\lambda}{n+\lambda} \\
w_c^{[0]} &= \frac{\lambda}{n+\lambda} + \left(1 - \alpha^2 + \beta\right) \\
w_m^{[i]} &= w_c^{[i]} = \frac{1}{2(n+\lambda)} \text{ for } i = 1, \dots, 2n
\end{aligned} \tag{2}$$

The parameter $\beta$ can be chosen to encode additional (higher-order) knowledge about the distribution underlying the Gaussian representation. If the distribution is an exact Gaussian, then $\beta = 2$ is the optimal choice. The sigma points are then passed through the non-linear model, thereby probing how the model changes the shape of the Gaussian.

# 3 Part 1

The state of the system is defined as $x$ where

$$x = \begin{bmatrix} p \\ q \\ \dot{p} \\ b_g \\ b_a \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{3}$$

Here, $p$ is the position of the body in the world, $l$ is the orientation, $\dot{p}$ is the linear velocity, and $b_g$ and $b_a$ are the bias of the gyroscope and accelerometer respectively.

## 3.1 Process Model

The process model is non-linear and is defined as

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{n}) = \begin{bmatrix} \dot{p} \\ G(q)^{-1} R(q)\left(\omega_m - b_g - n_g\right) \\ g + R(q)\left(a_m - b_a - n_a\right) \\ n_{b_g} \\ n_{b_a} \end{bmatrix} \tag{4}$$

g is the gravity vector acting in the negative Z-direction of the world frame. $w_m$ and $a_m$ are the measurements of the gyroscope and accelerometer respectively. They are in the body frame, we need to convert them to the world frame. That's why we need to multiply by the rotation matrix $R(q)$. The matrix $G(q)$ maps the Euler angle derivatives to the angular velocity.

We will use the unscented transform to linearize the transformation of our Gaussian state. Since our noise is non-additive i.e the output depends on the noise too, we will have to use an augmented state to estimate our state. We will also need to discretize it using the Euler approximation.

### 3.1.1 Step 1: Compute Sigma Points

Let the dimensionality of $x$ and $q$ be $n$ and $n_q$, respectively, and $n' = n + n_q$ **Note.** q here is the noise, not the orientation. Augment the state $\boldsymbol{x}_t$ as $\boldsymbol{x}_{aug} = \begin{pmatrix} \boldsymbol{x}_{t-1} \\ \boldsymbol{q}_t \end{pmatrix}$ and form the sigma points for the augmented random variable

$$\chi_{aug,t-1}^{(i)} = \boldsymbol{\mu}_{aug,t-1} \pm \sqrt{n' + \lambda'}\left[\sqrt{\boldsymbol{P}_{aug}}\right]_i \quad \boldsymbol{\mu}_{aug,t-1} = \begin{pmatrix} \boldsymbol{\mu}_{t-1} \\ \mathbf{0} \end{pmatrix} \quad \boldsymbol{P}_{aug} = \begin{pmatrix} \boldsymbol{\Sigma}_{t-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_t \end{pmatrix}$$

### 3.1.2 Step 2: Propagate Sigma Points through the nonlinear function $f$

$$\chi_t^{(i)} = f\left(\chi_{aug,t-1}^{(i),x}, u_t, \chi_{aug,t-1}^{(i),n}\right) \quad i = 0, \ldots 2n'$$

### 3.1.3 Step 3: Compute the mean and covariance

$$\overline{\boldsymbol{\mu}}_t = \sum_{i=0}^{2n'} W_i^{(m)} \chi_t^{(i)} \quad \overline{\boldsymbol{\Sigma}}_t = \sum_{i=0}^{2n'} W_i^{(c)\prime}\left(\chi_t^{(i)} - \overline{\boldsymbol{\mu}}_t\right)\left(\chi_t^{(i)} - \overline{\boldsymbol{\mu}}_t\right)^T$$

## 3.2 Measurement Model

The measurements in part 1 are the position and orientation of the body with respect to the world. The measurement model will be 6 dimensional.

$$\begin{aligned} \boldsymbol{z}_t &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{\eta} \\ \boldsymbol{\mu}_t &= \overline{\boldsymbol{\mu}}_t + \boldsymbol{K}_t\left(\mathbf{z}_t - C\overline{\boldsymbol{\mu}}_t\right) \\ \boldsymbol{\Sigma}_t &= \overline{\boldsymbol{\Sigma}}_t - \boldsymbol{K}_t \boldsymbol{C} \overline{\boldsymbol{\Sigma}}_t \\ \boldsymbol{K}_t &= \overline{\boldsymbol{\Sigma}}_t \boldsymbol{C}^T\left(\boldsymbol{C}\overline{\boldsymbol{\Sigma}}_t \boldsymbol{C}^T + \boldsymbol{R}\right)^{-1} \end{aligned} \tag{5}$$

Here C matrix is

$$C = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

And R is the covariance of the noise $\eta$. It's of size 6 x 6 since our measurement is of size 6 dimensions - position and orientation.

