AdMU ProgVar

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Contents

1. Data Structures

1.1. Dynamic Data Structures. Allows undoing of operations.

```
struct state {/*...*/};
struct operation {/*...*/};
struct data_struct {
--- state s;
--- void apply_operation(operation &op) {
------- /*...*/
------- s = /*...*/; }
--- void rollback(state &old_s) {
------- /*...*/
------- s = old_s; } };
struct segtree {
--- int l, r;
--- std::vector<operation> operations;
--- segtree *left, *right;
--- segtree(int l, int r) : l(l), r(r) {
------- if (l == r) {
----------- left = right = NULL;
------- } else {
----------- int m = (l + r) / 2;
----------- left = new segtree(l, m);
----------- right = new segtree(m + 1, r); } }
--- void add_operation(int _l, int _r, operation &op) {
------- if (_l <= l && r <= _r) {
----------- operations.push_back(op);
------- } else if (_r < l || r < _l) {
----------- return;
------- } else {
----------- left->add_operation(_l, _r, op);
----------- right->add_operation(_l, _r, op); } }
--- void solve(data_struct &ds, std::vector<int> &ans) {
------- state old_s = ds.s;
------- for (operation &op : operations)
----------- ds.apply_operation(op);
------- if (l == r) {
----------- ans[l] = /*...*/;
------- } else {
----------- left->solve(ds, ans);
----------- right->solve(ds, ans); }
------- ds.rollback(old_s); } };
```

1.2. Fenwick Tree.

```
struct fenwick {
- vi ar;
- fenwick(vi &_ar) : ar(_ar.size(), 0) {
--- for (int i = 0; i < ar.size(); ++i) {
----- ar[i] += _ar[i];
----- int j = i | (i+1);
----- if (j < ar.size())
------- ar[j] += ar[i]; } }
- int sum(int i) {
--- int res = 0;
--- for (; i >= 0; i = (i & (i+1)) - 1)
----- res += ar[i];
--- return res; }
- int sum(int i, int j) {  return sum(j) - sum(i-1);  }
- void add(int i, int val) {
--- for (; i < ar.size(); i |= i+1)
----- ar[i] += val; }
- int get(int i) {
--- int res = ar[i];
--- if (i) {
----- int lca = (i & (i+1)) - 1;
----- for (--i; i != lca; i = (i&(i+1))-1)
------- res -= ar[i]; }
--- return res; }
- void set(int i, int val) {  add(i, -get(i) + val);  }
- // range update, point query //
- void add(int i, int j, int val) {
--- add(i, val); add(j+1, -val); }
- int get1(int i) { return sum(i); } };
```

1.3. Leq Counter.

1.3.1. *Leq Counter Array.*

```
#include "segtree.cpp"
struct LeqCounter {
- segtree **roots;
- LeqCounter(int *ar, int n) {
--- std::vector<ii> nums;
--- for (int i = 0; i < n; ++i)
----- nums.push_back({ar[i], i});
--- std::sort(nums.begin(), nums.end());
--- roots = new segtree*[n];
--- roots[0] = new segtree(0, n);
--- int prev = 0;
--- for (ii &e : nums) {
----- for (int i = prev+1; i < e.first; ++i)
------- roots[i] = roots[prev];
----- roots[e.first] = roots[prev]->update(e.second, 1);
----- prev = e.first; }
--- for (int i = prev+1; i < n; ++i)
----- roots[i] = roots[prev]; }
- int count(int i, int j, int x) {
--- return roots[x]->query(i, j); } };
```

1.3.2. *Leq Counter Map.*

```
struct LeqCounter {
- std::map<int, segtree*> roots;
- std::set<int> neg_nums;
- LeqCounter(int *ar, int n) {
--- std::vector<ii> nums;
--- for (int i = 0; i < n; ++i) {
----- nums.push_back({ar[i], i});
----- neg_nums.insert(-ar[i]);
--- }
--- std::sort(nums.begin(), nums.end());
--- roots[0] = new segtree(0, n);
--- int prev = 0;
--- for (ii &e : nums) {
----- roots[e.first] = roots[prev]->update(e.second, 1);
----- prev = e.first; } }
- int count(int i, int j, int x) {
--- auto it = neg_nums.lower_bound(-x);
--- if (it == neg_nums.end())   return 0;
--- return roots[-*it]->query(i, j); } };
```

1.4. Misof Tree. A simple tree data structure for inserting, erasing, and querying the nth largest element.

```
#define BITS 15
struct misof_tree {
- int cnt[BITS][1<<BITS];
- misof_tree() { memset(cnt, 0, sizeof(cnt)); }
- void insert(int x) {
--- for (int i = 0; i < BITS; cnt[i++][x]++, x >>= 1); }
- void erase(int x) {
--- for (int i = 0; i < BITS; cnt[i++][x]--, x >>= 1); }
- int nth(int n) {
--- int res = 0;
--- for (int i = BITS-1; i >= 0; i--)
```

```
----- if (cnt[i][res <<= 1] <= n) n -= cnt[i][res], res |= 1;
--- return res; } };
```

1.5. **Mo's Algorithm.**

```
struct query {
- int id, l, r; ll hilbert_index;
- query(int id, int l, int r) : id(id), l(l), r(r) {
--- hilbert_index = hilbert_order(l, r, LOGN, 0); }
- ll hilbert_order(int x, int y, int pow, int rotate) {
--- if (pow == 0) return 0;
--- int hpow = 1 << (pow-1);
--- int seg = ((x<hpow) ? ((y<hpow)?0:3) : ((y<hpow)?1:2));
--- seg = (seg + rotate) & 3;
--- const int rotate_delta[4] = {3, 0, 0, 1};
--- int nx = x & (x ^ hpow), ny = y & (y ^ hpow);
--- int nrot = (rotate + rotate_delta[seg]) & 3;
--- ll sub_sq_size = ll(1) << (2*pow - 2);
--- ll ans = seg * sub_sq_size;
--- ll add = hilbert_order(nx, ny, pow-1, nrot);
--- ans += (seg==1 || seg==2) ? add : (sub_sq_size-add-1);
--- return ans; }
- bool operator<(const query& other) const {
--- return this->hilbert_index < other.hilbert_index; } };
std::vector<query> queries;
for(const query &q : queries) {  // [l,r] inclusive
- for(; r > q.r; r--)          update(r, -1);
- for(r = r+1; r <= q.r; r++) update(r);
- r--;
- for( ; l < q.l; l++)         update(l, -1);
- for(l = l-1; l >= q.l; l--) update(l);
- l++; }
```

1.6. **Ordered Statistics Tree.**

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;
template <typename T>
using index_set = tree<T, null_type, std::less<T>,
splay_tree_tag, tree_order_statistics_node_update>;
// indexed_set<int> t; t.insert(...);
// t.find_by_order(index); // 0-based
// t.order_of_key(key);
```

1.7. **Segment Tree.**

1.7.1. *Recursive, Point-update Segment Tree*.

1.7.2. *Iterative, Point-update Segment Tree.*

```
struct segtree {
- int n;
- int *vals;
- segtree(vi &ar, int n) {
--- this->n = n;
--- vals = new int[2*n];
--- for (int i = 0; i < n; ++i)
----- vals[i+n] = ar[i];
--- for (int i = n-1; i > 0; --i)
----- vals[i] = vals[i<<1] + vals[i<<1|1]; }
- void update(int i, int v) {
--- for (vals[i += n] += v; i > 1; i >>= 1)
----- vals[i>>1] = vals[i] + vals[i^1]; }
- int query(int l, int r) {
--- int res = 0;
--- for (l += n, r += n+1; l < r; l >>= 1, r >>= 1) {
----- if (l&1)  res += vals[l++];
----- if (r&1)  res += vals[--r]; }
--- return res; } };
```

1.7.3. *Pointer-based, Range-update Segment Tree.*

```
struct segtree {
- int i, j, val, temp_val = 0;
- segtree *l, *r;
- segtree(vi &ar, int _i, int _j) : i(_i), j(_j) {
--- if (i == j) {
----- val = ar[i];
----- l = r = NULL;
--- } else {
----- int k = (i + j) >> 1;
----- l = new segtree(ar, i, k);
----- r = new segtree(ar, k+1, j);
----- val = l->val + r->val; } }
- void visit() {
--- if (temp_val) {
----- val += (j-i+1) * temp_val;
----- if (l) {
------- l->temp_val += temp_val;
------- r->temp_val += temp_val; }
----- temp_val = 0; } }
- void increase(int _i, int _j, int _inc) {
--- visit();
--- if (_i <= i && j <= _j) {
----- temp_val += _inc;
----- visit();
--- } else if (_j < i or j < _i) {
----- // do nothing
--- } else {
----- l->increase(_i, _j, _inc);
----- r->increase(_i, _j, _inc);
----- val = l->val + r->val; } }
- int query(int _i, int _j) {
--- visit();
--- if (_i <= i and j <= _j)
----- return val;
--- else if (_j < i || j < _i)
----- return 0;
--- else
----- return l->query(_i, _j) + r->query(_i, _j);
} };
```

1.7.4. *Array-based, Range-update Segment Tree -.*

```
struct segtree {
- int n, *vals, *deltas;
- segtree(vi &ar) {
--- n = ar.size();
--- vals = new int[4*n];
--- deltas = new int[4*n];
--- build(ar, 1, 0, n-1); }
- void build(vi &ar, int p, int i, int j) {
--- deltas[p] = 0;
--- if (i == j)
----- vals[p] = ar[i];
--- else {
----- int k = (i + j) / 2;
----- build(ar, p<<1, i, k);
----- build(ar, p<<1|1, k+1, j);
----- pull(p); } }
- void pull(int p) { vals[p] = vals[p<<1] + vals[p<<1|1]; }
- void push(int p, int i, int j) {
--- if (deltas[p]) {
----- vals[p] += (j - i + 1) * deltas[p];
----- if (i != j) {
------- deltas[p<<1] += deltas[p];
------- deltas[p<<1|1] += deltas[p]; }
----- deltas[p] = 0; } }
- void update(int _i, int _j, int v, int p, int i, int j) {
--- push(p, i, j);
--- if (_i <= i && j <= _j) {
----- deltas[p] += v;
----- push(p, i, j);
--- } else if (_j < i || j < _i) {
----- // do nothing
--- } else {
----- int k = (i + j) / 2;
----- update(_i, _j, v, p<<1, i, k);
----- update(_i, _j, v, p<<1|1, k+1, j);
----- pull(p); } }
- int query(int _i, int _j, int p, int i, int j) {
--- push(p, i, j);
--- if (_i <= i and j <= _j)
----- return vals[p];
--- else if (_j < i || j < _i)
----- return 0;
--- else {
----- int k = (i + j) / 2;
----- return query(_i, _j, p<<1, i, k) +
----------- query(_i, _j, p<<1|1, k+1, j); } } };
```

1.7.5. *2D Segment Tree.*

```
struct segtree_2d {
- int n, m, **ar;
- segtree_2d(int n, int m) {
--- this->n = n;    this->m = m;
--- ar = new int[n];
--- for (int i = 0; i < n; ++i) {
----- ar[i] = new int[m];
----- for (int j = 0; j < m; ++j)
------- ar[i][j] = 0; } }
- void update(int x, int y, int v) {
--- ar[x + n][y + m] = v;
--- for (int i = x + n; i > 0; i >>= 1) {
----- for (int j = y + m; j > 0; j >>= 1) {
```

```
------- ar[i>>1][j] = min(ar[i][j], ar[i^1][j]);
------- ar[i][j>>1] = min(ar[i][j], ar[i][j^1]);
- }}} // just call update one by one to build
- int query(int x1, int x2, int y1, int y2) {
--- int s = INF;
--- if(~x2) for(int a=x1+n, b=x2+n+1; a<b; a>>=1, b>>=1) {
----- if (a & 1) s = min(s, query(a++, -1, y1, y2));
----- if (b & 1) s = min(s, query(--b, -1, y1, y2));
--- } else for (int a=y1+m, b=y2+m+1; a<b; a>>=1, b>>=1) {
----- if (a & 1) s = min(s, ar[x1][a++]);
----- if (b & 1) s = min(s, ar[x1][--b]);
--- } return s; } };
```

1.7.6. *Persistent Segment Tree.*

```
struct segtree {
- int i, j, val;
- segtree *l, *r;
- segtree(vi &ar, int _i, int _j) : i(_i), j(_j) {
--- if (i == j) {
----- val = ar[i];
----- l = r = NULL;
--- } else {
----- int k = (i+j) >> 1;
----- l = new segtree(ar, i, k);
----- r = new segtree(ar, k+1, j);
----- val = l->val + r->val;
- } }
- segtree(int i, int j, segtree *l, segtree *r, int val) :
--- i(i), j(j), l(l), r(r), val(val) {}
- segtree* update(int _i, int _val) {
--- if (_i <= i and j <= _i)
----- return new segtree(i, j, l, r, val + _val);
--- else if (_i < i or j < _i)
----- return this;
--- else {
----- segtree *nl = l->update(_i, _val);
----- segtree *nr = r->update(_i, _val);
----- return new segtree(i, j, nl, nr, nl->val + nr->val); } }
- int query(int _i, int _j) {
--- if (_i <= i and j <= _j)
----- return val;
--- else if (_j < i or j < _i)
----- return 0;
--- else
----- return l->query(_i, _j) + r->query(_i, _j); } };
```

1.8. **Sparse Table.**

1.8.1. *1D Sparse table.*

```
int lg[MAXN+1], spt[20][MAXN];
void build(vi &arr, int n) {
- lg[0] = lg[1] = 0;
- for (int i = 2; i <= n; ++i) lg[i] = lg[i>>1] + 1;
- for (int i = 0; i < n; ++i) spt[0][i] = arr[i];
- for (int j = 0; (2 << j) <= n; ++j)
--- for (int i = 0; i + (2 << j) <= n; ++i)
----- spt[j+1][i] = std::min(spt[j][i], spt[j][i+(1<<j)]); }
int query(int a, int b) {
- int k = lg[b-a+1], ab = b - (1<<k) + 1;
- return std::min(spt[k][a], spt[k][ab]); }
```

1.8.2. *2D Sparse Table*.

```
const int N = 100, LGN = 20;
int lg[N], A[N][N], st[LGN][LGN][N][N];
void build(int n, int m) {
- for(int k=2; k<=std::max(n,m); ++k) lg[k] = lg[k>>1]+1;
- for(int i = 0; i < n; ++i)
--- for(int j = 0; j < m; ++j)
----- st[0][0][i][j] = A[i][j];
- for(int bj = 0; (2 << bj) <= m; ++bj)
--- for(int j = 0; j + (2 << bj) <= m; ++j)
----- for(int i = 0; i < n; ++i)
------- st[0][bj+1][i][j] =
--------- std::max(st[0][bj][i][j],
----------------- st[0][bj][i][j + (1 << bj)]);
- for(int bi = 0; (2 << bi) <= n; ++bi)
--- for(int i = 0; i + (2 << bi) <= n; ++i)
----- for(int j = 0; j < m; ++j)
------- st[bi+1][0][i][j] =
--------- std::max(st[bi][0][i][j],
----------------- st[bi][0][i + (1 << bi)][j]);
- for(int bi = 0; (2 << bi) <= n; ++bi)
--- for(int i = 0; i + (2 << bi) <= n; ++i)
----- for(int bj = 0; (2 << bj) <= m; ++bj)
------- for(int j = 0; j + (2 << bj) <= m; ++j) {
--------- int ik = i + (1 << bi);
--------- int jk = j + (1 << bj);
--------- st[bi+1][bj+1][i][j] =
----------- std::max(std::max(st[bi][bj][i][j],
--------------------------- st[bi][bj][ik][j]),
------------------- std::max(st[bi][bj][i][jk],
--------------------------- st[bi][bj][ik][jk])); } }
int query(int x1, int x2, int y1, int y2) {
- int kx = lg[x2 - x1 + 1],   ky = lg[y2 - y1 + 1];
- int x12 = x2 - (1<<kx) + 1, y12 = y2 - (1<<ky) + 1;
- return std::max(std::max(st[kx][ky][x1][y1],
------------------------ st[kx][ky][x1][y12]),
---------------- std::max(st[kx][ky][x12][y1],
------------------------ st[kx][ky][x12][y12])); }
```

1.9. **Splay Tree.**

```
struct node *null;
struct node {
- node *left, *right, *parent;
- bool reverse; int size, value;
- node*& get(int d) {return d == 0 ? left : right;}
- node(int v=0): reverse(0), size(0), value(v) {
- left = right = parent = null ? null : this; } };
struct SplayTree {
- node *root;
- SplayTree(int arr[] = NULL, int n = 0) {
--- if (!null) null = new node();
--- root = build(arr, n); }
- node* build(int arr[], int n) {
--- if (n == 0) return null;
--- int mid = n >> 1;
--- node *p = new node(arr ? arr[mid] : 0);
--- link(p, build(arr, mid), 0);
--- link(p, build(arr? arr+mid+1 : NULL, n-mid-1), 1);
--- pull(p); return p; }
- void pull(node *p) {
--- p->size = p->left->size + p->right->size + 1; }
- void push(node *p) {
--- if (p != null && p->reverse) {
----- swap(p->left, p->right);
----- p->left->reverse ^= 1;
----- p->right->reverse ^= 1;
----- p->reverse ^= 1; } }
- void link(node *p, node *son, int d) {
--- p->get(d) = son;
--- son->parent = p; }
- int dir(node *p, node *son) {
--- return p->left == son ? 0 : 1; }
- void rotate(node *x, int d) {
--- node *y = x->get(d), *z = x->parent;
--- link(x, y->get(d ^ 1), d);
--- link(y, x, d ^ 1);
--- link(z, y, dir(z, x));
--- pull(x); pull(y); }
- node* splay(node *p) {
--- while (p->parent != null) {
----- node *m = p->parent, *g = m->parent;
----- push(g); push(m); push(p);
----- int dm = dir(m, p), dg = dir(g, m);
----- if (g == null) rotate(m, dm);
----- else if (dm == dg) rotate(g, dg), rotate(m, dm);
----- else rotate(m, dm), rotate(g, dg);
--- } return root = p; }
- node* get(int k) {
--- node *p = root;
--- while (push(p), p->left->size != k) {
------- if (k < p->left->size) p = p->left;
------- else k -= p->left->size + 1, p = p->right; }
--- return p == null ? null : splay(p); }
- void split(node *&r, int k) {
--- if (k == 0) { r = root; root = null; return; }
--- r = get(k - 1)->right;
--- root->right = r->parent = null;
--- pull(root); }
- void merge(node *r) {
--- if (root == null) {root = r; return;}
--- link(get(root->size - 1), r, 1);
--- pull(root); }
- void assign(int k, int val) {
--- get(k)->value = val; pull(root); }
- void reverse(int L, int R) {
--- node *m, *r; split(r, R + 1); split(m, L);
--- m->reverse ^= 1; push(m); merge(m); merge(r); }
- node* insert(int k, int v) {
--- node *r; split(r, k);
```

```
--- node *p = new node(v); p->size = 1;
--- link(root, p, 1); merge(r);
--- return p; }
- void erase(int k) {
--- node *r, *m;
--- split(r, k + 1); split(m, k);
--- merge(r); delete m; } };
```

1.10. Treap.

1.10.1. *Implicit Treap.*

```
struct cartree {
- typedef struct _Node {
--- int node_val, subtree_val, delta, prio, size;
--- _Node *l, *r;
--- _Node(int val) : node_val(val), subtree_val(val),
------- delta(0), prio((rand()<<16)^rand()), size(1),
------- l(NULL), r(NULL) {}
--- ~_Node() { delete l; delete r; }
- } *Node;
- int get_subtree_val(Node v) {
--- return v ? v->subtree_val : 0;  }
- int get_size(Node v) { return v ? v->size : 0; }
- void apply_delta(Node v, int delta) {
--- if (!v) return;
--- v->delta += delta;
--- v->node_val += delta;
--- v->subtree_val += delta * get_size(v); }
- void push_delta(Node v) {
--- if (!v) return;
--- apply_delta(v->l, v->delta);
--- apply_delta(v->r, v->delta);
--- v->delta = 0; }
- void update(Node v) {
--- if (!v) return;
--- v->subtree_val = get_subtree_val(v->l) + v->node_val
----------------- + get_subtree_val(v->r);
--- v->size = get_size(v->l) + 1 + get_size(v->r); }
- Node merge(Node l, Node r) {
--- push_delta(l);    push_delta(r);
--- if (!l || !r)   return l ? l : r;
--- if (l->size <= r->size) {
----- l->r = merge(l->r, r);
----- update(l);
----- return l;
--- } else {
----- r->l = merge(l, r->l);
----- update(r);
----- return r; } }
- void split(Node v, int key, Node &l, Node &r) {
--- push_delta(v);
--- l = r = NULL;
--- if (!v)     return;
--- if (key <= get_size(v->l)) {
----- split(v->l, key, l, v->l);
----- r = v;
--- } else {
----- split(v->r, key - get_size(v->l) - 1, v->r, r);
----- l = v; }
--- update(v); }
- Node root;
public:
- cartree() : root(NULL) {}
- ~cartree() { delete root; }
- int get(Node v, int key) {
--- push_delta(v);
--- if (key < get_size(v->l))
----- return get(v->l, key);
--- else if (key > get_size(v->l))
----- return get(v->r, key - get_size(v->l) - 1);
--- return v->node_val; }
- int get(int key) { return get(root, key); }
- void insert(Node item, int key) {
--- Node l, r;
--- split(root, key, l, r);
--- root = merge(merge(l, item), r); }
- void insert(int key, int val) {
--- insert(new _Node(val), key); }
- void erase(int key) {
--- Node l, m, r;
--- split(root, key + 1, m, r);
--- split(m, key, l, m);
--- delete m;
--- root = merge(l, r); }
- int query(int a, int b) {
--- Node l1, r1;
--- split(root, b+1, l1, r1);
--- Node l2, r2;
--- split(l1, a, l2, r2);
--- int res = get_subtree_val(r2);
--- l1 = merge(l2, r2);
--- root = merge(l1, r1);
--- return res; }
- void update(int a, int b, int delta) {
--- Node l1, r1;
--- split(root, b+1, l1, r1);
--- Node l2, r2;
--- split(l1, a, l2, r2);
--- apply_delta(r2, delta);
--- l1 = merge(l2, r2);
--- root = merge(l1, r1); }
- int size() { return get_size(root); } };
```

1.10.2. *Persistent Treap*.

1.11. Union Find.

