Assignment 10.1 (for Lecture 10A) Solutions

October 1, 2022

A1. If we let the empty set and a single element also be intervals, then from the definition of intersections, we have $A \cap B = \{x \mid (x \in A \wedge x \in B)$. There are four cases to considers. The first case involves an intersection in which no elements coincide, and thus we have an empty set for the intersection. The second case involves an intersection in which only one element is shared. The third case involves an intersection in which the sets overlap, with more than one element contained in the intersection. This is an interval. The fourth and final case involves an intersection in which one set completely envelopes the other set, and this is also an intersection.

A2.

(a) $(-\infty, 1) \cup (3, +\infty)$

(b) $(-\infty, 1] \cup (7, +\infty)$

(c) $(-\infty, 5] \cup (8, +\infty)$

(d) $(-\infty, 3] \cup (8, +\infty)$

(e) $[3, +\infty)$

(f) $(-\infty, \pi) \cup (\pi, +\infty)$

(g) $\{4\}$

(h) $\emptyset$

(i) $(-\infty, 7] \cup [8, +\infty)$

(j) $(5, 7]$

A3. If a set of integers/rationals/reals has an upper bound, call it $\alpha \in \mathbb{N}, \mathbb{Q}, \mathbb{R}$, then one can find another upper bound $\alpha+1$ which is greater than α. Since this can be done with any upper bound, there must be infinitely many upper bounds.

A4. If a set of integers/rationals/reals has a least upper bound, call it α, then for any upper bound β of the set , $\alpha \leq \beta$. Suppose there exist two least upper bounds, α_1 and α_2. Since they are least upper bounds, they are also upper bounds. Thus, they satisfy $(\alpha_1 \leq \alpha_2) \wedge$

($\alpha_2 \leq \alpha_1$), but this must mean $\alpha_1 = \alpha_2$; hence the least upper bound is unique.

A5. The definition of least upper bound states that b is the least upper bound if $a \leq b$, $\forall a \in A$ and for any other upper bound c, $b \leq c$. Property (a) is identical to the first part of the definition. Property (b) identical to saying that if there's an element which is strictly less than b, then it must be the case that there's an element in A that is strictly greater than it. Part two of the definition is the statement that if c is an upper bound, then $b \leq c$. The contrapositive is the statement that $c < b \Rightarrow \neg UpperBound(c)$. If c is not an upper bound, then it must be the case that $a > c$ for some $a \in A$. This is equivalent to what we needed to prove, and thus the proof is complete.

A6.

A7. The set of all natural numbers has no upper bound.

A8. If the set A is finite, then it can be enumerated in increasing order as follows, where $a_i \in A$, $\{a_1, a_2, a_3, \ldots, a_n\}$. Then a_n is the least upper bound because $a_n \geq a_i$,$\forall a_i \in A$ and $a_n \leq b$, where b is any other element in the $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ that is an upper bound.

A9. $LUB(a,b) = b$. $LUB[a,b] = b$. $Max(a,b)$ has no maximum element. $Max[a,b] = b$.

A10. If A is defined in such a way, then it's largest element may be constructed by letting $x = b - \epsilon, y = a + \epsilon$, such that $|x - y| = |(b - \epsilon) - (a + \epsilon)| = |(b - a) - 2\epsilon| = (b - a) - 2\epsilon$, where ϵ tends to 0 from above. If one takes $b - a$, we can show that it is an upper bound because $b - a \geq b - a - 2\epsilon \iff \epsilon \geq 0$. It is also the least upper bound because any element smaller than $b - a$ but greater than 0 will be inside A.

A11. A set A has a lower bound b if $b \leq a$, $\forall a \in A$.

A12. A set A has a greatest lower bound b if b is a lower bound and for any other lower bound c, $c \leq b$.

A13.

A14.

A15.

A16. The Completeness Property for the integers states that every non-empty set of integers that contains a lower/upper bound contains a greatest/least lower/upper bound.