### 3.3 Results

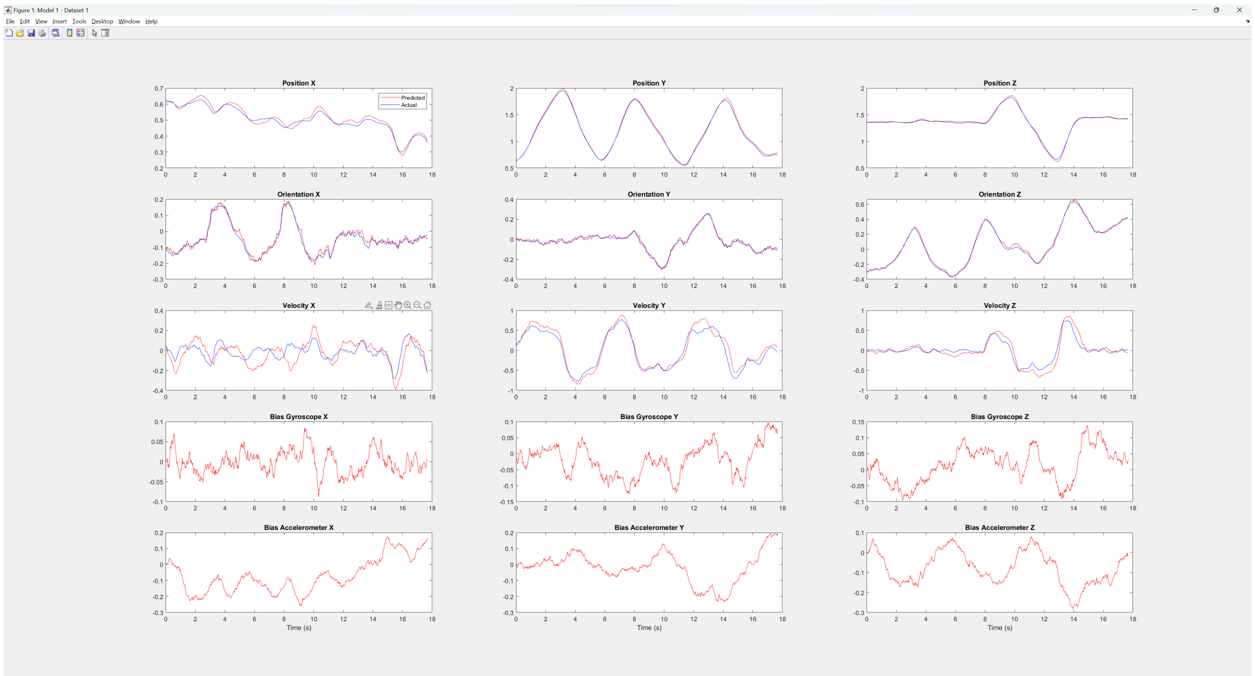


Figure 1: State Estimation results for Dataset 1, Q = 0.1, R = 0.01

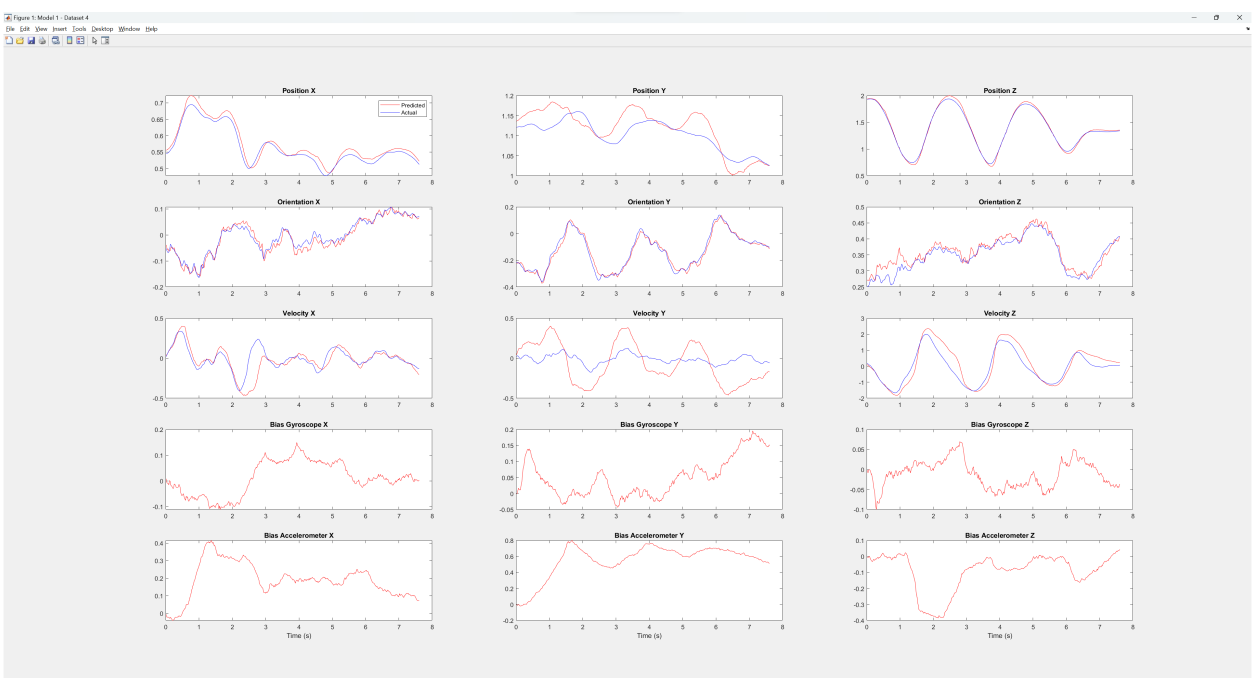


Figure 2: State Estimation results for Dataset 4, Q = 0.1, R = 0.01

Here we can see that since we are getting the measurement of the velocity of the camera, our body linear velocity estimates are quite accurate. This is expected because we are directly measuring the position and the orientation of the robot (body).

# 4 Part 2

## 4.1 Process Model

The process model is the same as the first part.

## 4.2 Measurement Model

The measurement in part 2 is the linear velocity of the camera in the camera frame. The measurement model will be 3 dimensional.

$$\boldsymbol{z}_t = g\left(\boldsymbol{x_t}\right) + \boldsymbol{r}_t \quad \boldsymbol{r_t} \sim N\left(\mathbf{0}, \boldsymbol{R}_t\right)$$

### 4.2.1 Step 1: Compute Sigma Points

Let the dimensionalities of $\boldsymbol{x}$ be $n$ and let $\boldsymbol{\mu}$ be the expected value of the state

$$\chi_t^{(0)} = \overline{\boldsymbol{\mu}}_t \quad \chi_t^{(i)} = \overline{\boldsymbol{\mu}}_t \pm \sqrt{n+\lambda}\left[\sqrt{\overline{\boldsymbol{\Sigma}}_t}\right]_i \quad i = 1, \dots n$$

### 4.2.2 Step 2: Propagate Sigma Points through the nonlinear measurement model function $g$

$$\begin{gathered} Z_t^{(i)} = g\left(\chi_t^{(i)}\right) \quad i = 0, \dots 2n \\ \begin{bmatrix} {}^{C}\dot{\boldsymbol{p}}_C^W \\ {}^{C}\boldsymbol{\omega}_C^W \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{\boldsymbol{B}}^C & -\boldsymbol{R}_{\boldsymbol{B}}^C \boldsymbol{S}\left(\boldsymbol{r}_{BC}^B\right) \\ \mathbf{0} & \boldsymbol{R}_{\boldsymbol{B}}^C \end{bmatrix} \begin{bmatrix} {}^{B}\dot{\boldsymbol{p}}_B^W \\ {}^{B}\boldsymbol{\omega}_B^W \end{bmatrix} \\ \begin{bmatrix} {}^{B}\dot{p}_{\boldsymbol{B}}^{W\,B} & \omega_B^W \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{\boldsymbol{C}}^{\boldsymbol{B}} & -\boldsymbol{R}_{\boldsymbol{C}}^{\boldsymbol{B}} \boldsymbol{S}\left(\boldsymbol{r}_{CB}^C\right) \\ \mathbf{0} & \boldsymbol{R}_{\boldsymbol{C}}^{\boldsymbol{B}} \end{bmatrix} \begin{bmatrix} \dot{p}_C^W \\ {}^{C}\boldsymbol{\omega}_C^W \end{bmatrix} \end{gathered} \tag{6}$$

Our state contains the linear velocity of the robot (body) in the world frame and the measurement is the linear velocity of the camera in the camera frame. We first need to convert the body velocity from the world frame to the body frame, following which we need to use the concept of adjoint to get the velocity of the camera. The following expression summarises this whole process. $z_\mu t$ is in the camera frame, which is the linear velocity of the camera obtained using the state in sigma points.

$$g(\cdot) = {}^{C}\dot{p}_C^W = \boldsymbol{R}_B^C {}^{B}\dot{\boldsymbol{p}}_B^W - \boldsymbol{R}_B^C \boldsymbol{S}\left(\boldsymbol{r}_{BC}^B\right) \boldsymbol{R}_C^B {}^{C}\omega_C^W \tag{7}$$