```
struct union_find {
- vi p; union_find(int n) : p(n, -1) { }
- int find(int x) { return p[x] < 0 ? x : p[x] = find(p[x]); }
- bool unite(int x, int y) {
--- int xp = find(x), yp = find(y);
--- if (xp == yp)       return false;
--- if (p[xp] > p[yp])  std::swap(xp,yp);
--- p[xp] += p[yp], p[yp] = xp; return true; }
- int size(int x) { return -p[find(x)]; } };
```

1.12. Unique Counter.

```
struct UniqueCounter {
- int *B; std::map<int, int> last; LeqCounter *leq_cnt;
- UniqueCounter(int *ar, int n) { // 0-index A[i]
--- B = new int[n+1];
--- B[0] = 0;
--- for (int i = 1; i <= n; ++i) {
----- B[i] = last[ar[i-1]];
----- last[ar[i-1]] = i; }
--- leq_cnt = new LeqCounter(B, n+1); }
- int count(int l, int r) {
--- return leq_cnt->count(l+1, r+1, l); } };
```

2. Dynamic Programming

2.1. Dynamic Convex Hull Trick.

```
// USAGE: hull.insert_line(m, b); hull.gety(x);
bool UPPER_HULL = true; // you can edit this
bool IS_QUERY = false, SPECIAL = false;
struct line {
- ll m, b; line(ll m=0, ll b=0): m(m), b(b) {}
- mutable std::multiset<line>::iterator it;
- const line *see(std::multiset<line>::iterator it)const;
- bool operator < (const line& k) const {
--- if (!IS_QUERY) return m < k.m;
--- if (!SPECIAL) {
----- ll x = k.m; const line *s = see(it);
----- if (!s) return 0;
----- return (b - s->b) < (x) * (s->m - m);
--- } else {
----- ll y = k.m; const line *s = see(it);
----- if (!s) return 0;
----- ll n1 = y - b, d1 = m;
----- ll n2 = b - s->b, d2 = s->m - m;
----- if (d1 < 0) n1 *= -1, d1 *= -1;
----- if (d2 < 0) n2 *= -1, d2 *= -1;
----- return (n1) * d2 > (n2) * d1; } } };
struct dynamic_hull : std::multiset<line> {
- bool bad(iterator y) {
--- iterator z = next(y);
--- if (y == begin()) {
----- if (z == end()) return 0;
----- return y->m == z->m && y->b <= z->b; }
--- iterator x = prev(y);
--- if (z == end()) return y->m == x->m && y->b <= x->b;
--- return (x->b - y->b)*(z->m - y->m)>=
--------- (y->b - z->b)*(y->m - x->m); }
- iterator next(iterator y) {return ++y;}
- iterator prev(iterator y) {return --y;}
- void insert_line(ll m, ll b) {
--- IS_QUERY = false;
--- if (!UPPER_HULL) m *= -1;
--- iterator y = insert(line(m, b));
--- y->it = y; if (bad(y)) {erase(y); return;}
```

```
--- while (next(y) != end() && bad(next(y)))
----- erase(next(y));
--- while (y != begin() && bad(prev(y)))
----- erase(prev(y)); }
- ll gety(ll x) {
--- IS_QUERY = true; SPECIAL = false;
--- const line& L = *lower_bound(line(x, 0));
--- ll y = (L.m) * x + L.b;
--- return UPPER_HULL ? y : -y; }
- ll getx(ll y) {
--- IS_QUERY = true; SPECIAL = true;
--- const line& l = *lower_bound(line(y, 0));
--- return /*floor*/ ((y - l.b + l.m - 1) / l.m); }
} hull;
const line* line::see(std::multiset<line>::iterator it)
const {return ++it == hull.end() ? NULL : &*it;}
```

2.2. **Divide and Conquer Optimization.** For DP problems of the form

$$dp(i,j) = min_{k \le j}\{dp(i-1,k) + C(k,j)\}$$

where $C(k,j)$ is some cost function.

```
ll dp[G+1][N+1];
void solve_dp(int g, int k_L, int k_R, int n_L, int n_R) {
- int n_M = (n_L+n_R)/2;
- dp[g][n_M] = INF;
- int best_k = -1;
- for (int k = k_L; k <= n_M && k <= k_R; k++)
--- if (dp[g-1][k]+cost(k+1,n_M) < dp[g][n_M]) {
----- dp[g][n_M] = dp[g-1][k]+cost(k+1,n_M);
----- best_k = k; }
- if (n_L <= n_M-1)
--- solve_dp(g, k_L, best_k, n_L, n_M-1);
- if (n_M+1 <= n_R)
--- solve_dp(g, best_k, k_R, n_M+1, n_R); }
```

3. Geometry

```
#include <complex>
#define x real()
#define y imag()
typedef std::complex<double> point; // 2D point only
const double PI = acos(-1.0), EPS = 1e-7;
```

3.1. Dots and Cross Products.

```
double dot(point a, point b) {
- return a.x * b.x + a.y * b.y; } // + a.z * b.z;
double cross(point a, point b) {
- return a.x * b.y - a.y * b.x; }
double cross(point a, point b, point c) {
- return cross(a, b) + cross(b, c) + cross(c, a); }
double cross3D(point a, point b) {
- return point(a.x*b.y - a.y*b.x, a.y*b.z -
-------------- a.z*b.y, a.z*b.x - a.x*b.z); }
```

3.2. Angles and Rotations.

```
double angle(point a, point b, point c) {
- // angle formed by abc in radians: PI < x <= PI
- return abs(remainder(arg(a-b) - arg(c-b), 2*PI)); }
point rotate(point p, point a, double d) {
- //rotate point a about pivot p CCW at d radians
- return p + (a - p) * point(cos(d), sin(d)); }
```

3.3. Spherical Coordinates.

$$x = r\cos\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\cos\theta\sin\phi \qquad \theta = \cos^{-1} x/r$$
$$z = r\sin\theta \qquad \phi = \operatorname{atan2}(y,x)$$

3.4. Point Projection.

```
point proj(point p, point v) {
- // project point p onto a vector v (2D & 3D)
- return dot(p, v) / norm(v) * v; }
point projLine(point p, point a, point b) {
- // project point p onto line ab (2D & 3D)
- return a + dot(p-a, b-a) / norm(b-a) * (b-a); }
point projSeg(point p, point a, point b) {
- // project point p onto segment ab (2D & 3D)
- double s = dot(p-a, b-a) / norm(b-a);
- return a + min(1.0, max(0.0, s)) * (b-a); }
point projPlane(point p, double a, double b,
--------------- double c, double d) {
- // project p onto plane ax+by+cz+d=0 (3D)
- // same as: o + p - project(p - o, n);
- double k = -d / (a*a + b*b + c*c);
- point o(a*k, b*k, c*k), n(a, b, c);
- point v(p.x-o.x, p.y-o.y, p.z-o.z);
- double s = dot(v, n) / dot(n, n);
- return point(o.x + p.x + s * n.x, o.y +
-------------- p.y +s * n.y, o.z + p.z + s * n.z); }
```

3.5. Great Circle Distance.

```
double greatCircleDist(double lat1, double long1,
--- double lat2, double long2, double R) {
- long1 *= PI / 180; lat1 *= PI / 180; // to radians
- long2 *= PI / 180; lat2 *= PI / 180;
- return R*acos(sin(lat1)*sin(lat2) +
-------- cos(lat1)*cos(lat2)*cos(abs(long1 - long2))); }
// another version, using actual (x, y, z)
double greatCircleDist(point a, point b) {
- return atan2(abs(cross3D(a, b)), dot3D(a, b)); }
```

3.6. Point/Line/Plane Distances.

```
double distPtLine(point p, double a, double b, double c) {
- // dist from point p to line ax+by+c=0
- return abs(a*p.x + b*p.y + c) / sqrt(a*a + b*b);}
double distPtLine(point p, point a, point b) {
- // dist from point p to line ab
- return abs((a.y - b.y) * (p.x - a.x) +
-------- (b.x - a.x) * (p.y - a.y)) /
-------- hypot(a.x - b.x, a.y - b.y);}
double distPtPlane(point p, double a, double b,
----- double c, double d) {
- // distance to 3D plane ax + by + cz + d = 0
- return (a*p.x+b*p.y+c*p.z+d)/sqrt(a*a+b*b+c*c); }
/*! // distance between 3D lines AB & CD (untested)
double distLine3D(point A,point B,point C,point D){
- point u = B - A, v = D - C, w = A - C;
- double a = dot(u, u), b = dot(u, v);
- double c = dot(v, v), d = dot(u, w);
- double e = dot(v, w), det = a*c - b*b;
- double s = det < EPS ? 0.0 : (b*e - c*d) / det;
- double t = det < EPS
--- ? (b > c ? d/b : e/c) // parallel
--- : (a*e - b*d) / det;
- point top = A + u * s, bot = w - A - v * t;
- return dist(top, bot);
} // dist<EPS: intersection     */
```

3.7. Intersections.

3.7.1. *Line-Segment Intersection.* Get intersection points of 2D lines/segments $\overline{ab}$ and $\overline{cd}$.

```
point null(HUGE_VAL, HUGE_VAL);
point line_inter(point a, point b, point c,
---------------- point d, bool seg = false) {
- point ab(b.x - a.x, b.y - a.y);
- point cd(d.x - c.x, d.y - c.y);
- point ac(c.x - a.x, c.y - a.y);
- double D = -cross(ab, cd); // determinant
- double Ds = cross(cd, ac);
- double Dt = cross(ab, ac);
- if (abs(D) < EPS) { // parallel
--- if (seg && abs(Ds) < EPS) { // collinear
----- point p[] = {a, b, c, d};
----- sort(p, p + 4, [](point a, point b) {
------- return a.x < b.x-EPS ||
-------------- (dist(a,b) < EPS && a.y < b.y-EPS); });
----- return dist(p[1], p[2]) < EPS ? p[1] : null; }
--- return null; }
- double s = Ds / D, t = Dt / D;
- if (seg && (min(s,t)<-EPS||max(s,t)>1+EPS)) return null;
- return point(a.x + s * ab.x, a.y + s * ab.y); }
/* double A = cross(d-a, b-a), B = cross(c-a, b-a);
return (B*d - A*c)/(B - A); */
```

3.7.2. *Circle-Line Intersection.* Get intersection points of circle at center c, radius r, and line $\overline{ab}$.

```
std::vector<point> CL_inter(point c, double r,
--- point a, point b) {
- point p = projLine(c, a, b);
- double d = abs(c - p); vector<point> ans;
- if (d > r + EPS); // none
- else if (d > r - EPS) ans.push_back(p); // tangent
- else if (d < EPS) { // diameter
--- point v = r * (b - a) / abs(b - a);
--- ans.push_back(c + v);
--- ans.push_back(c - v);
- } else {
--- double t = acos(d / r);
```

```
--- p = c + (p - c) * r / d;
--- ans.push_back(rotate(c, p, t));
--- ans.push_back(rotate(c, p, -t));
- } return ans; }
```

3.7.3. *Circle-Circle Intersection.*

```
std::vector<point> CC_intersection(point c1,
--- double r1, point c2, double r2) {
- double d = dist(c1, c2);
- vector<point> ans;
- if (d < EPS) {
--- if (abs(r1-r2) < EPS); // inf intersections
- } else if (r1 < EPS) {
--- if (abs(d - r2) < EPS) ans.push_back(c1);
- } else {
--- double s = (r1*r1 + d*d - r2*r2) / (2*r1*d);
--- double t = acos(max(-1.0, min(1.0, s)));
--- point mid = c1 + (c2 - c1) * r1 / d;
--- ans.push_back(rotate(c1, mid, t));
--- if (abs(sin(t)) >= EPS)
----- ans.push_back(rotate(c2, mid, -t));
- } return ans; }
```

3.8. **Areas.**

3.8.1. *Polygon Area.* Find the area of any 2D polygon given as points in $O(n)$.

```
double area(point p[], int n) {
- double a = 0;
- for (int i = 0, j = n - 1; i < n; j = i++)
--- a += cross(p[i], p[j]);
- return abs(a) / 2; }
```

3.8.2. *Triangle Area.* Find the area of a triangle using only their lengths. Lengths must be valid.

```
double area(double a, double b, double c) {
- double s = (a + b + c) / 2;
- return sqrt(s*(s-a)*(s-b)*(s-c)); }
```

3.8.3. *Cyclic Quadrilateral Area.* Find the area of a cyclic quadrilateral using only their lengths. A quadrilateral is cyclic if its inner angles sum up to 360°.

```
double area(double a, double b, double c, double d) {
- double s = (a + b + c + d) / 2;
- return sqrt((s-a)*(s-b)*(s-c)*(s-d)); }
```

3.9. **Polygon Centroid.** Get the centroid/center of mass of a polygon in $O(m)$.

```
point centroid(point p[], int n) {
- point ans(0, 0);
- double z = 0;
- for (int i = 0, j = n - 1; i < n; j = i++) {
--- double cp = cross(p[j], p[i]);
--- ans += (p[j] + p[i]) * cp;
--- z += cp;
- } return ans / (3 * z); }
```

3.10. **Convex Hull.**

3.10.1. *2D Convex Hull.* Get the convex hull of a set of points using Graham-Andrew's scan. This sorts the points at $O(n \log n)$, then performs the Monotonic Chain Algorithm at $O(n)$.

```
// counterclockwise hull in p[], returns size of hull
bool xcmp(const point& a, const point& b) {
- return a.x < b.x || (a.x == b.x && a.y < b.y); }
int convex_hull(point p[], int n) {
- std::sort(p, p + n, xcmp); if (n <= 1) return n;
- int k = 0; point *h = new point[2 * n];
- double zer = EPS; // -EPS to include collinears
- for (int i = 0; i < n; h[k++] = p[i++])
--- while (k >= 2 && cross(h[k-2],h[k-1],p[i]) < zer)
----- --k;
- for(int i = n-2, t = k; i >= 0; h[k++] = p[i--])
--- while (k > t && cross(h[k-2],h[k-1],p[i]) < zer)
----- --k;
- k -= 1 + (h[0].x==h[1].x&&h[0].y==h[1].y ? 1 : 0);
- copy(h, h + k, p); delete[] h; return k; }
```

3.10.2. *3D Convex Hull.* Currently $O(N^2)$, but can be optimized to a randomized $O(N \log N)$ using the Clarkson-Shor algorithm. Sauce: Efficient 3D Convex Hull Tutorial on CF.

```
typedef std::vector<bool> vb;
struct point3D {
- ll x, y, z;
- point3D(ll x = 0, ll y = 0, ll z = 0) : x(x), y(y), z(z) {}
- point3D operator-(const point3D &o) const {
--- return point3D(x - o.x, y - o.y, z - o.z); }
- point3D cross(const point3D &o) const {
--- return point3D(y*o.z-z*o.y, z*o.x-x*o.z, x*o.y-y*o.x); } -
- ll dot(const point3D &o) const {
--- return x*o.x + y*o.y + z*o.z; }
- bool operator==(const point3D &o) const {
--- return std::tie(x, y, z) == std::tie(o.x, o.y, o.z); } ---
- bool operator<(const point3D &o) const {
--- return std::tie(x, y, z) < std::tie(o.x, o.y, o.z); } }; -
struct face {
- std::vector<int> p_idx;
- point3D q; };
std::vector<face> convex_hull_3D(std::vector<point3D> &points) {
- int n = points.size();
- std::vector<face> faces;
- std::vector<vb> dead(points.size(), vb(points.size(), true));
- auto add_face = [&](int a, int b, int c) {
--- faces.push_back({{a, b, c},
----- (points[b] - points[a]).cross(points[c] - points[a])});
--- dead[a][b] = dead[b][c] = dead[c][a] = false; };
- add_face(0, 1, 2);
- add_face(0, 2, 1);
- for (int i = 3; i < n; ++i) {
--- std::vector<face> faces_inv;
--- for(face &f : faces) {
----- if ((points[i] - points[f.p_idx[0]]).dot(f.q) > 0)
------- for (int j = 0; j < 3; ++j)
--------- dead[f.p_idx[j]][f.p_idx[(j+1)%3]] = true;
----- else
------- faces_inv.push_back(f); }
--- faces.clear();
--- for(face &f : faces_inv) {
----- for (int j = 0; j < 3; ++j) {
------- int a = f.p_idx[j], b = f.p_idx[(j + 1) % 3];
------- if(dead[b][a])
--------- add_face(b, a, i); } }
--- faces.insert(
----- faces.end(), faces_inv.begin(), faces_inv.end()); }
- return faces; }
```

3.11. **Delaunay Triangulation**. Simply map each point (x, y) to $(x, y, x^2 + y^2)$, find the 3d convex hull, and drop the 3rd dimension.

3.12. **Point in Polygon.** Check if a point is strictly inside (or on the border) of a polygon in $O(n)$.

```
bool inPolygon(point q, point p[], int n) {
- bool in = false;
- for (int i = 0, j = n - 1; i < n; j = i++)
--- in ^= (((p[i].y > q.y) != (p[j].y > q.y)) &&
----- q.x < (p[j].x - p[i].x) * (q.y - p[i].y) /
----- (p[j].y - p[i].y) + p[i].x);
- return in; }
bool onPolygon(point q, point p[], int n) {
- for (int i = 0, j = n - 1; i < n; j = i++)
- if (abs(dist(p[i], q) + dist(p[j], q) -
--------- dist(p[i], p[j])) < EPS)
--- return true;
- return false; }
```

3.13. **Cut Polygon by a Line.** Cut polygon by line $\overline{ab}$ to its left in $O(n)$, such that $\angle abp$ is counter-clockwise.

```
vector<point> cut(point p[],int n,point a,point b) {
- vector<point> poly;
- for (int i = 0, j = n - 1; i < n; j = i++) {
--- double c1 = cross(a, b, p[j]);
--- double c2 = cross(a, b, p[i]);
--- if (c1 > -EPS) poly.push_back(p[j]);
--- if (c1 * c2 < -EPS)
----- poly.push_back(line_inter(p[j], p[i], a, b));
- } return poly; }
```

3.14. **Triangle Centers.**

```
point bary(point A, point B, point C,
---------- double a, double b, double c) {
- return (A*a + B*b + C*c) / (a + b + c); }
point trilinear(point A, point B, point C,
--------------- double a, double b, double c) {
- return bary(A,B,C,abs(B-C)*a,
------------- abs(C-A)*b,abs(A-B)*c); }
point centroid(point A, point B, point C) {
- return bary(A, B, C, 1, 1, 1); }
point circumcenter(point A, point B, point C) {
- double a=norm(B-C), b=norm(C-A), c=norm(A-B);
- return bary(A,B,C,a*(b+c-a),b*(c+a-b),c*(a+b-c)); }
point orthocenter(point A, point B, point C) {
- return bary(A,B,C, tan(angle(B,A,C)),
```

```
------------ tan(angle(A,B,C)), tan(angle(A,C,B))); } -------
point incenter(point A, point B, point C) { -----------------
- return bary(A,B,C,abs(B-C),abs(A-C),abs(A-B)); } ----------
// incircle radius given the side lengths a, b, c -----------
double inradius(double a, double b, double c) { -------------
- double s = (a + b + c) / 2; -------------------------------
- return sqrt(s * (s-a) * (s-b) * (s-c)) / s; } -------------
point excenter(point A, point B, point C) { -----------------
- double a = abs(B-C), b = abs(C-A), c = abs(A-B); ----------
- return bary(A, B, C, -a, b, c); } -------------------------
- // return bary(A, B, C, a, -b, c); -------------------------
- // return bary(A, B, C, a, b, -c); -------------------------
point brocard(point A, point B, point C) { ------------------
- double a = abs(B-C), b = abs(C-A), c = abs(A-B); ----------
- return bary(A,B,C,c/b*a,a/c*b,b/a*c); // CCW --------------
- // return bary(A,B,C,b/c*a,c/a*b,a/b*c); // CW } ----------
point symmedian(point A, point B, point C) { ----------------
- return bary(A,B,C,norm(B-C),norm(C-A),norm(A-B)); } -------
```

3.15. **Convex Polygon Intersection.** Get the intersection of two convex polygons in $O(n^2)$.

