Assignment 10.2 (for Lecture 10C) Solutions

November 23, 2024

A1. The set A consists of all positive rationals whose square is strictly greater than 3. Thus we can find a lower bound by finding a rational less than or equal to $\sqrt{3}$ i.e. a rational whose square is less than or equal to 3. One such lower bound is $^{1732}/_{1000}$, which is less than or equal to $\sqrt{3}$. Let's now show that one can always find a larger number that is still less than $\sqrt{3}$. Suppose we have one such lower bound denoted by $^p/_q \leq 3$. Actually, from earlier, there is no such rational such that $^p/_q = 3$. Thus, our lower bound is strictly less than 3. We can construct another number of the form $(^p/_q + 3)/2$. Let's check that it is greater than $^p/_q$, but strictly less than 3. $^p/_q < (^p/_q + 3)/2 = \frac{p+3q}{2q} \iff \frac{1}{2}\frac{p}{q} < \frac{3}{2} \iff \frac{p}{q} < 3$, hence it is strictly greater than $^p/_q$. $\frac{p+3q}{2q} < 3 \iff \frac{p}{2q} < \frac{3}{2} \iff \frac{p}{q} < 3$. Hence, it is strictly less than 3. Since we can do this for any $^p/_q < 3$, there is no greatest lower bound.

A2. Let $x = 1, y = 1/z > 0$, where $z = s - r$. Then by the Archimedean property, there exists an $n \in \mathbb{N}$ such that $n > 1/z \iff n \cdot z > 1 \iff n \cdot s - n \cdot r > 1$. Since $n \cdot s - n \cdot r$ is at least greater than one, we must have an integer between the two such that $n \cdot r < m < n \cdot s$, and so we have $r < \frac{m}{n} < s$, which was to be proved.

A3. The statement $a_n \to a$ means that $(\forall\epsilon > 0)(\exists n)[m > n \Rightarrow |a_m - a| < \epsilon]$, and it's negation means that $(\exists\epsilon > 0)(\forall n)[m > n) \wedge |a_m - a| \geq \epsilon]$.

A4. $\left(\frac{n}{n+1}\right)^2 \to 1$ means that $(\forall\epsilon > 0)(\exists n)[m > n \Rightarrow \left|\left(\frac{m}{m+1}\right)^2 - 1\right| < \epsilon$. If we manipulate the inside expression, we obtain that $\left|\left(\frac{m}{m+1}\right)^2 - 1\right| < \epsilon \iff \left|-\frac{2m+1}{(m+1)^2}\right| < \epsilon \iff \left|\frac{2m+1}{(m+1)^2}\right| < \epsilon$. Hence, let's try to find such an n for some arbitrarily given ϵ.

A5. $\frac{1}{n^2} \to 0$ means that $(\forall\epsilon > 0)(\exists n)[m > n \Rightarrow \frac{1}{m^2} < \epsilon]$. Let $n = \frac{1}{\sqrt{\epsilon}}$, then we have that $\epsilon = \frac{1}{n^2} > \frac{1}{m^2}$, which was to be proved.

A6. $\frac{1}{2^n} \to 0$ means that $(\forall\epsilon > 0)(\exists n)[m > n \Rightarrow \frac{1}{2^m} < \epsilon]$. Let $n = -log_2(\epsilon)$, then we have that $\epsilon = \frac{1}{2^n} > \frac{1}{2^m}$, which was to be proved.

A7. In simple terms, a sequence is said to tend to infinity if for every number I give you, you can find a term in the sequence that is bigger than it. Formally, this is $(\forall K \in \mathbb{R})(\exists n \in \mathbb{N})[m > n \Rightarrow a_m > K]$.

(a) Proof that $\{n\}_{n=1}^{\infty} = \infty$. Given a number K, choose $n = \lceil K \rceil$. Since we have that $n \geq K$, and that $m > n$, we must have $m > K$, which was to be proved.

(b) Proof that $\{2^n\}_{n=1}^{\infty} = \infty$. Given a number K, choose $n = \lceil \lg(K) \rceil$. Since we have that $2^n = 2^{\lceil \lg(K) \rceil} \geq K$, and that $m > n$, we must have that $2^m > K$, which was to be proved.

A8. A least upper bound of some set A (take LUB$\in \mathbb{R}$) will be greater than or equal to every element in A, and is less than or equal to any other upper bound. If a is to be a least upper bound of $\{a_n\}_{n \in \mathbb{N}}$, then we must necessarily have that $a \geq a_i \in \{a_n\}$.

A9. Let $\{a_n\}_{n=1}^{\infty}$be an increasing sequence bounded from above. Define $L = sup\{a_n : n \in \mathbb{N}\}$ and let $\epsilon > 0$ be arbitrary. Since L is the least upper bound of the sequence, $L-\epsilon$ is not an upper bound. Therefore, there exists an index $N \in \mathbb{N}$ such that $a_N > L - \epsilon$. Because the sequence is increasing, for all $n \geq N : a_n \geq a_N > L - \epsilon$. Also, since L is an upper bound for $\{a_n\}$, we have $a_n \leq L$ for all n. Combining these inequalities, we obtain for all $n \geq N : L - \epsilon < a_n \leq L$. This implies that $0 \leq L - a_n < \epsilon$ i.e. $|a_n - L| < \epsilon$. Since ϵ was chosen to be arbitrary, we can make it as small as we want, which is precisely the definition of the limit i.e. $\lim_{n \to \infty} a_n = L$.