### 4.2.3 Step 3: Compute the mean and covariance matrices

$$\begin{aligned} \boldsymbol{z}_{\mu,t} &= \sum_{i=0}^{2n} W_i^{(m)} Z_t^{(i)} \\ \boldsymbol{C}_t &= \sum_{i=0}^{2n} W_i^{(c)} \left(\chi_t^{(i),} - \overline{\boldsymbol{\mu}}_t\right) \left(z_t^{(i)} - \boldsymbol{z}_{\mu t}\right)^T \\ \boldsymbol{S}_t &= \sum_{i=0}^{2n} W_i^{(c)} \left(z_t^{(i)} - \boldsymbol{z}_{\mu t}\right) \left(z_t^{(i)} - \boldsymbol{z}_{\mu,t}\right)^T + \boldsymbol{R}_t \end{aligned} \tag{8}$$

Compute the filter gain and the filtered state mean and covariance, conditional to the measurement

$$
\begin{aligned}
\boldsymbol{\mu}_t &= \overline{\boldsymbol{\mu}}_t + \boldsymbol{K}_t\left(\mathbf{z}_t - \boldsymbol{z}_{\mu,t}\right) \\
\boldsymbol{\Sigma}_t &= \overline{\boldsymbol{\Sigma}}_t - \boldsymbol{K}_t \boldsymbol{S}_t \boldsymbol{K}_t^T \\
\boldsymbol{K}_t &= \boldsymbol{C}_t \boldsymbol{S}_t^{-1}
\end{aligned}
$$

## 4.3 Results


Figure 3: State Estimation Results for Dataset 1, Q = 0.01, R = 0.001

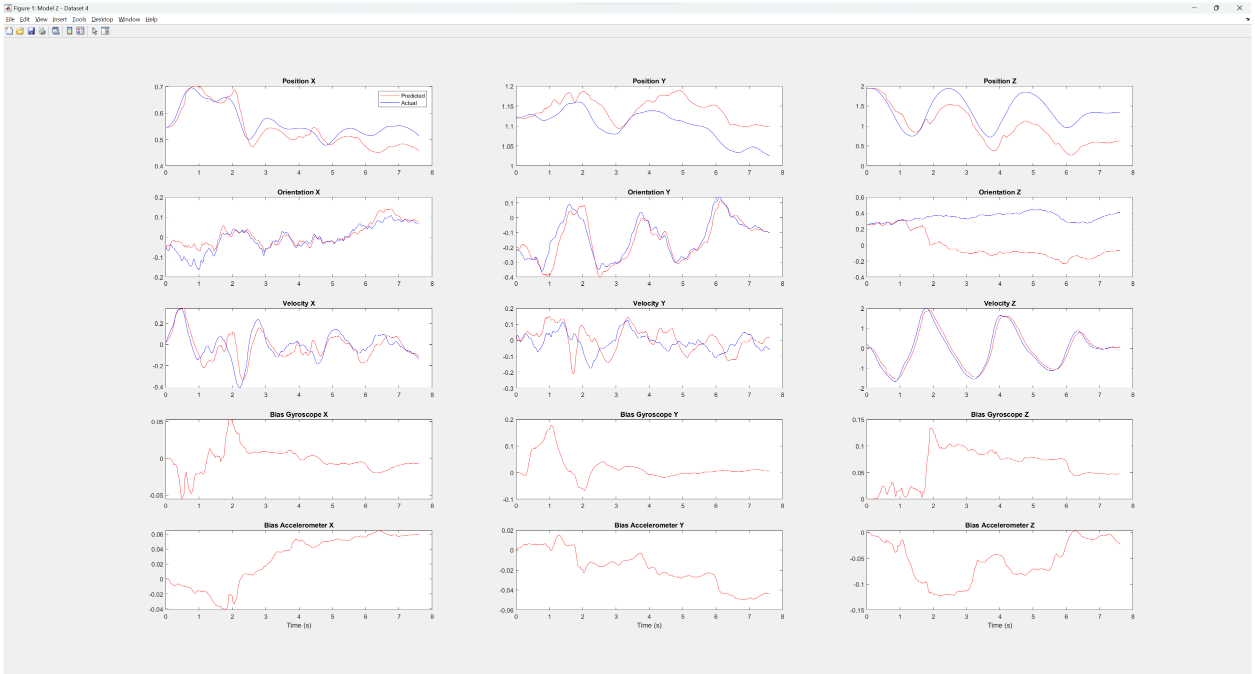


Figure 4: State Estimation Results for Dataset 4, Q = 0.0001, R = 0.0001

Here we can see that since we are getting the measurement of the velocity of the camera, our body linear velocity estimates aren't very correct. This is expected because we aren't directly measuring the linear velocity of the body.