```
std::vector<point> convex_polygon_inter( --------------------
--- point a[], int an, point b[], int bn) { -----------------
- point ans[an + bn + an*bn]; -------------------------------
- int size = 0; ---------------------------------------------
- for (int i = 0; i < an; ++i) ------------------------------
--- if (inPolygon(a[i],b,bn) || onPolygon(a[i],b,bn)) -------
----- ans[size++] = a[i]; -----------------------------------
- for (int i = 0; i < bn; ++i) ------------------------------
--- if (inPolygon(b[i],a,an) || onPolygon(b[i],a,an)) -------
----- ans[size++] = b[i]; -----------------------------------
- for (int i = 0, I = an - 1; i < an; I = i++) --------------
--- for (int j = 0, J = bn - 1; j < bn; J = j++) { ----------
----- try { -------------------------------------------------
------- point p=line_inter(a[i],a[I],b[j],b[J],true); -------
------- ans[size++] = p; ------------------------------------
----- } catch (exception ex) {} } ---------------------------
- size = convex_hull(ans, size); ----------------------------
- return vector<point>(ans, ans + size); } ------------------
```

3.16. **Pick's Theorem for Lattice Points.** Count points with integer coordinates inside and on the boundary of a polygon in $O(n)$ using Pick's theorem: Area $= I + B/2 - 1$.

```
int interior(point p[], int n) { ---------------------------
- return area(p,n) - boundary(p,n) / 2 + 1; } ---------------
int boundary(point p[], int n) { ----------------------------
- int ans = 0; ----------------------------------------------
- for (int i = 0, j = n - 1; i < n; j = i++) ----------------
--- ans += gcd(p[i].x - p[j].x, p[i].y - p[j].y); -----------
- return ans; } ---------------------------------------------
```

3.17. **Minimum Enclosing Circle.** Get the minimum bounding ball that encloses a set of points (2D or 3D) in Θn.

```
std::pair<point, double> bounding_ball(point p[], int n){ ----
- std::random_shuffle(p, p + n); ----------------------------
- point center(0, 0); double radius = 0; --------------------
- for (int i = 0; i < n; ++i) { -----------------------------
--- if (dist(center, p[i]) > radius + EPS) { ----------------
----- center = p[i]; radius = 0; ----------------------------
----- for (int j = 0; j < i; ++j) ---------------------------
------- if (dist(center, p[j]) > radius + EPS) { ------------
--------- center.x = (p[i].x + p[j].x) / 2; -----------------
--------- center.y = (p[i].y + p[j].y) / 2; -----------------
--------- // center.z = (p[i].z + p[j].z) / 2; --------------
--------- radius = dist(center, p[i]); // midpoint ----------
--------- for (int k = 0; k < j; ++k) -----------------------
----------- if (dist(center, p[k]) > radius + EPS) { --------
------------- center = circumcenter(p[i], p[j], p[k]); ------
------------- radius = dist(center, p[i]); } } } } ----------
- return {center, radius}; } --------------------------------
```

3.18. **Shamos Algorithm.** Solve for the polygon diameter in $O(n \log n)$.

```
double shamos(point p[], int n) { ---------------------------
- point *h = new point[n+1]; copy(p, p + n, h); -------------
- int k = convex_hull(h, n); if (k <= 2) return 0; ----------
- h[k] = h[0]; double d = HUGE_VAL; -------------------------
- for (int i = 0, j = 1; i < k; ++i) { ----------------------
--- while (distPtLine(h[j+1], h[i], h[i+1]) >= --------------
--------- distPtLine(h[j], h[i], h[i+1])) { -----------------
----- j = (j + 1) % k; } ------------------------------------
--- d = min(d, distPtLine(h[j], h[i], h[i+1])); -------------
- } return d; } ---------------------------------------------
```

3.19. **kD Tree.** Get the k-nearest neighbors of a point within pruned radius in $O(k \log k \log n)$.

```
#define cpoint const point& ---------------------------------
bool cmpx(cpoint a, cpoint b) {return a.x < b.x;} -----------
bool cmpy(cpoint a, cpoint b) {return a.y < b.y;} -----------
struct KDTree { ---------------------------------------------
- KDTree(point p[],int n): p(p), n(n) {build(0,n);} ---------
- priority_queue< pair<double, point*> > pq; ----------------
- point *p; int n, k; double qx, qy, prune; -----------------
- void build(int L, int R, bool dvx=false) { ----------------
--- if (L >= R) return; -------------------------------------
--- int M = (L + R) / 2; ------------------------------------
--- nth_element(p + L, p + M, p + R, dvx?cmpx:cmpy); --------
--- build(L, M, !dvx); build(M + 1, R, !dvx); } -------------
- void dfs(int L, int R, bool dvx) { ------------------------
--- if (L >= R) return; -------------------------------------
--- int M = (L + R) / 2; ------------------------------------
--- double dx = qx - p[M].x, dy = qy - p[M].y; --------------
--- double delta = dvx ? dx : dy; ---------------------------
--- double D = dx * dx + dy * dy; ---------------------------
--- if (D<=prune && (pq.size()<k||D<pq.top().first)) { ------
----- pq.push(make_pair(D, &p[M])); -------------------------
----- if (pq.size() > k) pq.pop(); } ------------------------
--- int nL = L, nR = M, fL = M + 1, fR = R; -----------------
--- if (delta > 0) {swap(nL, fL); swap(nR, fR);} ------------
--- dfs(nL, nR, !dvx); --------------------------------------
--- D = delta * delta; --------------------------------------
--- if (D<=prune && (pq.size()<k||D<pq.top().first)) --------
--- dfs(fL, fR, !dvx); } ------------------------------------
- // returns k nearest neighbors of (x, y) in tree ----------
- // usage: vector<point> ans = tree.knn(x, y, 2); ----------
- vector<point> knn(double x, double y, ---------------------
------------------- int k=1, double r=-1) { -----------------
--- qx=x; qy=y; this->k=k; prune=r<0?HUGE_VAL:r*r; ----------
--- dfs(0, n, false); vector<point> v; ----------------------
--- while (!pq.empty()) { -----------------------------------
----- v.push_back(*pq.top().second); ------------------------
----- pq.pop(); ---------------------------------------------
--- } reverse(v.begin(), v.end()); --------------------------
--- return v; } }; ------------------------------------------
```

3.20. **Line Sweep (Closest Pair).** Get the closest pair distance of a set of points in $O(n \log n)$ by sweeping a line and keeping a bounded rectangle. Modifiable for other metrics such as Minkowski and Manhattan distance. For external point queries, see kD Tree.

```
bool cmpy(const point& a, const point& b) { return a.y < b.y; }
double closest_pair_sweep(point p[], int n) { ---------------
- if (n <= 1) return HUGE_VAL; ------------------------------
- std::sort(p, p + n, cmpy); --------------------------------
- std::set<point> box; box.insert(p[0]); --------------------
- double best = 1e13; // infinity, but not HUGE_VAL ---------
- for (int L = 0, i = 1; i < n; ++i) { ----------------------
--- while(L < i && p[i].y - p[L].y > best) ------------------
----- box.erase(p[L++]); ------------------------------------
--- point bound(p[i].x - best, p[i].y - best); --------------
--- std::set<point>::iterator it = box.lower_bound(bound); ---
--- while (it != box.end() && p[i].x+best >= it->x){ --------
----- double dx = p[i].x - it->x; ---------------------------
----- double dy = p[i].y - it->y; ---------------------------
----- best = std::min(best, std::sqrt(dx*dx + dy*dy)); ------
----- ++it; } -----------------------------------------------
--- box.insert(p[i]); ---------------------------------------
- } return best; } ------------------------------------------
```

3.21. **Line upper/lower envelope.** To find the upper/lower envelope of a collection of lines $a_i + b_i x$, plot the points (b_i, a_i), add the point $(0, \pm\infty)$ (depending on if upper/lower envelope is desired), and then find the convex hull.

3.22. **Formulas.** Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two-dimensional vectors.

- $a \cdot b = |a||b| \cos\theta$, where θ is the angle between a and b.
- $a \times b = |a||b| \sin\theta$, where θ is the signed angle between a and b.
- $a \times b$ is equal to the area of the parallelogram with two of its sides formed by a and b. Half of that is the area of the triangle formed by a and b.
- The line going through a and b is $Ax + By = C$ where $A = b_y - a_y$, $B = a_x - b_x$, $C = Aa_x + Ba_y$.
- Two lines $A_1x + B_1y = C_1$, $A_2x + B_2y = C_2$ are parallel iff. $D = A_1B_2 - A_2B_1$ is zero. Otherwise their unique intersection is $(B_2C_1 - B_1C_2, A_1C_2 - A_2C_1)/D$.
- **Euler's formula:** $V - E + F = 2$
- Side lengths a, b, c can form a triangle iff. $a + b > c$, $b + c > a$ and $a + c > b$.
- Sum of internal angles of a regular convex n-gon is $(n-2)\pi$.
- **Law of sines:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:** $b^2 = a^2 + c^2 - 2ac \cos B$

- Internal tangents of circles $(c_1, r_1), (c_2, r_2)$ intersect at $(c_1 r_2 + c_2 r_1)/(r_1 + r_2)$, external intersect at $(c_1 r_2 - c_2 r_1)/(r_1 + r_2)$.

4. Graphs

4.1. Single-Source Shortest Paths.

4.1.1. *Dijkstra.*

```
#include "graph_template_adjlist.cpp"
// insert inside graph; needs n, dist[], and adj[]
void dijkstra(int s) {
- for (int u = 0; u < n; ++u)
--- dist[u] = INF;
- dist[s] = 0;
- std::priority_queue<ii, vii, std::greater<ii> > pq;
- pq.push({0, s});
- while (!pq.empty()) {
--- int u = pq.top().second;
--- int d = pq.top().first;
--- pq.pop();
--- if (dist[u] < d)
----- continue;
--- dist[u] = d;
--- for (auto &e : adj[u]) {
----- int v = e.first;
----- int w = e.second;
----- if (dist[v] > dist[u] + w) {
------- dist[v] = dist[u] + w;
------- pq.push({dist[v], v}); } } } }
```

4.1.2. *Bellman-Ford.*

```
#include "graph_template_adjlist.cpp"
// insert inside graph; needs n, dist[], and adj[]
void bellman_ford(int s) {
- for (int u = 0; u < n; ++u)
--- dist[u] = INF;
- dist[s] = 0;
- for (int i = 0; i < n-1; ++i)
--- for (int u = 0; u < n; ++u)
----- for (auto &e : adj[u])
------- if (dist[u] + e.second < dist[e.first])
--------- dist[e.first] = dist[u] + e.second; }
// you can call this after running bellman_ford()
bool has_neg_cycle() {
- for (int u = 0; u < n; ++u)
--- for (auto &e : adj[u])
----- if (dist[e.first] > dist[u] + e.second)
------- return true;
- return false; }
```

4.1.3. *Shortest Path Faster Algorithm.*

```
#include "graph_template_adjlist.cpp"
// insert inside graph;
// needs n, dist[], in_queue[], num_vis[], and adj[]
bool spfa(int s) {
- for (int u = 0; u < n; ++u) {
--- dist[u] = INF;
--- in_queue[u] = 0;
--- num_vis[u] = 0; }
- dist[s] = 0;
- in_queue[s] = 1;
- bool has_negative_cycle = false;
- std::queue<int> q;  q.push(s);
- while (not q.empty()) {
--- int u = q.front();  q.pop();  in_queue[u] = 0;
--- if (++num_vis[u] >= n)
----- dist[u] = -INF, has_negative_cycle = true;
--- for (auto &[v, c] : adj[u])
----- if (dist[v] > dist[u] + c) {
------- dist[v] = dist[u] + c;
------- if (!in_queue[v]) {
--------- q.push(v);
--------- in_queue[v] = 1; } } }
- return has_negative_cycle; }
```

4.2. All-Pairs Shortest Paths.

4.2.1. *Floyd-Washall.*

```
#include "graph_template_adjmat.cpp"
// insert inside graph; needs n and mat[][]
void floyd_warshall() {
- for (int k = 0; k < n; ++k)
--- for (int i = 0; i < n; ++i)
----- for (int j = 0; j < n; ++j)
------- if (mat[i][k] + mat[k][j] < mat[i][j])
--------- mat[i][j] = mat[i][k] + mat[k][j]; }
```

4.3. Strongly Connected Components.

4.3.1. *Kosaraju.*

```
struct kosaraju_graph {
- int n, *vis;
- vi **adj;
- std::vector<vi> sccs;
- kosaraju_graph(int n) {
--- this->n = n;
--- vis = new int[n];
--- adj = new vi*[2];
--- for (int dir = 0; dir < 2; ++dir)
----- adj[dir] = new vi[n]; }
- void add_edge(int u, int v) {
--- adj[0][u].push_back(v);
--- adj[1][v].push_back(u); }
- void dfs(int u, int p, int dir, vi &topo) {
--- vis[u] = 1;
--- for (int v : adj[dir][u])
----- if (!vis[v] && v != p) dfs(v, u, dir, topo);
--- topo.push_back(u); }
- void kosaraju() {
--- vi topo;
--- for (int u = 0; u < n; ++u)  vis[u] = 0;
--- for (int u = 0; u < n; ++u) if(!vis[u]) dfs(u, -1, 0, topo);
--- for (int u = 0; u < n; ++u)  vis[u] = 0;
--- for (int i = n-1; i >= 0; --i) {
----- if (!vis[topo[i]]) {
------- sccs.push_back({});
------- dfs(topo[i], -1, 1, sccs.back()); } } } };
```

4.3.2. *Tarjan's Offline Algorithm*.

```
int n, id[N], low[N], st[N], in[N], TOP, ID;
int scc[N], SCC_SIZE; // 0 <= scc[u] < SCC_SIZE
vector<int> adj[N]; // 0-based adjlist
void dfs(int u) {
- id[u] = low[u] = ID++;
- st[TOP++] = u; in[u] = 1;
- for (int v : adj[u]) {
--- if (id[v] == -1) {
----- dfs(v);
----- low[u] = min(low[u], low[v]);
--- } else if (in[v] == 1)
----- low[u] = min(low[u], id[v]); }
- if (id[u] == low[u]) {
--- int sid = SCC_SIZE++;
--- do {
----- int v = st[--TOP];
----- in[v] = 0; scc[v] = sid;
--- } while (st[TOP] != u); }}
void tarjan() { // call tarjan() to load SCC
- memset(id, -1, sizeof(int) * n);
- SCC_SIZE = ID = TOP = 0;
- for (int i = 0; i < n; ++i)
--- if (id[i] == -1) dfs(i); }
```

4.4. Minimum Mean Weight Cycle.

Run this for each strongly connected component

```
typedef std::vector<double> vd;
double min_mean_cycle(graph &g) {
- double mn = INF;
- std::vector<vd> dp(g.n+1, vd(g.n, mn));
- dp[0][0] = 0;
- for (int k = 1; k <= g.n; ++k)
--- for (int u = 0; u < g.n; ++u)
----- for (auto &[v, w]: g.adj[u])
------- dp[k][v] = std::min(ar[k][v], dp[k-1][u] + w);
- for (int k = 0; k < g.n; ++k) {
--- double mx = -INF;
--- for (int u = 0; u < g.n; ++u)
----- mx = std::max(mx, (dp[g.n][u] - dp[k][u]) / (g.n - k));
--- mn = std::min(mn, mx);  }
- return mn;  }
```

4.5. Biconnected Components.

4.5.1. *Bridges and Articulation Points.*

```
struct graph {
- int n, *disc, *low, TIME;
- vi *adj, stk, articulation_points;
- std::set<ii> bridges;
- vvi comps;
- graph (int n) : n(n) {
--- adj = new vi[n];
--- disc = new int[n];
```

```
--- low = new int[n]; }
- void add_edge(int u, int v) {
--- adj[u].push_back(v);
--- adj[v].push_back(u); }
- void _bridges_artics(int u, int p) {
--- disc[u] = low[u] = TIME++;
--- stk.push_back(u);
--- int children = 0;
--- bool has_low_child = false;
--- for (int v : adj[u]) {
----- if (disc[v] == -1) {
------- _bridges_artics(v, u);
------- children++;
------- if (disc[u] < low[v])
--------- bridges.insert({std::min(u, v), std::max(u, v)});
------- if (disc[u] <= low[v]) {
--------- has_low_child = true;
--------- comps.push_back({u});
--------- while (comps.back().back() != v and !stk.empty()) {
----------- comps.back().push_back(stk.back());
----------- stk.pop_back(); } }
------- low[u] = std::min(low[u], low[v]);
----- } else if (v != p)
------- low[u] = std::min(low[u], disc[v]); }
--- if ((p == -1 && children >= 2) ||
------- (p != -1 && has_low_child))
----- articulation_points.push_back(u); }
- void bridges_artics() {
--- for (int u = 0; u < n; ++u)   disc[u] = -1;
--- stk.clear();
--- articulation_points.clear();
--- bridges.clear();
--- comps.clear();
--- TIME = 0;
--- for (int u = 0; u < n; ++u) if (disc[u] == -1)
----- _bridges_artics(u, -1); } };
```

4.5.2. *Block Cut Tree.*

```
// insert inside code for finding articulation points
graph build_block_cut_tree() {
- int bct_n = articulation_points.size() + comps.size();
- vi block_id(n), is_art(n, 0);
- graph tree(bct_n);
- for (int i = 0; i < articulation_points.size(); ++i) {
--- block_id[articulation_points[i]] = i;
--- is_art[articulation_points[i]] = 1; }
- for (int i = 0; i < comps.size(); ++i) {
--- int id = i + articulation_points.size();
--- for (int u : comps[i])
----- if (is_art[u])  tree.add_edge(block_id[u], id);
----- else            block_id[u] = id; }
- return tree; }
```

4.5.3. *Bridge Tree.*

```
// insert inside code for finding bridges
// requires union_find and hasher
graph build_bridge_tree() {
- union_find uf(n);
- for (int u = 0; u < n; ++u) {
--- for (int v : adj[u]) {
----- ii uv = { std::min(u, v), std::max(u, v) };
----- if (bridges.find(uv) == bridges.end())
------- uf.unite(u, v); } }
- hasher h;
- for (int u = 0; u < n; ++u)
--- if (u == uf.find(u))  h.get_hash(u);
- int tn = h.h.size();
- graph tree(tn);
- for (int i = 0; i < M; ++i) {
--- int ui = h.get_hash(uf.find(u));
--- int vi = h.get_hash(uf.find(v));
--- if (ui != vi) tree.add_edge(ui, vi); }
- return tree; }
```

4.6. Minimum Spanning Tree.

4.6.1. *Kruskal.*

```
#include "graph_template_edgelist.cpp"
#include "union_find.cpp"
// insert inside graph; needs n, and edges
void kruskal(viii &res) {
- viii().swap(res); // or use res.clear();
- std::priority_queue<iii, viii, std::greater<iii> > pq;
- for (auto &edge : edges)
--- pq.push(edge);
- union_find uf(n);
- while (!pq.empty()) {
--- auto node = pq.top();   pq.pop();
--- int u = node.second.first;
--- int v = node.second.second;
--- if (uf.unite(u, v))
----- res.push_back(node); } }
```

4.6.2. *Prim.*

```
#include "graph_template_adjlist.cpp"
// insert inside graph; needs n, vis[], and adj[]
void prim(viii &res, int s=0) {
- res.clear();
- std::priority_queue<iii, viii, std::greater<iii>> pq;
- vis[s] = true;
- for (auto &[v, w] : adj[s])
--- if (!vis[v]) pq.push({w, {s, v}});
- while (!pq.empty()) {
--- auto edge = pq.top();    pq.pop();
--- int u = edge.second.second;
--- if (vis[u]) continue;
--- vis[u] = true;
--- res.push_back(edge);
--- for (auto &[v, w] : adj[u])
----- if (!vis[v]) pq.push({w, {u, v}});}}
```

4.7. Euler Path/Cycle.

4.7.1. *Euler Path/Cycle in a Directed Graph.*

```
#define MAXV 1000
#define MAXE 5000
int indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end(graph &g) {
- int start = -1, end = -1, any = 0, c = 0;
- for (int u = 0; u < n; ++u) {
--- if (outdeg[u] > 0) any = u;
--- if (indeg[u] + 1 == outdeg[u]) start = u, c++;
--- else if (indeg[u] == outdeg[u] + 1) end = u, c++;
--- else if (indeg[u] != outdeg[u]) return {-1, -1}; }
- if ((start == -1) != (end == -1) || (c != 2 && c != 0))
--- return {-1,-1};
- if (start == -1) start = end = any;
- return {start, end}; }
bool euler_path(graph &g) {
- ii se = start_end(g);
- int cur = se.first, at = g.edges.size() + 1;
- if (cur == -1) return false;
- std::stack<int> s;
- while (true) {
--- if (outdeg[cur] == 0) {
----- res[--at] = cur;
----- if (s.empty()) break;
----- cur = s.top(); s.pop();
--- } else s.push(cur), cur = g.adj[cur][--outdeg[cur]]; }
- return at == 0; }
```

4.7.2. *Euler Path/Cycle in an Undirected Graph.*

```
std::multiset<int> adj[1010];
std::list<int> L;
std::list<int>::iterator euler(
- int at, int to, std::list<int>::iterator it
) {
- if (at == to) return it;
- L.insert(it, at), --it;
- while (!adj[at].empty()) {
--- int nxt = *adj[at].begin();
--- adj[at].erase(adj[at].find(nxt));
--- adj[nxt].erase(adj[nxt].find(at));
--- if (to == -1) {
----- it = euler(nxt, at, it);
----- L.insert(it, at);
----- --it;
--- } else {
----- it = euler(nxt, to, it);
----- to = -1; } }
- return it; }
// euler(0,-1,L.begin())
```

4.8. Bipartite Matching.

4.8.1. *Alternating Paths Algorithm.*

```
vi* adj;
bool* done;   // initially all false
int* owner;   // initially all -1
```

```
int alternating_path(int left) {
- if (done[left]) return 0;
- done[left] = true;
- for (int right : adj[left]) {
--- if (owner[right] == -1 || alternating_path(owner[right])) {
----- owner[right] = left; return 1; } }
- return 0; }
```

4.8.2. *Hopcroft-Karp Algorithm*.

```
#define MAXN 5000
int dist[MAXN+1], q[MAXN+1];
#define dist(v) dist[v == -1 ? MAXN : v]
struct bipartite_graph {
- int n, m, *L, *R; vi *adj;
- bipartite_graph(int n, int m) : n(n), m(m),
--- L(new int[n]), R(new int[m]), adj(new vi[n]) {}
- ~bipartite_graph() { delete[] adj; delete[] L; delete[] R; }
- void add_edge(int u, int v) { adj[u].push_back(v); }
- bool bfs() {
--- int l = 0, r = 0;
--- for (int v = 0; v < n; ++v)
----- if(L[v] == -1) dist(v) = 0, q[r++] = v;
----- else dist(v) = INF;
--- dist(-1) = INF;
--- while(l < r) {
----- int v = q[l++];
----- if(dist(v) < dist(-1))
------- for (int u : adj[v])
--------- if(dist(R[u]) == INF) {
----------- dist(R[u]) = dist(v) + 1;
----------- q[r++] = R[u];  } }
--- return dist(-1) != INF; }
- bool dfs(int v) {
--- if(v != -1) {
----- for (int u : adj[v])
------- if(dist(R[u]) == dist(v) + 1)
--------- if(dfs(R[u])) { R[u] = v; L[v] = u; return true; }
----- dist(v) = INF;
----- return false; }
--- return true; }
- int maximum_matching() {
--- int matching = 0;
--- for (int u = 0; u < n; ++u)
----- L[u] = R[u] = -1;
--- while(bfs())
----- for (int u = 0; u < n; ++u)
------- matching += L[u] == -1 && dfs(u);
--- return matching; } };
```

4.8.3. *Minimum Vertex Cover in Bipartite Graphs*.

```
#include "hopcroft_karp.cpp"
std::vector<bool> alt;
void dfs(bipartite_graph &g, int u) {
- alt[u] = true;
- for (int v: g.adj[u]) {
--- alt[v + g.n] = true;
--- if (g.R[v] != -1 && !alt[g.R[v]])
----- dfs(g, g.R[v]); } }
vi mvc_bipartite(bipartite_graph &g) {
- vi res; g.maximum_matching();
- alt.assign(g.n + g.m, false);
- for (int i = 0; i<g.n; ++i) if (g.L[i] == -1) dfs(g, i);
- for (int i = 0; i<g.n; ++i) if (!alt[i]) res.push_back(i);
- for (int i = 0; i<g.m; ++i)
--- if (alt[g.n + i]) res.push_back(g.n + i);
- return res; }
```

4.9. Maximum Flow.

4.9.1. *Edmonds-Karp*. $O(VE^2)$

4.9.2. *Dinic*. $O(V^2E)$

```
struct flow_network_dinic {
- struct edge {
--- int u, v; ll c, f;
--- edge(int u, int v, ll c) : u(u), v(v), c(c), f(0) {} };
- int n;
- std::vector<int> adj_ptr, par, dist;
- std::vector<std::vector<int>> adj;
- std::vector<edge> edges;
- flow_network_dinic(int n) : n(n) {
--- std::vector<std::vector<int>>(n).swap(adj);
--- reset(); }
- void reset() {
--- std::vector<int>(n).swap(adj_ptr);
--- std::vector<int>(n).swap(par);
--- std::vector<int>(n).swap(dist);
--- for (edge &e : edges)  e.f = 0; }
- void add_edge(int u, int v, ll c, bool bi = false) {
--- adj[u].push_back(edges.size());
--- edges.push_back(edge(u, v, c));
--- adj[v].push_back(edges.size());
--- edges.push_back(edge(v, u, (bi ? c : 0LL))); }
- ll res(const edge &e) { return e.c - e.f; }
- bool make_level_graph(int s, int t) {
--- for (int u = 0; u < n; ++u)   dist[u] = -1;
--- dist[s] = 0;
--- std::queue<int> q;     q.push(s);
--- while (!q.empty()) {
----- int u = q.front();  q.pop();
----- for (int i : adj[u]) {
------- edge &e = edges[i];
------- if (dist[e.v] < 0 and res(e)) {
--------- dist[e.v] = dist[u] + 1;
--------- q.push(e.v); } } }
--- return dist[t] != -1; }
- bool is_next(int u, int v) {
--- return dist[v] == dist[u] + 1; }
- bool dfs(int u, int t) {
--- if (u == t)   return true;
--- for (int &ii = adj_ptr[u]; ii < adj[u].size(); ++ii) {
----- int i = adj[u][ii];
----- edge &e = edges[i];
----- if (is_next(u, e.v) and res(e) > 0 and dfs(e.v, t)) {
------- par[e.v] = i;
------- return true; } }
--- return false; }
- bool aug_path(int s, int t) {
--- for (int u = 0; u < n; ++u) par[u] = -1;
--- return dfs(s, t); }
- ll calc_max_flow(int s, int t) {
--- ll total_flow = 0;
--- while (make_level_graph(s, t)) {
----- for (int u = 0; u < n; ++u)   adj_ptr[u] = 0;
----- while (aug_path(s, t)) {
------- ll flow = pvl::LL_INF;
------- for (int i = par[t]; i != -1; i = par[edges[i].u])
--------- flow = std::min(flow, res(edges[i]));
------- for (int i = par[t]; i != -1; i = par[edges[i].u]) {
--------- edges[i].f   += flow;
--------- edges[i^1].f -= flow; }
------- total_flow += flow; } }
--- return total_flow; }
- std::vector<bool> min_cut(int s, int t) {
--- calc_max_flow(s, t);
--- assert(!make_level_graph(s, t));
--- std::vector<bool> cut_mem(n);
--- for (int u = 0; u < n; ++u)
----- cut_mem[u] = (dist[u] != -1);
--- return cut_mem; } };
```

4.9.3. *Push-relabel*. $\omega(VE + V^2\sqrt{E})$, $O(V^3)$

```
int n;
std::vector<vi> capacity, flow;
vi height, excess;
void push(int u, int v) {
- int d = min(excess[u], capacity[u][v] - flow[u][v]);
- flow[u][v] += d;    flow[v][u] -= d;
- excess[u] -= d;     excess[v] += d; }
void relabel(int u) {
- int d = INF;
- for (int i = 0; i < n; i++)
--- if (capacity[u][i] - flow[u][i] > 0)
----- d = min(d, height[i]);
- if (d < INF)  height[u] = d + 1; }
vi find_max_height_vertices(int s, int t) {
- vi max_height;
- for (int i = 0; i < n; i++) {
--- if (i != s && i != t && excess[i] > 0) {
----- if (!max_height.empty()&&height[i]>height[max_height[0]])
------- max_height.clear();
----- if (max_height.empty()||height[i]==height[max_height[0]])
------- max_height.push_back(i); } }
- return max_height; }
int max_flow(int s, int t) {
- flow.assign(n, vi(n, 0));
- height.assign(n, 0);    height[s] = n;
- excess.assign(n, 0);    excess[s] = INF;
- for (int i = 0; i < n; i++) if (i != s) push(s, i);
```

```
- vi current;
- while (!(current = find_max_height_vertices(s, t)).empty()) {
--- for (int i : current) {
----- bool pushed = false;
----- for (int j = 0; j < n && excess[i]; j++) {
------- if (capacity[i][j] - flow[i][j] > 0 &&
----------- height[i] == height[j] + 1) {
--------- push(i, j);
--------- pushed = true; } }
----- if (!pushed) relabel(i), break; } }
- int max_flow = 0;
- for (int i = 0; i < n; i++) max_flow += flow[i][t];
- return max_flow; }
```

4.9.4. *Gomory-Hu (All-pairs Maximum Flow).* $O(V^3E)$, possibly amortized $O(V^2E)$ with a big constant factor.

```
#include "dinic.cpp"
struct gomory_hu_tree {
- int n;
- std::vector<int> dep;
- std::vector<std::pair<int, ll>> par;
- explicit gomory_hu_tree(flow_network_dinic &g) : n(g.n) {
--- std::vector<std::pair<int, ll>>(n, {0, 0LL}).swap(par);
--- std::vector<int>(n, 0).swap(dep);
--- std::vector<int> temp_par(n, 0);
--- for (int u = 1; u < n; ++u) {
----- g.reset();
----- ll flow = g.calc_max_flow(u, temp_par[u]);
----- std::vector<bool> cut_mem = g.min_cut(u, temp_par[u]);
----- for (int v = u+1; v < n; ++v)
------- if (cut_mem[u] == cut_mem[v]
----------- and temp_par[u] == temp_par[v])
--------- temp_par[v] = u;
----- add_edge(temp_par[u], u, flow); } }
- void add_edge(int u, int v, ll w) {
--- par[v] = {u, w}; dep[v] = dep[u] + 1; }
- ll calc_max_flow(int s, int t) {
--- ll ans = pvl::LL_INF;
--- while (dep[s] > dep[t]) {
----- ans = std::min(ans, par[s].second); s = par[s].first; }
--- while (dep[s] < dep[t]) {
----- ans = std::min(ans, par[t].second); t = par[t].first; }
--- while (s != t) {
----- ans = std::min(ans, par[s].second); s = par[s].first;
----- ans = std::min(ans, par[t].second); t = par[t].first; }
--- return ans; } };
```

4.10. **Minimum Cost Maximum Flow.**

```
struct edge {
- int u, v; ll cost, cap, flow;
- edge(int u, int v, ll cap, ll cost) :
--- u(u), v(v), cap(cap), cost(cost), flow(0) {} };
struct flow_network {
- int n, s, t, *par, *in_queue, *num_vis; ll *dist, *pot;
- std::vector<edge> edges;
- std::vector<int> *adj;
- std::map<std::pair<int, int>, std::vector<int> > edge_idx;
- flow_network(int n, int s, int t) : n(n), s(s), t(t) {
--- adj = new std::vector<int>[n];
--- par = new int[n];
--- in_queue = new int[n];
--- num_vis = new int[n];
--- dist = new ll[n];
--- pot  = new ll[n];
--- for (int u = 0; u < n; ++u) pot[u] = 0; }
- void add_edge(int u, int v, ll cap, ll cost) {
--- adj[u].push_back(edges.size());
--- edge_idx[{u, v}].push_back(edges.size());
--- edges.push_back(edge(u, v, cap, cost));
--- adj[v].push_back(edges.size());
--- edge_idx[{v, u}].push_back(edges.size());
--- edges.push_back(edge(v, u, 0LL, -cost)); }
- ll get_flow(int u, int v) {
--- ll f = 0;
--- for (int i : edge_idx[{u, v}]) f += edges[i].flow;
--- return f; }
- ll res(edge &e) { return e.cap - e.flow; }
- void bellman_ford() {
--- for (int u = 0; u < n; ++u) pot[u] = INF;
--- pot[s] = 0;
--- for (int it = 0; it < n-1; ++it)
----- for (auto e : edges)
------- if (res(e) > 0)
--------- pot[e.v] = std::min(pot[e.v], pot[e.u] + e.cost); }
- bool spfa () {
--- std::queue<int> q;  q.push(s);
--- while (not q.empty()) {
----- int u = q.front();  q.pop();  in_queue[u] = 0;
----- if (++num_vis[u] >= n) {
------- dist[u] = -INF;
------- return false; }
----- for (int i : adj[u]) {
------- edge e = edges[i];
------- if (res(e) <= 0)  continue;
------- ll nd = dist[u] + e.cost + pot[u] - pot[e.v];
------- if (dist[e.v] > nd) {
--------- dist[e.v] = nd;
--------- par[e.v] = i;
--------- if (not in_queue[e.v]) {
----------- q.push(e.v);
----------- in_queue[e.v] = 1; } } } }
--- return dist[t] != INF; }
- bool aug_path() {
--- for (int u = 0; u < n; ++u) {
----- par[u]      = -1;
----- in_queue[u] = 0;
----- num_vis[u]  = 0;
----- dist[u]     = INF; }
--- dist[s] = 0;
--- in_queue[s] = 1;
--- return spfa();
- }
- pll calc_max_flow(bool do_bellman_ford=false) {
--- ll total_cost = 0, total_flow = 0;
--- if (do_bellman_ford)
----- bellman_ford();
--- while (aug_path()) {
----- ll f = INF;
----- for (int i = par[t]; i != -1; i = par[edges[i].u])
------- f = std::min(f, res(edges[i]));
----- for (int i = par[t]; i != -1; i = par[edges[i].u]) {
------- edges[i].flow   += f;
------- edges[i^1].flow -= f; }
----- total_cost += f * (dist[t] + pot[t] - pot[s]);
----- total_flow += f;
----- for (int u = 0; u < n; ++u)
------- if (par[u] != -1) pot[u] += dist[u]; }
--- return {total_cost, total_flow}; } };
```

4.10.1. *Hungarian Algorithm.*

```
int n, m; // size of A, size of B
int cost[N+1][N+1]; // input cost matrix, 1-indexed
int way[N+1], p[N+1]; // p[i]=j: Ai is matched to Bj
int minv[N+1], A[N+1], B[N+1]; bool used[N+1];
int hungarian() {
- for (int i = 0; i <= N; ++i)
--- A[i] = B[i] = p[i] = way[i] = 0; // init
- for (int i = 1; i <= n; ++i) {
--- p[0] = i; int R = 0;
--- for (int j = 0; j <= m; ++j)
----- minv[j] = INF, used[j] = false;
--- do {
----- int L = p[R], dR = 0;
----- int delta = INF;
----- used[R] = true;
----- for (int j = 1; j <= m; ++j)
------- if (!used[j]) {
--------- int c = cost[L][j] - A[L] - B[j];
--------- if (c < minv[j])      minv[j] = c, way[j] = R;
--------- if (minv[j] < delta)  delta = minv[j], dR = j;
------- }
----- for (int j = 0; j <= m; ++j)
--------- if (used[j])  A[p[j]] += delta, B[j] -= delta;
--------- else          minv[j] -= delta;
----- R = dR;
--- } while (p[R] != 0);
--- for (; R != 0; R = way[R])
----- p[R] = p[way[R]]; }
- return -B[0]; }
```

4.11. **Minimum Arborescence**. Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root r to each vertex. Returns a vector of size n, where the ith element is the edge for the ith vertex. The answer for the root is undefined!

```
#include "../data-structures/union_find.cpp"
struct arborescence {
- int n; union_find uf;
- vector<vector<pair<ii,int> > > adj;
- arborescence(int _n) : n(_n), uf(n), adj(n) { }
```

```
- void add_edge(int a, int b, int c) {
--- adj[b].push_back(make_pair(ii(a,b),c)); }
- vii find_min(int r) {
--- vi vis(n,-1), mn(n,INF); vii par(n);
--- rep(i,0,n) {
----- if (uf.find(i) != i) continue;
----- int at = i;
----- while (at != r && vis[at] == -1) {
------- vis[at] = i;
------- iter(it,adj[at]) if (it->second < mn[at] &&
----------- uf.find(it->first.first) != at)
--------- mn[at] = it->second, par[at] = it->first;
------- if (par[at] == ii(0,0)) return vii();
------- at = uf.find(par[at].first); }
----- if (at == r || vis[at] != i) continue;
----- union_find tmp = uf; vi seq;
----- do { seq.push_back(at); at = uf.find(par[at].first);
----- } while (at != seq.front());
----- iter(it,seq) uf.unite(*it,seq[0]);
----- int c = uf.find(seq[0]);
----- vector<pair<ii,int> > nw;
----- iter(it,seq) iter(jt,adj[*it])
------- nw.push_back(make_pair(jt->first,
------------- jt->second - mn[*it]));
----- adj[c] = nw;
----- vii rest = find_min(r);
----- if (size(rest) == 0) return rest;
----- ii use = rest[c];
----- rest[at = tmp.find(use.second)] = use;
----- iter(it,seq) if (*it != at)
------- rest[*it] = par[*it];
----- return rest; }
--- return par; } };
```

4.12. **Blossom algorithm**. Finds a maximum matching in an arbitrary graph in $O(|V|^4)$ time. Be vary of loop edges.

```
#define MAXV 300
bool marked[MAXV], emarked[MAXV][MAXV];
int S[MAXV];
vi find_augmenting_path(const vector<vi> &adj,const vi &m){
- int n = size(adj), s = 0;
- vi par(n,-1), height(n), root(n,-1), q, a, b;
- memset(marked,0,sizeof(marked));
- memset(emarked,0,sizeof(emarked));
- rep(i,0,n) if (m[i] >= 0) emarked[i][m[i]] = true;
----------- else root[i] = i, S[s++] = i;
- while (s) {
--- int v = S[--s];
--- iter(wt,adj[v]) {
----- int w = *wt;
----- if (emarked[v][w]) continue;
----- if (root[w] == -1) {
------- int x = S[s++] = m[w];
------- par[w]=v, root[w]=root[v], height[w]=height[v]+1;
------- par[x]=w, root[x]=root[w], height[x]=height[w]+1;
----- } else if (height[w] % 2 == 0) {
------- if (root[v] != root[w]) {
--------- while (v != -1) q.push_back(v), v = par[v];
--------- reverse(q.begin(), q.end());
--------- while (w != -1) q.push_back(w), w = par[w];
--------- return q;
------- } else {
--------- int c = v;
--------- while (c != -1) a.push_back(c), c = par[c];
--------- c = w;
--------- while (c != -1) b.push_back(c), c = par[c];
--------- while (!a.empty()&&!b.empty()&&a.back()==b.back())
----------- c = a.back(), a.pop_back(), b.pop_back();
--------- memset(marked,0,sizeof(marked));
--------- fill(par.begin(), par.end(), 0);
--------- iter(it,a) par[*it] = 1; iter(it,b) par[*it] = 1;
--------- par[c] = s = 1;
--------- rep(i,0,n) root[par[i] = par[i] ? 0 : s++] = i;
--------- vector<vi> adj2(s);
--------- rep(i,0,n) iter(it,adj[i]) {
----------- if (par[*it] == 0) continue;
----------- if (par[i] == 0) {
------------- if (!marked[par[*it]])  {
--------------- adj2[par[i]].push_back(par[*it]);
--------------- adj2[par[*it]].push_back(par[i]);
--------------- marked[par[*it]] = true; }
----------- } else adj2[par[i]].push_back(par[*it]); }
--------- vi m2(s, -1);
--------- if (m[c] != -1) m2[m2[par[m[c]]] = 0] = par[m[c]];
--------- rep(i,0,n) if(par[i]!=0&&m[i]!=-1&&par[m[i]]!=0)
----------- m2[par[i]] = par[m[i]];
--------- vi p = find_augmenting_path(adj2, m2);
--------- int t = 0;
--------- while (t < size(p) && p[t]) t++;
--------- if (t == size(p)) {
----------- rep(i,0,size(p)) p[i] = root[p[i]];
----------- return p; }
--------- if (!p[0] || (m[c] != -1 && p[t+1] != par[m[c]]))
----------- reverse(p.begin(), p.end()), t=(int)size(p)-t-1;
--------- rep(i,0,t) q.push_back(root[p[i]]);
--------- iter(it,adj[root[p[t-1]]]) {
----------- if (par[*it] != (s = 0)) continue;
----------- a.push_back(c), reverse(a.begin(), a.end());
----------- iter(jt,b) a.push_back(*jt);
----------- while (a[s] != *it) s++;
----------- if((height[*it]&1)^(s<(int)size(a)-(int)size(b)))
------------- reverse(a.begin(),a.end()), s=(int)size(a)-s-1;
----------- while(a[s]!=c)q.push_back(a[s]),s=(s+1)%size(a);
----------- q.push_back(c);
----------- rep(i,t+1,size(p)) q.push_back(root[p[i]]);
----------- return q; } } }
----- emarked[v][w] = emarked[w][v] = true; }
--- marked[v] = true; } return q; }
vii max_matching(const vector<vi> &adj) {
- vi m(size(adj), -1), ap; vii res, es;
- rep(i,0,size(adj)) iter(it,adj[i]) es.emplace_back(i,*it);
- random_shuffle(es.begin(), es.end());
- iter(it,es) if (m[it->first] == -1 && m[it->second] == -1)
--- m[it->first] = it->second, m[it->second] = it->first;
- do { ap = find_augmenting_path(adj, m);
------ rep(i,0,size(ap)) m[m[ap[i^1]] = ap[i]] = ap[i^1];
- } while (!ap.empty());
- rep(i,0,size(m)) if (i < m[i]) res.emplace_back(i, m[i]);
- return res; }
```

4.13. **Maximum Density Subgraph.** Given (weighted) undirected graph G. Binary search density. If g is current density, construct flow network: (S, u, m), $(u, T, m + 2g - d_u)$, $(u, v, 1)$, where m is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty S-component, then maximum density is smaller than g, otherwise it's larger. Distance between valid densities is at least $1/(n(n-1))$. Edge case when density is 0. This also works for weighted graphs by replacing d_u by the weighted degree, and doing more iterations (if weights are not integers).

4.14. **Maximum-Weight Closure.** Given a vertex-weighted directed graph G. Turn the graph into a flow network, adding weight ∞ to each edge. Add vertices S, T. For each vertex v of weight w, add edge (S, v, w) if $w \geq 0$, or edge $(v, T, -w)$ if $w < 0$. Sum of positive weights minus minimum $S - T$ cut is the answer. Vertices reachable from S are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

4.15. **Maximum Weighted Ind. Set in a Bipartite Graph.** This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges $(S, u, w(u))$ for $u \in L$, $(v, T, w(v))$ for $v \in R$ and (u, v, ∞) for $(u, v) \in E$. The minimum S,T-cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

4.16. **Synchronizing word problem.** A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

4.17. **Max flow with lower bounds on edges.** Change edge $(u, v, l \leq f \leq c)$ to $(u, v, f \leq c - l)$. Add edge (t, s, ∞). Create super-nodes S, T. Let $M(u) = \sum_v l(v, u) - \sum_v l(u, v)$. If $M(u) < 0$, add edge $(u, T, -M(u))$, else add edge $(S, u, M(u))$. Max flow from S to T. If all edges from S are saturated, then we have a feasible flow. Continue running max flow from s to t in original graph.

4.18. **Tutte matrix for general matching.** Create an $n \times n$ matrix A. For each edge (i, j), $i < j$, let $A_{ij} = x_{ij}$ and $A_{ji} = -x_{ij}$. All other entries are 0. The determinant of A is zero iff. the graph has a perfect matching. A randomized algorithm uses the Schwartz–Zippel lemma to check if it is zero.

4.19. **Heavy Light Decomposition.**

```
#include "segment_tree.cpp"
struct heavy_light_tree {
- int n, *par, *heavy, *dep, *path_root, *pos;
- std::vector<int> *adj;
- segtree *segment_tree;
- heavy_light_tree(int n) : n(n) {
--- this->adj = new std::vector<int>[n];
```

```
--- segment_tree = new segtree(0, n-1);
--- par = new int[n];
--- heavy = new int[n];
--- dep = new int[n];
--- path_root = new int[n];
--- pos = new int[n]; }
- void add_edge(int u, int v) {
--- adj[u].push_back(v);
--- adj[v].push_back(u); }
- void build(int root) {
--- for (int u = 0; u < n; ++u)
----- heavy[u] = -1;
--- par[root] = root;
--- dep[root] = 0;
--- dfs(root);
--- for (int u = 0, p = 0; u < n; ++u) {
----- if (par[u] == -1 or heavy[par[u]] != u) {
------- for (int v = u; v != -1; v = heavy[v]) {
--------- path_root[v] = u;
--------- pos[v] = p++; } } } }
- int dfs(int u) {
--- int sz = 1;
--- int max_subtree_sz = 0;
--- for (int v : adj[u]) {
----- if (v != par[u]) {
------- par[v] = u;
------- dep[v] = dep[u] + 1;
------- int subtree_sz = dfs(v);
------- if (max_subtree_sz < subtree_sz) {
--------- max_subtree_sz = subtree_sz;
--------- heavy[u] = v; }
------- sz += subtree_sz; } }
--- return sz; }
- int query(int u, int v) {
--- int res = 0;
--- while (path_root[u] != path_root[v]) {
----- if (dep[path_root[u]] > dep[path_root[v]])
------- std::swap(u, v);
----- res += segment_tree->sum(pos[path_root[v]], pos[v]);
----- v = par[path_root[v]]; }
--- res += segment_tree->sum(pos[u], pos[v]);
--- return res; }
- void update(int u, int v, int c) {
--- for (; path_root[u] != path_root[v]; v = par[path_root[v]]){
----- if (dep[path_root[u]] > dep[path_root[v]])
------- std::swap(u, v);
----- segment_tree->increase(pos[path_root[v]], pos[v], c); }
--- segment_tree->increase(pos[u], pos[v], c); } };
```

4.20. Centroid Decomposition.

```
#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
- path[MAXV][LGMAXV],
- sz[MAXV], seph[MAXV],
- shortest[MAXV];
struct centroid_decomposition {
- int n; vvi adj;
- centroid_decomposition(int _n) : n(_n), adj(n) { }
- void add_edge(int a, int b) {
--- adj[a].push_back(b); adj[b].push_back(a); }
- int dfs(int u, int p) {
--- sz[u] = 1;
--- for (int i = 0; i < adj[u].size())
----- if (adj[u][i] != p) sz[u] += dfs(adj[u][i], u);
--- return sz[u]; }
- void makepaths(int sep, int u, int p, int len) {
--- jmp[u][seph[sep]] = sep, path[u][seph[sep]] = len;
--- int bad = -1;
--- for (int i = 0; i < adj[u].size()) {
----- if (adj[u][i] == p) bad = i;
----- else makepaths(sep, adj[u][i], u, len + 1); }
--- if (p == sep)
----- swap(adj[u][bad], adj[u].back()), adj[u].pop_back(); }
- void separate(int h=0, int u=0) {
--- dfs(u, -1); int sep = u;
--- down:
--- for (int nxt : adj[sep])
----- if (sz[nxt] < sz[sep] && sz[nxt] > sz[u]/2)
------- sep = nxt, goto down;
--- seph[sep] = h, makepaths(sep, sep, -1, 0);
--- for (int i = 0; i < adj[sep].size())
----- separate(h+1, adj[sep][i]); }
- void paint(int u) {
--- for (int h = 0; h < seph[u] + 1)
----- shortest[jmp[u][h]] =
------- std::min(shortest[jmp[u][h]], path[u][h]); }
- int closest(int u) {
--- int mn = INF/2;
--- for (int h = 0; h < seph[u] + 1)
----- mn = std::min(mn, path[u][h] + shortest[jmp[u][h]]);
--- return mn; } };
```

4.21. Least Common Ancestor.

4.21.1. *Binary Lifting.*

```
struct graph {
- int n, logn, *dep, **par;
- std::vector<int> *adj;
- graph(int n, int logn=20) : n(n), logn(logn) {
--- adj = new std::vector<int>[n];
--- dep = new int[n];
--- par = new int*[n];
--- for (int i = 0; i < n; ++i) par[i] = new int[logn]; }
- void dfs(int u, int p, int d) {
--- dep[u] = d;
--- par[u][0] = p;
--- for (int v : adj[u])
----- if (v != p) dfs(v, u, d+1); }
- int ascend(int u, int k) {
--- for (int i = 0; i < logn; ++i)
----- if (k & (1 << i)) u = par[u][i];
--- return u; }
- int lca(int u, int v) {
--- if (dep[u] > dep[v])  u = ascend(u, dep[u] - dep[v]);
--- if (dep[v] > dep[u])  v = ascend(v, dep[v] - dep[u]);
--- if (u == v)           return u;
--- for (int k = logn-1; k >= 0; --k) {
----- if (par[u][k] != par[v][k]) {
------- u = par[u][k];  v = par[v][k]; } }
--- return par[u][0]; }
- bool is_anc(int u, int v) {
--- if (dep[u] < dep[v]) std::swap(u, v);
--- return ascend(u, dep[u] - dep[v]) == v; }
- void prep_lca(int root=0) {
--- dfs(root, root, 0);
--- for (int k = 1; k < logn; ++k)
----- for (int u = 0; u < n; ++u)
------- par[u][k] = par[par[u][k-1]][k-1]; } };
```

4.21.2. *Euler Tour Sparse Table.*

```
struct graph {
- int n, logn, *par, *dep, *first, *lg, **spt;
- vi *adj, euler; // spt size should be ~ 2n
- graph(int n, int logn=20) : n(n), logn(logn) {
--- adj = new vi[n];
--- par = new int[n];
--- dep = new int[n];
--- first = new int[n]; }
- void add_edge(int u, int v) {
--- adj[u].push_back(v); adj[v].push_back(u); }
- void dfs(int u, int p, int d) {
--- dep[u] = d; par[u] = p;
--- first[u] = euler.size();
--- euler.push_back(u);
--- for (int v : adj[u])
----- if (v != p) {
------- dfs(v, u, d+1);
------- euler.push_back(u); } }
- void prep_lca(int root=0) {
--- dfs(root, root, 0);
--- int en = euler.size();
--- lg = new int[en+1];
--- lg[0] = lg[1] = 0;
--- for (int i = 2; i <= en; ++i)
----- lg[i] = lg[i >> 1] + 1;
--- spt = new int*[en];
--- for (int i = 0; i < en; ++i) {
----- spt[i] = new int[lg[en]];
----- spt[i][0] = euler[i]; }
--- for (int k = 0; (2 << k) <= en; ++k)
----- for (int i = 0; i + (2 << k) <= en; ++i)
------- if (dep[spt[i][k]] < dep[spt[i+(1<<k)][k]])
--------- spt[i][k+1] = spt[i][k];
------- else
--------- spt[i][k+1] = spt[i+(1<<k)][k]; }
- int lca(int u, int v) {
--- int a = first[u], b = first[v];
--- if (a > b)    std::swap(a, b);
```

```
--- int k = lg[b-a+1], ba = b - (1 << k) + 1;
--- if (dep[spt[a][k]] < dep[spt[ba][k]]) return spt[a][k];
--- return spt[ba][k]; } };
```

4.21.3. *Tarjan Off-line LCA.*

```
#include "data-structures/union_find.cpp"
struct tarjan_olca {
- vi ancestor, answers;
- vvi adj;
- vvii queries;
- std::vector<bool> colored;
- union_find uf;
- tarjan_olca(int n, vvi &adj) : adj(adj), uf(n) {
--- vi(n).swap(ancestor);
--- vvii(n).swap(queries);
--- std::vector<bool>(n, false).swap(colored); }
- void query(int x, int y) {
--- queries[x].push_back(ii(y, size(answers)));
--- queries[y].push_back(ii(x, size(answers)));
--- answers.push_back(-1); }
- void process(int u) {
--- ancestor[u] = u;
--- for (int v : adj[u]) {
----- process(v);
----- uf.unite(u,v);
----- ancestor[uf.find(u)] = u; }
--- colored[u] = true;
--- for (auto &[a, b]: queries[u])
----- if (colored[a]) answers[b] = ancestor[uf.find(a)];
} };
```

4.22. **Counting Spanning Trees.** Kirchoff's Theorem: The number of spanning trees of any graph is the determinant of any cofactor of the Laplacian matrix in $O(n^3)$.

(1) Let A be the adjacency matrix.
(2) Let D be the degree matrix (matrix with vertex degrees on the diagonal).
(3) Get $D-A$ and delete exactly one row and column. Any row and column will do. This will be the cofactor matrix.
(4) Get the determinant of this cofactor matrix using Gauss-Jordan.
(5) Spanning Trees = $|\text{cofactor}(D-A)|$

4.23. **Erdős-Gallai Theorem.** A sequence of non-negative integers $d_1 \geq \cdots \geq d_n$ can be represented as the degree sequence of finite simple graph on n vertices if and only if $d_1 + \cdots + d_n$ is even and the following holds for $1 \leq k \leq n$:

$$\sum_{i=1}^{n} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

4.24. **Tree Isomorphism.**

```
// REQUIREMENT: list of primes pr[], see prime sieve
typedef long long LL;
int pre[N], q[N], path[N]; bool vis[N];
// perform BFS and return the last node visited
int bfs(int u, vector<int> adj[]) {
--- memset(vis, 0, sizeof(vis));
--- int head = 0, tail = 0;
--- q[tail++] = u; vis[u] = true; pre[u] = -1;
--- while (head != tail) {
------- u = q[head]; if (++head == N) head = 0;
------- for (int i = 0; i < adj[u].size(); ++i) {
----------- int v = adj[u][i];
----------- if (!vis[v]) {
--------------- vis[v] = true; pre[v] = u;
--------------- q[tail++] = v; if (tail == N) tail = 0;
----------- }}}
--- return u;
} // returns the list of tree centers
vector<int> tree_centers(int r, vector<int> adj[]) {
--- int size = 0;
--- for (int u=bfs(bfs(r, adj), adj); u!=-1; u=pre[u])
------- path[size++] = u;
--- vector<int> med(1, path[size/2]);
--- if (size % 2 == 0) med.push_back(path[size/2-1]);
--- return med;
} // returns "unique hashcode" for tree with root u
LL rootcode(int u, vector<int> adj[], int p=-1, int d=15){
--- vector<LL> k; int nd = (d + 1) % primes;
--- for (int i = 0; i < adj[u].size(); ++i)
------- if (adj[u][i] != p)
----------- k.push_back(rootcode(adj[u][i], adj, u, nd));
--- sort(k.begin(), k.end());
--- LL h = k.size() + 1;
--- for (int i = 0; i < k.size(); ++i)
------- h = h * pr[d] + k[i];
--- return h;
} // returns "unique hashcode" for the whole tree
LL treecode(int root, vector<int> adj[]) {
--- vector<int> c = tree_centers(root, adj);
--- if (c.size()==1)
------- return (rootcode(c[0], adj) << 1) | 1;
--- return (rootcode(c[0],adj)*rootcode(c[1],adj))<<1;
} // checks if two trees are isomorphic
bool isomorphic(int r1, vector<int> adj1[], int r2,
--------------- vector<int> adj2[], bool rooted = false) {
--- if (rooted)
------- return rootcode(r1, adj1) == rootcode(r2, adj2);
--- return treecode(r1, adj1) == treecode(r2, adj2); }
```

5. Math I - Algebra

5.1. **Generating Function Manager.**

```
const int DEPTH = 19;
const int ARR_DEPTH = (1<<DEPTH); //approx 5x10^5
const int SZ = 12;
ll temp[SZ][ARR_DEPTH+1];
const ll MOD = 998244353;
struct GF_Manager {
- int tC = 0;
- std::stack<int> to_be_freed;
- const static ll DEPTH = 23;
- ll prim[DEPTH+1], prim_inv[DEPTH+1], two_inv[DEPTH+1];
- ll mod_pow(ll base, ll exp) {
--- if(exp==0) return 1;
--- if(exp&1) return (base*mod_pow(base,exp-1))%MOD;
--- else return mod_pow((base*base)%MOD, exp/2); }
- void set_up_primitives() {
--- prim[DEPTH] = 31;
--- prim_inv[DEPTH] = mod_pow(prim[DEPTH], MOD-2);
--- two_inv[DEPTH] = mod_pow(1<<DEPTH,MOD-2);
--- for(int n = DEPTH-1; n >= 0; n--) {
----- prim[n] = (prim[n+1]*prim[n+1])%MOD;
----- prim_inv[n] = mod_pow(prim[n],MOD-2);
----- two_inv[n] = mod_pow(1<<n,MOD-2); } }
- GF_Manager(){ set_up_primitives(); }
- void start_claiming(){ to_be_freed.push(0); }
- ll* claim(){
--- ++to_be_freed.top(); assert(tC < SZ); return temp[tC++]; }
- void end_claiming(){tC-=to_be_freed.top(); to_be_freed.pop();}
- void NTT(ll A[], int n, ll t[],
--------- bool is_inverse=false, int offset=0) {
--- if (n==0)  return;
--- //Put the evens first, then the odds
--- for (int i = 0; i < (1<<(n-1)); i++) {
----- t[i] = A[offset+2*i];
----- t[i+(1<<(n-1))] = A[offset+2*i+1]; }
--- for(int i = 0; i < (1<<n); i++)
----- A[offset+i] = t[i];
--- NTT(A, n-1, t, is_inverse, offset);
--- NTT(A, n-1, t, is_inverse, offset+(1<<(n-1)));
--- ll w1 = (is_inverse ? prim_inv[n] : prim[n]), w = 1;
--- for (int i = 0; i < (1<<(n-1)); i++, w=(w*w1)%MOD) {
----- t[i] = (A[offset+i] + w*A[offset+(1<<(n-1))+i])%MOD;
----- t[i+(1<<(n-1))] = (A[offset+i]-
----------------------- w*A[offset+(1<<(n-1))+i])%MOD; }
--- for (int i = 0; i < (1<<n); i++) A[offset+i] = t[i];
- }
- int add(ll A[], int an, ll B[], int bn, ll C[]) {
--- int cn = 0;
--- for(int i = 0; i < max(an,bn); i++) {
----- C[i] = A[i]+B[i];
----- if(C[i] <= -MOD)  C[i] += MOD;
----- if(MOD <= C[i])   C[i] -= MOD;
----- if(C[i]!=0)       cn = i; }
--- return cn; }
- int subtract(ll A[], int an, ll B[], int bn, ll C[]) {
--- int cn = 0;
--- for(int i = 0; i < max(an,bn); i++) {
----- C[i] = A[i]-B[i];
----- if(C[i] <= -MOD)  C[i] += MOD;
----- if(MOD <= C[i])   C[i] -= MOD;
----- if(C[i]!=0)       cn = i; }
--- return cn+1; }
- int scalar_mult(ll v, ll A[], int an, ll C[]) {
--- for(int i = 0; i < an; i++) C[i] = (v*A[i])%MOD;
--- return v==0 ? 0 : an; }
- int mult(ll A[], int an, ll B[], int bn, ll C[]) {
--- start_claiming();
--- // make sure you've called setup prim first
--- // note: an and bn refer to the *number of items in
```

```
--- // each array*, NOT the degree of the largest term
--- int n, degree = an+bn-1;
--- for(n=0; (1<<n) < degree; n++);
--- ll *tA = claim(), *tB = claim(), *t = claim();
--- copy(A,A+an,tA); fill(tA+an,tA+(1<<n),0);
--- copy(B,B+bn,tB); fill(tB+bn,tB+(1<<n),0);
--- NTT(tA,n,t);
--- NTT(tB,n,t);
--- for(int i = 0; i < (1<<n); i++)
----- tA[i] = (tA[i]*tB[i])%MOD;
--- NTT(tA,n,t,true);
--- scalar_mult(two_inv[n],tA,degree,C);
--- end_claiming();
--- return degree; }
- int reciprocal(ll F[], int fn, ll R[]) {
--- start_claiming();
--- ll *tR = claim(), *tempR = claim();
--- int n;  for(n=0; (1<<n) < fn; n++);
--- fill(tempR,tempR+(1<<n),0);
--- tempR[0] = mod_pow(F[0],MOD-2);
--- for (int i = 1; i <= n; i++) {
----- mult(tempR,1<<i,F,1<<i,tR);
----- tR[0] -= 2;
----- scalar_mult(-1,tR,1<<i,tR);
----- mult(tempR,1<<i,tR,1<<i,tempR); }
--- copy(tempR,tempR+fn,R);
--- end_claiming();
--- return n; }
- int quotient(ll F[], int fn, ll G[], int gn, ll Q[]) {
--- start_claiming();
--- ll* revF = claim();
--- ll* revG = claim();
--- ll* tempQ = claim();
--- copy(F,F+fn,revF); reverse(revF,revF+fn);
--- copy(G,G+gn,revG); reverse(revG,revG+gn);
--- int qn = fn-gn+1;
--- reciprocal(revG,qn,revG);
--- mult(revF,qn,revG,qn,tempQ);
--- reverse(tempQ,tempQ+qn);
--- copy(tempQ,tempQ+qn,Q);
--- end_claiming();
--- return qn; }
- int mod(ll F[], int fn, ll G[], int gn, ll R[]) {
--- start_claiming();
--- ll *Q = claim(), *GQ = claim();
--- int qn = quotient(F, fn, G, gn, Q);
--- int gqn = mult(G, gn, Q, qn, GQ);
--- int rn = subtract(F, fn, GQ, gqn, R);
--- end_claiming();
--- return rn; }
- ll horners(ll F[], int fn, ll xi) {
--- ll ans = 0;
--- for(int i = fn-1; i >= 0; i--)
----- ans = (ans*xi+F[i]) % MOD;
--- return ans; } };
GF_Manager gfManager;
ll split[DEPTH+1][2*(ARR_DEPTH)+1];
ll Fi[DEPTH+1][2*(ARR_DEPTH)+1];
int bin_splitting(ll a[], int l, int r, int s=0, int offset=0) {
- if(l == r) {
--- split[s][offset] = -a[l]; //x^0
--- split[s][offset+1] = 1; //x^1
--- return 2; }
- int m = (l+r)/2;
- int sz = m-l+1;
- int da = bin_splitting(a, l, m, s+1, offset);
- int db = bin_splitting(a, m+1, r, s+1, offset+(sz<<1));
- return gfManager.mult(split[s+1]+offset, da,
--- split[s+1]+offset+(sz<<1), db, split[s]+offset); }
void multipoint_eval(ll a[], int l, int r, ll F[], int fn,
- ll ans[], int s=0, int offset=0) {
--- if(l == r) {
----- ans[l] = gfManager.horners(F,fn,a[l]);
----- return; }
--- int m = (l+r)/2;
--- int sz = m-l+1;
--- int da = gfManager.mod(F, fn, split[s+1]+offset,
----- sz+1, Fi[s]+offset);
--- int db = gfManager.mod(F, fn, split[s+1]+offset+(sz<<1),
----- r-m+1, Fi[s]+offset+(sz<<1));
--- multipoint_eval(a,l,m,Fi[s]+offset,da,ans,s+1,offset);
--- multipoint_eval(a,m+1,r,Fi[s]+offset+(sz<<1),
----- db,ans,s+1,offset+(sz<<1));
}
```

5.2. **Fast Fourier Transform.** Compute the Discrete Fourier Transform (DFT) of a polynomial in $O(n \log n)$ time.

```
struct poly {
--- double a, b;
--- poly(double a=0, double b=0): a(a), b(b) {}
--- poly operator+(const poly& p) const {
------- return poly(a + p.a, b + p.b);}
--- poly operator-(const poly& p) const {
------- return poly(a - p.a, b - p.b);}
--- poly operator*(const poly& p) const {
------- return poly(a*p.a - b*p.b, a*p.b + b*p.a);}
};
void fft(poly in[], poly p[], int n, int s) {
--- if (n < 1) return;
--- if (n == 1) {p[0] = in[0]; return;}
--- n >>= 1; fft(in, p, n, s << 1);
--- fft(in + s, p + n, n, s << 1);
--- poly w(1), wn(cos(M_PI/n), sin(M_PI/n));
--- for (int i = 0; i < n; ++i) {
------- poly even = p[i], odd = p[i + n];
------- p[i] = even + w * odd;
------- p[i + n] = even - w * odd;
------- w = w * wn;
--- }
}
void fft(poly p[], int n) {
--- poly *f = new poly[n]; fft(p, f, n, 1);
--- copy(f, f + n, p); delete[] f;
}
void inverse_fft(poly p[], int n) {
--- for(int i=0; i<n; i++) {p[i].b *= -1;} fft(p, n);
--- for(int i=0; i<n; i++) {p[i].a/=n; p[i].b/= -1*n;}
}
```

5.3. **FFT Polynomial Multiplication.** Multiply integer polynomials a, b of size an, bn using FFT in $O(n \log n)$. Stores answer in an array c, rounded to the nearest integer (or double).

```
// note: c[] should have size of at least (an+bn)
int mult(int a[],int an,int b[],int bn,int c[]) {
--- int n, degree = an + bn - 1;
--- for (n = 1; n < degree; n <<= 1); // power of 2
--- poly *A = new poly[n], *B = new poly[n];
--- copy(a, a + an, A); fill(A + an, A + n, 0);
--- copy(b, b + bn, B); fill(B + bn, B + n, 0);
--- fft(A, n); fft(B, n);
--- for (int i = 0; i < n; i++) A[i] = A[i] * B[i];
--- inverse_fft(A, n);
--- for (int i = 0; i < degree; i++)
------- c[i] = int(A[i].a + 0.5); // same as round(A[i].a)
--- delete[] A, B; return degree;
}
```

5.4. **Number Theoretic Transform.** Other possible moduli: $2113929217 (2^{25}), 2013265920268435457 (2^{28}, with g = 5)$

```
#include "../mathematics/primitive_root.cpp"
int mod = 998244353, g = primitive_root(mod),
- ginv = mod_pow<ll>(g, mod-2, mod),
- inv2 = mod_pow<ll>(2, mod-2, mod);
#define MAXN (1<<22)
struct Num {
- int x;
- Num(ll _x=0) { x = (_x%mod+mod)%mod; }
- Num operator +(const Num &b) { return x + b.x; }
- Num operator -(const Num &b) const { return x - b.x; }
- Num operator *(const Num &b) const { return (ll)x * b.x; }
- Num operator /(const Num &b) const {
--- return (ll)x * b.inv().x; }
- Num inv() const { return mod_pow<ll>((ll)x, mod-2, mod); }
- Num pow(int p) const { return mod_pow<ll>((ll)x, p, mod); }
} T1[MAXN], T2[MAXN];
void ntt(Num x[], int n, bool inv = false) {
- Num z = inv ? ginv : g;
- z = z.pow((mod - 1) / n);
- for (ll i = 0, j = 0; i < n; i++) {
--- if (i < j) swap(x[i], x[j]);
--- ll k = n>>1;
--- while (1 <= k && k <= j) j -= k, k >>= 1;
--- j += k; }
- for (int mx = 1, p = n/2; mx < n; mx <<= 1, p >>= 1) {
--- Num wp = z.pow(p), w = 1;
--- for (int k = 0; k < mx; k++, w = w*wp) {
----- for (int i = k; i < n; i += mx << 1) {
------- Num t = x[i + mx] * w;
------- x[i + mx] = x[i] - t;
```

```
------- x[i] = x[i] + t; } } }
- if (inv) {
--- Num ni = Num(n).inv();
--- rep(i,0,n) { x[i] = x[i] * ni; } } }
void inv(Num x[], Num y[], int l) {
- if (l == 1) { y[0] = x[0].inv(); return; }
- inv(x, y, l>>1);
- // NOTE: maybe l<<2 instead of l<<1
- rep(i,l>>1,l<<1) T1[i] = y[i] = 0;
- rep(i,0,l) T1[i] = x[i];
- ntt(T1, l<<1); ntt(y, l<<1);
- rep(i,0,l<<1) y[i] = y[i]*2 - T1[i] * y[i] * y[i];
- ntt(y, l<<1, true); }
void sqrt(Num x[], Num y[], int l) {
- if (l == 1) { assert(x[0].x == 1); y[0] = 1; return; }
- sqrt(x, y, l>>1);
- inv(y, T2, l>>1);
- rep(i,l>>1,l<<1) T1[i] = T2[i] = 0;
- rep(i,0,l) T1[i] = x[i];
- ntt(T2, l<<1); ntt(T1, l<<1);
- rep(i,0,l<<1) T2[i] = T1[i] * T2[i];
- ntt(T2, l<<1, true);
- rep(i,0,l) y[i] = (y[i] + T2[i]) * inv2; }
// vim: cc=60 ts=2 sts=2 sw=2:
```

5.5. **Polynomial Long Division.** Divide two polynomials A and B to get Q and R, where $\frac{A}{B} = Q + \frac{R}{B}$.

```
typedef vector<double> Poly;
Poly Q, R; // quotient and remainder
void trim(Poly& A) { // remove trailing zeroes
--- while (!A.empty() && abs(A.back()) < EPS)
--- A.pop_back();
}
void divide(Poly A, Poly B) {
--- if (B.size() == 0) throw exception();
--- if (A.size() < B.size()) {Q.clear(); R=A; return;}
--- Q.assign(A.size() - B.size() + 1, 0);
--- Poly part;
--- while (A.size() >= B.size()) {
------- int As = A.size(), Bs = B.size();
------- part.assign(As, 0);
------- for (int i = 0; i < Bs; i++)
----------- part[As-Bs+i] = B[i];
------- double scale = Q[As-Bs] = A[As-1] / part[As-1];
------- for (int i = 0; i < As; i++)
----------- A[i] -= part[i] * scale;
------- trim(A);
--- } R = A; trim(Q); }
```

5.6. **Matrix Multiplication.** Multiplies matrices $A_{p\times q}$ and $B_{q\times r}$ in $O(n^3)$ time, modulo MOD.

```
long[][] multiply(long A[][], long B[][]) {
--- int p = A.length, q = A[0].length, r = B[0].length;
--- // if(q != B.length) throw new Exception(":(((");
--- long AB[][] = new long[p][r];
--- for (int i = 0; i < p; i++)
--- for (int j = 0; j < q; j++)
--- for (int k = 0; k < r; k++)
------- (AB[i][k] += A[i][j] * B[j][k]) %= MOD;
--- return AB; }
```

5.7. **Matrix Power.** Computes for B^e in $O(n^3 \log e)$ time. Refer to Matrix Multiplication.

```
long[][] power(long B[][], long e) {
--- int n = B.length;
--- long ans[][]= new long[n][n];
--- for (int i = 0; i < n; i++) ans[i][i] = 1;
--- while (e > 0) {
------- if (e % 2 == 1) ans = multiply(ans, b);
------- b = multiply(b, b); e /= 2;
--- } return ans;}
```

5.8. **Fibonacci Matrix.** Fast computation for nth Fibonacci $\{F_1, F_2, \ldots, F_n\}$ in $O(\log n)$:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \times \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

5.9. **Gauss-Jordan/Matrix Determinant.** Row reduce matrix A in $O(n^3)$ time. Returns `true` if a solution exists.

```
boolean gaussJordan(double A[][]) {
--- int n = A.length, m = A[0].length;
--- boolean singular = false;
--- // double determinant = 1;
--- for (int i=0, p=0; i<n && p<m; i++, p++) {
------- for (int k = i + 1; k < n; k++) {
----------- if (Math.abs(A[k][p]) > EPS) { // swap
----------- // determinant *= -1;
----------- double t[]=A[i]; A[i]=A[k]; A[k]=t;
----------- break;
----------- }
------- }
------- // determinant *= A[i][p];
------- if (Math.abs(A[i][p]) < EPS)
----------- { singular = true; i--; continue; }
------- for (int j = m-1; j >= p; j--) A[i][j]/= A[i][p];
------- for (int k = 0; k < n; k++) {
----------- if (i == k) continue;
----------- for (int j = m-1; j >= p; j--)
--------------- A[k][j] -= A[k][p] * A[i][j];
------- }
--- } return !singular; }
```

6. Math II - Combinatorics

6.1. **Lucas Theorem.** Compute $\binom{n}{k} \bmod p$ in $O(p + \log_p n)$ time, where p is a prime.

```
LL f[P], lid; // P: biggest prime
LL lucas(LL n, LL k, int p) {
--- if (k == 0) return 1;
--- if (n < p && k < p) {
------- if (lid != p) {
----------- lid = p; f[0] = 1;
----------- for (int i = 0; i < p; ++i) f[i]=f[i-1]*i%p;
------- }
------- return f[n] * modpow(f[n-k]*f[k]%p, p-2, p) % p;}
--- return lucas(n/p,k/p,p) * lucas(n%p,k%p,p) % p; }
```

6.2. **Granville's Theorem.** Compute $\binom{n}{k} \bmod m$ (for any m) in $O(m^2 \log^2 n)$ time.

```
def fprime(n, p):
--- # counts the number of prime divisors of n!
--- pk, ans = p, 0
--- while pk <= n:
------- ans += n // pk
------- pk *= p
--- return ans


def granville(n, k, p, E):
--- # n choose k (mod p^E)
--- prime_pow = fprime(n, p) - fprime(k, p) - fprime(n - k, p)
--- if prime_pow >= E:
------- return 0
--- e = E - prime_pow
--- pe = p**e
--- r, f = n - k, [1] * pe
--- for i in range(1, pe):
------- x = i
------- if x % p == 0:
----------- x = 1
------- f[i] = f[i - 1] * x % pe
--- numer, denom, negate, ptr = 1, 1, 0, 0
--- while n:
------- if f[-1] != 1 and ptr >= e:
----------- negate ^= (n & 1) ^ (k & 1) ^ (r & 1)
------- numer = numer * f[n % pe] % pe
------- denom = denom * f[k % pe] % pe * f[r % pe] % pe
------- n, k, r = n // p, k // p, r // p
------- ptr += 1
--- ans = numer * modinv(denom, pe) % pe
--- if negate and (p != 2 or e < 3):
------- ans = (pe - ans) % pe
--- return mod(ans * p**prime_pow, p**E)


def choose(n, k, m): # generalized (n choose k) mod m
--- factors, x, p = [], m, 2
--- while p * p <= x:
------- e = 0
------- while x % p == 0:
----------- e += 1
----------- x //= p
------- if e:
----------- factors.append((p, e))
------- p += 1
--- if x > 1:
------- factors.append((x, 1))
--- crt_array = [granville(n, k, p, e) for p, e in factors]
--- mod_array = [p**e for p, e in factors]
--- return chinese_remainder(crt_array, mod_array)[0]
```

6.3. **Derangements.** Compute the number of permutations with n elements such that no element is at their original position:

$$D(n) = (n-1)\left(D(n-1) + D(n-2)\right) = nD(n-1) + (-1)^n$$

6.4. **Factoradics.** Convert a permutation of n items to factoradics and vice versa in $O(n \log n)$.

```
// use fenwick tree add, sum, and low code
typedef long long LL;
void factoradic(int arr[], int n) { // 0 to n-1
--- for (int i = 0; i <=n; i++) fen[i] = 0;
--- for (int i = 1; i < n; i++) add(i, 1);
--- for (int i = 0; i < n; i++) {
--- int s = sum(arr[i]);
--- add(arr[i], -1); arr[i] = s;
--- }}
void permute(int arr[], int n) { // factoradic to perm
--- for (int i = 0; i <=n; i++) fen[i] = 0;
--- for (int i = 1; i < n; i++) add(i, 1);
--- for (int i = 0; i < n; i++) {
--- arr[i] = low(arr[i] - 1);
--- add(arr[i], -1);
--- }}
```

6.5. **kth Permutation.** Get the next kth permutation of n items, if exists, using factoradics. All values should be from 0 to $n-1$. Use factoradics methods as discussed above.

```
std::vector<int> nth_permutation(int cnt, int n) {
- std::vector<int> idx(cnt), per(cnt), fac(cnt);
- for (int i = 0; i < cnt; ++i) idx[i] = i;
- for (int i = 1; i < cnt+1; ++i) fac[i - 1] = n % i, n /= i;
- for (int i = cnt - 1; i >= 0; --i)
--- per[cnt - i - 1] = idx[fac[i]],
--- idx.erase(idx.begin() + fac[i]);
- return per; }
```

6.6. **Catalan Numbers.**

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$$

(1) The number of non-crossing partitions of an n-element set
(2) The number of expressions with n pairs of parentheses
(3) The number of ways $n+1$ factors can be parenthesized
(4) The number of full binary trees with $n+1$ leaves
(5) The number of monotonic lattice paths of an $n \times n$ grid (5-SAT problem)
(6) The number of triangulations of a convex polygon with $n+2$ sides (non-rotational)
(7) The number of permutations $\{1, \ldots, n\}$ without a 3-term increasing subsequence
(8) The number of ways to form a mountain range with n ups and n downs

6.7. **Stirling Numbers.** s_1: Count the number of permutations of n elements with k disjoint cycles

s_2: Count the ways to partition a set of n elements into k nonempty subsets

$$s_1(n,k) = \begin{cases} 1 & n = k = 0 \\ s_1(n-1,k-1) - (n-1)s_1(n-1,k) & n,k > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$s_2(n,k) = \begin{cases} 1 & n = k = 0 \\ s_2(n-1,k-1) + ks_2(n-1,k) & n,k > 0 \\ 0 & \text{elsewhere} \end{cases}$$

6.8. **Partition Function.** Pregenerate the number of partitions of positive integer n with n positive addends.

$$p(n,k) = \begin{cases} 1 & n = k = 0 \\ 0 & n < k \\ p(n-1,k-1) + p(n-k,k) & n \geq k \end{cases}$$

7. Math III - Number Theory

7.1. **Linear Prime Sieve.**

```
std::bitset<N> isc; // #include <bitset>
std::vector<int> p;
void sieve() {
- for (int i = 2; i < N; i++) {
--- if (!isc[i]) p.push_back(i);
----- for (int j = 0; j < p.size() && i*p[j] < N; j++) {
------- isc[i*p[j]] = 1;
------- if (i%p[j] == 0) break; } } }
```

7.2. **Number/Sum of Divisors.** If a number n is prime factorized where $n = {p_1}^{e_1} \times {p_2}^{e_2} \times \cdots \times {p_k}^{e_k}$, where σ_0 is the number of divisors while σ_1 is the sum of divisors:

$$\sum_{d|n} d^k = \sigma_k(n) = \prod \frac{{p_i}^{k(e_i)+1} - 1}{p_i - 1}$$

$$\text{Product: } \prod_{d|n} d = n^{\frac{\sigma_1(n)}{2}}$$

7.3. **Möbius Sieve.** The Möbius function μ is the Möbius inverse of e such that $e(n) = \sum_{d|n} \mu(d)$.

```
std::bitset<N> is; int mu[N];
void mobiusSieve() {
- for (int i = 1; i < N; ++i) mu[i] = 1;
- for (int i = 2; i < N; ++i) if (!is[i]) {
--- for (int j = i; j < N; j += i) is[j] = 1, mu[j] *= -1;
--- for (ll j = 1LL*i*i; j < N; j += i*i) mu[j] = 0; } }
```

7.4. **Möbius Inversion.** Given arithmetic functions f and g:

$$g(n) = \sum_{d|n} f(d) \quad \Leftrightarrow \quad f(n) = \sum_{d|n} \mu(d)\, g\left(\frac{n}{d}\right)$$

7.5. **GCD Subset Counting.** Count number of subsets $S \subseteq A$ such that $\gcd(S) = g$ (modifiable).

```
int f[MX+1]; // MX is maximum number of array
long long gcnt[MX+1]; // gcnt[G]: answer when gcd==G
long long C(int f) {return (1ll << f) - 1;}
// f: frequency count
// C(f): # of subsets of f elements (YOU CAN EDIT)
// Usage: int subsets_with_gcd_1 = gcnt[1];
void gcd_counter(int a[], int n) {
- memset(f, 0, sizeof f);
- memset(gcnt, 0, sizeof gcnt);
- int mx = 0;
- for (int i = 0; i < n; ++i) {
----- f[a[i]] += 1;
----- mx = max(mx, a[i]); }
- for (int i = mx; i >= 1; --i) {
--- int add = f[i];
--- long long sub = 0;
--- for (int j = 2*i; j <= mx; j += i) {
----- add += f[j];
----- sub += gcnt[j]; }
--- gcnt[i] = C(add) - sub; }}
```

7.6. **Euler Totient.** Counts all integers from 1 to n that are relatively prime to n in $O(\sqrt{n})$ time.

```
ll totient(ll n) {
- if (n <= 1) return 1;
- ll tot = n;
- for (int i = 2; i * i <= n; i++) {
--- if (n % i == 0) tot -= tot / i;
--- while (n % i == 0) n /= i; }
- if (n > 1) tot -= tot / n;
- return tot; }
```

7.7. **Extended Euclidean.** Assigns x, y such that $ax + by = \gcd(a,b)$ and returns $\gcd(a,b)$.

```
ll mod(ll x, ll m) { // use this instead of x % m
- if (m == 0) return 0;
- if (m < 0) m *= -1;
- return (x%m + m) % m; // always nonnegative
}
ll extended_euclid(ll a, ll b, ll &x, ll &y) {
- if (b==0) {x = 1; y = 0; return a;}
- ll g = extended_euclid(b, a%b, x, y);
- ll z = x - a/b*y;
- x = y; y = z; return g;
}
```

7.8. **Modular Exponentiation.** Find $b^e \pmod{m}$ in $O(loge)$ time.

```
template <class T>
T mod_pow(T b, T e, T m) {
- T res = T(1);
- while (e) {
--- if (e & T(1)) res = smod(res * b, m);
--- b = smod(b * b, m), e >>= T(1); }
- return res; }
```

7.9. Modular Inverse. Find unique x such that $ax \equiv 1 \pmod{m}$. Returns 0 if no unique solution is found. Please use modulo solver for the non-unique case.

```
ll modinv(ll a, ll m) {
- ll x, y; ll g = extended_euclid(a, m, x, y);
- if (g == 1 || g == -1) return mod(x * g, m);
- return 0; // 0 if invalid }
```

7.10. Modulo Solver. Solve for values of x for $ax \equiv b \pmod{m}$. Returns $(-1,-1)$ if there is no solution. Returns a pair (x, M) where solution is $x \bmod M$.

```
pll modsolver(ll a, ll b, ll m) {
- ll x, y; ll g = extended_euclid(a, m, x, y);
- if (b % g != 0) return {-1, -1};
- return {mod(x*b/g, m/g), abs(m/g)}; }
```

7.11. Linear Diophantine. Computes integers x and y such that $ax + by = c$, returns $(-1,-1)$ if no solution. Tries to return positive integer answers for x and y if possible.

```
pll null(-1, -1); // needs extended euclidean
pll diophantine(ll a, ll b, ll c) {
- if (!a && !b) return c ? null : {0, 0};
- if (!a) return c % b ? null : {0, c / b};
- if (!b) return c % a ? null : {c / a, 0};
- ll x, y; ll g = extended_euclid(a, b, x, y);
- if (c % g) return null;
- y = mod(y * (c/g), a/g);
- if (y == 0) y += abs(a/g); // prefer positive sol.
- return {(c - b*y)/a, y}; }
```

7.12. Chinese Remainder Theorem. Solves linear congruence $x \equiv b_i \pmod{m_i}$. Returns $(-1,-1)$ if there is no solution. Returns a pair (x, M) where solution is $x \bmod M$.

```
pll chinese(ll b1, ll m1, ll b2, ll m2) {
- ll x, y; ll g = extended_euclid(m1, m2, x, y);
- if (b1 % g != b2 % g) return ii(-1, -1);
- ll M = abs(m1 / g * m2);
- return {mod(mod(x*b2*m1+y*b1*m2, M*g)/g,M), M}; }
ii chinese_remainder(ll b[], ll m[], int n) {
- ii ans(0, 1);
- for (int i = 0; i < n; ++i) {
--- ans = chinese(b[i],m[i],ans.first,ans.second);
--- if (ans.second == -1) break; }
- return ans; }
```

7.12.1. *Super Chinese Remainder.* Solves linear congruence $a_i x \equiv b_i \pmod{m_i}$. Returns $(-1,-1)$ if there is no solution.

```
pll super_chinese(ll a[], ll b[], ll m[], int n) {
- pll ans(0, 1);
- for (int i = 0; i < n; ++i) {
--- pll two = modsolver(a[i], b[i], m[i]);
--- if (two.second == -1) return two;
--- ans = chinese(ans.first, ans.second,
--- two.first, two.second);
--- if (ans.second == -1) break; }
- return ans; }
```

7.13. Primitive Root.

```
#include "mod_pow.cpp"
ll primitive_root(ll m) {
- std::vector<ll> div;
- for (ll i = 1; i*i <= m-1; i++) {
--- if ((m-1) % i == 0) {
----- if (i < m) div.push_back(i);
----- if (m/i < m) div.push_back(m/i); } }
- for (int x = 2; x < m; ++x) {
--- bool ok = true;
--- for (int d : div) if (mod_pow<ll>(x, d, m) == 1) {
----- ok = false; break; }
--- if (ok) return x; }
- return -1; }
```

7.14. Josephus. Last man standing out of n if every kth is killed. Zero-based, and does not kill 0 on first pass.

```
int J(int n, int k) {
- if (n == 1) return 0;
- if (k == 1) return n-1;
- if (n < k) return (J(n-1,k)+k)%n;
- int np = n - n/k;
- return k*((J(np,k)+np-n%k%np)%np) / (k-1); }
```

7.15. Number of Integer Points under a Lines. Count the number of integer solutions to $Ax + By \leq C$, $0 \leq x \leq n$, $0 \leq y$. In other words, evaluate the sum $\sum_{x=0}^{n} \left\lfloor \frac{C - Ax}{B} + 1 \right\rfloor$. To count all solutions, let $n = \left\lfloor \frac{c}{a} \right\rfloor$. In any case, it must hold that $C - nA \geq 0$. Be very careful about overflows.

8. Math IV - Numerical Methods

8.1. Fast Square Testing. An optimized test for square integers.

```
long long M;
void init_is_square() {
- for (int i = 0; i < 64; ++i) M |= 1ULL << (63-(i*i)%64); }
inline bool is_square(ll x) {
- if (x == 0) return true; // XXX
- if ((M << x) >= 0) return false;
- int c = std::__builtin_ctz(x);
- if (c & 1) return false;
- x >>= c;
- if ((x&7) - 1) return false;
- ll r = std::sqrt(x);
- return r*r == x; }
```

8.2. Simpson Integration. Use to numerically calculate integrals

```
const int N = 1000 * 1000; // number of steps
double simpson_integration(double a, double b){
- double h = (b - a) / N;
- double s = f(a) + f(b); // a = x_0 and b = x_2n
- for (int i = 1; i <= N - 1; ++i) {
--- double x = a + h * i;
--- s += f(x) * ((i & 1) ? 4 : 2); }
- s *= h / 3;
- return s; }
```

9. Strings

9.1. Knuth-Morris-Pratt. Count and find all matches of string f in string s in $O(n)$ time.

```
int par[N]; // parent table
void buildKMP(string& f) {
- par[0] = -1, par[1] = 0;
- int i = 2, j = 0;
- while (i <= f.length()) {
--- if (f[i-1] == f[j]) par[i++] = ++j;
--- else if (j > 0) j = par[j];
--- else par[i++] = 0; } }
std::vector<int> KMP(string& s, string& f) {
- buildKMP(f); // call once if f is the same
- int i = 0, j = 0; vector<int> ans;
- while (i + j < s.length()) {
--- if (s[i + j] == f[j]) {
----- if (++j == f.length()) {
------- ans.push_back(i);
------- i += j - par[j];
------- if (j > 0) j = par[j]; }
--- } else {
----- i += j - par[j];
----- if (j > 0) j = par[j]; }
- } return ans; }
```

9.2. Trie.

```
template <class T>
struct trie {
- struct node {
--- map<T, node*> children;
--- int prefixes, words;
--- node() { prefixes = words = 0; } };
- node* root;
- trie() : root(new node()) {  }
- template <class I>
- void insert(I begin, I end) {
--- node* cur = root;
--- while (true) {
----- cur->prefixes++;
----- if (begin == end) { cur->words++; break; }
----- else {
------- T head = *begin;
------- typename map<T, node*>::const_iterator it;
------- it = cur->children.find(head);
------- if (it == cur->children.end()) {
--------- pair<T, node*> nw(head, new node());
--------- it = cur->children.insert(nw).first;
------- } begin++, cur = it->second; } } }
- template<class I>
- int countMatches(I begin, I end) {
--- node* cur = root;
--- while (true) {
----- if (begin == end) return cur->words;
----- else {
------- T head = *begin;
------- typename map<T, node*>::const_iterator it;
```

```
------- it = cur->children.find(head);
------- if (it == cur->children.end()) return 0;
------- begin++, cur = it->second; } } }
- template<class I>
- int countPrefixes(I begin, I end) {
--- node* cur = root;
--- while (true) {
----- if (begin == end) return cur->prefixes;
----- else {
------- T head = *begin;
------- typename map<T, node*>::const_iterator it;
------- it = cur->children.find(head);
------- if (it == cur->children.end()) return 0;
------- begin++, cur = it->second; } } } };
```

9.2.1. *Persistent Trie.*

```
const int MAX_KIDS = 2;
const char BASE = '0';  // 'a' or 'A'
struct trie {
- int val, cnt;
- std::vector<trie*> kids;
- trie () : val(-1), cnt(0), kids(MAX_KIDS, NULL) {}
- trie (int val) : val(val), cnt(0), kids(MAX_KIDS, NULL) {}
- trie (int val, int cnt, std::vector<trie*> &n_kids) :
--- val(val), cnt(cnt), kids(n_kids) {}
- trie *insert(std::string &s, int i, int n) {
--- trie *n_node = new trie(val, cnt+1, kids);
--- if (i == n) return n_node;
--- if (!n_node->kids[s[i]-BASE])
----- n_node->kids[s[i]-BASE] = new trie(s[i]);
--- n_node->kids[s[i]-BASE] =
----- n_node->kids[s[i]-BASE]->insert(s, i+1, n);
--- return n_node; } };
// max xor on a binary trie from version `a+1` to `b` (b > a):
int get_max_xor(trie *a, trie *b, int x) {
- int ans = 0;
- for (int i = MAX_BITS; i >= 0; --i) {
--- // don't flip the bit for min xor
--- int u = ((x & (1 << i)) > 0) ^ 1;
--- int res_cnt = (b and b->kids[u] ? b->kids[u]->cnt : 0) -
----------------- (a and a->kids[u] ? a->kids[u]->cnt : 0);
--- if (res_cnt == 0) u ^= 1;
--- ans ^= (u << i);
--- if (a) a = a->kids[u];
--- if (b) b = b->kids[u]; }
- return ans; }
```

9.3. **Suffix Array**. Construct a sorted catalog of all substrings of s in $O(n \log n)$ time using counting sort.

```
int n, equiv[N+1], suffix[N+1];
ii equiv_pair[N+1];
string T;
void make_suffix_array(string& s) {
- if (s.back()!='$') s += '$';
- n = s.length();
- for (int i = 0; i < n; i++)
--- suffix[i] = i;
- sort(suffix,suffix+n,[&s](int i, int j){return s[i] < s[j];});
- int sz = 0;
- for(int i = 0; i < n; i++){
--- if(i==0 || s[suffix[i]]!=s[suffix[i-1]])
----- ++sz;
--- equiv[suffix[i]] = sz; }
- for (int t = 1; t < n; t<<=1) {
--- for (int i = 0; i < n; i++)
----- equiv_pair[i] = {equiv[i],equiv[(i+t)%n]};
--- sort(suffix, suffix+n, [](int i, int j) {
------- return equiv_pair[i] < equiv_pair[j];});
--- int sz = 0;
--- for (int i = 0; i < n; i++) {
----- if(i==0 || equiv_pair[suffix[i]]!=equiv_pair[suffix[i-1]])
------- ++sz;
----- equiv[suffix[i]] = sz; } } }
int count_occurences(string& G) { // in string T
- int L = 0, R = n-1;
- for (int i = 0; i < G.length(); i++){
--- // lower/upper = first/last time G[i] is
--- // the ith character in suffixes from [L,R]
--- std::tie(L,R) = {lower(G[i],i,L,R), upper(G[i],i,L,R)};
--- if (L==-1 && R==-1) return 0; }
- return R-L+1; }
```

9.4. **Longest Common Prefix**. Find the length of the longest common prefix for every substring in $O(n)$.

```
int lcp[N]; // lcp[i] = LCP(s[sa[i]:], s[sa[i+1]:])
void buildLCP(std::string s) {// build suffix array first
- for (int i = 0, k = 0; i < n; i++) {
--- if (pos[i] != n - 1) {
----- for(int j = sa[pos[i]+1]; s[i+k]==s[j+k];k++);
----- lcp[pos[i]] = k; if (k > 0) k--;
- } else { lcp[pos[i]] = 0; } } }
```

9.5. **Aho-Corasick Trie**. Find all multiple pattern matches in $O(n)$ time. This is KMP for multiple strings.

```
class Node {
- HashMap<Character, Node> next = new HashMap<>();
- Node fail = null;
- long count = 0;
- public void add(String s) { // adds string to trie
--- Node node = this;
--- for (char c : s.toCharArray()) {
----- if (!node.contains(c))
------- node.next.put(c, new Node());
----- node = node.get(c);
--- } node.count++; }
- public void prepare() {
--- // prepares fail links of Aho-Corasick Trie
--- Node root = this; root.fail = null;
--- Queue<Node> q = new ArrayDeque<Node>();
--- for (Node child : next.values()) // BFS
----- { child.fail = root; q.offer(child); }
--- while (!q.isEmpty()) {
----- Node head = q.poll();
----- for (Character letter : head.next.keySet()) {
----- // traverse upwards to get nearest fail link
------- Node p = head;
------- Node nextNode = head.get(letter);
------- do { p = p.fail; }
------- while(p != root && !p.contains(letter));
------- if (p.contains(letter)) { // fail link found
--------- p = p.get(letter);
--------- nextNode.fail = p;
--------- nextNode.count += p.count;
------- } else { nextNode.fail = root; }
------- q.offer(nextNode); } } }
- public BigInteger search(String s) {
--- // counts the words added in trie present in s
--- Node root = this, p = this;
--- BigInteger ans = BigInteger.ZERO;
--- for (char c : s.toCharArray()) {
----- while (p != root && !p.contains(c)) p = p.fail;
----- if (p.contains(c)) {
------- p = p.get(c);
------- ans = ans.add(BigInteger.valueOf(p.count)); }
--- } return ans; }
- private Node get(char c) { return next.get(c); }
- private boolean contains(char c) {
--- return next.containsKey(c); }}
// Usage: Node trie = new Node();
// for (String s : dictionary) trie.add(s);
// trie.prepare(); BigInteger m = trie.search(str);
```

9.6. **Palimdromes.**

9.6.1. *Palindromic Tree*. Find lengths and frequencies of all palindromic substrings of a string in $O(n)$ time.

Theorem: there can only be up to n unique palindromic substrings for any string.

```
int par[N*2+1], child[N*2+1][128];
int len[N*2+1], node[N*2+1], cs[N*2+1], size;
long long cnt[N + 2]; // count can be very large
int newNode(int p = -1) {
- cnt[size] = 0; par[size] = p;
- len[size] = (p == -1 ? 0 : len[p] + 2);
- memset(child[size], -1, sizeof child[size]);
- return size++; }
int get(int i, char c) {
- if (child[i][c] == -1) child[i][c] = newNode(i);
- return child[i][c]; }
void manachers(char s[]) {
- int n = strlen(s), cn = n * 2 + 1;
- for (int i = 0; i < n; i++) {
--- cs[i * 2] = -1; cs[i * 2 + 1] = s[i]; }
- size = n * 2;
- int odd = newNode(), even = newNode();
- int cen = 0, rad = 0, L = 0, R = 0;
- size = 0; len[odd] = -1;
- for (int i = 0; i < cn; i++)
--- node[i] = (i % 2 == 0 ? even : get(odd, cs[i]));
- for (int i = 1; i < cn; i++) {
```

```
--- if (i > rad) { L = i - 1; R = i + 1; }
--- else {
----- int M = cen * 2 - i; // retrieve from mirror
----- node[i] = node[M];
----- if (len[node[M]] < rad - i) L = -1;
----- else {
------- R = rad + 1; L = i * 2 - R;
------- while (len[node[i]] > rad - i)
--------- node[i] = par[node[i]]; } } // expand palindrome
--- while (L >= 0 && R < cn && cs[L] == cs[R]) {
------- if (cs[L] != -1) node[i] = get(node[i],cs[L]);
------- L--, R++; }
--- cnt[node[i]]++;
--- if (i + len[node[i]] > rad) {
----- rad = i + len[node[i]]; cen = i; } }
- for (int i = size - 1; i >= 0; --i)
- cnt[par[i]] += cnt[i]; // update parent count }
int countUniquePalindromes(char s[]) {
- manachers(s); return size; }
int countAllPalindromes(char s[]) {
- manachers(s); int total = 0;
- for (int i = 0; i < size; i++) total += cnt[i];
- return total; }
// longest palindrome substring of s
std::string longestPalindrome(char s[]) {
- manachers(s);
- int n = strlen(s), cn = n * 2 + 1, mx = 0;
- for (int i = 1; i < cn; i++)
--- if (len[node[mx]] < len[node[i]])
----- mx = i;
- int pos = (mx - len[node[mx]]) / 2;
- return std::string(s + pos, s + pos + len[node[mx]]); }
```

9.6.2. *Eertree.*

```
struct node {
- int start, end, len, back_edge, *adj;
- node() {
--- adj = new int[26];
--- for (int i = 0; i < 26; ++i)  adj[i] = 0; }
- node(int start, int end, int len, int back_edge) :
------ start(start), end(end), len(len), back_edge(back_edge) {
--- adj = new int[26];
--- for (int i = 0; i < 26; ++i)  adj[i] = 0; } };
struct eertree {
- int ptr, cur_node;
- std::vector<node> tree;
- eertree () {
--- tree.push_back(node());
--- tree.push_back(node(0, 0, -1, 1));
--- tree.push_back(node(0, 0,  0, 1));
--- cur_node = 1;
--- ptr = 2; }
- int get_link(int temp, std::string &s, int i) {
--- while (true) {
----- int cur_len = tree[temp].len;
----- // don't return immediately if you want to
----- // get all palindromes; not recommended
----- if (i-cur_len-1 >= 0 and s[i] == s[i-cur_len-1])
------- return temp;
----- temp = tree[temp].back_edge; }
--- return temp; }
- void insert(std::string &s, int i) {
--- int temp = cur_node;
--- temp = get_link(temp, s, i);
--- if (tree[temp].adj[s[i] - 'a'] != 0) {
----- cur_node = tree[temp].adj[s[i] - 'a'];
----- return; }
--- ptr++;
--- tree[temp].adj[s[i] - 'a'] = ptr;
--- int len = tree[temp].len + 2;
--- tree.push_back(node(i-len+1, i, len, 0));
--- temp = tree[temp].back_edge;
--- cur_node = ptr;
--- if (tree[cur_node].len == 1) {
----- tree[cur_node].back_edge = 2;
----- return; }
--- temp = get_link(temp, s, i);
--- tree[cur_node].back_edge = tree[temp].adj[s[i]-'a']; }
- void insert(std::string &s) {
--- for (int i = 0; i < s.size(); ++i)
----- insert(s, i); } };
```

9.7. **Z Algorithm.** Find the longest common prefix of all substrings of s with itself in $O(n)$ time.

```
int z[N]; // z[i] = lcp(s, s[i:])
void compute_z(string s) {
- int n = s.length(); z[0] = 0;
- for (int i = 1, L = 0, R = 0; i < n; i++) {
--- if (i <= R) z[i] = min(R-i+1, z[i-L]);
--- while(i+z[i] < n && s[z[i]] == s[i+z[i]]) z[i]++;
--- if (i+z[i]-1 > R) L = i, R = i+z[i]-1; }
--- z[0] = n; }
```

9.8. **Booth's Minimum String Rotation**. Booth's Algo: Find the index of the lexicographically least string rotation in $O(n)$ time.

```
int f[N * 2];
int booth(string S) {
- S.append(S); // concatenate itself
- int n = S.length(), i, j, k = 0;
- memset(f, -1, sizeof(int) * n);
- for (j = 1; j < n; j++) {
--- i = f[j-k-1];
--- while (i != -1 && S[j] != S[k + i + 1]) {
----- if (S[j] < S[k + i + 1]) k = j - i - 1;
----- i = f[i];
--- } if (i == -1 && S[j] != S[k + i + 1]) {
----- if (S[j] < S[k + i + 1]) k = j;
----- f[j - k] = -1;
--- } else f[j - k] = i + 1;
- } return k; }
```

9.9. **Hashing.**

9.9.1. *Rolling Hash.*

```
int MAXN = 1e5+1, MOD = 1e9+7;
struct hasher {
- int n;
- std::vector<ll> *p_pow, *h_ans;
- hash(vi &s, vi primes) : n(primes.size()) {
--- p_pow = new std::vector<ll>[n];
--- h_ans = new std::vector<ll>[n];
--- for (int i = 0; i < n; ++i) {
----- p_pow[i] = std::vector<ll>(MAXN);
----- p_pow[i][0] = 1;
----- for (int j = 0; j+1 < MAXN; ++j)
------- p_pow[i][j+1] = (p_pow[i][j] * primes[i]) % MOD;
----- h_ans[i] = std::vector<ll>(MAXN);
----- h_ans[i][0] = 0;
----- for (int j = 0; j < s.size(); ++j)
------- h_ans[i][j+1] = (h_ans[i][j] +
----------------------- s[j] * p_pow[i][j]) % MOD; } } };
```

10. Other Algorithms

10.1. **2SAT.** Build the implication graph of the input by converting ORs $A \vee B$ to $!A \to B$ and $!B \to A$. This forms a bipartite graph. If there exists X such that both X and $!X$ are in the same strongly connected component, then there is no solution. Otherwise, iterate through the literals, arbitrarily assign a truth value to unassigned literals and propagate the values to its neighbors.

10.2. **DPLL Algorithm.** A SAT solver that can solve a random 1000-variable SAT instance within a second.

```
#define IDX(x) ((abs(x)-1)*2+((x)>0))
struct SAT {
- int n;
- vi cl, head, tail, val;
- vii log; vvi w, loc;
- SAT() : n(0) { }
- int var() { return ++n; }
- void clause(vi vars) {
--- set<int> seen; iter(it,vars) {
----- if (seen.find(IDX(*it)^1) != seen.end()) return;
----- seen.insert(IDX(*it)); }
--- head.push_back(cl.size());
--- iter(it,seen) cl.push_back(*it);
--- tail.push_back((int)cl.size() - 2); }
- bool assume(int x) {
--- if (val[x^1]) return false;
--- if (val[x]) return true;
--- val[x] = true; log.push_back(ii(-1, x));
--- rep(i,0,w[x^1].size()) {
----- int at = w[x^1][i], h = head[at], t = tail[at];
----- log.push_back(ii(at, h));
----- if (cl[t+1] != (x^1)) swap(cl[t], cl[t+1]);
----- while (h < t && val[cl[h]^1]) h++;
----- if ((head[at] = h) < t) {
------- w[cl[h]].push_back(w[x^1][i]);
------- swap(w[x^1][i--], w[x^1].back());
------- w[x^1].pop_back();
```

```
------- swap(cl[head[at]++], cl[t+1]);
----- } else if (!assume(cl[t])) return false; }
--- return true; }
- bool bt() {
--- int v = log.size(), x; ll b = -1;
--- rep(i,0,n) if (val[2*i] == val[2*i+1]) {
----- ll s = 0, t = 0;
----- rep(j,0,2) { iter(it,loc[2*i+j])
------- s+=1LL<<max(0,40-tail[*it]+head[*it]); swap(s,t); }
----- if (max(s,t) >= b) b = max(s,t), x = 2*i + (t>=s); }
--- if (b == -1 || (assume(x) && bt())) return true;
--- while (log.size() != v) {
----- int p = log.back().first, q = log.back().second;
----- if (p == -1) val[q] = false; else head[p] = q;
----- log.pop_back(); }
--- return assume(x^1) && bt(); }
- bool solve() {
--- val.assign(2*n+1, false);
--- w.assign(2*n+1, vi()); loc.assign(2*n+1, vi());
--- rep(i,0,head.size()) {
----- if (head[i] == tail[i]+2) return false;
----- rep(at,head[i],tail[i]+2) loc[cl[at]].push_back(i); }
--- rep(i,0,head.size()) if (head[i] < tail[i]+1) rep(t,0,2)
----- w[cl[tail[i]+t]].push_back(i);
--- rep(i,0,head.size()) if (head[i] == tail[i]+1)
----- if (!assume(cl[head[i]])) return false;
--- return bt(); }
- bool get_value(int x) { return val[IDX(x)]; } };
```

10.3. **Stable Marriage.** The Gale-Shapley algorithm for solving the stable marriage problem.

```
vi stable_marriage(int n, vvi &m, vvi &w) {
- std::queue<int> q;
- vi at(n, 0), eng(n, -1), res(n, -1); vvi inv(n, vi(n));
- for (int i = 0; i < n; ++i) {
--- for (int j = 0; j < n; ++j)
----- inv[i][w[i][j]] = j;
--- q.push(i); }
- while (!q.empty()) {
--- int curm = q.front(); q.pop();
--- for (int &i = at[curm]; i < n; i++) {
----- int curw = m[curm][i];
----- if (eng[curw] == -1) { }
----- else if (inv[curw][curm] < inv[curw][eng[curw]])
------- q.push(eng[curw]);
----- else continue;
----- res[eng[curw] = curm] = curw, ++i; break; } }
- return res; }
```

10.4. **Cycle-Finding.** An implementation of Floyd's Cycle-Finding algorithm.

```
ii find_cycle(int x0, int (*f)(int)) {
- int t = f(x0), h = f(t), mu = 0, lam = 1;
- while (t != h) t = f(t), h = f(f(h));
- h = x0;
- while (t != h) t = f(t), h = f(h), mu++;
- h = f(t);
- while (t != h) h = f(h), lam++;
- return ii(mu, lam); }
```

10.5. **Longest Increasing Subsequence.**

```
vi lis(vi &arr) {
- if (arr.empty()) return vi();
- vi seq, back(arr.size()), ans;
- for (int i = 0; i < arr.size()) {
--- int res = 0, lo = 1, hi = seq.size();
--- while (lo <= hi) {
----- int mid = (lo + hi) / 2;
----- if (arr[seq[mid-1]] < arr[i]) res = mid, lo = mid + 1;
----- else hi = mid - 1; }
--- if (res < seq.size()) seq[res] = i;
--- else seq.push_back(i);
--- back[i] = res == 0 ? -1 : seq[res-1]; }
- int at = seq.back();
- while (at != -1) ans.push_back(at), at = back[at];
- std::reverse(ans.begin(), ans.end());
- return ans; }
```

10.6. **Dates.** Functions to simplify date calculations.

```
int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
- return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
--- 367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
--- 3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
--- d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
- int x, n, i, j;
- x = jd + 68569;
- n = 4 * x / 146097;
- x -= (146097 * n + 3) / 4;
- i = (4000 * (x + 1)) / 1461001;
- x -= 1461 * i / 4 - 31;
- j = 80 * x / 2447;
- d = x - 2447 * j / 80;
- x = j / 11;
- m = j + 2 - 12 * x;
- y = 100 * (n - 49) + i + x; }
```

10.7. **Simulated Annealing.** An example use of Simulated Annealing to find a permutation of length n that maximizes $\sum_{i=1}^{n-1} |p_i - p_{i+1}|$.

```
double curtime() {
- return static_cast<double>(clock()) / CLOCKS_PER_SEC; }
int simulated_annealing(int n, double seconds) {
- default_random_engine rng;
- uniform_real_distribution<double> randfloat(0.0, 1.0);
- uniform_int_distribution<int> randint(0, n - 2);
- // random initial solution
- vi sol(n);
- for (int i = 0; i < n; ++i) sol[i] = i + 1;
- std::random_shuffle(sol.begin(), sol.end());
- // initialize score
- int score = 0;
- for (int i = 1; i < n; ++i)
--- score += std::abs(sol[i] - sol[i-1]);
- int iters = 0;
- double T0 = 100.0, T1 = 0.001,
----- progress = 0, temp = T0,
----- starttime = curtime();
- while (true) {
--- if (!(iters & ((1 << 4) - 1))) {
----- progress = (curtime() - starttime) / seconds;
----- temp = T0 * std::pow(T1 / T0, progress);
----- if (progress > 1.0) break; }
--- // random mutation
--- int a = std::randint(rng);
--- // compute delta for mutation
--- int delta = 0;
--- if (a > 0) delta += std::abs(sol[a+1] - sol[a-1])
------------------- - std::abs(sol[a] - sol[a-1]);
--- if (a+2 < n) delta += std::abs(sol[a] - sol[a+2])
--------------------- - std::abs(sol[a+1] - sol[a+2]);
--- // maybe apply mutation
--- if (delta >= 0 || randfloat(rng) < exp(delta / temp)) {
----- std::swap(sol[a], sol[a+1]);
----- score += delta;
----- // if (score >= target) return;
--- }
--- iters++; }
- return score; }
```

10.8. **Simplex.**

```
// Two-phase simplex algorithm for solving linear programs
// of the form
//     maximize     c^T x
//     subject to   Ax <= b
//                  x >= 0
// INPUT: A -- an m x n matrix
//        b -- an m-dimensional vector
//        c -- an n-dimensional vector
//        x -- a vector where the optimal solution will be
//             stored
// OUTPUT: value of the optimal solution (infinity if
//                    unbounded above, nan if infeasible)
// To use this code, create an LPSolver object with A, b,
// and c as arguments.  Then, call Solve(x).

typedef long double DOUBLE;
typedef vector<DOUBLE> VD;
typedef vector<VD> VVD;
typedef vector<int> vi;
const DOUBLE EPS = 1e-9;
struct LPSolver {
 int m, n;
 vi B, N;
 VVD D;
 LPSolver(const VVD &A, const VD &b, const VD &c) :
 - m(b.size()), n(c.size()),
 - N(n + 1), B(m), D(m + 2, VD(n + 2)) {
 - for (int i = 0; i < m; i++) for (int j = 0; j < n; j++)
 --- D[i][j] = A[i][j];
```

```
- for (int i = 0; i < m; i++) { B[i] = n + i; D[i][n] = -1;
--- D[i][n + 1] = b[i]; }
- for (int j = 0; j < n; j++) { N[j] = j; D[m][j] = -c[j]; }
- N[n] = -1; D[m + 1][n] = 1; }
 void Pivot(int r, int s) {
- double inv = 1.0 / D[r][s];
- for (int i = 0; i < m + 2; i++) if (i != r)
-- for (int j = 0; j < n + 2; j++) if (j != s)
--- D[i][j] -= D[r][j] * D[i][s] * inv;
- for (int j = 0; j < n + 2; j++) if (j != s) D[r][j] *= inv;
- for (int i = 0; i < m + 2; i++) if (i != r) D[i][s] *= -inv;
- D[r][s] = inv;
- swap(B[r], N[s]); }
 bool Simplex(int phase) {
- int x = phase == 1 ? m + 1 : m;
- while (true) {
-- int s = -1;
-- for (int j = 0; j <= n; j++) {
--- if (phase == 2 && N[j] == -1) continue;
--- if (s == -1 || D[x][j] < D[x][s] ||
------ D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
-- if (D[x][s] > -EPS) return true;
-- int r = -1;
-- for (int i = 0; i < m; i++) {
--- if (D[i][s] < EPS) continue;
--- if (r == -1 || D[i][n + 1] / D[i][s] < D[r][n + 1] /
------ D[r][s] || (D[i][n + 1] / D[i][s]) == (D[r][n + 1] /
------ D[r][s]) && B[i] < B[r]) r = i; }
-- if (r == -1) return false;
-- Pivot(r, s); } }
 DOUBLE Solve(VD &x) {
- int r = 0;
- for (int i = 1; i < m; i++) if (D[i][n + 1] < D[r][n + 1])
--- r = i;
- if (D[r][n + 1] < -EPS) {
-- Pivot(r, n);
-- if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
---- return -numeric_limits<DOUBLE>::infinity();
-- for (int i = 0; i < m; i++) if (B[i] == -1) {
--- int s = -1;
--- for (int j = 0; j <= n; j++)
---- if (s == -1 || D[i][j] < D[i][s] ||
------- D[i][j] == D[i][s] && N[j] < N[s])
------ s = j;
--- Pivot(i, s); } }
- if (!Simplex(2)) return numeric_limits<DOUBLE>::infinity();
- x = VD(n);
- for (int i = 0; i < m; i++) if (B[i] < n)
--- x[B[i]] = D[i][n + 1];
- return D[m][n + 1]; } };
```

10.9. **Fast Input Reading.** If input or output is huge, sometimes it is beneficial to optimize the input reading/output writing. This can be achieved by reading all input in at once (using fread), and then parsing it manually. Output can also be stored in an output buffer and then dumped once in the end (using fwrite). A simpler, but still effective, way to achieve speed is to use the following input reading method.

```
void readn(register int *n) {
- int sign = 1;
- register char c;
- *n = 0;
- while((c = getc_unlocked(stdin)) != '\n') {
--- switch(c) {
----- case '-': sign = -1; break;
----- case ' ': goto hell;
----- case '\n': goto hell;
----- default: *n *= 10; *n += c - '0'; break; } }
hell:
- *n *= sign; }
```

10.10. **128-bit Integer.** GCC has a 128-bit integer data type named `__int128`. Useful if doing multiplication of 64-bit integers, or something needing a little more than 64-bits to represent. There's also `__float128`.

10.11. **Bit Hacks.**

```
int snoob(int x) {
- int y = x & -x, z = x + y;
- return z | ((x ^ z) >> 2) / y; }
```

11. Misc

11.1. **Debugging Tips.**

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
  - Getting `NaN`? Make sure `acos` etc. are not getting values out of their range (perhaps `1+eps`).
  - Rounding negative numbers?
  - Outputting in scientific notation?
- Wrong Answer?
  - Read the problem statement again!
  - Are multiple test cases being handled correctly? Try repeating the same test case many times.
  - Integer overflow?
  - Think very carefully about boundaries of all input parameters
  - Try out possible edge cases:
    * $n = 0, n = -1, n = 1, n = 2^{31} - 1$ or $n = -2^{31}$
    * List is empty, or contains a single element
    * n is even, n is odd
    * Graph is empty, or contains a single vertex
    * Graph is a multigraph (loops or multiple edges)
    * Polygon is concave or non-simple
  - Is initial condition wrong for small cases?
  - Are you sure the algorithm is correct?
  - Explain your solution to someone.
  - Are you using any functions that you don't completely understand? Maybe STL functions?
  - Maybe you (or someone else) should rewrite the solution?
  - Can the input line be empty?
- Run-Time Error?
  - Is it actually Memory Limit Exceeded?

11.2. **Solution Ideas.**

- Dynamic Programming
  - Parsing CFGs: CYK Algorithm
  - Drop a parameter, recover from others
  - Swap answer and a parameter
  - When grouping: try splitting in two
  - 2^k trick
  - When optimizing
    * Convex hull optimization
      · $\mathrm{dp}[i] = \min_{j<i}\{\mathrm{dp}[j] + b[j] \times a[i]\}$
      · $b[j] \geq b[j+1]$
      · optionally $a[i] \leq a[i+1]$
      · $O(n^2)$ to $O(n)$
    * Divide and conquer optimization
      · $\mathrm{dp}[i][j] = \min_{k<j}\{\mathrm{dp}[i-1][k] + C[k][j]\}$
      · $A[i][j] \leq A[i][j+1]$
      · $O(kn^2)$ to $O(kn \log n)$
      · sufficient: $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$, $a \leq b \leq c \leq d$ (QI)
    * Knuth optimization
      · $\mathrm{dp}[i][j] = \min_{i<k<j}\{\mathrm{dp}[i][k] + \mathrm{dp}[k][j] + C[i][j]\}$
      · $A[i][j-1] \leq A[i][j] \leq A[i+1][j]$
      · $O(n^3)$ to $O(n^2)$
      · sufficient: QI and $C[b][c] \leq C[a][d]$, $a \leq b \leq c \leq d$
- Greedy
- Randomized
- Optimizations
  - Use bitset (/64)
  - Switch order of loops (cache locality)
- Process queries offline
  - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
  - Mo's algorithm
  - Sqrt decomposition
  - Store 2^k jump pointers
- Data structure techniques
  - Sqrt buckets
  - Store 2^k jump pointers
  - 2^k merging trick
- Counting
  - Inclusion-exclusion principle
  - Generating functions
- Graphs
  - Can we model the problem as a graph?
  - Can we use any properties of the graph?
  - Strongly connected components
  - Cycles (or odd cycles)
  - Bipartite (no odd cycles)
    * Bipartite matching
    * Hall's marriage theorem
    * Stable Marriage
  - Cut vertex/bridge
  - Biconnected components
  - Degrees of vertices (odd/even)
  - Trees
    * Heavy-light decomposition
    * Centroid decomposition
    * Least common ancestor

    - Centers of the tree
  - Eulerian path/circuit
  - Chinese postman problem
  - Topological sort
  - (Min-Cost) Max Flow
  - Min Cut
    - Maximum Density Subgraph
  - Huffman Coding
  - Min-Cost Arborescence
  - Steiner Tree
  - Kirchoff's matrix tree theorem
  - Prüfer sequences
  - Lovász Toggle
  - Look at the DFS tree (which has no cross-edges)
  - Is the graph a DFA or NFA?
    - Is it the Synchronizing word problem?
- Mathematics
  - Is the function multiplicative?
  - Look for a pattern
  - Permutations
    - Consider the cycles of the permutation
  - Functions
    - Sum of piecewise-linear functions is a piecewise-linear function
    - Sum of convex (concave) functions is convex (concave)
  - Modular arithmetic
    - Chinese Remainder Theorem
    - Linear Congruence
  - Sieve
  - System of linear equations
  - Values too big to represent?
    - Compute using the logarithm
    - Divide everything by some large value
  - Linear programming
    - Is the dual problem easier to solve?
  - Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
  - 2-SAT
  - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ($\log(n)$)
- Strings
  - Trie (maybe over something weird, like bits)
  - Suffix array
  - Suffix automaton (+DP?)
  - Aho-Corasick
  - eerTree
  - Work with $S+S$
- Hashing
- Euler tour, tree to array
- Segment trees
  - Lazy propagation
  - Persistent
  - Implicit
  - Segment tree of X
- Geometry
  - Minkowski sum (of convex sets)
  - Rotating calipers
  - Sweep line (horizontally or vertically?)
  - Sweep angle
  - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem
  - Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)
- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

12. Formulas

- **Legendre symbol:** $\left(\frac{a}{b}\right) = a^{(b-1)/2} \pmod b$, b odd prime.
- **Heron's formula:** A triangle with side lengths a, b, c has area $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.
- **Pick's theorem:** A polygon on an integer grid strictly containing i lattice points and having b lattice points on the boundary has area $i + \frac{b}{2} - 1$. (Nothing similar in higher dimensions)
- **Euler's totient:** The number of integers less than n that are coprime to n are $n\prod_{p|n}\left(1 - \frac{1}{p}\right)$ where each p is a distinct prime factor of n.
- **König's theorem:** In any bipartite graph $G = (L \cup R, E)$, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover. Let U be the set of unmatched vertices in L, and Z be the set of vertices that are either in U or are connected to U by an alternating path. Then $K = (L \setminus Z) \cup (R \cap Z)$ is the minimum vertex cover.
- A minumum Steiner tree for n vertices requires at most $n-2$ additional Steiner vertices.
- The number of vertices of a graph is equal to its minimum vertex cover number plus the size of a maximum independent set.
- **Lagrange polynomial** through points $(x_0, y_0), \ldots, (x_k, y_k)$ is $L(x) = \sum_{j=0}^{k} y_j \prod_{\substack{0 \le m \le k \\ m \ne j}} \frac{x - x_m}{x_j - x_m}$
- **Hook length formula:** If λ is a Young diagram and $h_\lambda(i,j)$ is the hook-length of cell (i,j), then then the number of Young tableux $d_\lambda = n!/\prod h_\lambda(i,j)$.
- **Möbius inversion formula:** If $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(d) f(n/d)$. If $f(n) = \sum_{m=1}^{n} g(\lfloor n/m \rfloor)$, then $g(n) = \sum_{m=1}^{n} \mu(m) f(\lfloor \frac{n}{m} \rfloor)$.
- #primitive pythagorean triples with hypotenuse $< n$ approx $n/(2\pi)$.
- **Frobenius Number:** largest number which can't be expressed as a linear combination of numbers $a_1, \ldots, a_n$ with non-negative coefficients. $g(a_1, a_2) = a_1a_2 - a_1 - a_2$, $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d-1)$. An integer $x > (\max_i a_i)^2$ can be expressed in such a way iff. $x \mid \gcd(a_1, \ldots, a_n)$.

12.1. Physics.

- **Snell's law:** $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$

12.2. Markov Chains.

A Markov Chain can be represented as a weighted directed graph of states, where the weight of an edge represents the probability of transitioning over that edge in one timestep. Let $P^{(m)} = (p_{ij}^{(m)})$ be the probability matrix of transitioning from state i to state j in m timesteps, and note that $P^{(1)}$ is the adjacency matrix of the graph. **Chapman-Kolmogorov:** $p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$. It follows that $P^{(m+n)} = P^{(m)}P^{(n)}$ and $P^{(m)} = P^m$. If $p^{(0)}$ is the initial probability distribution (a vector), then $p^{(0)}P^{(m)}$ is the probability distribution after m timesteps.

The return times of a state i is $R_i = \{m \mid p_{ii}^{(m)} > 0\}$, and i is *aperiodic* if $\gcd(R_i) = 1$. A MC is aperiodic if any of its vertices is aperiodic. A MC is *irreducible* if the corresponding graph is strongly connected.

A distribution π is stationary if $\pi P = \pi$. If MC is irreducible then $\pi_i = 1/\mathbb{E}[T_i]$, where T_i is the expected time between two visits at i. π_j/π_i is the expected number of visits at j in between two consecutive visits at i. A MC is *ergodic* if $\lim_{m\to\infty} p^{(0)}P^m = \pi$. A MC is ergodic iff. it is irreducible and aperiodic.

A MC for a random walk in an undirected weighted graph (unweighted graph can be made weighted by adding 1-weights) has $p_{uv} = w_{uv}/\sum_x w_{ux}$. If the graph is connected, then $\pi_u = \sum_x w_{ux}/\sum_v \sum_x w_{vx}$. Such a random walk is aperiodic iff. the graph is not bipartite.

An *absorbing* MC is of the form $P = \begin{pmatrix} Q & R \\ 0 & I_r \end{pmatrix}$. Let $N = \sum_{m=0}^{\infty} Q^m = (I_t - Q)^{-1}$. Then, if starting in state i, the expected number of steps till absorption is the i-th entry in $N1$. If starting in state i, the probability of being absorbed in state j is the (i,j)-th entry of NR.

Many problems on MC can be formulated in terms of a system of recurrence relations, and then solved using Gaussian elimination.

12.3. Burnside's Lemma.

Let G be a finite group that acts on a set X. For each g in G let X^g denote the set of elements in X that are fixed by g. Then the number of orbits

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

$$Z(S_n) = \frac{1}{n}\sum_{l=1}^{n} a_l Z(S_{n-l})$$

12.4. Bézout's identity.

If (x, y) is any solution to $ax + by = d$ (e.g. found by the Extended Euclidean Algorithm), then all solutions are given by

$$\left(x + k\frac{b}{\gcd(a,b)}, y - k\frac{a}{\gcd(a,b)}\right)$$

12.5. Misc.

12.5.1. *Determinants and PM.*

$$det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$
$$perm(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$
$$\begin{aligned} pf(A) &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)} \\ &= \sum_{M \in \text{PM}(n)} \text{sgn}(M) \prod_{(i,j) \in M} a_{i,j} \end{aligned}$$

12.5.2. *BEST Theorem.* Count directed Eulerian cycles. Number of OST given by Kirchoff's Theorem (remove r/c with root) $\#\text{OST}(G, r) \cdot \prod_v (d_v - 1)!$

12.5.3. *Primitive Roots.* Only exists when n is $2, 4, p^k, 2p^k$, where p odd prime. Assume n prime. Number of primitive roots $\phi(\phi(n))$ Let g be primitive root. All primitive roots are of the form g^k where $k, \phi(p)$ are coprime.
k-roots: $g^{i \cdot \phi(n)/k}$ for $0 \le i < k$

12.5.4. *Sum of primes.* For any multiplicative f:

$$S(n, p) = S(n, p-1) - f(p) \cdot (S(n/p, p-1) - S(p-1, p-1))$$

12.5.5. *Floor.*

$$\begin{aligned} \lfloor \lfloor x/y \rfloor / z \rfloor &= \lfloor x/(yz) \rfloor \\ x\%y &= x - y \lfloor x/y \rfloor \end{aligned}$$

13. Other Combinatorics Stuff

Catalan	$C_0 = 1,\ C_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \frac{4n-2}{n+1} C_{n-1}$	
Stirling 1st kind	$\left[{0 \atop 0}\right] = 1,\ \left[{n \atop 0}\right] = \left[{0 \atop n}\right] = 0,\ \left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$	#perms of n objs with exactly k cycles
Stirling 2nd kind	$\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1,\ \left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$	#ways to partition n objs into k nonempty sets
Euler	$\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1,\ \left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$	#perms of n objs with exactly k ascents
Euler 2nd Order	$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-k-1)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$	#perms of $1, 1, 2, 2, ..., n, n$ with exactly k ascents
Bell	$B_1 = 1,\ B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^{n} \left\{{n \atop k}\right\}$	#partitions of $1..n$ (Stirling 2nd, no limit on k)

#labeled rooted trees	n^{n-1}
#labeled unrooted trees	n^{n-2}
#forests of k rooted trees	$\frac{k}{n}\binom{n}{k} n^{n-k}$
$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$	$\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$
$!n = n \times !(n-1) + (-1)^n$	$!n = (n-1)(!(n-1) + !(n-2))$
$\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n}$	$\sum_i \binom{n-i}{i} = F_{n+1}$
$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$	$x^k = \sum_{i=0}^{k} i! \left\{{k \atop i}\right\} \binom{x}{i} = \sum_{i=0}^{k} \left\langle{k \atop i}\right\rangle \binom{x+i}{k}$
$a \equiv b \pmod{x, y} \Rightarrow a \equiv b \pmod{\operatorname{lcm}(x, y)}$	$\sum_{d \mid n} \phi(d) = n$
$ac \equiv bc \pmod{m} \Rightarrow a \equiv b \pmod{\frac{m}{\gcd(c,m)}}$	$(\sum_{d \mid n} \sigma_0(d))^2 = \sum_{d \mid n} \sigma_0(d)^3$
p prime $\Leftrightarrow (p-1)! \equiv -1 \pmod{p}$	$\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$
$\sigma_x(n) = \prod_{i=0}^{r} \frac{p_i^{(a_i+1)x} - 1}{p_i^x - 1}$	$\sigma_0(n) = \prod_{i=0}^{r} (a_i + 1)$
$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$	
$2^{\omega(n)} = O(\sqrt{n})$	$\sum_{i=1}^{n} 2^{\omega(i)} = O(n \log n)$
$d = v_i t + \frac{1}{2} a t^2$	$v_f^2 = v_i^2 + 2ad$
$v_f = v_i + at$	$d = \frac{v_i + v_f}{2} t$

13.1. **The Twelvefold Way.** Putting n balls into k boxes.

Balls	same	distinct	same	distinct	
Boxes	same	same	distinct	distinct	Remarks
-	$\mathrm{p}_k(n)$	$\sum_{i=0}^{k} \left\{{n \atop i}\right\}$	$\binom{n+k-1}{k-1}$	k^n	$\mathrm{p}_k(n)$: #partitions of n into $\le k$ positive parts
size ≥ 1	$\mathrm{p}(n, k)$	$\left\{{n \atop k}\right\}$	$\binom{n-1}{k-1}$	$k! \left\{{n \atop k}\right\}$	$\mathrm{p}(n, k)$: #partitions of n into k positive parts
size ≤ 1	$[n \le k]$	$[n \le k]$	$\binom{k}{n}$	$n! \binom{k}{n}$	$[cond]$: 1 if $cond = true$, else 0